Smoothable Gorenstein points via marked schemes and double-generic initial ideals

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Abstract

Over an infinite field K with $\mathrm{char}(K) \neq 2, 3$, we investigate smoothable Gorenstein K-points in a punctual Hilbert scheme from a new point of view, which is based on properties of double-generic initial ideals and of marked schemes. We obtain the following results: (i) points defined by graded Gorenstein K-algebras with Hilbert function $(1, 7, 7, 1)$ are smoothable, in the further hypothesis that K is algebraically closed; (ii) the Hilbert scheme Hilb^7_{16} has at least three irreducible components. The properties of marked schemes give us a simple method to compute the Zariski tangent space to a Hilbert scheme at a given K-point, which is very useful in this context. Over an algebraically closed field of characteristic 0, we also test our tools to find the already known result that points defined by graded Gorenstein K-algebras with Hilbert function $(1, 5, 5, 1)$ are smoothable. In characteristic zero, all the results about smoothable points also hold for local Artin Gorenstein K-algebras.

Introduction

Let K be an infinite field of characteristic other than 2 and 3. For any positive integer n and an admissible Hilbert polynomial $p(t)$, we denote by $\mathrm{Hilb}^n_{p(t)}$ the Hilbert scheme parameterizing the projective subschemes of $\mathbb{P}^n_K$ with Hilbert polynomial $p(t)$. We deal with punctual Hilbert schemes, hence with constant Hilbert polynomials, and when we take a point we mean a K-valued point (K-point, for short), i.e. a closed point with residue field K.

Let $p(t) = d$ be the Hilbert polynomial of d points. The *smoothable component* $\mathcal{R}^n_d$ *of* $\mathrm{Hilb}^n_{p(t)}$ is the closure of the open set of points corresponding to ideals of d distinct points, i.e. the rational component of $\mathrm{Hilb}^n_{p(t)}$ containing the point corresponding to the lex-segment ideal.

A zero-dimensional subscheme X is *smoothable* if it belongs to the smoothable component $\mathcal{R}^n_d$ or, equivalently, the K-algebra A defining $X = \mathrm{Proj}\,(A)$ is isomorphic to the special fiber of a flat one-parameter family of K-algebras with smooth general point (e.g. [Cartwright et al., 2009, Lemma 4.1], see also [Iarrobino and Kanev, 1999, Definitions 5.16 and 6.20]).

As noted in [Cartwright et al., 2009, Remark 1.6] for Hilbert schemes of points, every point in $\mathrm{Hilb}^n_{p(t)}$ has an open neighborhood that can be studied by suitable "affine" techniques. In the same context, a similar approach is also used in [Miller and Sturmfels, 2005, Chapter 18].

So, up to a suitable change of coordinates, we can identify every point of a punctual Hilbert scheme Hilb_d^n with an ideal in $K[x_1, \dots, x_n]$, non-necessarily homogeneous, with affine Hilbert polynomial $p(t) = d$.

Due to the structure theorem of Artin rings and to the fact that direct sums commute with limits of flat families, a zero-dimensional subscheme X is smoothable if and only if the same is true for all its irreducible components (e.g. [Casnati and Notari, 2011, page 1245], [Cartwright et al., 2009, Section 4]). This observation motivates interest for the so-called *elementary components* of a punctual Hilbert scheme, i.e. components whose points parameterize zero-dimensional subschemes with support of cardinality one (see [Jelisiejew, 2017] for a very recent contribution in this context). Hence, the problem of detecting smoothable points is connected to the study of ideals I such that R/I is a local K-algebra.

This paper is devoted to investigate when Gorenstein points in a punctual Hilbert scheme are smoothable. In particular, we are interested in studying Gorenstein points defined by graded (Artin) K-algebras with Hilbert function $(1, 7, 7, 1)$, that is the only case not treated in the range considered in [Iarrobino and Kanev, 1999, Lemma 6.21] for the detection of nonsmoothable points in a punctual Hilbert scheme in characteristic 0. Observe that a graded Artin K-algebra is necessarily local, in particular defines a scheme supported on a single point. The vice versa holds on an algebraically closed field of characteristic zero, that is every local Artin Gorenstein K-algebra is graded, due to [Elias and Rossi, 2012, Theorem 3.3].

The study of Gorenstein smoothable points is strictly related to the study of the irreducibility of the Gorenstein locus in a Hilbert scheme. In this context, it is well-known that a punctual Hilbert scheme Hilb_d^n is irreducible if $n = 2$ (see [Fogarty, 1968]) and if $d \leq 7$ for $n \geq 3$ (see [Cartwright et al., 2009]). Moreover, the Gorenstein locus of Hilb_d^n is irreducible if $d \leq 13$ for every n (see [Casnati and Notari, 2009, 2011, 2014; Casnati et al., 2015] and the references therein). Other relevant and also more general results about irreducibility in a Hilbert scheme are due to Ellingsrud and Iarrobino.

We prove the following results:

(i) The graded Gorenstein K-algebras with Hilbert function $(1, 7, 7, 1)$ are smoothable, in the further hypothesis that K is algebraically closed (see Theorem 3.6).

(ii) There are at least three irreducible components in Hilb_{16}^7 (see Section 4).

Moreover, we show how our arguments apply to prove the now known result that graded Gorenstein K-algebras with Hilbert function $(1, 5, 5, 1)$ are smoothable, on an algebraically closed field of characterisic 0 (see Theorem 5.7).

An outline of some results of the present paper was described by ansatz in [Bertone et al., 2012] as an application of the constructive methods about marked bases in an affine framework that were lately deeply studied and completely described in [Bertone et al., 2017a]. Here, we give an extensive description of the outcome of our study. Different proofs of the case $(1, 5, 5, 1)$ were presented in [Jelisiejew, 2014], contemporary to our first version given in [Bertone et al., 2012], and later in [Casnati et al., 2015] when $\mathrm{char}(K) \neq 2, 3$.

We obtain our results facing the problem from a new point of view: we apply the notion of double-generic initial ideal (see [Bertone et al., 2017b]) and constructive methods that are based on marked schemes (see [Bertone et al., 2017a] and the references therein). These methods are also useful to compute the Zariski tangent space to a Hilbert scheme at a given point (see Corollary 1.6 and Remark 1.7). Essentially, step by step we alternate experimental results and theoretical properties of our tools, which have been applied in this context for the first time.

The paper is organized in the following way. In Section 1, we describe the results about marked schemes that we need in our arguments. These results also give information on the computation of the Zariski tangent space to a Hilbert scheme via marked schemes (Corollary 1.6). In Section 2, we focus on Gorenstein schemes and their relation with double-generic initial ideals when the Hilbert function is of type $(1, n, n, 1)$ (Proposition 2.5). In Sections 3 and 5 we prove that graded Artin Gorenstein K-algebras with Hilbert function either $(1, 7, 7, 1)$ or $(1, 5, 5, 1)$ are smoothable. In characteristic 0, this result also holds for local Artin Gorenstein K-algebras, due to [Elias and Rossi, 2012, Theorem 3.3]. In Section 4 we prove the existence of three different components in Hilb^7_{16} that we explicitly describe (in the Appendix, we list the outputs of some of the computations involved in this proof).

1 Backgroud: marked schemes and Zariski tangent space

In this paper, we work in an affine framework and apply the affine computational techniques developed in [Bertone et al., 2017a]. Then, in this section, we set some notations and recall the main notions involved in these techniques. Moreover, we give some new insights for the computation of the Zariski tangent space to a Hilbert scheme at a given point.

We will consider the rings of polynomials $R = K[x_1, \dots, x_n] \subset S = R[x_0]$, with $x_0 < x_1 < \dots < x_n$. For a term $x^\alpha = x_0^{\alpha_0} x_1^{\alpha_1} \dots x_n^{\alpha_n}$ we set $|\alpha| := \sum_i \alpha_i$, $\max(x^\alpha) := \max\{x_i \mid \alpha_i \neq 0\}$ and $\min(x^\alpha) := \min\{x_i \mid \alpha_i \neq 0\}$. For a non-null polynomial f we denote by $\mathrm{Supp}(f)$ its *support*, that is the set of terms that appears in f with a non-null coefficient. If f is a polynomial in R then we denote by $f^h := x_0^{deg(f)} f(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})$ its homogenization, and if F is a polynomial in S then we denote by $F^a := F(1, x_1, \dots, x_n)$ its dehomogenization.

Given a monomial ideal $\mathfrak{j} \subset R$ (resp. $J \subset S$), we denote by $\mathcal{N}(\mathfrak{j})$ (resp. $\mathcal{N}(J)$) the set of terms of R outside $\mathfrak{j}$ (resp. of S outside J) and by $B_\mathfrak{j}$ (resp. B_J) its minimal monomial basis. We refer to [Kreuzer and Robbiano, 2000, 2005; Mora, 2005] for results concerning Gröbner bases and Hilbert functions.

The results we are going to recall use the notion of *quasi-stable* ideal. It is well-known that a monomial ideal J is quasi-stable if and only if it has a so-called Pommaret basis [Seiler, 2009, Definition 4.3 and Proposition 4.4]. In general, a Pommaret basis $\mathcal{P}(J)$ strictly contains the minimal monomial basis B_J. The quasi-stable ideals having $\mathcal{P}(J) = B_J$ are called *stable* ideals. Stable ideals have a nice combinatorial characterization: for each term in a stable ideal, replacing the variable of smallest index with a variable of larger index produces another term in the ideal. In our setting we will only consider a special set of stable ideals, namely *strongly stable* ideals: in each term in a strongly stable ideal, we may replace any variable with a variable of larger index to get another term in the ideal. In characteristic 0, Borel-fixed ideals are strongly stable (see for example [Bayer and Stillman, 1987]). Although when J is strongly stable we have $\mathcal{P}(J) = B_J$, in the following we will use the notation of Pommaret bases, according to papers [Ceria et al., 2015; Bertone et al., 2017a].

Recall that a *marked polynomial* is a polynomial F together with a specified term of $\mathrm{Supp}(F)$ that will be called *head term of* F and denoted by $\mathrm{Ht}(F)$ (see [Reeves and Sturmfels, 1993]).

Definition 1.1. [Ceria et al., 2015, Definition 5.1] Let $J \subset S$ be a quasi-stable ideal.

A $\mathcal{P}(J)$*-marked set* (or marked set over $\mathcal{P}(J)$) G is a set of homogeneous monic marked polynomials F_α in S such that the head terms $Ht(F_\alpha) = x^\alpha$ are pairwise different and form the Pommaret basis $\mathcal{P}(J)$ of J, and $\mathrm{Supp}(F_\alpha - x^\alpha) \subset \mathcal{N}(J)$.

A $\mathcal{P}(J)$*-marked basis* (or marked basis over $\mathcal{P}(J)$) G is a $\mathcal{P}(J)$-marked set such that $\mathcal{N}(J)$ is a basis of $S/(G)$ as a K-module, i.e. $S = (G) \oplus \langle \mathcal{N}(J) \rangle$ as a K-module.

Let $\mathfrak{j} \subset R$ be a quasi-stable ideal and m a non-negative integer. Setting $J := \mathfrak{j} \cdot S$, we now recall the affine counterpart of Definition 1.1.

Definition 1.2. [Bertone et al., 2017a, Definition 4.1]

A $[\mathcal{P}(\mathfrak{j}), m]$-*marked set* $\mathfrak{G}$ is a set of monic marked polynomials f_α of R such that the head terms $\mathrm{Ht}(f_\alpha) = x^\alpha$ are pairwise different and form the Pommaret basis $\mathcal{P}(\mathfrak{j})$ of $\mathfrak{j}$, and $\mathrm{Supp}(f_\alpha - x^\alpha) \subseteq \mathcal{N}(\mathfrak{j})_{\leq t}$ with $t = \max\{m, |\alpha|\}$.

A $[\mathcal{P}(\mathfrak{j}), m]$-marked set $\mathfrak{G} = \{f_\alpha\}_{x^\alpha \in B_\mathfrak{j}}$ is a $[\mathcal{P}(\mathfrak{j}), m]$-*marked basis* if there exists a $\mathcal{P}(J_{\geq m})$-marked basis G such that for every $x^\alpha \in B_\mathfrak{j}$ the term $x_0^{k_\alpha} f_\alpha^h$ belongs to G for a suitable integer k_α.

The $[\mathcal{P}(\mathfrak{j}), m]$-*marked family* is the set of all the ideals $I \subseteq R$ that are generated by a $[\mathcal{P}(\mathfrak{j}), m]$-marked basis.

Lemma 1.3. *[Bertone et al., 2017a, Lemma 6.1(i) and Definition 6.2] An ideal $I \subset R$ belongs to a $[\mathcal{P}(\mathfrak{j}), m]$-marked family if and only if $R_{\leq t} = I_{\leq t} \bigoplus \langle \mathcal{N}(\mathfrak{j})_{\leq t}\rangle$, for every $t \geq m$.*

Theorem 1.4. *[Bertone et al., 2017a, Theorem 6.6 and Proposition 6.13] A $[\mathcal{P}(\mathfrak{j}), m]$-marked family is parameterized by a locally closed subscheme $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}),m}$ of the Hilbert scheme $\mathrm{Hilb}^n_{p(t)}$, where $p(t)$ is the affine Hilbert polynomial of $R/\mathfrak{j}$. If ρ is the satiety of $\mathfrak{j}$ and $m \geq \rho - 1$, then $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}),m}$ is an open subscheme of $\mathrm{Hilb}^n_{p(t)}$.*

The scheme $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}),m}$ of Theorem 1.4 is called $[\mathcal{P}(\mathfrak{j}), m]$-*marked scheme*.

Theorem 1.5. *[Bertone et al., 2017a, Section 6] The scheme $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}),m}$ is the spectrum $\mathrm{Spec}\ (K[C]/\mathfrak{U})$, where C is the set of parameters corresponding to the possible coefficients in the polynomials of a $[\mathcal{P}(\mathfrak{j}), m]$-marked basis, and the ideal $\mathfrak{U}$ is generated by the relations that are satisfied by these coefficients and which can be computed by [Bertone et al., 2017a, Algorithm* MarkedScheme$(\mathfrak{j}, m)$*].*

In next statement, we denote by X a point of $\mathrm{Hilb}^n_{p(t)}$ and let $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}),m}$ be a $[\mathcal{P}(\mathfrak{j}), m]$-marked scheme containing X up to a suitable change of coordinates, where $m \geq \rho - 1$ and ρ is the satiety of $\mathfrak{j}$.

Corollary 1.6. *The Zariski tangent space to $\mathrm{Hilb}^n_{p(t)}$ at X is equal to the Zariski tangent space to $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}),m}$ at X and it can be explicitly computed by marked bases techniques.*

Proof. It is enough to observe that $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}),m}$ is an open subscheme of $\mathrm{Hilb}^n_{p(t)}$ due to Theorem 1.4 and that the Zariski tangent space to $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}),m}$ at X can be computed by the generators of the ideal $\mathfrak{U}$ of Theorem 1.5. □

Remark 1.7. Concerning an effective computation of the Zariski tangent space to a marked scheme $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}),m}$ at the origin $\mathfrak{j}$, we can use the same techniques that are described in [Lella and Roggero, 2011, Sections 3 and 4] for the so-called Gröbner strata. If we want to compute the Zariski tangent space at another point, we perform the change of coordinates that brings this point in the origin, as usual.

We end this section with the following result that is inferred from [Ferrarese and Roggero, 2009] and is analogous to results contained in [Lella and Roggero, 2011] for the homogeneous case. We first need to describe an adjustment to the affine case of the notion of segment (for details on segments see [Cioffi et al., 2011]).

Definition 1.8. Let $\mathfrak{j}$ be a strongly stable ideal in R, m a positive integer. The ideal $\mathfrak{j}$ is an *affine m-segment* if there is a weight vector $\omega \in \mathbb{N}^n$ such that for every $x^\alpha \in B_\mathfrak{j}$, $\deg_\omega(x^\alpha) > \deg_\omega(x^\gamma)$ for every $x^\gamma \in \mathcal{N}(\mathfrak{j})_{\leq t}$, with $t = \max\{m, |\alpha|\}$.

Theorem 1.9. *If $\mathfrak{j} \subset R$ is an affine m-segment, then every irreducible component $\mathcal{M}$ of $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}),m}$ contains $\mathfrak{j}$, hence $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}),m}$ is a connected scheme. If moreover the point corresponding to $\mathfrak{j}$ is smooth on $\mathcal{M}$, then $\mathcal{M}$ is isomorphic to an affine space.*

Proof. Let $\omega \in \mathbb{N}^n$ be a weight vector with respect to whom $\mathfrak{j}$ is an affine m-segment. Then, $\mathrm{Mf}(\mathfrak{j}, m)$ is a ω-cone with vertex in $\mathfrak{j}$ by [Ferrarese and Roggero, 2009, Corollary 2.7] and the thesis follows. □

2 Gorenstein points and double-generic initial ideals

In this section, we highlight a relation between the locus of Gorenstein points defined by graded Gorenstein K-algebras with Hilbert function of type $(1, n, n, 1)$ and the notion of double-generic initial ideal.

