

COHOMOLOGY AND COQUASI-BIALGEBRAS IN THE CATEGORY OF YETTER-DRINFELD MODULES

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ABSTRACT. We prove that a finite-dimensional Hopf algebra with the dual Chevalley Property over a field of characteristic zero is quasi-isomorphic to a Radford-Majid bosonization whenever the third Hochschild cohomology group in the category of Yetter-Drinfeld modules of its diagram with coefficients in the base field vanishes. Moreover we show that this vanishing occurs in meaningful examples where the diagram is a Nichols algebra.

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INTRODUCTION

Let A be a finite-dimensional Hopf algebra over a field \mathbb{k} of characteristic zero such that the coradical H of A is a sub-Hopf algebra (i.e. A has the dual Chevalley Property). Denote by $\mathcal{D}(A)$ the diagram of A . The main aim of this paper (see Theorem 5.6) is to prove that, if the third Hochschild cohomology group in ${}^H_H\mathcal{YD}$ of the algebra $\mathcal{D}(A)$ with coefficients in \mathbb{k} vanishes, in symbols $H^3_{\mathcal{YD}}(\mathcal{D}(A), \mathbb{k}) = 0$, then A is quasi-isomorphic to the Radford-Majid bosonization $E\#H$ of some connected bialgebra E in ${}^H_H\mathcal{YD}$ with $\text{gr}(E) \cong \mathcal{D}(A)$ as bialgebras in ${}^H_H\mathcal{YD}$.

The paper is organized as follows. Let H be a Hopf algebra over a field \mathbb{k} . In Section 1 we investigate the properties of coalgebras with multiplication and unit in the category ${}^H_H\mathcal{YD}$ (in particular of coquasi-bialgebras) and their associated graded coalgebra. The main result of this section, Theorem 1.5, establishes that the associated graded coalgebra $\text{gr}Q$ of a connected coquasi-bialgebra in ${}^H_H\mathcal{YD}$ is a connected bialgebra in ${}^H_H\mathcal{YD}$.

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In Section 2 we study the deformation of coquasi-bialgebras in ${}^H_H\mathcal{YD}$ by means of gauge transformations. In Proposition 2.5 we investigate its behaviour with respect to bosonization while in Proposition 2.6 with respect to the associated graded coalgebra.

In Section 3 we consider the associated graded coalgebra in case the Hopf algebra H is semisimple and cosemisimple (e.g. H is finite-dimensional cosemisimple over a field of characteristic zero). In particular, in Theorem 3.2, we prove that a f.d. connected coquasi-bialgebra Q in ${}^H_H\mathcal{YD}$ is gauge equivalent to a connected bialgebra in ${}^H_H\mathcal{YD}$ whenever $H^3_{\mathcal{YD}}(\text{gr}Q, \mathbb{k}) = 0$. This result is inspired by [EG, Proposition 2.3].

In Section 4, we focus on the link between $H^3_{\mathcal{YD}}(B, \mathbb{k})$ and the invariants of $H^n(B, \mathbb{k})$, where B is a bialgebra in $H^3_{\mathcal{YD}}(B, \mathbb{k})$. In particular, in Proposition 4.7 we show that $H^3_{\mathcal{YD}}(B, \mathbb{k})$ is isomorphic to $H^n(B, \mathbb{k})^{D(H)}$, which is a subspace of $H^n(B, \mathbb{k})^H \cong H^n(B \# H, \mathbb{k})$, see Corollary 4.3.

Section 5 is devoted to the proof of the main result of the paper, the aforementioned Theorem 5.6.

In Section 6 we provide examples where $H^3_{\mathcal{YD}}(B, \mathbb{k}) = 0$ in case B is the Nichols algebra $\mathcal{B}(V)$ of a Yetter-Drinfeld module V . In particular we show that that $H^3_{\mathcal{YD}}(\mathcal{B}(V), \mathbb{k})$ can be zero although $H^3(\mathcal{B}(V) \# H, \mathbb{k})$ is non-trivial.

PRELIMINARIES

Given a category \mathcal{C} and objects $M, N \in \mathcal{C}$, the notation $\mathcal{C}(M, N)$ stands for the set of morphisms in \mathcal{C} . This notation will be mainly applied to the case \mathcal{C} is the category of vector space $\mathbf{Vect}_{\mathbb{k}}$ over a field \mathbb{k} or \mathcal{C} is the category of Yetter-Drinfeld modules ${}^H_H\mathcal{YD}$ over a Hopf algebra H . The set of natural numbers including 0 is denoted by \mathbb{N}_0 while \mathbb{N} denotes the same set without 0.

1. YETTER-DRINFELD

DEFINITION 1.1. Let C be a coalgebra. Denote by C_n the n -th term of the coradical filtration of C and set $C_{-1} := 0$. For every $x \in C$, we set

$$|x| := \min\{i \in \mathbb{N}_0 : x \in C_i\} \quad \text{and} \quad \bar{x} := x + C_{|x|-1}.$$

Note that, for $x = 0$, we have $|x| = 0$. One can define the associated graded coalgebra

$$\text{gr}C := \bigoplus_{i \in \mathbb{N}_0} \frac{C_i}{C_{i-1}}$$

with structure given, for every $x \in C$, by

$$(1) \quad \Delta_{\text{gr}C}(\bar{x}) = \sum_{0 \leq i \leq |x|} (x_1 + C_{i-1}) \otimes (x_2 + C_{|x|-i-1}),$$

$$(2) \quad \varepsilon_{\text{gr}C}(\bar{x}) = \delta_{|x|, 0} \varepsilon_C(x).$$

1.2. For every $i \in \mathbb{N}_0$, take a basis $\{\overline{x^{i,j}} \mid j \in B_i\}$ of the \mathbb{k} -module C_i/C_{i-1} with $\overline{x^{i,j}} \neq \overline{x^{i,l}}$ for $j \neq l$ and

$$|x^{i,j}| = i.$$

Then $\{x^{i,j} \mid 0 \leq i \leq n, j \in B_i\}$ is a basis of C_n and $\{x^{i,j} \mid i \in \mathbb{N}_0, j \in B_i\}$ is a basis of C . Assume that C has a distinguished grouplike element $1 = 1_C \neq 0$ and take $i > 0$. If $\varepsilon(x^{i,j}) \neq 0$ then we have that

$$\overline{x^{i,j} - \varepsilon(x^{i,j})1} = \overline{x^{i,j}}$$

so that we can take $x^{i,j} - \varepsilon(x^{i,j})1$ in place of $x^{i,j}$. In other words we can assume

$$(3) \quad \varepsilon(x^{i,j}) = 0, \text{ for every } i > 0, j \in B_i.$$

It is well-known there is a \mathbb{k} -linear isomorphism $\varphi : C \rightarrow \text{gr}C$ defined on the basis by $\varphi(x^{i,j}) := \overline{x^{i,j}}$.

We compute

$$\varepsilon_{\text{gr}C} \varphi(x^{i,j}) = \varepsilon_{\text{gr}C}(\overline{x^{i,j}}) \stackrel{(2)}{=} \delta_{i,0} \varepsilon(x^{0,j}) \stackrel{(3)}{=} \varepsilon(x^{i,j}).$$

Hence we obtain

$$(4) \quad \varepsilon_{\text{gr}C} \circ \varphi = \varepsilon.$$

Let H be a Hopf algebra. A **coalgebra with multiplication and unit** in ${}^H_H\mathcal{YD}$ is a datum $(Q, m, u, \Delta, \varepsilon)$ where (Q, Δ, ε) is a coalgebra in ${}^H_H\mathcal{YD}$, $m : Q \otimes Q \rightarrow Q$ is a coalgebra morphism in ${}^H_H\mathcal{YD}$ called multiplication (which may fail to be associative) and $u : \mathbb{k} \rightarrow Q$ is a coalgebra morphism in ${}^H_H\mathcal{YD}$ called unit. In this case we set $1_Q := u(1_{\mathbb{k}})$.

Note that, for every $h \in H, k \in \mathbb{k}$, we have

$$(5) \quad h1_Q = hu(1_{\mathbb{k}}) = u(h1_{\mathbb{k}}) = u(\varepsilon_H(h)1_{\mathbb{k}}) = \varepsilon_H(h)u(1_{\mathbb{k}}) = \varepsilon_H(h)1_Q,$$

$$(6) \quad (1_Q)_{-1} \otimes (1_Q)_0 = (u(1_{\mathbb{k}}))_{-1} \otimes (u(1_{\mathbb{k}}))_0 = (1_{\mathbb{k}})_{-1} \otimes u((1_{\mathbb{k}})_0) = 1_H \otimes u(1_{\mathbb{k}}) = 1_H \otimes 1_Q.$$

PROPOSITION 1.3. *Let H be a Hopf algebra and let $(Q, m, u, \Delta, \varepsilon)$ be a coalgebra with multiplication and unit in ${}^H_H\mathcal{YD}$. If Q_0 is a subcoalgebra of Q in ${}^H_H\mathcal{YD}$ such that $Q_0 \cdot Q_0 \subseteq Q_0$, then Q_n is a subcoalgebra of Q in ${}^H_H\mathcal{YD}$ for every $n \in \mathbb{N}_0$. Moreover $Q_a \cdot Q_b \subseteq Q_{a+b}$ for every $a, b \in \mathbb{N}_0$ and the graded coalgebra $\text{gr}Q$, associated with the coradical filtration of Q , is a coalgebra with multiplication and unit in ${}^H_H\mathcal{YD}$ with respect to the usual coalgebra structure and with multiplication and unit defined by*

$$(7) \quad \begin{aligned} m_{\text{gr}Q}((x + Q_{a-1}) \otimes (y + Q_{b-1})) &: = xy + Q_{a+b-1}, \\ u_{\text{gr}Q}(k) &: = k1_Q + Q_{-1} \end{aligned}$$

Proof. The coalgebra structure of Q induces a coalgebra structure on $\text{gr}Q$. Since Q_0 is a subcoalgebra of Q in ${}^H_H\mathcal{YD}$ and, for $n \geq 1$, one has $Q_n = Q_{n-1} \wedge_Q Q_0$, then inductively one proves that Q_n is a subcoalgebra of Q in ${}^H_H\mathcal{YD}$. As a consequence one gets that $\text{gr}Q$ is a coalgebra in ${}^H_H\mathcal{YD}$ (this construction can be performed in the setting of monoidal categories under suitable assumptions, see e.g. [AM, Theorem 2.10]). Let us prove that $\text{gr}Q$ inherits also a multiplication and unit. Let us check that $Q_a \cdot Q_b \subseteq Q_{a+b}$ for every $a, b \in \mathbb{N}_0$. We proceed by induction on $n = a + b$. If $n = 0$ there is nothing to prove. Let $n \geq 1$ and assume that $Q_i \cdot Q_j \subseteq Q_{i+j}$ for every $i, j \in \mathbb{N}_0$ such that $0 \leq i + j \leq n - 1$. Let $a, b \in \mathbb{N}_0$ be such that $n = a + b$. Since $\Delta(Q_a) \subseteq \sum_{i=0}^a Q_i \otimes Q_{a-i}$ and $c_{Q,Q}(Q_u \otimes Q_v) \subseteq Q_v \otimes Q_u$, where $c_{Q,Q}$ denotes the braiding in ${}^H_H\mathcal{YD}$, using the compatibility condition between Δ and m , one easily gets that $\Delta(Q_a \cdot Q_b) \subseteq Q_{a+b-1} \otimes Q + Q \otimes Q_0$.

Therefore $Q_a \cdot Q_b \subseteq Q_{a+b}$. This property implies we have a well-defined map in ${}^H_H\mathcal{YD}$

$$m_{\text{gr}Q}^{a,b} : \frac{Q_a}{Q_{a-1}} \otimes \frac{Q_b}{Q_{b-1}} \rightarrow \frac{Q_{a+b}}{Q_{a+b-1}}$$

defined, for $x \in Q_a$ and $y \in Q_b$, by (7). This can be seen as the graded component of a morphism in ${}^H_H\mathcal{YD}$ that we denote by $m_{\text{gr}Q} : \text{gr}Q \otimes \text{gr}Q \rightarrow \text{gr}Q$. Let us check that $m_{\text{gr}Q}$ is a coalgebra morphism in ${}^H_H\mathcal{YD}$. Consider a basis of Q with terms of the form $x^{i,j}$ as in 1.2. Hence we can write the comultiplication in the form

$$\Delta(x^{a,u}) = \sum_{s+t \leq a} \sum_{l,m} \eta_{s,t,l,m}^{a,u} x^{s,l} \otimes x^{t,m}.$$

Now, using (1), one gets that

$$(8) \quad \Delta_{\text{gr}Q}(\overline{x^{a,u}}) = \sum_{0 \leq i \leq a} \sum_{l,m} \eta_{i,a-i,l,m}^{a,u} \overline{x^{i,l}} \otimes \overline{x^{a-i,m}}.$$

Using that $\Delta_{\text{gr}Q \otimes \text{gr}Q} = (\text{gr}Q \otimes c_{\text{gr}Q, \text{gr}Q} \otimes \text{gr}Q)(\Delta_{\text{gr}Q} \otimes \Delta_{\text{gr}Q})$ and (8), it is straightforward to check that $(m_{\text{gr}Q} \otimes m_{\text{gr}Q}) \Delta_{\text{gr}Q \otimes \text{gr}Q}(\overline{x^{a,u}} \otimes \overline{x^{b,v}}) = \Delta_{\text{gr}Q} m_{\text{gr}Q}(\overline{x^{a,u}} \otimes \overline{x^{b,v}})$.

Moreover, since $\varepsilon_{\text{gr}Q \otimes \text{gr}Q} = \varepsilon_{\text{gr}Q} \otimes \varepsilon_{\text{gr}Q}$, we get that $\varepsilon_{\text{gr}Q} m_{\text{gr}Q}(\overline{x^{a,u}} \otimes \overline{x^{b,v}}) = \varepsilon_{\text{gr}Q \otimes \text{gr}Q}(\overline{x^{a,u}} \otimes \overline{x^{b,v}})$.

This proves that $m_{\text{gr}Q}$ is a coalgebra morphism in ${}^H_H\mathcal{YD}$.

The fact that $u_{\text{gr}Q} : \mathbb{k} \rightarrow \text{gr}Q$, defined by $u_{\text{gr}Q}(k) := k1_Q + Q_{-1}$ is a coalgebra morphism in ${}^H_H\mathcal{YD}$ easily follows by means of (5) and (6). \square

DEFINITION 1.4 ([ABM, Definition 5.2]). Let H be a Hopf algebra. Recall that a *coquasi-bialgebra* $(Q, m, u, \Delta, \varepsilon, \alpha)$ in the pre-braided monoidal category ${}^H_H\mathcal{YD}$ is a coalgebra (Q, Δ, ε) in ${}^H_H\mathcal{YD}$ together with coalgebra homomorphisms $m : Q \otimes Q \rightarrow Q$ and $u : \mathbb{k} \rightarrow Q$ in ${}^H_H\mathcal{YD}$ and a convolution invertible element $\alpha \in {}^H_H\mathcal{YD}(Q^{\otimes 3}, \mathbb{k})$ (*braided reassociator*) such that

$$(9) \quad \alpha(Q \otimes Q \otimes m) * \alpha(m \otimes Q \otimes Q) = (\varepsilon \otimes \alpha) * \alpha(Q \otimes m \otimes Q) * (\alpha \otimes \varepsilon),$$

$$(10) \quad \alpha(Q \otimes u \otimes Q) = \alpha(u \otimes Q \otimes Q) = \alpha(Q \otimes Q \otimes u) = \varepsilon_{Q \otimes Q},$$

$$(11) \quad m(Q \otimes m) * \alpha = \alpha * m(m \otimes Q),$$

$$(12) \quad m(u \otimes Q) = \text{Id}_Q = m(Q \otimes u).$$

Here $*$ denotes the convolution product, where $Q^{\otimes 3}$ is the tensor product of coalgebras in ${}^H_H\mathcal{YD}$ whence it depends on the braiding of this category. Note that in (10) any of the three equalities such as $\alpha(u \otimes Q \otimes Q) = \varepsilon_{Q \otimes Q}$ implies that α is unital.

THEOREM 1.5. *Let H be a Hopf algebra and let $(Q, m, u, \Delta, \varepsilon, \omega)$ be a connected coquasi-bialgebra in ${}^H_H\mathcal{YD}$. Then $\text{gr}Q$ is a connected bialgebra in ${}^H_H\mathcal{YD}$.*

Proof. By Proposition 1.3, we know that $\text{gr}Q$ is a coalgebra with multiplication and unit in ${}^H_H\mathcal{YD}$. We have to check that the multiplication is associative and unitary.

Given two coalgebras D, E in ${}^H_H\mathcal{YD}$ endowed with coalgebras filtration $(D_{(n)})_{n \in \mathbb{N}_0}$ and $(E_{(n)})_{n \in \mathbb{N}_0}$ in ${}^H_H\mathcal{YD}$ such that $D_{(0)}$ and $E_{(0)}$ are one-dimensional, let us check that $C_{(n)} := \sum_{0 \leq i \leq n} D_{(i)} \otimes E_{(n-i)}$ gives a coalgebra filtration on $C := D \otimes E$ in ${}^H_H\mathcal{YD}$. First note that the coalgebra structure of C depends on the braiding. Thus, we have

$$\begin{aligned} \Delta_C(C_{(n)}) &= (D \otimes c_{D,E} \otimes E) (\Delta_D \otimes \Delta_E) \left(\sum_{i=0}^n D_{(i)} \otimes E_{(n-i)} \right) \\ &\subseteq (D \otimes c_{D,E} \otimes E) \left(\sum_{i=0}^n \sum_{a=0}^i \sum_{b=0}^{n-i} D_{(a)} \otimes D_{(i-a)} \otimes E_{(b)} \otimes E_{(n-i-b)} \right) \\ &\subseteq \sum_{i=0}^n \sum_{a=0}^i \sum_{b=0}^{n-i} D_{(a)} \otimes c_{D,E} (D_{(i-a)} \otimes E_{(b)}) \otimes E_{(n-i-b)} \\ &\subseteq \sum_{i=0}^n \sum_{a=0}^i \sum_{b=0}^{n-i} D_{(a)} \otimes c_{D_{(i-a)}, E_{(b)}} (D_{(i-a)} \otimes E_{(b)}) \otimes E_{(n-i-b)} \\ &\subseteq \sum_{i=0}^n \sum_{a=0}^i \sum_{b=0}^{n-i} D_{(a)} \otimes E_{(b)} \otimes D_{(i-a)} \otimes E_{(n-i-b)} \\ &\subseteq \sum_{i=0}^n \sum_{w=0}^n \sum_{\substack{0 \leq a \leq i, \\ 0 \leq b \leq n-i, \\ a+b=w}} D_{(a)} \otimes E_{(b)} \otimes D_{(i-a)} \otimes E_{(n-i-b)} \\ &\subseteq \sum_{w=0}^n C_{(w)} \otimes C_{(n-w)}. \end{aligned}$$

Moreover, by [Sw, Proposition 11.1.1], we have that the coradical of C is contained in $D_{(0)} \otimes E_{(0)}$ and hence it is one-dimensional.

This argument can be used to produce a coalgebra filtration on $C := Q \otimes Q \otimes Q$ using as a filtration on Q the coradical filtration. Let $n > 0$ and let $w \in C_{(n)} = \sum_{i+j+k \leq n} Q_i \otimes Q_j \otimes Q_k$. By [AMS1, Lemma 3.69], we have that

$$\Delta_C(w) - w \otimes (1_Q)^{\otimes 3} - (1_Q)^{\otimes 3} \otimes w \in C_{(n-1)} \otimes C_{(n-1)}.$$

Thus we get

$$w_1 \otimes w_2 \otimes w_3 - \Delta_C(w) \otimes (1_Q)^{\otimes 3} - \Delta_C\left((1_Q)^{\otimes 3}\right) \otimes w \in \Delta_C(C_{(n-1)}) \otimes C_{(n-1)}$$

and hence, tensoring the first relation by $(1_Q)^{\otimes 3}$ on the right and adding it to the second one, we get

$$w_1 \otimes w_2 \otimes w_3 - w \otimes (1_Q)^{\otimes 3} \otimes (1_Q)^{\otimes 3} - (1_Q)^{\otimes 3} \otimes w \otimes (1_Q)^{\otimes 3} - (1_Q)^{\otimes 6} \otimes w \in C_{(n-1)} \otimes C_{(n-1)} \otimes C_{(n-1)}.$$

For shortness, we set $\nu_n(z) := m(Q \otimes m)(z) + Q_{n-1}$ for every $z \in C$. Thus, by applying to the last displayed relation $C_{(n-1)} \otimes m(Q \otimes m) \otimes C_{(n-1)}$ and factoring out the middle term by Q_{n-1} ,

we get

$$\begin{aligned} & \left[\begin{array}{l} w_1 \otimes \nu_n(w_2) \otimes w_3 - w \otimes \nu_n((1_Q)^{\otimes 3}) \otimes (1_Q)^{\otimes 3} + \\ - (1_Q)^{\otimes 3} \otimes \nu_n(w) \otimes (1_Q)^{\otimes 3} - (1_Q)^{\otimes 3} \otimes \nu_n((1_Q)^{\otimes 3}) \otimes w \end{array} \right] \\ & \in C_{(n-1)} \otimes \left(\frac{\nu_n(C_{(n-1)})}{Q_{n-1}} \right) \otimes C_{(n-1)} \subseteq C_{(n-1)} \otimes \frac{Q_{n-1}}{Q_{n-1}} \otimes C_{(n-1)} = 0. \end{aligned}$$

Thus we can express the first term with respect to the remaining ones as follows

$$\begin{aligned} & w_1 \otimes \nu_n(w_2) \otimes w_3 \\ &= w \otimes \nu_n((1_Q)^{\otimes 3}) \otimes (1_Q)^{\otimes 3} + (1_Q)^{\otimes 3} \otimes \nu_n(w) \otimes (1_Q)^{\otimes 3} + (1_Q)^{\otimes 3} \otimes \nu_n((1_Q)^{\otimes 3}) \otimes w \\ &= w \otimes (1_Q + Q_{n-1}) \otimes (1_Q)^{\otimes 3} + (1_Q)^{\otimes 3} \otimes \nu_n(w) \otimes (1_Q)^{\otimes 3} + (1_Q)^{\otimes 3} \otimes (1_Q + Q_{n-1}) \otimes w \\ &\stackrel{n \geq 0}{=} (1_Q)^{\otimes 3} \otimes \nu_n(w) \otimes (1_Q)^{\otimes 3}. \end{aligned}$$

We have so proved that for $n > 0$ and $w \in C_{(n)}$

$$(13) \quad w_1 \otimes \nu_n(w_2) \otimes w_3 = (1_Q)^{\otimes 3} \otimes \nu_n(w) \otimes (1_Q)^{\otimes 3}.$$

The same equation trivially holds also in the case $n = 0$ as $C_{(n)}$ is one-dimensional.

