Towards analytic local sector subtraction at NNLO

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A new method for local subtraction at next-to-next-to-leading order in QCD is sketched, attempting to conjugate the minimal counterterm structure arising from a sector partition of the radiation phase space with the simplifications following from analytic integration of the counterterms.

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arXiv:1801.06458v2 [hep-ph] 24 Jul 2018

1. Introduction

The next-to-next-to-leading perturbative order (NNLO) in QCD is rapidly becoming the new accuracy standard for fixed-order cross-section predictions at colliders. Calculations beyond leading order (LO) receive contributions from virtual and real radiation; when considered separately, these contributions generate infrared and collinear (IRC) singularities, that however cancel upon combination into physical cross sections[1]. Since the complexity of the processes under consideration requires numerical techniques to evaluate the relevant amplitudes, it becomes necessary to address the problem of getting rid of IRC singularities before the final numerical evaluation.

The subtraction technique achieves this goal systematically and with no approximations, by adding and subtracting to the cross sections a set of local counterterms with the same singular behaviour as the real matrix elements in all unresolved corners of phase space. Upon analytic integration, these give rise to the same singularities as the virtual corrections. The universality of the IRC behaviour of gauge-theory amplitudes ensures the existence of such counterterms, and thus the applicability of a subtraction method.

At next-to-leading order (NLO) the problem has been solved two decades ago, and the main recipes developed in the literature are the FKS [1] and the CS [2] methods. At NNLO, a number of subtraction methods have been proposed and employed to produce important phenomenological results, among them antenna subtraction [3], sector-improved residue subtraction [4, 5], colourful subtraction [6], $\mathscr{E}$-prescription [7], projection to Born [8][2]. Still, the considerable increase in complexity in the proposed schemes in comparison with the available NLO solutions motivates further investigation, especially considering that, for some of them, complexity implies forgoing desirable features such as locality or analyticity.

In view of trying to solve the NNLO problem in full generality and at minimal computational cost, and eventually in the hope of extending the procedure to yet higher orders, we believe it is worthwhile to re-examine some of the fundamental questions about the nature of the subtraction mechanisms, such as what are the simplest possible structures capable of achieving a local subtraction, how the available freedom in the definition of counterterms can be fully exploited, and what are the ideas, among those successfully applied at NLO, that can be advantageously exported to the next order(s). In this contribution we present the preliminary results of this investigation [3], for now limited to processes featuring only final-state massless QCD partons.

2. NLO analysis

At NLO, for a generic $2 \to n$ process, the differential cross section with respect to an IRC-safe observable X can be schematically written as

$$(d\sigma_{\mathrm{NLO}} - d\sigma_{\mathrm{LO}})/dX = \int d\Phi_n V \, \delta_n + \int d\Phi_{n+1} R \, \delta_{n+1} \,, \qquad (2.1)$$

where R and V are the real and UV-renormalised virtual corrections, and $\delta_i \equiv \delta(X - X_i)$. V features up to a double $1/\varepsilon$ pole (ε being the dimensional regulator, $d = 4 - 2\varepsilon$), while R is finite

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[1]UV renormalisation and collinear factorisation are understood.

[2]Slicing methods are also available at NNLO. The main ones are q_T [9] and N-jettiness [10, 11].

[3]For further developments and details see [12].

in $d = 4$, but features up to two singular limits in the radiation phase space. The subtraction procedure amounts to adding and subtracting a counterterm $\int d\tilde{\Phi}_{n+1} K \delta_n$, with $d\tilde{\Phi}_{n+1} K$ featuring the same phase-space singularities as $d\Phi_{n+1} R$, but at the same time being sufficiently simple to be analytically integrated in d dimensions. Denoting the integrated counterterm with

$$I = \int \frac{d\tilde{\Phi}_{n+1}}{d\Phi_n} K, \qquad (2.2)$$

the subtracted cross section becomes

$$(d\sigma_{\mathrm{NLO}} - d\sigma_{\mathrm{LO}})/dX = \int d\Phi_n (V + I) \delta_n + \int \left(d\Phi_{n+1} R \delta_{n+1} - d\tilde{\Phi}_{n+1} K \delta_n \right), \qquad (2.3)$$

which is manifestly finite in $d = 4$ and integrable numerically.

2.1 FKS subtraction

The problem of finding and integrating the function K is considerably simplified by introducing, as done by FKS, a partition of the radiation phase space in sectors, in each of which only up to two identified partons can give rise to IRC singularities. This is achieved by introducing sector functions $\mathcal{W}_{ij}$ (with ij the singular pair, $j \neq i$), normalised through $\sum_{ij} \mathcal{W}_{ij} = 1$, which dampen all real-radiation singularities except the ones stemming from configurations where i becomes soft ($\mathbf{S}_i$ limit) and ij become collinear ($\mathbf{C}_{ij}$ limit). Moreover one requires that the following properties be satisfied

$$\mathbf{S}_i \sum_k \mathcal{W}_{ik} = 1, \qquad \mathbf{C}_{ij} (\mathcal{W}_{ij} + \mathcal{W}_{ji}) = 1, \qquad (2.4)$$

implying that, by summing over the sectors whose functions do not vanish in the $\mathbf{S}_i$ and $\mathbf{C}_{ij}$ limits, the functions disappear. This feature is crucial for analytic counterterm integration: since the integrated counterterm is to be eventually combined in (2.1) with the virtual contribution, which is not split into sectors, it is convenient to sum over sectors before analytic integration, thus getting rid of the explicit (and potentially complicated) functional form of $\mathcal{W}_{ij}$, by means of (2.4). The sectors are thus useful when combining the real correction with the counterterm into a finite quantity in $d = 4$, but their presence must not complicate the analytic part of the computation.

