RESTRICTED LIE ALGEBRAS VIA MONADIC DECOMPOSITION

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Abstract. We give a description of the category of restricted Lie algebras over a field $\Bbbk$ of prime characteristic by means of monadic decomposition of the functor that computes the $\Bbbk$-vector space of primitive elements of a $\Bbbk$-bialgebra.

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Introduction

Let $\Bbbk$ be a field and let $\mathfrak{M}$ denote the category of $\Bbbk$-vector spaces. Denoting $\mathsf{Alg}(\mathfrak{M})$ the category of unital, associative $\Bbbk$-algebras, there is the obvious forgetful functor $\Omega : \mathsf{Alg}(\mathfrak{M}) \to \mathfrak{M}$, which has a left adjoint T. The composition ΩT defines a monad on $\mathfrak{M}$ and the comparison functor Ω_1 from $\mathsf{Alg}(\mathfrak{M})$ to ${}_{\Omega T}\mathfrak{M}$ -the Eilenberg-Moore category associated to the monad ΩT- can be shown to have a left adjoint T_1 such that the adjunction (T_1, Ω_1) becomes an equivalence of categories, i.e. Ω is a monadic functor.

It is well-known that for any $V \in \mathfrak{M}$, TV can be given moreover a $\Bbbk$-bialgebra structure, thus inducing a functor $\widetilde{T} : \mathfrak{M} \to \mathsf{Bialg}(\mathfrak{M})$. Now, a right adjoint for $\widetilde{T}$ is provided by the functor P that computes the space of primitive elements of any bialgebra. This adjunction furnishes $\mathfrak{M}$ with a (different) monad $P\widetilde{T}$. This time, P fails to be monadic, alas. Indeed, P_1 -the comparison functor associated to the monad $P\widetilde{T}$- still allows for a left adjoint $\widetilde{T}_1$, but the adjunction $(\widetilde{T}_1, P_1)$ is not an equivalence anymore. Yet, something can be done with it. Using the notation $\mathfrak{M}_2$ for the Eilenberg-Moore category of the monad $P_1\widetilde{T}_1$ on ${}_{P\widetilde{T}}\mathfrak{M}$ (the Eilenberg-Moore category of the monad $P\widetilde{T}$), it was proven in [AGM] that there exists a functor

$$\widetilde{T}_2 : \mathfrak{M}_2 \to \mathsf{Bialg}(\mathfrak{M})$$

that allows a right adjoint P_2 and which is moreover full and faithful. This means that the functor P has so-called "monadic decomposition of length at most 2".

In case the characteristic of the ground field $\Bbbk$ is zero, the above result was further refined in [AM]. Indeed, amongst other things in the cited article, it is proven that the category $\mathfrak{M}_2$ is equivalent with $\mathsf{Lie}(\mathfrak{M})$, the category of $\Bbbk$-Lie algebras. This theorem is actually obtained as a consequence

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of a more general statement ([AM, Theorem 7.2]) that is proven for Lie algebras in abelian symmetric monoidal categories that satisfy the so-called Milnor-Moore condition. It is then verified, using a result of Kharchenko ([Kh, Lemma 6.2]), that the category of vector spaces over a field of characteristic zero satisfies this condition. In concreto, there exists a functor Γ such that $(P_2\widetilde{U},\Gamma)$ gives the afore-mentioned equivalence, $\widetilde{U}$ being the functor that computes the universal enveloping bialgebra of any Lie algebra in characteristic zero.

In case the characteristic of $\Bbbk$ is a prime number p, things appear to be slightly different. Of course, one can still work with the ordinary definition of Lie algebra and consider its universal enveloping algebra (which is still a $\Bbbk$-bialgebra, also in finite characteristic), but, in general, the latter has primitive elements which are not contained in the Lie algebra. Hence this does not seem to be the appropriate notion if we wish to imitate the above-mentioned equivalence between $\mathfrak{M}_2$ and $\mathsf{Lie}(\mathfrak{M})$ we had in case $\mathrm{char}(\Bbbk)=0$.
The aim of this note is to provide an appropriate equivalence in case of prime characteristic. In order to do so, we will use a slightly different approach than the one in [AM] (as Kharchenko's mentioned lemma is not at hand in prime characteristic). Therefore, recall that a *restricted Lie algebra* in characteristic p (which is a notion due to Jacobson, see [Ja1]) is a triple $(L,[-,-],-^{[p]})$ where $(L,[,])$ is an ordinary $\Bbbk$-Lie algebra, endowed with a map $-^{[p]}:L\to L$ satisfying three conditions. These restricted Lie algebras in many respects bear a closer relation to Lie algebras in characteristic 0 than ordinary Lie algebras in characteristic p.
Now, restricted Lie algebras cannot be seen as Lie algebras in some abelian symmetric monoidal category, at least not to the authors' knowledge. However, in this short article we show that restricted Lie algebras allow for an interpretation using monadic decomposition of the functor P. Indeed, Theorem 2.3 states that one can construct a functor $\Lambda:\mathfrak{M}_2\to\mathsf{Lie}_\mathsf{p}$, Lie_p being the category of restricted Lie algebras over $\Bbbk$, such that $(P_2\widetilde{\mathfrak{u}},\Lambda)$ defines an equivalence between Lie_p and $\mathfrak{M}_2$. Here $\widetilde{\mathfrak{u}}$ is the functor computing the restricted universal enveloping algebra of a restricted Lie algebra.
The article is organized as follows. In the preliminary section we recall some notation and results concerning monadic decomposition. Along the way, we address to the interested reader by stating two questions which seem to be of independent interest.
In the second section, we prove Theorem 2.3, using a lemma due to Berger.
In the last section, we provide an alternative way to arrive at the conclusion of Theorem 2.3, by using so-called adjoint squares. These categorical tools actually allows us to refine our main result. Indeed, in Remark 3.8 we obtain that the functor $\mathcal{P}\widetilde{T}$ is left adjoint to the forgetful functor $H_{\mathsf{Lie}_\mathsf{p}}:\mathsf{Lie}_\mathsf{p}\to\mathfrak{M}$. The left adjoint of $H_{\mathsf{Lie}_\mathsf{p}}$ already appeared in literature, for some particular base field $\Bbbk$ (e.g. $\Bbbk=\mathbb{Z}_2$), under the name of "free restricted Lie algebra functor" and several constructions of this functor can be found. We note that here, in our approach, no additional requirement on $\Bbbk$ is needed other than the finite characteristic. Finally, in Remark 3.10, using adjoint squares, it is shown that the adjunction $(\widetilde{T}_2,P_2)$ turns out to identify with $(\widetilde{\mathfrak{u}},\mathcal{P})$ via Λ, $\mathcal{P}$ being the functor that computes the restricted primitive elements of a bialgebra in characteristic p.

1. Preliminary results

In this section, we shall fix some basic notation and terminology.

Notation 1.1. *Throughout this note $\Bbbk$ will denote a field. $\mathfrak{M}$ will denote the category of vector spaces over $\Bbbk$. Unadorned tensor product are to taken over $\Bbbk$ unless stated otherwise.*
When X is an object in a category $\mathcal{C}$, we will denote the identity morphism on X by 1_X or X for short. For categories $\mathcal{C}$ and $\mathcal{D}$, a functor $F:\mathcal{C}\to\mathcal{D}$ will be the name for a covariant functor; it will only be a contravariant one if it is explicitly mentioned. By $\mathsf{id}_\mathcal{C}$ we denote the identity functor on $\mathcal{C}$. For any functor $F:\mathcal{C}\to\mathcal{D}$, we denote Id_F (or sometimes -in order to lighten notation in some computations- just F, if the context doesn't allow for confusion) the natural transformation defined by $\mathsf{Id}_{FX}=1_{FX}$.