Let $x, y, z \in Q$. Then $x \otimes y \otimes z \in C_{(|x|+|y|+|z|)}$ so that

$$\begin{aligned} (\bar{x} \cdot \bar{y}) \cdot \bar{z} &= ((x + Q_{|x|-1}) \cdot (y + Q_{|y|-1})) \cdot (z + Q_{|z|-1}) \\ &= ((xy) + Q_{|x|+|y|-1}) \cdot (z + Q_{|z|-1}) \\ &= (xy)z + Q_{|x|+|y|+|z|-1} \\ &= \omega^{-1}((x \otimes y \otimes z)_1) \nu_{|x|+|y|+|z|}((x \otimes y \otimes z)_2) \omega((x \otimes y \otimes z)_3) \\ &\stackrel{(13)}{=} \omega^{-1}(1_Q \otimes 1_Q \otimes 1_Q) \nu_{|x|+|y|+|z|}(x \otimes y \otimes z) \omega(1_Q \otimes 1_Q \otimes 1_Q) \\ &= \nu_{|x|+|y|+|z|}(x \otimes y \otimes z) \\ &= x(yz) + Q_{|x|+|y|+|z|-1} = \bar{x} \cdot (\bar{y} \cdot \bar{z}). \end{aligned}$$

Therefore the multiplication is associative. It is also unitary as

$$\bar{x} \cdot \overline{1_Q} = (x + Q_{|x|-1}) \cdot (1_Q + Q_{-1}) = x \cdot 1_Q + Q_{|x|-1} = x + Q_{|x|-1} = \bar{x}$$

and similarly $\overline{1_Q} \cdot \bar{x} = \bar{x}$ for every $x \in Q$. \square

2. GAUGE DEFORMATION

DEFINITION 2.1. Let H be a Hopf algebra and let $(Q, m, u, \Delta, \varepsilon, \omega)$ be a coquasi-bialgebra in ${}^H_H\mathcal{YD}$. A **gauge transformation** for Q is a morphism $\gamma : Q \otimes Q \rightarrow \mathbb{k}$ in ${}^H_H\mathcal{YD}$ which is convolution invertible in ${}^H_H\mathcal{YD}$ and which is also unitary on both entries.

REMARK 2.2. For γ as above, let us check that γ^{-1} is unitary whence a gauge transformation too.

First note that for all $x \in Q$, by means of (6) and (5), one gets

$$(14) \quad (1_Q \otimes x)_1 \otimes (1_Q \otimes x)_2 = 1_Q \otimes x_1 \otimes 1_Q \otimes x_2$$

$$(15) \quad (x \otimes 1_Q)_1 \otimes (x \otimes 1_Q)_2 = x_1 \otimes 1_Q \otimes x_2 \otimes 1_Q$$

Thus

$$\gamma^{-1}(1_Q \otimes x) = \gamma^{-1}(1_Q \otimes x_1) \varepsilon(x_2) = \gamma^{-1}(1_Q \otimes x_1) \gamma(1_Q \otimes x_2) = (\gamma^{-1} * \gamma)(1_Q \otimes x) = \varepsilon(x)$$

and similarly $\gamma^{-1}(x \otimes 1_Q) = \varepsilon(x)$.

LEMMA 2.3. Let H be a Hopf algebra and let C be a coalgebra in ${}^H_H\mathcal{YD}$. Given a map $\gamma \in {}^H_H\mathcal{YD}(C, \mathbb{k})$, we have that γ is convolution invertible in ${}^H_H\mathcal{YD}(C, \mathbb{k})$ if and only if it is convolution invertible in $\mathbf{Vec}_{\mathbb{k}}(C, \mathbb{k})$. Moreover the inverse is the same.

Proof. Assume there is a \mathbb{k} -linear map $\gamma^{-1} : C \rightarrow \mathbb{k}$ which is a convolution inverse of γ in $\mathbf{Vec}_{\mathbb{k}}(C, \mathbb{k})$. By [ABM1, Remark 2.4(ii)], γ^{-1} is left H -linear. Let us check that γ^{-1} is left H -colinear:

$$\begin{aligned} c_{-1} \otimes \gamma^{-1}(c_0) &= (c_1)_{-1} 1_H \otimes \gamma^{-1}((c_1)_0) \gamma(c_2) \gamma^{-1}(c_3) \\ &= (c_1)_{-1} (c_2)_{-1} \otimes \gamma^{-1}((c_1)_0) \gamma((c_2)_0) \gamma^{-1}(c_3) \\ &\stackrel{(*)}{=} (c_1)_{-1} \otimes \gamma^{-1}(((c_1)_0)_1) \gamma(((c_1)_0)_2) \gamma^{-1}(c_2) \\ &= (c_1)_{-1} \otimes (\gamma^{-1} * \gamma)((c_1)_0) \gamma^{-1}(c_2) \\ &= (c_1)_{-1} \otimes \varepsilon_C((c_1)_0) \gamma^{-1}(c_2) \\ &\stackrel{(*)}{=} 1_H \otimes \varepsilon_C(c_1) \gamma^{-1}(c_2) = 1_H \otimes \gamma^{-1}(c) \end{aligned}$$

where in (*) we used that the comultiplication or the counit of C is left H -colinear. Thus γ is convolution invertible in ${}^H_H\mathcal{YD}(C, \mathbb{k})$. The other implication is obvious. \square

PROPOSITION 2.4. *Let H be a Hopf algebra and let $(Q, m, u, \Delta, \varepsilon, \omega)$ be a coquasi-bialgebra in ${}^H_H\mathcal{YD}$. Let $\gamma : Q \otimes Q \rightarrow \mathbb{k}$ be a gauge transformation in ${}^H_H\mathcal{YD}$. Then*

$$Q^\gamma := (Q, m^\gamma, u, \Delta, \varepsilon, \omega^\gamma)$$

is a coquasi-bialgebra in ${}^H_H\mathcal{YD}$, where

$$\begin{aligned} m^\gamma &:= \gamma * m * \gamma^{-1} \\ \omega^\gamma &:= (\varepsilon \otimes \gamma) * \gamma(Q \otimes m) * \omega * \gamma^{-1}(m \otimes Q) * (\gamma^{-1} \otimes \varepsilon). \end{aligned}$$

Proof. The proof is analogue to [K, Proposition XV.3.2] in its dual version. We include some details for the reader's sake. Note that Q^γ has the same underlying coalgebra of Q which is a coalgebra in ${}^H_H\mathcal{YD}$. The unit is also the same and hence it is a coalgebra map in ${}^H_H\mathcal{YD}$. Since m^γ is the convolution product of morphisms in ${}^H_H\mathcal{YD}$, it results that m^γ is in ${}^H_H\mathcal{YD}$ as well.

Since m is a coalgebra map in ${}^H_H\mathcal{YD}$ and γ is convolution invertible with convolution inverse γ^{-1} , it follows that m^γ is a coalgebra map in ${}^H_H\mathcal{YD}$.

By means of (14) and (15), one gets that $m^\gamma(1_Q \otimes x) = x = m^\gamma(x \otimes 1_Q)$.

Let us consider now ω^γ . Since it is the convolution product of morphisms in ${}^H_H\mathcal{YD}$, it results that ω^γ is in ${}^H_H\mathcal{YD}$ as well.

Let us check that ω^γ is unitary. Consider the map $\alpha_2 : Q \otimes Q \rightarrow Q \otimes Q \otimes Q$ defined by $\alpha_2(x \otimes y) = x \otimes 1_Q \otimes y$. The equalities (15) and (6) yield

$$\begin{aligned} (\alpha_2(x \otimes y))_1 \otimes (\alpha_2(x \otimes y))_2 &= \alpha_2(x_1 \otimes (x_2)_{-1} y_1) \otimes \alpha_2((x_2)_0 \otimes y_2) \\ &= \alpha_2((x \otimes y)_1) \otimes \alpha_2((x \otimes y)_2) \end{aligned}$$

so that α_2 is comultiplicative.

Thus

$$\omega^\gamma \alpha_2 := (\varepsilon \otimes \gamma) \alpha_2 * \gamma(Q \otimes m) \alpha_2 * \omega \alpha_2 * \gamma^{-1}(m \otimes Q) \alpha_2 * (\gamma^{-1} \otimes \varepsilon) \alpha_2$$

and computing the factors of this convolution products one gets

$$\begin{aligned} (\varepsilon \otimes \gamma) \alpha_2 &= \varepsilon \otimes \varepsilon, \quad \gamma(Q \otimes m) \alpha_2 = \gamma, \quad \omega \alpha_2 = \varepsilon \otimes \varepsilon, \\ \gamma^{-1}(m \otimes Q) \alpha_2 &= \gamma^{-1}, \quad (\gamma^{-1} \otimes \varepsilon) \alpha_2 = \varepsilon \otimes \varepsilon \end{aligned}$$

and hence $\omega^\gamma \alpha_2 = \gamma * \gamma^{-1} = \varepsilon \otimes \varepsilon$, which means that $\omega^\gamma(x \otimes 1_Q \otimes y) = \varepsilon(x) \varepsilon(y)$ for every $x, y \in Q$.

Similarly, considering $\alpha_1 : Q \otimes Q \rightarrow Q \otimes Q \otimes Q$ defined by $\alpha_1(x \otimes y) = 1_Q \otimes x \otimes y$, one proves that $\omega^\gamma(1_Q \otimes x \otimes y) = \varepsilon(x) \varepsilon(y)$. A symmetric argument shows that $\omega^\gamma(x \otimes y \otimes 1_Q) = \varepsilon(x) \varepsilon(y)$.

Note that, by Lemma 2.3, ω^γ is convolution invertible in ${}^H_H\mathcal{YD}(D, \mathbb{k})$ as it is convolution invertible in $\mathbf{Vec}_{\mathbb{k}}(D, \mathbb{k})$.

Let us check that the multiplication is quasi-associative. By [ABM, Lemma 2.10 formula (2.7)], we have

$$m^\gamma(Q \otimes \gamma * m * \gamma^{-1}) = (\varepsilon \otimes \gamma) * m^\gamma(Q \otimes m) * (\varepsilon \otimes \gamma^{-1}),$$

$$\begin{aligned}
(\varepsilon \otimes \gamma^{-1}) * (\varepsilon \otimes \gamma) &= \varepsilon \otimes (\gamma^{-1} * \gamma) = \varepsilon \otimes \varepsilon \otimes \varepsilon, \\
m^\gamma (m^\gamma \otimes Q) &= m^\gamma (\gamma * m * \gamma^{-1} \otimes Q) = (\gamma \otimes \varepsilon) * m^\gamma (m * \gamma^{-1} \otimes Q) \\
&= (\gamma \otimes \varepsilon) * m^\gamma (m \otimes Q) * (\gamma^{-1} \otimes \varepsilon), \\
(\gamma^{-1} \otimes \varepsilon) * (\gamma \otimes \varepsilon) &= ((\gamma^{-1} * \gamma) \otimes \varepsilon) = \varepsilon \otimes \varepsilon \otimes \varepsilon.
\end{aligned}$$

By using these equalities one obtains

$$\begin{aligned}
m^\gamma (Q \otimes m^\gamma) * \omega^\gamma &= (\varepsilon \otimes \gamma) * \gamma (Q \otimes m) * m (Q \otimes m) * \omega * \gamma^{-1} (m \otimes Q) * (\gamma^{-1} \otimes \varepsilon), \\
\omega^\gamma * m^\gamma (m^\gamma \otimes Q) &= (\varepsilon \otimes \gamma) * \gamma (Q \otimes m) * \omega * m (m \otimes Q) * \gamma^{-1} (m \otimes Q) * (\gamma^{-1} \otimes \varepsilon)
\end{aligned}$$

so that $\omega^\gamma * m^\gamma (m^\gamma \otimes Q) = m^\gamma (Q \otimes m^\gamma) * \omega^\gamma$.

It remains to check that ω^γ is a reassociator. By [ABM, Lemma 2.10 formula (2.7)], we have

$$\begin{aligned}
\omega^\gamma (Q \otimes Q \otimes \gamma * m * \gamma^{-1}) &= (\varepsilon \otimes \varepsilon \otimes \gamma) * \omega^\gamma (Q \otimes Q \otimes m) * (\varepsilon \otimes \varepsilon \otimes \gamma^{-1}), \\
\omega^\gamma (\gamma * m * \gamma^{-1} \otimes Q \otimes Q) &= (\gamma \otimes \varepsilon \otimes \varepsilon) * \omega^\gamma (m \otimes Q \otimes Q) * (\gamma^{-1} \otimes \varepsilon \otimes \varepsilon), \\
(\gamma \otimes \varepsilon \otimes \varepsilon) * (\varepsilon \otimes \varepsilon \otimes \gamma) &= \gamma \otimes \gamma = (\varepsilon \otimes \varepsilon \otimes \gamma) * (\gamma \otimes \varepsilon \otimes \varepsilon).
\end{aligned}$$

By using these equalities one obtains

$$\begin{aligned}
&\omega^\gamma (Q \otimes Q \otimes m^\gamma) * \omega^\gamma (m^\gamma \otimes Q \otimes Q) \\
&= \left[\begin{array}{c} (\varepsilon \otimes \varepsilon \otimes \gamma) * (\varepsilon \otimes \gamma (Q \otimes m)) * \gamma (Q \otimes m (Q \otimes m)) \\ * \omega (Q \otimes Q \otimes m) * \omega (m \otimes Q \otimes Q) \\ * \gamma^{-1} (m (m \otimes Q) \otimes Q) * (\gamma^{-1} (m \otimes Q) \otimes \varepsilon) * (\gamma^{-1} \otimes \varepsilon \otimes \varepsilon) \end{array} \right]
\end{aligned}$$

and

$$\begin{aligned}
&(\varepsilon \otimes \omega^\gamma) * \omega^\gamma (Q \otimes m^\gamma \otimes Q) * (\omega^\gamma \otimes \varepsilon) \\
&= \left[\begin{array}{c} (\varepsilon \otimes \varepsilon \otimes \gamma) * (\varepsilon \otimes \gamma (Q \otimes m)) * \gamma (Q \otimes m (Q \otimes m)) \\ * (\varepsilon \otimes \omega) * \omega (Q \otimes m \otimes Q) * (\omega \otimes \varepsilon) \\ * \gamma^{-1} (m (m \otimes Q) \otimes Q) * (\gamma^{-1} (m \otimes Q) \otimes \varepsilon) * (\gamma^{-1} \otimes \varepsilon \otimes \varepsilon) \end{array} \right].
\end{aligned}$$

Therefore

$$\omega^\gamma (Q \otimes Q \otimes m^\gamma) * \omega^\gamma (m^\gamma \otimes Q \otimes Q) = (\varepsilon \otimes \omega^\gamma) * \omega^\gamma (Q \otimes m^\gamma \otimes Q) * (\omega^\gamma \otimes \varepsilon).$$

□

In analogy to the case of Hopf algebras, one can define the bosonization $E\#H$ of a coquasi-bialgebra in ${}^H_H\mathcal{YD}$ by a Hopf algebra H , see [ABM, Definition 5.4] for further details on the structure. The following result was originally stated for E a Hopf algebra. Yorck Sommerhäuser suggested the present more general form which investigates the behaviour of the bosonization under a suitable gauge transformation.

PROPOSITION 2.5. *Let H be a Hopf algebra and let $(E, m, u, \Delta, \varepsilon, \omega)$ be a coquasi-bialgebra in ${}^H_H\mathcal{YD}$. Let $\gamma : E \otimes E \rightarrow \mathbb{k}$ be a gauge transformation in ${}^H_H\mathcal{YD}$. Set*

$$\Gamma : (E\#H) \otimes (E\#H) \rightarrow \mathbb{k} : (x\#h) \otimes (x'\#h') \mapsto \gamma(x \otimes hx') \varepsilon_H(h').$$

Then Γ is a gauge transformation and $(E\#H)^\Gamma = E^\gamma\#H$ as ordinary coquasi-bialgebras.

Proof. By [ABM, Lemma 2.15 and what follows], we have that Γ is convolution invertible H -bilinear and H -balanced. Moreover $\Gamma^{-1}((x\#h) \otimes (x'\#h')) = \gamma^{-1}(x \otimes hx') \varepsilon_H(h')$. If $\alpha : (E\#H) \otimes (E\#H) \rightarrow E\#H$ is H -bilinear and H -balanced, it is easy to check that $\Gamma * \alpha * \Gamma^{-1}$ is H -bilinear and H -balanced too.

In particular, since

$$m_{E\#H}((x\#h) \otimes (x'\#h')) = m(x \otimes h_1x') \otimes h_2h'$$

we have that $m_{E\#H}$ is H -bilinear and H -balanced where $E\#H$ carries the left H -diagonal action and the right regular action over H .

Thus $m_{(E\#H)^\Gamma} = \Gamma * m_{E\#H} * \Gamma^{-1}$ is H -bilinear and H -balanced. Moreover, since E^γ is also a coquasi-bialgebra in ${}^H_H\mathcal{YD}$ we have that $m_{E^\gamma\#H} : (E\#H) \otimes (E\#H) \rightarrow E\#H$ is H -bilinear and H -balanced too.

Therefore, in order to check that $m_{(E\#H)^\Gamma} = m_{E^\gamma\#H}$, it suffices to prove that they coincide on elements of the form $(x\#1_H) \otimes (x'\#1_H)$.

Let us consider the multiplication

$$\begin{aligned} & m_{(E\#H)^\Gamma}((x\#1_H) \otimes (x'\#1_H)) \\ &= (\Gamma * m_{E\#H} * \Gamma^{-1})((x\#1_H) \otimes (x'\#1_H)) \\ &= \Gamma((x\#1_H)_1 \otimes (x'\#1_H)_1) \cdot m_{E\#H}((x\#1_H)_2 \otimes (x'\#1_H)_2) \cdot \Gamma^{-1}((x\#1_H)_3 \otimes (x'\#1_H)_3). \end{aligned}$$

Now, from

$$\Delta_{E\#H}(x\#h) = \sum \left(x^{(1)}\#x^{(2)}\langle_{-1}h_1 \right) \otimes \left(x^{(2)}\langle_0\#h_2 \right)$$

we get

$$\begin{aligned} & (x\#1_H)_1 \otimes (x\#1_H)_2 \otimes (x\#1_H)_3 \\ &= \sum \left(x^{(1)}\#x^{(2)}\langle_{-1}x^{(3)}\langle_{-2} \right) \otimes \left(x^{(2)}\langle_0\#x^{(3)}\langle_{-1} \right) \otimes \left(x^{(3)}\langle_0\#1_H \right) \end{aligned}$$

so that

$$\begin{aligned} & m_{(E\#H)^\Gamma}((x\#1_H) \otimes (x'\#1_H)) \\ &= \Gamma((x\#1_H)_1 \otimes (x'\#1_H)_1) \cdot m_{E\#H}((x\#1_H)_2 \otimes (x'\#1_H)_2) \cdot \Gamma^{-1}((x\#1_H)_3 \otimes (x'\#1_H)_3) \\ &= \left[\begin{array}{c} \sum \Gamma \left(x^{(1)}\#x^{(2)}\langle_{-1}x^{(3)}\langle_{-2} \otimes x'^{(1)}\#x'^{(2)}\langle_{-1}x'^{(3)}\langle_{-2} \right) \\ \cdot m_{E\#H} \left(x^{(2)}\langle_0\#x^{(3)}\langle_{-1} \otimes x'^{(2)}\langle_0\#x'^{(3)}\langle_{-1} \right) \\ \cdot \Gamma^{-1} \left(x^{(3)}\langle_0\#1_H \otimes x'^{(3)}\langle_0\#1_H \right) \end{array} \right] \\ &= \left[\begin{array}{c} \sum \gamma \left(x^{(1)} \otimes x^{(2)}\langle_{-1}x^{(3)}\langle_{-2}x'^{(1)} \right) \\ \cdot m_{E\#H} \left(x^{(2)}\langle_0\#x^{(3)}\langle_{-1} \otimes x'^{(2)}\#x'^{(3)}\langle_{-1} \right) \\ \cdot \gamma^{-1} \left(x^{(3)}\langle_0 \otimes x'^{(3)}\langle_0 \right) \end{array} \right] \\ &= \left[\begin{array}{c} \sum \gamma \left(x^{(1)} \otimes x^{(2)}\langle_{-1}x^{(3)}\langle_{-2}x'^{(1)} \right) \\ \cdot m \left(x^{(2)}\langle_0 \otimes x^{(3)}\langle_{-2}x'^{(2)} \right) \otimes x^{(3)}\langle_{-1}x'^{(3)}\langle_{-1} \\ \cdot \gamma^{-1} \left(x^{(3)}\langle_0 \otimes x'^{(3)}\langle_0 \right) \end{array} \right] \\ &= \left[\begin{array}{c} \sum \gamma \left(x^{(1)} \otimes x^{(2)}\langle_{-1}x^{(3)}\langle_{-2}x'^{(1)} \right) \\ \cdot m \left(x^{(2)}\langle_0 \otimes x^{(3)}\langle_{-1}x'^{(2)} \right) \otimes \left(x^{(3)}\langle_0 \otimes x'^{(3)}\langle_{-1} \right) \\ \cdot \gamma^{-1} \left(x^{(3)}\langle_0 \otimes x'^{(3)}\langle_0 \right) \end{array} \right] \\ &\stackrel{\gamma^{-1} \text{ colin.}}{=} \left[\begin{array}{c} \sum \gamma \left(x^{(1)} \otimes x^{(2)}\langle_{-1}x^{(3)}\langle_{-2}x'^{(1)} \right) \cdot m \left(x^{(2)}\langle_0 \otimes x^{(3)}\langle_{-1}x'^{(2)} \right) \otimes 1_H \\ \cdot \gamma^{-1} \left(x^{(3)}\langle_0 \otimes x'^{(3)}\langle_0 \right) \end{array} \right] \\ &= \left[\begin{array}{c} \sum \gamma \left(x^{(1)} \otimes x^{(2)}\langle_{-1}x^{(3)}\langle_{-2}x'^{(1)} \right) m \left(x^{(2)}\langle_0 \otimes x^{(3)}\langle_{-1}x'^{(2)} \right) \\ \gamma^{-1} \left(x^{(3)}\langle_0 \otimes x'^{(3)}\langle_0 \right) \end{array} \right] \otimes 1_H. \end{aligned}$$

Now we have

$$\sum (x \otimes y)^{(1)} \otimes (x \otimes y)^{(2)} = \sum x^{(1)} \otimes x^{(2)}\langle_{-1}y^{(1)} \otimes x^{(2)}\langle_0 \otimes y^{(2)}$$

so that

$$\begin{aligned} & \sum (x \otimes y)^{(1)} \otimes (x \otimes y)^{(2)} \otimes (x \otimes y)^{(3)} \\ &= \sum \left(x^{(1)} \otimes x^{(2)}\langle_{-1}x^{(3)}\langle_{-2}y^{(1)} \right) \otimes \left(x^{(2)}\langle_0 \otimes x^{(3)}\langle_{-1}y^{(2)} \right) \otimes \left(x^{(3)}\langle_0 \otimes y^{(3)} \right). \end{aligned}$$

Using this equality we can proceed in our computation:

$$m_{(E\#H)^\Gamma}((x\#1_H) \otimes (x'\#1_H))$$

$$\begin{aligned}
&= \left[m(x^{(2)}_{\langle 0 \rangle} \otimes x^{(3)}_{\langle -1 \rangle} x'^{(2)}) \gamma^{-1}(x^{(3)}_{\langle 0 \rangle} \otimes x'^{(3)}) \right] \otimes 1_H \\
&= \left[\sum \gamma((x \otimes x')^{(1)}) \cdot m((x \otimes x')^{(2)}) \cdot \gamma^{-1}((x \otimes x')^{(3)}) \right] \# 1_H \\
&= (\gamma * m * \gamma^{-1})(x \otimes x') \# 1_H \\
&= m_{E^\gamma}(x \otimes x') \# 1_H \\
&= m_{E^\gamma \# H}((x \# 1_H) \otimes (x' \# 1_H)).
\end{aligned}$$

Finally $u_{(E \# H)^\Gamma} = u_{E \# H} = 1_E \# 1_H = 1_{E^\gamma} \# 1_H = u_{E^\gamma \# H}$.

