ON THE AUTOMORPHISM GROUP OF A CLOSED G_2-STRUCTURE

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ABSTRACT. We study the automorphism group of a compact 7-manifold M endowed with a closed non-parallel G_2-structure, showing that its identity component is abelian with dimension bounded by $\min\{6, b_2(M)\}$. This implies the non-existence of compact homogeneous manifolds endowed with an invariant closed non-parallel G_2-structure. We also discuss some relevant examples.

1. Introduction

A seven-dimensional smooth manifold M admits a G_2-structure if the structure group of its frame bundle can be reduced to the exceptional Lie group $G_2 \subset SO(7)$. Such a reduction is characterized by the existence of a global 3-form $\varphi \in \Omega^3(M)$ satisfying a suitable non-degeneracy condition and giving rise to a Riemannian metric g_φ and to a volume form dV_φ on M via the identity

$$g_\varphi(X,Y)\, dV_\varphi = \frac{1}{6}\, \iota_X\varphi \wedge \iota_Y\varphi \wedge \varphi,$$

for all $X, Y \in \mathfrak{X}(M)$ (see e.g. [2, 13]).

By [10], the intrinsic torsion of a G_2-structure φ can be identified with the covariant derivative $\nabla^{g_\varphi}\varphi$, and it vanishes identically if and only if both $d\varphi = 0$ and $d *_\varphi \varphi = 0$, $*_\varphi$ being the Hodge operator defined by g_φ and dV_φ. On a compact manifold, this last fact is equivalent to $\Delta_\varphi \varphi = 0$, where $\Delta_\varphi = d^*d + dd^*$ is the Hodge Laplacian of g_φ. A G_2-structure φ satisfying any of these conditions is said to be *parallel* and its associated Riemannian metric g_φ has holonomy contained in G_2. Consequently, g_φ is Ricci-flat and the automorphism group $\mathrm{Aut}(M,\varphi) := \{f \in \mathrm{Diff}(M) \mid f^*\varphi = \varphi\}$ of (M,φ) is finite when M is compact and $\mathrm{Hol}(g_\varphi) = G_2$.

Parallel G_2-structures play a central role in the construction of compact manifolds with holonomy G_2, and various known methods to achieve this result involve *closed* G_2-structures, i.e., those whose defining 3-form φ satisfies $d\varphi = 0$ (see e.g. [2, 3, 15, 19]).

A G_2-structure whose defining 3-form φ satisfies the equation $d *_\varphi \varphi = 0$ is called *co-closed*. On every compact 7-manifold admitting G_2-structures there exists a co-closed one (cf. [6]), while general results on the existence of closed G_2-structures are not known.

Due to the recent developments on the G_2-Laplacian flow and related open problems [11, 14, 16, 17, 19, 20, 21], it is of foremost interest to provide examples of compact manifolds admitting closed G_2-structures. Most of the known examples consist of simply connected Lie groups endowed with a left-invariant closed G_2-form φ [5, 8, 9, 12, 17]. Compact

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locally homogeneous examples can be obtained considering the quotient of such groups by a co-compact discrete subgroup, whenever this exists. Further non-homogeneous closed G_2-structures on the 7-torus can be constructed starting from the symplectic half-flat SU(3)-structure on $\mathbb{T}^6$ described in [7, Ex. 5.1] (see Example 2.4 for details).

Up to now, the existence of compact homogeneous 7-manifolds admitting an invariant closed non-parallel G_2-structure was not known (cf. [17, Question 3.1] and [18, 26]). Moreover, among the G_2-manifolds acted on by a cohomogeneity one simple group of automorphisms studied in [4] no compact examples admitting a closed G_2-structure occur.

In this short note, we investigate the properties of the automorphism group $\mathrm{Aut}(M,\varphi)$ of a compact 7-manifold M endowed with a closed non-parallel G_2-structure φ. Our main results are contained in Theorem 2.1, where we show that the identity component $\mathrm{Aut}^0(M,\varphi)$ is necessarily abelian with dimension bounded by $\min\{6, b_2(M)\}$. In particular, this answers negatively [17, Question 3.1] and explains why compact examples cannot occur in [4]. Moreover, we also prove some interesting properties of the automorphism group action, and we describe some relevant examples.