1.2. **Monadic decomposition.** Recall that a *monad* on a category $\mathcal{A}$ is a triple $\mathbb{Q} := (Q, m, u)$ consisting of a functor $Q : \mathcal{A} \to \mathcal{A}$ and natural transformations $m : QQ \to Q$ and $u : \mathcal{A} \to Q$ satisfying the associativity and the unitality conditions $m \circ mQ = m \circ Qm$ and $m \circ Qu = \mathsf{Id}_Q = m \circ uQ$. An *algebra* over a monad $\mathbb{Q}$ on $\mathcal{A}$ (or simply a $\mathbb{Q}$-*algebra*) is a pair (X, μ) where $X \in \mathcal{A}$ and $\mu : QX \to X$ is a morphism in $\mathcal{A}$ such that $\mu \circ Q\mu = \mu \circ mX$ and $\mu \circ uX = X$. A *morphism between two* $\mathbb{Q}$-*algebras* (X, μ) and (X', μ') is a morphism $f : X \to X'$ in $\mathcal{A}$ such that $\mu' \circ Qf = f \circ \mu$. For the time being, we will denote by ${}_{\mathbb{Q}}\mathcal{A}$ the category of $\mathbb{Q}$-algebras and their morphisms. This is the so-called *Eilenberg-Moore category* of the monad $\mathbb{Q}$. We denote ${}_{Q}U : {}_{Q}\mathcal{A} \to \mathcal{A}$ the forgetful functor. When the multiplication and unit of the monad are clear from the context, we will just write Q instead of $\mathbb{Q}$.

Let $(L : \mathcal{B} \to \mathcal{A}, R : \mathcal{A} \to \mathcal{B})$ be an adjunction with unit η and counit ϵ. Then $(RL, R\epsilon L, \eta)$ is a monad on $\mathcal{B}$ and we can consider the so-called *comparison functor* $K : \mathcal{A} \to {}_{RL}\mathcal{B}$ of the adjunction (L, R) which is defined by $KX := (RX, R\epsilon X)$ and $Kf := Rf$. Note that ${}_{RL}U \circ K = R$.

Definition 1.3. An adjunction $(L : \mathcal{B} \to \mathcal{A}, R : \mathcal{A} \to \mathcal{B})$ is called *monadic* (tripleable in Beck's terminology [Be, Definition 3]) whenever the comparison functor $K : \mathcal{A} \to {}_{RL}\mathcal{B}$ is an equivalence of categories. A functor R is called *monadic* if it has a left adjoint L such that the adjunction (L, R) is monadic, see [Be, Definition 3'].

Definition 1.4. ([AT, page 231]) A monad (Q, m, u) is called *idempotent* if m is an isomorphism. An adjunction (L, R) is called *idempotent* whenever the associated monad is idempotent.

The interested reader can find results on idempotent monads in [AT, MS]. Here we just note that the fact that (L, R) being idempotent is equivalent to requiring that ηR is a natural isomorphism. The notion of idempotent monad is tightly connected with the following.

Definition 1.5. (See [AGM, Definition 2.7], [AHT, Definition 2.1] and [MS, Definitions 2.10 and 2.14]) Fix a $N \in \mathbb{N}$. A functor R is said to have a *monadic decomposition of monadic length* N when there exists a sequence $(R_n)_{n \leq N}$ of functors R_n such that

1) $R_0 = R$;
2) for $0 \leq n \leq N$, the functor R_n has a left adjoint functor L_n;
3) for $0 \leq n \leq N-1$, the functor R_{n+1} is the comparison functor induced by the adjunction (L_n, R_n) with respect to its associated monad;
4) L_N is full and faithful while L_n is not full and faithful for $0 \leq n \leq N-1$.

A functor R having monadic length N is equivalent to requiring that the forgetful functor $U_{N,N+1}$ defines an isomorphism of categories and that no $U_{n,n+1}$ has this property for $\leq n \leq N-1$ (see [AGM, Remark 2.4]).
Note that for a functor $R : \mathcal{A} \to \mathcal{B}$ having a monadic decomposition of monadic length N, we thus have a diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[d, "R_0"] & \mathcal{A} \arrow[l, "\mathrm{Id}_{\mathcal{A}}"'] \arrow[d, "R_1"] & \mathcal{A} \arrow[l, "\mathrm{Id}_{\mathcal{A}}"'] \arrow[d, "R_2"] & \cdots \arrow[l, "\mathrm{Id}_{\mathcal{A}}"'] & \cdots & \mathcal{A} \arrow[l, "\mathrm{Id}_{\mathcal{A}}"'] \arrow[d, "R_N"] \\
\mathcal{B}_0 \arrow[u, dotted, "L_0"] & \mathcal{B}_1 \arrow[l, "U_{0,1}"'] \arrow[u, dotted, "L_1"] & \mathcal{B}_2 \arrow[l, "U_{1,2}"'] \arrow[u, dotted, "L_2"] & \cdots \arrow[l, "U_{2,3}"'] & \cdots & \mathcal{B}_N \arrow[l, "U_{N-1,N}"'] \arrow[u, dotted, "L_N"]
\end{tikzcd} \tag{1}$$

where $\mathcal{B}_0 = \mathcal{B}$ and, for $1 \leq n \leq N$,

- $\mathcal{B}_n$ is the category of $(R_{n-1}L_{n-1})$-algebras ${}_{R_{n-1}L_{n-1}}\mathcal{B}_{n-1}$;
- $U_{n-1,n} : \mathcal{B}_n \to \mathcal{B}_{n-1}$ is the forgetful functor ${}_{R_{n-1}L_{n-1}}U$.

We will denote by $\eta_n : \mathsf{id}_{\mathcal{B}_n} \to R_nL_n$ and $\epsilon_n : L_nR_n \to \mathsf{id}_{\mathcal{A}}$ the unit, resp. counit of the adjunction (L_n, R_n) for $0 \leq n \leq N$. Note that one can introduce the forgetful functor $U_{m,n} : \mathcal{B}_n \to \mathcal{B}_m$ for all $m \leq n$ with $0 \leq m, n \leq N$.

We recall the following from [AGM]:

Proposition 1.6. [AGM, Proposition 2.9] *Let $(L : \mathcal{B} \to \mathcal{A}, R : \mathcal{A} \to \mathcal{B})$ be an idempotent adjunction. Then $R : \mathcal{A} \to \mathcal{B}$ has a monadic decomposition of monadic length at most* 1.