A Gorenstein scheme $X \in \mathrm{Hilb}^n_{p(t)}$ is a scheme such that the stalk of the ideal sheaf in every point $x \in X$ is Gorenstein. Recall that the locus of points in $\mathrm{Hilb}^n_{p(t)}$ representing Gorenstein schemes is an open subset [Stoia, 1975; Greco and Marinari, 1978].

We will consider zero-dimensional Gorenstein schemes in an open neighborhood, which can be studied by our affine techniques [Bertone et al., 2017a]. Hence, following [Iarrobino and Kanev, 1999, Definition 2.1] and [Bruns and Herzog, 1993], we now recall some main notions and already known results for Artin K-algebras.

Let A be a local Artin K-algebra and M its maximal ideal. The *socle* of A is the annihilator $Soc(A) := (0 :_A M) = \{h \in A \mid hM = 0\}$. Then, A is called *Gorenstein* if $\dim_K Soc(A) = 1$. An Artin K-algebra is *Gorenstein* if its localization at every maximal ideal is a Gorenstein (local) K-algebra. The *socle degree* of a graded Artin Gorenstein K-algebra A is the maximum degree j such that $A_j \neq 0$.

The following result is due to Macaulay, as observed in [Iarrobino and Kanev, 1999] which we refer to.

Lemma 2.1. *[Iarrobino and Kanev, 1999, Definiton 1.11 and Lemma 2.12] There is a bijection between the hypersurfaces of degree j in $\mathbb{P}^n_K$ and the set of graded Artin Gorenstein quotient rings of R of socle degree j. This correspondence associates to a form F of degree j the quotient $A_F := R/Ann(F)$, where $Ann(F)$ is computed by apolarity.*

Theorem 2.2. ([Iarrobino and Emsalem, 1978, Theorem 3.31], [Iarrobino, 1984, Theorem I], [Casnati and Notari, 2011, Theorem 3.1]) *The set of cubic hypersurfaces, which determine all the graded Artin Gorenstein K-algebras with Hilbert function $(1, n, n, 1)$ by apolarity, is a non-empty irreducible subset of $\mathbb{P}(R_3)$.*

Let $\mathrm{Gor}(T)$ denote the subset of the projective space consisting of the hypersurfaces F such that the Hilbert function of A_F is a given function T. By Theorem 2.2, $\mathrm{Gor}(T)$ can be embedded in a Hilbert scheme as an irreducible locally closed subset and we denote by $\overline{\mathrm{Gor}}(T)$ its closure. The following definitions and results are crucial in our study of $\overline{\mathrm{Gor}}(1, n, n, 1)$.

Definition 2.3. [Bertone et al., 2017b] An irreducible closed subset Y of a Hilbert scheme is called a *GL-stable subset* if is invariant under the action of the general linear group.

Every GL-stable subset Y of a Hilbert scheme contains at least one point corresponding to a strongly stable ideal. Given a term order, among the strongly stable ideals that define points of Y, we can find a special strongly stable ideal which is the saturation of the generic initial ideal of the generic (and general) point of Y [Bertone et al., 2017b, Proposition 4(b)].

Definition 2.4. [Bertone et al., 2017b, Definition 5] The saturation of the generic initial ideal of the generic (and general) point of a GL-stable subset Y is called the *double-generic initial ideal of* Y and is denoted by G_Y.

The notion of double-generic initial ideal has been introduced and investigated for the first time in [Bertone et al., 2017b], also in the more general setting of Grassmannian, with the terminology of extensors.

Proposition 2.5. $\overline{\mathrm{Gor}}(1,n,n,1) \subseteq \mathrm{Hilb}^n_{2n+2}$ *is a GL-stable subset, in particular it has a double-generic initial ideal.*

Proof. Theorem 2.2 implies that the closure $\overline{\mathrm{Gor}}(1,n,n,1)$ in Hilb^n_{2n+2} is a closed irreducible subset, hence it is GL-stable because it is also invariant under the action of the general linear group, by construction. $\square$

Next result contains some of the main properties of a double-generic initial ideal. First, we need to recall the following definition.

Definition 2.6. [Bertone et al., 2017b, Definition 6] Let J and H be monomial ideals in S such that S/J and S/H have the same Hilbert polynomial $p(t)$. Let r be the Gotzmann number of $p(t)$ and consider the set of generators $B_{J_{\geq r}} = \{\tau_1, \dots, \tau_q\}$ and $B_{I_{\geq r}} = \{\sigma_1, \dots, \sigma_q\}$ ordered by a term order $>$, where $q = \binom{n+r}{n} - p(r)$. We write $J \gg H$ if $\tau_i \geq \sigma_i$ for every $i \in \{1, \dots, q\}$.

Lemma 2.7. *[Bertone et al., 2017a, Propositions 2 and 3, Definition 5, Theorems 3 and 4, Remark 5] Let $>$ be a term order, Y a GL-stable subset of $\mathrm{Hilb}^n_{p(t)}$ and G_Y its double-generic initial ideal.*

(i) For every ideal I defining a point of Y, $\mathrm{gin}(I)$ and $\mathrm{in}(I)$ define points of Y.

(ii) There exists the maximum among all the Borel ideals defining points of Y with respect to the partial order $\gg$, and this maximum is G_Y.

(iii) There is a non-empty open subset V of Y such that $\mathrm{gin}(I) = \mathrm{in}(I) = G_Y$ for every saturated ideal I defining a point in V.

3 Graded Artin Gorenstein K-algebras with Hilbert function $(1,7,7,1)$ define smoothable points

In this section, we consider the Hilbert scheme Hilb^7_{16} parameterizing zero-dimensional subschemes of $\mathbb{P}^7_K$ of length 16. Recall that, up to a generic change of coordinates, we can identify every point of Hilb^7_{16} with an ideal in $R = K[x_1, \dots, x_7]$, not necessarily homogeneous. Hence, we consider the polynomial ring R and the ideals in R with affine Hilbert polynomial $p(t) = 16$. A double-generic initial ideal J will be considered in its *affine* version too, that is its dehomogenization $\mathfrak{j} := J^a = B_J \cdot R$.

The lex-point of Hilb^7_{16} corresponds to the following lex-segment ideal in R:

$$\mathfrak{j}_{\mathrm{lex}} = (x_7, x_6, x_5, x_4, x_3, x_2, x_1^{16}).$$

It is well-known that $\mathfrak{j}_{\mathrm{lex}}$ is a smooth point of the smoothable component $\mathcal{R}^7_{16}$ of dimension $7 \cdot 16 = 112$, because the general point of $\mathcal{R}^7_{16}$ is a reduced scheme of 16 distinct points.

We can compute the complete list of 561 strongly stable ideals of R lying in Hilb^7_{16} by the algorithm described in [Cioffi et al., 2011] (and further developed and implemented in [Lella, 2012] and generalized for quasi-stable ideals and Borel-fixed ideals in positive characteristic in [Bertone, 2015]). Among them, we focus on the following one:

$$\begin{aligned}\mathfrak{j}_G =&(x_7^2, x_7x_6, x_7x_5, x_7x_4, x_7x_3, x_7x_2, x_7x_1, x_6^2, x_6x_5, x_6x_4, x_6x_3, x_6x_2, x_6x_1, x_5^2, x_5x_4, x_5x_3,\\ &x_5x_2, x_5x_1, x_4^2, x_4x_3, x_4x_2, x_4x_1^2, x_3^3, x_3^2x_2, x_3^2x_1, x_3x_2^2, x_3x_2x_1, x_3x_1^2, x_2^3, x_2^2x_1, x_2x_1^2, x_1^4).\end{aligned}$$

By the constructive tools of [Bertone et al., 2017a] and by theoretical results on the double-generic initial ideal, we now show that $\mathfrak{j}_G$ is the generic initial ideal w.r.t. lex term order of a general ideal defining a graded (Artin) Gorenstein K-algebra with Hilbert function $(1, 7, 7, 1)$.

Theorem 3.1. *The strongly stable ideal $\mathfrak{j}_G$ is the double-generic initial ideal of the GL-stable subset $\overline{\mathrm{Gor}}(1, 7, 7, 1)$ w.r.t. the lex order. In particular, it is the generic initial ideal w.r.t. the lex order of a general point of* $\mathrm{Gor}(1, 7, 7, 1)$.

Proof. We explicitly construct a random ideal defining a graded Artin Gorenstein K-algebra with Hilbert function $(1, 7, 7, 1)$ by apolarity, thanks to the already cited correspondence with cubic hypersurfaces (see Lemma 2.1). We randomly choose the following cubic form F in $K[x_1, \dots, x_7]$

$$\begin{aligned}F = &\,2\,x_1^3 - 3\,x_1^2x_2 - 6\,x_1^2x_4 - 6\,x_1^2x_5 - 3\,x_1^2x_7 + 9\,x_1x_2^2 + 12\,x_1x_2x_3 + 12\,x_2x_1x_4+\\ &+12\,x_2x_1x_5 + 12\,x_2x_1x_6 + 6\,x_2x_1x_7 + 6\,x_1x_3^2 + 6\,x_1x_3x_4 + 6\,x_1x_3x_5 + 12\,x_1x_3x_6+\\ &+6\,x_1x_3x_7 + 6\,x_1x_4^2 + 12\,x_1x_4x_5 + 6\,x_1x_4x_6 + 6\,x_1x_4x_7 + 6\,x_1x_5^2 + 6\,x_1x_5x_6+\\ &+6\,x_1x_5x_7 + 6\,x_1x_6^2 + 6\,x_1x_6x_7 + 3\,x_1x_7^2 - x_2^3 + 3\,x_2^2x_3 - 9\,x_2^2x_4 - 6\,x_2^2x_5+\\ &-3\,x_2^2x_6 - 6\,x_2^2x_7 + 3\,x_2x_3^2 - 12\,x_2x_3x_4 - 6\,x_2x_3x_5 - 6\,x_2x_3x_6 - 12\,x_2x_3x_7 - 3\,x_2x_4^2+\\ &-12\,x_2x_4x_5 - 6\,x_2x_5^2 - 6\,x_2x_5x_6 - 6\,x_2x_5x_7 + 3\,x_2x_6^2 - 6\,x_3^2x_4 - 3\,x_3^2x_5 - 3\,x_3^2x_6+\\ &-6\,x_3^2x_7 - 6\,x_3x_4x_5 - 3\,x_3x_5^2 - 6\,x_3x_5x_6 - 6\,x_3x_5x_7 + 3\,x_3x_6^2 - 5\,x_4^3 - 6\,x_4^2x_5+\\ &-6\,x_4^2x_6 - 3\,x_4^2x_7 - 6\,x_4x_5^2 - 6\,x_4x_5x_6 - 6\,x_4x_5x_7 - 6\,x_4x_6^2 - 12\,x_4x_6x_7 - 9\,x_4x_7^2+\\ &-3\,x_5^3 - 3\,x_5^2x_6 - 3\,x_5^2x_7 - 3\,x_5x_6^2 - 6\,x_5x_6x_7 - 3\,x_5x_7^2 - 2\,x_6^3 - 6\,x_6^2x_7+\\ &-6\,x_6x_7^2 - 2\,x_7^3.\end{aligned}$$

Let $\mathrm{Ann}(F) \subset K[x_1, \dots, x_7]$ be the ideal that we obtain by apolarity from F. We check that $A_F := K[x_1, \dots, x_7]/\mathrm{Ann}(F)$ is a graded Gorenstein K-algebra with Hilbert function $(1, 7, 7, 1)$. We can also observe that A_F is local as we expected, because A_F is Artin and graded. The reduced Gröbner basis w.r.t. the lex order of the ideal $\mathrm{Ann}(F)$ is given by the following 32 polynomials (in bold the initial term of each polynomial):

$\mathfrak{f}_1 = \mathbf{x_7^2} - 4x_1x_4 - 2x_3^2 + x_2x_3 - 2x_1x_3 - x_2^2 + 4x_1x_2$,

$\mathfrak{f}_2 = \mathbf{x_6x_7} - x_1x_4 - x_2x_3 + x_1x_2$,

$\mathfrak{f}_3 = \mathbf{x_5x_7} + x_1x_4 + x_1x_3 - x_1x_2$,

$\mathfrak{f}_4 = \mathbf{x_4x_7} + 2x_1x_4 + 2x_3^2 - 2x_2x_3 + 5x_1x_3 - 5x_1x_2 - x_1^2$,

$\mathfrak{f}_5 = \mathbf{x_3x_7} + 3x_1x_4 + x_2x_3 + 2x_1x_3 - 3x_1x_2$,

$\mathfrak{f}_6 = \mathbf{x_2x_7} + 3x_1x_4 + x_2x_3 + 2x_1x_3 - 3x_1x_2$,

$\mathfrak{f}_7 = \mathbf{x_1x_7} - x_1x_4 - x_1x_3 + x_1x_2$,

$\mathfrak{f}_8 = \mathbf{x_6^2} + x_1x_4 + x_3^2 - 3x_2x_3 - x_1x_3 + x_2^2 + x_1x_2 + x_1^2$,

$\mathfrak{f}_9 = \mathbf{x_5x_6} + x_1x_4 + x_1x_3 - x_1x_2$,

$\mathfrak{f}_{10} = \mathbf{x_4x_6} - x_1x_4 - x_2x_3 + x_1x_2$,

$\mathfrak{f}_{11} = \mathbf{x_3x_6} + 4x_1x_4 + x_2x_3 + 2x_1x_3 - 4x_1x_2$,

$\mathfrak{f}_{12} = \mathbf{x_2x_6} + 4x_1x_4 + x_2x_3 + 2x_1x_3 - 4x_1x_2$,

$\mathfrak{f}_{13} = \mathbf{x_1x_6} - x_1x_3$,

$\mathfrak{f}_{14} = \mathbf{x_5^2} + 2x_1x_4 + 2x_3^2 - 3x_2x_3 + 2x_2^2 - x_1^2$,

$\mathfrak{f}_{15} = \mathbf{x_4x_5} + x_1x_4$,

$\mathfrak{f}_{16} = \mathbf{x_3x_5} + x_1x_4 + x_1x_3 - x_1x_2$,

$\mathfrak{f}_{17} = \mathbf{x_2 x_5} + x_1 x_4,$
$\mathfrak{f}_{18} = \mathbf{x_1 x_5} - x_1 x_4,$
$\mathfrak{f}_{19} = \mathbf{x_4^2} - 7x_1x_4 - 4x_3^2 + 2x_2x_3 - 8x_1x_3 - x_2^2 + 11x_1x_2 + x_1^2,$
$\mathfrak{f}_{20} = \mathbf{x_3 x_4} + 3x_1x_4 + x_2x_3 + 2x_1x_3 - 3x_1x_2,$
$\mathfrak{f}_{21} = \mathbf{x_2 x_4} + 3x_1x_4 + x_2x_3 + x_1x_3 - 2x_1x_2,$
$\mathfrak{f}_{22} = \mathbf{x_1^2 x_4} + x_1^3,$
$\mathfrak{f}_{23} = \mathbf{x_3^3}$, $\mathfrak{f}_{24} = \mathbf{x_2 x_3^2} - \frac{1}{2}x_1^3$, $\mathfrak{f}_{25} = \mathbf{x_1 x_3^2} - x_1^3$, $\mathfrak{f}_{26} = \mathbf{x_2^2 x_3} - \frac{1}{2}x_1^3$,
$\mathfrak{f}_{27} = \mathbf{x_1 x_2 x_3} - x_1^3$, $\mathfrak{f}_{28} = \mathbf{x_1^2 x_3}$, $\mathfrak{f}_{29} = \mathbf{x_2^3} + \frac{1}{2}x_1^3$, $\mathfrak{f}_{30} = \mathbf{x_1 x_2^2} - \frac{3}{2}x_1^3$,
$\mathfrak{f}_{31} = \mathbf{x_1^2 x_2} + \frac{1}{2}x_1^3$, $\mathfrak{f}_{32} = \mathbf{x_1^4}$.
Then, the monomial ideal $\mathfrak{j}_G$ is the initial ideal of $\mathrm{Ann}(F)$ w.r.t. lex order. Moreover, $\mathfrak{j}_G$ is the generic initial ideal of $\mathrm{Ann}(F)$ by [Bertone et al., 2017b, Theorem 3], because $\mathfrak{j}_G$ is the maximum w.r.t. the order of Definition 2.6 among all the strongly stable ideals with Hilbert function $(1, 7, 7, 1)$ (see [Eisenbud, 1995, Theorem 15.18]). By [Bertone et al., 2017b, Theorem 3], this fact also implies that $\mathfrak{j}_G$ is the double-generic initial ideal of $\overline{\mathrm{Gor}}(1, 7, 7, 1)$. By [Bertone et al., 2017b, Proposition 4] we can now conclude that $\mathfrak{j}_G$ is the generic initial ideal w.r.t. the lex order of a general point of $\mathrm{Gor}(1, 7, 7, 1)$. □

Proposition 3.2. $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}_G),3} \cap \mathrm{Gor}(1, 7, 7, 1) \neq \emptyset$.