Sectors however do not uniquely define the subtraction scheme: freedom is left in the parametrisation of the radiation phase space in each sector, in the kinematic mapping that allows to factorise exactly the Born result from the radiation phase space, so as to integrate the countertem only in the latter, and in the choice of the non-singular contributions to be included in the definition of the counterterm.

In FKS, the radiation phase space in each sector ij is parametrised independently, in terms of the rescaled energy $\xi_i = 2E_i/\sqrt{s}$ of parton i ($\mathbf{S}_i = \lim_{\xi_i \to 0}$), and of the cosine $y_{ij} = \cos \theta_{ij}$ of the angle between partons ij ($\mathbf{C}_{ij} = \lim_{y_{ij} \to 1}$)[4], where all quantities are defined in the center-of-mass frame of the collision, with energy $\sqrt{s}$. The kinematic mapping is defined once the sector is specified, by means of an appropriate common Lorentz boost of all particles but i, j. With this parametrisation, the counterterm in sector ij is defined as the collection of the singular terms in the Laurent expansion of the real correction around the IRC limits,

$$d\tilde{\Phi}_{n+1}^{(ij)} K_{ij} = \left(\mathbf{S}_i + \mathbf{C}_{ij} - \mathbf{S}_i \mathbf{C}_{ij} \right) d\Phi_{n+1} R \mathcal{W}_{ij}, \qquad (2.5)$$

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[4] A third variable, the azimuth ϕ_i of parton i with respect to a given reference direction, is understood.

and the full counterterm is

$$d\tilde{\Phi}_{n+1} K = \sum_{ij} d\tilde{\Phi}_{n+1}^{(ij)} K_{ij} \,. \tag{2.6}$$

In (2.5), the ordering of the limits in the third term has been chosen arbitrarily, as the latter do commute (ξ_i and $1 - y_{ij}$ are allowed to tend to 0 independently).

2.2 Bottlenecks of FKS in view of NNLO

FKS defines a natural and compact subtraction scheme, however some of its features are not optimal towards analytical simplicity; all of them are fully manageable at NLO, owing to the straightforward structure of the relevant IRC kernels, but these seeds of complication may eventually hamper an analytic treatment of counterterms at the next perturbative orders.

As an example, by parametrising *before* defining the counterterm, FKS looses some freedom in its analytic integration: the soft limit $\mathbf{S}_i$ features an eikonal double sum $\sum_{kl} \frac{s_{kl}}{s_{ik}s_{il}}$ that results in

$$\int \mathbf{S}_i \sum_j d\Phi_{n+1} R \mathcal{W}_{ij} \propto \sum_{kl} \int d\Omega_i \frac{1 - \cos\theta_{kl}}{(1 - \cos\theta_{ki})(1 - \cos\theta_{il})} \,, \tag{2.7}$$

where $s_{ab} = 2p_a \cdot p_b$, and Ω_i is the solid angle of parton i. This is not immediately trivial because the eikonal kernel involves invariants that do not belong to the sector for which the parametrisation has been devised, and the freedom in re-parametrising them is reduced after the soft variable ξ_i has been pulled out in the limit (see right-hand side of (2.7)).

This difficulty is also partly related to the non-Lorentz-invariance of the FKS variables, which may represent a bottleneck at NNLO: the double-unresolved kernels [13, 14] are compact in terms of s_{ab}, but hardly manageable analytically if parametrised with energies and angles.

Finally, the d-dimensional radiation phase space in the FKS parametrisation is

$$\frac{d\Phi_{n+1}}{d\Phi_n} \propto d\xi_i \, dy_{ij} \left[2 - \xi_i(1 - y_{ij})\right]^{2\varepsilon} \,, \tag{2.8}$$

which is immediately integrated only because the parenthesis trivialises in all IRC limits relevant to sector ij; at NNLO it may be problematic to find a parametrisation with such a feature, that still respects the commutation properties of the composite limits (see comments below (2.6)).

2.3 Modified sector subtraction at NLO

The above bottlenecks can be alleviated by means of the following considerations. First, the singularities in sector ij are known once the identity of partons i and j is given, hence a local counterterm can be defined without referring to any specific parametrisation, by collecting the singular limits of the real-radiation matrix element, written in terms of dot products s_{ab} of four-momenta. Second, it is not necessary that all contributions to the counterterm in a sector feature the same parametrisation or kinematic mapping: the latter can be chosen so as to maximally simplify the integration of the selected contribution.

The first of these considerations allows us to introduce K_{ij} through the following procedure.

- Define the behaviour of (functions of) invariants in the singular limits:

$$\text{soft } i, \ \mathbf{S}_i: \quad s_{ia}/s_{ib} \rightarrow \text{constant}, \quad s_{ia}/s_{bc} \rightarrow 0, \quad \forall a,b,c \neq i, \tag{2.9}$$

$$\text{collinear } ij, \ \mathbf{C}_{ij}: \quad s_{ij}/s_{ab} \rightarrow 0, \quad s_{ia}/s_{ja} \rightarrow \text{independent of } a, \quad \forall ab \neq ij. \tag{2.10}$$

- Define $\mathbf{S}_i R \mathscr{W}_{ij}$ and $\mathbf{C}_{ij} R \mathscr{W}_{ij}$ as the most singular terms in the Laurent expansion of $R \mathscr{W}_{ij}$ around the IRC limits, according to the scaling in (2.9) and (2.10). In particular

$$\mathbf{S}_i R \; \propto \; -\delta_{ig} \sum_{kl} \frac{s_{kl}}{s_{ki} s_{il}} B_{kl} , \qquad \mathbf{C}_{ij} R \; \propto \; \frac{1}{s_{ij}} P(z_{ij}) B , \qquad (2.11)$$

where δ_{ig} forces the soft parton to be a gluon, B and B_{kl} are the Born and colour-linked Born squared matrix elements, P is the relevant Altarelli-Parisi collinear kernel, and $z_{ij} = s_{ir}/(s_{ir} + s_{jr})$, with arbitrary $r \neq ij$.

