SYMMETRY IN THE COMPOSITE PLATE PROBLEM

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ABSTRACT. In this paper we deal with the *composite plate problem*, namely the following optimization eigenvalue problem

$$\inf_{\rho \in \mathrm{P}} \inf_{u \in \mathcal{W} \setminus \{0\}} \frac{\int_\Omega (\Delta u)^2}{\int_\Omega \rho u^2},$$

where P is a class of admissible densities, $\mathcal{W} = H_0^2(\Omega)$ for Dirichlet boundary conditions and $\mathcal{W} = H^2(\Omega) \cap H_0^1(\Omega)$ for Navier boundary conditions. The associated Euler-Lagrange equation is a fourth-order elliptic PDE governed by the biharmonic operator Δ^2. In the spirit of [11], we study qualitative properties of the optimal pairs (u, ρ). In particular, we prove existence and regularity and we find the explicit expression of ρ. When Ω is a ball, we can also prove uniqueness of the optimal pair, as well as positivity of u and radial symmetry of both u and ρ.

CONTENTS

1. INTRODUCTION

In a series of papers during the 2000's, many mathematicians (see e.g. [10–14, 37]) studied an eigenvalue optimization problem that arises in Continuum Mechanics, usually referred to as *composite membrane problem*. In physical terms, quoting [11], it can be stated as follows:

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Build a body of prescribed shape out of given materials (of varying densities) in such a way that the body has a prescribed mass and so that the basic frequency of the resulting membrane (with fixed boundary) is as small as possible.

This problem has a long history, without aiming at completeness, we just mention here the existence result proved in [23] and the qualitative results proved in [18]. We refer the interested reader to the monograph [28] and the references therein for more results concerning this and related problems.

In mathematical terms, the composite membrane problem can be described in a variational way. Throughout the paper, for any measurable set $S \subset \Omega$, we denote by χ_S its characteristic function and by $|S|$ its n-dimensional Lebesgue measure. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary $\partial\Omega$, $0 \le h < H$ be two fixed constants, and $M \in [h\,|\Omega|, H\,|\Omega|]$. Define the class of *admissible densities* as

$$\mathrm{P} := \left\{ \rho : \Omega \to \mathbb{R} : \int_\Omega \rho(x)\,dx = M,\ h \le \rho \le H \ \text{ in } \Omega, \text{ and } \rho \not\equiv 0 \text{ a.e. in } \Omega \right\}. \tag{1.1}$$

The composite membrane problem is given by

$$\Theta(h, H, M) := \inf_{\rho \in \mathrm{P}} \inf_{u \in H^1_0(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega \rho\, u^2},$$

and a couple (u, ρ) which realizes the double infimum is called a *optimal pair*. The first results proved in [11] and [12] were however obtained for a slightly more general eigenvalue optimization problem, which we briefly describe: let $A \in [0, |\Omega|]$ and $\alpha > 0$ be real numbers, and let

$$\mathcal{S} := \{S \subset \Omega : |S| = A\}$$

be the class of *admissible sets*. The minimization problem is

$$\Lambda(\alpha, A) := \inf_{S \in \mathcal{S}} \inf_{u \in H^1_0(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 + \alpha \int_\Omega \chi_S u^2}{\int_\Omega u^2}.$$

In this case, we call *optimal pair* any couple (u, S) which realizes the infimum. Let us spend a few words concerning the results proved in [11] for the last problem. First of all, one is interested in proving existence of optimal pairs, and it can be done relying on a sort of *bathtub principle*, [29]. It is not possible however to expect uniqueness of such solutions, unless assuming some kind of symmetry on the domain Ω. We will come back to this aspect later on, because symmetry properties will be at the core of our investigation along this paper. The second aspect concerns the regularity of the minimizers u and the description of the optimal set S, which can be considered as a free boundary. Concerning the regularity of the function u, one can rely on classical elliptic regularity theory [26] and get the sharpest regularity. A much more delicate issue is the study of the free boundary.

More recently, in [10], the author pointed out a close relation between the composite membrane problem and a problem in conformal geometry (see Section 6 for more details) while an extension of the composite membrane problem to the case governed by the p-Laplacian operator can be found in [3, 20, 32].

The aim of this paper is to study a fourth-order analogue of the composite membrane problem, that can be called *composite plate problem*. Similar problems have been recently investigated for instance in [4, 15, 19], see also [17] for an analogous problem involving the polyharmonic operator. We now introduce our problem. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^4-boundary $\partial\Omega$, $0 \le h < H$ be two fixed constants, and $M \in [h\,|\Omega|, H\,|\Omega|]$. Here we consider the dimensions $n \ge 2$, we refer to [5, 15] for the unidimensional case. Define the class P of *admissible densities* ρ as in (1.1) and let the functional space $\mathcal{W}$ be

$$\text{either} \quad \mathcal{W} := H^2_0(\Omega) \quad \text{or} \quad \mathcal{W} := H^2(\Omega) \cap H^1_0(\Omega),$$

depending on the boundary conditions one wants to consider. The composite plate problem is given by

$$\Theta(h, H, M) := \inf_{\rho\in \mathrm{P}} \inf_{u\in\mathcal{W}\setminus\{0\}} \frac{\int_\Omega |\Delta u|^2}{\int_\Omega \rho\, u^2}, \tag{CP}$$

and the associated Euler-Lagrange equation is the fourth-order problem

$$\begin{cases} \Delta^2 u = \Theta\, \rho\, u, & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

when $\mathcal{W} = H^2(\Omega) \cap H_0^1(\Omega)$, and

$$\begin{cases} \Delta^2 u = \Theta\, \rho\, u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

when $\mathcal{W} = H_0^2(\Omega)$.

Definition 1.1. A couple $(u, \rho) \in \mathcal{W} \times \mathrm{P}$ which realizes the double infimum in (CP) is called *CP-optimal pair*.

As for its second-order analogue, this problem has a physical interpretation in Continuum Mechanics for inhomogeneous linear elastic plates (cf. Section 2) and is related to the following more general variational problem. Let $\Omega \subset \mathbb{R}^n$ be as in (CP), $\alpha > 0$ and $A \in [0, |\Omega|]$ be real numbers. Let $\lambda_N = \lambda_N(\alpha, S)$ be the lowest eigenvalue of the following boundary value problem with Navier boundary conditions:

$$\begin{cases} \Delta^2 u + \alpha\chi_S u = \lambda\, u, & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \qquad \lambda \in \mathbb{R}, \tag{P_N}$$

whose variational characterization is given by

$$\lambda_N(\alpha, S) = \inf\left\{R(u, \alpha, S) : u \in H^2(\Omega) \cap H_0^1(\Omega), u \not\equiv 0\right\},$$

where

$$R(u, \alpha, S) := \frac{\int_\Omega (\Delta u)^2 dx + \alpha \int_\Omega \chi_S u^2 dx}{\int_\Omega u^2 dx}$$

denotes the Rayleigh quotient.
An analogous problem appears when considering Dirichlet boundary conditions. Let $\lambda_D = \lambda_D(\alpha, S)$ be the lowest eigenvalue of the following Dirichlet boundary value problem:

$$\begin{cases} \Delta^2 u + \alpha\chi_S u = \lambda\, u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \qquad \lambda \in \mathbb{R}. \tag{P_D}$$

The variational characterization of λ_D is now given by

$$\lambda_D(\alpha, S) = \inf\left\{R(u, \alpha, S) : u \in H_0^2(\Omega), u \not\equiv 0\right\},$$

with $R(u, \alpha, S)$ defined as above. In both cases, we consider the following generalized problem

$$\Lambda_j(\alpha, A) := \inf_{S\in\mathcal{S}} \lambda_j(\Omega, \alpha, S) \quad \text{for } j = N, D, \tag{G}$$

where $\mathcal{S} = \{S \subset \Omega \,:\, |S| = A\}$ as above.

For notational ease, hereafter we will drop all subscripts j, D, N of the eigenvalues.

Definition 1.2. A couple $(u, S) \in \mathcal{W} \times \mathcal{S}$ which realizes the double infimum in (G) is called *G-optimal pair*.

We observe that the set S is defined up to zero-measure sets.

Our first result for (G) reads as follows.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^4-boundary $\partial\Omega$. For any positive $\alpha > 0$ and every $A \in [0, |\Omega|]$, there exists a G-optimal pair (u, S). Furthermore, every G-optimal pair (u, S) satisfies*

(a) *$u \in C^{3,\gamma}(\overline{\Omega}) \cap W^{4,q}(\Omega)$, for every $\gamma \in (0, 1)$ and $q \geq 1$;*
(b) *there exists a non-negative number $t \geq 0$ such that $S = \{u^2 \leq t\}$.*

We stress that, due to the presence of a characteristic function in the equation in (G), $C^{3,\gamma}(\overline{\Omega})$ is the the sharpest regularity we can obtain for u, see Remark 3.2.

We say that problem (G) is a generalization of (CP) because there exists a positive number $\bar{\alpha}(A)$ such that the two problems are in one-to-one correspondence for every $\alpha \in (0, \bar{\alpha}(A)]$. The explicit form of the optimal set S for (G) allows in turn to give a complete description of the optimal density ρ of (CP), as stated in the following theorem.

Theorem 1.4. *Under the structural assumptions on (G) and (CP), the following properties hold.*

(a) *Let (u, ρ) be a CP-optimal pair, then ρ has the following form:*
$$\rho = h\,\chi_S + H\,\chi_{S^c},$$
for a set of the form $S = \{u^2 \leq t\}$.
(b) *The pair (u, ρ) is a CP-optimal pair with parameters (h, H, M) if and only if (u, S) is a G-optimal pair for (G) with parameters (α, A) given by*
$$\alpha = (H - h)\Theta, \tag{1.4}$$
$$A = \tfrac{H|\Omega| - M}{H - h}. \tag{1.5}$$
Moreover, the two minimal eigenvalues are related by
$$\Lambda = H\Theta. \tag{1.6}$$
(c) *When h and H vary in their ranges, the corresponding α takes value in $(0, \bar{\alpha}(A)]$ if $A < |\Omega|$, and in $(0, \infty)$ if $A = |\Omega|$. In the first case, the value $\bar{\alpha}(A)$ occurs when $h = 0$.*

The physical interpretation of Theorem 1.4 is that the plate can be made only out of two materials, whose densities are given by the constants h and H. Moreover, the denser material is farther from the boundary $\partial\Omega$. We mentioned that there are two main issues in the composite membrane problem, namely *symmetry and symmetry breaking phenomena* and *regularity of the free boundary* of the generalized problem. The same lines of investigation arise naturally in our context. As a first step, we will study positivity and symmetry properties of optimal pairs when Ω is a ball B. The assumption $\Omega = B$ could apparently be very restrictive, especially when compared with the results available for the composite membrane problem. The main reason behind this request can be roughly explained as follows. Symmetry properties of solutions of second-order elliptic equations can be proved by means of the *moving plane method* introduced by Serrin in [36], as a refinement of the *reflection principle* of Aleksandrov [1,2]. One of the main ingredients of this technique is the maximum principle. The situation changes completely when dealing with fourth-order elliptic equations. For example, symmetry and monotonicity results of Gidas-Ni-Nirenberg-type [25] for semilinear biharmonic problems cannot hold even in the ball if the nonlinearity does not have the right sign, cf. [6,38]. Moreover, there is a striking difference between Dirichlet and Navier boundary conditions. Indeed, for Navier it is possible to reduce the fourth-order equation to a second-order elliptic system, where one recovers the main properties holding in the scalar case, we refer to [24] and references therein for a comprehensive survey of existing results on the topic. In particular, the first eigenfunction of Δ^2 with Navier boundary conditions is not sign-changing, while the same result does not hold in general domains under Dirichlet boundary conditions, cf. [27]. A second difficulty arises due to the fact that higher-order Sobolev spaces $W^{2,p}(\Omega)$ are not invariant under symmetric rearrangements, i.e., $u \in W^{2,p}(\Omega)$ does not imply that its symmetric rearrangement u^* belongs to $W^{2,p}(\Omega)$, see [16, 24]. Nevertheless, there are instances where it is possible to bypass the structural problems appearing in the fourth-order context, e.g. [22, 40].