Letting $\Bbbk$ be a field, and B a $\Bbbk$-bialgebra, the set $P(B)$ of primitive elements of B is defined as

$$P(B) = \{x \in B | \Delta(x) = 1 \otimes x + x \otimes 1\},$$

where Δ is the comultiplication of B. $P(B)$ forms a $\Bbbk$-vector space, yielding a functor

$$P : \mathsf{Bialg}(\mathfrak{M}) \to \mathfrak{M} \tag{2}$$

Theorem 3.4 from loc. cit. asserts that the functor P has monadic decomposition at most 2, by showing that the comparison functor P_1 of the adjunction $(\widetilde{T}, P)$ admits a left adjoint $\widetilde{T}_1$ such that the adjunction $(\widetilde{T}_1, P_1)$ is idempotent. For the sake of completeness, we recall here that $\widetilde{T}$ is the functor from $\mathfrak{M}$ to $\mathsf{Bialg}(\mathfrak{M})$, assigning to any vector space V the tensor algebra $T(V)$ (which can be endowed with a bialgebra structure $\widetilde{T}(V)$, as is known).
Intriguingly, it is not known to the authors whether the bound provided by this above-mentioned Theorem 3.4 is sharp. It would thus be satisfying to have an answer to the following question -of independent interest- the interested reader is evidently invited to think about.

Question 1.7. Is the functor $\widetilde{T}_1$ fully faithful?

As mentioned in the Introduction, it is known -by combining Theorems 7.2 and 8.1 from [AM]- that in case char$(\Bbbk) = 0$, the category $\mathfrak{M}_2$ is equivalent to the category of $\Bbbk$-Lie algebras. It is the aim of this note to handle the case of finite characteristic. Before doing so, we would like to round off this preliminary section by the following.

Definition 1.8. We say that a functor R is *comparable* whenever there exists a sequence $(R_n)_{n\in\mathbb{N}}$ of functors R_n such that $R_0 = R$ and, for $n \in \mathbb{N}$,

1) the functor R_n has a left adjoint functor L_n;

2) the functor R_{n+1} is the comparison functor induced by the adjunction (L_n, R_n) with respect to its associated monad.

In this case we have a diagram as (1) but not necessarily stationary. Hence we can consider the forgetful functors $U_{m,n} : \mathcal{B}_n \to \mathcal{B}_m$ for all $m \leq n$ with $m, n \in \mathbb{N}$.

Remark 1.9. Fix a $N \in \mathbb{N}$. A functor R having a monadic decomposition of monadic length N is comparable, see [AGM, Remark 2.10].

By the proof of Beck's Theorem [Be, Proof of Theorem 1], one gets the following result.

Lemma 1.10 ([AM]). *Let $\mathcal{A}$ be a category such that, for any (reflexive) pair (f, g) (*[BW, 3.6, page 98]*) where $f, g : X \to Y$ are morphisms in $\mathcal{A}$, one can choose a specific coequalizer. Then the comparison functor $K : \mathcal{A} \to {}_{RL}\mathcal{B}$ of an adjunction (L, R) is a right adjoint. Thus any right adjoint $R : \mathcal{A} \to \mathcal{B}$ is comparable.*

Let $\Bbbk$ again be a field of characteristic zero. Dually to $\Bbbk$-Lie algebras, recall that $\Bbbk$-Lie coalgebras, as introduced by Michaelis in [Mi1], are precisely Lie algebras in the abelian symmetric monoidal category $(\mathfrak{M}^{op}, \otimes^{op}, \Bbbk)$, where associativity and unit constraints are taken to be trivial.
$Q(B)$ denotes the $\Bbbk$-Lie coalgebra of indecomposables of a $\Bbbk$-bialgebra, more precisely, $Q(B) = I/I^2$, where $I = \ker \varepsilon$, ε being the counit of B. This construction is functorial and, composed with the forgetful functor F from the category of $\Bbbk$-Lie coalgebras to $\mathfrak{M}$, yields the following functor

$$Q : \mathsf{Bialg}(\mathfrak{M}) \to \mathfrak{M}.$$

In [Mi1, page 18], it is asserted that a right adjoint for the functor F is provided by the functor L^c that computes the so-called "cofree Lie coalgebra" on a vector space. Finally, Q has a right adjoint given by the cofree coalgebra functor $\widetilde{T}^c$. In fact $Q = F \circ \widetilde{Q}$, where $\widetilde{Q}$ is the functor sending a bialgebra B to the $\Bbbk$-Lie coalgebra $Q(B)$, while, by [Mi1, page 24], we have $\widetilde{T}^c = \widetilde{U}^c_H \circ L^c$, where $\widetilde{U}^c_H(C)$ is the universal coenveloping bialgebra of a Lie coalgebra C (by the bialgebra version of [Mi1, Theorem, page 37], the functor $\widetilde{U}^c_H$ is right adjoint to $\widetilde{Q}$). Lemma 1.10 now guarantees that the functor

$$Q^{op} : \mathsf{Bialg}(\mathfrak{M}^{op}) \to \mathfrak{M}^{op}$$

is comparable. Strangely enough, at the moment, we don't have an answer to the following question, which is -again- of independent interest in the authors' opinion.

QUESTION 1.11. Does the functor Q^{op} also have monadic decomposition length at most 2? If so, is the resulting category $\mathfrak{M}_2$ equivalent to the category of $\Bbbk$-Lie coalgebras?

2. The category of restricted Lie algebras

For the sake of the reader's comfort, we include a result due to Berger, here presented in a slightly different form.

LEMMA 2.1 (cf. [Ber, Lemma 1.2]). *Consider the following diagram*

$$\begin{tikzcd} & \mathcal{E}' \arrow[dl, shift left, "\Psi"] \arrow[dr, shift left, "\Psi'"] & \\ \mathcal{M} \arrow[ur, dotted, shift left, "\Phi"] \arrow[dr, "U"'] & & \mathcal{M}' \arrow[ul, dotted, shift left, "\Phi'"] \arrow[dl, "U'"] \\ & \mathcal{E} & \end{tikzcd}$$

where

- $U' \circ \Psi' = U \circ \Psi$,
- U *and* U' *are conservative,*
- Ψ *and* Ψ' *are coreflections i.e. functors having fully faithful left adjoints* Φ *and* Φ' *respectively (i.e. the units* $\eta : \mathrm{id}_{\mathcal{M}} \to \Psi\Phi$ *and* $\eta' : \mathrm{id}_{\mathcal{M}'} \to \Psi'\Phi'$ *are invertible).*

Then $(\Psi'\Phi, \Psi\Phi')$ *is an adjoint equivalence of categories with unit* $\widehat{\eta}$ *and counit* $\widehat{\epsilon}$ *defined by*

$$(\widehat{\eta})^{-1} := \Psi\Phi'\Psi'\Phi \xrightarrow{\Psi\epsilon'\Phi} \Psi\Phi \xrightarrow{\eta^{-1}} \mathrm{id}_{\mathcal{M}},$$

$$\widehat{\epsilon} := \Psi'\Phi\Psi\Phi' \xrightarrow{\Psi'\epsilon\Phi'} \Psi'\Phi' \xrightarrow{(\eta')^{-1}} \mathrm{id}_{\mathcal{M}'}.$$

Fix an arbitrary field $\Bbbk$ such that $\mathrm{char}\,(\Bbbk)$ is a prime p and recall that $\mathfrak{M}$ denotes the category of vector spaces over $\Bbbk$.