- Define the counterterm in sector ij as

$$d\Phi_{n+1} K_{ij} \; = \; d\Phi_{n+1} \big(\mathbf{S}_i + \mathbf{C}_{ij} - \mathbf{S}_i \mathbf{C}_{ij} \big) R \mathscr{W}_{ij} \; = \; d\Phi_{n+1} \left[1 - (1 - \mathbf{S}_i)(1 - \mathbf{C}_{ij}) \right] R \mathscr{W}_{ij} . \quad (2.12)$$

The order in which the $\mathbf{S}_i$ and $\mathbf{C}_{ij}$ operators appear in the composite limit is arbitrary. While in FKS the chosen parametrisation must explicitly realise such a commutation of limits, in order for composite residues to be defined, in this modified framework commutation naturally stems from fundamental properties of the soft and collinear limits, which are physically independent. Once the counterterm is defined as in (2.12), a subsequent parametrisation of the latter in terms of non-independent variables is allowed, and does not spoil any of its properties.

Equations (2.5) and (2.12) are structurally very similar and clearly share the same singular terms, showing that the modified scheme defines as minimal a local subtraction procedure as the original FKS; the two prescriptions differ by finite contributions, precisely those that make the counterterm in (2.12) parametrisation-independent. Moreover, in (2.12) the phase space associated with the counterterm is exact, namely the soft and collinear limits are applied only to matrix elements and sector functions. While this property is not crucial, and could immediately be lifted if required by computational convenience, it displays the enhanced flexibility of the modified scheme: as a parametrisation has not been chosen at this point yet, one has still the freedom to select one in which the phase space is trivial everywhere, without being compelled to evaluate the latter in the IRC limits.

The second of the above considerations allows to choose kinematic mappings and parametrisation independently of the sector. A particularly convenient choice of mapping is the one introduced by CS, where the $n+1$ real momenta p_i are mapped on n Born-like momenta $\bar{p}_j$ (the latter entering the computation as arguments of B and B_{kl} in (2.11)) through

$$\bar{p}_c = \frac{1}{1-y} p_c , \qquad \bar{p}_{[ab]} = p_a + p_b - \frac{y}{1-y} p_c , \qquad \bar{p}_l = p_l , \quad \forall l \neq a, b, c , \qquad (2.13)$$

$$y = y_{abc} = \frac{s_{ab}}{s_{ab} + s_{ac} + s_{bc}} , \qquad z = z_{abc} = \frac{s_{ac}}{s_{ac} + s_{bc}} . \qquad (2.14)$$

In the hard-collinear counterterm in sector ij, $(\mathbf{C}_{ij} - \mathbf{S}_i \mathbf{C}_{ij}) R \mathscr{W}_{ij}$ in (2.12), labels are assigned as $a = i$, $b = j$, $c = r$, where i and j define the sector, while r appears in the definition of z_{ij} in (2.11). In the soft counterterm, $\mathbf{S}_i R \mathscr{W}_{ij}$, *each term* of the sum over kl *is mapped differently*, with $a = i$, $b = k$, $c = l$. The phase space is parametrised in terms of variables y_{abc} and z_{abc} defined in (2.14), with labels abc assigned according to the relevant kinematic mapping, as just described. In particular, in a given sector, not all contributions to the counterterm are parametrised in the same way, the latter indeed being the feature that complicates the integration of the soft counterterm in FKS.

4

Each term in the eikonal double sum is now straightforwardly integrated:

$$
\int \frac{d\Phi_{n+1}}{d\Phi_n} \frac{s_{kl}}{s_{ki}s_{il}} \propto (\bar{p}_{[ik]} \cdot \bar{p}_l)^{-\varepsilon} \int_0^1 dz \int_0^1 dy \left[y(1-y)^2 z(1-z) \right]^{-\varepsilon} \frac{(1-y)(1-z)}{yz},
$$
$$
= (\bar{p}_{[ik]} \cdot \bar{p}_l)^{-\varepsilon} B(-\varepsilon, 2-\varepsilon) B(-\varepsilon, 2-2\varepsilon), \qquad (2.15)
$$

where $z = z_{ikl}$, $y = y_{ikl}$, and B is the Euler beta function, a result valid to all orders in ε.

The modified sector subtraction outlined in this section successfully works at NLO, as the integrated counterterm can be shown to analytically reproduce all virtual poles. The method, to some extent, bridges the FKS and CS approaches, retaining the strengths of both, in particular sector partition and minimal counterterm structure from FKS, and Lorentz invariance and phase-space mappings from CS. We believe this approach to be more easily exportable to NNLO, since it achieves the maximal possible simplification as far as analytic integration is concerned.

