Research Article

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Coclosed G₂-structures inducing nilsolitons

Abstract: We show obstructions to the existence of a coclosed G_2-structure on a Lie algebra $\mathfrak{g}$ of dimension seven with non-trivial center. In particular, we prove that if there exists a Lie algebra epimorphism from $\mathfrak{g}$ to a six-dimensional Lie algebra $\mathfrak{h}$, with the kernel contained in the center of $\mathfrak{g}$, then any coclosed G_2-structure on $\mathfrak{g}$ induces a closed and stable three form on $\mathfrak{h}$ that defines an almost complex structure on $\mathfrak{h}$. As a consequence, we obtain a classification of the 2-step nilpotent Lie algebras which carry coclosed G_2-structures. We also prove that each one of these Lie algebras has a coclosed G_2-structure inducing a nilsoliton metric, but this is not true for 3-step nilpotent Lie algebras with coclosed G_2-structures. The existence of contact metric structures is also studied.

Keywords: Cocalibrated G_2 forms, half-flat structures, nilpotent Lie algebras, nilsolitons, contact forms

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1 Introduction

A 7-dimensional smooth manifold M admits a G_2-structure if there is a reduction of the structure group of its frame bundle from $\mathrm{Gl}(7, \mathbb{R})$ to the exceptional Lie group G_2, which can actually be viewed naturally as a subgroup of $\mathrm{SO}(7)$. Therefore, a G_2-structure determines a Riemannian metric and an orientation on M. The presence of a G_2-structure is equivalent to the existence of a positive 3-form, where the positivity is a natural nondegeneracy condition (see Section 2 for details). Such a 3-form φ defines a unique Riemannian metric g_φ and an orientation on M.

Whenever this 3-form φ is covariantly constant with respect to the Levi-Civita connection of g_φ or, equivalently, the intrinsic torsion of the G_2-structure vanishes [25], the holonomy group is contained in G_2, and this happens if and only if the 3-form φ is closed and coclosed [12]. A G_2-structure is called *closed* if the 3-form φ is closed, and a G_2-structure is said to be *coclosed* if the 3-form φ is coclosed. Usually these two classes of G_2-structures are very different in nature, being that the closed condition is much more restrictive; for example, coclosed G_2-structures always exist on closed spin manifolds and satisfy the parametric h-principle [7].

In [5], Conti and the second author classified the nilmanifolds endowed with an invariant closed G_2-structure (see also [10]). By a nilmanifold M we mean a compact manifold which is a quotient $M = \Gamma \setminus G$, where G is a connected, simply connected and nilpotent Lie group, and $\Gamma \subset G$ is a lattice. By Mal'cev theorem [23], a lattice $\Gamma \subset G$ exists if and only if the Lie algebra $\mathfrak{g}$ of G has a basis such that the structure constants of $\mathfrak{g}$ are rational numbers. Therefore, any nilmanifold is parallelizable, and so spin for any Riemannian metric. Then, by [7, Theorem 1.8], every nilmanifold has a coclosed G_2-structure.

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In this paper we study the existence of invariant coclosed G$_2$-structures on 2-step nilmanifolds $\Gamma \setminus G$. Since invariant differential forms on $\Gamma \setminus G$ are uniquely determined by forms on the Lie algebra $\mathfrak{g}$ of G, we can restrict our attention to 7-dimensional nilpotent Lie algebras, which were classified in [14]. We show two necessary conditions that a Lie algebra must satisfy if it supports a coclosed G$_2$-structure (see Corollary 3.2 and Lemma 3.3). It turns out that if the 7-dimensional nilpotent Lie algebra $\mathfrak{g}$ is 2-step decomposable, then it admits a coclosed G$_2$-structure (Corollary 4.4), but if $\mathfrak{g}$ is indecomposable, up to isomorphism, only 7 of the 9 indecomposable 2-step nilpotent Lie algebras carry a coclosed G$_2$-structure (Theorem 5.1). As a result we obtain that there exist 2-step nilmanifolds which admit a coclosed G$_2$-structure, but not an invariant one.

Nilpotent Lie groups cannot admit left invariant Einstein metrics. Natural generalizations of Einstein metrics are given by Ricci solitons, which have been introduced by Hamilton in [16]. All known examples of non-trivial homogeneous Ricci solitons are left invariant metrics on simply connected solvable Lie groups, whose Ricci operator satisfies the condition $\mathrm{Ric}(g) = \lambda I + D$ for some $\lambda \in \mathbb{R}$ and some derivation D of the corresponding Lie algebra. The left invariant metrics satisfying the previous condition are called nilsolitons if the Lie groups are nilpotent [22]. Not all nilpotent Lie groups admit nilsoliton metrics, but if a nilsoliton exists, then it is unique up to automorphism and scaling [22].

Closed G$_2$-structures inducing nilsolitons have been studied in [11]. A natural question is thus to see how is restrictive to impose that a left invariant coclosed G$_2$-structure on a nilpotent Lie group induces a nilsoliton metric. In Section 6 we prove that all the 2-step nilpotent Lie groups admitting a left invariant coclosed G$_2$-structure have a coclosed G$_2$-structure inducing a nilsoliton (see Theorems 6.1 and 6.3). However, this property is not true for higher steps. Indeed, in Example 6.4, we show that there exists a 3-step nilpotent Lie group admitting a nilsoliton and a left invariant coclosed G$_2$-structure, but without having any coclosed G$_2$-structure inducing the nilsoliton.