Let us now briefly describe our specific case.
For Navier boundary conditions, it is possible to rewrite (1.2) as the second-order elliptic system

$$\begin{cases} -\Delta u = v, & \text{in } B, \\ -\Delta v = \Theta\, \rho\, u, & \text{in } B, \\ u = v = 0, & \text{on } \partial B. \end{cases} \tag{1.7}$$

Symmetry results for second-order elliptic systems on balls are available in the literature, starting from the results by Troy [40] where the author considers C^2-solutions of the following system of PDE's

$$\begin{cases} -\Delta u_i = f_i(u_1, \dots, u_n), & \text{in } B, \\ u_i > 0, & \text{in } B, \\ u_i = 0, & \text{on } \partial B, \end{cases} \qquad i = 1, \dots, m. \tag{1.8}$$

The nonlinearities f_i are supposed to be of class C^1 and non-decreasing as functions of u_j for every $j \neq i$. It is clear that once you fix an optimal configuration, (1.7) becomes a cooperative elliptic system which presents a non-autonomous right hand side $g(x, u) = \rho(x)u$ with no a priori symmetry assumptions on the first entry. Hence, it does not satisfy the same assumptions of (1.8), due to the expression of ρ which is the sum of two characteristic functions, and therefore not smooth enough to allow the existence of classical solutions. In particular, this implies that we deal with weak solutions in the appropriate Sobolev space. Despite these differences, the very specific structure of (1.7), combined with the special form of the optimal ρ, allows to adapt the proof of Troy even in our case, yielding the symmetry of the weak solutions of (1.7).

The situation is in general more complicated when dealing with Dirichlet boundary conditions, since much less symmetry results are available in the literature. Nevertheless, when $\Omega = B$, Ferrero, Gazzola and Weth in [22] prove the radial symmetry for minimizers of subcritical Sobolev inequalities, by means of *polarization*, cf. Section 5. This technique was introduced by Brock and Solynin in [8] to avoid rearrangements methods. Indeed, since higher-order Sobolev spaces are not closed under symmetrization, in those spaces it is not possible to have estimates of the form

$$\|\Delta u^*\|_{L^2} \leq \|\Delta u\|_{L^2},$$

useful in proving that the infimum of the Rayleigh ratio is achieved at a radial symmetric function. Again, the method used in [22] exploits the continuity of the nonlinearity there involved, while in our case we cannot rely on such regularity. Here the fact that $\Theta\rho u$ can be regarded as a non-decreasing function of u plays a crucial role in adapting the polarization technique in [22] to get the desired symmetry.

Throughout the paper we write *increasing* and *decreasing* meaning the *strict* monotonicity property.

We state below our main result when Ω is a ball B. Without loss of generality we take $B := \{x \in \mathbb{R}^n : |x| < 1\}$.

Theorem 1.5. *Let $\Omega = B$, then there exists a unique CP-optimal pair (u, ρ). Furthermore, u is positive, radial, and radially decreasing. The set S for which $\rho = h\chi_S + H\chi_{S^c}$ is the unique shell region $\{x : r(A) < |x| < 1\}$ of measure A (i.e., $r(A) > 0$ is the unique positive constant for which $|\{x : r(A) < |x| < 1\}| = A$).*

We point out that if (u, ρ) is a CP-optimal pair then, for every $\mu \in \mathbb{R} \setminus \{0\}$, $(\mu u, \rho)$ is a CP-optimal pair as well, cf. Remark 4.1. This means that uniqueness of CP-optimal pairs in Theorem 1.5 has to be intended up to a multiplicative constant in u.

A few comments on the existing literature are now in order. Theorem 1.5 is morally stated in [28, Remark 11.4.2] as a direct consequence of the technique introduced by Pólya-Szegö in [33, Section F.5] for the biharmonic Faber-Krahn problem. Nevertheless, it came out that this technique is not suitable for higher-order problems [1]. Furthermore, a symmetry result for a problem somehow related to ours is stated in [4].

The paper is organized as follows. In Section 2 we describe the physical interpretation of the problem and we recall some known results that are useful in the rest of the paper. In Section 3 we prove Theorem 1.3 and study the dependence of Λ on the parameters α and A. In Section 4, we show the relation between the two problems (G) and (CP) proving Theorem 1.4, while in Section 5 we prove Theorem 1.5. Finally, in Section 6, we present an application to a problem in conformal geometry.

2. Preliminaries and known results

We start this section with a detailed physical interpretation of problems (1.2) and (1.3). As already mentioned in the introduction, when $n = 2$ these problems are related to Continuum Mechanics for inhomogeneous linear elastic plates. Plates are plane structural elements with a small thickness compared to the planar dimensions. For a transversely loaded plate without axial deformations, the governing equation is given by the Germain-Lagrange equation

$$\Delta^2 u(x) = \frac{q(x)}{D},$$

where $u(x)$ is the transverse displacement of the plate at x, q is the imposed stress, which is supposed to be a distributed external load that is normal to the mid-surface, and D is the flexural rigidity, supposed to be constant. The constant D depends on the material of the plate and its geometry as follows

$$D = \frac{Eh^3}{12(1-\nu)},$$

where E is the Young modulus, h the thickness of the plate, and ν is the Poisson coefficient. In particular, the units of D are $[D] = \mathrm{N} \cdot \mathrm{m}$. We can always write the stress q as

$$q = \rho \cdot a,$$

where ρ is the surface density and a an acceleration. We suppose that the acceleration is proportional to the displacement

$$q = \rho \cdot a = \beta \rho u,$$

with $[\beta] = \mathrm{s}^{-2}$. Therefore, if we include all the constants in Θ in the equation in (1.2) (or (1.3)), we get $\Theta = \beta/D$ and its units are $[\Theta] = \mathrm{kg}^{-1}\mathrm{m}^{-2}$, the same as the ones of

[1]Quoting [24, p. 72]: *(...) u^* may not be twice weakly differentiable even if u is very smooth. In their monograph, Pólya-Szegö [33, Section F.5] claim that they can extend the Faber-Krahn result to the Dirichlet biharmonic operator among domains having a first eigenfunction of fixed sign. Not only this assumption does not cover all domains (...) but also their argument is not correct. They deal with the Laplacian of a symmetrised smooth function and implicitly claim that it belongs to L^2, which is false in general. (...) This shows that standard symmetrisation methods are not available for higher order problems.*

an eigenvalue of Δ^2 (i.e., m^{-4}), divided by a surface density. Finally, Dirichlet boundary conditions are meant to describe a clamped plate, while Navier boundary conditions, when Ω is a polygonal domain in $\mathbb{R}^2$, describe a hinged plate, cf. [31].

Throughout the paper, unless differently stated, $\Omega \subset \mathbb{R}^n$, with $n \geq 2$, will denote a bounded domain (i.e., open and connected) with C^4-boundary $\partial\Omega$. This regularity assumption is needed to prove the regularity result in Theorem 1.3 up to the boundary, but it can be weakened if we look for less regular solutions, cf. Remark 3.2. We assume throughout the paper that $0 \in \Omega$, this can be done without loss of generality, since the problems we are considering are invariant under translation. Furthermore, with a slight abuse of notation, we denote

$$\{f < t\} := \{x \in \Omega : \, f(x) < t\} \,,$$

and analogously for $\{f \leq t\}$.

As already mentioned in the introduction, we work with two Sobolev spaces: we use $H^2_0(\Omega)$ for the *clamped plate*, and $H^2(\Omega) \cap H^1_0(\Omega)$ for the *hinged plate*. Some of our results will be proved in the same way either for the hinged plate or the clamped one. In these cases, to simplify the notation, we will denote both spaces by $\mathcal{W}$. In both cases, we consider the space equipped with the following norm

$$\|u\|^2_{\mathcal{W}} := \int_\Omega (\Delta u)^2 \, dx, \quad u \in \mathcal{W}$$

which is equivalent to the standard Sobolev one. The proof of the equivalence in $H^2_0(\Omega)$ relies on the Poincaré and Calderón-Zygmund inequalities, see for instance [24, Chapter 2.7], while in $H^2(\Omega) \cap H^1_0(\Omega)$ it is a consequence of the equivalence in $H^2_0(\Omega)$ and the continuous embedding $H^2_0(\Omega) \hookrightarrow H^2(\Omega) \cap H^1_0(\Omega)$. We stress that $\mathcal{W}$ endowed with $\|\cdot\|_{\mathcal{W}}$ is a Hilbert space. There is a huge literature dealing with best constants of the critical embeddings of these spaces, e.g. [41]. We refer to the monograph [24] for a comprehensive introduction to the subject.

We recall here two classical embedding theorems that will be useful in what follows.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with Lipschitz boundary $\partial\Omega$. Let $1 \leq p < +\infty$ and let $m \in \mathbb{N}^+$. Then, the following continuous embeddings hold*

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \text{for any } q \in \begin{cases} [1, \frac{np}{n-mp}], & \text{if } n > mp, \\ [1, \infty), & \text{if } n \leq mp. \end{cases}$$

An improvement of Theorem 2.1 when $n < mp$ is given by the following

Theorem 2.2 (Theorem 2.6 of [24])**.** *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with Lipschitz boundary $\partial\Omega$. Let $1 \leq p < +\infty$ and let $m \in \mathbb{N}^+$ and assume that there exists $k \in \mathbb{N}$ such that $n < (m-k)p$. Then*

$$W^{m,p}(\Omega) \hookrightarrow C^{k,\gamma}(\overline{\Omega}), \quad \text{for every } \gamma \in \big(0, m-k-\frac{n}{p}\big] \cap (0,1),$$

with compact embedding if $\gamma < m - k - \frac{n}{p}$.

The following maximum principle for a forth-order problem set in a ball will be useful in Section 5.

Lemma 2.3 (Lemma 1 of [22])**.** *Let $\Omega = B := \{x \in \mathbb{R}^n : |x| < 1\}$ and $\mathcal{C}^+ := \{w \in \mathcal{W} : w \geq 0$ a.e. in $B\}$. Assume that $u \in \mathcal{W}(B)$ is such that*

$$\int_B \Delta u \Delta v \geq 0 \quad \text{for every } v \in \mathcal{C}^+,$$

then $u \in \mathcal{C}^+$. Moreover, either $u \equiv 0$ or $u > 0$ a.e. in B.

Remark 2.4. We stated Lemma 2.3 just in the case of the ball B but, for Navier boundary conditions, it is actually possible to consider more general domains Ω with Lipschitz boundary, and the proof is precisely the same as in [22, Lemma 1].

We introduce now some notation, definitions and preliminary results on the polarization of a function. This technique will be useful when dealing with the symmetry properties in the problem with Dirichlet boundary conditions.

Definition 2.5. Let $\mathcal{H} \subset \mathbb{R}^n$ be a half-space with boundary $\partial\mathcal{H}$, and for every $x \in \mathbb{R}^n$, let $\bar{x}$ denote the reflection of x with respect to $\partial\mathcal{H}$. For every function $v : \mathbb{R}^n \to \mathbb{R}$, we define its *polarization relative to* $\mathcal{H}$ as $v_{\mathcal{H}} : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$v_{\mathcal{H}}(x) := \begin{cases} \max\{v(x), v(\bar{x})\}, & \text{if } x \in \mathcal{H}, \\ \min\{v(x), v(\bar{x})\}, & \text{if } x \in \mathbb{R}^n \setminus \mathcal{H}. \end{cases}$$

It is straightforward to check that polarization preserves continuity and moreover, if v is a compact supported continuous function, then also $v_{\mathcal{H}} \in C_c(\mathbb{R}^n)$. Furthermore, every polarization preserves the L^p-norms ($1 \le p \le +\infty$) and the following pointwise identity holds

$$v(x) + v(\bar{x}) = v_{\mathcal{H}}(x) + v_{\mathcal{H}}(\bar{x}) \quad \text{for every } x \in \mathbb{R}^n. \tag{2.1}$$

Proposition 2.6 (Ex. 2.4 of [9], [29])**.** *Let $\mathcal{H} \subset \mathbb{R}^n$ be a half-space and $f \in L^1(\mathbb{R}^n)$ be non-negative, then*

$$|\{x : f(x) > s\}| = |\{x : f_{\mathcal{H}}(x) > s\}| \quad \textit{for every } s > 0.$$

We will use the following characterization for radial, radially non-increasing functions.