3. NNLO analysis

At NNLO, the differential cross section with respect to IRC-safe observable X is

$$
(d\sigma_{\mathrm{NNLO}} - d\sigma_{\mathrm{NLO}})/dX = \int d\Phi_n VV \, \delta_n + \int d\Phi_{n+1} RV \, \delta_{n+1} + \int d\Phi_{n+2} RR \, \delta_{n+2}, \qquad (3.1)
$$

where RR, VV, RV, are the double-real and UV renormalised double-virtual and real-virtual corrections. VV features up to a quadruple $1/\varepsilon$ pole, RR is finite in $d = 4$, but features up to four phase-space singularities, and RV has up to a double $1/\varepsilon$ pole and diverges doubly in the radiation phase space. The subtraction procedure amounts to adding and subtracting $\int d\tilde{\Phi}_{n+2} \left[K^{(\mathbf{1})} \delta_{n+1} + (K^{(\mathbf{12})} + K^{(\mathbf{2})}) \delta_n \right]$, as well as $\int d\tilde{\Phi}_{n+1} K^{(\mathbf{RV})} \delta_n$, where $K^{(\mathbf{1})}$ and $(K^{(\mathbf{12})} + K^{(\mathbf{2})})$ are the single- and double-unresolved counterterms, containing all singularities of RR in the limits where one or two partons become unresolved[5], while $K^{(\mathbf{RV})}$ is the real-virtual counterterm, featuring the same phase-space singularities as RV. Denoting the corresponding integrated counterterms with

$$
I^{(\mathbf{p})} = \int \frac{d\tilde{\Phi}_{n+2}}{d\Phi_{n+2-p}} K^{(\mathbf{p})}, \qquad I^{(\mathbf{12})} = \int \frac{d\tilde{\Phi}_{n+2}}{d\Phi_{n+1}} K^{(\mathbf{12})}, \qquad I^{(\mathbf{RV})} = \int \frac{d\tilde{\Phi}_{n+1}}{d\Phi_n} K^{(\mathbf{RV})}, \qquad p = 1, 2,
$$
$$
(3.2)
$$

the subtracted cross section becomes

$$
(d\sigma_{\mathrm{NNLO}} - d\sigma_{\mathrm{NLO}})/dX = \int d\Phi_n (VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})}) \delta_n
$$
$$
+ \int \left[(d\Phi_{n+1} RV + d\tilde{\Phi}_{n+1} I^{(\mathbf{1})}) \delta_{n+1} - d\tilde{\Phi}_{n+1} (K^{(\mathbf{RV})} - I^{(\mathbf{12})}) \delta_n \right]
$$
$$
+ \int \left[d\Phi_{n+2} RR \, \delta_{n+2} - d\tilde{\Phi}_{n+2} K^{(\mathbf{1})} \delta_{n+1} - d\tilde{\Phi}_{n+2} (K^{(\mathbf{12})} + K^{(\mathbf{2})}) \delta_n \right]. \quad (3.3)
$$

$I^{(\mathbf{1})}$ features the same $1/\varepsilon$ poles as RV, $I^{(\mathbf{12})}$ features the same $1/\varepsilon$ poles as $K^{(\mathbf{RV})}$, while the sum $I^{(\mathbf{2})} + I^{(\mathbf{RV})}$ has the same $1/\varepsilon$ poles as VV, ensuring all contributions are finite in $d = 4$ and integrable numerically.

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[5] In the case of the double-unresolved counterterm, $K^{(\mathbf{2})}$ collects all 'homogeneous' double-unresolved configurations, namely the ones where two partons become unresolved with the same scaling, while $K^{(\mathbf{12})}$ contains all 'hierarchical' double-unresolved configurations, namely the ones where two partons become unresolved in a strongly-ordered manner.

3.1 Modified sector subtraction at NNLO

In order to define an analytic subtraction procedure at NNLO, it is convenient to divide the phase space in sectors, in each of which only up to four identified partons can give rise to IRC singularities. Each sector function $\mathscr{W}_{abcd}$ (*abcd* being the singular combinations, $b \neq a$, $c \neq a$, $d \neq a, c$), normalised through $\sum_{abcd} \mathscr{W}_{abcd} = 1$, dampens all double-real singularities, except a single-soft and a single-collinear ($\mathbf{S}_i$, $\mathbf{C}_{ij}$ in sectors *ijkj*, *ijjk*, and *ijkl*), a double-soft ($\mathbf{S}_{ik}$ in sectors *ijkj* and *ijkl*, $\mathbf{S}_{ij}$ in sector *ijjk*), a double-collinear ($\mathbf{C}_{ikj}$ in sectors *ijkj* and *ijjk*, $\mathbf{C}_{ijkl}$ in sector *ijkl*), and a soft-collinear ($\mathbf{SC}_{ijk}$ in sector *ijkj*, $\mathbf{SC}_{ijk}$ and $\mathbf{CS}_{ijk}$ in sector *ijjk*, $\mathbf{SC}_{ikl}$ and $\mathbf{CS}_{ijk}$ in sector *ijkl*). To clarify: in configuration $\mathbf{S}_{ab}$, partons *ab* are all soft; in $\mathbf{C}_{abc}$, partons *abc* are all collinear, while in $\mathbf{C}_{abcd}$ the four partons become collinear in pairs; in $\mathbf{SC}_{abc}$, hierarchically, *a* becomes soft, then *bc* become collinear; in $\mathbf{CS}_{abc}$, *ab* become collinear, then *c* becomes soft. $\mathbf{SC}_{abc}$ and $\mathbf{CS}_{bca}$ are the same projector when acting on matrix elements, but not on sector functions. Roughly speaking, sectors *ijkj* and *ijjk* select singularities associated with splitting $a \to ijk$, while sector *ijkl* is associated with independent splittings $a_1 \to ij$, and $a_2 \to kl$.