As a coalgebra $(E \# H)^\Gamma$ coincides with $E \# H$ and hence with $E^\gamma \# H$.

Finally let us check that $\omega_{E^\gamma \# H}$ and $\omega_{(E \# H)^\Gamma}$ coincide. To this aim, let us use the maps $\mathcal{U}_{H,-}^*$ of [ABM, Lemma 2.15]. First note that $\omega_{E^\gamma \# H} = \mathcal{U}_{H,E^\gamma}^3(\omega_{E^\gamma})$ by [ABM, Proposition 5.3]. Now

$$\begin{aligned}
\omega_{(E \# H)^\Gamma} &= (\varepsilon_{E \# H} \otimes \Gamma) * \Gamma(E \# H \otimes m_{E \# H}) * \omega_{E \# H} * \Gamma^{-1}(m_{E \# H} \otimes E \# H) * (\Gamma^{-1} \otimes \varepsilon_{E \# H}) \\
&= (\mathcal{U}_{H,E}^1(\varepsilon) \otimes \mathcal{U}_{H,E}^2(\gamma)) * \mathcal{U}_{H,E}^2(\gamma)(E \# H \otimes m_{E \# H}) * \mathcal{U}_{H,E}^3(\omega) \\
&\quad * \mathcal{U}_{H,E}^2(\gamma^{-1})(m_{E \# H} \otimes E \# H) * (\mathcal{U}_{H,E}^2(\gamma^{-1}) \otimes \mathcal{U}_{H,E}^1(\varepsilon))
\end{aligned}$$

One easily checks that

$$\begin{aligned}
\mathcal{U}_{H,E}^1(\varepsilon) \otimes \mathcal{U}_{H,E}^2(\gamma) &= \mathcal{U}_{H,E^\gamma}^3(\varepsilon \otimes \gamma), \\
\mathcal{U}_{H,E}^2(\gamma)(E \# H \otimes m_{E \# H}) &= \mathcal{U}_{H,E^\gamma}^3(\gamma(E \otimes m)), \\
\mathcal{U}_{H,E}^2(\gamma^{-1})(m_{E \# H} \otimes E \# H) &= \mathcal{U}_{H,E^\gamma}^3(\gamma^{-1}(m \otimes E)), \\
\mathcal{U}_{H,E}^2(\gamma^{-1}) \otimes \mathcal{U}_{H,E}^1(\varepsilon E) &= \mathcal{U}_{H,E^\gamma}^3(\gamma^{-1} \otimes \varepsilon).
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
\omega_{(E \# H)^\Gamma} &= \mathcal{U}_{H,E^\gamma}^3(\varepsilon \otimes \gamma) * \mathcal{U}_{H,E^\gamma}^3(\gamma(E \otimes m)) * \mathcal{U}_{H,E}^3(\omega) * \mathcal{U}_{H,E^\gamma}^3(\gamma^{-1}(m \otimes E)) * \mathcal{U}_{H,E^\gamma}^3(\gamma^{-1} \otimes \varepsilon) \\
&= \mathcal{U}_{H,E^\gamma}^3[(\varepsilon \otimes \gamma) * \gamma(E \otimes m) * \omega * \gamma^{-1}(m \otimes E) * (\gamma^{-1} \otimes \varepsilon)] \\
&= \mathcal{U}_{H,E^\gamma}^3(\omega_{E^\gamma}) = \omega_{E^\gamma \# H}.
\end{aligned}$$

□

PROPOSITION 2.6. *Let H be a Hopf algebra and let $(Q, m, u, \Delta, \varepsilon, \omega)$ be a connected coquasi-bialgebra in ${}^H_H\mathcal{YD}$. Let $\gamma : Q \otimes Q \rightarrow \mathbb{k}$ be a gauge transformation in ${}^H_H\mathcal{YD}$. Then $\text{gr}(Q^\gamma)$ and $\text{gr}(Q)$ coincide as bialgebras in ${}^H_H\mathcal{YD}$.*

Proof. By Proposition 2.4, Q^γ is a coquasi-bialgebra in ${}^H_H\mathcal{YD}$. It is obviously connected as it coincides with Q as a coalgebra. By Theorem 1.5, both $\text{gr}Q$ and $\text{gr}(Q^\gamma)$ are connected bialgebras in ${}^H_H\mathcal{YD}$. Let us check they coincide.

Note that, by Remark 2.2, we have that γ^{-1} is a gauge transformation, hence it is trivial on $\mathbb{k}1_Q \otimes 1_Q$. Let $C := Q \otimes Q$. Let $n > 0$ and let $w \in C_{(n)} = \sum_{i+j \leq n} Q_i \otimes Q_j$. By [AMS1, Lemma 3.69], we have that $\Delta_C(w) - w \otimes (1_Q)^{\otimes 2} - (1_Q)^{\otimes 2} \otimes w \in C_{(n-1)} \otimes C_{(n-1)}$. Thus we get

$$w_1 \otimes w_2 \otimes w_3 - \Delta_C(w) \otimes (1_Q)^{\otimes 2} - \Delta_C((1_Q)^{\otimes 2}) \otimes w \in \Delta_C(C_{(n-1)}) \otimes C_{(n-1)}$$

and hence

$$w_1 \otimes w_2 \otimes w_3 - w \otimes (1_Q)^{\otimes 2} \otimes (1_Q)^{\otimes 2} - (1_Q)^{\otimes 2} \otimes w \otimes (1_Q)^{\otimes 2} - (1_Q)^{\otimes 4} \otimes w \in C_{(n-1)} \otimes C_{(n-1)} \otimes C_{(n-1)}.$$

Since $m(C_{(n-1)}) \subseteq Q_{n-1}$ we get

$$w_1 \otimes m(w_2) \otimes w_3 - w \otimes 1_Q \otimes (1_Q)^{\otimes 2} - (1_Q)^{\otimes 2} \otimes m(w) \otimes (1_Q)^{\otimes 2} - (1_Q)^{\otimes 3} \otimes w \in C_{(n-1)} \otimes Q_{n-1} \otimes C_{(n-1)}$$

and hence

$$(16) \quad w_1 \otimes (m(w_2) + Q_{n-1}) \otimes w_3 = (1_Q)^{\otimes 2} \otimes (m(w) + Q_{n-1}) \otimes (1_Q)^{\otimes 2}.$$

Let $x, y \in Q$. We compute

$$\begin{aligned}
\bar{x} \cdot_\gamma \bar{y} &= (x + Q_{|x|-1}) \cdot_\gamma (y + Q_{|y|-1}) \\
&= (x \cdot_\gamma y) + Q_{|x|+|y|-1} \\
&= \gamma((x \otimes y)_1) m((x \otimes y)_2) \gamma^{-1}((x \otimes y)_3) + Q_{|x|+|y|-1} \\
&= \gamma((x \otimes y)_1) (m((x \otimes y)_2) + Q_{|x|+|y|-1}) \gamma^{-1}((x \otimes y)_3) \\
&\stackrel{(16)}{=} \gamma((1_Q)^{\otimes 2}) (m(x \otimes y) + Q_{|x|+|y|-1}) \gamma^{-1}((1_Q)^{\otimes 2}) \\
&= m(x \otimes y) + Q_{|x|+|y|-1} = (x \cdot y) + Q_{|x|+|y|-1} = \bar{x} \cdot \bar{y}.
\end{aligned}$$

Note that Q^γ and Q have the same unit so that $\text{gr}Q$ and $\text{gr}Q^\gamma$ have. \square

3. (CO)SEMISIMPLE CASE

Assume H is a semisimple and cosemisimple Hopf algebra (e.g. H is finite-dimensional cosemisimple over a field of characteristic zero). Note that H is then separable (see e.g. [Stf, Corollary 3.7] or [AMS1, Theorem 2.34]) whence finite-dimensional. Let $(Q, m, u, \Delta, \varepsilon)$ be a f.d. coalgebra with multiplication and unit in ${}^H_H\mathcal{YD}$. Assume that the coradical Q_0 is a subcoalgebra of Q in ${}^H_H\mathcal{YD}$ such that $Q_0 \cdot Q_0 \subseteq Q_0$. Let $y^{n,i}$ with $1 \leq i \leq \dim(Q_n/Q_{n-1})$ be a basis for Q_n/Q_{n-1} . Consider, for every $n > 0$, the exact sequence in ${}^H_H\mathcal{YD}$ given by

$$0 \longrightarrow Q_{n-1} \xrightarrow{s_n} Q_n \xrightarrow{\pi_n} \frac{Q_n}{Q_{n-1}} \longrightarrow 0$$

Now, since H is semisimple and cosemisimple, by [Ra2, Proposition 7] the Drinfeld double $D(H)$ is semisimple. By a result essentially due to Majid (see [Mo, Proposition 10.6.16]) and by [RT, Proposition 6], we get that the category ${}^H_H\mathcal{YD} \cong {}_{D(H)}\mathfrak{M}$ is a semisimple category. Therefore π_n cosplits i.e. there is a morphism $\sigma_n : (Q_n/Q_{n-1}) \rightarrow Q_n$ in ${}^H_H\mathcal{YD}$ such that $\pi_n \sigma_n = \text{Id}$. Let $u_n : \mathbb{k} \rightarrow Q_n$ be the corestriction of the unit $u : \mathbb{k} \rightarrow Q$ and let $\varepsilon_n = \varepsilon|_{Q_n} : Q_n \rightarrow \mathbb{k}$ be the counit of the subcoalgebra Q_n . Set

$$\sigma'_n := \sigma_n - u_n \circ \varepsilon_n \circ \sigma_n$$

This is a morphism in ${}^H_H\mathcal{YD}$. Moreover

$$\begin{aligned}
\pi_n \circ \sigma'_n &= \pi_n \circ \sigma_n - \pi_n \circ u_n \circ \varepsilon_n \circ \sigma_n \stackrel{n \geq 0}{=} \text{Id}_{Q_n/Q_{n-1}} - 0 = \text{Id}_{Q_n/Q_{n-1}}, \\
\varepsilon_n \circ \sigma'_n &= \varepsilon_n \circ \sigma_n - \varepsilon_n \circ u_n \circ \varepsilon_n \circ \sigma_n = \varepsilon_n \circ \sigma_n - \varepsilon_n \circ \sigma_n = 0.
\end{aligned}$$

Therefore, without loss of generality we can assume that $\varepsilon_n \circ \sigma_n = 0$. A standard argument on split short exact sequences shows that there exists a morphism $p_n : Q_n \rightarrow Q_{n-1}$ in ${}^H_H\mathcal{YD}$ such that $s_n p_n + \sigma_n \pi_n = \text{Id}_{Q_n}$, $p_n s_n = \text{Id}_{Q_{n-1}}$ and $p_n \sigma_n = 0$. We set

$$x^{n,i} := \sigma_n(y^{n,i}).$$

Therefore

$$y^{n,i} = \pi_n \sigma_n(y^{n,i}) = \pi_n(x^{n,i}) = x^{n,i} + Q_{n-1} = \overline{x^{n,i}}.$$

These terms $x^{n,i}$ define a \mathbb{k} -basis for Q . As Q is finite-dimensional, there exists $d \in \mathbb{N}_0$ such that $Q = Q_d$; we fix d minimal. For all $0 \leq a \leq b$, define the maps

$$\begin{aligned}
p_{a,b} &: Q_b \rightarrow Q_a, & p_{a,b} &:= p_{a+1} \circ p_{a+2} \circ \cdots \circ p_{b-1} \circ p_b, \\
s_{b,a} &: Q_a \rightarrow Q_b, & s_{b,a} &:= s_b \circ s_{b-1} \circ \cdots \circ s_{a+2} \circ s_{a+1}.
\end{aligned}$$

Clearly one has

$$p_{a,b} \circ s_{b,a} = \text{Id}_{Q_a}.$$

Thus, for $0 \leq i, a \leq b$ we have

$$(17) \quad p_{i,b} \circ s_{b,a} = \begin{cases} p_{i,b} \circ s_{b,i} \circ s_{i,a} & i > a \\ p_{i,a} \circ p_{a,b} \circ s_{b,a} & i \leq a \end{cases} = \begin{cases} s_{i,a} & i > a \\ p_{i,a} & i \leq a \end{cases}$$

Thus we get an isomorphism $\varphi : Q \rightarrow \text{gr}Q$ of objects in ${}^H_H\mathcal{YD}$ given by

$$\varphi(x) := p_{0,d}(x) + \pi_1 p_{1,d}(x) + \pi_2 p_{2,d}(x) + \cdots + \pi_{d-2} p_{d-2,d}(x) + \pi_{d-1} p_{d-1,d}(x) + \pi_d(x)$$

$$= \sum_{0 \leq t \leq d} \pi_t p_{t,d}(x), \text{ for every } x \in Q,$$

where we set

$$\pi_0 = \text{Id}_{Q_0}, \quad p_{d,d} = \text{Id}_{Q_d}.$$

For $0 \leq n \leq d$, we have

$$\begin{aligned} \varphi(x^{n,i}) &= \varphi(s_{d,n}(x^{n,i})) = \varphi(s_{d,n}\sigma_n(y^{n,i})) = \sum_{0 \leq t \leq d} \pi_t p_{t,d} s_{d,n}(\sigma_n(y^{n,i})) \\ &= \sum_{n < t \leq d} \pi_t p_{t,d} s_{d,n}(\sigma_n(y^{n,i})) + \sum_{0 \leq t \leq n} \pi_t p_{t,d} s_{d,n}(\sigma_n(y^{n,i})) \\ &\stackrel{(17)}{=} \sum_{n < t \leq d} \pi_t s_{t,n}(\sigma_n(y^{n,i})) + \sum_{0 \leq t < n} \pi_t p_{t,n}(\sigma_n(y^{n,i})) + \pi_n p_{n,d} s_{d,n}(\sigma_n(y^{n,i})) \\ &= \sum_{n < t \leq d} \pi_t s_{t,t-1} s_{t-1,n}(\sigma_n(y^{n,i})) + \sum_{0 \leq t < n} \pi_t p_{t,n-1} p_{n-1,n}(\sigma_n(y^{n,i})) + \\ &\quad + \pi_n p_{n,d} s_{d,n}(\sigma_n(y^{n,i})) \\ &= \sum_{n < t \leq d} \pi_t s_{t,t-1,n} \sigma_n(y^{n,i}) + \sum_{0 \leq t < n} \pi_t p_{t,n-1} p_n \sigma_n(y^{n,i}) + \pi_n \sigma_n(y^{n,i}) \\ &= 0 + 0 + y^{n,i} = y^{n,i}. \end{aligned}$$

Hence $\varphi(x^{n,i}) = y^{n,i}$. Since $y^{n,i}$ with $1 \leq i \leq \dim(Q_n/Q_{n-1}) =: d_n$ form a basis for Q_n/Q_{n-1} we have that

$$hy^{n,i} \in \frac{Q_n}{Q_{n-1}}, \quad (y^{n,i})_{-1} \otimes (y^{n,i})_0 \in H \otimes \frac{Q_n}{Q_{n-1}}.$$

Therefore there are $\chi_{t,i}^n \in H^*$ and $h_{t,i}^n \in H$ such that

$$(18) \quad hy^{n,i} = \sum_{1 \leq t \leq d_n} \chi_{t,i}^n(h) y^{n,t}, \quad (y^{n,i})_{-1} \otimes (y^{n,i})_0 = \sum_{1 \leq t \leq d_n} h_{t,i}^n \otimes y^{n,t}.$$

We have

$$\begin{aligned} h(h'y^{n,i}) &= \sum_{1 \leq s \leq d_n} \chi_{s,i}^n(h') hy^{n,s} = \sum_{1 \leq s \leq d_n} \chi_{s,i}^n(h') \sum_{1 \leq t \leq d_n} \chi_{t,s}^n(h) y^{n,t} \\ &= \sum_{1 \leq s \leq d_n} \sum_{1 \leq t \leq d_n} \chi_{t,s}^n(h) \chi_{s,i}^n(h') y^{n,t}, \\ (hh')y^{n,i} &= \sum_{1 \leq t \leq d_n} \chi_{t,i}^n(hh') y^{n,t} \end{aligned}$$

and hence

$$\chi_{t,i}^n(hh') = \sum_{1 \leq s \leq d_n} \chi_{t,s}^n(h) \chi_{s,i}^n(h').$$

Moreover

$$y^{n,i} = 1_H y^{n,i} = \sum_{1 \leq t \leq d_n} \chi_{t,i}^n(1_H) y^{n,t}$$

and hence

$$\chi_{t,i}^n(1_H) = \delta_{t,i}.$$

We also have

$$\begin{aligned} (y^{n,i})_{-1} \otimes ((y^{n,i})_0)_{-1} \otimes ((y^{n,i})_0)_0 &= \sum_{1 \leq s \leq d_n} h_{i,s}^n \otimes (y^{n,s})_{-1} \otimes (y^{n,s})_0 \\ &= \sum_{1 \leq s \leq d_n} h_{i,s}^n \otimes \sum_{1 \leq t \leq d_n} h_{s,t}^n \otimes y^{n,t} \\ &= \sum_{1 \leq s \leq d_n} \sum_{1 \leq t \leq d_n} h_{i,s}^n \otimes h_{s,t}^n \otimes y^{n,t}, \\ ((y^{n,i})_{-1})_1 \otimes ((y^{n,i})_{-1})_2 \otimes (y^{n,i})_0 &= \sum_{1 \leq t \leq d_n} \Delta_H(h_{t,i}^n) \otimes y^{n,t} \end{aligned}$$

so that

$$\Delta_H(h_{t,i}^n) = \sum_{1 \leq s \leq d_n} h_{i,s}^n \otimes h_{s,t}^n.$$

Moreover

$$y^{n,i} = \varepsilon_H((y^{n,i})_{-1}) (y^{n,i})_0 = \sum_{1 \leq t \leq d_n} \varepsilon_H(h_{t,i}^n) y^{n,t}$$

and hence

$$\varepsilon_H(h_{t,i}^n) = \delta_{t,i}.$$

Finally

$$\begin{aligned}
(h_1 y^{n,i})_{-1} h_2 \otimes (h_1 y^{n,i})_0 &= \sum_{1 \leq s \leq d_n} \chi_{s,i}^n(h_1) (y^{n,s})_{-1} h_2 \otimes (y^{n,s})_0 \\
&= \sum_{1 \leq s \leq d_n} \chi_{s,i}^n(h_1) \sum_{1 \leq t \leq d_n} h_{s,t}^n h_2 \otimes y^{n,t} \\
&= \sum_{1 \leq s \leq d_n} \sum_{1 \leq t \leq d_n} h_{s,t}^n \chi_{s,i}^n(h_1) h_2 \otimes y^{n,t}, \\
h_1 (y^{n,i})_{-1} \otimes h_2 (y^{n,i})_0 &= \sum_{1 \leq s \leq d_n} h_1 h_{i,s}^n \otimes h_2 y^{n,s} = \sum_{1 \leq s \leq d_n} h_1 h_{i,s}^n \otimes \sum_{1 \leq t \leq d_n} \chi_{t,s}^n(h_2) y^{n,t} \\
&= \sum_{1 \leq s \leq d_n} \sum_{1 \leq t \leq d_n} h_1 \chi_{t,s}^n(h_2) h_{i,s}^n \otimes y^{n,t}
\end{aligned}$$

Therefore, we get

$$\sum_{1 \leq s \leq d_n} h_{s,t}^n \chi_{s,i}^n(h_1) h_2 = \sum_{1 \leq s \leq d_n} h_1 \chi_{t,s}^n(h_2) h_{i,s}^n.$$

We have

$$\begin{aligned}
hx^{n,i} &= h\sigma_n(y^{n,i}) = \sigma_n(hy^{n,i}) = \sigma_n\left(\sum_{1 \leq t \leq d_n} \chi_{t,i}^n(h) y^{n,t}\right) = \sum_{1 \leq t \leq d_n} \chi_{t,i}^n(h) x^{n,t}, \\
(x^{n,i})_{-1} \otimes (x^{n,i})_0 &= (\sigma_n(y^{n,i}))_{-1} \otimes (\sigma_n(y^{n,i}))_0 = (y^{n,i})_{-1} \otimes \sigma_n((y^{n,i})_0) = \sum_{1 \leq t \leq d_n} h_{i,t}^n \otimes x^{n,t}, \\
\varepsilon_Q(x^{n,i}) &= \varepsilon_n(x^{n,i}) = \varepsilon_n \sigma_n(y^{n,i}) = 0 \text{ for } n > 0.
\end{aligned}$$