Given a smooth manifold M of dimension seven, one can ask the existence not only of a G$_2$-structure but also of a contact structure. By [2], every manifold with G$_2$-structure admits an almost contact structure, and two types of compatibility between contact and G$_2$-structures have been studied. In Section 7, we show that if a 2-step nilmanifold $\Gamma \setminus G$ admits a contact metric (g, η) such that the metric g is induced by a coclosed G$_2$-structure, then G is isomorphic to the 7-dimensional Heisenberg Lie group (Proposition 7.1), and therefore $\Gamma \setminus G$ has an invariant Sasakian structure. In higher step, this property is not true. Indeed, in Example 7.2, we construct a 3-step (not Sasakian) 7-dimensional nilpotent Lie algebra admitting a K-contact metric structure (η, g) such that the metric g is determined by a coclosed G$_2$-structure.

2 Algebraic preliminaries on stable forms

In this section we collect some results about stable forms on an n-dimensional real vector space. We focus on the existence of stable forms in dimension $n = 6$ and $n = 7$ [6, 9, 17, 24, 26].

Let V be a real vector space of dimension n. Consider the representation of the general linear group $\mathrm{GL}(V)$ on the space $\Lambda^k(V^*)$ of k-forms on V. An element $\rho \in \Lambda^k(V^*)$ is said to be *stable* if its orbit under $\mathrm{GL}(V)$ is open in $\Lambda^k(V^*)$.

In the following proposition, we recall the values of k and n for which there exist open orbits in $\Lambda^k(V^*)$ under the action of $\mathrm{GL}(V)$, and so for which there exist stable k-forms on V.

Proposition 2.1 ([6, 18]). *Let V be an n-dimensional real vector space. For $1 \leq k \leq [\frac{n}{2}]$, the general linear group $\mathrm{GL}(V)$ has an open orbit in $\Lambda^k(V^*)$ if and only if $k \leq 2$, or if $k = 3$ and $n = 6$, 7 or 8. Moreover, the number of open orbits in $\Lambda^k(V^*)$ is finite.*

Note that any non-zero 1-form on V is stable. In fact, if α is such a 1-form, then the orbit of α is open in V^*, since $\mathrm{GL}(V) \cdot \alpha = V^* \setminus \{0\}$. Moreover, according to the following result due to Hictchin [17, 18], for any stable k-form ρ on V, there is a *dual* $(n - k)$-form $\hat{\rho} \in \Lambda^{n-k}(V^*)$, which is also stable and such that $\hat{\rho} \wedge \rho$ defines a volume form on V.

Proposition 2.2 ([6, 17, 18, 26]). *Let V be an n-dimensional oriented vector space. Assume that $k \in \{2, n-2\}$ and n even, or $k \in \{3, n-3\}$ and $n = 6, 7$ or 8. Then, there exists a $\mathrm{GL}(V)$-equivariant map*

$$\varepsilon \colon \Lambda^k(V^*) \to \Lambda^n(V^*), \tag{2.1}$$

homogeneous of degree $\frac{n}{k}$, such that it assigns a volume form to a stable k-form and it vanishes on non-stable forms. Given a stable k-form ρ, the derivative of ε in ρ defines a dual $(n-k)$-form $\widehat{\rho} \in \Lambda^{n-k}(V^*)$ *by the following property:*

$$d\varepsilon_\rho(\alpha) = \widehat{\rho} \wedge \alpha$$

for all $\alpha \in \Lambda^k(V^*)$. Moreover, the dual form $\widehat{\rho}$ is also stable, the identity component of its stabiliser is equal to the stabiliser of ρ, and the forms ρ, $\widehat{\rho}$ and the volume form $\varepsilon(\rho)$ are related by

$$\widehat{\rho} \wedge \rho = \frac{n}{k}\varepsilon(\rho).$$

In the following subsections, we recall the explicit description of the spaces of open orbits in $\Lambda^k(V^*)$ when $n = 6$ and when $n = 7$.

2.1 Stable forms in dimension six

Let V be a 6-dimensional oriented vector space.

Theorem 2.3 ([17–19]). *There is a unique $\mathrm{GL}(V)$ open orbit in $\Lambda^2(V^*)$, which can be characterized as follows:*

$$\Lambda_0(V^*) = \{\omega \in \Lambda^2(V^*) \mid \omega^3 \neq 0\}.$$

Therefore, if $\omega \in \Lambda_0(V^*)$, then its stabiliser is isomorphic to $\mathrm{Sp}(6, \mathbb{R})$, its volume form $\varepsilon(\omega)$ (where ε is the map (2.1)) can be chosen to be equal to $\frac{1}{3!}\omega^3$, and its dual form $\widehat{\omega}$ (in the sense of Proposition 2.2) is equal to $\frac{1}{2}\omega^2$. Moreover, there exists a suitable coframe $\{f^1, \ldots, f^6\}$ of V^* such that

$$\omega = f^{12} + f^{34} + f^{56},$$
$$\widehat{\omega} = f^{1234} + f^{1256} + f^{3456},$$
$$\varepsilon(\omega) = f^{123456},$$

where f^{12} stands for $f^1 \wedge f^2$, and so on.