The next step, in analogy with (2.4), is to enforce the constraint that sector functions disappear upon summation over the sectors whose functions do not vanish in double-unresolved limits. This requirement, crucial for the analytic integration of $K^{(2)}$, reads

$$\mathbf{S}_{ik} \sum_{d \neq i,k} \left(\sum_{b \neq i} \mathscr{W}_{ibkd} + \sum_{b \neq k} \mathscr{W}_{kbid} \right) = 1, \qquad \mathbf{C}_{ikj} \sum_{abc \,\in\, \mathrm{perm}\, ijk} \left(\mathscr{W}_{abbc} + \mathscr{W}_{abcb} \right) = 1, \qquad (3.4)$$

and analogously for projectors $\mathbf{C}_{ijkl}$, $\mathbf{SC}_{abc}$, and $\mathbf{CS}_{abc}$, where perm $ik = ik$, ki, while perm $ijk = ijk$, ikj, jik, jki, kij, kji. At NNLO, however, one more constraint has to be satisfied: as *RV* is split into NLO-type sectors $\mathscr{W}_{ij}$, since it has single-real kinematics, $I^{(1)}$ must feature the same $1/\varepsilon$ poles as *RV*, *NLO sector by NLO sector*, in order for $d\Phi_{n+1} RV + d\tilde{\Phi}_{n+1} I^{(1)}$ to be finite for each *ij* independently. This is achieved by requiring the NNLO sector functions to factorise the NLO ones in the single-unresolved limits, as

$$\mathbf{C}_{ij} \mathscr{W}_{ijkj} \sim \bar{\mathscr{W}}_{k[ij]} \, \mathbf{C}_{ij} \mathscr{W}_{ij}, \qquad \mathbf{S}_i \mathscr{W}_{ijkj} \sim \bar{\mathscr{W}}_{kj} \, \mathbf{S}_i \mathscr{W}_{ij}, \qquad (3.5)$$

$$\mathbf{C}_{ij} \mathscr{W}_{ijjk} \sim \bar{\mathscr{W}}_{[ij]k} \, \mathbf{C}_{ij} \mathscr{W}_{ij}, \qquad \mathbf{S}_i \mathscr{W}_{ijjk} \sim \bar{\mathscr{W}}_{jk} \, \mathbf{S}_i \mathscr{W}_{ij}, \qquad (3.6)$$

$$\mathbf{C}_{ij} \mathscr{W}_{ijkl} \sim \bar{\mathscr{W}}_{kl} \, \mathbf{C}_{ij} \mathscr{W}_{ij}, \qquad \mathbf{S}_i \mathscr{W}_{ijkl} \sim \bar{\mathscr{W}}_{kl} \, \mathbf{S}_i \mathscr{W}_{ij}, \qquad (3.7)$$

where the bars denote kinematic mappings analogous to the ones described in (2.13).

The local counterterms are defined in analogy with (2.12), as

$$K^{(1)}_{ijkj} + K^{(12)}_{ijkj} + K^{(2)}_{ijkj} = \left[1 - (1 - \mathbf{S}_i)(1 - \mathbf{C}_{ij})(1 - \mathbf{S}_{ik})(1 - \mathbf{C}_{ikj})(1 - \mathbf{SC}_{ijk}) \right] RR \, \mathscr{W}_{ijkj}, \qquad (3.8)$$

$$K^{(1)}_{ijjk} + K^{(12)}_{ijjk} + K^{(2)}_{ijjk} = \left[1 - (1 - \mathbf{S}_i)(1 - \mathbf{C}_{ij})(1 - \mathbf{S}_{ij})(1 - \mathbf{C}_{ikj})(1 - \mathbf{SC}_{ijk})(1 - \mathbf{CS}_{ijk}) \right] RR \, \mathscr{W}_{ijjk},$$

$$K^{(1)}_{ijkl} + K^{(12)}_{ijkl} + K^{(2)}_{ijkl} = \left[1 - (1 - \mathbf{S}_i)(1 - \mathbf{C}_{ij})(1 - \mathbf{S}_{ik})(1 - \mathbf{C}_{ijkl})(1 - \mathbf{SC}_{ikl})(1 - \mathbf{CS}_{ijk}) \right] RR \, \mathscr{W}_{ijkl}.$$

The kernels $\mathbf{S}_{ik}$, $\mathbf{C}_{ikj}$, $\mathbf{SC}_{ijk}$, and $\mathbf{CS}_{ijk}$ are universal, and have been computed in [13, 14, 15]. The order of the various operators in the composite limits is arbitrary, as all limits commute.

Equations (3.8) are appropriate to define local counterterms, but redundant: in particular, *RR* can feature at most four singularities, hence not all operators that appear in those equations are

6

'primary', namely carry independent information on the singularity structure of *RR*. These redundancies are readily eliminated by considering the idempotence of projection operators: for instance, once $\mathbf{SC}_{iab}$ has been applied to a given quantity, further acting on it with $\mathbf{S}_i$ does not produce any effect, and similarly for the action of $\mathbf{CS}_{ijk}$ after $\mathbf{C}_{ij}$ has been applied. The same is true for $\mathbf{C}_{ijkl}$ and $\mathbf{C}_{ij}$ when acting on matrix elements, but not on sector functions. One thus has $\mathbf{S}_i\mathbf{SC}_{ijk} = \mathbf{SC}_{ijk}$, $\mathbf{C}_{ij}\mathbf{CS}_{ijk} = \mathbf{CS}_{ijk}$, which implies

$$(1-\mathbf{S}_i)\mathbf{SC}_{ijk} \;=\; (1-\mathbf{S}_i)\mathbf{SC}_{ikl} \;=\; (1-\mathbf{C}_{ij})\mathbf{C}_{ijkl} \;=\; 0. \tag{3.9}$$