If Q is connected, then $d_0 = 1$ so we may assume $y^{0,0} := 1_Q + Q_{-1}$. Since $\pi_0 = \text{Id}_{Q_0}$ we get

$$\sigma_0 = \text{Id}_{Q_0} \circ \sigma_0 = \pi_0 \circ \sigma_0 = \text{Id}_{Q_0}$$

and hence

$$x^{0,0} = \sigma_0(y^{0,0}) = \sigma_0(1_Q + Q_{-1}) = 1_Q.$$

Since, by Proposition 1.3, $Q_a \cdot Q_{a'} \subseteq Q_{a+a'}$ for every $a, a' \in \mathbb{N}_0$, we can write the product of two elements of the basis in the form

$$(19) \quad x^{a,l} x^{a',l'} = \sum_{u \leq a+a'} \sum_v \mu_{u,v}^{a,l,a',l'} x^{u,v}.$$

We compute

$$\begin{aligned}
\overline{x^{a,l}} \cdot \overline{x^{a',l'}} &= (x^{a,l} + Q_{a-1}) (x^{a',l'} + Q_{a'-1}) \\
&= (x^{a,l} x^{a',l'}) + Q_{a+a'-1} \\
&\stackrel{(19)}{=} \left(\sum_{u \leq a+a'} \sum_v \mu_{u,v}^{a,l,a',l'} x^{u,v} \right) + Q_{a+a'-1} \\
&= \left(\sum_v \mu_{a+a',v}^{a,l,a',l'} x^{a+a',v} \right) + Q_{a+a'-1} \\
&= \sum_v \mu_{a+a',v}^{a,l,a',l'} (x^{a+a',v} + Q_{a+a'-1}) \\
&= \sum_v \mu_{a+a',v}^{a,l,a',l'} \overline{x^{a+a',v}}.
\end{aligned}$$

which gives

$$(20) \quad \overline{x^{a,l}} \cdot \overline{x^{a',l'}} = \sum_v \mu_{a+a',v}^{a,l,a',l'} \overline{x^{a+a',v}}.$$

REMARK 3.1. Let H be a Hopf algebra and let (A, m_A, u_A) be an algebra in ${}^H_H\mathcal{YD}$. Let $\varepsilon_A : A \rightarrow \mathbb{k}$ be an algebra map in ${}^H_H\mathcal{YD}$. The Hochschild cohomology in a monoidal category is known, see e.g. [AMS2]. Consider \mathbb{k} as an A -bimodule in ${}^H_H\mathcal{YD}$ through ε_A . Then, following [AMS2, 1.24], we can consider an analogue of the standard complex

$${}^H_H\mathcal{YD}(\mathbb{k}, \mathbb{k}) \xrightarrow{\partial^0} {}^H_H\mathcal{YD}(A, \mathbb{k}) \xrightarrow{\partial^1} {}^H_H\mathcal{YD}(A^{\otimes 2}, \mathbb{k}) \xrightarrow{\partial^2} {}^H_H\mathcal{YD}(A^{\otimes 3}, \mathbb{k}) \xrightarrow{\partial^3} \dots$$

Explicitly, given f in the corresponding domain of ∂^n , for $n = 0, 1, 2, 3$, we have

$$\partial^0(f) = f(1)\varepsilon_A - \varepsilon_A f(1) = 0,$$

$$\begin{aligned}
\partial^1(f) &= f \otimes \varepsilon_A - f m_A + \varepsilon_A \otimes f, \\
\partial^2(f) &= f \otimes \varepsilon_A - f(A \otimes m_A) + f(m_A \otimes A) - \varepsilon_A \otimes f, \\
\partial^3(f) &= f \otimes \varepsilon_A - f(A \otimes A \otimes m_A) + f(A \otimes m_A \otimes A) - f(m_A \otimes A \otimes A) + \varepsilon_A \otimes f.
\end{aligned}$$

For every $n \geq 1$ denote by

$$Z_{\mathcal{YD}}^n(A, \mathbb{k}) := \ker(\partial^n), \quad B_{\mathcal{YD}}^n(A, \mathbb{k}) := \text{Im}(\partial^{n-1}) \quad \text{and} \quad H_{\mathcal{YD}}^n(A, \mathbb{k}) := \frac{Z_{\mathcal{YD}}^n(A, \mathbb{k})}{B_{\mathcal{YD}}^n(A, \mathbb{k})}$$

the abelian groups of n -cocycles, of n -coboundaries and the n -th Hochschild cohomology group in ${}^H_H\mathcal{YD}$ of the algebra A with coefficients in \mathbb{k} . We point out that the construction above works for an arbitrary A -bimodule M in ${}^H_H\mathcal{YD}$ instead of \mathbb{k} .

Next result is inspired by [EG, Proposition 2.3]. Two coquasi-bialgebras Q and Q' in ${}^H_H\mathcal{YD}$ will be called **gauge equivalent** whenever there is some gauge transformation $\gamma : Q \otimes Q \rightarrow \mathbb{k}$ in ${}^H_H\mathcal{YD}$ such that $Q^\gamma \cong Q'$ as coquasi-bialgebras in ${}^H_H\mathcal{YD}$, see Proposition 2.4 for the structure of Q^γ .

THEOREM 3.2. *Let H be a semisimple and cosemisimple Hopf algebra and let $(Q, m, u, \Delta, \varepsilon, \omega)$ be a f.d. connected coquasi-bialgebra in ${}^H_H\mathcal{YD}$. If $H_{\mathcal{YD}}^3(\text{gr}Q, \mathbb{k}) = 0$ then Q is gauge equivalent to a connected bialgebra in ${}^H_H\mathcal{YD}$.*

Proof. For $t \in \mathbb{N}_0$, and x, y, z in the basis of Q , we set

$$\omega_t(x \otimes y \otimes z) := \delta_{|x|+|y|+|z|, t} \omega(x \otimes y \otimes z).$$

Let us check it defines a morphism $\omega_t : Q \otimes Q \otimes Q \rightarrow \mathbb{k}$ in ${}^H_H\mathcal{YD}$. It is left H -linear as, by means of (18), the definition of ω_t and the H -linearity of ω , we can prove that $\omega_t \left(h \left(x^{n,i} \otimes x^{n',i'} \otimes x^{n'',i''} \right) \right) = \varepsilon_H(h) \omega_t \left(x^{n,i} \otimes x^{n',i'} \otimes x^{n'',i''} \right)$.

Moreover it is left H -colinear as, by means of (18), the definition of ω_t and the H -colinearity of ω , we can prove that

$$\left(x^{n,i} \otimes x^{n',i'} \otimes x^{n'',i''} \right)_{\langle -1 \rangle} \otimes \omega_t \left(\left(x^{n,i} \otimes x^{n',i'} \otimes x^{n'',i''} \right)_{\langle 0 \rangle} \right) = 1_H \otimes \omega_t \left(x^{n,i} \otimes x^{n',i'} \otimes x^{n'',i''} \right).$$

Clearly, for $x, y, z \in Q$ in the basis, one has

$$\sum_{t \in \mathbb{N}_0} \omega_t(x \otimes y \otimes z) = \sum_{t \in \mathbb{N}_0} \delta_{|x|+|y|+|z|, t} \omega(x \otimes y \otimes z) = \omega(x \otimes y \otimes z)$$

so that we can formally write

$$(21) \quad \omega = \sum_{t \in \mathbb{N}_0} \omega_t.$$

Since ε is trivial on elements in the basis of strictly positive degree, one gets

$$(22) \quad \omega_0 = \varepsilon \otimes \varepsilon \otimes \varepsilon.$$

If $\omega = \omega_0$ then Q is a (connected) bialgebra in ${}^H_H\mathcal{YD}$ and the proof is finished. Thus we can assume $\omega \neq \omega_0$ and set

$$\begin{aligned}
s &: = \min \{ i \in \mathbb{N} : \omega_i \neq 0 \}, \\
\bar{\omega}_s &: = \omega_s(\varphi^{-1} \otimes \varphi^{-1} \otimes \varphi^{-1}), \\
\bar{Q} &: = \text{gr}Q.
\end{aligned}$$

Note that $\bar{\omega}_s$ is a morphism in ${}^H_H\mathcal{YD}$ as a composition of morphisms in ${}^H_H\mathcal{YD}$.

Let $n \in \mathbb{N}_0$, let $C^4 = Q \otimes Q \otimes Q \otimes Q$ and let $u \in C_{(n)}^4 = \sum_{i+j+k+l \leq n} Q_i \otimes Q_j \otimes Q_k \otimes Q_l$.

A direct computation rewriting the cocycle condition using (21) proves that, for every $n \in \mathbb{N}_0$, and $u \in C_{(n)}^4$

$$(23) \quad \sum_{0 \leq i+j \leq n} [\omega_i(Q \otimes Q \otimes m) * \omega_j(m \otimes Q \otimes Q)](u)$$

$$= \sum_{0 \leq a+b+c \leq n} [(\varepsilon \otimes \omega_a) * \omega_b(Q \otimes m \otimes Q) * (\omega_c \otimes \varepsilon)](u).$$

Next aim is to check that $[\overline{\omega}_s] \in H_{\mathcal{YD}}^3(\text{gr}Q, \mathbb{k})$ i.e. that

$$\overline{\omega}_s(m_{\overline{Q}} \otimes \overline{Q} \otimes \overline{Q}) + \overline{\omega}_s(\overline{Q} \otimes \overline{Q} \otimes m_{\overline{Q}}) = (\varepsilon_{\overline{Q}} \otimes \overline{\omega}_s) + \overline{\omega}_s(\overline{Q} \otimes m_{\overline{Q}} \otimes \overline{Q}) + (\overline{\omega}_s \otimes \varepsilon_{\overline{Q}}).$$

This is achieved by evaluating the two sides of the equality above on $\overline{u} := \overline{x} \otimes \overline{y} \otimes \overline{z} \otimes \overline{t}$ where x, y, z, t are elements in the basis and using (20). If \overline{u} has homogeneous degree greater than s , then both terms are zero. Otherwise, i.e. if \overline{u} has homogeneous degree at most s , one has $\overline{\omega}_s(m_{\overline{Q}} \otimes \overline{Q} \otimes \overline{Q})(\overline{u}) = \omega_s(m_Q \otimes Q \otimes Q)(u)$ and similarly for the other pieces so that one has to check that

$$\omega_s(m \otimes Q \otimes Q)(u) + \omega_s(Q \otimes Q \otimes m)(u) = (\varepsilon \otimes \omega_s)(u) + \omega_s(Q \otimes m \otimes Q)(u) + (\omega_s \otimes \varepsilon)(u).$$

This equality follows by using (23) and the definition of s .

By assumption $H_{\mathcal{YD}}^3(\text{gr}Q, \mathbb{k}) = 0$ so that there exists a morphism $\overline{v} : \overline{Q} \otimes \overline{Q} \rightarrow \mathbb{k}$ in ${}^H_H\mathcal{YD}$ such that

$$\overline{\omega}_s = \partial^2 \overline{v} = \overline{v} \otimes \varepsilon_{\overline{Q}} - \overline{v}(\overline{Q} \otimes m_{\overline{Q}}) + \overline{v}(m_{\overline{Q}} \otimes \overline{Q}) - \varepsilon_{\overline{Q}} \otimes \overline{v}.$$

Explicitly, on elements in the basis we get

$$\overline{\omega}_s(\overline{x} \otimes \overline{y} \otimes \overline{z}) = \overline{v}(\overline{x} \otimes \overline{y}) \varepsilon_{\overline{Q}}(\overline{z}) - \overline{v}(\overline{x} \otimes \overline{y} \cdot \overline{z}) + \overline{v}(\overline{x} \cdot \overline{y} \otimes \overline{z}) - \varepsilon_{\overline{Q}}(\overline{x}) \overline{v}(\overline{y} \otimes \overline{z}).$$

Define $\overline{\zeta} : \overline{Q} \otimes \overline{Q} \rightarrow \mathbb{k}$ on the basis by setting

$$\overline{\zeta}(\overline{x} \otimes \overline{y}) := \delta_{|x|+|y|,s} \overline{v}(\overline{x} \otimes \overline{y}).$$

As we have done for ω_t , one can check that $\overline{\zeta}$ is a morphism in ${}^H_H\mathcal{YD}$.

Moreover on elements in the basis we get

$$\begin{aligned} & (\partial^2 \overline{\zeta})(\overline{x} \otimes \overline{y} \otimes \overline{z}) \\ &= (\overline{\zeta} \otimes \varepsilon_{\overline{Q}})(\overline{x} \otimes \overline{y} \otimes \overline{z}) - \overline{\zeta}(\overline{Q} \otimes m_{\overline{Q}})(\overline{x} \otimes \overline{y} \otimes \overline{z}) + \overline{\zeta}(m_{\overline{Q}} \otimes \overline{Q})(\overline{x} \otimes \overline{y} \otimes \overline{z}) - (\varepsilon_{\overline{Q}} \otimes \overline{\zeta})(\overline{x} \otimes \overline{y} \otimes \overline{z}) \\ &= \overline{\zeta}(\overline{x} \otimes \overline{y}) \varepsilon_{\overline{Q}}(\overline{z}) - \overline{\zeta}(\overline{x} \otimes \overline{y} \cdot \overline{z}) + \overline{\zeta}(\overline{x} \cdot \overline{y} \otimes \overline{z}) - \varepsilon_{\overline{Q}}(\overline{x}) \overline{\zeta}(\overline{y} \otimes \overline{z}). \end{aligned}$$

By using (20), one gets

$$\overline{\zeta}(\overline{x} \otimes \overline{y} \cdot \overline{z}) = \delta_{|x|+|y|+|z|,s} \overline{v}(\overline{x} \otimes \overline{y} \cdot \overline{z}) \quad \text{and} \quad \overline{\zeta}(\overline{x} \cdot \overline{y} \otimes \overline{z}) = \delta_{|x|+|y|+|z|,s} \overline{v}(\overline{x} \cdot \overline{y} \otimes \overline{z}).$$

By means of these equalities one gets

$$\begin{aligned} (\partial^2 \overline{\zeta})(\overline{x} \otimes \overline{y} \otimes \overline{z}) &= \delta_{|x|+|y|+|z|,s} (\partial^2 \overline{v})(\overline{x} \otimes \overline{y} \otimes \overline{z}) = \delta_{|x|+|y|+|z|,s} \overline{\omega}_s(\overline{x} \otimes \overline{y} \otimes \overline{z}) \\ &= \delta_{|x|+|y|+|z|,s} \omega_s(x \otimes y \otimes z) = \delta_{|x|+|y|+|z|,s} \delta_{|x|+|y|+|z|,s} \omega(x \otimes y \otimes z) \\ &= \delta_{|x|+|y|+|z|,s} \omega(x \otimes y \otimes z) = \omega_s(x \otimes y \otimes z) = \overline{\omega}_s(\overline{x} \otimes \overline{y} \otimes \overline{z}). \end{aligned}$$

Therefore $\partial^2 \overline{\zeta} = \overline{\omega}_s$. This means that we can assume that $\overline{v}(\overline{x} \otimes \overline{y}) = 0$ for $|x| + |y| \neq s$. Equivalently

$$(24) \quad \overline{v}(\overline{x} \otimes \overline{y}) = \delta_{|x|+|y|,s} \overline{v}(\overline{x} \otimes \overline{y}) \text{ for } x, y \text{ in the basis.}$$

Set

$$v := \overline{v} \circ (\varphi \otimes \varphi) \quad \text{and} \quad \gamma := (\varepsilon \otimes \varepsilon) + v.$$

In particular, one gets

$$(25) \quad v(x \otimes y) = \delta_{|x|+|y|,s} v(x \otimes y) \text{ for } x, y \text{ in the basis.}$$

Note also that both v and γ are morphisms in ${}^H_H\mathcal{YD}$ as they are obtained as composition or sum of morphisms in this category. Let us check that γ is a gauge transformation on Q in ${}^H_H\mathcal{YD}$.

Recall that $x^{0,0} = 1_Q$ is in the basis. For x in the basis, we have $\gamma(x \otimes 1_Q) = \varepsilon(x) + v(x \otimes 1_Q)$. Note that

$$\begin{aligned} 0 &= \delta_{|x|,s} \varepsilon(x) = \delta_{|x|+|1_Q|+|1_Q|,s} \omega(x \otimes 1_Q \otimes 1_Q) \\ &= \omega_s(x \otimes 1_Q \otimes 1_Q) = \overline{\omega}_s(\overline{x} \otimes \overline{1_Q} \otimes \overline{1_Q}) \end{aligned}$$

$$\begin{aligned}
&= \bar{v}(\bar{x} \otimes \bar{1}_Q) \varepsilon_{\bar{Q}}(\bar{1}_Q) - \bar{v}(\bar{x} \otimes \bar{1}_Q \cdot \bar{1}_Q) + \bar{v}(\bar{x} \cdot \bar{1}_Q \otimes \bar{1}_Q) - \varepsilon_{\bar{Q}}(\bar{x}) \bar{v}(\bar{1}_Q \otimes \bar{1}_Q) \\
&\stackrel{(24)}{=} \bar{v}(\bar{x} \otimes \bar{1}_Q) - \bar{v}(\bar{x} \otimes \bar{1}_Q) + \bar{v}(\bar{x} \otimes \bar{1}_Q) - \varepsilon_{\bar{Q}}(\bar{x}) \delta_{|1_Q|+|1_Q|,s} \bar{v}(\bar{1}_Q \otimes \bar{1}_Q) \\
&= v(x \otimes 1_Q)
\end{aligned}$$

so that $v(x \otimes 1_Q) = 0$ and hence $\gamma(x \otimes 1_Q) = \varepsilon(x) + v(x \otimes 1_Q) = \varepsilon(x)$. Similarly one proves $\gamma(1_Q \otimes x) = \varepsilon(x)$. Hence γ is unital. Note that the coalgebra $C = Q \otimes Q$ is connected as Q is. Thus, in order to prove that $\gamma : Q \otimes Q \rightarrow \mathbb{k}$ is convolution invertible it suffices to check (see [Mo, Lemma 5.2.10]) that $\gamma|_{\mathbb{k}1_Q \otimes \mathbb{k}1_Q}$ is convolution invertible. But for $k, k' \in \mathbb{k}$ we have

$$\gamma(k1_Q \otimes k'1_Q) = kk' \gamma(1_Q \otimes 1_Q) = kk' \varepsilon(1_Q) = kk' = (\varepsilon \otimes \varepsilon)(k1_Q \otimes k'1_Q)$$

Hence $\gamma|_{\mathbb{k}1_Q \otimes \mathbb{k}1_Q} = (\varepsilon \otimes \varepsilon)|_{\mathbb{k}1_Q \otimes \mathbb{k}1_Q}$ which is convolution invertible. Thus there is a \mathbb{k} -linear map $\gamma^{-1} : Q \otimes Q \rightarrow \mathbb{k}$ and such that

$$\gamma * \gamma^{-1} = \varepsilon \otimes \varepsilon = \gamma^{-1} * \gamma.$$

Note that, by Lemma 2.3, $\gamma \in {}^H_H\mathcal{YD}$ implies $\gamma^{-1} \in {}^H_H\mathcal{YD}$.

Therefore γ is a gauge transformation in ${}^H_H\mathcal{YD}$. By Proposition 2.4, Q^γ is a coquasi-bialgebra in ${}^H_H\mathcal{YD}$. By Proposition 2.6, we have that $\text{gr}Q^\gamma$ and $\text{gr}Q$ coincide as bialgebras in ${}^H_H\mathcal{YD}$. Hence $\text{H}_{\mathcal{YD}}^3(\text{gr}Q^\gamma, \mathbb{k}) = \text{H}_{\mathcal{YD}}^3(\text{gr}Q, \mathbb{k}) = 0$. Therefore Q^γ fulfills the same requirement of Q as in the statement. Let us check that $(\omega^\gamma)_t = 0$ for $1 \leq t \leq s$ (this will complete the proof by an induction process as Q is finite-dimensional).

Note that the definition of γ and (25) imply

$$(26) \quad \gamma(x \otimes y) = \delta_{|x|+|y|,0} \gamma(x \otimes y) + \delta_{|x|+|y|,s} \gamma(x \otimes y) \text{ for } x, y \text{ in the basis.}$$

Let $C^2 = Q \otimes Q$ and let $C_{(n)}^2 = \sum_{i+j \leq n} Q_i \otimes Q_j$. For $u \in C_{(2s-1)}^2$ we have

$$[\gamma * ((\varepsilon \otimes \varepsilon) - v)](u) = (\varepsilon \otimes \varepsilon)(u) - v(u) + v(u) - v(u_1)v(u_2) \stackrel{(25)}{=} (\varepsilon \otimes \varepsilon)(u).$$

Therefore $[\gamma * ((\varepsilon \otimes \varepsilon) - v)]|_{C_{(2s-1)}^2} = (\varepsilon \otimes \varepsilon)|_{C_{(2s-1)}^2}$. By uniqueness of the convolution inverse, we deduce

$$(27) \quad \gamma^{-1}(u) = (\varepsilon \otimes \varepsilon)(u) - v(u), \text{ for } u \in C_{(2s-1)}^2.$$

Let x, y, z be in the basis. Set $\bar{u} := \bar{x} \otimes \bar{y} \otimes \bar{z}$ and $u := x \otimes y \otimes z$. We compute

$$\begin{aligned}
(\omega^\gamma)_s(u) &= \delta_{|x|+|y|+|z|,s} \omega^\gamma(u) \\
&= \delta_{|x|+|y|+|z|,s} [(\varepsilon \otimes \gamma) * \gamma(Q \otimes m) * \omega * \gamma^{-1}(m \otimes Q) * (\gamma^{-1} \otimes \varepsilon)](u) \\
&= \delta_{|x|+|y|+|z|,s} [(\varepsilon \otimes \gamma) * \gamma(Q \otimes m) * (\omega_0 + \omega_s) * \gamma^{-1}(m \otimes Q) * (\gamma^{-1} \otimes \varepsilon)](u) \\
&\stackrel{(22)}{=} \delta_{|x|+|y|+|z|,s} \left[\begin{array}{l} (\varepsilon \otimes \gamma) * \gamma(Q \otimes m) * \gamma^{-1}(m \otimes Q) * (\gamma^{-1} \otimes \varepsilon) + \\ (\varepsilon \otimes \gamma) * \gamma(Q \otimes m) * \omega_s * \gamma^{-1}(m \otimes Q) * (\gamma^{-1} \otimes \varepsilon) \end{array} \right](u) \\
&= \left[\begin{array}{l} \delta_{|x|+|y|+|z|,s} (\varepsilon \otimes \gamma)(u_1) \cdot \gamma(Q \otimes m)(u_2) \cdot \gamma^{-1}(m \otimes Q)(u_3) \cdot (\gamma^{-1} \otimes \varepsilon)(u_4) + \\ \delta_{|x|+|y|+|z|,s} (\varepsilon \otimes \gamma)(u_1) \cdot \gamma(Q \otimes m)(u_2) \cdot \omega_s(u_3) \cdot \gamma^{-1}(m \otimes Q)(u_4) \cdot (\gamma^{-1} \otimes \varepsilon)(u_5) \end{array} \right].
\end{aligned}$$

Now, all terms appearing in the last two lines, excepted ω_s , vanish out of degrees 0 and s and coincide with $\varepsilon \otimes \varepsilon \otimes \varepsilon$ on degree 0. On the other hand ω_s vanishes out of s . Since $\gamma := (\varepsilon \otimes \varepsilon) + v$ and in view of (27), the term $\delta_{|x|+|y|+|z|,s}$ forces the following simplification

$$(\omega^\gamma)_s(u) = \left[\begin{array}{l} \delta_{|x|+|y|+|z|,s} [(\varepsilon \otimes v)(u) + v(Q \otimes m)(u) - v(m \otimes Q)(u) - (v \otimes \varepsilon)(u)] + \\ + \delta_{|x|+|y|+|z|,s} \omega_s(u) \end{array} \right].$$

Now $\omega_s(u) = \bar{\omega}_s(\bar{u})$ while one proves that $(\varepsilon \otimes v)(u) = (\varepsilon_{\bar{Q}} \otimes \bar{v})(\bar{u})$, $\delta_{|x|+|y|+|z|,s} v(m \otimes Q)(u) = \delta_{|x|+|y|+|z|,s} \bar{v}(m_{\bar{Q}} \otimes \bar{Q})(\bar{u})$ and similarly for the other pieces of the equality.