Proof. From the proof of Theorem 3.1 we see that the 32 polynomials $\mathfrak{f}_i$, which generate the ideal $\mathrm{Ann}(F)$, form a Gröbner basis that is also a $[\mathcal{P}(\mathfrak{j}_G), 3]$-marked basis. Thus, $\mathrm{Ann}(F)$ belongs to the family of ideals defining the $[\mathcal{P}(\mathfrak{j}_G), 3]$-marked scheme $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}_G),3}$. □

With a suitable choice of values for the parameters occurring in the defining ideal $\mathfrak{U}$ of $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}_G),3}$, we obtain that the following polynomials form a $[\mathcal{P}(\mathfrak{j}_G), 3]$-marked basis $\mathfrak{G}_\tau$, for every $\tau \in \mathbb{A}^1_K$:
$F_1 = \mathfrak{f}_1 - 9\,\tau x_7 + 16\,\tau x_4 + \frac{11}{2}\,\tau x_3 - 7\,\tau x_2 + 2\,\tau x_1 - 8\,\tau^2,$
$F_2 = \mathfrak{f}_2 - \frac{1}{2}\,\tau x_6 + \tau x_4 + \frac{1}{2}\,\tau x_3 - \tau x_2 + \frac{1}{2}\,\tau x_1 - \frac{1}{2}\,\tau^2,$
$F_3 = \mathfrak{f}_3 - \tau x_7 - \frac{1}{2}\,\tau x_5 - \tau x_4 - \tau x_3 + \tau x_2 - \frac{1}{2}\,\tau x_1 + \tau^2,$
$F_4 = \mathfrak{f}_5 + 11\,\tau x_7 - \frac{45\,\tau}{2} x_4 - 8\,\tau x_3 + 5\,\tau x_2 - \frac{3}{2}\,\tau x_1 + 17\,\tau^2,$
$F_5 = \mathfrak{f}_5 - 3\,\tau x_4 - 3\,\tau x_3 + 3\,\tau x_2 - \frac{3}{2}\,\tau x_1 + \frac{3}{2}\,\tau^2,$
$F_6 = \mathfrak{f}_6 - \frac{1}{2}\,\tau x_7 - 3\,\tau x_4 - \frac{5}{2}\,\tau x_3 + \frac{5}{2}\,\tau x_2 - \frac{3}{2}\,\tau x_1 + \frac{7}{4}\,\tau^2,$
$F_7 = \mathfrak{f}_7 - \tau x_2 + \tau x_3 + \tau x_4 - \tau x_7,$
$F_8 = \mathfrak{f}_8 + 2\,\tau x_6 - \tau x_4 + \frac{1}{2}\,\tau x_3 + 2\,\tau x_2 + \frac{1}{2}\,\tau x_1 - \frac{13\,\tau^2}{4},$
$F_9 = \mathfrak{f}_9 - \tau x_6 - \tau x_4 - \tau x_3 + \tau x_2 - \frac{1}{2}\,\tau x_1 + \frac{1}{2}\,\tau^2,$
$F_{10} = \mathfrak{f}_{10} - \tau x_6 + \tau x_4 + \frac{1}{2}\,\tau x_3 - \tau x_2 + \frac{1}{2}\,\tau x_1 - \frac{1}{2}\,\tau^2,$
$F_{11} = \mathfrak{f}_{11} - 4\,\tau x_4 - \frac{5}{2}\,\tau x_3 + 4\,\tau x_2 - 2\,\tau x_1 + 2\,\tau^2,$
$F_{12} = \mathfrak{f}_{12} - \frac{1}{2}\,\tau x_6 - 4\,\tau x_4 - \frac{5}{2}\,\tau x_3 + 4\,\tau x_2 - 2\,\tau x_1 + 2\,\tau^2,$
$F_{13} = \mathfrak{f}_{13} + \tau x_3 - \tau x_6 - x_3 x_1,$
$F_{14} = \mathfrak{f}_{14} - 14\,\tau x_5 - 2\,\tau x_4 - \frac{5}{2}\,\tau x_3 + 6\,\tau x_2 - 4\,\tau x_1 + \frac{29\,\tau^2}{2},$
$F_{15} = \mathfrak{f}_{15} + 2\,\tau^2 - \tau x_1 - 2\,\tau x_4 - \tau x_5,$
$F_{16} = \mathfrak{f}_{16} - \tau x_4 - 2\,\tau x_3 + \tau x_2 - \frac{1}{2}\,\tau x_1 + \frac{1}{2}\,\tau^2,$
$F_{17} = \mathfrak{f}_{17} - \frac{1}{2}\,\tau x_5 - \tau x_4 - \tau x_2 - \tau x_1 + \frac{3}{2}\,\tau^2,$
$F_{18} = \mathfrak{f}_{18} + \tau x_4 - \tau x_5,$
$F_{19} = \mathfrak{f}_{19} - 20\,\tau x_7 + 37\,\tau x_4 + 15\,\tau x_3 - 14\,\tau x_2 + \frac{7}{2}\,\tau x_1 - \frac{95\,\tau^2}{4},$
$F_{20} = \mathfrak{f}_{20} - 3\,\tau x_4 - \frac{7}{2}\,\tau x_3 + 3\,\tau x_2 - \frac{3}{2}\,\tau x_1 + \frac{3}{2}\,\tau^2,$
$F_{21} = \mathfrak{f}_{21} - \frac{7}{2}\,\tau x_4 - \frac{3}{2}\,\tau x_3 + \tau x_2 - 2\,\tau x_1 + \frac{5}{2}\,\tau^2,$
$F_{22} = \mathfrak{f}_{22} + 4\,\tau^3 - 5\,\tau^2 x_1 - 4\,\tau^2 x_2 + \tau^2 x_4 + 4\,\tau x_1 x_2 - 2\,\tau x_1 x_4,$
$F_{23} = \mathfrak{f}_{23} - 2\,\tau^3 + 2\,\tau^2 x_1 - 4\,\tau^2 x_2 - 3\,\tau^2 x_3 + 4\,\tau^2 x_4 + 4\,\tau x_1 x_2 - 4\,\tau x_1 x_4 + 6\,\tau x_2 x_3 - 2\,\tau {x_3}^2,$

$F_{24} = \mathfrak{f}_{24} - \frac{1}{2}\,\tau x_3{}^2 + 4\,\tau x_2 x_3 - 6\,\tau x_1 x_4 - 6\,\tau x_3 x_1 + 8\,\tau x_1 x_2 - \frac{1}{2}\,\tau x_1{}^2 + 6\,\tau^2 x_4 +$
$+ 4\,\tau^2 x_3 - 8\,\tau^2 x_2 + \frac{9}{2}\,\tau^2 x_1 - \frac{7}{2}\,\tau^3,$
$F_{25} = \mathfrak{f}_{25} - 5\,\tau^3 + 7\,\tau^2 x_1 - 12\,\tau^2 x_2 + 8\,\tau^2 x_3 + 8\,\tau^2 x_4 - \tau x_1{}^2 + 12\,\tau x_1 x_2 - 8\,\tau x_3 x_1 +$
$- 8\,\tau x_1 x_4 - \tau x_3{}^2,$
$F_{26} = \mathfrak{f}_{26} + 3\,\tau x_2 x_3 - 6\,\tau x_1 x_4 - 6\,\tau x_3 x_1 + 8\,\tau x_1 x_2 - \frac{1}{2}\,\tau x_1{}^2 + 6\,\tau^2 x_4 + \frac{17\,\tau^2 x_3}{4} +$
$- 8\,\tau^2 x_2 + \frac{9}{2}\,\tau^2 x_1 - \frac{7}{2}\,\tau^3,$
$F_{27} = \mathfrak{f}_{27} - \tau x_2 x_3 - 8\,\tau x_1 x_4 - \frac{17}{2}\,\tau x_3 x_1 + 12\,\tau x_1 x_2 - \tau x_1{}^2 + 8\,\tau^2 x_4 + \frac{17}{2}\,\tau^2 x_3 +$
$- 12\,\tau^2 x_2 + 7\,\tau^2 x_1 - 5\,\tau^3,$
$F_{28} = \mathfrak{f}_{28} - 6\,\tau^3 + 6\,\tau^2 x_1 - 12\,\tau^2 x_2 + 5\,\tau^2 x_3 + 12\,\tau^2 x_4 + 12\,\tau x_1 x_2 - 6\,\tau x_3 x_1 - 12\,\tau x_1 x_4,$
$F_{29} = \mathfrak{f}_{29} + \frac{5}{2}\,\tau x_2{}^2 - 2\,\tau x_1 x_4 - 6\,\tau x_3 x_1 + 12\,\tau x_1 x_2 + 1/2\,\tau x_1{}^2 + 2\,\tau^2 x_4 + 6\,\tau^2 x_3 +$
$- \frac{61\,\tau^2 x_2}{4} - \frac{13}{2}\,\tau^2 x_1 + \frac{51\,\tau^3}{8},$
$F_{30} = \mathfrak{f}_{30} - \tau x_2{}^2 - 10\,\tau x_1 x_4 - 6\,\tau x_3 x_1 + 7\,\tau x_1 x_2 - \frac{3}{2}\,\tau x_1{}^2 + 10\,\tau^2 x_4 + 6\,\tau^2 x_3 +$
$- 7\,\tau^2 x_2 + \frac{55\,\tau^2 x_1}{4} - \frac{43\,\tau^3}{4},$
$F_{31} = \mathfrak{f}_{31} - 10\,\tau x_1 x_4 - 6\,\tau x_3 x_1 + 14\,\tau x_1 x_2 + 10\,\tau^2 x_4 + 6\,\tau^2 x_3 - 15\,\tau^2 x_2 + \frac{1}{2}\,\tau^2 x_1 - \tau^3,$
$F_{32} = \mathfrak{f}_{32} + 45\,\tau^4 - 40\,\tau^3 x_1 + 32\,\tau^3 x_2 - 64\,\tau^3 x_4 - 6\,\tau^2 x_1{}^2 - 32\,\tau^2 x_1 x_2 + 64\,\tau^2 x_1 x_4.$
From now, for every $\tau \in \mathbb{A}^1_K$ we denote by $\mathfrak{i}_\tau \subset K[x_1,\dots,x_7]$ the ideal generated by the $[\mathcal{P}(\mathfrak{j}_G),3]$-marked basis $\mathfrak{G}_\tau = \{F_1,\dots,F_{32}\}$. Note that for $\tau = 0$ we obtain the ideal $\mathfrak{i}_0 = \mathrm{Ann}(F)$, which defines a scheme with support in a single point, as we have already observed in the proof of Theorem 3.1. For every $\tau \in \mathbb{A}^1_K \setminus \{0\}$ we have a different situation because every ideal $\mathfrak{i}_\tau$ defines a scheme whose support contains at least the following 8 distinct affine points:

$$\left(\tau, \frac{1}{2}\tau, 2\tau, \tau, \tau, 0, \frac{1}{2}\tau\right), \quad \left(\tau, \frac{-7}{2}\tau, 0, \tau, \tau, 0, \frac{1}{2}\tau\right), \quad \left(-3\tau, \frac{1}{2}\tau, 0, \tau, \tau, 0, \frac{1}{2}\tau\right),$$
$$\left(\tau, \frac{1}{2}\tau, 0, \tau, \tau, 0, \frac{1}{2}\tau\right), \quad \left(-7\tau, \frac{-45}{6}\tau, -8\tau, \tau, \tau, -8\tau, \frac{1}{2}\tau\right), \quad \left(\tau, \frac{1}{2}\tau, 0, \tau, \tau, -2\tau, \frac{1}{2}\tau\right),$$
$$\left(\tau, \frac{1}{2}\tau, 0, \tau, 13\tau, 0, \frac{1}{2}\tau\right), \quad \left(-7\tau, \frac{17}{2}\tau, 0, 9\tau, 9\tau, 0, \frac{1}{2}\tau\right).$$

This observation is crucial for next result.

Lemma 3.3. *Over an algebraically closed field K with $\mathrm{char}(K) \neq 2,3$, there exists a flat family of ideals which is contained in $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}_G),3} \cap \mathcal{R}^7_{16}$ such that the special fiber corresponds to a Gorenstein point defined by a graded K-algebra with Hilbert function $(1,7,7,1)$.*

Proof. We prove that the family of ideals $\{\mathfrak{i}_\tau\}_\tau$, which are generated by the $[\mathcal{P}(\mathfrak{j}_G),3]$-marked bases $\mathfrak{G}_\tau$, is contained in the smoothable component $\mathcal{R}^7_{16}$. By construction, the ideals $\mathfrak{i}_\tau$ belong to the marked scheme $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}_G),3}$ which embeds as an open subset in the punctual Hilbert scheme Hilb^7_{16}. So, these ideals define a flat family over $\mathbb{A}^1$. As we have already recalled in Section 2, the locus of points in a Hilbert scheme representing all the Gorenstein schemes is an open subset. Hence, the intersection of this locus with the flat family $\{\mathfrak{i}_\tau\}_\tau$, which is non-empty because $\mathfrak{i}_0$ represents a Gorenstein point, is an open subset of the family. Thus, we find at least a value $\overline{\tau} \neq 0$ such that $\mathfrak{i}_{\overline{\tau}}$ represents a Gorenstein point.

By computational tools, we have already found that for every $\tau \neq 0$ the ideal $\mathfrak{i}_\tau$ defines a scheme whose support contains at least 8 distinct points and hence components of multiplicity at most 9. So, the ideal $\mathfrak{i}_{\overline{\tau}}$ is smoothable due to the fact that for $d \leq 9$ the locus of Gorenstein points is a Hilbert scheme of d points is irreducible (see [Casnati and Notari, 2009, Theorem A]). Thus, every ideal $\mathfrak{i}_\tau$ belongs to the smoothable component $\mathcal{R}^7_{16}$ because the family is irreducible. In particular, the special fiber $\mathfrak{i}_0$ belongs to $\mathcal{R}^7_{16}$, because $\mathcal{R}^7_{16}$ is closed and irreducible. □

Remark 3.4. Although the ideal $\mathfrak{i}_0$ corresponds to a Gorenstein point in $\mathrm{Gor}(1,7,7,1)$, for every $\tau \neq 0$ the ideal $\mathfrak{i}_\tau$ corresponds to a point which does not belong to $\mathrm{Gor}(1,7,7,1)$ because its support consists of more than one point. We constructed the family of ideals $\{\mathfrak{i}_\tau\}_\tau$ with this property letting the term x_7 have a non-null coefficient in the polynomial F_{19}. Indeed, the term x_7 is higher than the head term x_4^2 of F_{19} with respect to lex term order. This fact implies that the initial ideal of $\mathfrak{i}_\tau$ is not $\mathfrak{j}_G$ and is not comparable with $\mathfrak{j}_G$ w.r.t. the order of Definition 2.6 (see [Bertone et al., 2017b, Theorem 3 and Proposition 8]). Thus, the generic initial ideal of $\mathfrak{i}_\tau$ is different from $\mathfrak{j}_G$. Recalling that $\mathfrak{j}_G$ is the double-generic initial ideal of $\overline{\mathrm{Gor}}(1,7,7,1)$, we obtain our claim. We can also observe that the initial ideal of $\mathfrak{i}_\tau$ is *closer* to the lex-segment ideal than $\mathfrak{j}_G$ w.r.t. the order of Definition 2.6.

Lemma 3.5. *Over an algebraically closed field K with $\mathrm{char}(K) \neq 2,3$, there exists a smooth Gorenstein point defined by a graded K-algebra with Hilbert function $(1,7,7,1)$ belonging to the smoothable component $\mathcal{R}_{16}^7$.*

Proof. We prove that the ideal $\mathfrak{i}_0$ defines a smooth Gorenstein point in the smoothable component $\mathcal{R}_{16}^7$. We already know that $\mathfrak{i}_0$ defines a Gorenstein point with Hilbert function $(1,7,7,1)$. Moreover, by Lemma 3.3 and by construction, the ideal $\mathfrak{i}_0$ represents a point that belongs to $\mathcal{R}_{16}^7 \cap \mathrm{Mf}_{\mathcal{P}(\mathfrak{j}_G),3}$. Due to Corollary 1.6, we can compute the Zariski tangent space to Hilb_{16}^7 at $\mathfrak{i}_0$ from the polynomials of the $[\mathcal{P}(\mathfrak{j}_G),3]$-marked basis of $\mathfrak{i}_0$, obtaining that the dimension of the Zariski tangent space to Hilb_{16}^7 at the point $\mathfrak{i}_0$ is $112 = 16 \times 7$, i.e. the dimension of the smoothable component. □

Theorem 3.6. *Over an algebraically closed field K with $\mathrm{char}(K) \neq 2,3$, every Gorenstein point defined by a graded K-algebra with Hilbert function $(1,7,7,1)$ is smoothable.*

Proof. By Lemmas 3.3 and 3.5, there exists a smooth Gorenstein point with Hilbert function $(1,7,7,1)$ in the smoothable component $\mathcal{R}_{16}^7$. These facts imply that also all the other Gorenstein points with the same Hilbert function belong to $\mathcal{R}_{16}^7$, i.e. are smoothable, because the locus $\mathrm{Gor}(1,n,n,1)$ in $\mathrm{Hilb}_{p(t)}^n$ of the schemes parameterizing homogeneous Gorenstein ideals with Hilbert function $(1,n,n,1)$ is irreducible (see Theorem 2.2 and Proposition 2.5). □

Remark 3.7. Recall that Theorem 3.6 covers the unique case not treated in the range considered by [Iarrobino and Kanev, 1999, Lemma 6.21] about the study of non-smoothable Gorenstein points.