Lemma 2.7 (Lemma 6.3 of [8])**.** *A function $v \in C_c(\mathbb{R}^n)$ is radial and radially non-increasing if and only if $v = v_{\mathcal{H}}$ for every half-space $\mathcal{H} \subset \mathbb{R}^n$ such that $0 \in \mathrm{int}(\mathcal{H})$.*

Lemma 2.8 (Lemma 3 of [22])**.** *Let $\mathcal{H}$ be a half-space such that $0 \in \mathrm{int}(\mathcal{H})$ and $G = G(x, y)$ the Green function of Δ^2 in B relative to Dirichlet boundary conditions. Then, for every x, $y \in \mathcal{H}$, $x \neq y$ the following inequalities hold*

(i) $G(x, y) \ge \max\{G(x, \bar{y}), G(\bar{x}, y)\}$*;*
(ii) $G(x, y) - G(\bar{x}, \bar{y}) \ge |G(x, \bar{y}) - G(\bar{x}, y)|$.

Moreover, if x, $y \in \mathrm{int}(B \cap \mathcal{H})$, the inequalities in (i) and (ii) are strict.

3. Proof of Theorem 1.3

The aim of this section is to prove Theorem 1.3. Besides the regularity of the solutions of either (P_N) or (P_D), Theorem 1.3 provides an explicit description of the optimal set S as a sub-level set of u^2. Once established the connection between (G) and (CP), the knowledge of the optimal set S will be crucial to provide also a description of any optimal density ρ, which in turn will play a crucial role in the study of the symmetry properties of u.

Before proving regularity in our case, we recall the following result in a more general setting.

Proposition 3.1 (Theorem 2.20 of [24])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, with C^4-smooth boundary $\partial\Omega$, and let $f \in L^p(\Omega)$ for $p \in (1, \infty)$. Then*

$$\Delta^2 u = f \quad \textit{in } \Omega, \tag{3.1}$$

coupled either with Dirichlet or with Navier boundary conditions, admits a unique strong solution[2] *in $W^{4,p}(\Omega)$.*

[2] For *strong solution* we mean a function u that satisfies the equation (3.1) almost everywhere.

Now, we are ready to prove Part (a) of Theorem 1.3.

• *Proof of Part (a).* Let us go back to the fourth-order PDE

$$\Delta^2 u + \alpha\chi_S u = \lambda u \quad \text{in } \Omega$$

and define the function

$$f(x) := \lambda u(x) - \alpha\chi_S(x)u(x). \tag{3.2}$$

Since u is a weak solution of either (P_N) or (P_D), it holds that $u \in H^2(\Omega)$. This implies that $f \in L^2(\Omega)$, and so $f \in L^p(\Omega)$ for every $p \in [1,2]$, being Ω bounded. Therefore, by Proposition 3.1, we know that $u \in W^{4,p}(\Omega)$ for every $p \in [1,2]$.
Now, by Theorem 2.1

$$H^4(\Omega) \subset L^q(\Omega) \quad \text{for all } q \in \begin{cases} [1,2^*], & \text{if } n > 8, \\ [1,2^*), & \text{if } n \le 8, \end{cases} \qquad 2^* := \begin{cases} \frac{2n}{n-8}, & \text{if } n > 8, \\ +\infty, & \text{if } n \le 8. \end{cases}$$

If $2^* = +\infty$, then u and f belong to $L^p(\Omega)$ for all $p \in [1,\infty)$. By Proposition 3.1, $u \in W^{4,p}(\Omega)$ for all $p \in [1,\infty)$. In particular, $u \in W^{4,p}(\Omega)$ for all $p > n$, hence $u \in C^{3,\gamma}(\overline{\Omega})$ for all $\gamma \in (0,1)$, by Theorem 2.2. If $2^* < \infty$, we use a bootstrap argument. For every $j \in \mathbb{N}$, we define

$$2^*_j := \begin{cases} \frac{2n}{n-8j}, & \text{if } n > 8j, \\ +\infty, & \text{if } n \le 8j. \end{cases}$$

It is straightforward to verify, by induction on $j \ge 1$, that $2^*_{j+1} = (2^*_j)^*$. Since $u \in L^{2^*}(\Omega)$, also $f \in L^{2^*}(\Omega)$ and, again by Proposition 3.1, we have $u \in W^{4,2^*}(\Omega)$. Iterating the application of both Proposition 3.1 and Theorem 2.1 j-times, as long as $2^*_j < \infty$, we get that $u \in W^{4,2^*_{j-1}}(\Omega)$ and

$$W^{4,2^*_{j-1}}(\Omega) \subset L^{2^*_j}(\Omega).$$

Now, for every $n \in \mathbb{N}$, there exists $\bar{j} \in \mathbb{N}$ such that $n \le 8\bar{j}$ and so, $2^*_{\bar{j}} = +\infty$. After $\bar{j}$ iterations, we can conclude by using Theorem 2.2, as already done in the case $2^* = \infty$. $\square$

Remark 3.2. The regularity of u cannot be improved up to $C^4(\Omega)$, at least in the more relevant cases when $\emptyset \neq S \subsetneq \Omega$, due to the presence of the characteristic function.
We want also to stress another fact: from the modeling point of view, it is more reasonable to work with a Lipschitz boundary $\partial\Omega$. In this case, we can obtain the same regularity result of Theorem 1.3-(a), but only in the *interior*, mainly due to the fact that the argument provided by [24, Theorem 2.20] requires a smooth enough boundary. Therefore, if we restrict our attention to interior regularity, we can use the same bootstrap argument presented in the proof of Theorem 1.3-(a) to prove that a weak solution u of (P_N) (or (P_D)) is such that

$$u \in W^{4,q}_{\text{loc}}(\Omega) \cap C^{3,\gamma}(\Omega)$$

for every $q \in [1,\infty)$ and for every $\gamma \in (0,1)$.

Let us prove now the existence of a G-optimal pair. As for the regularity, the strategy of the proof of existence is independent of the boundary conditions imposed. Therefore, we will adopt the compact notation $\mathcal{W}$ for the Sobolev space over which we consider the infimum.

We first prove an auxiliary result.

Proposition 3.3. *Let $A \ge 0$ be a fixed non-negative constant,*

$$\mathcal{A} := \left\{ \eta : \Omega \to \mathbb{R} : 0 \le \eta \le 1 \ a.e. \ in \ \Omega, \ \int_\Omega \eta = A \right\},$$

and $u \in \mathcal{W}$ such that $\|u\|_{L^2(\Omega)} = 1$. If we define the functional $I : \mathcal{A} \to \mathbb{R}$ as

$$I(\eta) := \int_\Omega \eta(x)\, u^2(x)\, dx,$$

then the minimization problem

$$\inf_{\eta \in \mathcal{A}} I(\eta) \tag{3.3}$$

admits a solution $\eta = \chi_S$, with S belonging to the following set

$$\begin{aligned} &\mathcal{S}_t := \Big\{ S \subset \Omega \ : \ |S| = A,\ \{u^2 < t\} \subset S \subset \{u^2 \le t\} \Big\}, \\ &\textit{where} \quad t := \sup\{s > 0 : |\{u^2 < s\}| < A\}. \end{aligned} \tag{3.4}$$

In particular, for every $\alpha > 0$

$$\Lambda(\alpha, A) = \inf_{\eta \in \mathcal{A}} \inf_{u \in \mathcal{W}\setminus\{0\}} \frac{\int_\Omega (\Delta u)^2 + \alpha \int_\Omega \eta u^2}{\int_\Omega u^2} \tag{3.5}$$

and the set S, that realizes Λ, belongs to $\mathcal{S}_t$.

Proof. We observe that for every set S of measure A, its characteristic function χ_S belongs to $\mathcal{A}$. Hence, it is enough to prove that $I(\chi_S) \le I(\eta)$ for every $S \subset \Omega$ satisfying (3.4) and for every $\eta \in \mathcal{A}$. A simple splitting of the domain of integration yields

$$\begin{aligned} \int_\Omega u^2(\chi_S - \eta)dx &= \int_{\{u^2<t\}} u^2(\chi_S - \eta)dx + \int_{\{u^2>t\}} u^2(\chi_S - \eta)dx + \int_{\{u^2=t\}} u^2(\chi_S - \eta)dx \\ &\le t \int_{\{u^2<t\}} (\chi_S - \eta)dx - t \int_{\{u^2>t\}} \eta dx + t \int_{\{u^2=t\}} (\chi_S - \eta)dx \\ &= t \left(\int_{\{u^2<t\}} (\chi_S - \eta)dx + \int_{\{u^2>t\}} (\chi_S - \eta)dx + \int_{\{u^2=t\}} (\chi_S - \eta)dx \right) \\ &= t \int_\Omega (\chi_S - \eta)dx = 0. \end{aligned}$$

This closes the proof of the first part of the statement and easily gives

$$\inf_{\eta \in \mathcal{A}} \inf_{u \in \mathcal{W}\setminus\{0\}} \frac{\int_\Omega (\Delta u)^2 + \alpha \int_\Omega \eta u^2}{\int_\Omega u^2} = \inf_{S \in \mathcal{S}_t} \inf_{u \in \mathcal{W}\setminus\{0\}} \frac{\int_\Omega (\Delta u)^2 + \alpha \int_\Omega \chi_S u^2}{\int_\Omega u^2}. \tag{3.6}$$

Indeed, since

$$\{\chi_S \ : \ S \in \mathcal{S}_t\} \subseteq \mathcal{A},$$

we get

$$\inf_{\eta \in \mathcal{A}} \inf_{u \in \mathcal{W}\setminus\{0\}} \frac{\int_\Omega (\Delta u)^2 + \alpha \int_\Omega \eta u^2}{\int_\Omega u^2} \le \inf_{S \in \mathcal{S}_t} \inf_{u \in \mathcal{W}\setminus\{0\}} \frac{\int_\Omega (\Delta u)^2 + \alpha \int_\Omega \chi_S u^2}{\int_\Omega u^2}.$$

The opposite inequality follows directly from the previous computation.

Now, on one hand,

$$\inf_{\eta \in \mathcal{A}} \inf_{u \in \mathcal{W}\setminus\{0\}} \frac{\int_\Omega (\Delta u)^2 + \alpha \int_\Omega \eta u^2}{\int_\Omega u^2} \le \Lambda(\alpha, A),$$

being $\chi_S \in \mathcal{A}$ for all S of measure A. On the other hand, by definition (G) of Λ

$$\Lambda(\alpha, A) \le \inf_{S \in \mathcal{S}_t} \inf_{u \in \mathcal{W}\setminus\{0\}} \frac{\int_\Omega (\Delta u)^2 + \alpha \int_\Omega \chi_S u^2}{\int_\Omega u^2}.$$

Together with (3.6), this implies (3.5) and concludes the proof. ∎

We are now ready to proceed with the proof of the existence of a G-optimal pair. In what follows we fix $\alpha > 0$ and $A \in [0, |\Omega|]$ and we simplify the notation by writing

$$\Lambda := \Lambda(\alpha, A) \quad \text{and} \quad \lambda(S) := \lambda(\alpha, S) \text{ for every } S \subset \Omega.$$

• *Proof of existence.* Let $(S_k)_k$ be a minimizing sequence, meaning $|S_k| = A$ for every $k \in \mathbb{N}$ and

$$\lambda(S_k) \longrightarrow \Lambda \quad \text{as } k \to \infty.$$

For every $k \in \mathbb{N}$, we consider a first eigenfunction $u_k \in \mathcal{W}$ of $\Delta^2 + \alpha\chi_{S_k}$. Without loss of generality, we can assume that $\|u_k\|_{L^2(\Omega)} = 1$ for every $k \in \mathbb{N}$.
Now, the sequences $(\chi_{S_k})_k \subset L^2(\Omega)$ and $(\lambda(S_k))_k$ are bounded. Keeping in mind that the norm used is $\|u\|^2_{\mathcal{W}} = \int_\Omega (\Delta u)^2$, the previous considerations imply that $(u_k)_k$ is a bounded sequence in $\mathcal{W}$. Since both the spaces $L^2(\Omega)$ and $\mathcal{W}$ are Hilbert spaces, we can extract two sub-sequences, still denoted $(\chi_{S_k})_k$ and $(u_k)_k$, and we can find two functions $\eta \in L^2(\Omega)$ and $u \in \mathcal{W}$, such that

$$\begin{aligned} &\chi_{S_k} \rightharpoonup \eta \quad \text{in } L^2(\Omega), \quad \text{as } k \to \infty, \\ &u_k \rightharpoonup u \quad \text{in } \mathcal{W}, \quad \text{as } k \to \infty. \end{aligned}$$

Hence, up to a subsequence, we have the following:

(i) $u_k \to u$ in $L^2(\Omega)$, as $k \to \infty$;

(ii) $\int_\Omega \chi_{S_k} u_k \psi dx \to \int_\Omega \eta u \psi dx$ for every $\psi \in C_0^\infty(\Omega)$, as $k \to \infty$;

(iii) $\int_\Omega \eta(x) dx = A$.