These results shed some light on the structure of compact 7-manifolds admitting closed G_2-structures and can be of some help in the construction of new examples.

2. The automorphism group

Let M be a seven-dimensional manifold endowed with a closed G_2-structure φ, and consider its automorphism group

$$\mathrm{Aut}(M,\varphi) \coloneqq \{f \in \mathrm{Diff}(M) \mid f^*\varphi = \varphi\}.$$

Notice that $\mathrm{Aut}(M,\varphi)$ is a closed Lie subgroup of the isometry group $\mathrm{Iso}(M, g_\varphi)$ of g_φ, and that the Lie algebra of its identity component $\mathrm{G} \coloneqq \mathrm{Aut}^0(M,\varphi)$ is

$$\mathfrak{g} = \{X \in \mathfrak{X}(M) \mid \mathcal{L}_X\varphi = 0\}.$$

In particular, every $X \in \mathfrak{g}$ is a Killing vector field for the metric g_φ (cf. [19, Lemma 9.3]).

When M is compact, the Lie group $\mathrm{Aut}(M,\varphi) \subset \mathrm{Iso}(M, g_\varphi)$ is also compact, and we can show the following.

Theorem 2.1. *Let M be a compact seven-dimensional manifold endowed with a closed non-parallel G_2-structure φ. Then, there exists an injective map*

$$F : \mathfrak{g} \to \mathscr{H}^2(M), \quad X \mapsto \iota_X\varphi,$$

where $\mathscr{H}^2(M)$ is the space of Δ_φ-harmonic 2-forms. As a consequence, the following properties hold:

1) $\dim(\mathfrak{g}) \leq b_2(M)$*;*
2) $\mathfrak{g}$ *is abelian with* $\dim(\mathfrak{g}) \leq 6$*;*
3) for every $p \in M$, the isotropy subalgebra $\mathfrak{g}_p$ has dimension $\dim(\mathfrak{g}_p) \leq 2$*, with equality only when* $\dim(\mathfrak{g}) = 2, 3$*;*
4) the G*-action is free when* $\dim(\mathfrak{g}) \geq 5$*. Moreover, when* $\dim(\mathfrak{g}) = 6$ *the manifold M is diffeomorphic to* $\mathbb{T}^7$*.*

Proof. Let $X \in \mathfrak{g}$. Then, $0 = \mathcal{L}_X\varphi = d(\iota_X\varphi)$, as φ is closed. We claim that $\iota_X\varphi$ is co-closed (see also [19, Lemma 9.3]). Indeed, for every $p \in M$ the 2-form $\iota_X\varphi|_p$ belongs to the unique seven-dimensional G_2-irreducible submodule $\Lambda^2_7(T^*_pM) \subset \Lambda^2(T^*_pM)$, and therefore (see e.g. [1, p. 541]) we have

$$\iota_X\varphi \wedge \varphi = 2 *_\varphi (\iota_X\varphi),$$

from which it follows that

$$0 = d(\iota_X\varphi \wedge \varphi) = 2\, d *_\varphi (\iota_X\varphi).$$

Consequently, the 2-form $\iota_X\varphi$ is Δ_φ-harmonic and F is the restriction of the injective map $Z \mapsto \iota_Z\varphi$ to $\mathfrak{g}$. From this 1) follows.

As for 2), we begin observing that $\mathcal{L}_Y(\iota_X\varphi) = 0$ for all $X, Y \in \mathfrak{g}$, since every Killing field on a compact manifold preserves every harmonic form. Hence, we have

$$0 = \mathcal{L}_Y(\iota_X\varphi) = \iota_{[Y,X]}\varphi + \iota_X(\mathcal{L}_Y\varphi) = \iota_{[Y,X]}\varphi.$$

This proves that $\mathfrak{g}$ is abelian, the map $Z \mapsto \iota_Z\varphi$ being injective. Now, G is compact abelian and it acts effectively on the compact manifold M. Therefore, the principal isotropy is trivial and $\dim(\mathfrak{g}) \leq 7$. When $\dim(\mathfrak{g}) = 7$, M can be identified with the 7-torus $\mathbb{T}^7$ endowed with a left-invariant metric, which is automatically flat. Hence, if φ is closed non-parallel, then $\dim(\mathfrak{g}) \leq 6$.