In order to describe the open orbits in $\Lambda^3(V^*)$, we proceed as follows. For any $\rho \in \Lambda^3(V^*)$, we consider the map $k_\rho \colon V \mapsto \Lambda^5(V^*)$ defined by

$$k_\rho(x) = \iota_x \rho \wedge \rho,$$

where ι_x denotes the contraction by the vector $x \in V$. Clearly, we have $\Lambda^5(V^*) \cong V \otimes \Lambda^6(V^*)$. Indeed, the map $\mu \colon \Lambda^5(V^*) \mapsto V \otimes \Lambda^6(V^*)$ given by $\mu(\xi) = x \otimes \alpha \in V \otimes \Lambda^6(V^*)$, with $\iota_x(\alpha) = \xi$, is an isomorphism. Thus, we have the linear map

$$K_\rho = \mu \circ k_\rho \colon V \mapsto V \otimes \Lambda^6(V^*). \tag{2.2}$$

Then, one can also define the quadratic function

$$\lambda \colon \Lambda^3(V^*) \mapsto (\Lambda^6(V^*))^{\otimes 2}$$

on the space $\Lambda^3(V^*)$ by

$$6\,\lambda(\rho) = \mathrm{trace}(K_\rho{}^2) \in (\Lambda^6(V^*))^{\otimes 2}. \tag{2.3}$$

    An element $\lambda(\rho) \in (\Lambda^6(V^*))^{\otimes 2}$, where $\rho \in \Lambda^3(V^*)$, is said to be *positive*, and we write $\lambda(\rho) > 0$, if $\lambda(\rho) = v \otimes v$ with $v \in \Lambda^6(V^*)$. In this case, the form ρ is called *positive*. An element $\lambda(\rho) \in (\Lambda^6(V^*))^{\otimes 2}$ is said to be *negative*, and we write $\lambda(\rho) < 0$, if $-\lambda(\rho)$ is positive, that is, $-\lambda(\rho) = v \otimes v$. Then, the form ρ is called *negative*.

Now, let us suppose that $\rho \in \Lambda^3(V^*)$ is such that $\lambda(\rho) \neq 0$. Then, we can consider the linear map $J_\rho \colon V \to V$ such that for $x \in V$, $J_\rho(x)$ is defined by

$$J_\rho(x) = \frac{1}{\sqrt{\lambda(\rho)}} K_\rho(x) \in V \tag{2.4}$$

if $\lambda(\rho) > 0$, and

$$J_\rho(x) = \frac{1}{\sqrt{-\lambda(\rho)}} K_\rho(x) \in V \tag{2.5}$$

if $\lambda(\rho) < 0$.

Theorem 2.4 ([17–19]). *The unique* $\mathrm{GL}(V)$ *open orbits in* $\Lambda^3(V^*)$ *are the two following sets*

$$\Lambda_+(V^*) = \{\rho \in \Lambda^3 V^* \mid \lambda(\rho) > 0\} \quad \text{and} \quad \Lambda_-(V^*) = \{\rho \in \Lambda^3 V^* \mid \lambda(\rho) < 0\},$$

where $\lambda(\rho)$ *is given by* (2.3). *If* $\rho \in \Lambda_+(V^*)$, *then the identity component of its stabiliser is isomorphic to* $\mathrm{SL}(3, \mathbb{R}) \times \mathrm{SL}(3, \mathbb{R})$, *and there is a coframe* $\{f^1, \ldots, f^6\}$ *of* V^* *such that* ρ *has the following expression:*

$$\rho = f^{123} + f^{456}.$$

If $\rho \in \Lambda_-(V^*)$, *then the identity component of its stabiliser is isomorphic to* $\mathrm{SL}(3, \mathbb{C})$ *and there is a coframe* $\{f^1, \ldots, f^6\}$ *of* V^* *such that*

$$\rho = -f^{246} + f^{136} + f^{145} + f^{235}.$$

Moreover, $\rho \in \Lambda_+(V^*)$ *if and only if the map* J_ρ, *defined by* (2.4), *is a paracomplex structure on* V; *and* $\rho \in \Lambda_-(V^*)$ *if and only if the map* J_ρ, *defined by* (2.5), *is a complex structure on* V. *In both cases the dual form* $\widehat{\rho}$ *of* ρ *is given by* $\widehat{\rho} = -J_\rho^* \rho$.

Remark 2.5. Note that a 3-form ρ on a 6-dimensional oriented vector space V is stable if and only if $\lambda(\rho) \neq 0$. Moreover, if $\rho \in \Lambda^3(V^*)$ is a 3-form on V, then $k_\rho = k_{-\rho}$, so $K_\rho = K_{-\rho}$ and $\lambda(\rho) = \lambda(-\rho)$. Thus, if $\rho \in \Lambda_+(V^*)$, then $-\rho \in \Lambda_+(V^*)$; and if $\rho \in \Lambda_-(V^*)$, then $-\rho \in \Lambda_-(V^*)$.

2.2 SU(3)-structures as pairs of stable forms in $\Lambda_0(V^*) \times \Lambda_-(V^*)$

We recall the notion of SU(3)-structure on a vector space V of (real) dimension 6. An SU(3)-*structure* on V is a triple (g, J, ψ) such that (g, J) is an almost Hermitian structure on V, and $\psi = \psi_+ + i\psi_-$ is a complex $(3, 0)$-form which satisfies

$$\psi_+ \wedge \psi_- = \frac{2}{3}\omega^3, \tag{2.6}$$

where ω is the Kähler form of (g, J), and ψ_+ and ψ_- are the real part and the imaginary part of ψ, respectively. It is clear that $\omega \wedge \psi_+ = \omega \wedge \psi_- = 0$ and $\psi_- = J\psi_+$.