Indeed, (i) follows from the compact embedding $\mathcal{W} \hookrightarrow L^2(\Omega)$; (ii) follows from (i) and Hölder's inequality in the direct computation

$$\begin{aligned} \left| \int_\Omega (\chi_{S_k} u_k - \eta u)\, \psi dx \right| &\le \left| \int_\Omega \chi_{S_k}(u_k - u)\psi dx \right| + \left| \int_\Omega (\chi_{S_k} - \eta) u \psi dx \right| \\ &\le \|u_k - u\|_{L^2(\Omega)} \|\psi\|_{L^2(\Omega)} + \left| \int_\Omega (\chi_{S_k} - \eta) u \psi dx \right| \to 0 \end{aligned}$$

for every $\psi \in C_0^\infty(\Omega)$. To prove (iii) we argue as follows: since $\chi_{S_k} \rightharpoonup \eta$ in $L^2(\Omega)$ and Ω is bounded, we have in particular

$$A = \int_\Omega \chi_{S_k} \cdot 1\, dx \to \int_\Omega \eta \cdot 1\, dx,$$

which gives (iii) by uniqueness of the limit.
By definition, any pair (u_k, S_k) satisfies

$$\Delta^2 u_k + \alpha \chi_{S_k} u_k = \lambda_{S_k} u_k \tag{3.7}$$

and so

$$\int_\Omega \Delta u_k \Delta \psi + \alpha \int_\Omega \chi_{S_k} u_k \psi = \lambda_{S_k} \int_\Omega u_k \psi \quad \text{for all } \psi \in C_0^\infty(\Omega). \tag{3.8}$$

By previous remarks, we can pass to the limit in (3.8) as $k \to \infty$, finding

$$\int_\Omega \Delta u \Delta \psi + \alpha \int_\Omega \eta u \psi = \Lambda \int_\Omega u \psi.$$

Integrating by parts, we recover the variational formulation of the eigenvalue equation associated to Λ, which implies that $u \in \mathcal{W}$ solves the equation

$$\Delta^2 u + \alpha \eta u = \Lambda u, \tag{3.9}$$

in the weak sense. Now, the sets

$$P_a := \{w \in L^2(\Omega) : w(x) \le 1 \quad \text{for a.e. } x \in \Omega\}$$
$$P_b := \{w \in L^2(\Omega) : w(x) \ge 0 \quad \text{for a.e. } x \in \Omega\}$$

are strongly closed in the L^2-topology and convex, then weakly closed. Since $(\chi_{S_k})_k \subset P_a \cap P_b$ and $\chi_{S_k} \rightharpoonup \eta$ in $L^2(\Omega)$,

$$0 \le \eta(x) \le 1 \text{ for a.e. } x \in \Omega.$$

Thus, $\eta \in \mathcal{A}$. In order to end the proof, we need to show that we can replace the function η with a characteristic function of a suitable set $S \subset \mathbb{R}^n$ of measure A. To this aim, let us multiply (3.9) by u and let us integrate it over Ω. Since by (i) we have $\|u\|_{L^2(\Omega)} = 1$, it follows that

$$\int_\Omega (\Delta u)^2 + \alpha \int_\Omega \eta u^2 = \Lambda.$$

By Proposition 3.3, we have that there exists a set $S \subset \Omega$ satisfying (3.4) such that

$$\Lambda = \int_\Omega (\Delta u)^2 + \alpha \int_\Omega \eta u^2 \ge \int_\Omega (\Delta u)^2 + \alpha \int_\Omega \chi_S u^2.$$

Hence, from the definition of Λ as an infimum, we have that

$$\int_\Omega (\Delta u)^2 + \alpha \int_\Omega \chi_S u^2 = \Lambda,$$

and therefore the pair (u, S) is a G-optimal pair. □

We can now give a precise description of the optimal set S in terms of a sub-level of u^2.

• *Proof of Part (b).* Let (u, S) be a G-optimal pair. By the proof of the existence result, we know that $S \in \mathcal{S}_t$, with t defined as in (3.4). Hence, it is enough to prove that $\mathcal{N}_t := \{u^2 = t\} \subset S$. Now, if $t > 0$, $\mathcal{N}_t = \{u = \sqrt{t}\} \cup \{u = -\sqrt{t}\}$. By [26, Lemma 7.7], we have that $\Delta^2 u = 0$ a.e. in $\mathcal{N}_t$, being u constant in both $\{u = \sqrt{t}\}$ and $\{u = -\sqrt{t}\}$. Therefore, the Euler-Lagrange equation associated to (G) reduces to

$$(\Lambda \cdot \mathrm{Id} - \alpha \chi_S) u = 0 \quad \text{a.e. in } \mathcal{N}_t.$$

Since $u \neq 0$ in $\mathcal{N}_t$, this implies that $\Lambda \cdot \mathrm{Id} = \alpha \chi_S$ a.e. in $\mathcal{N}_t$, which yields in turn $\mathcal{N}_t \subset S$, being $\Lambda, \alpha > 0$. This concludes the proof in the case $t > 0$.

If $t = 0$, we have to prove that $\mathcal{N}_0 = \{u = 0\} \subset S$. By (3.4), we know that $S \subset \{u = 0\}$, thus $\chi_S u = 0$ in Ω and the equation reduces to $\Delta^2 u = \Lambda u$ in Ω. Thus, $\Lambda = \mu(\Omega)$, where $\mu(\Omega)$ is the first eigenvalue of Δ^2 in Ω with either Navier or Dirichlet boundary conditions, and u is the corresponding first eigenfunction. Since $\Delta^2 - \Lambda \cdot \mathrm{Id}$ has elliptic principal part and constant coefficients, it is analytic hypoelliptic, see [39, Chapter 3]. Hence, u is a real analytic function and by [30, Proposition 0], its zero set has zero measure. The proof of this last statement relies on the Weierstrass preparation theorem. In conclusion, $0 \le A \le |\{u = 0\}| = 0$ and since S is defined up to zero-measure sets, we can put $S = \{u = 0\}$. □

As a consequence of the previous result, we know in particular that S contains a neighborhood of $\partial\Omega$.

The next proposition deals with the dependence of Λ on the parameters α and A. This is the analogue of [11, Proposition 10]. For notational ease, in what follows we write S^c instead of $S^c \cap \Omega$.

Proposition 3.4. *The following properties hold*

- *for $A > 0$, $\Lambda(\alpha, A)$ is increasing in α;*

- $\Lambda(\alpha, A)$ *is Lipschitz continuous in* α *with Lipschitz constant* $A|\Omega|^{-1}$*;*
- *for* $A < |\Omega|$*, there exists a unique value of* α*, denoted by* $\bar{\alpha}(A)$*, such that* $\Lambda(\alpha, A) = \alpha$*;*
- *for* $A < |\Omega|$*,* $\Lambda(\alpha, A) - \alpha$ *is decreasing in* α*;*
- $\Lambda(\alpha, A)$ *is continuous and non-decreasing in* A*.*

Proof. Let $A \in [0, |\Omega|]$ and take $0 < \alpha_1 < \alpha_2$ to fix the ideas. Denote (u_1, S_1) and (u_2, S_2) G-optimal pairs corresponding to (α_1, A) and (α_2, A) respectively. Without loss of generality, we can assume $\|u_1\|_{L^2(\Omega)} = \|u_2\|_{L^2(\Omega)} = 1$. Then, by the optimality of (u_1, S_1) for the data (α_1, A), we get

$$\begin{aligned}\Lambda(\alpha_1, A) &= \int_\Omega (\Delta u_1)^2 + \alpha_1 \int_{S_1} u_1^2 \leq \int_\Omega (\Delta u_2)^2 + \alpha_1 \int_{S_2} u_2^2 \\ &\leq \int_\Omega (\Delta u_2)^2 + \alpha_2 \int_{S_2} u_2^2 = \Lambda(\alpha_2, A)\end{aligned}$$

where the last inequality is strict if $A > 0$, since u_2 cannot be zero a.e. in S_2. Indeed, if by contradiction $u_2 = 0$ a.e. in S_2, since S_2 is of the form $\{u_2^2 \leq t\}$ for some $t \geq 0$, it results that $t = 0$. By the discussion in the proof of Theorem 1.3-(b), this implies that $A = 0$, which is a contradiction. Hence, if $A > 0$, $\Lambda(\alpha, A)$ is increasing in α. On the other hand, by the optimality of (u_2, S_2) for the data (α_2, A), we obtain

$$\begin{aligned}\Lambda(\alpha_2, A) &= \int_\Omega (\Delta u_2)^2 + \alpha_2 \int_{S_2} u_2^2 \leq \int_\Omega (\Delta u_1)^2 + \alpha_2 \int_{S_1} u_1^2 \\ &= \Lambda(\alpha_1, A) + (\alpha_2 - \alpha_1) \int_{S_1} u_1^2 \leq \Lambda(\alpha_1, A) + (\alpha_2 - \alpha_1) \frac{A}{|\Omega|},\end{aligned}$$

where the last estimate comes from

$$\frac{\int_{\{u^2 \leq t\}} u^2}{|\{u^2 \leq t\}|} \leq \frac{\int_\Omega u^2}{|\Omega|},$$

which in turn is a simple consequence of $\{u^2 \leq t\} \cup \{u^2 > t\} = \Omega$, $\{u^2 \leq t\} \cap \{u^2 > t\} = \emptyset$, and

$$⨍_{\{u^2 > t\}} u^2 \geq ⨍_{\{u^2 \leq t\}} u^2.$$

Altogether, we get for $\alpha_1 < \alpha_2$

$$0 \leq \Lambda(\alpha_2, A) - \Lambda(\alpha_1, A) \leq (\alpha_2 - \alpha_1) \frac{A}{|\Omega|}.$$

Analogously, if $\alpha_1 > \alpha_2$ we have

$$0 \leq \Lambda(\alpha_1, A) - \Lambda(\alpha_2, A) \leq (\alpha_1 - \alpha_2) \frac{A}{|\Omega|},$$

and so for all α_1, $\alpha_2 > 0$

$$|\Lambda(\alpha_1, A) - \Lambda(\alpha_2, A)| \leq \frac{A}{|\Omega|} |\alpha_1 - \alpha_2|,$$

that is $\Lambda(\cdot, A)$ is Lipschitz continuous with Lipschitz constant $A|\Omega|^{-1}$. In particular, for $A < |\Omega|$, $\Lambda(\cdot, A)$ is a contraction mapping and, by the Banach fixed-point Theorem, it admits a unique fixed-point $\bar{\alpha}(A)$.

Now, suppose that $A < |\Omega|$ and $0 < \alpha_1 < \alpha_2$, and estimate in the same notation as above

$$\begin{aligned}\Lambda(\alpha_2, A) - \alpha_2 &\le \int_\Omega (\Delta u_1)^2 + \alpha_2 \int_{S_1} u_1^2 - \alpha_2 \\ &= \Lambda(\alpha_1, A) - \alpha_1 - (\alpha_2 - \alpha_1)\left(\int_\Omega u_1^2 - \int_{S_1} u_1^2\right).\end{aligned}$$

In order to prove that $\Lambda(\alpha, A) - \alpha$ is decreasing in α it remains to show that

$$\int_\Omega u_1^2 - \int_{S_1} u_1^2 > 0.$$

We argue by contradiction and suppose

$$\int_\Omega u_1^2 - \int_{S_1} u_1^2 = \int_{S_1^c} u_1 = 0,$$

that is $u_1 = 0$ a.e in S_1^c. Since $S_1^c = \{u_1^2 > t\}$ and u_1 is continuous, S_1^c is open and, up to a translation, we can assume that $0 \in S_1^c$. Furthermore, $|S_1^c| = |\Omega| - A > 0$ and $|\Delta^2 u_1| = |\Lambda(\alpha_1, A) \cdot \mathrm{Id} - \alpha_1 \chi_{S_1}| \cdot |u_1| \le (\Lambda(\alpha_1, A) + \alpha_1)|u_1|$. Hence, by the Unique Continuation Theorem in [34], $u_1 \equiv 0$ in Ω. This is impossible being $\|u_1\|_{L^2(\Omega)} = 1$ and concludes the proof of this part.