DEFINITION 2.2. (due to Jacobson, see [Ja1, page 210]) A *restricted Lie algebra over* $\Bbbk$ (also called *p-Lie algebra* by some authors) is a triple $(L, [-,-], -^{[p]})$ consisting of a (ordinary) Lie algebra $(L, [-,-])$ (i.e. a $\Bbbk$-vector space L endowed with a $\Bbbk$-bilinear map $[-,-]$ satisfying the antisymmetry and Jacobi condition) and a (set-theoretical) map $-^{[p]} : L \to L$ satisfying

$$\begin{aligned} (\alpha x)^{[p]} &= \alpha^p x^{[p]} \text{ for all } x \in L, \alpha \in \Bbbk; \\ \mathrm{ad}(x^{[p]}) &= (\mathrm{ad}(x))^p \text{ for all } x \in L; \\ (x+y)^{[p]} &= x^{[p]} + y^{[p]} + s(x,y) \text{ for all } x, y \in L, \end{aligned}$$

where ad is the adjoint representation of L;

$$\mathrm{ad} : L \to \mathrm{End}(L), x \mapsto \mathrm{ad}_x \text{ where } \mathrm{ad}_x(y) = [x,y],$$

and $s(x,y) = \sum_{i=1}^{p-1} \frac{s_i(x,y)}{i}$, where $s_i(x,y)$ is the coefficient of β^{i-1} in the expansion of $(\mathrm{ad}(\beta x + y))^{p-1}(x)$.
A map $f : (L, [-,-], -^{[p]}) \to (L', [-,-]', -^{[p']})$ is a morphism of restricted Lie algebras if f is a morphism of (ordinary) Lie algebras $f : (L, [-,-]) \to (L', [-,-]')$ such that $f(x^{[p]}) = (f(x))^{[p']}$, for all $x \in L$.
The category of restricted Lie algebras with their morphisms will be denoted by $\mathsf{Lie_p}$.

There is an adjunction $(\widetilde{\mathfrak{u}} : \mathsf{Lie_p} \to \mathsf{Bialg}(\mathfrak{M}), \mathcal{P} : \mathsf{Bialg}(\mathfrak{M}) \to \mathsf{Lie_p})$, given by the following functors (see [Gru] or [Mi2, Appendix], e.g.):

- $\widetilde{\mathfrak{u}} : \mathsf{Lie_p} \to \mathsf{Bialg}(\mathfrak{M})$; the restricted universal enveloping algebra functor.
  Explicitly, $\widetilde{\mathfrak{u}}(L, [-,-], -^{[p]}) = \frac{\widetilde{U}(L,[-,-])}{I}$, where I is the ideal in $\widetilde{U}(L, [-,-])$ generated by elements of the form $x^p - x^{[p]}$.

- $\mathcal{P} : \mathsf{Bialg}(\mathfrak{M}) \to \mathsf{Lie_p}$; the restricted primitive functor.
  Explicitly, for $B \in \mathsf{Bialg}(\mathfrak{M})$, the space $P(B)$ becomes a Lie algebra for the commutator bracket $[-,-]$ and can moreover be endowed with the map $-^{[p]}$ sending an element $x \in L$ to x^p such that $\mathcal{P}(B) := (P(B), [-,-], -^{[p]})$ becomes a restricted Lie algebra.

We denote by $\widetilde{\eta}_{\mathrm{L}}$ the unit and by $\widetilde{\epsilon}_{\mathrm{L}}$ the counit of the adjunction $(\widetilde{\mathfrak{u}}, \mathcal{P})$. By [MM, Theorem 6.11(1)], we know that $\widetilde{\eta}_{\mathrm{L}} : \mathsf{id}_{\mathsf{Lie_p}} \to \mathcal{P}\widetilde{\mathfrak{u}}$ is an isomorphism, see also [Ja2, Theorem 1]. We also use the notation $H_{\mathsf{Lie_p}} : \mathsf{Lie_p} \to \mathfrak{M}$ for the forgetful functor. We obviously have that $H_{\mathsf{Lie_p}}\mathcal{P} = P : \mathsf{Bialg}(\mathfrak{M}) \to \mathfrak{M}$ is the usual primitive functor (cf. (2)).
Before stating the main result, we notice that in case we wish to stress the algebra nature of objects and morphisms in $\mathsf{Alg}(\mathfrak{M})$, resp. the bialgebra nature of objects and morphisms in $\mathsf{Bialg}(\mathfrak{M})$, we will do so by simply overlining, resp. over and underlining things. Please mind as well that we denote η (resp. $\widetilde{\eta}$) and ϵ (resp. $\widetilde{\epsilon}$) the unit and counit of the adjunction (T, Ω) (resp. $(\widetilde{T}, P)$).

Theorem 2.3. *We have the following diagram.*

(3)
$$\begin{tikzcd}
\mathsf{Bialg}(\mathfrak{M}) \arrow[dd, shift left, "P"] & \mathsf{Bialg}(\mathfrak{M}) \arrow[l, "\mathsf{id}_{\mathsf{Bialg}(\mathfrak{M})}"'] \arrow[dd, shift left, "P_1"] & & \mathsf{Bialg}(\mathfrak{M}) \arrow[ll, "\mathsf{id}_{\mathsf{Bialg}(\mathfrak{M})}"'] \arrow[dl, "\mathsf{id}_{\mathsf{Bialg}(\mathfrak{M})}"] \arrow[dd, shift left, "P_2"] \\
& & \mathsf{Bialg}(\mathfrak{M}) \arrow[ull, "\mathsf{id}_{\mathsf{Bialg}(\mathfrak{M})}"] \arrow[dd, shift left, "\mathcal{P}"] & \\
\mathfrak{M} \arrow[uu, dotted, shift left, "\widetilde{T}"] & \mathfrak{M}_1 \arrow[l, "U_{0,1}"'] \arrow[uu, dotted, shift left, "\widetilde{T}_1"] & & \mathfrak{M}_2 \arrow[ll, "U_{1,2}"'] \arrow[uu, dotted, shift left, "\widetilde{T}_2"] \arrow[dl, "\Lambda"] \\
& & \mathsf{Lie_p} \arrow[ull, "H_{\mathsf{Lie_p}}"] \arrow[uu, dotted, shift left, "\widetilde{\mathfrak{u}}"] &
\end{tikzcd}$$

The functor P is comparable so that we can use the notation of Definition 1.8. There is a functor $\Lambda : \mathfrak{M}_2 \to \mathsf{Lie_p}$ such that $\Lambda \circ P_2 = \mathcal{P}$ and $H_{\mathsf{Lie_p}} \circ \Lambda = U_{0,2}$.

- *The adjunction $(\widetilde{T}_1, P_1)$ is idempotent, we can choose $\widetilde{T}_2 := \widetilde{T}_1 U_{1,2}$, $\pi_2 = \mathsf{Id}_{\widetilde{T}_2}$ and $\widetilde{T}_2$ is full and faithful i.e. $\widetilde{\eta}_2$ is an isomorphism. The functor P has a monadic decomposition of monadic length at most two.*
- *The pair $(P_2\widetilde{\mathfrak{u}}, \Lambda)$ is an adjoint equivalence of categories with unit $\widetilde{\eta}_{\mathrm{L}}$ and counit $(\widetilde{\eta}_2)^{-1} \circ P_2\left(\widetilde{\epsilon}_{\mathrm{L}}\widetilde{T}_2 \circ \widetilde{\mathfrak{u}}\Lambda\widetilde{\eta}_2\right)$.*

Proof. By [AGM, Theorem 3.4], the functor P has monadic decomposition of monadic length at most 2. Moreover, the adjunction $(\widetilde{T}_1, P_1)$ is idempotent and we can define a functor $\Lambda : \mathfrak{M}_2 \to \mathsf{Lie_p}$. Indeed, letting $V_2 = ((V_0, \mu_0), \mu_1)$ be an object in $\mathfrak{M}_2$, we can define an object $\Lambda V_2 \in \mathsf{Lie_p}$ as follows:

$$\Lambda V_2 = \left(V_0, [-,-], -^{[p]}\right),$$

where $[-,-] : V_0 \otimes V_0 \to V_0$ is defined by setting $[x,y] := \mu_0 (x \otimes y - y \otimes x)$, for every $x, y \in V_0$, while $-^{[p]} : V_0 \to V_0$ is defined by setting $x^{[p]} := \mu_0 (x^{\otimes p})$, for every $x \in V_0$.