Theorem 2.6 ([17, 26]). *Let* $(\omega, \psi_-) \in \Lambda_0(V^*) \times \Lambda_-(V^*)$ *be such that*

$$\omega \wedge \psi_- = 0.$$

Let J_{ψ_-} *be the complex structure on* V *defined by* (2.5), *and let* $h \colon V \times V \to \mathbb{R}$ *be the map given by*

$$h(x, y) = \omega(x, J_{\psi_-} y)$$

for $x, y \in V$. *Then, if* h *is positive definite, the stabiliser of the pair* (ω, ψ_-) *is a subgroup of* $\mathrm{SO}(V, h)$ *isomorphic to* $\mathrm{SU}(3)$, *that is, the pair* (ω, ψ_-) *defines an* $\mathrm{SU}(3)$-*structure for which* J_{ψ_-} *is the complex structure,* $\psi = -J_{\psi_-}^* \psi_- + i\psi_-$ *is the complex volume form and* h *is the underlying Hermitian metric. Furthermore, any other* $\mathrm{SU}(3)$-*structure is obtained in this way.*

If (ω, ψ_-) is normalized, *that is, the condition (2.6) is satisfied, then the dual form $\widehat{\omega}$ and the real part ψ_+ of ψ are given by*

$$\widehat{\omega} = \star_h \omega, \quad \psi_+ = \star_h \psi_-,$$

where $\star_h$ is the Hodge star operator of h; and there exists a suitable (h-orthonormal) coframe $\{f^1, \ldots, f^6\}$ of V^* such that

$$\omega = f^{12} + f^{34} + f^{56}, \quad \psi_- = -f^{246} + f^{136} + f^{145} + f^{235}.$$

2.3 Stable forms in dimension seven

Let V be a 7-dimensional oriented vector space. For each 3-form $\varphi \in \Lambda^3(V^*)$ on V, we can define the symmetric quadratic form $b_\varphi \colon V \times V \to \Lambda^7(V^*)$ by

$$6 b_\varphi(x, y) = \iota_x \varphi \wedge \iota_y \varphi \wedge \varphi,$$

where $x, y \in V$. If φ is stable, then (see [17, 26]) the volume form $\varepsilon(\varphi)$ defined by the map (2.1) is given by

$$\varepsilon(\varphi) = \sqrt[9]{\det(b_\varphi)}.$$

Then, we can consider the symmetric map $g_\varphi \colon V \times V \to \mathbb{R}$ given by

$$g_\varphi(x, y)\varepsilon(\varphi) = b_\varphi(x, y), \tag{2.7}$$

where $x, y \in V$.

Definition 2.7. Let V be a 7-dimensional oriented vector space. A 3-form φ on V is called *positive* if g_φ is positive definite; and φ is said to be *negative* if g_φ is indefinite.

The following result states that the positive and negative forms on V are the unique stable 3-forms on V.

Theorem 2.8 ([17, 26]). *There are exactly two* $\mathrm{GL}(V)$ *open orbits in* $\Lambda^3(V^*)$ *which are defined by*

$$\Pi_+(V^*) = \{\varphi \in \Lambda^3(V^*) \mid g_\varphi \text{ is positive definite}\}$$

and

$$\Pi_-(V^*) = \{\varphi \in \Lambda^3(V^*) \mid g_\varphi \text{ is indefinite}\}.$$

If $\varphi \in \Pi_+(V^*)$, then the stabiliser of φ is a subgroup of $\mathrm{O}(V, g_\varphi)$ isomorphic to G_2, and the dual form $\phi = \widehat{\varphi}$ of φ is given by $\phi = \star_\varphi \varphi$, where $\star_\varphi$ is the Hodge star operator of the metric g_φ. Moreover, in a suitable g_φ-orthonormal coframe $\{f^1, \ldots, f^7\}$ of V^*, the forms φ and $\phi = \widehat{\varphi}$ have the following expressions:

$$\varphi = f^{127} + f^{347} + f^{567} + f^{135} - f^{146} - f^{236} - f^{245},$$
$$\phi = f^{1234} + f^{1256} + f^{1367} + f^{1457} + f^{2357} - f^{2467} + f^{3456}.$$

If $\varphi \in \Pi_-(V^*)$, then the metric g_φ has signature $(3, 4)$, and the stabiliser of φ is the non-compact group G_2^*. In this case, there exists a g_φ-orthonormal coframe $\{f^1, \ldots, f^7\}$ of V^* such that the forms φ and $\phi = \widehat{\varphi}$ are given by

$$\varphi = -f^{127} + f^{347} + f^{567} + f^{135} - f^{146} - f^{236} - f^{245},$$
$$\phi = \star_\varphi \varphi = f^{3456} - f^{1234} - f^{1256} - f^{2467} + f^{1367} + f^{1457} + f^{2357}.$$

Remark 2.9. The correspondence $\{\varphi \mapsto \star_\varphi \varphi\}$ from the set of stable 3-forms to the one of stable 4-forms is $2 : 1$, preserves both positivity and negativity, and verifies $g_\varphi = g_{\star_\varphi \varphi}$. A section of this map is completely determined by an orientation on V (see [3]).

3 Coclosed G$_2$-structures and obstructions

In this section we show obstructions to the existence of a coclosed G$_2$ form on a Lie algebra with non-trivial center, but not necessarily nilpotent. First, we recall some definitions and results about G$_2$-structures.

A 7-dimensional smooth manifold M is said to admit a G$_2$-*structure* if there is a reduction of the structure group of its frame bundle from GL(7, $\mathbb{R}$) to the exceptional Lie group G$_2$, which can actually be viewed naturally as a subgroup of SO(7). Thus, a G$_2$-structure determines a Riemannian metric and an orientation on M. In fact, the presence of a G$_2$-structure is equivalent to the existence of a 3-form φ (the G$_2$ form) on M, which is *positive* (in the sense of Definition 2.7) on the tangent space T_pM of M at every point $p \in M$.