Finally, let $0 \le A_1 < A_2 \le |\Omega|$ and $\alpha > 0$. Denote (u_1, S_1) and (u_2, S_2) G-optimal pairs corresponding to the data (α, A_1) and (α, A_2) respectively. Let $S_2' \subset \Omega$ be such that $|S_2'| = A_2$ and $S_2' \supset S_1$. Then, by the optimality of $\Lambda(\alpha, A_2)$ we get

$$\begin{aligned}\Lambda(\alpha, A_1) &= \int_\Omega (\Delta u_1)^2 + \alpha \int_{S_2'} u_1^2 - \alpha \int_{S_2' \setminus S_1} u_1^2 \\ &\ge \int_\Omega (\Delta u_2)^2 + \alpha \int_{S_2} u_2^2 - \alpha \int_{S_2' \setminus S_1} u_1^2 = \Lambda(\alpha, A_2) - \alpha \int_{S_2' \setminus S_1} u_1^2.\end{aligned}$$

On the other hand, denoting by S_1' a subset of S_2 having $|S_1'| = A_1$ and using the optimality of $\Lambda(\alpha, A_1)$, we have

$$\Lambda(\alpha, A_1) \le \int_\Omega (\Delta u_2)^2 + \alpha \int_{S_1'} u_2^2 \le \int_\Omega (\Delta u_2)^2 + \alpha \int_{S_2} u_2^2 = \Lambda(\alpha, A_2).$$

Therefore,

$$0 \le \Lambda(\alpha, A_2) - \Lambda(\alpha, A_1) \le \alpha \int_{S_2' \setminus S_1} u_1^2,$$

and so $\Lambda(\alpha, \cdot)$ is non-decreasing and

$$|\Lambda(\alpha, A_1) - \Lambda(\alpha, A_2)| \le \alpha \int_{S_2' \setminus S_1} u_1^2 \to 0 \quad \text{as } A_1 \to A_2.$$

□

Proposition 3.5. *Every set $\{u^2 = s\}$, $s \ge 0$, has zero measure, except possibly $\{u^2 = t\}$ when $\alpha = \bar{\alpha}(A)$.*

Proof. We use the same notation as in the proof of Theorem 1.3-(b). The argument is similar to the one contained in [11, Theorem 1-(c)], but we present it here for the sake of completeness. If $s > t$, $\mathcal{N}_s \subset S^c$. Hence,

$$0 = \Delta^2 u = (\Lambda \cdot \mathrm{Id} - \alpha \chi_S) u = \Lambda u \quad \text{a.e. on } \mathcal{N}_s.$$

Since $\Lambda > 0$ and $u \ne 0$ on $\mathcal{N}_s$, $|\mathcal{N}_s| = 0$. Now, if $s = t$, $\mathcal{N}_s \subset S$ and so

$$0 = (\Lambda - \alpha) u \quad \text{a.e. on } \mathcal{N}_s.$$

Thus, if $\alpha \neq \bar{\alpha}(A) = \Lambda$, we can conclude again that $|\mathcal{N}_t| = 0$. Finally, if $s < t$, again $\mathcal{N}_s \subset S$ and $\Delta^2 u = (\Lambda - \alpha)u$ in the open set $\{u^2 < t\}$. The function $v := u - s$ solves the equation

$$\Delta^2 v = (\Lambda - \alpha)v + (\Lambda - \alpha)s \quad \text{in } \{u^2 < t\}.$$

Therefore, v is a real analytic function and so $|\{v = 0\}| = |\{u = s\}| = 0$. $\square$

4. Proof of Theorem 1.4

In this section, as in [11], we highlight the relations between the two problems (G) and (CP), which will be useful in proving the symmetry results later on.

• *Proof of Theorem 1.4.* For (a), let us consider a CP-minimizer (u, ρ). We write any $\rho \in \mathrm{P}$ as $\rho = H + (\rho - H)$ and so the PDE in (1.2) (or (1.3)) reads as

$$\Delta^2 u + \Theta(H - \rho)u = \Theta H u \quad \text{in } \Omega. \tag{4.1}$$

<u>*Claim*</u>: it is possible to choose $\alpha > 0$ and $A \in [0, |\Omega|]$ for which (4.1) can be seen in the form

$$\Delta^2 u + \alpha\eta u = \Lambda(\alpha, A)u \quad \text{in } \Omega$$

for some $\eta \in \mathcal{A} := \{\eta : \Omega \to \mathbb{R} \,:\, 0 \le \eta \le 1,\, \int_\Omega \eta = A\}$.

In order to prove the claim, we put

$$\alpha := \Theta(H - h) > 0, \quad \eta := \frac{H - \rho}{H - h}, \quad \text{and consequently} \quad A := \frac{H|\Omega| - M}{H - h}. \tag{4.2}$$

Thus, we need to show that

(i) $0 \le \frac{H-\rho}{H-h} \le 1$;

(ii) $0 \le \frac{H\,|\Omega|-M}{H-h} \le |\Omega|$;

(iii) $\Lambda\left(\Omega, \Theta(H-h), \frac{H\,|\Omega|-M}{H-h}\right) = \Theta H$.

Now, (i) follows immediately by the bounds $h \le \rho \le H$, while (ii) follows from the assumption $M \in [h|\Omega|, H|\Omega|]$. Hence, Proposition 3.3 applies and we know that

$$\Lambda(\alpha, A) = \inf_{\eta \in \mathcal{A}} \inf_{u \in \mathcal{W}\setminus\{0\}} \frac{\int_\Omega (\Delta u)^2 + \alpha \int_\Omega \eta u^2}{\int_\Omega u^2},$$

with α, η, A as in (4.2). In terms of ρ, by (4.2) this reads as

$$\Lambda(\alpha, A) = \Theta H + \inf_{\rho \in \mathrm{P}} \inf_{u \in \mathcal{W}\setminus\{0\}} \frac{\int_\Omega (\Delta u)^2 - \Theta \int_\Omega \rho u^2}{\int_\Omega u^2}. \tag{4.3}$$

By the definition of Θ as an infimum, $\int_\Omega (\Delta u)^2 - \Theta \int_\Omega \rho u^2 \ge 0$, hence

$$\inf_{\rho \in \mathrm{P}} \inf_{u \in \mathcal{W}\setminus\{0\}} \frac{\int_\Omega (\Delta u)^2 - \Theta \int_\Omega \rho u^2}{\int_\Omega u^2} \ge 0. \tag{4.4}$$

On the other hand, since $\rho \le H$, and using again the definition of Θ, we get

$$\begin{aligned} \inf_{\rho \in \mathrm{P}} \inf_{u \in \mathcal{W}\setminus\{0\}} \frac{\int_\Omega (\Delta u)^2 - \Theta \int_\Omega \rho u^2}{\int_\Omega u^2} &= \inf_{\rho \in \mathrm{P}} \inf_{u \in \mathcal{W}\setminus\{0\}} \left(\frac{\int_\Omega (\Delta u)^2}{\int_\Omega \rho u^2} - \Theta \right) \frac{\int_\Omega \rho u^2}{\int_\Omega u^2} \\ &\le H \inf_{\rho \in \mathrm{P}} \inf_{u \in \mathcal{W}\setminus\{0\}} \left(\frac{\int_\Omega (\Delta u)^2}{\int_\Omega \rho u^2} - \Theta \right) = 0. \end{aligned} \tag{4.5}$$

Combining together (4.4) and (4.5), we obtain

$$\inf_{\rho\in \mathrm{P}} \inf_{u\in\mathcal{W}\setminus\{0\}} \frac{\int_\Omega(\Delta u)^2 - \Theta\int_\Omega \rho u^2}{\int_\Omega u^2} = 0,$$

and, by (4.3), (iii) is proved. This concludes the proof of the claim.

Now, by Proposition 3.3, we know that $\eta = \chi_S$ for a set $S \in \mathcal{S}_t$ as in (3.4). Hence, by (4.2),

$$\rho = H - (H-h)\eta = H - (H-h)\chi_S = h\chi_S + H\chi_{S^c},$$

which closes the proof of part (a).

We are now ready to prove (b). Here and in what follows, ρ and S are as in the statement of part (a). We first observe that the "only if" part and the fact that $\Lambda(\alpha, A) = \Theta(h,H,M)H$ for α, A as in (1.4)-(1.5) have been shown in the proof of (a). Hence, it remains to prove that if (u,S) realizes $\Lambda := \Lambda(\alpha, A)$, with α as in (1.4), and A as in (1.5), then (u,ρ) realizes $\Theta := \Theta(h,H,M)$. By assumption, we have

$$\Lambda = \Theta H = \frac{\int_\Omega(\Delta u)^2 + \Theta(H-h)\int_\Omega \chi_S u^2}{\int_\Omega u^2},$$

thus

$$\Theta = \frac{\int_\Omega(\Delta u)^2 - \Theta\int_\Omega \rho u^2 + \Theta H\int_\Omega(\chi_S+\chi_{S^c})u^2}{H\int_\Omega u^2} = \frac{\int_\Omega(\Delta u)^2 - \Theta\int_\Omega \rho u^2}{H\int_\Omega u^2} + \Theta.$$

Therefore,

$$\Theta\int_\Omega \rho u^2 = \int_\Omega (\Delta u)^2,$$

that is (u,ρ) realizes Θ.

For part (c), we observe that, by $h < H$, $\Lambda > 0$ and (1.4), we immediately get $\alpha > 0$. Furthermore, if $A \in [0, |\Omega|)$,

$$\alpha = \frac{H-h}{H}\Lambda(\alpha, A)\begin{cases} < \Lambda(\alpha, A) & \text{if } h > 0, \\ = \Lambda(\alpha, A) & \text{if } h = 0. \end{cases}$$

While, if $A = |\Omega|$, $S = \Omega$ and $\rho \equiv h$ by part (a). Thus, $\Theta(H,h,M) = \mu(\Omega)/h$ for any $H > h$. Hence, by (1.4), $\alpha = (H-h)\mu(\Omega)/h$ can take any value in $(0,\infty)$ varying $H \in (h,\infty)$. □

Remark 4.1. We end this section by noting explicitly that, by part (a) of the previous theorem it follows in particular that, if $h = 0$, $\rho \equiv 0$ in S. Now, since $|\{\rho = 0\}| = 0$ by the definition of admissible densities $\rho \in \mathrm{P}$, $|S| = 0$. Moreover, since S is defined up to a zero-measure set, $S^c = \Omega$. Therefore, when $h = 0$ problem (CP) reduces to the standard eigenvalue problem for the biharmonic operator.
We further observe that by the very definition of $t = t(u)$ in (3.4), denoting by (u, ρ_u) a CP-optimal pair, since $\rho_u = h\chi_{\{u^2\le t(u)\}} + H\chi_{\{u^2>t(u)\}}$, we have that $\rho_{\mu u} = \rho_u$. Indeed $t(\mu u) = \mu^2 t(u)$ and so $\{u^2 \le t(u)\} = \{(\mu u)^2 \le t(\mu u)\}$.

5. Proof of Theorem 1.5

The aim of this section is to address qualitative properties of the CP-optimal pairs (u,ρ), such as positivity and radial symmetry in the case $\Omega = B := \{x \in \mathbb{R}^n : |x| < 1\}$.

We start with the positivity of u.