Let $f_2 : V_2 \to V_2'$ be a morphism in $\mathfrak{M}_2$ and set $f_1 := U_{1,2}f_2$ and $f_0 := U_{0,1}f_1$. Then, for every $x, y \in V_0$

$$\begin{aligned}
f_0([x,y]) &= f_0\mu_0 (x \otimes y - y \otimes x) = \mu_0' \left(P\widetilde{T}f_0\right)(x \otimes y - y \otimes x) \\
&= \mu_0' \left(\left(\widetilde{T}f_0\right)(x \otimes y - y \otimes x)\right) = \mu_0' \left[(f_0(x) \otimes f_0(y) - f_0(y) \otimes f_0(x))\right] \\
&= [f_0(x), f_0(y)]'
\end{aligned}$$

and

$$\begin{aligned}
f_0\left(x^{[p]}\right) &= f_0\mu_0 (x^{\otimes p}) = \mu_0' \left(P\widetilde{T}f_0\right)(x^{\otimes p}) = \mu_0' \left(\left(\widetilde{T}f_0\right)(x^{\otimes p})\right) \\
&= \mu_0' \left(f_0(x)^{\otimes p}\right) = f_0(x)^{[p]'}.
\end{aligned}$$

Thus f_2 induces a unique morphism Λf_2 such that $H_{\mathsf{Lie_p}}(\Lambda f_2) = f_0$. This defines a functor $\Lambda : \mathfrak{M}_2 \to \mathsf{Lie_p}$, as claimed above. By construction $H_{\mathsf{Lie_p}} \circ \Lambda = U_{0,2}$. Moreover,

$$(\Lambda \circ P_2)\left(\underline{\overline{B}}\right) = \Lambda\left(P_2\left(\underline{\overline{B}}\right)\right).$$

In order to proceed, we have to compute $\Lambda\left(P_2\left(\underline{\overline{B}}\right)\right)$. We have that

$$P_1\left(\underline{\overline{B}}\right) = \left(P\left(\underline{\overline{B}}\right), P\widetilde{\epsilon}_{\underline{\overline{B}}} : P\widetilde{T}P\left(\underline{\overline{B}}\right) \to P\left(\underline{\overline{B}}\right)\right)$$

so the brackets $[-,-]$ and $-^{[p]}$ are, for every $x, y \in P\left(\underline{\overline{B}}\right)$, given by the following:

$$\begin{aligned}
[x,y] &= \left(P\widetilde{\epsilon}_{\underline{\overline{B}}}\right)(x\otimes y - y\otimes x) = \left(\widetilde{\epsilon}_{\underline{\overline{B}}}\right)(x\otimes y - y\otimes x) = \left(\epsilon_{\overline{B}}\right)(x\otimes y - y\otimes x) = xy - yx,\\
x^{[p]} &= \left(P\widetilde{\epsilon}_{\underline{\overline{B}}}\right)\left(x^{\otimes p}\right) = \left(\widetilde{\epsilon}_{\underline{\overline{B}}}\right)\left(x^{\otimes p}\right) = \left(\epsilon_{\overline{B}}\right)\left(x^{\otimes p}\right) = x^p
\end{aligned}$$

so that $(\Lambda \circ P_2)\left(\underline{\overline{B}}\right) = \mathcal{P}\left(\underline{\overline{B}}\right)$. For the morphisms, we have

$$\left(H_{\mathsf{Lie_p}}\right)\Lambda P_2\left(\underline{\overline{f}}\right) = U_{0,2}P_2\left(\underline{\overline{f}}\right) = P\left(\underline{\overline{f}}\right) = \left(H_{\mathsf{Lie_p}}\right)\mathcal{P}\left(\underline{\overline{f}}\right).$$

Since $H_{\mathsf{Lie_p}}$ is faithful, we conclude that $\Lambda P_2\left(\underline{\overline{f}}\right) = \mathcal{P}\left(\underline{\overline{f}}\right)$ and hence $\Lambda \circ P_2 = \mathcal{P}$.
Now, since the adjunction $(\widetilde{T}_1, P_1)$ is idempotent, by [AGM, Proposition 2.3], we can choose $\widetilde{T}_2 := \widetilde{T}_1 U_{1,2}$ with $\widetilde{\eta}_1 U_{1,2} = U_{1,2}\widetilde{\eta}_2$ and $\widetilde{\epsilon}_1 = \widetilde{\epsilon}_2$. Since $\widetilde{T}_2$ is full and faithful, we have that $\widetilde{\eta}_2$ is an isomorphism. We already observed that $\widetilde{\eta}_{\mathrm{L}} : \mathsf{id}_{\mathsf{Lie_p}} \to \mathcal{P}\widetilde{\mathfrak{u}}$ is an isomorphism.

Since $U_{0,2} = H_{\mathsf{Lie_p}} \circ \Lambda$ and $U_{0,2}$ is conservative, so is Λ. We can apply Lemma 2.1 to the following diagram

$$\begin{tikzcd}
 & \mathsf{Bialg}(\mathfrak{M}) \arrow[dl, "\mathcal{P}"'] \arrow[dr, "P_2"] & \\
\mathsf{Lie_p} \arrow[ur, dotted, "\widetilde{\mathfrak{u}}"] \arrow[dr, "H_{\mathsf{Lie_p}}"'] & & \mathfrak{M}_2 \arrow[ul, dotted, "\widetilde{T}_2"] \arrow[dl, "U_{0,2}"] \\
 & \mathfrak{M} &
\end{tikzcd}$$

to deduce that $\left(P_2\widetilde{\mathfrak{u}}, \mathcal{P}\widetilde{T}_2\right)$ is an adjoint equivalence with unit $\widehat{\eta}$ and counit $\widehat{\epsilon}$ defined by $\left(\widehat{\eta}\right)^{-1} := \widetilde{\eta}_{\mathrm{L}}^{-1} \circ \mathcal{P}\widetilde{\epsilon}_2\widetilde{\mathfrak{u}}$ and $\widehat{\epsilon} := \widetilde{\eta}_2^{-1} \circ P_2\widetilde{\epsilon}_{\mathrm{L}}\widetilde{T}_2$. By the first part of the proof, we have $\mathcal{P} = \Lambda P_2$. Thus we can use the isomorphism $\Lambda\widetilde{\eta}_2 : \Lambda \to \Lambda P_2\widetilde{T}_2 = \mathcal{P}\widetilde{T}_2$ to replace $\mathcal{P}\widetilde{T}_2$ by Λ in the adjunction. Thus we obtain that the pair $(P_2\widetilde{\mathfrak{u}}, \Lambda)$ is an equivalence of categories with unit $\widetilde{\eta}_{\mathrm{L}}$ and counit $\widetilde{\eta}_2^{-1} \circ P_2\left(\widetilde{\epsilon}_{\mathrm{L}}\widetilde{T}_2 \circ \widetilde{\mathfrak{u}}\Lambda\widetilde{\eta}_2\right)$ by the following computations