If φ is a G$_2$-form on M, then by (2.7) φ induces both an orientation and a Riemannian metric g_φ on M given by

$$6g_\varphi(X, Y) \, \mathrm{vol} = \iota_X\varphi \wedge \iota_Y\varphi \wedge \varphi$$

for any vector fields X, Y on M, where vol is the volume form on M, and ι_X denotes the contraction by X. Let $\star_\varphi$ be the Hodge star operator determined by g_φ and the orientation induced by φ. We will always write ϕ to mean the dual 4-form of a G$_2$ form, that is,

$$\phi = \star_\varphi\varphi.$$

We say that a manifold M has a *coclosed* G$_2$-*structure* if there is a G$_2$-structure on M such that the G$_2$ form φ is coclosed, that is, $d\phi = 0$.

Now, let G be a 7-dimensional simply connected Lie group with Lie algebra $\mathfrak{g}$. Then a G$_2$-structure on G is *left invariant* if and only if the corresponding 3-form is left invariant. According to Theorem 2.8, a left invariant G$_2$-structure on G is defined by a positive 3-form $\varphi \in \Pi_+(\mathfrak{g}^*)$, which can be written as

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}, \tag{3.1}$$

with respect to some orthonormal coframe $\{e^1, \ldots, e^7\}$ of $\mathfrak{g}^*$. So the dual form $\phi = \star_\varphi\varphi$ has the following expression:

$$\phi = e^{1234} + e^{1256} + e^{1367} + e^{1457} + e^{2357} - e^{2467} + e^{3456}, \tag{3.2}$$

where e^{127} stands for $e^1 \wedge e^2 \wedge e^7$, and so on.

A G$_2$-structure on $\mathfrak{g}$ is said to be *coclosed* (or *cocalibrated*) if φ is coclosed, that is,

$$d\phi = 0,$$

where d denotes the Chevalley-Eilenberg differential on $\mathfrak{g}^*$. By [23], we know that if $\mathfrak{g}$ is nilpotent with rational structure constants, then the associated simply connected nilpotent Lie group G admits a uniform discrete subgroup Γ. Therefore, a G$_2$-structure on $\mathfrak{g}$ determines a G$_2$-structure on the compact manifold $\Gamma \backslash G$, which is called a compact nilmanifold; and if $\mathfrak{g}$ has a coclosed G$_2$-structure, the G$_2$-structure on $\Gamma \backslash G$ is also coclosed.

In order to show obstructions to the existence of a coclosed G$_2$-structure on a Lie algebra $\mathfrak{g}$, let us consider first the case that $\mathfrak{g}$ is a direct sum of two ideals $\mathfrak{h}$ and $\mathbb{R}$, i.e.,

$$\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R},$$

where $\mathfrak{h}$ is a 6-dimensional Lie algebra. If ϕ is a 4-form defining a G$_2$-structure on $\mathfrak{g}$, and the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$ is *orthogonal with respect to the underlying metric on* $\mathfrak{g}$, then

$$\phi = \frac{1}{2}\omega^2 + \psi_- \wedge dt,$$

where the pair (ω, ψ_-) defines an SU(3)-structure on $\mathfrak{h}$, and t is a coordinate on $\mathbb{R}$. Now, the condition that ϕ is closed is equivalent to both ω^2 and ψ_- are closed. This means that the SU(3)-structure is *half-flat*. There are exactly 24 nilpotent Lie algebras of dimension six that admit a half-flat structure [4]. Hence, if we focus our attention on decomposable nilpotent Lie algebras, then there are at least 24 nilpotent Lie algebras, of dimension seven, with a coclosed G$_2$-structure. In Theorem 4.3, we show that those are exactly the decomposable nilpotent Lie algebras admitting coclosed G$_2$-structures.

Let $\mathfrak{g}$ be a 7-dimensional Lie algebra with non-trivial center. If $X \in \mathfrak{g}$ belongs to the center of $\mathfrak{g}$, then the quotient $\mathfrak{h} = \mathfrak{g}/\mathrm{Span}\{X\}$ has a unique Lie algebra structure that makes the projection map $\pi\colon \mathfrak{g} \to \mathfrak{h}$ a Lie algebra morphism. Thus, we have the following short exact sequence of Lie algebras:

$$0 \longrightarrow \mathbb{R}X \longrightarrow \mathfrak{g} \overset{\pi}{\longrightarrow} \mathfrak{h} \longrightarrow 0.$$

Moreover, if we assume that $\mathfrak{g}$ is nilpotent, then every epimorphism $\mathfrak{g} \to \mathfrak{h}$, with $\mathfrak{h}$ of dimension six, is of this form.

We need also to recall the following result due to Schulte [26, Proposition 4.5]. If φ is a G$_2$-structure on a 7-dimensional Lie algebra and we choose a vector $X \in \mathfrak{g}$ of length one with respect to the metric g_φ determined by φ, then on the orthogonal complement of the span of X one has an SU(3)-structure (ω, ψ_-) given by the 2-form $\omega = \iota_X \varphi$ and the 3-form $\psi_- = -\iota_X \phi$, where ϕ is the dual 4-form of φ, that is, $\phi = \star_\varphi \varphi$.