Corollary 3.8. *Over an algebraically closed field K of characteristic 0, every local Gorenstein K-algebra with Hilbert function $(1,7,7,1)$ is smoothable.*

Proof. This is a consequence of Theorem 3.6 and [Elias and Rossi, 2012, Theorem 3.3]. □

4 Hilb_{16}^7 has at least three irreducible components

In this section, we now obtain interesting information about the components of Hilb_{16}^7 from a study of the irreducible components of $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}_G),3}$. Indeed, by construction the marked scheme $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}_G),3}$ is the open subscheme of Hilb_{16}^7 where the Plücker coordinate corresponding to the monomial ideal ${\mathfrak{j}_G}^h$ is invertible. So, the closures of the components of $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}_G),3}$ are irreducible components of Hilb_{16}^7. Our first result is a consequence of Theorem 1.9.

Proposition 4.1. *The marked scheme $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}_G),3}$ is a connected open subset of Hilb_{16}^7 with irreducible components containing $\mathfrak{j}_G$.*

Proof. The marked scheme $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}_G),3}$ is an open subset of Hilb^7_{16} due to [Bertone et al., 2017b, Proposition 6.12(ii)]. Furthermore, the ideal $\mathfrak{j}_G$ is an affine 3-segment with respect to the weight vector $\omega = [11, 10, 9, 8, 6, 5, 4]$. Hence, by Theorem 1.9, $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}_G),3}$ is connected and every its irreducible component contains $\mathfrak{j}_G$. □

Remark 4.2. From the fact that $\mathfrak{j}_G$ is an affine 3-segment with respect to the weight vector $\omega := [11, 10, 9, 8, 6, 5, 4]$ we obtain that $\mathcal{M}_2$ is a cone, with vertex in $\mathfrak{j}_G$, with respect to a positive non-standard grading (see [Ferrarese and Roggero, 2009, Corollary 2.7]). Thus, there is a projection in the Zariski tangente space to $\mathcal{M}_2$ at the origin which induces an isomorphism of $\mathcal{M}_2$ with its image (see [Ferrarese and Roggero, 2009, Theorem 3.2]). This projection identifies a set of eliminable variables which is very useful, for example, in order to enhance the performance of the computations in this context.

We denote by $\mathcal{M}_1 := \mathrm{Mf}_{\mathcal{P}(\mathfrak{j}_G),3} \cap \mathcal{R}^7_{16}$ the irreducible component of $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}_G),3}$ that is obtained by intersecting $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}_G),3}$ with the smoothable component $\mathcal{R}^7_{16}$. Thus, the dimension of $\mathcal{M}_1$ is 112. We now highlight the existence of other two components of $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}_G),3}$. The result of Proposition 4.1 suggests us to look for the irreducible components other that $\mathcal{M}_1$ containing the ideal $\mathfrak{j}_G$.

By the techniques described in [Bertone et al., 2017a] and briefly recalled in Section 1, we obtain $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}_G),3}$ as the affine scheme defined by an ideal $\mathfrak{A}$ generated by 2160 polynomials of degrees $d = 3, 4, 5$ in the polynomial ring $K[C]$ in 512 variables. The computation of a primary decomposition of the ideal $\mathfrak{A}$ is unaffordable with Gröbner bases techniques. Thus, we look for other strategies.

Recalling the construction of a $[\mathcal{P}(\mathfrak{j}_G), 3]$-marked set, we consider the terms outside $\mathfrak{j}_G$ of degree up to 3 in the following order:

$$x_1^3, x_3^2, x_3x_2, x_2^2, x_4x_1, x_3x_1, x_2x_1, x_1^2, x_7, x_6, x_5, x_4, x_3, x_2, x_1, 1.$$

For example, the polynomial of a $[\mathcal{P}(\mathfrak{j}_G), 3]$-marked set with head term x_7^2 has the following shape:
$x_7^2 - (c_{1,1}x_1^3 + c_{1,2}x_3^2 + c_{1,3}x_2x_3 + c_{1,4}x_2^2 + c_{1,5}x_1x_4 + c_{1,6}x_1x_3 + c_{1,7}x_1x_2 + c_{1,8}x_1^2 + c_{1,9}x_7 +$
$+c_{1,10}x_6 + c_{1,11}x_5 + c_{1,12}x_4 + c_{1,13}x_3 + c_{1,14}x_2 + c_{1,15}x_1 + c_{1,16})$,
and, observing that $(\mathfrak{j}_G)_4 = R_4$, the polynomial with head term x_1^4 is
$x_1^4 - (c_{32,1}x_1^3 + c_{32,2}x_3^2 + c_{32,3}x_2x_3 + c_{32,4}x_2^2 + c_{32,5}x_1x_4 + c_{32,6}x_1x_3 + c_{32,7}x_1x_2 + c_{32,8}x_1^2 + c_{32,9}x_7 +$
$c_{32,10}x_6 + c_{32,11}x_5 + c_{32,12}x_4 + c_{32,13}x_3 + c_{32,14}x_2 + c_{32,15}x_1 + c_{32,16})$.

Theorem 4.3. *There is an irreducible component $\mathcal{M}_2$ of the marked scheme $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}_G),3}$ that is rational and has dimension* 161*.*

Proof. Let $\overline{C}$ be the set of the parameters $c_{i,j}$ that are coefficients in a $[\mathcal{P}(\mathfrak{j}_G), 3]$-marked set and have indexes $j \geq 9$ or $i \geq 22$ and $j \geq 2$. Note that the parameters in $\overline{C}$ are the coefficients of the terms of degree lower than the degree of the corresponding head term, except for $i = 32$. Consider the family of $[\mathcal{P}(\mathfrak{j}_G), 3]$-marked sets in which the parameters in $\overline{C}$ are null. The remaining parameters are $179 = 8 \cdot 21 + 11$ and have indexes either $i \leq 21$ and $j \leq 8$ or $i \geq 22$ and $j = 1$. This choice guarantees that we are considering points of the Hilbert scheme corresponding to schemes with a singularity in $[0, \dots, 0, 1] \in \mathbb{P}^7_K$

Intersecting the marked scheme $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}_G),3}$ with the linear variety L defined by the vanishing of the parameters in $\overline{C}$, we obtain that the generators of the ideal defining $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}_G),3}$ become polynomials, many of which are divisible by $c_{32,1}$. Removing this factor, we obtain a set of polynomials defining a particular family $\mathcal{F}_2$ of ideals in $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}_G),3} \cap L$. Interreducing the polynomials

defining $\mathcal{F}_2$ we obtain 25 polynomials $u_{i,j}$, which are listed in the Appendix and form a complete intersection of dimension $154 = 179 - 25$ in $K[C]/(\overline{C})$, hence the family $\mathcal{F}_2$ has dimension 154. We can observe that by these 25 polynomials the following 25 parameters are eliminable, in the sense that they can be replaced by polynomials in the remaining parameters:
$c_{1,8}, c_{2,8}, c_{3,8}, c_{4,8}, c_{5,8}, c_{6,8}, c_{8,8}, c_{9,8}, c_{10,8}, c_{11,8}, c_{12,8}, c_{14,8}, c_{15,8}, c_{16,8}, c_{17,8}, c_{19,8}, c_{20,8}, c_{21,8}, c_{23,1}, c_{24,1}, c_{25,1}, c_{26,1}, c_{27,1}, c_{29,1}, c_{30,1}$.

We denote by C_0 the set of the remaining $154 = 179 - 25$ parameters. Of course, all the polynomials $u_{i,j}$ defining the family $\mathcal{F}_2$ vanish after the elimination of the above 25 parameters.

Allowing translations on the variables $x_1, \dots, x_7$, the family $\mathcal{F}_2$ spreads to a larger family $\widetilde{\mathcal{F}}_2$ which depends on $161 = 154 + 7$ parameters and is still contained in $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}_G),3}$. Denote by $\mathcal{M}_2$ the subscheme of points corresponding to the ideals in $\widetilde{\mathcal{F}}_2$. By construction, $\mathcal{M}_2$ is a complete intesection too and has dimension 161. Moreover, it is rational because it depends on exactly 161 parameters.

Now, we observe that $\mathcal{M}_2$ is an irreducible component of $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}_G),3}$. We randomly choose particular values for the 154 parameters in C_0 in order to obtain the $[\mathcal{P}(\mathfrak{j}_G), 3]$-marked basis of an ideal $\mathfrak{a}$ corresponding to a points of $\mathcal{M}_2$ (please, see the Appendix for a possible choice of the values for the 154 parameters in C_0, the consequent values for the 25 eliminable variables and the generators of the ideal $\mathfrak{a}$).

Due to Corollary 1.6, we compute the Zariski tangent space to Hilb^7_{16} at the point corresponding to $\mathfrak{a}$ finding that it has dimension 161, that is the dimension of $\mathcal{M}_2$. Thus, $\overline{\mathcal{M}_2}$ is an irreducible component of Hilb^7_{16}. □

Remark 4.4. The ideal $\mathfrak{a}$ of Theorem 4.3 defines a general point of $\mathcal{M}_2$ and the corresponding scheme is the union of a simple point and of a non-reduced structure of multiplicity 15 on a different point. Observe that the ideal $\mathfrak{a}$ is only one of the possible points we can consider in order to check that the dimension of $\mathcal{M}_2$ is 161.

Theorem 4.5. *There is an irreducible component $\mathcal{M}_3$ of $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}_G),3}$ which is different from $\mathcal{M}_1$ and $\mathcal{M}_2$. This component $\mathcal{M}_3$ ha dimension ≥ 116 and contains a subscheme of $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}_G),3}$ which is isomorphic to an affine space of dimension 116.*

Proof. Referring to the construction of a $[\mathcal{P}(\mathfrak{j}_G), 3]$-marked set, consider the family $\mathcal{F}_3$ of $[\mathcal{P}(\mathfrak{j}_G), 3]$-marked sets in which the parameters $c_{i,j}$ are null if they are outside the set $\widetilde{C}$ of remaining 109 parameters that is listed in the Appendix.

The marked sets that we obtain with the above setting are actually marked bases for every value of the remaining 109 parameters (see the Appendix for details about these marked bases). Thus, allowing translations on the variables $x_1, \dots, x_7$ the family $\mathcal{F}_3$ spreads to a larger family $\widetilde{\mathcal{F}}_3$ depending on $116 = 109 + 7$ free parameters, which is still contained in $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}_G),3}$. Denote by $\widetilde{\mathcal{M}}_3$ the subscheme of points corresponding to the ideals in $\widetilde{\mathcal{F}}_3$. Thus $\widetilde{\mathcal{M}}_3$ is isomorphic to an affine space of dimension 116 and then it is different from $\mathcal{M}_1$ which has dimension 112.

We choose a particular point of $\widetilde{\mathcal{M}}_3$ at which the dimension of the Zariski tangent space to the Hilbert scheme Hilb^7_{16} is 153 (see the Appendix for the explicit description of one of these possible points which defines a scheme with support in the origin). Thus, the dimension of $\widetilde{\mathcal{M}}_3$ is ≤ 153 and, above all, there is a point of $\widetilde{\mathcal{M}}_3$ which cannot belong to $\mathcal{M}_2$, because the dimension of $\mathcal{M}_2$ is $161 > 153$. This observation proves the existence of an irreducible component $\mathcal{M}_3 \supseteq \widetilde{\mathcal{M}}_3$ of $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}_G),3}$ other than $\mathcal{M}_1$ and $\mathcal{M}_2$. □

Remark 4.6. In the proof of Theorem 4.5 the family $\mathcal{F}_3$ is constructed by setting $c_{32,1} = 0$, where $c_{32,1}$ is the parameter already considered in the proof of Theorem 4.3. Note that the ideals of

$\mathcal{F}_3$ do not belong to $\mathrm{Gor}(1,7,7,1)$ because a general point of $\widetilde{\mathcal{M}}_3$ corresponds to a non-reduced structure over a point. Otherwise, $\mathcal{M}_3$ should be a second component containing $\mathrm{Gor}(1,7,7,1)$, that is impossible by the irreducibility of the locus of these points.

Corollary 4.7. *There are at least the three irreducible components $\overline{\mathcal{M}_1}$, $\overline{\mathcal{M}_2}$ and $\overline{\mathcal{M}_3}$ of Hilb^7_{16} passing through the point corresponding to the ideal $\mathfrak{j}_G$.*

Proof. This is an immediate consequence of Proposition 4.1 and Theorems 4.3 and 4.5. □

5 Graded Artin Gorenstein K-algebras with Hilbert function $(1,5,5,1)$ define smoothable points

In this section we apply the same techniques of Section 3 to prove the smoothability of graded Gorenstein K-algebras with Hilbert function $(1,5,5,1)$ over an algebraically closed field of characteristic 0. We recall that different proofs of this case $(1,5,5,1)$ were presented in [Jelisiejew, 2014], contemporary to our first version given in [Bertone et al., 2012], and later in [Casnati et al., 2015] when $\mathrm{char}(K) \neq 2,3$.

We consider the Hilbert scheme Hilb^5_{12} parameterizing zero-dimensional subschemes of $\mathbb{P}^5$ of length 12. As before, we identify every point of Hilb^5_{12} with an ideal in $R = K[x_1,\dots,x_5]$, non-necessarily homogeneous, with affine Hilbert polynomial $p(t) = 12$. The lex-point of Hilb^5_{12} corresponds to the following lex-segment ideal in R:

$$\mathfrak{j}_{\mathrm{lex}} = (x_5, x_4, x_3, x_2, x_1^{12}).$$

It is well-known that $\mathfrak{j}_{\mathrm{lex}}$ is a smooth point of the smoothable component $\mathcal{R}^7_{16}$ of dimension $5 \cdot 12 = 60$, because the general poin of $\mathcal{R}^7_{16}$ is the reduced scheme of 12 distinct points.

As in Section 3, we compute the complete list of 92 strongly stable ideals of R lying on Hilb^5_{12}. Among them, we focus on the following one:

$$\mathfrak{j}_G = (x_5^2, x_4x_5, x_3x_5, x_2x_5, x_5x_1, x_4^2, x_3x_4, x_2x_4, x_1x_4, x_3^2, x_2^2x_3, x_1x_2x_3, x_1^2x_3, x_2^3, x_1x_2^2, x_1^2x_2, x_1^4)$$

By the constructive tools of [Bertone et al., 2017a] and by theoretical results on the double-generic initial ideal, we now show that $\mathfrak{j}_G$ is the generic initial ideal w.r.t. lex term order of a general ideal defining a graded Gorenstein K-algebra with Hilbert function $(1,5,5,1)$.

Theorem 5.1. *The ideal $\mathfrak{j}_G$ is the generic initial ideal w.r.t. the lex term order of a general ideal defining a graded Gorenstein K-algebra with Hilbert function $(1,5,5,1)$.*

Proof. We explicitly construct a random ideal defining a graded Gorenstein K-algebra with Hilbert function $(1,5,5,1)$ by apolarity, thanks to the already cited correspondence with cubic hypersurfaces (see [Iarrobino and Kanev, 1999, Lemma 2.12]). We randomly choose a cubic form F in $K[x_1,\dots,x_5]$ and from F compute the ideal $\mathrm{Ann}(F) \subset K[x_1,\dots,x_5]$ by apolarity. The reduced Gröbner basis w.r.t. the lex order of the ideal $\mathrm{Ann}(F)$ is given by the following 17 polynomials (in bold the initial term of each polynomial):
$\mathfrak{f}_1 := \mathbf{x_5^2} + 4x_1^2 + \frac{17}{3}x_1x_2 - \frac{83}{12}x_1x_3 - \frac{23}{4}x_2x_3,$
$\mathfrak{f}_2 := \mathbf{x_4x_5} - \frac{3}{4}x_2x_3 - \frac{5}{4}x_1x_3 + x_1x_2,$
$\mathfrak{f}_3 := \mathbf{x_4^2} + \frac{25}{6}x_2x_3 + x_2^2 + \frac{71}{18}x_1x_3 - \frac{28}{9}x_1x_2 - 5x_1^2,$
$\mathfrak{f}_4 := \mathbf{x_3x_5} - \frac{3}{4}x_2x_3 + \frac{3}{4}x_1x_3 - x_1x_2,$
$\mathfrak{f}_5 := \mathbf{x_3x_4} - x_2x_3,$
$\mathfrak{f}_6 := \mathbf{x_3^2} - \frac{85}{24}x_2x_3 - \frac{317}{72}x_1x_3 + \frac{71}{18}x_1x_2 + 2x_1^2,$

$\mathfrak{f}_7 := \mathbf{x_2x_5} - \frac{3}{4}x_2x_3 - \frac{5}{4}x_1x_3 + x_1x_2$,
$\mathfrak{f}_8 := \mathbf{x_2x_4} - x_2x_3 - x_1x_3 + x_1x_2$,
$\mathfrak{f}_9 := \mathbf{x_1x_5} - \frac{1}{4}x_2x_3 + \frac{1}{4}x_1x_3 - x_1x_2$,
$\mathfrak{f}_{10} := \mathbf{x_1x_4} - x_1x_2$, $\mathfrak{f}_{11} := \mathbf{x_2^2x_3} + x_1^3$, $\mathfrak{f}_{12} := \mathbf{x_2^3} + \frac{5}{9}x_1^3$, $\mathfrak{f}_{13} := \mathbf{x_2x_1x_3} - \frac{11}{9}x_1^3$,
$\mathfrak{f}_{14} := \mathbf{x_1x_2^2} - \frac{8}{9}x_1^3$, $\mathfrak{f}_{15} := \mathbf{x_1^2x_3} + x_1^3$, $\mathfrak{f}_{16} := \mathbf{x_1^2x_2} + \frac{2}{3}x_1^3$, $\mathfrak{f}_{17} := \mathbf{x_1^4}$.
Then, we check that $A_F := K[x_1, \ldots, x_5]/\mathrm{Ann}(F)$ is a (graded) Gorenstein K-algebra with Hilbert function $(1, 5, 5, 1)$. We can also observe that A_F is local as we expected, because A_F is Artin and graded. By further computations, we obtain that:

- $\mathfrak{j}_G$ is the initial ideal of $\mathrm{Ann}(F)$ w.r.t. lex
- $\mathfrak{j}_G$ is the maximum w.r.t. the order of Definition 2.6 among all the strongly stable ideals with Hilbert function $(1, 5, 5, 1)$.