In order to prove 3), we fix a point p of M and we observe that the image of the isotropy representation $\rho : \mathrm{G}_p \to \mathrm{O}(7)$ is conjugate into G_2. Since G_2 has rank two and G_p is abelian, the dimension of $\mathfrak{g}_p$ is at most two. If $\dim(\mathfrak{g}_p) = 2$, then the image of ρ is conjugate to a maximal torus of G_2 and its fixed point set in T_pM is one-dimensional. As $T_p(\mathrm{G} \cdot p) \subseteq (T_pM)^{\mathrm{G}_p}$, the dimension of the orbit $\mathrm{G} \cdot p$ is at most one, which implies that $\dim(\mathfrak{g})$ is either two or three.

The first assertion in 4) is equivalent to proving that G_p is trivial for every $p \in M$ whenever $\dim(\mathfrak{g}) \geq 5$. In this case, $\dim(\mathfrak{g}_p) \leq 1$ by 3) and therefore the dimension of the orbit $\mathrm{G} \cdot p$ is at least four. Then, for every $h \in \mathrm{G}_p$, the element $\rho(h) \in \mathrm{G}_2$ has a fixed point set containing $T_p(\mathrm{G} \cdot p)$, hence with dimension at least four. On the other hand, a non-trivial element in G_2 is easily seen to have a fixed point set in $\mathbb{R}^7$ of dimension at most three. Indeed, every $u \in \mathrm{G}_2$ is conjugate to an element of a maximal torus of G_2 contained in the maximal rank subgroup $\mathrm{SU}(3) \subset \mathrm{G}_2$, i.e., it can be supposed to be of the form

$$\mathrm{diag}\,(z, w, \overline{z} \cdot \overline{w}) \in \mathrm{SU}(3),$$

for some $z, w \in \mathbb{C}$ of unit norm. Thus, u fixes at least the real line $V \subset \mathbb{R}^7$ that is fixed by $\mathrm{SU}(3)$. Moreover, if u is non-trivial, its fixed point set in the $\mathrm{SU}(3)$-module $V^\perp$ has complex dimension at most one. This shows that $\mathrm{G}_p = \{1_\mathrm{G}\}$. The last assertion follows from [23]. □

The following corollary answers negatively a question posed by Lauret in [17].

Corollary 2.2. *There are no compact homogeneous 7-manifolds endowed with an invariant closed non-parallel* G_2*-structure.*

Proof. The assertion follows immediately from point 2) of Theorem 2.1. □

In contrast to the last result, it is possible to exhibit non-compact homogeneous examples. Consider for instance a six-dimensional non-compact homogeneous space H/K endowed with an invariant symplectic half-flat SU(3)-structure, namely an SU(3)-structure (ω, ψ) such that $d\omega = 0$ and $d\psi = 0$ (see [25] for the classification of such spaces when H is semisimple and for more information on symplectic half-flat structures). If (ω, ψ) is not torsion-free, i.e., if $d(J\psi) \neq 0$, then the non-compact homogeneous space $(\mathrm{H}\times\mathbb{S}^1)/\mathrm{K}$ admits an invariant closed non-parallel G_2-structure defined by the 3-form

$$\varphi \coloneqq \omega \wedge ds + \psi,$$

where ds denotes the global 1-form on $\mathbb{S}^1$.

Remark 2.3. In [4], the authors investigated G_2-manifolds acted on by a cohomogeneity one simple group of automorphisms. Theorem 2.1 explains why compact examples in the case of closed non-parallel G_2-structures do not occur.

The next example shows that G can be non-trivial, that the upper bound on its dimension given in 2) can be attained, and that 4) is only a sufficient condition.