Thus one gets

$$(\omega^\gamma)_s(u) = \left[\begin{array}{l} \delta_{|x|+|y|+|z|,s} \left[(\varepsilon_{\bar{Q}} \otimes \bar{v})(\bar{u}) + \bar{v}(\bar{Q} \otimes m_{\bar{Q}})(\bar{u}) - \bar{v}(m_{\bar{Q}} \otimes \bar{Q})(\bar{u}) - (\bar{v} \otimes \varepsilon_{\bar{Q}})(\bar{u}) \right] + \\ + \delta_{|x|+|y|+|z|,s} \bar{\omega}_s(\bar{u}) \end{array} \right]$$

$$= -\delta_{|x|+|y|+|z|,s} \partial^2 \bar{v} + \delta_{|x|+|y|+|z|,s} \bar{\omega}_s(\bar{u}) = 0.$$

For $0 \leq t \leq s-1$, analogously to the above, we compute

$$\begin{aligned} (\omega^\gamma)_t(u) &= \delta_{|x|+|y|+|z|,t} \omega^\gamma(u) \\ &= \delta_{|x|+|y|+|z|,t} [(\varepsilon \otimes \gamma) * \gamma(Q \otimes m) * \omega * \gamma^{-1}(m \otimes Q) * (\gamma^{-1} \otimes \varepsilon)](u) \\ &= \delta_{|x|+|y|+|z|,t} [(\varepsilon \otimes \gamma) * \gamma(Q \otimes m) * \omega_0 * \gamma^{-1}(m \otimes Q) * (\gamma^{-1} \otimes \varepsilon)](u) \\ &\stackrel{(22)}{=} \delta_{|x|+|y|+|z|,t} [(\varepsilon \otimes \gamma) * \gamma(Q \otimes m) * \gamma^{-1}(m \otimes Q) * (\gamma^{-1} \otimes \varepsilon)](u) \\ &= \delta_{|x|+|y|+|z|,t} (\varepsilon \otimes \varepsilon \otimes \varepsilon)(u) = \delta_{0,t} (\varepsilon \otimes \varepsilon \otimes \varepsilon)(u). \end{aligned}$$

Therefore we can now repeat the argument on ω^γ instead of ω . Deforming several times we will get a reassociator, say ω' , whose first non trivial component ω'_t , with $t \neq 0$, exceeds the dimension of Q . In other words $\omega' = \omega'_0$ which is trivial. Hence Q is gauge equivalent to a connected bialgebra in ${}^H_H\mathcal{YD}$. \square

4. INVARIANTS

Given a \mathbb{k} -algebra A , we denote by $H^n(A, -)$ the n -th right derived functor of $\text{Hom}_{A,A}(A, -)$ in the category of A -bimodules. In other words, for every A -bimodule M , $H^n(A, M)$ is the Hochschild cohomology group of A with coefficients in M . Denote by $Z^n(A, M)$ and $B^n(A, M)$ the abelian groups of n -cocycles and of n -coboundaries respectively.

Let H be a Hopf algebra, let B be a left H -module algebra and let M be a $B\#H$ -bimodule, where $B\#H$ denotes the smash product algebra, see e.g. [Mo, Definition 4.1.3]. Then $H^n(B, M)$ becomes an H -bimodule as follows. Its structure of left H -module is given via ε_H and its structure of right H -module is defined, for every $f \in Z^n(B, M)$ and $h \in H$, by setting

$$[f]h := [\chi_n^h(M)(f)]$$

where, for every $k \in \mathbb{k}, b_1, \dots, b_n \in B$, we set

$$\begin{aligned} \chi_0^h(M)(f)(k) &:= (1_B \# S(h_1)) f(k) (1_B \# h_2) \text{ for } n=0 \text{ while and for } n \geq 1 \\ \chi_n^h(M)(f)(b_1 \otimes b_2 \otimes \dots \otimes b_n) &:= (1_B \# S(h_1)) f(h_2 b_1 \otimes h_3 b_2 \otimes \dots \otimes h_{n+1} b_n) (1_B \# h_{n+2}). \end{aligned}$$

Moreover

$$(28) \quad \partial^n \circ \chi_n^h(M) = \chi_{n+1}^h(M) \circ \partial^n, \text{ for every } n \geq -1,$$

where $\partial^n : \text{Hom}_{\mathbb{k}}(B^{\otimes n}, M) \rightarrow \text{Hom}_{\mathbb{k}}(B^{\otimes(n+1)}, M)$ denotes the differential of the usual Hochschild cohomology.

Denote by $H^n(B, M)^H$ the space of H -invariant elements of $H^n(B, M)$.

PROPOSITION 4.1. *Let H be a semisimple Hopf algebra and let B be a left H -module algebra. Denote by $A := B\#H$. Then, for each $n \in \mathbb{N}_0$ and for every A -bimodule M*

$$H^n(B\#H, M) \cong H^n(B, M)^H.$$

Proof. We will apply [Stf, Equation (3.6.1)]. To this aim we have to prove first that A/B is an H -Galois extension such that A is flat as left and right B -module. Now, $A = B\#_\xi H$ for $\xi : H \otimes H \rightarrow B$ defined by $\xi(x, y) = \varepsilon_H(x) \varepsilon_H(y) 1_A$, cf. [Mo, Definition 7.1.1]. Moreover a direct computation shows that $\iota : B \rightarrow A : b \mapsto b\#1_H$ is a right H -extension where A is regarded as a right H -comodule via $\rho : A \rightarrow A \otimes H : b\#h \mapsto (b\#h_1) \otimes h_2$. Thus, by [Mo, Proposition 7.2.7], we know that $\iota : B \rightarrow A$ is H -cleft and hence, by [Mo, Theorem 8.2.4], it is H -Galois. The B -bimodule structure of A is induced by ι so that, explicitly, we have

$$\begin{aligned} b'(b\#h) &= (b'\#1_H)(b\#h) = b'b\#h, \\ (b\#h)b' &= (b\#h)(b'\#1_H) = b(h_1 b')\#h_2. \end{aligned}$$

Note that $A = B\#H$ is flat as a left B -module as H is a free \mathbb{k} -module (\mathbb{k} is a field). Now consider the map $\alpha : H \otimes B \rightarrow A$ defined by setting $\alpha(h \otimes b) := h_1 b \otimes h_2$ (note that it is defined as the braiding in ${}^H_H\mathcal{YD}$). We have

$$\alpha(h \otimes bb') = h_1(bb') \otimes h_2 = (h_1 b)(h_2 b') \otimes h_3 = (h_1 b\#h_2)b' = \alpha(h \otimes b)b'$$

so that α is right B -linear where $H \otimes B$ is regarded as a right module via $(h\#b)b' := h\#bb'$. Now H is semisimple and hence separable (see [Stf, Corollary 3.7]). Thus H is finite-dimensional and hence it has bijective antipode S_H . Thus α is invertible with inverse given by $\alpha^{-1}(b\#h) := h_2 \otimes S_H^{-1}(h_1)b$. Therefore α is an isomorphism of right B -modules and hence A is flat as a right B -module as $H \otimes B$ is.

We have now the hypotheses necessary to apply [Stf, Equation (3.6.1)] and obtain

$$\mathrm{H}^n(A, M) \cong \mathrm{Hom}_{-,H}(\mathbb{k}, \mathrm{H}^n(B, M)) = \mathrm{Hom}_{\mathbb{k}}(\mathbb{k}, \mathrm{H}^n(B, M))^H \cong \mathrm{H}^n(B, M)^H.$$

□

REMARK 4.2. Proposition 4.1 in the particular case when $M = \mathbb{k}$ and B is finite-dimensional is [SV, Theorem 2.17]. Note that in the notations therein, one has $E(B) = \bigoplus_{n \in \mathbb{N}_0} E_n(B, \mathbb{k})$ where $E_n(B, \mathbb{k}) = \mathrm{Ext}_{\mathbb{D}}^n(\mathbb{k}, \mathbb{k}) \cong \mathrm{H}^n(B, \mathbb{k})$. The latter isomorphism is [CE, Corollary 4.4, page 170].

Let H be a Hopf algebra and let B be a bialgebra in the braided category ${}^H_H\mathcal{YD}$. Denote by $A := B\#H$ the Radford-Majid bosonization of B by H , see e.g. [Ra3, Theorem 1]. Note that A is endowed with an algebra map $\varepsilon_A : A \rightarrow \mathbb{k}$ defined by $\varepsilon_A(b\#h) = \varepsilon_B(b)\varepsilon_H(h)$ so that we can regard \mathbb{k} as an A -bimodule via ε_A . Then we can consider $\mathrm{H}^n(B, \mathbb{k})$ as an H -bimodule as follows. Its structure of left H -module is given via ε_H and its structure of right H -module is defined, for every $f \in \mathrm{Z}^n(B, \mathbb{k})$ and $h \in H$, by setting

$$[f]h := [fh],$$

where $(fh)(z) = f(hz)$, for every $z \in B^{\otimes n}$. The latter is the usual right H -module structure of $\mathrm{Hom}_{\mathbb{k}}(B^{\otimes n}, \mathbb{k})$. Indeed, for every $n \geq -1$, the vector space $\mathrm{Hom}_{\mathbb{k}}(B^{\otimes n}, \mathbb{k})$ is an H -bimodule with respect to this right H -module structure and the left one induced by ε_H .

COROLLARY 4.3. *Let H be a semisimple Hopf algebra and let B be a bialgebra in the braided category ${}^H_H\mathcal{YD}$. Set $A := B\#H$. Then, for each $n \in \mathbb{N}_0$*

$$\mathrm{H}^n(B\#H, \mathbb{k}) \cong \mathrm{H}^n(B, \mathbb{k})^H$$

and the differential $\partial^n : \mathrm{Hom}_{\mathbb{k}}(B^{\otimes n}, \mathbb{k}) \rightarrow \mathrm{Hom}_{\mathbb{k}}(B^{\otimes(n+1)}, \mathbb{k})$ of the usual Hochschild cohomology is H -bilinear.

Proof. In the particular case $M = \mathbb{k}$, the right module H -structure used in Proposition 4.1 simplifies as follows. It is defined, for every $f \in \mathrm{Z}^n(B, \mathbb{k})$ and $h \in H$, by setting

$$[f]h := [\chi_n^h(\mathbb{k})(f)]$$

where, for every $k \in \mathbb{k}, b_1, \dots, b_n \in B$, we set

$$\begin{aligned} \chi_0^h(\mathbb{k})(f)(k) & : = \varepsilon_H(h)f(k) \text{ for } n = 0 \text{ while and for } n \geq 1 \\ \chi_n^h(\mathbb{k})(f)(b_1 \otimes b_2 \otimes \dots \otimes b_n) & : = f(h_1 b_1 \otimes h_2 b_2 \otimes \dots \otimes h_n b_n). \end{aligned}$$

More concisely $\chi_n^h(\mathbb{k})(f)(z) = f(hz)$ for every $n \in \mathbb{N}_0$ and $z \in B^{\otimes n}$ i.e. $[f]h := [fh]$ where $fh := \chi_n^h(\mathbb{k})(f)$.

Now consider the differential $\partial^n : \mathrm{Hom}_{\mathbb{k}}(B^{\otimes n}, \mathbb{k}) \rightarrow \mathrm{Hom}_{\mathbb{k}}(B^{\otimes(n+1)}, \mathbb{k})$ of the usual Hochschild cohomology. Note that for each $n \in \mathbb{N}_0$, $\mathrm{Hom}_{\mathbb{k}}(B^{\otimes n}, \mathbb{k})$ is regarded as a bimodule over H using the left H -module structures of its arguments. By (28), we have

$$\partial^n \chi_n^h(\mathbb{k})(f) = \chi_{n+1}^h(\mathbb{k})\partial^n(f)$$

Since $\chi_n^h(\mathbb{k})(f) = fh$, the last displayed equality becomes $\partial^n(fh) = \partial^n(f)h$ for every $n \in \mathbb{N}_0$. Thus ∂^n is right H -linear. Since $hf = \varepsilon_H(h)f$ for every $f \in \mathrm{Hom}_{\mathbb{k}}(B^{\otimes n}, \mathbb{k}), h \in H$, we get that ∂^n is also left H -linear whence H -bilinear. □

REMARK 4.4. Note that, in the context of the proof of [EG, Proposition 5.1], one has

$$\mathrm{H}^3(\mathcal{B}(V) \# \mathbb{C}[\mathbb{Z}_p], \mathbb{C}) \cong \mathrm{H}^3(\mathcal{B}(V), \mathbb{C})^{\mathbb{Z}_p}.$$

This is a particular case of Corollary 4.3 where $H = \mathbb{C}[\mathbb{Z}_p]$, $V \in {}_H^H\mathcal{YD}$ and $B = \mathcal{B}(V)$.

PROPOSITION 4.5. *Let \mathcal{C} and \mathcal{D} be abelian categories. Let $r, \omega : \mathcal{C} \rightarrow \mathcal{D}$ be exact functors such that r is a subfunctor of ω i.e. there is a natural transformation $\eta : r \rightarrow \omega$ which is a monomorphism when evaluated on objects. If X is a subobject of Y then $r(X) = \omega(X) \cap r(Y)$. Moreover, for every morphism $f : X \rightarrow Y$ in \mathcal{C} one has*

$$\begin{aligned} \ker(r(f)) &= r(\ker(f)) = \omega(\ker(f)) \cap r(X) = \ker(\omega(f)) \cap r(X), \\ \mathrm{Im}(r(f)) &= \mathrm{Im}(\omega(f)) \cap r(Y) = r(\mathrm{Im}(f)). \end{aligned}$$

Proof. The proof is similar to [Stn, Proposition 1.7, page 138]. □

REMARK 4.6. From Corollary 4.3, we have

$$\begin{aligned} \mathrm{H}^n(B, \mathbb{k})^H &= \{[f] \mid f \in Z^n(B, \mathbb{k}), \varepsilon_H(h)[f] = [f]h, \text{ for every } h \in H\} \\ &= \{[f] \mid f \in Z^n(B, \mathbb{k}), [\varepsilon_H(h)f] = [fh], \text{ for every } h \in H\} \end{aligned}$$

where, for every $z \in B^{\otimes n}$, we have

$$(fh)(z) = f(hz).$$

Note that, for any H -bimodule M one has

$$\mathrm{Hom}_{H,H}(H, M) \cong M^H = \{m \in M \mid hm = mh, \text{ for every } h \in H\}.$$

Note also that H is a separable \mathbb{k} -algebra whence it is projective in the category of H -bimodules. As a consequence $\mathrm{Hom}_{H,H}(H, -) \cong (-)^H : {}_H\mathfrak{M}_H \rightarrow \mathfrak{M}$ is an exact functor (here ${}_H\mathfrak{M}_H$ is the category of H -bimodules and \mathfrak{M} the category of \mathbb{k} -vector spaces). By Proposition 4.5 applied to the case when $r := (-)^H : {}_H\mathfrak{M}_H \rightarrow \mathfrak{M}$ and ω is the forgetful functor, for every morphism $f : X \rightarrow Y$ of H -bimodules one has

$$\ker(f^H) = \ker(f) \cap X^H = (\ker(f))^H \quad \text{and} \quad \mathrm{Im}(f^H) = \mathrm{Im}(f) \cap Y^H = (\mathrm{Im}(f))^H.$$

Still by Corollary 4.3, we know that the differential $\partial^n : \mathrm{Hom}_{\mathbb{k}}(B^{\otimes n}, \mathbb{k}) \rightarrow \mathrm{Hom}_{\mathbb{k}}(B^{\otimes(n+1)}, \mathbb{k})$ of the usual Hochschild cohomology is H -bilinear. Thus we can apply the argument above to get

$$\begin{aligned} \ker\left((\partial^n)^H\right) &= \ker(\partial^n) \cap \mathrm{Hom}_{\mathbb{k}}(B^{\otimes n}, \mathbb{k})^H = (\ker(\partial^n))^H \quad \text{and} \\ \mathrm{Im}\left((\partial^{n-1})^H\right) &= \mathrm{Im}(\partial^{n-1}) \cap \mathrm{Hom}_{\mathbb{k}}(B^{\otimes n}, \mathbb{k})^H = (\mathrm{Im}(\partial^{n-1}))^H. \end{aligned}$$

Now $\mathrm{Hom}_{\mathbb{k}}(B^{\otimes n}, \mathbb{k})^H = \mathrm{Hom}_{H,-}(B^{\otimes n}, \mathbb{k})$ so that we get

$$\begin{aligned} Z_{H\text{-Mod}}^n(B, \mathbb{k}) &= Z^n(B, \mathbb{k}) \cap \mathrm{Hom}_{H,-}(B^{\otimes n}, \mathbb{k}) = Z^n(B, \mathbb{k})^H \quad \text{and} \\ B_{H\text{-Mod}}^n(B, \mathbb{k}) &= B^n(B, \mathbb{k}) \cap \mathrm{Hom}_{H,-}(B^{\otimes n}, \mathbb{k}) = B^n(B, \mathbb{k})^H. \end{aligned}$$

where $Z_{H\text{-Mod}}^n(B, \mathbb{k})$ and $B_{H\text{-Mod}}^n(B, \mathbb{k})$ denotes the the abelian groups of n -cocycles, of n -coboundaries for the cohomology of the algebra B with coefficients in \mathbb{k} computed in the monoidal category $H\text{-Mod}$ of left H -modules. The corresponding n -th Hochschild cohomology group is

$$\mathrm{H}_{H\text{-Mod}}^n(B, \mathbb{k}) := \frac{Z_{H\text{-Mod}}^n(B, \mathbb{k})}{B_{H\text{-Mod}}^n(B, \mathbb{k})} = \frac{Z^n(B, \mathbb{k})^H}{B^n(B, \mathbb{k})^H} \cong \left(\frac{Z^n(B, \mathbb{k})}{B^n(B, \mathbb{k})} \right)^H = \mathrm{H}^n(B, \mathbb{k})^H.$$

Denote by $D(H)$ the Drinfeld double, see e.g. the first structure of [Ma], Theorem 7.1.1].

PROPOSITION 4.7. *In the setting of Corollary 4.3 assume that H is also cosemisimple. Then, for $n \in \mathbb{N}_0$*

$Z_{\mathcal{YD}}^n(B, \mathbb{k}) = Z^n(B, \mathbb{k})^{D(H)}$, $B_{\mathcal{YD}}^n(B, \mathbb{k}) = B^n(B, \mathbb{k})^{D(H)}$ and $\mathrm{H}_{\mathcal{YD}}^n(B, \mathbb{k}) \cong \mathrm{H}^n(B, \mathbb{k})^{D(H)}$.
where $Z^n(B, \mathbb{k})$ and $B^n(B, \mathbb{k})$ are regarded as $D(H)$ -subbimodules of $\mathrm{Hom}_{\mathbb{k}}(B^{\otimes n}, \mathbb{k})$ whose structure is induced by the left $D(H)$ -module structures of its arguments.

Moreover $H^n(B, \mathbb{k})^{D(H)}$ is a subspace of $H^n(B, \mathbb{k})^H$.