Then, by Lemma 2.7, $\mathfrak{j}_G$ is the double-generic initial ideal of $\overline{\mathrm{Gor}}(1, 5, 5, 1)$ and, hence, is the generic initial ideal of a general point of $\mathrm{Gor}(1, 5, 5, 1)$ by [Bertone et al., 2017b, Proposition 4].

Moreover, we can observe that the spectrum of $R/\mathrm{Ann}(F)$ is supported on a single point which is Gorenstein with Hilbert function $(1, 5, 5, 1)$ and hence $R/\mathrm{Ann}(F)$ is a local Gorenstein algebra with Hilbert function $(1, 5, 5, 1)$. □

Remark 5.2. The strongly stable ideal $\mathfrak{j}_G$ is an affine 3-segment with respect to the weight vector $\omega = [8, 7, 5, 4, 3]$. Hence, we can apply Theorem 1.9 to $\mathfrak{j}_G$.

The following straightforward consequence of Theorem 5.1 suggests that the marked scheme $\mathrm{Mf}(\mathcal{P}(\mathfrak{j}_G), 3)$ is the right place in which smoothable Gorenstein points can be.

Proposition 5.3. $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}_G),3} \cap \mathrm{Gor}(1, 5, 5, 1)$ *is a non-empty open subset.*

Proof. From the proof of Theorem 5.1 we deduce that the ideal $\mathrm{Ann}(F)$ belongs to the family of ideals having a $[\mathcal{P}(\mathfrak{j}_G), 3]$-marked basis, hence to the family of ideals defining the $[\mathcal{P}(\mathfrak{j}_G), 3]$-scheme $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}_G),3}$. □

By the techniques described in [Bertone et al., 2017a], we obtain $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}_G),3}$ as the affine scheme defined by an ideal $\mathfrak{U}$ generated by 576 polynomials in the polynomial ring $K[C]$ in $204 = 12 \cdot 17$ variables. By a suitable choice of values for the parameters C, we find a family $\{\mathfrak{G}_T\}_T$ of $[\mathcal{P}(\mathfrak{j}_G), 3]$-marked bases consisting of the following polynomials whose coefficients depend on the parameter T:
$F_1 := \mathfrak{f}_1$, $F_2 := \mathfrak{f}_2$, $F_3 := \mathfrak{f}_3 - Tx_4 + x_2T$, $F_4 := \mathfrak{f}_4$, $F_5 := \mathfrak{f}_5$, $F_6 := \mathfrak{f}_6$, $F_7 := \mathfrak{f}_7$, $F_8 := \mathfrak{f}_8$,
$F_9 := \mathfrak{f}_9$, $F_{10} := \mathfrak{f}_{10}$, $F_{11} := \mathfrak{f}_{11}$, $F_{12} := \mathfrak{f}_{12} - x_2x_3T - x_3x_1T + Tx_2^2 + x_2x_1T$, $F_{13} := \mathfrak{f}_{13}$,
$F_{14} := \mathfrak{f}_{14}$, $F_{15} := \mathfrak{f}_{15}$, $F_{16} := \mathfrak{f}_{16}$, $F_{17} := \mathfrak{f}_{17}$.
From now, for every $T \in \mathbb{A}^1_K$ we denote by $\mathfrak{i}_T \subset K[x_1, \ldots, x_7]$ the ideal generated by the $[\mathcal{P}(\mathfrak{j}_G), 3]$-marked basis $\mathfrak{G}_T$. For $T = 0$ we obtain the ideal $\mathfrak{i}_0 = \mathrm{Ann}(F)$ that we considered in the proof of Theorem 5.1 and which defines a scheme with support in a single point. For every $T \in \mathbb{A}^1_K \setminus \{0\}$ we have a different situation because every ideal $\mathfrak{i}_T$ defines a scheme whose support contains at least the following 3 distinct affine points:

$$(0, 0, 0, T, 0), \qquad (0, 0, 0, 0, 0), \qquad (0, -T, 0, 0, 0).$$

Lemma 5.4. *Over an algebraically closed field K of characteristic 0, there exists a flat family of ideals which is contained in $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}_G),3} \cap \mathcal{R}^5_{12}$ such that the special fiber corresponds to a Gorenstein point defined by a graded K-algebra with Hilbert function $(1, 5, 5, 1)$.*

Proof. We prove that the family of ideals $\{\mathfrak{i}_\tau\}_\tau$, which are generated by the $[\mathcal{P}(\mathfrak{j}_G), 3]$-marked bases $\mathfrak{G}_\tau$, is contained in the smoothable component $\mathcal{R}^5_{12}$.

By construction, the ideals $\mathfrak{j}_T$ belong to the marked scheme $\mathrm{Mf}_{\mathcal{P}(\mathfrak{j}_G),3}$ which embeds as an open subset in the punctual Hilbert scheme Hilb^5_{12}. Hence, these ideals define a flat family over $\mathbb{A}^1$. As we have already recalled in Section 2, the locus of points in a Hilbert scheme representing all the Gorenstein schemes is an open subset. Hence, the intersection of this locus with the flat family $\{\mathfrak{i}_T\}_T$, which is non-empty because $\mathfrak{j}_0$ represents a Gorenstein point, is an open subset of the family. So, we find at least a value $\overline{T} \neq 0$ such that $\mathfrak{i}_{\overline{T}}$ represents a Gorenstein point.

By computational tools, we find that for every $T \neq 0$ the ideal $\mathfrak{i}_T$ defines a scheme whose supports contain at least 3 distinct points and hence components of multiplicity at most 10. So, the ideal $\mathfrak{i}_{\overline{T}}$ is smoothable by the fact that for $d \leq 10$ the locus of Gorenstein points is a Hilbert scheme of d points is irreducible (see [Casnati and Notari, 2011]). Thus, every ideal $\mathfrak{i}_T$ belongs to the smoothable component $\mathcal{R}^5_{12}$ because the family is irreducible. In particular, the limit of this family, that is the ideal $\mathfrak{j}_0$, belongs to $\mathcal{R}^5_{12}$ too. □

Remark 5.5. We now highlight the following fact, which is analogous to that described in Remark 3.4. The ideal $\mathfrak{i}_0$ is the ideal $\mathrm{Ann}(F)$ of the proof of Theorem 5.1, hence it defines a Gorenstein point in $\mathrm{Gor}(1,5,5,1)$. Nevertheless, for every $T \neq 0$, the ideal $\mathfrak{i}_T$ defines a Gorenstein point which does not belong to $\mathrm{Gor}(1,5,5,1)$ because it is supported on more than one point. Moreover, in the polynomial $\mathfrak{f}_{14}$ the term x_3x_2 has a non-null coefficient and is higher than x_2^3 with respect to lex term order. This fact implies that the initial ideal of $\mathfrak{i}_T$ with respect to lex order is not $\mathfrak{j}_G$.

Lemma 5.6. *Over an algebraically closed field K of characteristic 0, there exists a smooth Gorenstein point defined by a graded K-algebra with Hilbert function $(1,5,5,1)$ belonging to the smoothable component $\mathcal{R}^5_{12}$.*

Proof. We prove that the ideal $\mathfrak{i}_0$ defines a smooth Gorenstein point in the smoothable component $\mathcal{R}^5_{12}$. We already know that $\mathfrak{i}_0$ defines a Gorenstein point. Moreover, by Lemma 5.4 and by construction, the ideal $\mathfrak{i}_0$ represents a point that belongs to $\mathcal{R}^5_{12} \cap \mathrm{Mf}_{\mathcal{P}(\mathfrak{j}_G),3}$. Due to Corollary 1.6, we can compute the tangent space from the polynomials of its $[\mathcal{P}(\mathfrak{j}_G), 3]$-marked basis obtaining that the dimension of the Zariski tangent space to Hilb^5_{12} at the point $\mathfrak{j}_0$ is 60, i.e. the dimension of the smoothable component. □

Theorem 5.7. *Over an algebraically closed field K of characteristic 0, every Gorenstein point defined by a graded K-algebra with Hilbert function $(1,5,5,1)$ is smoothable.*

Proof. By Lemmas 5.4 and 5.6, we have a Gorenstein point of $\mathrm{Gor}(1,5,5,1)$ that belongs to the smoothable component $\mathcal{R}^5_{12}$ and that is smooth in the Hilbert scheme. These facts imply that also all the other points of $\mathrm{Gor}(1,5,5,1)$ belong to $\mathcal{R}^5_{12}$, i.e. are smoothable, because the locus $\mathrm{Gor}(1,n,n,1)$ is irreducible, as we have already recalled. □

Corollary 5.8. *Over an algebraically closed field K of characteristic 0, every local Gorenstein K-algebra with Hilbert function $(1,5,5,1)$ is smoothable.*

Proof. This is a consequence of Theorem 5.7 and [Elias and Rossi, 2012, Theorem 3.3]. □

Acknowledgments

The authors are members of GNSAGA (INdAM). Part of the research was conducted during a visit of the first and third author at the Dipartimento di Matematica e Applicazioni of Università di Napoli Federico II, which also financially supported that visit.

Appendix

The computations supporting our results often produce huge polynomials. In this Appendix we list the polynomials that are involved in the proofs described in Section 4.

Concerning the proof of Theorem 4.3

Here are the 25 polynomials $u_{i,j}$ which highlight the presence of 25 eliminable parameters for the ideal defining the family $\mathcal{F}_2$ that is contained in $\mathrm{Mf}(\mathrm{j}_G, 3)$ and is constructed in the cited proof:

$$u_{1,8} = \mathbf{c_{1,8}} - (c_{7,2}^2 c_{28,1}^4 + 2\,c_{7,2}c_{7,3}c_{28,1}^3 c_{31,1} + 2\,c_{7,2}c_{7,4}c_{28,1}^2 c_{31,1}^2 + c_{7,3}^2 c_{28,1}^2 c_{31,1}^2 + 2\,c_{7,3}c_{7,4}c_{28,1}c_{31,1}^3 + c_{7,4}^2 c_{31,1}^4 + 2\,c_{7,1}c_{7,2}c_{28,1}^2 c_{32,1} + 2\,c_{7,1}c_{7,3}c_{28,1}c_{31,1}c_{32,1} + 2\,c_{7,1}c_{7,4}c_{31,1}^2 c_{32,1} + 2\,c_{7,2}c_{7,5}c_{22,1}c_{28,1}^2 + 2\,c_{7,2}c_{7,6}c_{28,1}^3 + 2\,c_{7,2}c_{7,7}c_{28,1}^2c_{31,1} + 2\,c_{7,3}c_{7,5}c_{22,1}c_{28,1}c_{31,1} + 2\,c_{7,3}c_{7,6}c_{28,1}^2c_{31,1} + 2\,c_{7,3}c_{7,7}c_{28,1}c_{31,1}^2 + 2\,c_{7,4}c_{7,5}c_{22,1}c_{31,1}^2 + 2\,c_{7,4}c_{7,6}c_{28,1}c_{31,1}^2 + 2\,c_{7,4}c_{7,7}c_{31,1}^3 + c_{7,1}^2c_{32,1}^2 + 2\,c_{7,1}c_{7,5}c_{22,1}c_{32,1} + 2\,c_{7,1}c_{7,6}c_{28,1}c_{32,1} + 2\,c_{7,1}c_{7,7}c_{31,1}c_{32,1} + 2\,c_{7,2}c_{7,8}c_{28,1}^2 + 2\,c_{7,3}c_{7,8}c_{28,1}c_{31,1} + 2\,c_{7,4}c_{7,8}c_{31,1}^2 + c_{7,5}^2c_{22,1}^2 + 2\,c_{7,5}c_{7,6}c_{22,1}c_{28,1} + 2\,c_{7,5}c_{7,7}c_{22,1}c_{31,1} + c_{7,6}^2c_{28,1}^2 + 2\,c_{7,6}c_{7,7}c_{28,1}c_{31,1} + c_{7,7}^2c_{31,1}^2 - c_{1,2}c_{28,1}^2 - c_{1,3}c_{28,1}c_{31,1} - c_{1,4}c_{31,1}^2 + 2\,c_{7,1}c_{7,8}c_{32,1} + 2\,c_{7,5}c_{7,8}c_{22,1} + 2\,c_{7,6}c_{7,8}c_{28,1} + 2\,c_{7,7}c_{7,8}c_{31,1} - c_{1,1}c_{32,1} - c_{1,5}c_{22,1} - c_{1,6}c_{28,1} - c_{1,7}c_{31,1} + c_{7,8}^2),$$

$$u_{2,8} = \mathbf{c_{2,8}} - (c_{7,5}c_{13,5}c_{22,1}^2 - c_{2,3}c_{28,1}c_{31,1} + c_{7,2}c_{13,6}c_{28,1}^3 + c_{7,6}c_{13,6}c_{28,1}^2 + c_{13,2}c_{28,1}^4c_{7,2} + c_{13,2}c_{28,1}^3c_{7,6} + c_{13,2}c_{28,1}^2c_{7,8} + c_{7,4}c_{13,7}c_{31,1}^3 + c_{7,7}c_{13,7}c_{31,1}^2 + c_{13,4}c_{31,1}^4c_{7,4} + c_{13,4}c_{31,1}^3c_{7,7} + c_{13,4}c_{31,1}^2c_{7,8} + c_{7,4}c_{13,8}c_{31,1}^2 + c_{7,2}c_{13,8}c_{28,1}^2 - c_{2,4}c_{31,1}^2 - c_{2,2}c_{28,1}^2 + c_{32,1}c_{7,1}c_{13,3}c_{28,1}c_{31,1} + c_{7,3}c_{13,5}c_{22,1}c_{28,1}c_{31,1} + c_{7,5}c_{13,3}c_{22,1}c_{28,1}c_{31,1} + c_{7,1}c_{13,1}c_{32,1}^2 + c_{7,3}c_{13,8}c_{28,1}c_{31,1} + c_{7,3}c_{13,7}c_{28,1}c_{31,1}^2 + c_{7,5}c_{13,7}c_{22,1}c_{31,1} + c_{7,6}c_{13,7}c_{28,1}c_{31,1} + c_{13,4}c_{31,1}^2c_{7,2}c_{28,1}^2 + c_{13,4}c_{31,1}^3c_{7,3}c_{28,1} + c_{13,4}c_{31,1}^2c_{7,5}c_{22,1} + c_{13,4}c_{31,1}^2c_{7,6}c_{28,1} + c_{7,5}c_{13,6}c_{22,1}c_{28,1} + c_{7,7}c_{13,6}c_{28,1}c_{31,1} + c_{7,2}c_{13,3}c_{28,1}^3c_{31,1} + c_{7,3}c_{13,3}c_{28,1}^2c_{31,1}^2 + c_{7,4}c_{13,3}c_{28,1}c_{31,1}^3 + c_{7,6}c_{13,3}c_{28,1}^2c_{31,1} + c_{7,7}c_{13,3}c_{28,1}c_{31,1}^2 + c_{7,8}c_{13,3}c_{28,1}c_{31,1} + c_{13,2}c_{28,1}^3c_{7,3}c_{31,1} + c_{13,2}c_{28,1}^2c_{7,4}c_{31,1}^2 + c_{13,2}c_{28,1}^2c_{7,5}c_{22,1} + c_{13,2}c_{28,1}^2c_{7,7}c_{31,1} + c_{7,2}c_{13,7}c_{28,1}^2c_{31,1} + c_{7,4}c_{13,5}c_{22,1}c_{31,1}^2 + c_{7,6}c_{13,5}c_{22,1}c_{28,1} + c_{7,7}c_{13,5}c_{22,1}c_{31,1} + c_{7,3}c_{13,6}c_{28,1}^2c_{31,1} + c_{7,4}c_{13,6}c_{28,1}c_{31,1}^2 + c_{7,2}c_{13,5}c_{22,1}c_{28,1}^2 - c_{2,1}c_{32,1} - c_{2,5}c_{22,1} - c_{2,6}c_{28,1} - c_{2,7}c_{31,1} + c_{32,1}c_{7,3}c_{13,1}c_{28,1}c_{31,1} + c_{32,1}c_{7,4}c_{13,1}c_{31,1}^2 + c_{13,4}c_{31,1}^2c_{7,1}c_{32,1} + c_{13,2}c_{28,1}^2c_{7,1}c_{32,1} + c_{32,1}c_{7,2}c_{13,1}c_{28,1}^2 + c_{7,8}c_{13,7}c_{31,1} + c_{7,1}c_{13,8}c_{32,1} + c_{7,8}c_{13,1}c_{32,1} + c_{7,8}c_{13,5}c_{22,1} + c_{7,8}c_{13,6}c_{28,1} + c_{7,5}c_{13,1}c_{22,1}c_{32,1} + c_{7,6}c_{13,1}c_{28,1}c_{32,1} + c_{7,7}c_{13,1}c_{31,1}c_{32,1} + c_{7,1}c_{13,5}c_{22,1}c_{32,1} + c_{7,1}c_{13,6}c_{28,1}c_{32,1} + c_{7,1}c_{13,7}c_{31,1}c_{32,1} + c_{7,8}c_{13,8} + c_{7,5}c_{13,8}c_{22,1} + c_{7,6}c_{13,8}c_{28,1} + c_{7,7}c_{13,8}c_{31,1}),$$