As a consequence, all soft-collinear double-unresolved limits, $\mathbf{SC}_{ijk}$, $\mathbf{SC}_{ikl}$, and $\mathbf{CS}_{ijk}$, completely disappear from the sum $K^{(12)}+K^{(2)}$ (see also [16] about the redundancy of the soft-collinear limit)[6]. Equations (3.8) finally can be simplified to (with $T = ijjk,\, ijkj,\, ijkl$)

$$\begin{aligned}
K_T^{(1)} &= [\mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)]RR\,\mathscr{W}_T, \tag{3.10}\\
K_{ijjk}^{(2)} &= [\mathbf{S}_{ij} + \mathbf{C}_{ikj}(1-\mathbf{S}_{ij}) + \mathbf{SC}_{ijk}(1-\mathbf{S}_{ij})(1-\mathbf{C}_{ikj})]RR\,\mathscr{W}_{ijjk},\\
K_{ijkj}^{(2)} &= [\mathbf{S}_{ik} + \mathbf{C}_{ikj}(1-\mathbf{S}_{ik}) + (\mathbf{SC}_{ijk}+\mathbf{CS}_{ijk})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ikj})]RR\,\mathscr{W}_{ijkj},\\
K_{ijkl}^{(2)} &= [\mathbf{S}_{ik} + \mathbf{C}_{ijkl}(1-\mathbf{S}_{ik}) + (\mathbf{SC}_{ikl}+\mathbf{CS}_{ijk})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijkl})]RR\,\mathscr{W}_{ijkl},\\
K_{ijjk}^{(12)} &= -\{[\mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)][\mathbf{S}_{ij} + \mathbf{C}_{ikj}(1-\mathbf{S}_{ij})] + \mathbf{SC}_{ijk}(1-\mathbf{S}_{ij})(1-\mathbf{C}_{ikj})\}RR\,\mathscr{W}_{ijjk},\\
K_{ijkj}^{(12)} &= -\{[\mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)][\mathbf{S}_{ik} + \mathbf{C}_{ikj}(1-\mathbf{S}_{ik})] + (\mathbf{SC}_{ijk}+\mathbf{CS}_{ijk})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ikj})\}RR\,\mathscr{W}_{ijkj},\\
K_{ijkl}^{(12)} &= -\{[\mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)][\mathbf{S}_{ik} + \mathbf{C}_{ijkl}(1-\mathbf{S}_{ik})] + (\mathbf{SC}_{ikl}+\mathbf{CS}_{ijk})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijkl})\}RR\,\mathscr{W}_{ijkl},
\end{aligned}$$

where we have separated the counterterms according to the type of singularities they feature: single-unresolved in $K^{(1)}$, pure double-unresolved in $K^{(2)}$, overlaps of single- and double-unresolved projectors in $K^{(12)}$.

3.2 Counterterm integration

The integration of the double-unresolved counterterm proceeds from the definitions of $K^{(2)}$ and crucially benefits from the defining properties in (3.4), which allow to completely get rid of sector functions before analytical integration. Indeed one gets

$$\begin{aligned}
I^{(2)} = \int \frac{d\Phi_{n+2}}{d\Phi_n} \sum_i \Big[&\sum_{j>i}\mathbf{S}_{ik} + \sum_{j>i}\sum_{k>j}\mathbf{C}_{ikj}(1-\mathbf{S}_{ik}-\mathbf{S}_{ij}-\mathbf{S}_{jk}) \\
&+ \sum_{j>i}\sum_{\substack{k>i\\k\neq j}}\sum_{\substack{l>k\\l\neq j}}\mathbf{C}_{ijkl}(1-\mathbf{S}_{ik}-\mathbf{S}_{jk}-\mathbf{S}_{il}-\mathbf{S}_{jl}) \\
&+ \sum_{j\neq i}\sum_{\substack{k\neq i\\k>j}}\mathbf{SC}_{ijk}(1-\mathbf{S}_{ik}-\mathbf{S}_{ij})\Big(1-\mathbf{C}_{ikj}-\sum_{l\neq i,j,k}\mathbf{C}_{iljk}\Big) \\
&+ \sum_{j>i}\sum_{k\neq i,j}\mathbf{CS}_{ijk}(1-\mathbf{S}_{ik}-\mathbf{S}_{jk})\Big(1-\mathbf{C}_{ikj}-\sum_{l\neq i,j,k}\mathbf{C}_{ijkl}\Big)\Big]RR. \tag{3.11}
\end{aligned}$$

Since the sector functions have disappeared from the integrand, and only the singular kernels are left over, the integration can be managed analytically in $d = 4 - 2\varepsilon$ dimensions. As an explicit

---

[6]The integrals $I^{(2)}$ and $I^{(12)}$ have to be evaluated separately, see Eq. (3.3), hence the kernels $\mathbf{SC}$ and $\mathbf{CS}$ do contribute in that case, even if they would cancel in the sum.