Proof. For shortness, in this proof, we denote $D(H)$ by D . Consider the analogue of the standard complex as in Remark 3.1

$${}^H_H\mathcal{YD}(\mathbb{k}, \mathbb{k}) \xrightarrow{\partial^0} {}^H_H\mathcal{YD}(B, \mathbb{k}) \xrightarrow{\partial^1} {}^H_H\mathcal{YD}(B^{\otimes 2}, \mathbb{k}) \xrightarrow{\partial^2} \dots$$

where ∂^n is induced by the differential $\partial^n : \text{Hom}_{\mathbb{k}}(B^{\otimes n}, \mathbb{k}) \rightarrow \text{Hom}_{\mathbb{k}}(B^{\otimes(n+1)}, \mathbb{k})$ of the ordinary Hochschild cohomology. Now, since H is semisimple, it is finite-dimensional (whence it has bijective antipode) so that, by a result essentially due to Majid (see [Mo, Proposition 10.6.16]) and by [RT, Proposition 6], we get a category isomorphism ${}^H_H\mathcal{YD} \cong {}_D\mathfrak{M}$. Thus the complex above can be rewritten as follows

$$\text{Hom}_{D,-}(\mathbb{k}, \mathbb{k}) \xrightarrow{\partial^0} \text{Hom}_{D,-}(B, \mathbb{k}) \xrightarrow{\partial^1} \text{Hom}_{D,-}(B^{\otimes 2}, \mathbb{k}) \xrightarrow{\partial^2} \dots$$

Now, since, for each $n \in \mathbb{N}_0$, we have $\text{Hom}_{D,-}(B^{\otimes n}, \mathbb{k}) = \text{Hom}_{\mathbb{k}}(B^{\otimes n}, \mathbb{k})^D$, we obtain the complex

$$\text{Hom}_{\mathbb{k}}(\mathbb{k}, \mathbb{k})^D \xrightarrow{\partial^0} \text{Hom}_{\mathbb{k}}(B, \mathbb{k})^D \xrightarrow{\partial^1} \text{Hom}_{\mathbb{k}}(B^{\otimes 2}, \mathbb{k})^D \xrightarrow{\partial^2} \dots$$

We will write $(\partial^n)^D$ instead of ∂^n when we would like to stress that the map considered is the one induced on invariants. Thus we will write equivalently

$$\text{Hom}_{\mathbb{k}}(\mathbb{k}, \mathbb{k})^D \xrightarrow{(\partial^0)^D} \text{Hom}_{\mathbb{k}}(B, \mathbb{k})^D \xrightarrow{(\partial^1)^D} \text{Hom}_{\mathbb{k}}(B^{\otimes 2}, \mathbb{k})^D \xrightarrow{(\partial^2)^D} \dots$$

Now, assume H is also cosemisimple. Since H is both semisimple and cosemisimple, by [Ra2, Proposition 7] the Hopf algebra D is semisimple as an algebra. Thus, as in Remark 4.6 in case of H , the functor $(-)^D : {}_D\mathfrak{M}_D \rightarrow \mathfrak{M}$ is exact (here ${}_D\mathfrak{M}_D$ is the category of D -bimodules and \mathfrak{M} the category of \mathbb{k} -vector spaces). By Proposition 4.5 applied to the case when $r := (-)^D : {}_D\mathfrak{M}_D \rightarrow \mathfrak{M}$ and ω is the forgetful functor, for every morphism $f : X \rightarrow Y$ of D -bimodules one has

$$\ker(f^D) = \ker(f) \cap X^D = (\ker(f))^D \quad \text{and} \quad \text{Im}(f^D) = \text{Im}(f) \cap Y^D = (\text{Im}(f))^D.$$

In particular we get

$$\begin{aligned} \ker\left((\partial^n)^D\right) &= \ker(\partial^n) \cap \text{Hom}_{\mathbb{k}}(B^{\otimes n}, \mathbb{k})^D = \ker(\partial^n)^D \quad \text{and} \\ \text{Im}\left((\partial^{n-1})^D\right) &= \text{Im}(\partial^{n-1}) \cap \text{Hom}_{\mathbb{k}}(B^{\otimes n}, \mathbb{k})^D = \text{Im}(\partial^{n-1})^D \end{aligned}$$

and hence

$$\begin{aligned} Z_{\mathcal{YD}}^n(B, \mathbb{k}) &= Z^n(B, \mathbb{k}) \cap \text{Hom}_{D,-}(B^{\otimes n}, \mathbb{k}) = Z^n(B, \mathbb{k})^D \quad \text{and} \\ B_{\mathcal{YD}}^n(B, \mathbb{k}) &= B^n(B, \mathbb{k}) \cap \text{Hom}_{D,-}(B^{\otimes n}, \mathbb{k}) = B^n(B, \mathbb{k})^D \end{aligned}$$

Then we obtain

$$H_{\mathcal{YD}}^n(B, \mathbb{k}) = \frac{Z_{\mathcal{YD}}^n(B, \mathbb{k})}{B_{\mathcal{YD}}^n(B, \mathbb{k})} = \frac{Z^n(B, \mathbb{k})^D}{B^n(B, \mathbb{k})^D} \cong H^n(B, \mathbb{k})^D.$$

Let us prove the last part of the statement. The correspondence between the left D -module structure and the structure of Yetter-Drinfeld module over H is written explicitly in [Maj, Proposition 7.1.6]. In particular $D = H^* \otimes H$ and given $V \in {}^H_H\mathcal{YD}$, the two structures are related by the following equality $(f \otimes h) \triangleright v = f((h \triangleright v)_{-1})(h \triangleright v)_0$ for every $f \in H^*, h \in H, v \in V$. Thus $(\varepsilon_H \otimes h) \triangleright v = h \triangleright v$. Moreover H is a Hopf subalgebra of D via $h \mapsto \varepsilon_H \otimes h$, where D is considered with the first structure of [Maj, Theorem 7.1.1]. Since the D -bimodule structure of $H^n(B, \mathbb{k})$ is induced by the one of $\text{Hom}_{\mathbb{k}}(B^{\otimes n}, \mathbb{k})$ which comes from the left D -module structures of its arguments and similarly for the H -bimodule structure of $H^n(B, \mathbb{k})$, we deduce that $H^n(B, \mathbb{k})^D$ is a subspace of $H^n(B, \mathbb{k})^H$. \square

EXAMPLE 4.8. In the setting of the proof of [An, Theorem 4.1.3], a Nichols algebra $\mathcal{B}(V)$ such that $H^3(\mathcal{B}(V), \mathbb{k})^{\mathbb{Z}^m} = 0$ is considered where \mathbb{k} is a field of characteristic zero. By Proposition 4.7 applied in the case $H = \mathbb{k}\mathbb{Z}_m$ and $B = \mathcal{B}(V)$, we have that $H_{\mathcal{YD}}^3(\mathcal{B}(V), \mathbb{k}) \cong H^3(\mathcal{B}(V), \mathbb{k})^{D(H)}$ is a subspace of $H^3(\mathcal{B}(V), \mathbb{k})^H = H^3(\mathcal{B}(V), \mathbb{k})^{\mathbb{Z}^m} = 0$. Thus we get $H_{\mathcal{YD}}^3(\mathcal{B}(V), \mathbb{k}) = 0$. Therefore, in view of Theorem 3.2, if $(Q, m, u, \Delta, \varepsilon, \omega)$ is a f.d. connected coquasi-bialgebra in ${}^H_H\mathcal{YD}$ such that $\text{gr}Q \cong \mathcal{B}(V)$ (as above) as augmented algebras in ${}^H_H\mathcal{YD}$ (the counit must be the same in order to have the same Yetter-Drinfeld module structure on \mathbb{k}), then we can conclude that Q is gauge equivalent to a connected bialgebra in ${}^H_H\mathcal{YD}$.

REMARK 4.9. Let A be a finite-dimensional coquasi-bialgebra with the dual Chevalley property i.e. the coradical H of A is a coquasi-subbialgebra of A (in particular H is cosemisimple). Assume the coquasi-bialgebra structure of H has trivial reassociator (i.e. it is an ordinary bialgebra) and also assume it has an antipode (i.e. it is a Hopf algebra). Then, by [AP, Corollary 6.4], $\text{gr}A$ is isomorphic to $R\#H$ as a coquasi-bialgebra, where R is a suitable connected bialgebra in ${}^H_H\mathcal{YD}$. Note that $R\#H$ is the usual Radford-Majid bosonization as H has trivial reassociator, see [AP, Definition 5.4]. Hence we can compute

$$H^3(\text{gr}A, \mathbb{k}) = H^3(R\#H, \mathbb{k}).$$

Assume further that H is semisimple. Then, by Corollary 4.3, we have

$$H^n(R\#H, \mathbb{k}) \cong H^n(R, \mathbb{k})^H$$

so that $H^3(\text{gr}A, \mathbb{k}) \cong H^3(R, \mathbb{k})^H$. Thus, if $H^3(R, \mathbb{k})^H = 0$, one gets $H^3(\text{gr}A, \mathbb{k}) = 0$ which is the analogue of the condition [EG, Proposition 2.3] (note that our A is the dual of the one considered therein) which guarantees that A is gauge equivalent to an ordinary Hopf algebra, if A has a quasi-antipode and $\mathbb{k} = \mathbb{C}$. Next we will give another approach to arrive at the same conclusion but just requiring $H_{\mathcal{YD}}^3(R, \mathbb{k}) = 0$. Note that a priori $H_{\mathcal{YD}}^3(R, \mathbb{k}) \cong H^3(R, \mathbb{k})^{D(H)}$ is smaller than $H^3(R, \mathbb{k})^H$.

5. DUAL CHEVALLEY

The main aim of this section is to prove Theorem 5.6. Let A be a Hopf algebra over a field \mathbb{k} of characteristic zero such that the coradical H of A is a sub-Hopf algebra (i.e. A has the dual Chevalley Property). Assume H is finite-dimensional so that H is semisimple. By [ABM, Theorem I], there is a gauge transformation $\zeta : A \otimes A \rightarrow \mathbb{k}$ such that A^ζ is isomorphic, as a coquasi-bialgebra, to the bosonization $Q\#H$ of a connected coquasi-bialgebra Q in ${}^H_H\mathcal{YD}$ by H . By construction ζ is H -bilinear and H -balanced: this follows from [ABM, Proposition 5.7] (note that gauge transformation $v_B : B \otimes B \rightarrow \mathbb{k}$, used therein for $B := R\#_\xi H$, is H -bilinear and H -balanced, as observed in the proof) and the fact that there is an H -bilinear Hopf algebra isomorphism $\psi : B \rightarrow A$ (see [ABM, Proof of Theorem I, page 36 and Theorem 6.1] which is a consequence of [AMS1, Theorem 3.64]) where (R, ξ) is a suitable connected pre-bialgebra with cocycle in ${}^H_H\mathcal{YD}$ (note that $\zeta = v_B \circ (\psi^{-1} \otimes \psi^{-1})$): here by connected pre-bialgebra we mean that the coradical R_0 of R is $\mathbb{k}1_R$ (by the properties of 1_R this implies that R_0 is a subcoalgebra in ${}^H_H\mathcal{YD}$ of R). Assume that A is finite-dimensional. Then $Q\#H$ and hence Q is finite dimensional.

Thus, by Theorem 3.2, if $H_{\mathcal{YD}}^3(\text{gr}Q, \mathbb{k}) = 0$, then Q is gauge equivalent to a connected bialgebra in ${}^H_H\mathcal{YD}$.

First let us check which condition on A guarantee that $H_{\mathcal{YD}}^3(\text{gr}Q, \mathbb{k}) = 0$. Note that by construction $Q = R^v$ (see [ABM, Proposition 5.7]) where $v := (\lambda\xi)^{-1}$, the convolution inverse of $\lambda\xi$ and $\lambda : H \rightarrow \mathbb{k}$ denotes the total integral on H . Thus we can rewrite $\text{gr}(Q)$ as $\text{gr}(R^v)$.

Moreover v_B is given by $v_B((r\#h) \otimes (r'\#h')) = v(r \otimes hr') \varepsilon_H(h')$ for every $r, r' \in R, h, h' \in H$. By [AMStu, Proposition 2.5], $\text{gr}(R)$ inherits the pre-bialgebra structure in ${}^H_H\mathcal{YD}$ of R . This is proved by checking that $R_i \cdot R_j \subseteq R_{i+j}$ for every $i, j \in \mathbb{N}_0$, where R_i denotes the i -th term of the coradical filtration of R . Moreover R_i is a subcoalgebra of R in ${}^H_H\mathcal{YD}$.

LEMMA 5.1. *Keep the above hypotheses and notations. Then $\text{gr}(R^v)$ and $\text{gr}(R)$ coincide as bialgebras in ${}^H_H\mathcal{YD}$ where the structures of $\text{gr}(R)$ are induced by the ones of (R, ξ) .*

Proof. By Theorem 1.5, $\text{gr}(R^v) = \text{gr}(Q)$ is a connected bialgebras in ${}^H_H\mathcal{YD}$.

Note that R^v and R coincide as coalgebras in ${}^H_H\mathcal{YD}$ so that $\text{gr}(R^v)$ and $\text{gr}(R)$ coincide as coalgebras in ${}^H_H\mathcal{YD}$. They also have the same unit. It remains to check that their two multiplications coincide too.

Since ξ is unital, by [AMS1, Proposition 4.8], we have that v is unital and this is equivalent to v^{-1} unital (see the proof therein).

Let $C := R \otimes R$. Let $n > 0$ and let $w \in C_{(n)} = \sum_{i+j \leq n} R_i \otimes R_j$. By [AMS1, Lemma 3.69], we have that

$$\Delta_C(w) - w \otimes (1_R)^{\otimes 2} - (1_R)^{\otimes 2} \otimes w \in C_{(n-1)} \otimes C_{(n-1)}.$$

Thus we get

$$w_1 \otimes w_2 \otimes w_3 - \Delta_C(w) \otimes (1_R)^{\otimes 2} - \Delta_C\left((1_R)^{\otimes 2}\right) \otimes w \in \Delta_C(C_{(n-1)}) \otimes C_{(n-1)}$$

and hence

$$w_1 \otimes w_2 \otimes w_3 - w \otimes (1_R)^{\otimes 2} \otimes (1_R)^{\otimes 2} - (1_R)^{\otimes 2} \otimes w \otimes (1_R)^{\otimes 2} - (1_R)^{\otimes 4} \otimes w \in C_{(n-1)} \otimes C_{(n-1)} \otimes C_{(n-1)}.$$

Since $m(C_{(n-1)}) \subseteq \sum_{i+j \leq n} m(R_i \otimes R_j) \subseteq R_{n-1}$ we get

$$w_1 \otimes m(w_2) \otimes w_3 - w \otimes 1_R \otimes (1_R)^{\otimes 2} - (1_R)^{\otimes 2} \otimes m(w) \otimes (1_R)^{\otimes 2} - (1_R)^{\otimes 3} \otimes w \in C_{(n-1)} \otimes R_{n-1} \otimes C_{(n-1)}$$

and hence

$$(29) \quad w_1 \otimes (m(w_2) + R_{n-1}) \otimes w_3 = (1_R)^{\otimes 2} \otimes (m(w) + R_{n-1}) \otimes (1_R)^{\otimes 2}.$$

Let $x, y \in R$. We compute

$$\begin{aligned} \bar{x} \cdot_v \bar{y} &= (x + R_{|x|-1}) \cdot_v (y + R_{|y|-1}) \\ &= (x \cdot_v y) + R_{|x|+|y|-1} = m^v(x \otimes y) + R_{|x|+|y|-1} \\ &= v((x \otimes y)_1) m((x \otimes y)_2) v^{-1}((x \otimes y)_3) + R_{|x|+|y|-1} \\ &= v((x \otimes y)_1) (m((x \otimes y)_2) + R_{|x|+|y|-1}) v^{-1}((x \otimes y)_3) \\ &\stackrel{(29)}{=} v\left((1_R)^{\otimes 2}\right) (m(x \otimes y) + R_{|x|+|y|-1}) v^{-1}\left((1_R)^{\otimes 2}\right) \\ &= m(x \otimes y) + R_{|x|+|y|-1} = (x \cdot y) + R_{|x|+|y|-1} = \bar{x} \cdot \bar{y}. \end{aligned}$$

□

The following result is inspired by [AMS1, Theorem 3.71].

LEMMA 5.2. *Let H be a cosemisimple Hopf algebra. Let C be a left H -comodule coalgebra such that C_0 is a one-dimensional left H -comodule subcoalgebra of C . Let $B = C \# H$ be the smash coproduct of C by H i.e. the coalgebra defined by*

$$(30) \quad \begin{aligned} \Delta_B(c \# h) &= \sum \left(c_1 \# (c_2)_{\langle -1 \rangle} h_1 \right) \otimes \left((c_2)_{\langle 0 \rangle} \# h_2 \right), \\ \varepsilon_B(c \# h) &= \varepsilon_C(c) \varepsilon_H(h). \end{aligned}$$

Then, for every $n \in \mathbb{N}_0$ we have $B_n = C_n \# H$.

Proof. Since C_0 is a subcoalgebra of C in ${}^H\mathfrak{M}$ and, for $n \geq 1$, one has $C_n = C_{n-1} \wedge_C C_0$, then inductively one proves that C_n is a subcoalgebra of C in ${}^H\mathfrak{M}$. Set $B_{(n)} := C_n \# H$ for every $n \in \mathbb{N}_0$. Let us check that $B_{(n)} = B_n$ by induction on $n \in \mathbb{N}_0$.

Let $n = 0$. First note $B = \cup_{m \in \mathbb{N}_0} B_{(m)}$ and, since $\Delta_C(C_m) \subseteq \sum_{0 \leq i \leq m} C_i \otimes C_{m-i}$, we also have

$$\begin{aligned} \Delta_B(B_{(m)}) &= \Delta_B(C_m \# H) \subseteq \sum_{0 \leq i \leq m} \sum \left(C_i \# (C_{m-i})_{\langle -1 \rangle} (H)_1 \right) \otimes \left((C_{m-i})_{\langle 0 \rangle} \# (H)_2 \right) \\ &\subseteq \sum_{0 \leq i \leq m} (C_i \# H) \otimes (C_{m-i} \# H) = \sum_{0 \leq i \leq m} B_{(i)} \otimes B_{(m-i)}. \end{aligned}$$

Therefore $(B_{(m)})_{m \in \mathbb{N}_0}$ is a coalgebra filtration for B and hence, by [Sw, Proposition 11.1.1], we get that $B_{(0)} \supseteq B_0$. Since C_0 is one-dimensional, there is a grouplike element $1_C \in C_0$ such that $C_0 = \mathbb{k}1_C$. Moreover one checks that C_0 is a subcoalgebra of C in ${}^H\mathfrak{M}$ implies $\sum (1_C)_{\langle -1 \rangle} \otimes (1_C)_{\langle 0 \rangle} = 1_H \otimes 1_C$.

Let $\sigma : H \rightarrow C \otimes H : h \mapsto 1_C \otimes h$ be the canonical injection. We have

$$\begin{aligned} \Delta_B \sigma(h) &= \Delta_B(1_C \otimes h) = \sum \left(1_C \# (1_C)_{\langle -1 \rangle} h_1 \right) \otimes \left((1_C)_{\langle 0 \rangle} \# h_2 \right) \\ &= \sum (1_C \# 1_H h_1) \otimes (1_C \# h_2) = \sum \sigma(h_1) \otimes \sigma(h_2) = (\sigma \otimes \sigma) \Delta_H(h), \\ \varepsilon_B \sigma(h) &= \varepsilon_B(1_C \otimes h) = \varepsilon_C(1_C) \varepsilon_H(h) = \varepsilon_H(h) \end{aligned}$$

so that σ is a coalgebra map. Since H is cosemisimple and σ an injective coalgebra map we deduce that also $\sigma(H) = C_0 \otimes H = B_{(0)}$ is a cosemisimple subcoalgebra of B whence $B_{(0)} \subseteq B_0$.

Let $n > 0$ and assume that $B_i = B_{\langle i \rangle}$ for $0 \leq i \leq n-1$. Let $\sum_{i \in I} c_i \# h_i \in B_n$. Then

$$\Delta_B \left(\sum_{i \in I} c_i \# h_i \right) \in B_{n-1} \otimes B + B \otimes B_0 = C_{n-1} \otimes H \otimes C \otimes H + C \otimes H \otimes C_0 \otimes H.$$

Let $p_n : C \rightarrow \frac{C}{C_n}$ be the canonical projection. If we apply $(p_{n-1} \otimes \varepsilon_H \otimes p_0 \otimes H)$ we get

$$\begin{aligned} 0 &= (p_{n-1} \otimes \varepsilon_H \otimes p_0 \otimes H) \Delta_B \left(\sum_{i \in I} c_i \# h_i \right) \\ &= (p_{n-1} \otimes \varepsilon_H \otimes p_0 \otimes H) \left(\sum_{i \in I} \left((c_i)_1 \# ((c_i)_2)_{\langle -1 \rangle} (h_i)_1 \right) \otimes \left(((c_i)_2)_{\langle 0 \rangle} \# (h_i)_2 \right) \right) \\ &= (p_{n-1} \otimes p_0 \otimes H) \left(\sum_{i \in I} (c_i)_1 \otimes (c_i)_2 \otimes h_i \right) = ((p_{n-1} \otimes p_0) \Delta_C \otimes H) \left(\sum_{i \in I} c_i \# h_i \right). \end{aligned}$$

Thus $\sum_{i \in I} c_i \# h_i \in \ker((p_{n-1} \otimes p_0) \Delta_C \otimes H) = [\ker((p_{n-1} \otimes p_0) \Delta_C)] \otimes H = C_n \otimes H = B_{(n)}$. Thus $B_n \subseteq B_{(n)}$. On the other hand, from $\Delta_C(C_n) \subseteq C_{n-1} \otimes C + C \otimes C_0$ we deduce

$$\begin{aligned} \Delta_B(B_{(n)}) &= \Delta_B(C_n \otimes H) \\ &\subseteq \sum \left((C_n)_1 \# ((C_n)_2)_{\langle -1 \rangle} (H)_1 \right) \otimes \left(((C_n)_2)_{\langle 0 \rangle} \# (H)_2 \right) \\ &\subseteq \sum \left(C_{n-1} \# (C)_{\langle -1 \rangle} H \right) \otimes \left((C)_{\langle 0 \rangle} \# H \right) + \sum \left(C \# (C_0)_{\langle -1 \rangle} H \right) \otimes \left((C_0)_{\langle 0 \rangle} \# H \right) \\ &\subseteq (C_{n-1} \# H) \otimes (C \# H) + (C \# H) \otimes (C_0 \# H) \\ &= B_{(n-1)} \otimes B + B \otimes B_{(0)} = B_{n-1} \otimes B + B \otimes B_0 \end{aligned}$$

and hence $B_{(n)} \subseteq B_n$. \square

DEFINITION 5.3. Let A be a Hopf algebra over a field \mathbb{k} such that the coradical H of A is a sub-Hopf algebra (i.e. A has the dual Chevalley Property). Set $G := \text{gr}(A)$. There are two canonical Hopf algebra maps

$$\begin{aligned} \sigma_G &: H \rightarrow \text{gr}(A) : h \mapsto h + A_{-1}, \\ \pi_G &: \text{gr}(A) \rightarrow H : a + A_{n-1} \mapsto a \delta_{n,0}, \quad n \in \mathbb{N}_0. \end{aligned}$$

The diagram of A (see [AS1, page 659]) is the vector space

$$\mathcal{D}(A) := \left\{ d \in \text{gr}(A) \mid \sum d_1 \otimes \pi_G(d_2) = d \otimes 1_H \right\}.$$

It is a bialgebra in ${}^H_H\mathcal{YD}$ as follows. $\mathcal{D}(A)$ is a subalgebra of G . The left H -action, the left H -coaction of $\mathcal{D}(A)$, the comultiplication and counit are given respectively by

$$h \triangleright d := \sum \sigma_G(h_1) d \sigma_G S(h_2), \quad \rho(d) = \sum \pi_G(d_1) \otimes d_2,$$

$$\Delta_{D(A)}(d) := \sum d_1 \sigma_G S_H \pi_G(d_2) \otimes d_3, \quad \varepsilon_{D(A)}(d) = \varepsilon_G(d).$$

Although the following result seems to be folklore, we include here its statement for future references.