$$\begin{gathered}
\left(\widehat{\eta}\right)^{-1} \circ \Lambda\widetilde{\eta}_2 P_2\widetilde{\mathfrak{u}} = \widetilde{\eta}_{\mathrm{L}}^{-1} \circ \mathcal{P}\widetilde{\epsilon}_2\widetilde{\mathfrak{u}} \circ \Lambda\widetilde{\eta}_2 P_2\widetilde{\mathfrak{u}} = \widetilde{\eta}_{\mathrm{L}}^{-1} \circ \Lambda P_2\widetilde{\epsilon}_2\widetilde{\mathfrak{u}} \circ \Lambda\widetilde{\eta}_2 P_2\widetilde{\mathfrak{u}} = \widetilde{\eta}_{\mathrm{L}}^{-1},\\
\widehat{\epsilon} \circ P_2\widetilde{\mathfrak{u}}\Lambda\widetilde{\eta}_2 = \widetilde{\eta}_2^{-1} \circ P_2\widetilde{\epsilon}_{\mathrm{L}}\widetilde{T}_2 \circ P_2\widetilde{\mathfrak{u}}\Lambda\widetilde{\eta}_2 = \widetilde{\eta}_2^{-1} \circ P_2\left(\widetilde{\epsilon}_{\mathrm{L}}\widetilde{T}_2 \circ \widetilde{\mathfrak{u}}\Lambda\widetilde{\eta}_2\right).
\end{gathered}$$

□

3. An alternative approach via adjoint squares

The main aim of this section is to give an alternative approach to Theorem 2.3 by means of some results that -in our opinion- could have an interest in their own right.

Definition 3.1. Recall from [Gra, Definition I,6.7, page 144] that an *adjoint square* consists of a (not necessarily commutative) diagram of functors as depicted below ((L,R) and (L',R') being adjunctions with units η resp. η' and counits ϵ resp. ϵ') together with a matrix of natural transformations "inside":

(4)
$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "F"] \arrow[d, shift left, "R"] \arrow[drr, phantom, "\begin{pmatrix}\zeta_{11} & \zeta_{12}\\ \zeta_{21} & \zeta_{22}\end{pmatrix}"] & & \mathcal{A}' \arrow[d, shift left, "R'"] \\
\mathcal{B} \arrow[rr, "G"'] \arrow[u, shift left, "L"] & & \mathcal{B}' \arrow[u, shift left, "L'"]
\end{tikzcd}
\qquad
\begin{array}{ll}
\zeta_{11} : L'G \to FL, & \zeta_{12} : L'GR \to F,\\
\zeta_{21} : G \to R'FL, & \zeta_{22} : GR \to R'F,
\end{array}$$

These ingredients are required to be subject to the following equalities:

$$\begin{aligned}
&(5) && \zeta_{11} &=& \zeta_{12}L \circ L'G\eta = \epsilon' FL \circ L'\zeta_{21} = \epsilon' FL \circ L'\zeta_{22}L \circ L'G\eta,\\
&(6) && \zeta_{12} &=& F\epsilon \circ \zeta_{11}R = \epsilon' F\epsilon \circ L'\zeta_{21}R = \epsilon' F \circ L'\zeta_{22},\\
&(7) && \zeta_{21} &=& R'\zeta_{11} \circ \eta' G = R'\zeta_{12}L \circ \eta' G\eta = \zeta_{22}L \circ G\eta,\\
&(8) && \zeta_{22} &=& R'F\epsilon \circ R'\zeta_{11}R \circ \eta' GR = R'\zeta_{12} \circ \eta' GR = R'F\epsilon \circ \zeta_{21}R.
\end{aligned}$$

We call such natural transformations *transposes* of each other. If only one of the entries of the matrix is given, its transposes can be defined by means of the equalities above.

Example 3.2. Let $(L : \mathcal{B} \to \mathcal{A}, R : \mathcal{A} \to \mathcal{B})$ be an adjunction with unit η and counit ϵ. Assume that the comparison functor $R_1 : \mathcal{A} \to \mathcal{B}_1$ has a left adjoint L_1 with unit η_1 and counit ϵ_1. We then have an adjoint square

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\mathsf{id}_{\mathcal{A}}"] \arrow[d, shift left, "R_1"] \arrow[dr, phantom, "\begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix}"] & \mathcal{A} \arrow[d, shift left, "R"] \\
\mathcal{B}_1 \arrow[u, shift left, "L_1"] \arrow[r, "G"'] & \mathcal{B} \arrow[u, shift left, "L"]
\end{tikzcd}$$

where

$$\pi_{11} \overset{(5)}{=} \epsilon L_1 \circ LU_{01}\eta_1 : LU_{01} \to L_1 \quad \text{and} \quad \pi_{22} = \mathsf{Id}_{U_{01}R_1}$$

so that, by the proof of [Be, Theorem 1], π_{11} is the canonical projection defining L_1. More explicitly, for every $(B, \mu) \in \mathcal{B}_1$ we have the following coequalizer of a reflexive pair in $\mathcal{A}$

$$\begin{tikzcd}
LRLB \arrow[r, shift left, "L\mu"] \arrow[r, shift right, "\epsilon_{LB}"'] & LB = LU_{01}(B, \mu) \arrow[r, "\pi_{11}(B,\mu)"] & L_1(B, \mu)\ .
\end{tikzcd}$$

As a consequence we get

$$\begin{aligned}
&(9) && \epsilon_1 \circ \pi_{11}R_1 \overset{(6)}{=} \epsilon,\\
& && R\pi_{11} \circ \eta U_{01} \overset{(7)}{=} U_{01}\eta_1.
\end{aligned}$$

Remark 3.3. Given two adjoint squares

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[d, shift left, "R"] \arrow[dr, phantom, "\begin{pmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{21} & \zeta_{22} \end{pmatrix}"] & \mathcal{A}' \arrow[d, shift left, "R'"] \\
\mathcal{B} \arrow[u, shift left, "L"] \arrow[r, "G"'] & \mathcal{B}' \arrow[u, shift left, "L'"]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathcal{A}' \arrow[r, "F'"] \arrow[d, shift left, "R'"] \arrow[dr, phantom, "\begin{pmatrix} \zeta'_{11} & \zeta'_{12} \\ \zeta'_{21} & \zeta'_{22} \end{pmatrix}"] & \mathcal{A}'' \arrow[d, shift left, "R''"] \\
\mathcal{B}' \arrow[u, shift left, "L'"] \arrow[r, "G'"'] & \mathcal{B}'' \arrow[u, shift left, "L''"]
\end{tikzcd}$$

their horizontal composition is given by

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F'F"] \arrow[d, shift left, "R"] \arrow[dr, phantom, "\begin{pmatrix} \zeta'_{11} * \zeta_{11} & \zeta'_{12} * \zeta_{12} \\ \zeta'_{21} * \zeta_{21} & \zeta'_{22} * \zeta_{22} \end{pmatrix}"] & \mathcal{A}'' \arrow[d, shift left, "R''"] \\
\mathcal{B} \arrow[u, shift left, "L"] \arrow[r, "G'G"'] & \mathcal{B}'' \arrow[u, shift left, "L''"]
\end{tikzcd}$$

where

$$\begin{aligned}
\zeta'_{11} * \zeta_{11} &= F'\zeta_{11} \circ \zeta'_{11}G,\\
\zeta'_{12} * \zeta_{12} &= \zeta'_{12}\zeta_{12} \circ L''G'\eta' GR,\\
\zeta'_{21} * \zeta_{21} &= R''F'\epsilon' FL \circ \zeta'_{21}\zeta_{21},\\
\zeta'_{22} * \zeta_{22} &= \zeta'_{22}F \circ G'\zeta_{22}.
\end{aligned}$$

The following result should be compared with [Gra, Proposition I,6.9].