$$u_{3,8} = \mathbf{c_{3,8}} - (c_{7,5}c_{18,5}c_{22,1}^2 - c_{3,3}c_{28,1}c_{31,1} + c_{7,2}c_{18,6}c_{28,1}^3 + c_{7,6}c_{18,6}c_{28,1}^2 + c_{18,2}c_{28,1}^4c_{7,2} + c_{18,2}c_{28,1}^3c_{7,6} + c_{18,2}c_{28,1}^2c_{7,8} + c_{7,4}c_{18,7}c_{31,1}^3 + c_{7,7}c_{18,7}c_{31,1}^2 + c_{18,4}c_{31,1}^4c_{7,4} + c_{18,4}c_{31,1}^3c_{7,7} + c_{18,4}c_{31,1}^2c_{7,8} + c_{7,4}c_{18,8}c_{31,1}^2 - c_{3,4}c_{31,1}^2 - c_{3,2}c_{28,1}^2 + c_{7,3}c_{18,5}c_{22,1}c_{28,1}c_{31,1} + c_{7,5}c_{18,3}c_{22,1}c_{28,1}c_{31,1} + c_{7,2}c_{18,8}c_{28,1}^2 + c_{7,1}c_{18,8}c_{32,1} + c_{7,8}c_{18,1}c_{32,1} + c_{7,8}c_{18,5}c_{22,1} + c_{7,8}c_{18,6}c_{28,1} + c_{7,8}c_{18,7}c_{31,1} + c_{7,1}c_{18,1}c_{32,1}^2 - c_{3,1}c_{32,1} - c_{3,5}c_{22,1} - c_{3,6}c_{28,1} - c_{3,7}c_{31,1} + c_{7,3}c_{18,8}c_{28,1}c_{31,1} + c_{18,2}c_{28,1}^2c_{7,4}c_{31,1}^2 + c_{18,2}c_{28,1}^2c_{7,5}c_{22,1} + c_{18,2}c_{28,1}^2c_{7,7}c_{31,1} + c_{7,2}c_{18,7}c_{28,1}^2c_{31,1} + c_{7,3}c_{18,7}c_{28,1}c_{31,1}^2 + c_{7,5}c_{18,7}c_{22,1}c_{31,1} + c_{7,6}c_{18,7}c_{28,1}c_{31,1} + c_{18,4}c_{31,1}^2c_{7,2}c_{28,1}^2 + c_{18,4}c_{31,1}^3c_{7,3}c_{28,1} + c_{18,4}c_{31,1}^2c_{7,5}c_{22,1} + c_{18,4}c_{31,1}^2c_{7,6}c_{28,1} + c_{7,6}c_{18,5}c_{22,1}c_{28,1} + c_{7,7}c_{18,5}c_{22,1}c_{31,1} + c_{7,3}c_{18,6}c_{28,1}^2c_{31,1} + c_{7,4}c_{18,6}c_{28,1}c_{31,1}^2 + c_{7,5}c_{18,6}c_{22,1}c_{28,1} + c_{7,7}c_{18,6}c_{28,1}c_{31,1} + c_{7,2}c_{18,3}c_{28,1}^3c_{31,1} + c_{7,3}c_{18,3}c_{28,1}^2c_{31,1}^2 + c_{7,4}c_{18,3}c_{28,1}c_{31,1}^3 + c_{7,6}c_{18,3}c_{28,1}^2c_{31,1} + c_{7,7}c_{18,3}c_{28,1}c_{31,1}^2 + c_{7,8}c_{18,3}c_{28,1}c_{31,1} + c_{18,2}c_{28,1}^3c_{7,3}c_{31,1} + c_{7,4}c_{18,5}c_{22,1}c_{31,1}^2 + c_{7,2}c_{18,5}c_{22,1}c_{28,1}^2 + c_{7,1}c_{18,5}c_{22,1}c_{32,1} + c_{7,1}c_{18,6}c_{28,1}c_{32,1} + c_{7,1}c_{18,7}c_{31,1}c_{32,1} + c_{7,8}c_{18,8} + c_{7,5}c_{18,1}c_{22,1}c_{32,1} + c_{7,6}c_{18,1}c_{28,1}c_{32,1} + c_{7,7}c_{18,1}c_{31,1}c_{32,1} + c_{32,1}c_{7,1}c_{18,3}c_{28,1}c_{31,1} + c_{32,1}c_{7,3}c_{18,1}c_{28,1}c_{31,1} + c_{32,1}c_{7,4}c_{18,1}c_{31,1}^2 + c_{18,2}c_{28,1}^2c_{7,1}c_{32,1} + c_{18,4}c_{31,1}^2c_{7,1}c_{32,1} + c_{32,1}c_{7,2}c_{18,1}c_{28,1}^2 + c_{7,5}c_{18,8}c_{22,1} + c_{7,6}c_{18,8}c_{28,1} + c_{7,7}c_{18,8}c_{31,1}),$$

$$u_{4,8} = \mathbf{c_{4,8}} - (c_{7,2}c_{22,1}c_{28,1}^2 + c_{7,3}c_{22,1}c_{28,1}c_{31,1} + c_{7,4}c_{22,1}c_{31,1}^2 - c_{4,2}c_{28,1}^2 - c_{4,3}c_{28,1}c_{31,1} - c_{4,4}c_{31,1}^2 + c_{7,1}c_{22,1}c_{32,1} + c_{7,5}c_{22,1}^2 + c_{7,6}c_{22,1}c_{28,1} + c_{7,7}c_{22,1}c_{31,1} - c_{4,1}c_{32,1} - c_{4,5}c_{22,1} - c_{4,6}c_{28,1} - c_{4,7}c_{31,1} + c_{7,8}c_{22,1}),$$

$$u_{5,8} = \mathbf{c_{5,8}} - (c_{7,2}c_{28,1}^3 + c_{7,3}c_{28,1}^2c_{31,1} + c_{7,4}c_{28,1}c_{31,1}^2 - c_{5,2}c_{28,1}^2 - c_{5,3}c_{28,1}c_{31,1} - c_{5,4}c_{31,1}^2 + c_{7,1}c_{28,1}c_{32,1} + c_{7,5}c_{22,1}c_{28,1} + c_{7,6}c_{28,1}^2 + c_{7,7}c_{28,1}c_{31,1} - c_{5,1}c_{32,1} - c_{5,5}c_{22,1} - c_{5,6}c_{28,1} - c_{5,7}c_{31,1} + c_{7,8}c_{28,1}),$$

$$u_{6,8} = \mathbf{c_{6,8}} - (c_{7,2}c_{28,1}^2c_{31,1} + c_{7,3}c_{28,1}c_{31,1}^2 + c_{7,4}c_{31,1}^3 - c_{6,2}c_{28,1}^2 - c_{6,3}c_{28,1}c_{31,1} - c_{6,4}c_{31,1}^2 + c_{7,1}c_{31,1}c_{32,1} + c_{7,5}c_{22,1}c_{31,1} + c_{7,6}c_{28,1}c_{31,1} + c_{7,7}c_{31,1}^2 - c_{6,1}c_{32,1} - c_{6,5}c_{22,1} - c_{6,6}c_{28,1} - c_{6,7}c_{31,1} + c_{7,8}c_{31,1}),$$

$$u_{8,8} = \mathbf{c_{8,8}} - (c_{13,2}^2c_{28,1}^4 + 2\,c_{13,2}c_{13,3}c_{28,1}^3c_{31,1} + 2\,c_{13,2}c_{13,4}c_{28,1}^2c_{31,1}^2 + c_{13,3}^2c_{28,1}^2c_{31,1}^2 + 2\,c_{13,3}c_{13,4}c_{28,1}c_{31,1}^3 + c_{13,4}^2c_{31,1}^4 + 2\,c_{13,1}c_{13,2}c_{28,1}^2c_{32,1} + 2\,c_{13,1}c_{13,3}c_{28,1}c_{31,1}c_{32,1} + 2\,c_{13,1}c_{13,4}c_{31,1}^2c_{32,1} + 2\,c_{13,2}c_{13,5}c_{22,1}c_{28,1}^2 + 2\,c_{13,2}c_{13,6}c_{28,1}^3 + 2\,c_{13,2}c_{13,7}c_{28,1}^2c_{31,1} + 2\,c_{13,3}c_{13,5}c_{22,1}c_{28,1}c_{31,1} + 2\,c_{13,3}c_{13,6}c_{28,1}^2c_{31,1} + 2\,c_{13,3}c_{13,7}c_{28,1}c_{31,1}^2 +$$