Proposition 5.1. *Let $\Omega = B$ and let (u,ρ) be a CP-optimal pair, then $u > 0$ in B.*

Proof. Let w be a solution of

$$\Delta^2 w = \Theta\rho|u| \quad \text{in } B, \tag{5.1}$$

coupled either with Navier or with Dirichlet boundary conditions. By Lemma 2.3, $w > 0$ a.e. in B, otherwise we would have $u \equiv 0$ in B which is impossible. Now, suppose by contradiction that u is sign-changing and consider the functions $w - u$ and $w + u$. Then

$$\Delta^2(w-u) = 2\Theta\rho u^- \quad \text{and} \quad \Delta^2(w+u) = 2\Theta\rho u^+ \quad \text{in } B.$$

Hence,

$$\int_B \Delta(w-u)\Delta v dx \geq 0 \quad \text{and} \quad \int_B \Delta(w+u)\Delta v dx \geq 0 \quad \text{for all } v \in \mathcal{C}^+.$$

Again by Lemma 2.3, we get that either $\pm u \equiv w$ or $|u| < w$ a.e. in B. In the first case, being $w > 0$, up to a change of sign of u, we are done. In the latter, we multiply (5.1) by w, integrate over B and get

$$\int_B (\Delta w)^2 = \Theta \int_B \rho|u|w < \Theta \int_B \rho w^2,$$

which implies

$$\frac{\int_B (\Delta w)^2}{\int_B \rho w^2} < \Theta.$$

This contradicts the minimality of Θ and concludes the proof. □

Remark 5.2. As for Lemma 2.3, if we deal with Navier boundary conditions, we can consider more general open sets in Proposition 5.1.
There is a simple consequence of the positivity result in Proposition 5.1: for $\alpha \leq \bar{\alpha}(A)$ we have an equivalence between (CP) and (G) therefore, recalling Theorem 1.4, the optimal set S can be written as a sub-level set of the function u itself, i.e.

$$S = \{u \leq \sqrt{t}\}.$$

For the symmetry issues we need to distinguish the case with Dirichlet boundary conditions from the one with Navier boundary conditions.

Before proving the symmetry result for *Dirichlet boundary conditions*, we need to prove some preliminary lemmas.

In the rest of the section we consider a CP-optimal pair (u, ρ) and we extend $u \in C_0(\overline{B}) := \{\varphi \in C(\overline{B}) : \varphi = 0 \text{ on } \partial B\}$ by defining it to be zero outside B. We must consider an extension of ρ as well. We will denote it by

$$\rho_u := h\chi_{\{u \leq \sqrt{t}\}} + H\chi_{\{u > \sqrt{t}\}},$$

where we are considering sub-level sets of the extended function u.

Lemma 5.3. *Let $\mathcal{H} \subset \mathbb{R}^n$ be a half-space. Then*

$$[\rho_u\, u]_{\mathcal{H}} \equiv \rho_{u_{\mathcal{H}}}\, u_{\mathcal{H}}.$$

Proof. We prove this lemma by using the definitions of the two functions involved, namely

$$[\rho_u\, u]_{\mathcal{H}}(x) = \begin{cases} \max\{\rho_u(x)u(x), \rho_u(\bar{x})u(\bar{x})\}, & \text{if } x \in \mathcal{H}, \\ \min\{\rho_u(x)u(x), \rho_u(\bar{x})u(\bar{x})\}, & \text{if } x \in \mathbb{R}^n \setminus \mathcal{H}, \end{cases}$$

and

$$\rho_{u_{\mathcal{H}}}(x)u_{\mathcal{H}}(x) = \begin{cases} hu_{\mathcal{H}}(x), & \text{if } u_{\mathcal{H}}(x) \leq \sqrt{t}, \\ Hu_{\mathcal{H}}(x), & \text{if } u_{\mathcal{H}}(x) > \sqrt{t}. \end{cases}$$

Now, for every $x \in \mathbb{R}^n$ four cases may occur:

- $x \in \{u \le \sqrt{t}\}$ and $\bar{x} \in \{u \le \sqrt{t}\}$;
- $x \in \{u \le \sqrt{t}\}$ and $\bar{x} \notin \{u \le \sqrt{t}\}$;
- $x \notin \{u \le \sqrt{t}\}$ and $\bar{x} \in \{u \le \sqrt{t}\}$;
- $x \notin \{u \le \sqrt{t}\}$ and $\bar{x} \notin \{u \le \sqrt{t}\}$.

We start with considering $x \in \mathcal{H}$. In the first case

$$[\rho_u\, u]_{\mathcal{H}}(x) = \max\{hu(x), hu(\bar{x})\} = hu_{\mathcal{H}}(x).$$

Furthermore, since $u(x) \le \sqrt{t}$ and $u(\bar{x}) \le \sqrt{t}$, also $u_{\mathcal{H}}(x) = \max\{u(x), u(\bar{x})\} \le \sqrt{t}$ and so

$$\rho_{u_{\mathcal{H}}}(x)u_{\mathcal{H}}(x) = hu_{\mathcal{H}}(x).$$

If the second case occurs, we know that $u(\bar{x}) > u(x)$ and consequently

$$[\rho_u\, u]_{\mathcal{H}}(x) = \max\{hu(x), Hu(\bar{x})\} = Hu(\bar{x}) = Hu_{\mathcal{H}}(x).$$

On the other hand, since $u(\bar{x}) > \sqrt{t}$, also $u_{\mathcal{H}}(x) = \max\{u(x), u(\bar{x})\} > \sqrt{t}$, which implies

$$\rho_{u_{\mathcal{H}}}(x)u_{\mathcal{H}}(x) = Hu_{\mathcal{H}}(x)$$

and concludes the proof also in this case. With similar arguments it is possible to check the remaining cases both for $x \in \mathcal{H}$ and $x \in \mathbb{R}^n \setminus \mathcal{H}$. $\square$

Lemma 5.4. *Let (u, ρ_u) be a CP-optimal pair in the ball B and $\mathcal{H} \subset \mathbb{R}^n$ a half-space. If $\rho_u u = [\rho_u u]_{\mathcal{H}}$, then $u = u_{\mathcal{H}}$.*

Proof. Suppose first that $h > 0$. By hypothesis and Lemma 5.3, we know that for every $x \in \mathbb{R}^n$

$$\begin{cases} hu(x) & \text{if } u(x) \le \sqrt{t} \\ Hu(x) & \text{if } u(x) > \sqrt{t} \end{cases} = \begin{cases} hu_{\mathcal{H}}(x) & \text{if } u_{\mathcal{H}}(x) \le \sqrt{t} \\ Hu_{\mathcal{H}}(x) & \text{if } u_{\mathcal{H}}(x) > \sqrt{t}. \end{cases} \tag{5.2}$$

Suppose by contradiction that there exists $x \in \mathbb{R}^n$ such that $u_{\mathcal{H}}(x) \ne u(x)$. Then, if $u(x) \le \sqrt{t}$ and $u_{\mathcal{H}}(x) \le \sqrt{t}$, by (5.2), $hu(x) = hu_{\mathcal{H}}(x)$ which is absurd. Analogously, the case $u(x) > \sqrt{t}$ and $u_{\mathcal{H}}(x) > \sqrt{t}$ cannot occur if $u_{\mathcal{H}}(x) \ne u(x)$. Now, if $u(x) \le \sqrt{t}$ and $u_{\mathcal{H}}(x) > \sqrt{t}$, by (5.2), we get $hu(x) = Hu_{\mathcal{H}}(x)$ and so clearly $u(x) \ne 0$. Hence,

$$\sqrt{t} < u_{\mathcal{H}}(x) = \frac{h}{H}u(x) < u(x) \le \sqrt{t}$$

which is a contradiction. Analogously, we can rule out the opposite case $u(x) > \sqrt{t}$ and $u_{\mathcal{H}}(x) \le \sqrt{t}$, and conclude the proof for $h > 0$.

If $h = 0$, by Remark 4.1, $u > \sqrt{t}$ in B, and consequently $u_{\mathcal{H}}$ can attain values only in $\{0\} \cup (\sqrt{t}, \infty)$. Now, if $x \in \mathbb{R}^n \setminus B$, $u(x) = 0$. Then, in view of (5.2), for every $x \in \mathbb{R}^n \setminus B$

$$0 = \begin{cases} 0, & \text{if } u_{\mathcal{H}}(x) \le \sqrt{t}, \\ Hu_{\mathcal{H}}(x), & \text{if } u_{\mathcal{H}}(x) > \sqrt{t}, \end{cases}$$

which implies that $u_{\mathcal{H}}(x) \le \sqrt{t}$ and therefore $u_{\mathcal{H}}(x) = 0$. Hence, $u(x) = 0 = u_{\mathcal{H}}(x)$ for every $x \in \mathbb{R}^n \setminus B$. Analogously it can be seen that $u \equiv u_{\mathcal{H}}$ in B and the proof is concluded. $\square$

Let $G : B \times B \to \mathbb{R}$ be the Green function for the biharmonic operator with Dirichlet boundary conditions on the ball. We recall that G has an explicit representation due to Boggio [7]. We are now considering the trivial zero extension of G to the whole of $\mathbb{R}^n \times \mathbb{R}^n$. We define $\tilde{u} : \mathbb{R}^n \to \mathbb{R}$ as

$$\tilde{u}(x) := \Theta \int_{\mathbb{R}^n} G(x, y)\rho_u(y)u(y)\, dy,$$

then $\tilde{u} \equiv 0$ in $\mathbb{R}^n \setminus B$ and $\tilde{u}|_B$ is the unique solution of the problem

$$\begin{cases} \Delta^2 v = \Theta \left(h\chi_{\{u\leq\sqrt{t}\}} + H\chi_{\{u>\sqrt{t}\}} \right) u, & \text{in } B, \\ v = \frac{\partial v}{\partial \nu} = 0, & \text{on } \partial B. \end{cases}$$

By uniqueness and by the trivial extension of u, if (u,ρ) is a CP-optimal pair, $\tilde{u} \equiv u$.

Lemma 5.5. *Let $\mathcal{H}$ be a half-space such that $0 \in \operatorname{int}(\mathcal{H})$, and for every $x \in \mathbb{R}^n$*

$$w(x) := \Theta \int_{\mathbb{R}^n} G(x,y)\rho_{u_{\mathcal{H}}}(y)u_{\mathcal{H}}(y)dx. \tag{5.3}$$

Then the following inequalities hold

(i) $w(x) \geq w(\bar{x})$ *for every* $x \in \mathcal{H}$;
(ii) $w(x) \geq u_{\mathcal{H}}(x)$ *for every* $x \in \mathcal{H}$;
(iii) $w(x) + w(\bar{x}) \geq u_{\mathcal{H}}(x) + u_{\mathcal{H}}(\bar{x})$ *for every* $x \in \mathbb{R}^n$.

Moreover, if $\rho_u u \not\equiv [\rho_u u]_{\mathcal{H}}$, then (iii) is strict for every $x \in \operatorname{int}(B \cap \mathcal{H})$.

Proof. For the proofs of (i), (ii), and (iii), we refer to [22, Lemma 4]. We now show the last part of the statement whose proof is slightly different from the one contained in [22], since in our case the function f is $\rho_u u$ which is not continuous. However, formula (4.18) of [22] still holds, namely for every $x \in \mathbb{R}^n$

$$\begin{aligned} &w(x) + w(\bar{x}) - [u_{\mathcal{H}}(x) + u_{\mathcal{H}}(\bar{x})] \\ &= \Theta \int_{\mathcal{H}} \Big(G(x,y) + G(\bar{x},y) - [G(x,\bar{y}) + G(\bar{x},\bar{y})] \Big) \Big(\rho_{u_{\mathcal{H}}}(y)u_{\mathcal{H}}(y) - \rho_u(y)u(y) \Big) dy \geq 0. \end{aligned} \tag{5.4}$$

By Lemma 2.8, we know that if $x,\, y \in \operatorname{int}(B \cap \mathcal{H})$,

$$G(x,y) + G(\bar{x},y) > G(x,\bar{y}) + G(\bar{x},\bar{y}),$$

thus, by (5.4) and by Lemma 5.3, it is enough to prove that we can find a positive-measure subset of $\operatorname{int}(B \cap \mathcal{H})$ in which

$$\rho_{u_{\mathcal{H}}} u_{\mathcal{H}} > \rho_u u.$$

We first observe that

$$\rho_{u_{\mathcal{H}}} u_{\mathcal{H}} \equiv \rho_u u \equiv 0 \quad \text{in } \mathbb{R}^n \setminus B. \tag{5.5}$$