example of the computation, consider the case in which a $q\bar{q}$ pair becomes soft, which leads to the double-soft kernel [14]

$$\mathbf{S}_{ik}RR \propto (\alpha_s \mu^{2\varepsilon})^2 \, T_R \sum_{l,m=1}^{n} B_{lm} \frac{s_{il}s_{km} + s_{im}s_{kl} - s_{ik}s_{lm}}{s_{ik}^2(s_{il} + s_{kl})(s_{im} + s_{km})}, \tag{3.12}$$

with μ the renormalisation scale. Each term in the double sum in (3.12) is associated with a different CS mapping, as was the case for the soft term at NLO, in order to optimise the parametrisation for each addend separately. Denoting with z', y' the CS variables relevant to dipole (ik, l), and with z, y those relevant to dipole $([ik]l, m)$, the double-soft integrand for $l \neq m$ (for $l = m$ the result is trivial) after azimuthal integration is

$$\frac{s_{il}s_{km} + s_{im}s_{kl} - s_{ik}s_{lm}}{s_{ik}^2(s_{il} + s_{kl})(s_{im} + s_{km})} \propto \frac{z'(1-z')}{y^2 y'^2} \frac{z - y'(1-z)}{z + y'(1-z)}, \tag{3.13}$$

to be integrated with the measure $\int_0^1 dy' dz' dy dz \left[y'(1-y')^2 y^2 (1-y)^2 z(1-z) \right]^{-\varepsilon} (1-y')y(1-y)$. The final result for $n = 2$ Born-level particles, integrated over the Born phase space, and with prefactors reinstated, reads

$$\int d\Phi_{n+2} \mathbf{S}_{ik}RR = \sigma_{\text{LO}} \left(\frac{\alpha_s}{2\pi} \right)^2 T_R C_F \left(\frac{\mu^2}{s} \right)^{2\varepsilon}$$
$$\times \left[-\frac{1}{3\varepsilon^3} - \frac{17}{9\varepsilon^2} + \frac{1}{\varepsilon} \left(\frac{7}{18}\pi^2 - \frac{232}{27} \right) + \frac{38}{9}\zeta_3 + \frac{131}{54}\pi^2 - \frac{2948}{81} \right] + \mathcal{O}(\varepsilon). \tag{3.14}$$

The double-collinear limit relevant for a splitting $q \to qq'\bar{q}'$ is mapped and parametrised in a similar fashion, resulting in an integral of comparable complexity. One finds

$$\int d\Phi_{n+2} \mathbf{C}_{ikj}RR = \sigma_{\text{LO}} \left(\frac{\alpha_s}{2\pi} \right)^2 T_R C_F \left(\frac{\mu^2}{s} \right)^{2\varepsilon}$$
$$\times \left[-\frac{1}{3\varepsilon^3} - \frac{31}{18\varepsilon^2} + \frac{1}{\varepsilon} \left(\frac{1}{2}\pi^2 - \frac{889}{108} \right) + \frac{80}{9}\zeta_3 + \frac{31}{12}\pi^2 - \frac{23941}{648} \right] + \mathcal{O}(\varepsilon). \tag{3.15}$$

It has to be noted that double-unresolved limits involving gluons are more complicated than the one detailed here, but still manageable analytically.

3.3 Proof-of-concept example

Considering the $T_R C_F$ contribution to the NNLO total cross section for $e^+e^- \to q(1)\bar{q}(2)$, the double-real process is $e^+e^- \to q(1)\bar{q}(2)q'(3)\bar{q}'(4)$. All relevant matrix elements can be found in [17, 18, 19]. Limits $\mathbf{S}_{34}$, $\mathbf{C}_{134}$, $\mathbf{C}_{234}$, and $\mathbf{C}_{34}$ are non-zero, and the integrated counterterms read

$$\int d\Phi_n I^{(2)} = \int d\Phi_{n+2} \left[\mathbf{S}_{34} + \mathbf{C}_{134}(1 - \mathbf{S}_{34}) + \mathbf{C}_{234}(1 - \mathbf{S}_{34}) \right] RR$$
$$= \sigma_{\text{LO}} \left(\frac{\alpha_s}{2\pi} \right)^2 T_R C_F \left(\frac{\mu^2}{s} \right)^{2\varepsilon} \left[-\frac{1}{3\varepsilon^3} - \frac{14}{9\varepsilon^2} + \frac{1}{\varepsilon} \left(\frac{11}{18}\pi^2 - \frac{425}{54} \right) + \frac{122}{9}\zeta_3 + \frac{74}{27}\pi^2 - \frac{12149}{324} \right],$$
$$I^{(1)} = I^{(1)}_{12} + I^{(1)}_{1[34]} + I^{(1)}_{2[34]},$$
$$I^{(1)}_{hq} = \mathcal{W}_{hq} \int \frac{d\Phi_{n+2}}{d\Phi_{n+1}} \mathbf{C}_{34} RR = -\frac{\alpha_s}{2\pi} \left(\frac{\mu^2}{s} \right)^{\varepsilon} \frac{2}{3} T_R \left[\frac{1}{\varepsilon} - \ln \frac{\bar{s}_{[34]r}}{s} + \frac{8}{3} \right] R\mathcal{W}_{hq} + \mathcal{O}(\varepsilon). \tag{3.16}$$

The structure of $\mathscr{W}$ functions appearing in the addends of $I^{(1)}$ is precisely the one of the subtracted real-virtual contribution, split into NLO sectors. The sums $RV_{hq}^{\text{fin}} \equiv RV \mathscr{W}_{hq} + I_{hq}^{(1)}$ are finite in $d = 4$

$$RV_{hq}^{\text{fin}} = -\frac{\alpha_{\text{s}}}{2\pi} \frac{2}{3} T_R \left(\ln \frac{\mu^2}{\bar{s}_{r[34]}} + \frac{8}{3} \right) R \mathscr{W}_{hq} + \mathcal{O}(\varepsilon), \tag{3.17}$$

with $r = 1$ or 2 when $hq = 12$, while $r = 3 - h$ in the other cases. Analogously, the sum $K_{hq}^{(\text{RV})} - I_{hq}^{(12)}$ is finite in $d = 4$, and reads

$$K_{hq}^{(\text{RV})} - I_{hq}^{(12)} = -\frac{\alpha_{\text{s}}}{2\pi} \frac{2}{3} T_R \left(\ln \frac{\mu^2}{\bar{s}_{r[34]}} + \frac{8}{3} \right) [\mathbf{S}_h + \mathbf{C}_{hq}(1 - \mathbf{S}_h)] R \mathscr{W}_{hq} + \mathcal{O}(\varepsilon). \tag{3.18}$$