PROPOSITION 5.4. *Let A be a Hopf algebra over a field \mathbb{k} such that the coradical H of A is a sub-Hopf algebra. Let A' be a Hopf algebra over a field \mathbb{k} . Let $f : A' \rightarrow A$ be an isomorphism of Hopf algebras. Then $H' := f^{-1}(H) \cong H$ is the coradical of A' and it is a sub-Hopf algebra of A' . Thus we can identify H' with H . Moreover f induces an isomorphism $\mathcal{D}(f) : \mathcal{D}(A') \rightarrow \mathcal{D}(A)$ of bialgebras in ${}^H_H\mathcal{YD}$.*

PROPOSITION 5.5. *Keep the hypotheses and notations of the beginning of the section. Then $\mathcal{D}(A) \cong \mathcal{D}(R\#_\xi H) \cong \text{gr}(R)$ as bialgebras in ${}^H_H\mathcal{YD}$.*

Proof. Apply Proposition 5.4 to the canonical isomorphism $\psi : B := R\#_\xi H \rightarrow A$ that we recalled at the beginning of the section to get that $\mathcal{D}(R\#_\xi H) \cong \mathcal{D}(A)$. Note that, by H -linearity we have

$$\psi(1_R\#h) = \psi((1_R\#1_H)(1_R\#h)) = \psi((1_R\#1_H)h) = \psi(1_R\#1_H)h = h$$

so that $\psi(\mathbb{k}1_R \otimes H) = H$ and hence $H' = \psi^{-1}(H) = \mathbb{k}1_R \otimes H$ with the notation of Proposition 5.4. Thus $B_0 = \mathbb{k}1_R \otimes H = R_0 \otimes H$ so that we can identify B_0 with H via the canonical isomorphism $H \rightarrow R_0 \otimes H : h \mapsto 1_R \otimes h$. Its inverse is $R_0 \otimes H \rightarrow H : r \otimes h \mapsto \varepsilon_R(r)h$. With this identification and by setting $G := \text{gr}(B)$, we can consider the canonical bialgebra maps

$$\begin{aligned} \sigma_G & : H \rightarrow \text{gr}(B) : h \mapsto 1_R\#h + (R\#_\xi H)_{-1}, \\ \pi_G & : \text{gr}(B) \rightarrow H : r\#h + (R\#_\xi H)_{n-1} \mapsto \varepsilon_R(r)h\delta_{n,0}, \text{ where } r\#h \in (R\#_\xi H)_n, n \in \mathbb{N}_0. \end{aligned}$$

Since the underlying coalgebra of B is exactly the smash coproduct of R by H and (R, ξ) is a connected pre-bialgebra with cocycle in ${}^H_H\mathcal{YD}$, by Lemma 5.2, we have that $B_n = R_n \otimes H$. Let us compute $\mathcal{D} := \mathcal{D}(B)$. As a vector space it is

$$\mathcal{D} := \left\{ d \in G \mid \sum d_1 \otimes \pi_G(d_2) = d \otimes 1_H \right\}.$$

By [AS1, Lemma 2.1], we have that $\mathcal{D} = \bigoplus_{n \in \mathbb{N}_0} \mathcal{D}^n$ where $\mathcal{D}^n = \mathcal{D} \cap G^n = \mathcal{D} \cap \frac{B_n}{B_{n-1}}$. Let $d := \sum_{i \in I} \overline{r_i\#h_i} \in \mathcal{D}^n$ where we can assume $\sum_{i \in I} r_i\#h_i \in B_n \setminus B_{n-1}$ and, for every $i \in I$, $r_i\#h_i \in B_n \setminus B_{n-1}$.

Then $\sum_{i \in I} \overline{r_i\#h_i} = \sum_{i \in I} \overline{r_i\#h_i}$ and hence the fact that d is coinvariant rewrites as

$$(31) \quad \sum_{i \in I} (\overline{r_i\#h_i})_1 \otimes \pi_G((\overline{r_i\#h_i})_2) = \sum_{i \in I} \overline{r_i\#h_i} \otimes 1_H.$$

By definition of π_G and (1), the left-hand side becomes

$$\sum_{i \in I} (\overline{r_i\#h_i})_1 \otimes \pi_G((\overline{r_i\#h_i})_2) = \sum_{i \in I} ((r_i\#(h_i)_1) + B_{n-1}) \otimes (h_i)_2$$

so that (31) becomes

$$\sum_{i \in I} ((r_i\#(h_i)_1) + B_{n-1}) \otimes (h_i)_2 = \sum_{i \in I} \overline{r_i\#h_i} \otimes 1_H = \sum_{i \in I} (r_i\#h_i + B_{n-1}) \otimes 1_H$$

i.e.

$$\sum_{i \in I} (r_i\#(h_i)_1) \otimes (h_i)_2 - \sum_{i \in I} r_i\#h_i \otimes 1_H \in B_{n-1} \otimes H = R_{n-1} \otimes H \otimes H.$$

If we apply $R \otimes \varepsilon_H \otimes H$, we get

$$\sum_{i \in I} r_i \otimes h_i - \sum_{i \in I} r_i \varepsilon_H(h_i) \otimes 1_H \in R_{n-1} \otimes H = B_{n-1}.$$

Thus $\sum_{i \in I} \overline{r_i\#h_i} = \sum_{i \in I} \overline{r_i\#h_i} = \sum_{i \in I} (r_i\#h_i + B_{n-1}) = \sum_{i \in I} (r_i \varepsilon_H(h_i) \otimes 1_H) + B_{n-1}$.

Since $\sum_{i \in I} r_i \# h_i \in B_n \setminus B_{n-1}$ we get that $\left(\sum_{i \in I} r_i \varepsilon_H(h_i) \right) \otimes 1_H \notin B_{n-1}$ and hence $\sum_{i \in I} r_i \varepsilon_H(h_i) \notin R_{n-1}$ and we can write

$$\overline{\sum_{i \in I} r_i \# h_i} = \overline{\left(\sum_{i \in I} r_i \varepsilon_H(h_i) \right) \otimes 1_H}.$$

Therefore we have proved that the map

$$\varphi_n : \frac{R_n}{R_{n-1}} \rightarrow \mathcal{D}^n : \bar{r} \mapsto \overline{r \otimes 1_H},$$

which is well-defined as $\mathcal{D}^n = \mathcal{D} \cap G^n = \mathcal{D} \cap \frac{B_n}{B_{n-1}} = \mathcal{D} \cap \frac{R_n \otimes H}{R_{n-1} \otimes H}$, is also surjective.

It is also injective as $\varphi_n(\bar{r}) = \varphi_n(\bar{s})$ implies $r \otimes 1_H - s \otimes 1_H \in B_{n-1} = R_{n-1} \otimes H$ and hence, by applying $R \otimes \varepsilon_H$, we get $r - s \in R_{n-1}$ i.e. $\bar{r} = \bar{s}$. Therefore φ_n is an isomorphism such that $\overline{\sum_{i \in I} r_i \# h_i} = \varphi_n \left(\overline{\sum_{i \in I} r_i \varepsilon_H(h_i)} \right)$ and hence

$$\varphi_n^{-1} \left(\overline{\sum_{i \in I} r_i \# h_i} \right) = \overline{\sum_{i \in I} r_i \varepsilon_H(h_i)}.$$

Clearly this extends to a graded \mathbb{k} -linear isomorphism

$$\varphi : \text{gr}(R) \rightarrow \mathcal{D}.$$

Let us check that φ is a morphism in ${}^H_H\mathcal{YD}$. First note that, for every $r \in R_n$, we have

$$\begin{aligned} \varphi(r + R_{n-1}) &= \delta_{|r|,n} \varphi(r + R_{n-1}) = \delta_{|r|,n} \varphi_n(r + R_{n-1}) = \delta_{|r|,n} \varphi_n(\bar{r}) \\ &= \delta_{|r|,n} \overline{r \otimes 1_H} = \delta_{|r|,n} \left(r \otimes 1_H + (R \#_\xi H)_{n-1} \right) = r \otimes 1_H + (R \#_\xi H)_{n-1}. \end{aligned}$$

Thus

$$(32) \quad \varphi(r + R_{n-1}) = r \otimes 1_H + (R \#_\xi H)_{n-1}, \text{ for every } r \in R_n.$$

For every $r \in R_n \setminus R_{n-1}$, by using (32), it is straightforward to prove that $h \triangleright \varphi(\bar{r}) = \varphi(h\bar{r})$.

Moreover, by applying (1), (30), the definition of π_G and (32), we get that $\rho\varphi(\bar{r}) = (H \otimes \varphi)\rho(\bar{r})$.

Let us check that φ is a morphism of bialgebras in ${}^H_H\mathcal{YD}$. Fix $r \in R_n \setminus R_{n-1}$.

Using the definition of $\Delta_{\mathcal{D}}$, (1), (30), the definition of π_G , the definition of σ_G , (32) and (1) again, we obtain $\Delta_{\mathcal{D}}\varphi(\bar{r}) = (\varphi \otimes \varphi)\Delta_{\text{gr}(R)}(\bar{r})$.

Let us check φ is counitary:

$$\begin{aligned} \varepsilon_{\mathcal{D}}\varphi(\bar{r}) &= \varepsilon_G\varphi(\bar{r}) = \varepsilon_G(\overline{r \otimes 1_H}) \stackrel{(2)}{=} \delta_{n,0} \varepsilon_B(r \otimes 1_H) \\ &= \delta_{n,0} \varepsilon_R(r) \stackrel{(2)}{=} \varepsilon_{\text{gr}(R)}(\bar{r}). \end{aligned}$$

Let us check φ is multiplicative. Let $s \in R_m \setminus R_{m-1}$. Then, by definition of φ , of $m_{\mathcal{D}}$ and of the multiplication of $R \#_\xi H$, we have that

$$m_{\mathcal{D}}(\varphi \otimes \varphi)(\bar{s} \otimes \bar{r}) = \sum \left(s^{(1)} \left(\left(s^{(2)} \right)_{\langle -1 \rangle} r^{(1)} \right) \#_\xi \left(\left(s^{(2)} \right)_{\langle 0 \rangle} \otimes r^{(2)} \right) \right) + (R \#_\xi H)_{m+n-1}.$$

Now write $\sum s^{(1)} \otimes s^{(2)} = \sum_{0 \leq i \leq m} s_i \otimes s'_{m-i}$ for some $s_i, s'_i \in R_i$ and similarly $\sum r^{(1)} \otimes r^{(2)} = \sum_{0 \leq j \leq n} r_j \otimes r'_{n-j}$ for some $r_j, r'_j \in R_j$. Then

$$\begin{aligned} m_{\mathcal{D}}(\varphi \otimes \varphi)(\bar{s} \otimes \bar{r}) &= \sum_{\substack{0 \leq i \leq m \\ 0 \leq j \leq n}} \left(s_i \left(\left(s'_{m-i} \right)_{\langle -1 \rangle} r_j \right) \#_\xi \left(\left(s'_{m-i} \right)_{\langle 0 \rangle} \otimes r'_{n-j} \right) \right) + (R \#_\xi H)_{m+n-1} \\ &= \sum_{\substack{0 \leq i \leq m \\ 0 \leq j \leq n}} \delta_{i,m} \delta_{j,n} \left(s_i \left(\left(s'_{m-i} \right)_{\langle -1 \rangle} r_j \right) \#_\xi \left(\left(s'_{m-i} \right)_{\langle 0 \rangle} \otimes r'_{n-j} \right) \right) + (R \#_\xi H)_{m+n-1} \\ &= \sum \left(s_m \left(\left(s'_0 \right)_{\langle -1 \rangle} r_n \right) \#_\xi \left(\left(s'_0 \right)_{\langle 0 \rangle} \otimes r'_0 \right) \right) + (R \#_\xi H)_{m+n-1} \\ &\stackrel{R_0 = \mathbb{k}1_R}{=} \sum s_m \left(\left(s'_0 \right)_{\langle -1 \rangle} r_n \right) \#_{\varepsilon_R} \left(\left(s'_0 \right)_{\langle 0 \rangle} \right) \varepsilon_R(r'_0) 1_H + (R \#_\xi H)_{m+n-1} \end{aligned}$$

$$\begin{aligned}
&= \sum s_m \varepsilon_R(s'_0) r_n \varepsilon_R(r'_0) \# 1_H + (R \#_\xi H)_{m+n-1} \\
&= \sum_{\substack{0 \leq i \leq m \\ 0 \leq j \leq n}} \delta_{i,m} \delta_{j,n} (s_i \varepsilon_R(s'_{m-i}) r_j \varepsilon_R(r'_{n-j}) \# 1_H) + (R \#_\xi H)_{m+n-1} \\
&= \sum_{\substack{0 \leq i \leq m \\ 0 \leq j \leq n}} (s_i \varepsilon_R(s'_{m-i}) r_j \varepsilon_R(r'_{n-j}) \# 1_H) + (R \#_\xi H)_{m+n-1} \\
&= \sum (s^{(1)} \varepsilon_R(s^{(2)}) r^{(1)} \varepsilon_R(r^{(2)}) \# 1_H) + (R \#_\xi H)_{m+n-1} \\
&= (sr \# 1_H) + (R \#_\xi H)_{m+n-1} \stackrel{(32)}{=} \varphi(sr + R_{m+n-1}) \\
&= \varphi((s + R_{m-1})(r + R_{n-1})) = \varphi m_{\text{gr}(R)}(\bar{s} \otimes \bar{r}).
\end{aligned}$$

Let us check φ is unitary. We have

$$\varphi(1_{\text{gr}(R)}) = \varphi(1_R + R_{-1}) = \varphi(\overline{1_R}) = \overline{1_R \otimes 1_H} = (1_R \otimes 1_H) + (R \#_\xi H)_{-1} = 1_B + B_{-1} = 1_G.$$

□

Summing up we have proved that

$$\text{gr}(Q) \stackrel{Q=R^v}{=} \text{gr}(R^v) \stackrel{\text{Lem. 5.1}}{\cong} \text{gr}(R) \stackrel{\text{Pro. 5.5}}{\cong} \mathcal{D}(R \#_\xi H) \stackrel{\text{Pro. 5.4}}{\cong} \mathcal{D}(A)$$

as bialgebras in ${}^H_H\mathcal{YD}$. Therefore $\text{H}_{\mathcal{YD}}^3(\mathcal{D}(A), \mathbb{k}) = 0$ (the Hochschild cohomology in ${}^H_H\mathcal{YD}$ of the algebra $\mathcal{D}(A)$ with values in \mathbb{k}) if, and only if, $\text{H}_{\mathcal{YD}}^3(\text{gr}Q, \mathbb{k}) = 0$. In this case, by the foregoing, we get that Q is gauge equivalent to a connected bialgebra in ${}^H_H\mathcal{YD}$.

Now let E be a connected bialgebra in ${}^H_H\mathcal{YD}$ and let $\gamma : E \otimes E \rightarrow \mathbb{k}$ be a gauge transformation in ${}^H_H\mathcal{YD}$ such that $Q = E^\gamma$. We proved that $A^\zeta \cong Q \# H \cong E^\gamma \# H$ as coquasi-bialgebras. By Proposition 2.5, we have that $(E \# H)^\Gamma = E^\gamma \# H$ as an ordinary coquasi-bialgebras. Recall that two coquasi-bialgebras A and A' are called **gauge equivalent** or **quasi-isomorphic** whenever there is some gauge transformation $\gamma : Q \otimes Q \rightarrow \mathbb{k}$ in $\mathbf{Vect}_{\mathbb{k}}$ such that $A^\gamma \cong A'$ as coquasi-bialgebras. We point out that, if A and A' are ordinary bialgebras and $A^\gamma \cong A'$, then γ comes out to be a unitary cocycle. This is encoded in the triviality of the reassociators of A and A' .

THEOREM 5.6. *Let A be a finite-dimensional Hopf algebra over a field \mathbb{k} of characteristic zero such that the coradical H of A is a sub-Hopf algebra (i.e. A has the dual Chevalley Property). If $\text{H}_{\mathcal{YD}}^3(\mathcal{D}(A), \mathbb{k}) = 0$, then A is quasi-isomorphic to the Radford-Majid bosonization $E \# H$ of some connected bialgebra E in ${}^H_H\mathcal{YD}$ by H . Moreover $\text{gr}(E) \cong \mathcal{D}(A)$ as bialgebras in ${}^H_H\mathcal{YD}$.*

Proof. By the foregoing $A^\zeta \cong Q \# H \cong E^\gamma \# H = (E \# H)^\Gamma$ as coquasi-bialgebras. Now A is quasi-isomorphic to A^ζ which is quasi-isomorphic to $E \# H$ so that A is quasi-isomorphic to $E \# H$. Moreover

$$\text{gr}(E) = \text{gr}(E^\gamma) = \text{gr}(Q) \cong \mathcal{D}(A).$$

where the first equality holds by Proposition 2.6.

□

More generally, given A a (finite-dimensional) Hopf algebra whose coradical H is a sub-Hopf algebra, then if H is also semisimple, we expect that A is quasi-isomorphic to the Radford-Majid bosonization $E \# H$ of some connected bialgebra E in ${}^H_H\mathcal{YD}$ by H . See e.g. [GM, Corollary 3.4 and the proof therein] and [AAGMV, AAG] for a further clue in this direction.

6. EXAMPLES

We notice that the Hochschild cohomology of a finite-dimensional Nichols algebras has been computed in few examples. We consider here those Nichols algebras to compute $\text{H}_{\mathcal{YD}}^3(\mathcal{B}(V), \mathbb{k})$.

6.1. Braidings of Cartan type. Let $A = (a_{ij})_{1 \leq i, j \leq \theta}$ be a finite Cartan matrix, Δ the corresponding root system, $(\alpha_i)_{1 \leq i \leq \theta}$ a set of simple roots and W its Weyl group. Let $w_0 = s_{i_1} \cdots s_{i_M}$ be a reduced expression of the element $w_0 \in W$ of maximal length as a product of simple reflections, $\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$, $1 \leq j \leq M$. Then $\beta_j \neq \beta_k$ if $j \neq k$ and $\Delta^+ = \{\beta_j | 1 \leq j \leq M\}$, see [H, page 25 and Proposition 3.6].

Let Γ be a finite abelian group, $\widehat{\Gamma}$ its group of characters. $\mathcal{D} = (\Gamma, (g_i)_{1 \leq i \leq \theta}, (\chi_i)_{1 \leq i \leq \theta}, A)$ is a datum of finite Cartan type [AS2] associated to Γ and A if $g_i \in \Gamma$, $\chi_j \in \widehat{\Gamma}$, $1 \leq i, j \leq \theta$, satisfy $\chi_i(g_i) \neq 1$, $\chi_i(g_j)\chi_j(g_i) = \chi_i(g_i)^{a_{ij}}$ for all i, j . Set $\mathfrak{q} = (q_{ij})_{1 \leq i, j \leq \theta}$, where $q_{ij} = \chi_j(g_i)$.

In what follows V denotes the Yetter-Drinfeld module over $\mathbb{k}\Gamma$, $\dim V = \theta$, with a fixed basis x_1, \dots, x_θ , where the action and the coaction over each x_i is given by χ_i and g_i , respectively. Then the associated braiding is $c(x_i \otimes x_j) = q_{ij}x_j \otimes x_i$ for all i, j . Let $\mathcal{B}_{\mathfrak{q}} = \mathcal{B}(V)$. The tensor algebra $T(V)$ is \mathbb{N}_0^θ -graded with grading α_i for each x_i . For $\beta = \sum_{i=1}^\theta a_i \alpha_i \in \Delta^+$, set

$$g_\beta = g_1^{a_1} \cdots g_\theta^{a_\theta}, \quad \chi_\beta = \chi_1^{a_1} \cdots \chi_\theta^{a_\theta}, \quad q_\beta = \chi_\beta(g_\beta).$$

Given $\alpha, \beta \in \Delta^+$, we denote $q_{\alpha\beta} = \chi_\beta(g_\alpha)$.

We assume as in [AS2, MPSW] that *the order of q_{ii} is odd for all i , and not divisible by 3 for each connected component of the Dynkin diagram of A of type G_2* . Therefore the order of q_{ii} is the same for all the i in the same connected component J . Given $\beta \in J$, we denote by N_β the order of the corresponding q_{ii} in J , which is also the order of q_β .

By [L] there exist homogeneous elements x_β of degree β , $\beta \in \Delta^+$, such that the Nichols algebra $\mathcal{B}_{\mathfrak{q}}$ of V is presented by generators x_1, \dots, x_θ and relations

$$\begin{aligned} (\text{ad}_c x_i)^{1-a_{ij}} x_j &= 0, & 1 \leq i \neq j \leq \theta; \\ x_\beta^{N_\beta} &= 0, & \beta \in \Delta_+. \end{aligned}$$

Moreover $\{x_{\beta_1}^{n_1} \cdots x_{\beta_M}^{n_M} | 0 \leq n_i < N_{\beta_i}\}$ is a basis of $\mathcal{B}_{\mathfrak{q}}$.

We shall prove that $\text{Hy}_{\mathcal{D}}^3(\mathcal{B}_{\mathfrak{q}}, \mathbb{k}) = 0$. We need first some technical results.

LEMMA 6.1. *Let $\alpha, \beta \in \Delta_+$. Then either $g_\alpha g_\beta^{N_\beta} \neq e$, or else $\chi_\alpha \chi_\beta^{N_\beta} \neq \epsilon$.*

Proof. Suppose on the contrary that $g_\alpha g_\beta^{N_\beta} = e$, $\chi_\alpha \chi_\beta^{N_\beta} = \epsilon$. Then

$$q_\alpha = \chi_\alpha^{-1}(g_\alpha^{-1}) = \chi_\beta^{N_\beta}(g_\beta^{N_\beta}) = q_\beta^{N_\beta^2} = 1,$$

since q_β is a root of unity of order N_β . But this is a contradiction, since $q_\alpha \neq 1$. \square

LEMMA 6.2. *Let $\alpha, \beta, \gamma \in \Delta^+$ be pairwise different. Then either $g_\alpha g_\beta g_\gamma \neq e$, or else $\chi_\alpha \chi_\beta \chi_\gamma \neq \epsilon$.*

Proof. Suppose on the contrary that $g_\alpha g_\beta g_\gamma = e$ and $\chi_\alpha \chi_\beta \chi_\gamma = \epsilon$. Then

$$(33) \quad q_\alpha = \chi_\alpha^{-1}(g_\alpha^{-1}) = \chi_\beta \chi_\gamma (g_\beta g_\gamma) = q_\beta q_\gamma q_\beta q_\gamma, \quad q_\beta = q_\alpha q_\gamma q_\alpha q_\gamma, \quad q_\gamma = q_\alpha q_\beta q_\alpha q_\beta.$$

Notice that α, β, γ belong to the same connected component. Indeed, if γ belongs to a different connected component, then $q_{\beta\gamma} q_{\gamma\beta} = q_{\alpha\gamma} q_{\gamma\alpha} = 1$. Thus $q_\beta = q_\alpha q_\gamma = q_\beta q_\gamma^2$, so $q_\gamma^2 = 1$, which is a contradiction. Therefore we may assume that the Dynkin diagram is connected.