Indeed, since $u > 0$ in B and $u \equiv 0$ in $\mathbb{R}^n \setminus B$,

$$u_{\mathcal{H}}(x) = \begin{cases} \max\{0, u(\bar{x})\} = u(\bar{x}), & \text{if } x \in \mathcal{H}, \\ \min\{0, u(\bar{x})\} = 0, & \text{if } x \in \mathbb{R}^n \setminus \mathcal{H} \end{cases} \tag{5.6}$$

for every $x \in \mathbb{R}^n \setminus B$. Furthermore, since $0 \in \operatorname{int}(\mathcal{H})$, $|\bar{x}| \geq |x|$ for every $x \in \mathcal{H}$. Thus, $x \notin B$ implies $\bar{x} \notin B$, and so $u(\bar{x}) = 0$ in the first line of the definition (5.6), which yields (5.5). Moreover, $u_{\mathcal{H}} \equiv u$ on $B \cap \partial\mathcal{H}$, because for every $x \in \partial\mathcal{H}$ it holds $x = \bar{x}$. Therefore, $\rho_{u_{\mathcal{H}}} u_{\mathcal{H}} \not\equiv \rho_u u$ ensures that there exists $y \in B \setminus \partial\mathcal{H}$ for which $\rho_{u_{\mathcal{H}}}(y)u_{\mathcal{H}}(y) \neq \rho_u(y)u(y)$. We can always assume $y \in \operatorname{int}(B \cap \mathcal{H})$, since if this is not the case, $\bar{y}$ will do the job, being by (2.1) and Lemma 5.3

$$0 \neq \rho_{u_{\mathcal{H}}}(y)u_{\mathcal{H}}(y) - \rho_u(y)u(y) = \rho_u(\bar{y})u(\bar{y}) - \rho_{u_{\mathcal{H}}}(\bar{y})u_{\mathcal{H}}(\bar{y}).$$

Hence, there exists $y \in \operatorname{int}(B \cap \mathcal{H})$ such that

$$\begin{cases} hu(y) & \text{if } u(y) \leq \sqrt{t} \\ Hu(y) & \text{if } u(y) > \sqrt{t} \end{cases} \neq \begin{cases} hu_{\mathcal{H}}(y) & \text{if } u(y) \leq \sqrt{t} \\ Hu_{\mathcal{H}}(y) & \text{if } u(y) > \sqrt{t}. \end{cases} \tag{5.7}$$

Now, since $y \in \mathcal{H}$, if $u(y) > \sqrt{t}$, also $u_{\mathcal{H}}(y) > \sqrt{t}$, and so we have only the following three possible cases.

Case $u(y) > \sqrt{t}$. By (5.7) and the fact that $y \in \mathcal{H}$, we know that $Hu(y) < Hu_{\mathcal{H}}(y)$. Hence, by the continuity of u and $u_{\mathcal{H}}$ we can find a neighborhood U_y of y such that

$$U_y \subset \operatorname{int}(B \cap \mathcal{H}) \cap \{u > \sqrt{t}\} \cap \{u_{\mathcal{H}} > \sqrt{t}\}$$

and

$$\rho_{u_{\mathcal{H}}}(x) u_{\mathcal{H}}(x) > \rho_u(x) u(x) \quad \text{for every } x \in U_y. \tag{5.8}$$

Case $u(y) \le \sqrt{t}$ and $u_{\mathcal{H}}(y) > \sqrt{t}$. Again, by (5.7), we get $Hu_{\mathcal{H}}(y) \neq hu(y)$, and since $u_{\mathcal{H}} \ge u$ in $\mathcal{H}$ and $H > h$, this yields

$$Hu_{\mathcal{H}}(y) \ge Hu(y) > hu(y).$$

Now, if $u(y) < \sqrt{t}$, we can find a neighborhood U_y such that

$$U_y \subset \operatorname{int}(B \cap \mathcal{H}) \cap \{u < \sqrt{t}\} \cap \{u_{\mathcal{H}} > \sqrt{t}\}$$

and $Hu_{\mathcal{H}}(x) > hu(x)$ for every $x \in U_y$, that is to say (5.8) holds also in this case. If $u(y) = \sqrt{t}$ then clearly $u_{\mathcal{H}}(y) > u(y)$ and by continuity there exists a neighborhood $U_y \subset \operatorname{int}(B \cap \mathcal{H}) \cap \{u_{\mathcal{H}} > \sqrt{t}\}$ where $u_{\mathcal{H}} > u$. This implies that

$$Hu_{\mathcal{H}}(x) > Hu(x) > hu(x) \quad \text{for every } x \in U_y,$$

and in turn (5.8) holds for both $x \in \{u \le \sqrt{t}\}$ and $x \in \{u > \sqrt{t}\}$.

Case $u(y) \le \sqrt{t}$ and $u_{\mathcal{H}}(y) \le \sqrt{t}$. By (5.7), we get $hu_{\mathcal{H}}(y) > hu(y)$ and by continuity we can find $U_y \subset \operatorname{int}(B \cap \mathcal{H})$ where $u_{\mathcal{H}} > u$. Let $x \in U_y$. If $u_{\mathcal{H}}(x) \le \sqrt{t}$, then also $u(x) \le \sqrt{t}$, and so $hu_{\mathcal{H}}(x) > hu(x)$ is equivalent to $\rho_{u_{\mathcal{H}}}(x) u_{\mathcal{H}}(x) > \rho_u(x) u(x)$. If $u_{\mathcal{H}}(x) > \sqrt{t}$, then

$$Hu_{\mathcal{H}}(x) > Hu(x) > hu(x).$$

Hence, for both $x \in \{u \le \sqrt{t}\}$ and $x \in \{u > \sqrt{t}\}$,

$$\rho_{u_{\mathcal{H}}}(x) u_{\mathcal{H}}(x) > \rho_u(x) u(x).$$

Then, also in this case (5.8) holds, which concludes the proof. □

Lemma 5.6. *Let $\mathcal{H} \subset \mathbb{R}^n$ be a half-space with $0 \in \operatorname{int}(\mathcal{H})$, and w be defined as in (5.3). Then,*

$$\int_B w \rho_{u_{\mathcal{H}}} u_{\mathcal{H}} \le \int_B \rho_{u_{\mathcal{H}}} w^2. \tag{5.9}$$

Furthermore, if equality holds, then $\rho_u u \equiv [\rho_u u]_{\mathcal{H}}$.

Proof. By Lemma 5.5 we get

$$\begin{aligned} \int_B [\rho_{u_{\mathcal{H}}} w^2 - \rho_{u_{\mathcal{H}}} u_{\mathcal{H}} w] dx &= \int_{\mathcal{H}} \{\rho_{u_{\mathcal{H}}}(x) w(x)[w(x) - u_{\mathcal{H}}(x)] \\ &\qquad + \rho_{u_{\mathcal{H}}}(\bar{x}) w(\bar{x})[w(\bar{x}) - u_{\mathcal{H}}(\bar{x})]\} dx \\ &\ge \int_{\mathcal{H}} [w(x) - u_{\mathcal{H}}(x)] \cdot [\rho_{u_{\mathcal{H}}}(x) w(x) - \rho_{u_{\mathcal{H}}}(\bar{x}) w(\bar{x})] dx \ge 0. \end{aligned} \tag{5.10}$$

We stress that in the last inequality we have also used the fact that, if $x \in \mathcal{H}$ then $\bar{x} \notin \mathcal{H}$ and in particular

$$u_{\mathcal{H}}(x) = \max\{u(x), u(\bar{x})\} \ge \min\{u(x), u(\bar{x})\} = u_{\mathcal{H}}(\bar{x}).$$

Consequently, if $u_{\mathcal{H}}(x) \le \sqrt{t}$, then also $u_{\mathcal{H}}(\bar{x}) \le \sqrt{t}$, and so

$$\rho_{u_{\mathcal{H}}}(x) \ge \rho_{u_{\mathcal{H}}}(\bar{x}) \quad \text{for every } x \in \mathcal{H}.$$

We can now prove the last part of the statement as in [22, Lemma 5]. If equality holds in (5.9), then also in (5.10) we have equality. This is only possible in two situations: $w(x) - u_{\mathcal{H}}(x) = u_{\mathcal{H}}(\bar{x}) - w(\bar{x})$ for every $x \in \operatorname{int}(\mathcal{H} \cap B)$, or $\rho_{u_{\mathcal{H}}}(\bar{x})w(\bar{x}) = 0$ for all $x \in \operatorname{int}(\mathcal{H} \cap B)$. In the first case, we conclude by Lemma 5.5 that $\rho_u u \equiv [\rho_u u]_{\mathcal{H}}$. If the second case occurs, then since both w and $\rho_{u_{\mathcal{H}}}$ are positive in B, we conclude that $B \subset \mathcal{H}$ and so again $\rho_u u \equiv [\rho_u u]_{\mathcal{H}}$, being $u \equiv 0$ outside B. □

We are now ready to end the proof of Theorem 1.5 for Dirichlet boundary conditions.

• *Proof of Theorem 1.5 for Dirichlet.* Let (u, ρ_u) be a CP-optimal pair, with $u > 0$ in B, and let $\mathcal{H} \subset \mathbb{R}^n$ be a half-space such that $0 \in \operatorname{int}(\mathcal{H})$. Then, by the definition (5.3) of w we know that w solves the problem

$$\begin{cases} \Delta^2 v = \Theta \rho_{u_{\mathcal{H}}} u_{\mathcal{H}}, & \text{in } B, \\ v = \frac{\partial v}{\partial \nu} = 0, & \text{on } \partial B. \end{cases}$$

Thus, by Lemma 5.6 we get

$$\|\Delta w\|^2_{L^2(B)} = \Theta \int_B w\, \rho_{u_{\mathcal{H}}} u_{\mathcal{H}} \le \Theta \int_B \rho_{u_{\mathcal{H}}} w^2, \tag{5.11}$$

and so

$$\frac{\|\Delta w\|^2_{L^2(B)}}{\int_B \rho_{u_{\mathcal{H}}} w^2} \le \Theta.$$

By Proposition 2.6 also $\rho_{u_{\mathcal{H}}}$ is an admissible density (i.e. $\int_B \rho_{u_{\mathcal{H}}} = M$), then by the minimality of Θ equality must hold in (5.11), and so $u = u_{\mathcal{H}}$ by Lemmas 5.6 and 5.4. Therefore, by the arbitrariness of $\mathcal{H}$ and by Lemma 2.7, we get that u is a radial, radially non-increasing function and by its shape, S is radial and S^c is convex. In view of Proposition 3.5 and since S is defined up to a set of measure zero, S is the unique *open* shell region of measure A, $S = \{x : r(A) < |x| < 1\}$. In particular, S and S^c are of class C^∞. In conclusion, for $\Omega = B$ there is a unique CP-optimal pair (u, ρ). It remains to prove the strict monotonicity of the radial profile of u. To this aim, we observe that, thanks to the regularity of the boundaries of S and S^c and the fact that ρ is constant in both S and S^c, $u\big|_S$ and $u\big|_{S^c}$ are of class C^4 in $\operatorname{int}(S)$ and $\operatorname{int}(S^c)$ respectively, cf. [24, Theorem 2.20]. Now, we have just proved that u is radially non-increasing. Suppose by contradiction that there exists an open subset U of B where u is constant, consequently either $U \subset S = \{u < \sqrt{t}\}$ or $U \subset S^c = \{u > \sqrt{t}\}$. Thus, $\Delta^2 u = 0$ in U, which contradicts the positivity of u, being $\Delta^2 u = \Theta \rho u > 0$ in all of B. Here we are tacitly assuming $h > 0$, the case $h = 0$ being even simpler. □

We consider now the case of *Navier boundary conditions*. Here we can write our fourth-order problem (P_N) as the second-order system (1.7), that is

$$\begin{cases} -\Delta u = v, & \text{in } B, \\ -\Delta v = \Theta\, \rho\, u, & \text{in } B, \\ u = v = 0, & \text{on } \partial B. \end{cases}$$

Proposition 5.7. *Let (u, v) be a weak solution of (P_N) such that $u > 0$ and $v > 0$ in B. Then u and v are radial and radially decreasing in B.*

Proof. For the proof of this result we refer to the ones of [40, Theorem 1 and Lemmas 4.1-4.3] for system (1.8). We just skecth the proof below, and we highlight how we can overcome the lack of the regularity assumptions required in [40] to the solution $(u_i)_{i=1}^m$ (i.e., $u_i \in C^2(\overline{B})$) and the nonlinearity $(f_i)_{i=1}^m$ (i.e., $f_i \in C^1$) of (1.8), thanks to the special form of our system.