Lemma 3.4. *Consider an adjoint square as in the following diagram.*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[d, shift left, "R"] \arrow[r, phantom, shift right=6ex, "\begin{pmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{21} & \zeta_{22} \end{pmatrix}"] & \mathcal{A}' \arrow[d, shift left, "R'"] \\
\mathcal{B} \arrow[r, "G"'] \arrow[u, shift left, "L"] & \mathcal{B}' \arrow[u, shift left, "L'"]
\end{tikzcd}$$

Assume that F and G are equivalences of categories. Then the following assertions are equivalent.

(1) ζ_{11} *is an isomorphism.*
(2) ζ_{22} *is an isomorphism.*

Proof. Let $F' : \mathcal{A}' \to \mathcal{A}$ be a functor such that (F', F) is a category equivalence with (invertible) unit $\eta^{(F',F)}$ and counit $\epsilon^{(F',F)}$. Similarly we use the notation $\eta^{(G',G)}$ and $\epsilon^{(G',G)}$.
$(2) \Rightarrow (1)$. Consider the following adjoint squares, where the diagrams on the right-hand side are obtained by rotating clockwise by 90 degrees the ones on the left-hand side (the upper index c stands for "clockwise").

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[d, shift left, "R"] \arrow[r, phantom, shift right=6ex, "\begin{pmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{21} & \zeta_{22} \end{pmatrix}"] & \mathcal{A}' \arrow[d, shift left, "R'"] \\
\mathcal{B} \arrow[r, "G"'] \arrow[u, shift left, "L"] & \mathcal{B}' \arrow[u, shift left, "L'"]
\end{tikzcd}
\quad \overset{\circlearrowright}{\mapsto} \quad
\begin{tikzcd}
\mathcal{B} \arrow[r, "L"] \arrow[d, shift left, "G"] \arrow[r, phantom, shift right=6ex, "\begin{pmatrix} \zeta^c_{11} & \zeta^c_{12} \\ \zeta^c_{21} & \zeta^c_{22} = \zeta_{11} \end{pmatrix}"] & \mathcal{A} \arrow[d, shift left, "F"] \\
\mathcal{B}' \arrow[r, "L'"'] \arrow[u, shift left, "G'"] & \mathcal{A}' \arrow[u, shift left, "F'"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "R"] \arrow[d, shift left, "F"] \arrow[r, phantom, shift right=6ex, "\begin{pmatrix} \zeta^r_{11} & \zeta^r_{12} \\ \zeta^r_{21} & \zeta^r_{22} = (\zeta_{11})^{-1} \end{pmatrix}"] & \mathcal{B} \arrow[d, shift left, "G"] \\
\mathcal{A}' \arrow[r, "R'"'] \arrow[u, shift left, "F'"] & \mathcal{B}' \arrow[u, shift left, "G'"]
\end{tikzcd}
\quad \overset{\circlearrowright}{\mapsto} \quad
\begin{tikzcd}
\mathcal{A}' \arrow[r, "F'"] \arrow[d, shift left, "R'"] \arrow[r, phantom, shift right=6ex, "\begin{pmatrix} \zeta^{rc}_{11} & \zeta^{rc}_{12} \\ \zeta^{rc}_{21} & \zeta^{rc}_{22} = \zeta^r_{11} \end{pmatrix}"] & \mathcal{A} \arrow[d, shift left, "R"] \\
\mathcal{B}' \arrow[r, "G'"'] \arrow[u, shift left, "L'"] & \mathcal{B} \arrow[u, shift left, "L"]
\end{tikzcd}$$

Now apply [Gra, page 153], putting $(f, u) = (G', G)$, $(f', u') = (F', F)$, $(g, v) = (L, R)$, $(g', v') = (L', R')$, $\psi = \zeta_{22}$, $\widetilde{\psi} = \zeta_{11}$, $\widetilde{\widetilde{\psi}} = \zeta^c_{11}, \theta = \zeta^r_{22}$, $\widetilde{\theta} = \zeta^r_{11}$, $\widetilde{\widetilde{\theta}} = \zeta^{rc}_{11}$. Then we obtain that ζ^{rc}_{11} and ζ^c_{11} are mutual inverses. Note that

$$\zeta_{11} = \zeta^c_{22} \overset{(8)}{=} FL\epsilon^{(G',G)} \circ F\zeta^c_{11}G \circ \eta^{(F',F)}L'G$$

so that, as a composition of isomorphisms, ζ_{11} is an isomorphism.
$(1) \Rightarrow (2)$. This implication is shown in a very similar fashion, by applying the dual result of [Gra, page 153]. $\square$

Definition 3.5. Following [BMW, 1.4], an adjoint square as in (4) is called *exact* whenever both $\zeta_{11} : L'G \to FL$ and $\zeta_{22} : GR \to R'F$ are isomorphisms. Note that this implies that the given diagram commutes -up to isomorphism- when either the left adjoint functors or the right adjoint functors are omitted.

Remark 3.6. Consider a square of functors like in (4) and assume that $GR = R'F$. Then we can set $\zeta_{22} := \mathsf{Id}_{GR}$ and we get an adjoint square. This square is exact if and only if $(F, G) : (L, R) \to (L', R')$ is a *commutation datum* in the sense of [AM, Definition 2.3].

Proposition 3.7. *Consider two adjunctions (L, R, η, ϵ) and $(L', R', \eta', \epsilon')$ as in the diagram*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\mathsf{id}"] \arrow[d, shift left, "R"] & \mathcal{A} \arrow[d, shift left, "R'"] \\
\mathcal{B} \arrow[r, "G"'] \arrow[u, shift left, "L"] & \mathcal{B}' \arrow[u, shift left, "L'"]
\end{tikzcd}$$

where G is a functor such that $GR = R'$. Then the diagram above is an adjoint square with matrix $\left(\zeta_{ij}\right)$, where $\zeta_{22} : GR \to R'$ is the identity. If η is invertible, then $\left(RL', G, \eta', \eta^{-1} \circ R\zeta_{11}\right)$ is an adjunction too.