The integrated real-virtual counterterm is

$$
\begin{aligned}
\int d\Phi_n I^{(\text{RV})} &= \sum_{ij} \int d\Phi_{n+1} K_{ij}^{(\text{RV})} \\
&= \frac{\alpha_{\text{s}}}{2\pi} \frac{2}{3} \frac{T_R}{\varepsilon} \int d\Phi_{n+1} \left[\mathbf{S}_{[34]} + \mathbf{C}_{1[34]}(1 - \mathbf{S}_{[34]}) + \mathbf{C}_{2[34]}(1 - \mathbf{S}_{[34]}) \right] R \\
&= \sigma_{\text{LO}} \left(\frac{\alpha_{\text{s}}}{2\pi} \right)^2 T_R C_F \left(\frac{\mu^2}{s} \right)^{\varepsilon} \left[\frac{4}{3\varepsilon^3} + \frac{2}{\varepsilon^2} + \frac{1}{\varepsilon} \left(-\frac{7}{9}\pi^2 + \frac{20}{3} \right) - \frac{100}{9}\zeta_3 - \frac{7}{6}\pi^2 + 20 \right].
\end{aligned}
$$

Collecting all contributions, for instance setting $\mu = 0.35\sqrt{s}$, one gets

$$
\begin{aligned}
\int d\Phi_n \left(VV + I^{(2)} + I^{(\text{RV})} \right) &= \sigma_{\text{LO}} \left(\frac{\alpha_{\text{s}}}{2\pi} \right)^2 T_R C_F \left(\frac{8}{3}\zeta_3 - \frac{1}{9}\pi^2 - \frac{44}{9} - \frac{4}{3}\ln\frac{\mu^2}{s} \right) \\
&= \sigma_{\text{LO}} \left(\frac{\alpha_{\text{s}}}{2\pi} \right)^2 T_R C_F \times 0.01949914, \tag{3.19}
\end{aligned}
$$

$$\int d\Phi_{n+1} \left(RV + I^{(1)} - K^{(\text{RV})} \right) = \sigma_{\text{LO}} \left(\frac{\alpha_{\text{s}}}{2\pi} \right)^2 T_R C_F \times (-0.90635 \pm 0.00011), \tag{3.20}$$

$$\int d\Phi_{n+2} \left(RR - K^{(1)} - K^{(2)} \right) = \sigma_{\text{LO}} \left(\frac{\alpha_{\text{s}}}{2\pi} \right)^2 T_R C_F \times (+2.29491 \pm 0.00038), \tag{3.21}$$

where (3.19) is a fully analytic result, in (3.20) the cancellation of $1/\varepsilon$ poles is analytic, and the remaining finite integral is numerical, while (3.21) is fully numerical.

By summing (3.19) to (3.21), the NNLO correction obtained with the subtraction method is

$$\frac{1}{\left(\frac{\alpha_{\text{s}}}{2\pi} \right)^2 T_R C_F} \frac{\sigma_{\text{NNLO}} - \sigma_{\text{NLO}}}{\sigma_{\text{LO}}} = 1.40806 \pm 0.00040, \tag{3.22}$$

to be compared with the analytic result $-11/2 + 4\zeta_3 - \ln(\mu^2/s) = 1.40787186$. The plot below shows that the renormalisation-scale dependence is also correctly reproduced.
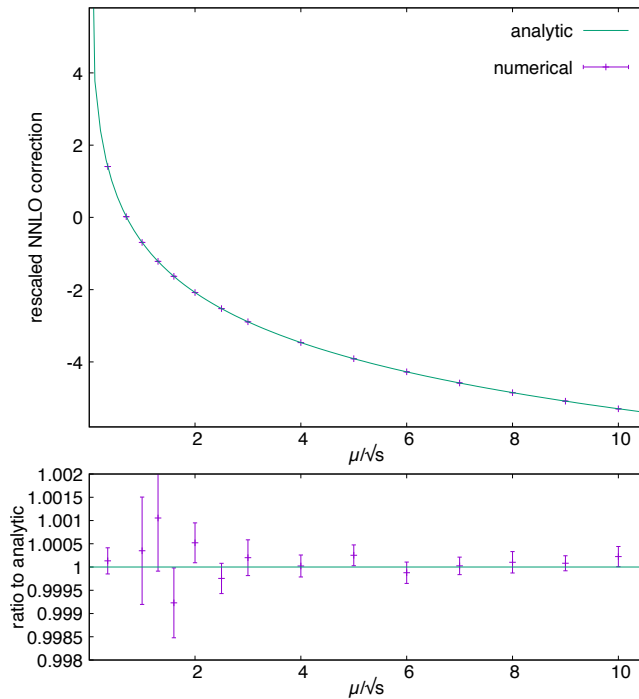
4. Conclusions

We have presented the theoretical basis of a new method for NNLO local sector subtraction, aiming at minimality in the definition of the counterterms, and analyticity in their integration. The method has been presented in the NLO case, and applied to a simplified case at NNLO, displaying the expected properties. Generalisations to the complete NNLO case are ongoing.

Acknowledgements

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