Example 2.4. In [7], the authors constructed a symplectic half-flat SU(3)-structure (ω, ψ) on the 6-torus $\mathbb{T}^6$ as follows. Let $(x^1, \ldots, x^6)$ be the standard coordinates on $\mathbb{R}^6$, and let $a(x^1)$, $b(x^2)$ and $c(x^3)$ be three smooth functions on $\mathbb{R}^6$ such that

$$\lambda_1 \coloneqq b(x^2) - c(x^3), \quad \lambda_2 \coloneqq c(x^3) - a(x^1), \quad \lambda_3 \coloneqq a(x^1) - b(x^2),$$

are $\mathbb{Z}^6$-periodic and non-constant. Then, the following pair of $\mathbb{Z}^6$-invariant differential forms on $\mathbb{R}^6$ induces an SU(3)-structure on $\mathbb{T}^6 = \mathbb{R}^6/\mathbb{Z}^6$:

$$\begin{aligned} \omega &= dx^{14} + dx^{25} + dx^{36}, \\ \psi &= -e^{\lambda_3}\, dx^{126} + e^{\lambda_2}\, dx^{135} - e^{\lambda_1}\, dx^{234} + dx^{456}, \end{aligned}$$

where $dx^{ijk\cdots}$ is a shorthand for the wedge product $dx^i \wedge dx^j \wedge dx^k \wedge \cdots$. It is immediate to check that both ω and ψ are closed and that $d(J\psi) \neq 0$ whenever at least one of the functions $a(x^1)$, $b(x^2)$, $c(x^3)$ is not identically zero. Thus, the pair (ω, ψ) defines a symplectic half-flat SU(3)-structure on the 6-torus. The automorphism group of $(\mathbb{T}^6, \omega, \psi)$ is $\mathbb{T}^3$ when $a(x^1)\, b(x^2)\, c(x^3) \not\equiv 0$, while it becomes $\mathbb{T}^4$ ($\mathbb{T}^5$) when one (two) of them vanishes identically.

Now, we can consider the closed G_2-structure on $\mathbb{T}^7 = \mathbb{T}^6 \times \mathbb{S}^1$ defined by the 3-form $\varphi = \omega \wedge ds + \psi$. Depending on the vanishing of none, one or two of the functions $a(x^1)$, $b(x^2)$, $c(x^3)$, φ is a closed non-parallel G_2-structure and the automorphism group of $(\mathbb{T}^7, \varphi)$ is $\mathbb{T}^4$, $\mathbb{T}^5$ or $\mathbb{T}^6$, respectively.

Finally, we observe that there exist examples where the upper bound on the dimension of $\mathfrak{g}$ given in 1) is more restrictive than the upper bound given in 2).

Example 2.5. In [5], the authors obtained the classification of seven-dimensional nilpotent Lie algebras admitting closed G_2-structures. An inspection of all possible cases shows that the Lie algebras whose second Betti number is lower than seven are those appearing in Table 1.

Let $\mathfrak{n}$ be one of the Lie algebras in Table 1, and consider a closed non-parallel G_2-structure φ on it. Then, left multiplication extends φ to a left-invariant G_2-structure of the same type on the simply connected nilpotent Lie group N corresponding to $\mathfrak{n}$. Moreover, as the

nilpotent Lie algebra $\mathfrak{n}$	$b_2(\mathfrak{n})$
$(0,0,e^{12},e^{13},e^{23},e^{15}+e^{24},e^{16}+e^{34})$	3
$(0,0,e^{12},e^{13},e^{23},e^{15}+e^{24},e^{16}+e^{34}+e^{25})$	3
$(0,0,e^{12},0,e^{13}+e^{24},e^{14},e^{46}+e^{34}+e^{15}+e^{23})$	5
$(0,0,e^{12},0,e^{13},e^{24}+e^{23},e^{25}+e^{34}+e^{15}+e^{16}-3e^{26})$	6

TABLE 1.

structure constants of $\mathfrak{n}$ are integers, there exists a co-compact discrete subgroup $\Gamma \subset \mathrm{N}$ giving rise to a compact nilmanifold $\Gamma\backslash\mathrm{N}$ [22]. The left-invariant 3-form φ on N passes to the quotient defining an invariant closed non-parallel G_2-structure on $\Gamma\backslash\mathrm{N}$. By Nomizu Theorem [24], the de Rham cohomology group $\mathrm{H}^k_{\mathrm{dR}}(\Gamma\backslash\mathrm{N})$ is isomorphic to the cohomology group $\mathrm{H}^k(\mathfrak{n}^*)$ of the Chevalley-Eilenberg complex of $\mathfrak{n}$. Hence, $b_2(\Gamma\backslash\mathrm{N}) = b_2(\mathfrak{n})$.

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