$2\,c_{13,4}c_{13,5}c_{22,1}c_{31,1}{}^2+2\,c_{13,4}c_{13,6}c_{28,1}c_{31,1}{}^2+2\,c_{13,4}c_{13,7}c_{31,1}{}^3+c_{13,1}{}^2c_{32,1}{}^2+2\,c_{13,1}c_{13,5}c_{22,1}c_{32,1}+2\,c_{13,1}c_{13,6}c_{28,1}c_{32,1}+$
$2\,c_{13,1}c_{13,7}c_{31,1}c_{32,1}+2\,c_{13,2}c_{13,8}c_{28,1}{}^2+2\,c_{13,3}c_{13,8}c_{28,1}c_{31,1}+2\,c_{13,4}c_{13,8}c_{31,1}{}^2+c_{13,5}{}^2c_{22,1}{}^2+2\,c_{13,5}c_{13,6}c_{22,1}c_{28,1}+$
$2\,c_{13,5}c_{13,7}c_{22,1}c_{31,1}+c_{13,6}{}^2c_{28,1}{}^2+2\,c_{13,6}c_{13,7}c_{28,1}c_{31,1}+c_{13,7}{}^2c_{31,1}{}^2-c_{8,2}c_{28,1}{}^2-c_{8,3}c_{28,1}c_{31,1}-c_{8,4}c_{31,1}{}^2+$
$2\,c_{13,1}c_{13,8}c_{32,1}+2\,c_{13,5}c_{13,8}c_{22,1}+2\,c_{13,6}c_{13,8}c_{28,1}+2\,c_{13,7}c_{13,8}c_{31,1}-c_{8,1}c_{32,1}-c_{8,5}c_{22,1}-c_{8,6}c_{28,1}-$
$c_{8,7}c_{31,1}+c_{13,8}{}^2),$
$u_{9,8}=\mathbf{c_{9,8}}-(c_{13,5}c_{18,5}c_{22,1}{}^2-c_{9,3}c_{28,1}c_{31,1}+c_{13,6}c_{18,2}c_{28,1}{}^3+c_{13,6}c_{18,6}c_{28,1}{}^2+c_{13,2}c_{28,1}{}^4c_{18,2}+c_{13,2}c_{28,1}{}^3c_{18,6}+$
$c_{13,2}c_{28,1}{}^2c_{18,8}+c_{13,8}c_{18,2}c_{28,1}{}^2+c_{13,8}c_{18,4}c_{31,1}{}^2+c_{13,7}c_{18,4}c_{31,1}{}^3+c_{13,7}c_{18,7}c_{31,1}{}^2+c_{13,4}c_{31,1}{}^4c_{18,4}+$
$c_{13,4}c_{31,1}{}^3c_{18,7}+c_{13,4}c_{31,1}{}^2c_{18,8}-c_{9,4}c_{31,1}{}^2-c_{9,2}c_{28,1}{}^2+c_{13,5}c_{18,3}c_{22,1}c_{28,1}c_{31,1}+c_{13,3}c_{18,5}c_{22,1}c_{28,1}c_{31,1}+$
$c_{13,5}c_{18,2}c_{22,1}c_{28,1}{}^2+c_{13,1}c_{18,8}c_{32,1}+c_{13,8}c_{18,1}c_{32,1}+c_{13,8}c_{18,5}c_{22,1}+c_{13,8}c_{18,6}c_{28,1}+c_{13,8}c_{18,7}c_{31,1}+c_{13,8}c_{18,3}c_{28,1}c_{31,1}+$
$c_{13,7}c_{18,2}c_{28,1}{}^2c_{31,1}+c_{13,7}c_{18,3}c_{28,1}c_{31,1}{}^2+c_{13,7}c_{18,5}c_{22,1}c_{31,1}+c_{13,7}c_{18,6}c_{28,1}c_{31,1}+c_{13,4}c_{31,1}{}^2c_{18,2}c_{28,1}{}^2+$
$c_{13,4}c_{31,1}{}^3c_{18,3}c_{28,1}+c_{13,4}c_{31,1}{}^2c_{18,5}c_{22,1}+c_{13,4}c_{31,1}{}^2c_{18,6}c_{28,1}+c_{13,6}c_{18,3}c_{28,1}{}^2c_{31,1}+c_{13,6}c_{18,4}c_{28,1}c_{31,1}{}^2+$
$c_{13,6}c_{18,5}c_{22,1}c_{28,1}+c_{13,6}c_{18,7}c_{28,1}c_{31,1}+c_{13,3}c_{18,2}c_{28,1}{}^3c_{31,1}+c_{13,3}c_{18,3}c_{28,1}{}^2c_{31,1}{}^2+c_{13,3}c_{18,4}c_{28,1}c_{31,1}{}^3+$
$c_{13,3}c_{18,6}c_{28,1}{}^2c_{31,1}+c_{13,3}c_{18,7}c_{28,1}c_{31,1}{}^2+c_{13,3}c_{18,8}c_{28,1}c_{31,1}+c_{13,2}c_{28,1}{}^3c_{18,3}c_{31,1}+c_{13,2}c_{28,1}{}^2c_{18,4}c_{31,1}{}^2+$
$c_{13,2}c_{28,1}{}^2c_{18,5}c_{22,1}+c_{13,2}c_{28,1}{}^2c_{18,7}c_{31,1}+c_{13,5}c_{18,6}c_{22,1}c_{28,1}+c_{13,5}c_{18,7}c_{22,1}c_{31,1}+c_{13,5}c_{18,4}c_{22,1}c_{31,1}{}^2+$
$c_{13,5}c_{18,8}c_{22,1}+c_{13,6}c_{18,8}c_{28,1}+c_{13,7}c_{18,8}c_{31,1}+c_{32,1}c_{13,3}c_{18,1}c_{28,1}c_{31,1}+c_{32,1}c_{13,1}c_{18,3}c_{28,1}c_{31,1}+c_{13,4}c_{31,1}{}^2c_{18,1}c_{32,1}+$
$c_{13,2}c_{28,1}{}^2c_{18,1}c_{32,1}+c_{32,1}c_{13,1}c_{18,2}c_{28,1}{}^2+c_{32,1}c_{13,1}c_{18,4}c_{31,1}{}^2+c_{13,5}c_{18,1}c_{22,1}c_{32,1}+c_{13,6}c_{18,1}c_{28,1}c_{32,1}+$
$c_{13,7}c_{18,1}c_{31,1}c_{32,1}+c_{13,1}c_{18,1}c_{32,1}{}^2-c_{9,1}c_{32,1}-c_{9,5}c_{22,1}-c_{9,6}c_{28,1}-c_{9,7}c_{31,1}+c_{13,8}c_{18,8}+c_{13,1}c_{18,5}c_{22,1}c_{32,1}+$
$c_{13,1}c_{18,6}c_{28,1}c_{32,1}+c_{13,1}c_{18,7}c_{31,1}c_{32,1}),$
$u_{10,8}=\mathbf{c_{10,8}}-(c_{13,2}c_{22,1}c_{28,1}{}^2+c_{13,3}c_{22,1}c_{28,1}c_{31,1}+c_{13,4}c_{22,1}c_{31,1}{}^2-c_{10,2}c_{28,1}{}^2-c_{10,3}c_{28,1}c_{31,1}-c_{10,4}c_{31,1}{}^2+$
$c_{13,1}c_{22,1}c_{32,1}+c_{13,5}c_{22,1}{}^2+c_{13,6}c_{22,1}c_{28,1}+c_{13,7}c_{22,1}c_{31,1}-c_{10,1}c_{32,1}-c_{10,5}c_{22,1}-c_{10,6}c_{28,1}-c_{10,7}c_{31,1}+$
$c_{13,8}c_{22,1}),$
$u_{11,8}=\mathbf{c_{11,8}}-(c_{13,2}c_{28,1}{}^3+c_{13,3}c_{28,1}{}^2c_{31,1}+c_{13,4}c_{28,1}c_{31,1}{}^2-c_{11,2}c_{28,1}{}^2-c_{11,3}c_{28,1}c_{31,1}-c_{11,4}c_{31,1}{}^2+$
$c_{13,1}c_{28,1}c_{32,1}+c_{13,5}c_{22,1}c_{28,1}+c_{13,6}c_{28,1}{}^2+c_{13,7}c_{28,1}c_{31,1}-c_{11,1}c_{32,1}-c_{11,5}c_{22,1}-c_{11,6}c_{28,1}-c_{11,7}c_{31,1}+$
$c_{13,8}c_{28,1}),$
$u_{12,8}=\mathbf{c_{12,8}}-(c_{13,2}c_{28,1}{}^2c_{31,1}+c_{13,3}c_{28,1}c_{31,1}{}^2+c_{13,4}c_{31,1}{}^3-c_{12,2}c_{28,1}{}^2-c_{12,3}c_{28,1}c_{31,1}-c_{12,4}c_{31,1}{}^2+$
$c_{13,1}c_{31,1}c_{32,1}+c_{13,5}c_{22,1}c_{31,1}+c_{13,6}c_{28,1}c_{31,1}+c_{13,7}c_{31,1}{}^2-c_{12,1}c_{32,1}-c_{12,5}c_{22,1}-c_{12,6}c_{28,1}-c_{12,7}c_{31,1}+$
$c_{13,8}c_{31,1}),$
$u_{14,8}=\mathbf{c_{14,8}}-(c_{18,2}{}^2c_{28,1}{}^4+2\,c_{18,2}c_{18,3}c_{28,1}{}^3c_{31,1}+2\,c_{18,2}c_{18,4}c_{28,1}{}^2c_{31,1}{}^2+c_{18,3}{}^2c_{28,1}{}^2c_{31,1}{}^2+2\,c_{18,3}c_{18,4}c_{28,1}c_{31,1}{}^3+$
$c_{18,4}{}^2c_{31,1}{}^4+2\,c_{18,1}c_{18,2}c_{28,1}{}^2c_{32,1}+2\,c_{18,1}c_{18,3}c_{28,1}c_{31,1}c_{32,1}+2\,c_{18,1}c_{18,4}c_{31,1}{}^2c_{32,1}+2\,c_{18,2}c_{18,5}c_{22,1}c_{28,1}{}^2+$
$2\,c_{18,2}c_{18,6}c_{28,1}{}^3+2\,c_{18,2}c_{18,7}c_{28,1}{}^2c_{31,1}+2\,c_{18,3}c_{18,5}c_{22,1}c_{28,1}c_{31,1}+2\,c_{18,3}c_{18,6}c_{28,1}{}^2c_{31,1}+2\,c_{18,3}c_{18,7}c_{28,1}c_{31,1}{}^2+$
$2\,c_{18,4}c_{18,5}c_{22,1}c_{31,1}{}^2+2\,c_{18,4}c_{18,6}c_{28,1}c_{31,1}{}^2+2\,c_{18,4}c_{18,7}c_{31,1}{}^3+c_{18,1}{}^2c_{32,1}{}^2+2\,c_{18,1}c_{18,5}c_{22,1}c_{32,1}+2\,c_{18,1}c_{18,6}c_{28,1}c_{32,1}+$
$2\,c_{18,1}c_{18,7}c_{31,1}c_{32,1}+2\,c_{18,2}c_{18,8}c_{28,1}{}^2+2\,c_{18,3}c_{18,8}c_{28,1}c_{31,1}+2\,c_{18,4}c_{18,8}c_{31,1}{}^2+c_{18,5}{}^2c_{22,1}{}^2+2\,c_{18,5}c_{18,6}c_{22,1}c_{28,1}+$
$2\,c_{18,5}c_{18,7}c_{22,1}c_{31,1}+c_{18,6}{}^2c_{28,1}{}^2+2\,c_{18,6}c_{18,7}c_{28,1}c_{31,1}+c_{18,7}{}^2c_{31,1}{}^2-c_{14,2}c_{28,1}{}^2-c_{14,3}c_{28,1}c_{31,1}-c_{14,4}c_{31,1}{}^2+$
$2\,c_{18,1}c_{18,8}c_{32,1}+2\,c_{18,5}c_{18,8}c_{22,1}+2\,c_{18,6}c_{18,8}c_{28,1}+2\,c_{18,7}c_{18,8}c_{31,1}-c_{14,1}c_{32,1}-c_{14,5}c_{22,1}-c_{14,6}c_{28,1}-$
$c_{14,7}c_{31,1}+c_{18,8}{}^2),$
$u_{15,8}=\mathbf{c_{15,8}}-(c_{18,2}c_{22,1}c_{28,1}{}^2+c_{18,3}c_{22,1}c_{28,1}c_{31,1}+c_{18,4}c_{22,1}c_{31,1}{}^2-c_{15,2}c_{28,1}{}^2-c_{15,3}c_{28,1}c_{31,1}-c_{15,4}c_{31,1}{}^2+$
$c_{18,1}c_{22,1}c_{32,1}+c_{18,5}c_{22,1}{}^2+c_{18,6}c_{22,1}c_{28,1}+c_{18,7}c_{22,1}c_{31,1}-c_{15,1}c_{32,1}-c_{15,5}c_{22,1}-c_{15,6}c_{28,1}-c_{15,7}c_{31,1}+$
$c_{18,8}c_{22,1}),$
$u_{16,8}=\mathbf{c_{16,8}}-(c_{18,2}c_{28,1}{}^3+c_{18,3}c_{28,1}{}^2c_{31,1}+c_{18,4}c_{28,1}c_{31,1}{}^2-c_{16,2}c_{28,1}{}^2-c_{16,3}c_{28,1}c_{31,1}-c_{16,4}c_{31,1}{}^2+$
$c_{18,1}c_{28,1}c_{32,1}+c_{18,5}c_{22,1}c_{28,1}+c_{18,6}c_{28,1}{}^2+c_{18,7}c_{28,1}c_{31,1}-c_{16,1}c_{32,1}-c_{16,5}c_{22,1}-c_{16,6}c_{28,1}-c_{16,7}c_{31,1}+$
$c_{18,8}c_{28,1}),$
$u_{17,8}=\mathbf{c_{17,8}}-(c_{18,2}c_{28,1}{}^2c_{31,1}+c_{18,3}c_{28,1}c_{31,1}{}^2+c_{18,4}c_{31,1}{}^3-c_{17,2}c_{28,1}{}^2-c_{17,3}c_{28,1}c_{31,1}-c_{17,4}c_{31,1}{}^2+$
$c_{18,1}c_{31,1}c_{32,1}+c_{18,5}c_{22,1}c_{31,1}+c_{18,6}c_{28,1}c_{31,1}+c_{18,7}c_{31,1}{}^2-c_{17,1}c_{32,1}-c_{17,5}c_{22,1}-c_{17,6}c_{28,1}-c_{17,7}c_{31,1}+$
$c_{18,8}c_{31,1}),$
$u_{19,8}=\mathbf{c_{19,8}}-(-c_{19,2}c_{28,1}{}^2-c_{19,3}c_{28,1}c_{31,1}-c_{19,4}c_{31,1}{}^2-c_{19,1}c_{32,1}-c_{19,5}c_{22,1}-c_{19,6}c_{28,1}-c_{19,7}c_{31,1}+$
$c_{22,1}{}^2),$
$u_{20,8}=\mathbf{c_{20,8}}-(-c_{20,2}c_{28,1}{}^2-c_{20,3}c_{28,1}c_{31,1}-c_{20,4}c_{31,1}{}^2-c_{20,1}c_{32,1}-c_{20,5}c_{22,1}-c_{20,6}c_{28,1}-c_{20,7}c_{31,1}+$
$c_{22,1}c_{28,1}),$

$u_{21,8} = \mathbf{c_{21,8}} - (-c_{21,2}{c_{28,1}}^2 - c_{21,3}c_{28,1}c_{31,1} - c_{21,4}{c_{31,1}}^2 - c_{21,1}c_{32,1} - c_{21,5}c_{22,1} - c_{21,6}c_{28,1} - c_{21,7}c_{31,1} + c_{22,1}c_{31,1})$,
$u_{23,1} = \mathbf{c_{23,1}} - ({c_{28,1}}^3)$,
$u_{24,1} = \mathbf{c_{24,1}} - ({c_{28,1}}^2 c_{31,1})$,
$u_{25,1} = \mathbf{c_{25,1}} - ({c_{28,1}}^2)$,
$u_{26,1} = \mathbf{c_{26,1}} - (c_{28,1}{c_{31,1}}^2)$,
$u_{27,1} = \mathbf{c_{27,1}} - (c_{28,1}c_{31,1})$,
$u_{29,1} = \mathbf{c_{29,1}} - ({c_{31,1}}^3)$,
$u_{30,1} = \mathbf{c_{30,1}} - ({c_{31,1}}^2)$.

Here are the values we choose for the 154 parameters in C_0 and the consequent values for the 25 variables that are eliminable variables due to the shape of the polynomials $u_{i,j}$, in order to obtain the generators of a particular ideal $\mathfrak{a}$ in $\mathrm{Mf}(\mathfrak{j}_G, 3)$:

values for the parameters in C_0
$c_{1,1} = -3$, $c_{1,2} = 2$, $c_{1,3} = -3$, $c_{1,4} = 0$, $c_{1,5} = 1$, $c_{1,6} = -2$, $c_{1,7} = -3$, $c_{2,1} = -1$, $c_{2,2} = -1$, $c_{2,3} = -1$, $c_{2,4} = -2$, $c_{2,5} = -1$, $c_{2,6} = -3$, $c_{2,7} = -3$, $c_{3,1} = 0$, $c_{3,2} = -2$, $c_{3,3} = -1$, $c_{3,4} = -2$, $c_{3,5} = -1$, $c_{3,6} = -2$, $c_{3,7} = 1$, $c_{4,1} = 1$, $c_{4,2} = -1$, $c_{4,3} = -2$, $c_{4,4} = 1$, $c_{4,5} = -1$, $c_{4,6} = -1$, $c_{4,7} = -3$, $c_{5,1} = -2$, $c_{5,2} = 2$, $c_{5,3} = 2$, $c_{5,4} = -2$, $c_{5,5} = 0$, $c_{5,6} = 1$, $c_{5,7} = -1$, $c_{6,1} = -1$, $c_{6,2} = -3$, $c_{6,3} = -2$, $c_{6,4} = 2$, $c_{6,5} = -2$, $c_{6,6} = -3$, $c_{6,7} = -2$, $c_{7,1} = -2$, $c_{7,2} = -1$, $c_{7,3} = 0$, $c_{7,4} = 0$, $c_{7,5} = -1$, $c_{7,6} = 1$, $c_{7,7} = -2$, $c_{7,8} = 1$, $c_{8,1} = 1$, $c_{8,2} = 2$, $c_{8,3} = 0$, $c_{8,4} = 0$, $c_{8,5} = -3$, $c_{8,6} = -1$, $c_{8,7} = -2$, $c_{9,1} = 1$, $c_{9,2} = -3$, $c_{9,3} = 0$, $c_{9,4} = 0$, $c_{9,5} = 0$, $c_{9,6} = -2$, $c_{9,7} = 0$, $c_{10,1} = 0$, $c_{10,2} = -2$, $c_{10,3} = -2$, $c_{10,4} = -1$, $c_{10,5} = -3$, $c_{10,6} = -3$, $c_{10,7} = 2$, $c_{11,1} = 2$, $c_{11,2} = -3$, $c_{11,3} = 0$, $c_{11,4} = 0$, $c_{11,5} = -1$, $c_{11,6} = -2$, $c_{11,7} = 1$, $c_{12,1} = -1$, $c_{12,2} = 2$, $c_{12,3} = -3$, $c_{12,4} = -2$, $c_{12,5} = 2$, $c_{12,6} = 1$, $c_{12,7} = -2$, $c_{13,1} = 1$, $c_{13,2} = 0$, $c_{13,3} = 0$, $c_{13,4} = 2$, $c_{13,5} = 0$, $c_{13,6} = 0$, $c_{13,7} = -1$, $c_{13,8} = 2$, $c_{14,1} = 2$, $c_{14,2} = -2$, $c_{14,3} = -3$, $c_{14,4} = 1$, $c_{14,5} = 0$, $c_{14,6} = -1$, $c_{14,7} = -2$, $c_{15,1} = 1$, $c_{15,2} = -1$, $c_{15,3} = 0$, $c_{15,4} = -1$, $c_{15,5} = -3$, $c_{15,6} = -2$, $c_{15,7} = -2$, $c_{16,1} = -2$, $c_{16,2} = 2$, $c_{16,3} = -1$, $c_{16,4} = -2$, $c_{16,5} = -3$, $c_{16,6} = -3$, $c_{16,7} = -3$, $c_{17,1} = -1$, $c_{17,2} = -2$, $c_{17,3} = -2$, $c_{17,4} = -3$, $c_{17,5} = -3$, $c_{17,6} = -2$, $c_{17,7} = -1$, $c_{18,1} = -3$, $c_{18,2} = -3$, $c_{18,3} = 2$, $c_{18,4} = -3$, $c_{18,5} = -2$, $c_{18,6} = 1$, $c_{18,7} = -2$, $c_{18,8} = 1$, $c_{19,1} = 0$, $c_{19,2} = -1$, $c_{19,3} = -3$, $c_{19,4} = 0$, $c_{19,5} = 0$, $c_{19,6} = -3$, $c_{19,7} = 2$, $c_{20,1} = 1$, $c_{20,2} = 0$, $c_{20,3} = -3$, $c_{20,4} = -2$, $c_{20,5} = 2$, $c_{20,6} = 0$, $c_{20,7} = 2$, $c_{21,1} = 1$, $c_{21,2} = 0$, $c_{21,3} = 2$, $c_{21,4} = 0$, $c_{21,5} = 0$, $c_{21,6} = 0$, $c_{21,7} = -1$, $c_{22,1} = -3$, $c_{28,1} = 1$, $c_{31,1} = -3$, $c_{32,1} = -1$;

values for the 25 eliminable variables
$c_{1,8} = 126$, $c_{2,8} = 270$, $c_{3,8} = -209$, $c_{4,8} = -60$, $c_{5,8} = 28$, $c_{6,8} = -67$, $c_{8,8} = 469$, $c_{9,8} = -412$, $c_{10,8} = -61$, $c_{11,8} = 29$, $c_{12,8} = -61$, $c_{14,8} = 342$, $c_{15,8} = 55$, $c_{16,8} = -23$, $c_{17,8} = 69$, $c_{19,8} = 10$, $c_{20,8} = 19$, $c_{21,8} = 13$, $c_{23,1} = 1$, $c_{24,1} = -3$, $c_{25,1} = 1$, $c_{26,1} = 9$, $c_{27,1} = -3$, $c_{29,1} = -27$, $c_{30,1} = 9$;

generators of $\mathfrak{a}$
$x_7^2 - \left(-3\,x_1^3 + 126\,x_1^2 - 3\,x_1x_2 - 2\,x_1x_3 + x_1x_4 - 3\,x_2x_3 + 2\,x_3^2\right)$,
$x_6x_7 - \left(-x_1^3 + 270\,x_1^2 - 3\,x_1x_2 - 3\,x_1x_3 - x_1x_4 - 2\,x_2^2 - x_2x_3 - x_3^2\right)$,
$x_5x_7 - \left(-209\,x_1^2 + x_1x_2 - 2\,x_1x_3 - x_1x_4 - 2\,x_2^2 - x_2x_3 - 2\,x_3^2\right)$,
$x_4x_7 - \left(x_1^3 - 60\,x_1^2 - 3\,x_1x_2 - x_1x_3 - x_1x_4 + x_2^2 - 2\,x_2x_3 - x_3^2\right)$,
$x_3x_7 - \left(-2\,x_1^3 + 28\,x_1^2 - x_1x_2 + x_1x_3 - 2\,x_2^2 + 2\,x_2x_3 + 2\,x_3^2\right)$,
$x_2x_7 - \left(-x_1^3 - 67\,x_1^2 - 2\,x_1x_2 - 3\,x_1x_3 - 2\,x_1x_4 + 2\,x_2^2 - 2\,x_2x_3 - 3\,x_3^2\right)$,
$x_1x_7 - \left(-2\,x_1^3 + x_1^2 - 2\,x_1x_2 + x_1x_3 - x_1x_4 - x_3^2\right)$,
$x_6^2 - \left(x_1^3 + 469\,x_1^2 - 2\,x_1x_2 - x_1x_3 - 3\,x_1x_4 + 2\,x_3^2\right)$,
$x_5x_6 - \left(x_1^3 - 412\,x_1^2 - 2\,x_1x_3 - 3\,x_3^2\right)$,
$x_4x_6 - \left(-61\,x_1^2 + 2\,x_1x_2 - 3\,x_1x_3 - 3\,x_1x_4 - x_2^2 - 2\,x_2x_3 - 2\,x_3^2\right)$,
$x_3x_6 - \left(2\,x_1^3 + 29\,x_1^2 + x_1x_2 - 2\,x_1x_3 - x_1x_4 - 3\,x_3^2\right)$,