One can prove that $q_{s_i(\alpha)} = q_\alpha$ for every $\alpha \in \Delta$. As we observed that $\Delta^+ = \{\beta_j | 1 \leq j \leq M\}$, we deduce that for every $\beta \in \Delta^+$ there is some j such that $q_\beta = q_j$. One can prove that there is some $q \in \mathbb{k}$ such that $q_\alpha = q^{(\alpha, \alpha)/2}$ and $q_{\alpha\gamma} q_{\gamma\alpha} = q^{(\alpha, \gamma)}$, where (\cdot, \cdot) is the invariant bilinear form on the simple Lie algebra \mathfrak{g} associated with the finite Cartan matrix [Bo, Ch. VI, §1, Proposition 3 and Definition 3] and the basis of the root systems given in [Bo, Ch. VI, §4] should be normalized in such a way that $q = q_\delta$, $(\delta, \delta) = 2$ for each short root $\delta \in \Delta$. Note that $q_\alpha = q^{(\alpha, \alpha)/2} \neq 1$ for all α as $(\alpha, \alpha) \neq 0$. Thus

- $q_\alpha = q_\beta = q_\gamma = q$ if the Dynkin diagram is simply laced,
- $q_\alpha, q_\beta, q_\gamma \in \{q, q^2\}$ if the Dynkin diagram has a double arrow,
- $q_\alpha, q_\beta, q_\gamma \in \{q, q^3\}$ if the Dynkin diagram is of type G_2 .

If the Dynkin diagram is simply laced, then, by (33), we have $q_{\beta\gamma}q_{\gamma\beta} = q_{\alpha\gamma}q_{\gamma\alpha} = q_{\alpha\beta}q_{\beta\alpha} = q^{-1}$. Then $q^{(\alpha,\gamma)} = q^{-1}$. Now set $n(\alpha, \beta) := 2(\alpha, \beta)/(\beta, \beta) = (\alpha, \beta)$. Then $n(\alpha, \beta)$ is symmetric whence, by [Bo, Ch. VI, §1, page 148] we have $(\alpha, \gamma) = -1$ as the order of q is odd, so $\alpha + \gamma \in \Delta^+$, by [Bo, VI, §1, Corollary, page 149]. Now the same argument we used above shows that also $(\alpha, \beta) = -1 = (\gamma, \beta)$ and hence $(\alpha + \gamma, \beta) = -2$, so $\alpha + \beta + \gamma \in \Delta^+$, since $\alpha + \gamma \neq -\beta$ (as $\alpha + \gamma$ and β are both in Δ^+). But $q_{\alpha+\beta+\gamma} = q_{\alpha}q_{\beta}q_{\gamma}q_{\beta\gamma}q_{\gamma\beta}q_{\alpha\gamma}q_{\gamma\alpha}q_{\alpha\beta}q_{\beta\alpha} = 1$, which is a contradiction.

If the Dynkin diagram has a double arrow, then $q_{\alpha}, q_{\beta}, q_{\gamma} \in \{q, q^2\}$. If $q_{\alpha} = q_{\beta} = q_{\gamma}$, then the proof follows as for the simply-laced case because $n(u, v) = n(v, u)$ for $u, v \in \{\alpha, \beta, \gamma\}$. If $q_{\alpha} = q_{\beta} = q$ and $q_{\gamma} = q^2$, then $q_{\beta\gamma}q_{\gamma\beta} = q_{\alpha\gamma}q_{\gamma\alpha} = q^{-2}$, and $q_{\alpha\beta}q_{\beta\alpha} = 1$, by (33). Then a simple calculation yields $(\beta, \gamma) = -2$ so that $\beta + \gamma \in \Delta^+$. One also gets $(\alpha, \beta) = 0$ and $(\alpha, \gamma) = -2$ so that $(\alpha, \beta + \gamma) = (\alpha, \beta) + (\alpha, \gamma) = -2 < 0$ by the conditions on the order of q , so again $\alpha + \beta + \gamma \in \Delta^+$; but again we obtain $q_{\alpha+\beta+\gamma} = 1$, which is a contradiction. The proof for $q_{\alpha} = q_{\beta} = q^2$ and $q_{\gamma} = q$ follows analogously.

Finally, if the Dynkin diagram is of type G_2 , then a similar analysis gives a contradiction. \square

For each $1 \leq k \leq M$, set $\mathcal{B}_q(k)$ as the subspace of \mathcal{B}_q spanned by $\{x_{\beta_1}^{n_1} \dots x_{\beta_k}^{n_k} | 0 \leq n_i < N_{\beta_i}\}$. By [DP] this gives an algebra filtration, and the graded algebra $\text{Gr } \mathcal{B}_q$ associated to this filtration is presented by generators \mathbf{x}_{β} , $\beta \in \Delta^+$, and relations

$$\mathbf{x}_{\beta}\mathbf{x}_{\gamma} = q_{\beta\gamma}\mathbf{x}_{\gamma}\mathbf{x}_{\beta}, \quad \mathbf{x}_{\beta}^{N_{\beta}} = 0, \quad \beta < \gamma \in \Delta_+.$$

In [MPSW] $\text{Gr } \mathcal{B}_q$ is viewed as an algebra in $\mathbb{k}\Gamma\mathcal{YD}$, which (as an algebra) is the Nichols algebra of Cartan type $A_1 \times \dots \times A_1$, M copies, with action and coaction on \mathbf{x}_{β} given by χ_{β}, g_{β} , respectively. By [MPSW, Theorem 4.1], $\text{H}^{\bullet}(\text{Gr } \mathcal{B}_q, \mathbb{k})$ is the algebra generated by $\xi_{\beta}, \eta_{\beta}$, $\beta \in \Delta^+$, where $\deg \xi_{\beta} = 2$, $\deg \eta_{\beta} = 1$, and relations

$$\xi_{\beta}\xi_{\gamma} = q_{\beta\gamma}^{N_{\beta}N_{\gamma}}\xi_{\gamma}\xi_{\beta}, \quad \eta_{\beta}\xi_{\gamma} = q_{\beta\gamma}^{N_{\gamma}}\xi_{\gamma}\eta_{\beta}, \quad \eta_{\beta}\eta_{\gamma} = -q_{\beta\gamma}\eta_{\gamma}\eta_{\beta}, \quad \beta, \gamma \in \Delta^+.$$

As we assume that all the q_{ii} have odd order, we deduce in particular from the last equality that $\eta_{\beta}^2 = 0$ for all $\beta \in \Delta^+$. As an algebra in $\mathbb{k}\Gamma\mathcal{YD}$, the action and coaction on ξ_{β} is given by $\chi_{\beta}^{-N_{\beta}}, g_{\beta}^{-N_{\beta}}$, while the action and coaction on η_{β} is given by $\chi_{\beta}^{-1}, g_{\beta}^{-1}$.

THEOREM 6.3. $\text{H}_{\mathcal{YD}}^3(\mathcal{B}_q, \mathbb{k}) = 0$.

Proof. First we will prove that $\text{H}^3(\text{Gr } \mathcal{B}_q, \mathbb{k})^D = 0$ for $D := D(\mathbb{k}\Gamma)$. Now, the invariants are with respect to the D -bimodule structure that $\text{H}^3(\text{Gr } \mathcal{B}_q, \mathbb{k})$ inherits from $\text{Hom}((\text{Gr } \mathcal{B}_q)^{\otimes 3}, \mathbb{k})$ (this is a D -bimodule as its arguments are left D -modules). Since the left D -module structure is induced by the one of \mathbb{k} , it is trivial. Thus the invariants of $\text{H}^3(\text{Gr } \mathcal{B}_q, \mathbb{k})$ as a D -bimodule reduce to the its invariants as a right D -module. Since right D -modules are equivalent to left D -modules, via the antipode of D which is invertible as D is finite-dimensional, the right D -module structure of $\text{H}^3(\text{Gr } \mathcal{B}_q, \mathbb{k})$ becomes the structure of object in $\mathbb{k}\Gamma\mathcal{YD}$ described above. Thus, in order to prove that $\text{H}^3(\text{Gr } \mathcal{B}_q, \mathbb{k})^D = 0$ we just have to check that the invariants of $\text{H}^3(\text{Gr } \mathcal{B}_q, \mathbb{k})$ as a left-left Yetter-Drinfeld modules are zero.

Now, by the defining relations of $\text{H}^{\bullet}(\text{Gr } \mathcal{B}_q, \mathbb{k})$, a basis B of $\text{H}^3(\text{Gr } \mathcal{B}_q, \mathbb{k})$ is given by $\{\xi_{\alpha}\eta_{\beta}\} \cup \{\eta_{\alpha}\eta_{\beta}\eta_{\gamma} | \alpha < \beta < \gamma\}$. If $v \in \text{H}^3(\text{Gr } \mathcal{B}_q, \mathbb{k})$ is invariant, then v is written as a linear combination of elements in the trivial component. Indeed, write $v = \sum_{b \in B} c_b b$ for some $c_b \in \mathbb{k}$, and let g_b, χ_b be the elements describing the component of $b \in B$. Then

$$\begin{aligned} v &= g \cdot v = \sum_{b \in B} c_b g \cdot b = \sum_{b \in B} c_b \chi_b(g) b, & \text{for all } g \in \Gamma, \\ 1 \otimes v &= \rho(v) = \sum_{b \in B} c_b \rho \cdot b = \sum_{b \in B} c_b g_b \otimes b. \end{aligned}$$

If $c_b \neq 0$, then $\chi_b(g) = 1$ for all $g \in \Gamma$ so $\chi_b = \epsilon$, and $g_b = 1$. Thus b is invariant. We have so proved that the existence of $v \neq 0$ invariant implies the existence of $b \in B$ invariant. Hence, if B has no invariant element then there is no invariant element at all. Note that, for all $h \in H$, we have $h \cdot (\xi_{\alpha}\eta_{\beta}) = (\chi_{\alpha}^{-N_{\alpha}}\chi_{\beta}^{-1})(h)\xi_{\alpha}\eta_{\beta}$ and $\rho(\xi_{\alpha}\eta_{\beta}) = g_{\alpha}^{-N_{\alpha}}g_{\beta}^{-1} \otimes \xi_{\alpha}\eta_{\beta}$ so that, by Lemma 6.1, the element

$\xi_\alpha \eta_\beta$ is not D -invariant. A similar argument, using Lemma 6.2, shows that also $\eta_\alpha \eta_\beta \eta_\gamma$ is not D -invariant. Thus the elements in B are not D -invariant, so $H^3(\text{Gr } \mathcal{B}_q, \mathbb{k})^D = 0$. Since the elements in $\{x_{\beta_1}^{n_1} \dots x_{\beta_k}^{n_k} \mid 0 \leq n_i < N_{\beta_i}\}$ are eigenvectors for D , we can mimic the argument in [MPSW, Section 5] by taking into account the spectral sequence associated to the filtration of algebras therein; see for example [MPSW, Corollary 5.5] for a similar argument. Thus $H_{\mathcal{YD}}^3(\mathcal{B}_q, \mathbb{k}) \cong H^3(\mathcal{B}_q, \mathbb{k})^D = 0$. \square

REMARK 6.4. Notice that $H_{\mathcal{YD}}^3(\mathcal{B}_q, \mathbb{k}) \cong H^3(\mathcal{B}_q, \mathbb{k})^{D(\mathbb{k}\Gamma)} = 0$ although $H^3(\mathcal{B}_q \# \mathbb{k}\Gamma, \mathbb{k}) \cong H^3(\mathcal{B}_q, \mathbb{k})^\Gamma$ can be non-trivial, see for example [MPSW, Example 5.8].

6.2. Braidings of non-diagonal type. For $n \geq 3$, \mathcal{FK}_n denotes the quadratic algebra [FK] with a presentation by generators $x_{(ij)}$, $1 \leq i < j \leq n$, and relations

$$\begin{aligned} x_{(ij)}^2 &= 0, & 1 \leq i < j \leq n, \\ x_{(ij)}x_{(jk)} &= x_{(jk)}x_{(ik)} + x_{(ik)}x_{(ij)}, & 1 \leq i < j < k \leq n, \\ x_{(jk)}x_{(ij)} &= x_{(ik)}x_{(jk)} + x_{(ij)}x_{(ik)}, & 1 \leq i < j < k \leq n, \\ x_{(ij)}x_{(kl)} &= x_{(kl)}x_{(ij)}, & \#\{i, j, k, l\} = 4. \end{aligned}$$

According to [MiS] each \mathcal{FK}_n is a graded bialgebra in the category of Yetter-Drinfeld modules over the symmetric group S_n , generated as an algebra by the vector space V_n with basis $\{x_{(ij)} \mid 1 \leq i < j \leq n\}$. The action is described by identifying (ij) with the corresponding transposition in S_n and then consider the conjugation twisted by the sign, while the coaction is given by declaring x_σ a homogeneous element of degree σ . Then the braiding on V_n becomes

$$c(x_\sigma \otimes x_\tau) = \chi(\sigma, \tau) x_{\sigma\tau\sigma^{-1}} \otimes x_\sigma, \quad \chi(\sigma, \tau) = \begin{cases} 1 & \sigma(i) < \sigma(j), \tau = (ij), i < j, \\ -1 & \text{otherwise,} \end{cases}$$

where σ and τ are transpositions. Moreover \mathcal{FK}_n projects onto the Nichols algebra $\mathcal{B}(V_n)$. For $n = 3, 4, 5$, it is known that $\mathcal{FK}_n = \mathcal{B}(V_n)$ and has dimension, respectively, 12, 576 and 8294400.

The Hochschild cohomology of \mathcal{FK}_3 is a consequence of the results in [SV] as follows.

THEOREM 6.5. $H_{\mathbb{k}S_3\text{-Mod}}^\bullet(\mathcal{FK}_3, \mathbb{k})$ is isomorphic to the graded algebra

$$\mathbb{k}[X, U, V]/(U^2V - VU^2), \quad \text{where } \deg U = \deg V = 2, \deg X = 4.$$

Proof. By [SV, Theorem 4.19], we have that $E(B \# \mathbb{k}S_3)$ is isomorphic to the algebra in the claim, where $B = \mathcal{FK}_3$. By [SV, Theorem 2.17], we know that $E(B \# \mathbb{k}S_3) \cong E(B)^{\mathbb{k}S_3}$ as graded algebras. As observed in Remark 4.2, we have that $E(B) \cong H^\bullet(B, \mathbb{k})$. By Remark 4.6, we have $H^\bullet(B, \mathbb{k})^{\mathbb{k}S_3} \cong H_{\mathbb{k}S_3\text{-Mod}}^\bullet(\mathcal{FK}_3, \mathbb{k})$. \square

From this result we get $H_{\mathbb{k}S_3\text{-Mod}}^3(\mathcal{FK}_3, \mathbb{k}) = 0$ so that, by Proposition 4.7 we conclude that

COROLLARY 6.6. $H_{\mathcal{YD}}^3(\mathcal{FK}_3, \mathbb{k}) = 0$.

REFERENCES

- [ABM1] A. Ardizzoni, M. Beattie and C. Menini, *Cocycle Deformations for Hopf Algebras with a Coalgebra Projection*, J. Algebra, Vol. **324**(4) (2010) 673–705.
- [ABM] A. Ardizzoni, M. Beattie and C. Menini, *Gauge Deformations for Hopf Algebras with the Dual Chevalley Property*, J. Algebra Appl., Vol. **11**(3) (2012) 1250051.
- [AAG] N. Andruskiewitsch, I. E. Angiono, A. García Iglesias, *Liftings of Nichols algebras of diagonal type I. Cartan type A*. ([arXiv:1509.01622](https://arxiv.org/abs/1509.01622))
- [AAGMV] N. Andruskiewitsch, I. E. Angiono, A. García Iglesias, A. Masuoka and C. Vay, *Liftings via cocycle deformations*. J. Pure Appl. Alg. **218** (2014), 684–703.
- [AM] A. Ardizzoni and C. Menini, *Associated Graded Algebras and Coalgebras*, Comm. Algebra, Vol. **30**(3) (2012), 862–896.
- [AMS1] A. Ardizzoni, C. Menini and D. Ştefan, *A Monoidal Approach to Splitting Morphisms of Bialgebras*, Trans. Amer. Math. Soc., **359** (2007), 991–1044.
- [AMS2] A. Ardizzoni, C. Menini and D. Ştefan, *Hochschild Cohomology And ‘Smoothness’ In Monoidal Categories*, J. Pure Appl. Algebra, Vol. **208** (2007), 297–330.

- [AMStu] A. Ardizzoni, C. Menini and F. Stumbo, *Small Bialgebras with a Projection*, J. Algebra Vol. **314**(2) (2007), 613-663.
- [An] I. E. Angiono, *Basic quasi-Hopf algebras over cyclic groups*. Adv. Math. **225** (2010), no. 6, 3545–3575.
- [AP] A. Ardizzoni, A. Pavarin, *Bosonization for Dual Quasi-Bialgebras and Preantipode*, J. Algebra, Vol. **390** (2013), Pages 126–159.
- [AS1] N. Andruskiewitsch, H.-J. Schneider, *Lifting of quantum linear spaces and pointed Hopf algebras of order p^3* . J. Algebra **209** (1998), no. 2, 658–691.
- [AS2] N. Andruskiewitsch, H.-J. Schneider, *On the classification of finite-dimensional pointed Hopf algebras*, Ann. of Math. **171** (2010), no. 1, 375–417.
- [Bo] N. BOURBAKI, *Groupes et algèbres de Lie*, Chapitres 4, 5 et 6, Hermann, Paris, 1968.
- [Br] K. S. Brown, *Cohomology of groups*. Graduate Texts in Mathematics, **87**. Springer-Verlag, New York-Berlin, 1982.
- [CE] H. Cartan, S. Eilenberg, *Homological algebra*. Princeton University Press, Princeton, N. J., 1956.
- [DNR] S. Dăscălescu, C. Năstăsescu, S. Raianu, *Hopf algebras. An introduction*. Monographs and Textbooks in Pure and Applied Mathematics, **235**. Marcel Dekker, Inc., New York, 2001.
- [DP] C. De Concini, C. Procesi, *Quantum groups*. D-modules, representation theory, and quantum groups, 31–140, Lecture Notes in Math. **1565**, Springer, 1993.
- [EG] P. Etingof, S. Gelaki, *Liftings of graded quasi-Hopf algebras with radical of prime codimension*. J. Pure Appl. Algebra **205** (2006), no. 2, 310–322.
- [FK] S. Fomin, A. N. Kirillov, *Quadratic algebras, Dunkl elements and Schubert calculus*, Progr. Math. **172** (1999), 146–182.
- [GiK] V. Ginzburg, S. Kumar, *Cohomology of quantum groups at roots of unity*. Duke Math. J. **69** (1993) 179–198.
- [GM] L. Grunenfelder; M. Mastnak, *Pointed Hopf Algebras As Cocycle Deformations*, preprint. ([arXiv:1010.4976v1](https://arxiv.org/abs/1010.4976v1))
- [H] H. Hiller, *Geometry of Coxeter groups*. Research Notes in Mathematics, **54**. Pitman (Advanced Publishing Program), Boston, Mass.-London, 1982.
- [L] G. Lusztig, *Quantum groups at roots of 1*. Geom. Dedicata **35** (1990), 89–113.
- [K] C. Kassel, *Quantum groups*. Graduate Texts in Mathematics, **155**. Springer-Verlag, New York, 1995.
- [Maj] S. Majid, *Foundations of quantum group theory*. Cambridge University Press, Cambridge, 1995.
- [MPSW] M. Mastnak, J. Pevtsova, P. Schauenburg, S. Witherspoon, *Cohomology of finite-dimensional pointed Hopf algebras*. Proc. Lond. Math. Soc. **100** (2010), 377–404.
- [MiS] A. Milinski, H.J. Schneider, *Pointed indecomposable Hopf algebras over Coxeter groups*, Contemp. Math. **267** (2000), 215–236.
- [Mo] S. Montgomery, *Hopf Algebras and their actions on rings*, CMBS Regional Conference Series in Mathematics **82**, 1993.
- [Ra1] D. E. Radford, *Hopf algebras*. Series on Knots and Everything, **49**. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012.
- [Ra2] D. E. Radford, *Minimal quasitriangular Hopf algebras*, J. Algebra **157** (1993), 285–315.
- [Ra3] D. E. Radford, *The structure of Hopf algebras with a projection*. J. Algebra **92** (1985), no. 2, 322–347.
- [RT] D. E. Radford, J. Towber, *Yetter-Drinfel'd categories associated to an arbitrary bialgebra*. J. Pure Appl. Algebra **87** (1993), no. 3, 259–279.
- [Sc] P. Schauenburg, *Faithful flatness over Hopf subalgebras: counterexamples*. Interactions between ring theory and representations of algebras (Murcia), 331–344, Lecture Notes in Pure and Appl. Math., **210**, Dekker, New York, 2000.
- [Stf] D. Ştefan, *Hochschild cohomology on Hopf Galois extensions*. J. Pure Appl. Algebra **103** (1995), no. 2, 221–233.
- [Stn] B. Stenström, *Rings of quotients Die Grundlehren der Mathematischen Wissenschaften, Band 217. An introduction to methods of ring theory*. Springer-Verlag, New York-Heidelberg, 1975.
- [SV] D. Ştefan, C. Vay, *The cohomology ring of the 12-dimensional Fomin-Kirillov algebra*. [arXiv:1404.5101](https://arxiv.org/abs/1404.5101).
- [Sw] M. E. Sweedler, *Hopf algebras*. Mathematics Lecture Note Series W. A. Benjamin, Inc., New York 1969.
- [W] X. Wang, *Semisimple connected Hopf Algebras*. ([arXiv:1212.0622v1](https://arxiv.org/abs/1212.0622v1))

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