As in [40], we arbitrarily choose the x_1 axis and denote by T_ξ the hyperplane $\mathrm{e}_1 \cdot x = \xi$. Since B is bounded, for sufficiently large $\xi > 0$, the plane T_ξ does not intersect $\overline{B}$. We decrease ξ (i.e., the plane T_ξ moves continuously toward B, preserving the normal) until ξ_0, that is the smallest value of ξ for which T_ξ begins to intersect B. From $\xi = \xi_0$ to $\xi = 0$, the plane T_ξ, cuts off from B an open set $\Sigma(\xi)$, which is the part of B that does not contain the origin. Let $\Sigma'(\xi)$ denote the reflection of $\Sigma(\xi)$ with respect to the plane T_ξ. For every $x \in \Sigma(\xi)$, we denote by x^ξ the reflection of x with respect to T_ξ.

The proof can be split into the following three steps.

Step 1. Let $x_0 \in \partial B$ be such that $\nu^{(1)}(x_0) > 0$. Then there exists $\delta > 0$ such that $\frac{\partial u}{\partial x_1} < 0$ and $\frac{\partial v}{\partial x_1} < 0$ in $B \cap B(x_0, \delta)$.

This can be proved as in [40, Lemma 4.1]. We observe that in our case $f_1(v) = v$ and $f_2(u) = \Theta\rho_u u$, hence $f_i(0) = 0$ for $i = 1, 2$. This allows to avoid the case (ii) in the proof of [40, Lemma 4.1] which would require the C^2-regularity of $v = \Delta u$.

Now, take $\xi \in (0, \xi_0)$ sufficiently close to ξ_0. Since $\nu^{(1)}(x) > 0$ for every $x \in \partial B \cap \partial(\Sigma(\xi))$, as a consequence of Step 1., it follows that for every $x \in \Sigma(\xi)$

$$\frac{\partial u}{\partial x_1}(x) < 0, \quad \frac{\partial v}{\partial x_1}(x) < 0, \quad u(x) < u(x^\xi), \quad v(x) < v(x^\xi). \tag{5.12}$$

As in the proof of [40, Lemma 4.3], decrease ξ below ξ_0 until a critical value $\bar{\xi} \ge 0$ beyond which (5.12) does not hold any more for u or v. Then, for every $x \in \Sigma(\bar{\xi})$

$$\frac{\partial u}{\partial x_1}(x) \le 0, \quad \frac{\partial v}{\partial x_1}(x) \le 0, \quad u(x) \le u(x^{\bar{\xi}}), \quad v(x) \le v(x^{\bar{\xi}}). \tag{5.13}$$

Step 2. Let $\xi \in (0, \xi_0)$, then

$$\begin{aligned} &u(x) < u(x^\xi), \quad v(x) < v(x^\xi) \quad \textit{for every } x \in \Sigma(\xi), \\ &\frac{\partial u}{\partial x_1}(x) < 0, \quad \frac{\partial v}{\partial x_1}(x) < 0 \quad \textit{for every } x \in B \cap T_\xi. \end{aligned}$$

This can be proved by using (5.12) and (5.13) as in [40, Lemma 4.2]. We observe that the special form of f_i, $i = 1, 2$ (i.e., the fact that f_1 does not depend on u and f_2 does not depend on v) allows us to avoid the use of the Mean Value Theorem in this proof. Furthermore, the proof of [40, Lemma 4.2] relies on the Hopf Lemma and the Strong Maximum Principle for C^2-solutions of second-order elliptic equations in domains with corners. In our case we can apply the Strong Maximum Principle and the Hopf Lemma in e.g. [21, Theorem 2.2] or [35, Theorem 2.5.1, Theorem 2.7.1 and comments on p. 40], which require only $C^1(\overline{B})$ regularity of the solution (u, v).

As a consequence of Step 1. and Step 2., it is possible to prove that the value $\bar{\xi} \ge 0$ is indeed equal to 0. This can be done by following the argument by contradiction proposed in [40, Lemma 4.3]-Case (i). Here again the use of the Mean Value Theorem can be avoided thanks to the special form of the f_i's in our problem.

Furthermore, by Step 2., we get

$$\frac{\partial u}{\partial x_1}(x) > 0, \quad \frac{\partial v}{\partial x_1}(x) > 0 \quad \text{for every } x \in B \cap \{x \in \mathbb{R}^n : x_1 < 0\}$$

and by continuity of partial derivatives of u and v,

$$\frac{\partial u}{\partial x_1}(x) = 0, \quad \frac{\partial v}{\partial x_1}(x) = 0 \quad \text{for every } x \in B \cap T_0. \tag{5.14}$$

Step 3. *The functions u and v are symmetric with respect to the plane T_0.*
This can be proved as in [40, Lemma 4.2], by using (5.14).

The conclusion of the proof then follows by the arbitrariness of the x_1 axis. □

• *Proof of Theorem 1.5 for Navier.* By Proposition 5.1 $u > 0$, this together with the strong maximum principle implies that $v > 0$. Therefore, we can apply Proposition 5.7. The conclusion of Theorem 1.5 for Navier, concerning the properties of S, can be repeated verbatim as in the case for Dirichlet boundary conditions. □

Remark 5.8. Let us denote by Θ_N and Θ_D the values of (CP) with Navier and Dirichlet boundary conditions, respectively. Since $H_0^2(\Omega) \subset H^2(\Omega) \cap H_0^1(\Omega)$, $\Theta_N \le \Theta_D$. We can follow the argument in [22] to prove that actually the strict inequality holds, namely

$$\Theta_N < \Theta_D.$$

Indeed, let $(u, \rho) \in H^2(\Omega) \cap H_0^1(\Omega) \times \mathrm{P}$ be a CP-optimal pair for Navier. Let us assume by contradiction that u does not have a sign in Ω. Consider now the problem

$$\begin{cases} -\Delta v = |\Delta u|, & \text{in } \Omega, \\ \quad\;\; v = 0, & \text{in } \partial\Omega. \end{cases} \tag{5.15}$$

By regularity theory, a solution v of (5.15) is such that $v \in H^2(\Omega) \cap H_0^1(\Omega)$, and therefore is an admissible candidate for the problem (CP) with Navier boundary conditions. On the other hand, we can argue as in the proof of Proposition 5.1 to get by the maximum principle that $v > |u|$ in Ω. Hence, being $\rho > 0$ a.e. Ω, we have

$$\frac{\int_\Omega (\Delta v)^2}{\int_\Omega \rho v^2} < \frac{\int_\Omega (\Delta u)^2}{\int_\Omega \rho u^2} = \Theta_N,$$

which contradicts the minimality of Θ_N. Thus, u has sign, and so we can take $u > 0$ in Ω. This, combined with $-\Delta u \ge 0$ (by maximum principle, being $\Delta^2 u = \Theta_N \rho u > 0$ in Ω and $\Delta u = 0$ on $\partial\Omega$), allows to employ the Hopf Boundary Point Lemma, which gives

$$\frac{\partial u}{\partial \nu} < 0 \quad \text{on } \partial\Omega.$$

In order to conclude, it is enough to notice that if (u, ρ) is a CP-optimal pair with Dirichlet boundary conditions, then $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, hence, it *cannot* be a CP-optimal pair with Navier boundary conditions as well.

6. A nonlinear eigenvalue minimization problem in conformal geometry

In [10], Chanillo showed the close relation between a nonlinear eigenvalue minimization problem for the Laplace-Beltrami operator $-\Delta_g$ and the composite membrane problem. More precisely, let (Ω, g_0) be a 2-dimensional bounded Riemannian manifold with smooth boundary $\partial\Omega$ and consider the conformal class of the metric g_0,

$$[g_0] := \left\{ g \text{ Riemannian metric on } \Omega : \exists\, f \text{ such that } g = e^{2f} g_0 \right\}. \tag{6.1}$$

Consider another class of Riemannian metrics which is strictly contained in $[g_0]$,

$$\mathcal{C} := \{ g \in [g_0] : g \text{ satisfies (6.3) and (6.4)} \}, \tag{6.2}$$

where

$$\text{there exists a positive constant } A > 0 \text{ such that } \|f\|_{L^\infty(\Omega)} \le A; \tag{6.3}$$

and

$$\text{there exists a positive constant } M > 0 \text{ such that } \int_\Omega dV_g = \int_\Omega e^{2f} dV_{g_0} = M. \tag{6.4}$$

The problem is now to minimize the first eigenvalue of the Laplace-Beltrami operator $-\Delta_g$ with Dirichlet boundary conditions, subject to the constraints provided by the class $\mathcal{C}$. In other words, find a couple (u, g) which realizes

$$\inf_{g\in\mathcal{C}} \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega \Delta_g u\, u\, dV_g}{\int_\Omega |u|^2\, dV_g}. \tag{6.5}$$

In the same paper, it is raised the question whether similar results can be obtained for higher order conformally invariant operators, with special attention devoted to the *Paneitz operator* $P^g_{n/2}$. The problem can be stated as follows. Let (Ω, g_0) be a 4-dimensional bounded Riemannian manifold with smooth boundary $\partial\Omega$. Inside the conformal class $[g_0]$, we want to consider the smaller class of Riemannian metrics,

$$\mathcal{C} := \{g \in [g_0] : g \text{ satisfies } (6.3) \text{ and } (6.7)\}, \tag{6.6}$$

where now

$$\text{there exists a positive constant } M > 0 \text{ such that } \int_\Omega dV_g = \int_\Omega e^{4f} dV_{g_0} = M. \tag{6.7}$$

The problem is now to minimize the first eigenvalue of P^g_2 with Dirichlet boundary conditions, subject to the constraints provided by the class $\mathcal{C}$. In other words, to find a pair (u, g) which realizes

$$\inf_{g\in\mathcal{C}} \inf_{u\in H_0^2(\Omega)\setminus\{0\}} \frac{\int_\Omega P^g_2 u\, u\, dV_g}{\int_\Omega |u|^2\, dV_g}. \tag{6.8}$$

We stress that the *Paneitz operator* P^g_2 has a leading term given by the fourth order differential operator $(-\Delta_g)^2$. In particular, if we are in the flat case (i.e. g is the standard Euclidean flat metric g_E),

$$P^{g_E}_2 = \Delta^2.$$

In [10, Proposition 4] it has been proved that the problem (6.8) is equivalent to

$$\inf_{\rho\in \mathrm{P}_{g_0}} \inf_{u\in H_0^2(\Omega)\setminus\{0\}} \frac{\int_\Omega P^g_2 u\, u\, dV_{g_0}}{\int_\Omega |u|^2\, \rho dV_{g_0}}, \tag{6.9}$$

where, for fixed $0 < h < H$, $M > 0$, we have defined

$$\mathrm{P}_{g_0} := \left\{\rho : \Omega \to \mathbb{R}^+ : h \le \rho \le H,\ \int_\Omega \rho\, dV_{g_0} = M\right\}.$$

For the sake of completeness, we recall here a few facts concerning the conformal change $g = e^{2f} g_0$. The volume forms are related by

$$dV_g = e^{nf} dV_{g_0},$$

where n is the dimension of the Riemannian manifold Ω, namely 4 in our case. The Paneitz operator related to g is given by

$$P^g_2(u) = e^{-4f} P^{g_0}_2(u) \quad \text{for every } u \in C^\infty(\Omega).$$

The first problem is to understand what happens in the flat case, i.e. for $g_0 = g_E$, where g_E denotes the standard Euclidean metric. We denote by dx the volume form associated with g_E. We can notice that (6.9) can be now written as

$$\inf_{\rho\in \mathrm{P}_{g_0}} \inf_{u\in H_0^2(\Omega)\setminus\{0\}} \frac{\int_\Omega (\Delta u)^2\, dx}{\int_\Omega \rho u^2 dx},$$

which coincides with (CP). Therefore we have the following

Theorem 6.1. *There exists a pair* $(u_\infty, \rho_\infty g_0)$ *which realizes* (6.8). *In particular,*

$$\rho_\infty = e^{f_\infty} = h\,\chi_S + H\,\chi_{S^c},$$

where

$$S = \{u_\infty^2 \le t\} \quad \text{for a certain } t > 0.$$

Furthermore,

$$u_\infty \in W^{4,q}(\Omega) \cap C^{3,\gamma}(\overline{\Omega}) \quad \text{for every } q \ge 1 \text{ and } \gamma \in (0,1).$$

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