Proposition 3.1. *Let $\mathfrak{g}$ be a 7-dimensional Lie algebra with non-trivial center and a 3-form φ defining a coclosed G$_2$-structure on $\mathfrak{g}$. If $\pi\colon \mathfrak{g} \to \mathfrak{h}$ is an epimorphism from $\mathfrak{g}$ to a 6-dimensional Lie algebra $\mathfrak{h}$, with kernel contained in the center of $\mathfrak{g}$, then φ determines an SU(3)-structure (ω, ψ_-) on $\mathfrak{h}$ such that the 3-form ψ_- is closed.*

Proof. Denote by g_φ the underlying metric on $\mathfrak{g}$ defined by the G$_2$-structure φ, and denote by ϕ the dual 4-form of φ, that is, $\phi = \star_\varphi \varphi$. Let X be a unit vector in the center of $\mathfrak{g}$, and let $\eta = \iota_X(g_\varphi)$ be the dual form of X with respect to g_φ. By the aforementioned result of [26, Proposition 4.5], we know that φ induces an SU(3)-structure $(\widetilde{\omega}, \widetilde{\psi}_-)$ on the orthogonal complement V of the span of X, that is, on $\ker(\eta)$, such that

$$\phi = \widetilde{\sigma} + \widetilde{\psi}_- \wedge \eta,$$

where $\widetilde{\sigma} \in \Lambda^4(V^*)$ is the dual 4-form of $\widetilde{\omega}$ with respect to the metric defined by $(\widetilde{\omega}, \widetilde{\psi}_-)$ on the space $\ker(\eta)$.

Now, consider the Lie algebra $\mathfrak{h} = \mathfrak{g}/\mathrm{Span}\{X\}$ and the projection map

$$\pi\colon \mathfrak{g} \to \mathfrak{h}.$$

Clearly, $\mathfrak{h}$ and V are isomorphic as vector spaces. Then, fixing an isomorphism between these spaces and doing the pullback of the SU(3)-structure $(\widetilde{\omega}, \widetilde{\psi}_-)$ on V, we have an SU(3)-structure (ω, ψ_-) on $\mathfrak{h}$ such that

$$\phi = \pi^* \sigma + \pi^* \psi_- \wedge \eta, \tag{3.3}$$

where $\widetilde{\sigma} \in \Lambda^4(\mathfrak{h}^*)$ is the dual 4-form of ω with respect to the metric defined by (ω, ψ_-) on $\mathfrak{h}$. Thus,

$$\psi_- = \pi_*(-\iota_X \phi).$$

Moreover, we see that

$$d\psi_- = 0, \quad d(\pi^* \sigma) = \pi^* \psi_- \wedge d\eta.$$

In fact, since d commutes with the pullback, from (3.3) we have

$$0 = d\phi = \pi^*(d\sigma) + \pi^*(d\psi_-) \wedge \eta - \pi^* \psi_- \wedge d\eta.$$

Taking the contraction by X, and using that X is in the center of $\mathfrak{g}$, that is, $\mathrm{ad}(X) = 0$, we have $d\psi_- = 0$ and $d(\pi^* \sigma) = \pi^* \psi_- \wedge d\eta$. □

As a consequence of Proposition 3.1, we have the following obstruction to existence of coclosed G$_2$-structures on Lie algebras with non-trivial center.

Corollary 3.2. *Let $\mathfrak{g}$ be a 7-dimensional oriented Lie algebra with non-trivial center. If there is an element X in the center of $\mathfrak{g}$ such that*

$$\pi_*(\iota_X \kappa) \notin \Lambda_-(\mathfrak{h}^*) \tag{3.4}$$

for every closed 4-form $\kappa \in \Lambda^4(\mathfrak{g}^*)$, where $\pi\colon \mathfrak{g} \to \mathfrak{h} = \mathfrak{g}/\mathrm{Span}\{X\}$ is the projection map, then $\mathfrak{g}$ does not admit any coclosed G$_2$-structure.

Proof. Let X be a non-zero vector in the center of $\mathfrak{g}$ such that condition (3.4) is satisfied for any closed 4-form on $\mathfrak{g}$. Suppose that there is a 4-form ϕ on $\mathfrak{g}$ defining a coclosed G_2-structure. Then, by Proposition 3.1, the 4-form ϕ determines an SU(3)-structure (ω, ψ_-) on $\mathfrak{h} = \mathfrak{g}/\mathrm{Span}\{X\}$, where $\psi_- = \pi_*(-\iota_X\phi)$ by (3.3). Now, Remark 2.5 and Theorem 2.6 imply that $\pi_*(\iota_X\phi) \in \Lambda_-(\mathfrak{h}^*)$, which contradicts condition (3.4). So, $\mathfrak{g}$ does not admit coclosed G_2-structures. $\square$

Another obstruction to the existence of a coclosed G_2-structure on a Lie algebra is given by the following lemma.

Lemma 3.3. *Let $\mathfrak{g}$ be a 7-dimensional Lie algebra. If there are non-zero vectors X and Y in $\mathfrak{g}$ such that $(\iota_X\iota_Y\kappa)^2 = 0$ for every closed 4-form κ on $\mathfrak{g}$, then $\mathfrak{g}$ does not admit coclosed G_2-structures.*

Proof. It follows from (3.2). $\square$

The following result, whilst straightforward, turns out effective to show that some Lie algebras do not admit coclosed G_2-structures (see Proposition 4.2).

Lemma 3.4. *Let (h, J) be an almost Hermitian structure on a 6-dimensional oriented vector space V, with orthogonal complex structure J, Hermitian metric h and fundamental two-form $\omega(\,\cdot\,,\cdot\,) = h(J\cdot\,,\cdot\,)$. Then, for any J-invariant 4-dimensional subspace W of V, we have that the restriction to W of the 4-form $(\star\omega)$ is non-zero, that is, $(\star\omega)|_W \neq 0$, where $\star$ denotes the Hodge $\star$-operator of the metric h.*

Proof. Since W is J-invariant, there exist non-zero vectors $x, y \in W$ such that $\{x, Jx, y, Jy\}$ is an orthonormal basis of the space W (with respect to h). Let $z \in V$ be a unit vector orthogonal to the space W. Then $\{x, Jx, y, Jy, z, Jz\}$ is a (real) h-orthonormal basis of V. Therefore,

$$(\star\omega)(x \wedge Jx \wedge y \wedge Jy) = \omega(z \wedge Jz) = \|z\|^2 = 1,$$

which proves that $(\star\omega)|_W \neq 0$. $\square$

4 Decomposable case

In this section we classify the 7-dimensional decomposable nilpotent Lie algebras which admit coclosed G_2-structures. Recall that a Lie algebra is called *decomposable* if it is the direct sum of two ideals.