$x_2x_6 - \big(-x_1^3 - 61\,x_1^2 - 2\,x_1x_2 + x_1x_3 + 2\,x_1x_4 - 2\,x_2^2 - 3\,x_2x_3 + 2\,x_3^2\big)$,
$x_1x_6 - \big(x_1^3 + 2\,x_1^2 - x_1x_2 + 2\,x_2^2\big)$,
$x_5^2 - \big(2\,x_1^3 + 342\,x_1^2 - 2\,x_1x_2 - x_1x_3 + x_2^2 - 3\,x_2x_3 - 2\,x_3^2\big)$,
$x_4x_5 - \big(x_1^3 + 55\,x_1^2 - 2\,x_1x_2 - 2\,x_1x_3 - 3\,x_1x_4 - x_2^2 - x_3^2\big)$,
$x_3x_5 - \big(-2\,x_1^3 - 23\,x_1^2 - 3\,x_1x_2 - 3\,x_1x_3 - 3\,x_1x_4 - 2\,x_2^2 - x_2x_3 + 2\,x_3^2\big)$,
$x_2x_5 - \big(-x_1^3 + 69\,x_1^2 - x_1x_2 - 2\,x_1x_3 - 3\,x_1x_4 - 3\,x_2^2 - 2\,x_2x_3 - 2\,x_3^2\big)$,
$x_1x_5 - \big(-3\,x_1^3 + x_1^2 - 2\,x_1x_2 + x_1x_3 - 2\,x_1x_4 - 3\,x_2^2 + 2\,x_2x_3 - 3\,x_3^2\big)$,
$x_4^2 - \big(10\,x_1^2 + 2\,x_1x_2 - 3\,x_1x_3 - 3\,x_2x_3 - x_3^2\big)$,
$x_3x_4 - \big(x_1^3 + 19\,x_1^2 + 2\,x_1x_2 + 2\,x_1x_4 - 2\,x_2^2 - 3\,x_2x_3\big)$,
$x_2x_4 - \big(x_1^3 + 13\,x_1^2 - x_1x_2 + 2\,x_2x_3\big)$,
$x_1^2x_4 - \big(-3\,x_1^3\big),\ x_3^3 - \big(x_1^3\big),\ x_2x_3^2 - \big(-3\,x_1^3\big),\ x_1x_3^2 - \big(x_1^3\big),\ x_2^2x_3 - \big(9\,x_1^3\big),\ x_1x_2x_3 - \big(-3\,x_1^3\big),\ x_1^2x_3 - \big(x_1^3\big)$,
t$x_2^3 - \big(-27\,x_1^3\big),\ x_1x_2^2 - \big(9\,x_1^3\big),\ x_1^2x_2 - \big(-3\,x_1^3\big),\ x_1^4 - \big(-x_1^3\big)$.

Concerning the proof of Theorem 4.5

Here is the list of the non-null parameters forming the set $\widetilde{C}$:
$c_{1,2}, c_{1,3}, c_{1,4}, c_{1,5}, c_{1,6}, c_{2,2}, c_{2,3}, c_{2,4}, c_{2,5}, c_{2,6}, c_{3,2}, c_{3,3}, c_{3,4}, c_{3,5}, c_{3,6}, c_{4,2}, c_{4,3}, c_{4,4}, c_{4,5}, c_{4,6}, c_{5,2},$
$c_{5,3}, c_{5,4}, c_{5,5}, c_{5,6}, c_{6,2}, c_{6,3}, c_{6,4}, c_{6,5}, c_{6,6}, c_{7,2}, c_{7,3}, c_{7,4}, c_{7,5}, c_{7,6}, c_{7,7}, c_{8,2}, c_{8,3}, c_{8,4}, c_{8,5}, c_{8,6}, c_{9,2},$
$c_{9,3}, c_{9,4}, c_{9,5}, c_{9,6}, c_{10,2}, c_{10,3}, c_{10,4}, c_{10,5}, c_{10,6}, c_{11,2}, c_{11,3}, c_{11,4}, c_{11,5}, c_{11,6}, c_{12,2}, c_{12,3}, c_{12,4}, c_{12,5},$
$c_{12,6}, c_{13,2}, c_{13,3}, c_{13,4}, c_{13,5}, c_{13,6}, c_{13,7}, c_{14,2}, c_{14,3}, c_{14,4}, c_{14,5}, c_{14,6}, c_{15,2}, c_{15,3}, c_{15,4}, c_{15,5}, c_{15,6},$
$c_{16,2}, c_{16,3}, c_{16,4}, c_{16,5}, c_{16,6}, c_{17,2}, c_{17,3}, c_{17,4}, c_{17,5}, c_{17,6}, c_{18,2}, c_{18,3}, c_{18,4}, c_{18,5}, c_{18,6}, c_{18,7}, c_{19,2},$
$c_{19,3}, c_{19,4}, c_{19,5}, c_{19,6}, c_{20,2}, c_{20,3}, c_{20,4}, c_{20,5}, c_{20,6}, c_{21,2}, c_{21,3}, c_{21,4}, c_{21,5}, c_{21,6}, c_{31,1}$.

Here are the polynomials forming the marked bases of the ideals in the family $\mathcal{F}_3$:
$x_7^2 - (c_{1,2}x_3^2 + c_{1,3}x_2x_3 + c_{1,4}x_2^2 + c_{1,5}x_1x_4 + c_{1,6}x_1x_3)$,
$x_6x_7 - (c_{2,2}x_3^2 + c_{2,3}x_2x_3 + c_{2,4}x_2^2 + c_{2,5}x_1x_4 + c_{2,6}x_1x_3)$,
$x_5x_7 - (c_{3,2}x_3^2 + c_{3,3}x_2x_3 + c_{3,4}x_2^2 + c_{3,5}x_1x_4 + c_{3,6}x_1x_3)$,
$x_4x_7 - (c_{4,2}x_3^2 + c_{4,3}x_2x_3 + c_{4,4}x_2^2 + c_{4,5}x_1x_4 + c_{4,6}x_1x_3)$,
$x_3x_7 - (c_{5,2}x_3^2 + c_{5,3}x_2x_3 + c_{5,4}x_2^2 + c_{5,5}x_1x_4 + c_{5,6}x_1x_3)$,
$x_2x_7 - (c_{6,2}x_3^2 + c_{6,3}x_2x_3 + c_{6,4}x_2^2 + c_{6,5}x_1x_4 + c_{6,6}x_1x_3)$,
$x_1x_7 - (c_{7,2}x_3^2 + c_{7,3}x_2x_3 + c_{7,4}x_2^2 + c_{7,5}x_1x_4 + c_{7,6}x_1x_3 + c_{7,7}x_1x_2)$,
$x_6^2 - (c_{8,2}x_3^2 + c_{8,3}x_2x_3 + c_{8,4}x_2^2 + c_{8,5}x_1x_4 + c_{8,6}x_1x_3)$,
$x_5x_6 - (c_{9,2}x_3^2 + c_{9,3}x_2x_3 + c_{9,4}x_2^2 + c_{9,5}x_1x_4 + c_{9,6}x_1x_3)$,
$x_4x_6 - (c_{10,2}x_3^2 + c_{10,3}x_2x_3 + c_{10,4}x_2^2 + c_{10,5}x_1x_4 + c_{10,6}x_1x_3)$,
$x_3x_6 - (c_{11,2}x_3^2 + c_{11,3}x_2x_3 + c_{11,4}x_2^2 + c_{11,5}x_1x_4 + c_{11,6}x_1x_3)$,
$x_2x_6 - (c_{12,2}x_3^2 + c_{12,3}x_2x_3 + c_{12,4}x_2^2 + c_{12,5}x_1x_4 + c_{12,6}x_1x_3)$,
$x_1x_6 - (c_{13,2}x_3^2 + c_{13,3}x_2x_3 + c_{13,4}x_2^2 + c_{13,5}x_1x_4 + c_{13,6}x_1x_3 + c_{13,7}x_1x_2)$,
$x_5^2 - (c_{14,2}x_3^2 + c_{14,3}x_2x_3 + c_{14,4}x_2^2 + c_{14,5}x_1x_4 + c_{14,6}x_1x_3)$,
$x_4x_5 - (c_{15,2}x_3^2 + c_{15,3}x_2x_3 + c_{15,4}x_2^2 + c_{15,5}x_1x_4 + c_{15,6}x_1x_3)$,
$x_3x_5 - (c_{16,2}x_3^2 + c_{16,3}x_2x_3 + c_{16,4}x_2^2 + c_{16,5}x_1x_4 + c_{16,6}x_1x_3)$,
$x_2x_5 - (c_{17,2}x_3^2 + c_{17,3}x_2x_3 + c_{17,4}x_2^2 + c_{17,5}x_1x_4 + c_{17,6}x_1x_3)$,
$x_1x_5 - (c_{18,2}x_3^2 + c_{18,3}x_2x_3 + c_{18,4}x_2^2 + c_{18,5}x_1x_4 + c_{18,6}x_1x_3 + c_{18,7}x_1x_2)$,
$x_4^2 - (c_{19,2}x_3^2 + c_{19,3}x_2x_3 + c_{19,4}x_2^2 + c_{19,5}x_1x_4 + c_{19,6}x_1x_3)$,
$x_3x_4 - (c_{20,2}x_3^2 + c_{20,3}x_2x_3 + c_{20,4}x_2^2 + c_{20,5}x_1x_4 + c_{20,6}x_1x_3)$,
$x_2x_4 - (c_{21,2}x_3^2 + c_{21,3}x_2x_3 + c_{21,4}x_2^2 + c_{21,5}x_1x_4 + c_{21,6}x_1x_3)$,
$x_1^2x_4, x_3^3, x_2x_3^2, x_1x_3^2, x_2^2x_3, x_1x_2x_3, x_1^2x_3, x_2^3, x_1x_2^2, x_1^2x_2 - (c_{31,1}x_1^3), x_1^4$.

According to the characteristic of the field K, here are two marked bases generating two ideals corresponding to points of the family $\widetilde{\mathcal{F}_3}$ at which we compute the Zariski tangent space to Hilb^7_{16}: for every characteristic different from 2 and 3, the Zariski tangent space to at least one of these points has dimension 153:

$x_7^2 - \left(-x_4x_1 + 2\,x_3^2 + 4\,x_3x_2 + 2\,x_3x_1 + 4\,x_2^2\right)$,
$x_7x_6 - \left(-x_4x_1 + 3\,x_3^2 - x_3x_2 + x_3x_1\right)$,
$x_7x_5 - \left(-2\,x_4x_1 + 3\,x_3^2 + 2\,x_3x_2 + 2\,x_3x_1\right)$,
$x_7x_4 - \left(-x_4x_1 - x_3^2 + x_3x_1 - x_2^2\right)$,
$x_7x_3 - \left(-x_4x_1 + 2\,x_3x_1 - 2\,x_2^2\right)$,
$x_7x_2 - \left(-x_4x_1 + x_3^2 + 2\,x_3x_2 + 3\,x_2^2\right)$,
$x_7x_1 - \left(x_4x_1 + 4\,x_3^2 - x_3x_2 - 2\,x_3x_1 + 3\,x_2^2 + 2\,x_2x_1\right)$,
$x_6^2 - \left(-2\,x_4x_1 - 2\,x_3x_2 + 3\,x_3x_1 + 3\,x_2^2\right)$,
$x_6x_5 - \left(4\,x_3^2 + x_3x_2 + 3\,x_3x_1 + 4\,x_2^2\right)$,
$x_6x_4 - \left(2\,x_3^2 + 4\,x_3x_2 + 2\,x_3x_1 + 4\,x_2^2\right)$,
$x_6x_3 - \left(-x_4x_1 + x_3^2 + 4\,x_2^2\right)$, $x_6x_2 - \left(-2\,x_4x_1 - 2\,x_3^2 - 2\,x_3x_2 + 2\,x_3x_1\right)$,
$x_6x_1 - \left(-x_3^2 - x_3x_2 - x_3x_1 - x_2^2 + x_2x_1\right)$,
$x_5^2 - \left(2\,x_4x_1 + 4\,x_3^2 - x_3x_2 - 2\,x_3x_1 - 2\,x_2^2\right)$,
$x_5x_4 - \left(2\,x_4x_1 + 4\,x_3^2 + x_3x_2 + x_2^2\right)$,
$x_5x_3 - \left(-x_4x_1 - x_3^2 + 4\,x_3x_2 + x_3x_1 - 2\,x_2^2\right)$,
$x_5x_2 - \left(-2\,x_4x_1 + 2\,x_3^2 + 4\,x_3x_1 + x_2^2\right)$,
$x_5x_1 - \left(-x_4x_1 - 2\,x_3^2 - 2\,x_3x_2 - x_2^2 + x_2x_1\right)$,
$x_4^2 - \left(3\,x_4x_1 + 4\,x_3^2 + 2\,x_3x_2 + x_3x_1 - 2\,x_2^2\right)$,
$x_4x_3 - \left(-2\,x_4x_1 - 2\,x_3^2 + 4\,x_3x_2 + 2\,x_3x_1 - x_2^2\right)$,
$x_4x_2 - \left(-2\,x_3^2 - 2\,x_3x_2 + 2\,x_3x_1 + x_2^2\right)$,
$x_4x_1^2,\quad x_3^3,\quad x_3^2x_2,\quad x_3^2x_1,\quad x_3x_2^2,\quad x_3x_2x_1,\quad x_3x_1^2,\quad x_2^3,\quad x_2^2x_1,\quad x_2x_1^2 - \left(4\,x_1^3\right),\quad x_1^4.$

$x_7^2 - \left(3\,x_4x_1 - x_3^2 + x_3x_2 + x_3x_1 + x_2^2\right)$,
$x_7x_6 - \left(-2\,x_4x_1 - x_3^2 + 2\,x_3x_2 + 3\,x_3x_1\right)$,
$x_7x_5 - \left(x_3^2 - 2\,x_3x_2 + 4\,x_3x_1 + 2\,x_2^2\right)$,
$x_7x_4 - \left(x_4x_1 + 2\,x_3^2 - x_3x_2 + 4\,x_3x_1 + 4\,x_2^2\right)$,
$x_7x_3 - \left(x_4x_1 + 3\,x_3^2\right)$,
$x_7x_2 - \left(3\,x_3^2 + x_3x_2 + x_3x_1 - 2\,x_2^2\right)$,
$x_7x_1 - \left(2\,x_4x_1 + x_3x_2 + 4\,x_3x_1 + 4\,x_2^2 + 3\,x_2x_1\right)$,
$x_6^2 - \left(-2\,x_4x_1 - 2\,x_3^2 + x_3x_2 + 2\,x_3x_1 - 2\,x_2^2\right)$,
$x_6x_5 - \left(-2\,x_3^2 - x_3x_1 + 4\,x_2^2\right)$,
$x_6x_4 - \left(2\,x_3^2 + 4\,x_3x_2 - 2\,x_3x_1 + 2\,x_2^2\right)$,
$x_6x_3 - \left(-2\,x_4x_1 + 4\,x_3^2 + x_3x_2 + 2\,x_3x_1 + 3\,x_2^2\right)$,
$x_6x_2 - \left(x_4x_1 - x_3^2 - 2\,x_3x_2 + x_3x_1 + 4\,x_2^2\right)$,
$x_6x_1 - \left(-2\,x_4x_1 + x_3^2 - 2\,x_3x_2 - 2\,x_3x_1 + x_2^2 + 2\,x_2x_1\right)$,
$x_5^2 - \left(-2\,x_4x_1 + 3\,x_3^2 + x_3x_1 + 2\,x_2^2\right)$,
$x_5x_4 - \left(x_4x_1 - 2\,x_3^2 - 2\,x_3x_2 - x_3x_1 + 3\,x_2^2\right)$,
$x_5x_3 - \left(4\,x_4x_1 + 4\,x_3^2 - x_3x_1 - x_2^2\right)$,
$x_5x_2 - \left(-x_4x_1 + 4\,x_3^2 - x_3x_2 - 2\,x_3x_1\right)$,
$x_5x_1 - \left(x_4x_1 + 4\,x_3^2 - x_3x_2 - 2\,x_3x_1 - 2\,x_2^2\right)$,
$x_4^2 - \left(-x_4x_1 - x_3^2 + 4\,x_3x_2 + x_3x_1 + 3\,x_2^2\right)$,
$x_4x_3 - \left(4\,x_4x_1 - x_3^2 + 4\,x_3x_2 - 2\,x_3x_1 - 2\,x_2^2\right)$,
$x_4x_2 - \left(3\,x_3^2 - 2\,x_3x_2 + x_3x_1 + 3\,x_2^2\right)$,
$x_4x_1^2, x_3^3, x_3^2x_2, x_3^2x_1, x_3x_2^2, x_3x_2x_1, x_3x_1^2, x_2^3, x_2^2x_1, x_2x_1^2 - \left(4\,x_1^3\right), x_1^4$

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