Proof. Assume η is invertible, set $F := RL'$ and

$$\begin{aligned}\alpha &:= \left[FG = RL'G \xrightarrow{R\zeta_{11}} RL \xrightarrow{\eta^{-1}} \mathrm{Id}_{\mathcal{B}}\right],\\ \beta &:= \left[\mathrm{Id}_{\mathcal{B}'} \xrightarrow{\eta'} R'L' = GRL' = GF\right].\end{aligned}$$

We compute that

$$\begin{aligned}\alpha F \circ F\beta &= \eta^{-1}F \circ R\zeta_{11}F \circ F\eta' = \eta^{-1}RL' \circ R\zeta_{11}RL' \circ RL'\eta'\\ &= R\epsilon L' \circ R\zeta_{11}RL' \circ RL'\eta' = R\left[(\epsilon \circ \zeta_{11}R)\, L' \circ L'\eta'\right]\\ &= R\left(\zeta_{12}L' \circ L'\eta'\right) = R\left((\epsilon' \circ L'\zeta_{22})\, L' \circ L'\eta'\right) = RL' = F\end{aligned}$$

and

$$G\alpha \circ \beta G = G\eta^{-1} \circ GR\zeta_{11} \circ \eta' G = G\eta^{-1} \circ R'\zeta_{11} \circ \eta' G = G\eta^{-1} \circ \zeta_{21} = G\eta^{-1} \circ \zeta_{22}L \circ G\eta = G,$$

so $\left(RL', G, \eta', \eta^{-1} \circ R\zeta_{11}\right)$ is an adjunction. □

Remark 3.8. Consider the following diagram

$$\begin{tikzcd} \mathsf{Bialg}(\mathfrak{M}) \arrow[r, "\mathsf{id}"] \arrow[d, shift left, "\mathcal{P}"] & \mathsf{Bialg}(\mathfrak{M}) \arrow[d, shift left, "P"] \\ \mathsf{Lie_p} \arrow[r, "H_{\mathsf{Lie_p}}"'] \arrow[u, shift left, "\widetilde{\mathfrak{u}}"] & \mathfrak{M} \arrow[u, shift left, "\widetilde{T}"] \end{tikzcd}$$

In the previous section we observed that $\widetilde{\eta}_{\mathrm{L}} : \mathsf{id}_{\mathsf{Lie_p}} \to \mathcal{P}\widetilde{\mathfrak{u}}$ is an isomorphism. By Proposition 3.7, the above is an adjoint square with matrix $\left(\zeta_{ij}\right)$, where $\zeta_{22} : H_{\mathsf{Lie_p}}\mathcal{P} \to P$ is the identity. Moreover, we also have that $\left(\mathcal{P}\widetilde{T}, H_{\mathsf{Lie_p}}, \widetilde{\eta}, \widetilde{\eta}_{\mathrm{L}}^{-1} \circ \mathcal{P}\zeta_{11}\right)$ is an adjunction. Thus $\mathcal{P}\widetilde{T}$ is a left adjoint of the functor $H_{\mathsf{Lie_p}}$. Let as write it explicitly on objects. For V in $\mathfrak{M}$, we have

$$\mathcal{P}\widetilde{T}V = \left(P\widetilde{T}V, [-,-], -^{[p]}\right)$$

where $[-,-]$ and $-^{[p]}$ are defined for every $x, y \in P\widetilde{T}V$ by $[x,y] = x \otimes y - y \otimes x$ and $x^{[p]} = x^{\otimes p}$ respectively.
The left adjoint of $H_{\mathsf{Lie_p}}$ appeared in the literature, for some particular base field $\Bbbk$ (e.g. $\Bbbk = \mathbb{Z}_2$), under the name of "free restricted Lie algebra functor" and several constructions of this functor can be found. We note that here, in our approach, no additional requirement on $\Bbbk$ is needed other than the finite characteristic. A similar description can be obtained even in characteristic zero.

Corollary 3.9. *In the setting of Proposition 3.7, assume that both η and η' are invertible and G is conservative. Then $\left(RL', G, \eta', \eta^{-1} \circ R\zeta_{11}\right)$ is an adjoint equivalence. Moreover, $\zeta_{11} : L'G \to L$ is invertible (so that the adjoint square considered in Proposition 3.7 is a commutation datum).*

Proof. We give two alternative proofs of the first part of the statement. Then the last part follows by Lemma 3.4 as ζ_{22} is the identity.

Proof I). By Proposition 3.7, $\left(RL', G, \eta', \eta^{-1} \circ R\zeta_{11}\right)$ is an adjunction too. Since η' is invertible, it remains to prove that $\eta^{-1} \circ R\zeta_{11}$ is invertible i.e. that $R\zeta_{11}$ is. Since $R'\zeta_{11} \circ \eta' G = \zeta_{21} = \zeta_{22}L \circ G\eta$ is invertible, so is $R'\zeta_{11}$. Thus $GR\zeta_{11} = R'\zeta_{11}$ is invertible. Since G is conservative, we conclude.

Proof II). Since η and η' are isomorphisms, we can apply Lemma 2.1 to the following diagram

$$\begin{tikzcd} & \mathcal{A} \arrow[dl, shift left, "R'"] \arrow[dr, shift left, "R"] & \\ \mathcal{B}' \arrow[ur, dotted, shift left, "L'"] \arrow[dr, "\mathrm{id}"'] & & \mathcal{B} \arrow[ul, dotted, shift left, "L"] \arrow[dl, "G"] \\ & \mathcal{B}' & \end{tikzcd}$$

to deduce that $(RL', R'L)$ is an adjoint equivalence. By hypothesis, we have $R' = GR$. Thus we can use the isomorphism $G\eta : G \to GRL = R'L$ to replace $R'L$ by G in the adjunction. $\square$

Remark 3.10. We are now able to provide a different closing for the proof of Theorem 2.3. Once proved that $\widetilde{\eta}_2$ and $\widetilde{\eta}_{\mathrm{L}}$ are isomorphisms, we can apply Corollary 3.9 to the following diagram

$$\begin{tikzcd} \mathsf{Bialg}(\mathfrak{M}) \arrow[rr, "\mathsf{id}"] \arrow[d, shift left, "P_2"] & & \mathsf{Bialg}(\mathfrak{M}) \arrow[d, shift left, "\mathcal{P}"] \\ \mathfrak{M}_2 \arrow[u, shift left, "\widetilde{T}_2"] \arrow[rr, "\Lambda"'] \arrow[urr, phantom, "\begin{pmatrix} \chi_{11} & \chi_{12} \\ \chi_{21} & \chi_{22} \end{pmatrix}"] & & \mathsf{Lie}_{\mathsf{p}} \arrow[u, shift left, "\widetilde{\mathfrak{u}}"] \end{tikzcd}$$

which is an adjoint square with matrix $\left(\chi_{ij}\right)$ where $\chi_{22} : \Lambda P_2 \to \mathcal{P}$ is the identity and $\chi_{11} := \widetilde{\epsilon}_{\mathrm{L}}\widetilde{T}_2 \circ \widetilde{\mathfrak{u}}\Lambda\widetilde{\eta}_2$. As a consequence $\left(P_2\widetilde{\mathfrak{u}}, \Lambda, \widetilde{\eta}_{\mathrm{L}}, \widetilde{\eta}_2^{-1} \circ P_2\chi_{11}\right)$ is an adjoint equivalence. Moreover $\chi_{11} : \widetilde{\mathfrak{u}}\Lambda \to \widetilde{T}_2$ is invertible so that $\widetilde{\mathfrak{u}}\Lambda \cong \widetilde{T}_2$. Since we already know that $\Lambda P_2 = \mathcal{P}$, we can identify $(\widetilde{T}_2, P_2)$ with $(\widetilde{\mathfrak{u}}, \mathcal{P})$ via Λ.

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