For convenience, from now on we will use the following notation. Suppose that $\mathfrak{g}$ is a 7-dimensional Lie algebra whose dual space $\mathfrak{g}^*$ is spanned by $\{e^1, \dots, e^7\}$ satisfying

$$de^i = 0, \quad 1 \leq i \leq 4, \qquad de^5 = e^{23}, \qquad de^6 = e^{34}, \qquad de^7 = e^{36}.$$

Then we will write

$$\mathfrak{g} = (0, 0, 0, 0, 23, 34, 36),$$

with the same meaning. Moreover, we will denote by $\{e_1, \dots, e_7\}$ the basis of $\mathfrak{g}$ dual to $\{e^1, \dots, e^7\}$.

Proposition 4.1. *If $\mathfrak{g}$ is one of the following 7-dimensional nilpotent Lie algebras:*

$\mathfrak{g}_1 = (0, 0, 0, 0, 12, 15, 0),$ $\qquad$ $\mathfrak{g}_2 = (0, 0, 0, 0, 23, 34, 36),$ $\qquad$ $\mathfrak{g}_3 = (0, 0, 0, 12, 13, 14, 0),$

$\mathfrak{g}_4 = (0, 0, 0, 12, 14, 24, 0),$ $\qquad$ $\mathfrak{g}_5 = (0, 0, 12, 13, 14, 23 + 15, 0),$ $\qquad$ $\mathfrak{g}_6 = (0, 0, 12, 13, 14, 15, 0),$

$\mathfrak{g}_7 = (0, 0, 12, 13, 14, 34 - 25, 0),$ $\quad$ $\mathfrak{g}_8 = (0, 0, 12, 13, 14 + 23, 34 - 25, 0),$

then $\mathfrak{g}$ carries no coclosed G_2-structures.

Proof. Using Lemma 3.3, we will prove that each Lie algebra $\mathfrak{g}_s$, $s \in \{1, \dots, 8\}$, listed in the statement, does not admit any coclosed G_2-structure. For this, we will show that there are non-zero vectors $X_s, Y_s \in \mathfrak{g}_s$ such that $(\iota_{X_s}(\iota_{Y_s}\kappa_s))^2 = 0$ for any closed 4-form κ_s on $\mathfrak{g}_s$.

$s = 1$. A generic closed 4-form κ_1 on $\mathfrak{g}_1 = (0, 0, 0, 0, 12, 15, 0)$ has the following expression:

$$\kappa_1 = c_{1234}e^{1234} + c_{1235}e^{1235} + c_{1236}e^{1236} + c_{1237}e^{1237} + c_{1245}e^{1245} + c_{1246}e^{1246} + c_{1247}e^{1247}$$
$$+ c_{1256}e^{1256} + c_{1257}e^{1257} + c_{1267}e^{1267} + c_{1345}e^{1345} + c_{1346}e^{1346} + c_{1347}e^{1347} + c_{1356}e^{1356}$$
$$+ c_{1357}e^{1357} + c_{1367}e^{1367} + c_{1456}e^{1456} + c_{1457}e^{1457} + c_{1467}e^{1467} + c_{1567}e^{1567} + c_{2345}e^{2345}$$
$$+ c_{2347}e^{2347} + c_{2356}e^{2356} + c_{2357}e^{2357} + c_{2456}e^{2456} + c_{2457}e^{2457} + c_{2567}e^{2567},$$

where c_{ijkl} are arbitrary real numbers. Now, one can check that if the coefficient c_{2356} of κ_1 vanishes, then

$$(\iota_{e_3}(\iota_{e_6}\kappa_1))^2 = 0,$$

that is, $\mathfrak{g}_1$ satisfies the hypothesis of Lemma 3.3 for $X_1 = e_3$ and $Y_1 = e_6$. If $c_{2456} = 0$, then

$$(\iota_{e_4}(\iota_{e_6}\kappa_1))^2 = 0.$$

But if c_{2356} and c_{2456} are both non-zero, then for $X_1 = c_{2356}e_4 - c_{2456}e_3$ and $Y_1 = e_6$, we have

$$(\iota_{c_{2356}e_4 - c_{2456}e_3}(\iota_{e_6}\kappa_1))^2 = 0.$$

$s = 2$. A generic closed 4-form κ_2 on $\mathfrak{g}_2 = (0, 0, 0, 0, 23, 34, 36)$ has the following expression:

$$\kappa_2 = c_{1234}e^{1234} + c_{1235}e^{1235} + c_{1236}e^{1236} + c_{1237}e^{1237} + c_{1245}e^{1245} + c_{1246}e^{1246} + c_{1247}e^{1247}$$
$$+ c_{1345}e^{1345} + c_{1346}e^{1346} + c_{1347}e^{1347} + c_{1356}e^{1356} + c_{1357}e^{1357} + c_{1367}e^{1367}$$
$$- c_{1247}e^{1456} + c_{1467}e^{1467} + c_{2345}e^{2345} + c_{2346}e^{2346} + c_{2347}e^{2347} + c_{2356}e^{2356}$$
$$+ c_{2357}e^{2357} + c_{2367}e^{2367} + c_{2456}e^{2456} + c_{2467}e^{2467} + c_{3456}e^{3456} + c_{3457}e^{3457}$$
$$+ c_{3467}e^{3467} + c_{3567}e^{3567},$$

where c_{ijkl} are arbitrary real numbers. Taking $X_2 = e_5$ and $Y_2 = e_7$, we have $(\iota_{e_5}(\iota_{e_7}\kappa_2))^2 = 0$.

$s = 3$. A generic closed 4-form κ_3 on $\mathfrak{g}_3 = (0, 0, 0, 12, 13, 14, 0)$ is expressed as follows:

$$\kappa_3 = c_{1234}e^{1234} + c_{1235}e^{1235} + c_{1236}e^{1236} + c_{1237}e^{1237} + c_{1245}e^{1245} + c_{1246}e^{1246} + c_{1247}e^{1247}$$
$$+ c_{1256}e^{1256} + c_{1257}e^{1257} + c_{1267}e^{1267} + c_{1345}e^{1345} + c_{1346}e^{1346} + c_{1347}e^{1347} + c_{1356}e^{1356}$$
$$+ c_{1357}e^{1357} + c_{1367}e^{1367} + c_{1456}e^{1456} + c_{1457}e^{1457} + c_{1467}e^{1467} + c_{1567}e^{1567} + c_{2345}e^{2345}$$
$$+ c_{2346}e^{2346} + c_{2347}e^{2347} + c_{2357}e^{2357} + c_{2367}(e^{2367} + e^{2457}) + c_{2467}e^{2467},$$

where c_{ijkl} are arbitrary real numbers. For $X_3 = e_5$ and $Y_3 = e_6$, we have $(\iota_{e_5}(\iota_{e_6}\kappa_3))^2 = 0$.

$s = 4$. A generic closed 4-form κ_4 on $\mathfrak{g}_4 = (0, 0, 0, 12, 14, 24, 0)$ is expressed as follows:

$$\kappa_4 = c_{1234}e^{1234} + c_{1235}e^{1235} + c_{1236}e^{1236} + c_{1237}e^{1237} + c_{1245}e^{1245} + c_{1246}e^{1246} + c_{1247}e^{1247}$$
$$+ c_{1256}e^{1256} + c_{1257}e^{1257} + c_{1267}e^{1267} + c_{1345}e^{1345} + c_{1346}e^{1346} + c_{1347}e^{1347} + c_{1357}e^{1357}$$
$$+ c_{1367}e^{1367} + c_{1456}e^{1456} + c_{1457}e^{1457} + c_{1467}e^{1467} + c_{2345}e^{2345} + c_{2346}e^{2346} + c_{2347}e^{2347}$$
$$+ c_{1367}e^{2357} + c_{2367}e^{2367} + c_{2456}e^{2456} + c_{2457}e^{2457} + c_{2467}e^{2467} + c_{3567}e^{3567},$$

where c_{ijkl} are arbitrary real numbers. Then, for $X_4 = e_5$ and $Y_4 = e_6$, we have $(\iota_{e_5}(\iota_{e_6}\kappa_4))^2 = 0$.

$s = 5$. A generic closed 4-form κ_5 on $\mathfrak{g}_5 = (0, 0, 12, 13, 14, 23 + 15, 0)$ has the following expression:

$$\kappa_5 = c_{1234}e^{1234} + c_{1235}e^{1235} + c_{1236}e^{1236} + c_{1237}e^{1237} + c_{1245}e^{1245} + c_{1246}e^{1246} + c_{1247}e^{1247}$$
$$+ c_{1256}e^{1256} + c_{1257}e^{1257} + c_{1267}e^{1267} + c_{1345}e^{1345} + c_{1346}e^{1346} + c_{1347}e^{1347} + c_{1356}e^{1356}$$
$$+ c_{1357}e^{1357} + c_{1367}e^{1367} + c_{1456}e^{1456} + c_{1457}e^{1457} + c_{1467}e^{1467} + c_{1567}e^{1567} + c_{2345}e^{2345}$$
$$+ c_{1456}e^{2346} + c_{2347}e^{2347} - c_{1467}e^{2357} + c_{2367}e^{2367} - (c_{1567} + c_{2367})e^{2457},$$

where c_{ijkl} are arbitrary real numbers. Then, for $X_5 = e_5$ and $Y_5 = e_6$, we have $(\iota_{e_5}(\iota_{e_6}\kappa_5))^2 = 0$.

$s = 6$. A generic closed 4-form κ_6 on $\mathfrak{g}_6 = (0, 0, 12, 13, 14, 15, 0)$ has the following expression:

$$
\begin{aligned}
\kappa_6 = {}& c_{1234}e^{1234} + c_{1235}e^{1235} + c_{1236}e^{1236} + c_{1237}e^{1237} + c_{1245}e^{1245} + c_{1246}e^{1246} + c_{1247}e^{1247} \\
& + c_{1256}e^{1256} + c_{1257}e^{1257} + c_{1267}e^{1267} + c_{1345}e^{1345} + c_{1346}e^{1346} + c_{1347}e^{1347} + c_{1356}e^{1356} \\
& + c_{1357}e^{1357} + c_{1367}e^{1367} + c_{1456}e^{1456} + c_{1457}e^{1457} + c_{1467}e^{1467} + c_{1567}e^{1567} + c_{2345}e^{2345} \\
& + c_{2347}e^{2347} + c_{2367}e^{2367} - c_{2367}e^{2457},
\end{aligned}
$$

where c_{ijkl} are arbitrary real numbers. So, for $X_6 = e_5$ and $Y_6 = e_7$, we have $(\iota_{e_5}(\iota_{e_7}\kappa_6))^2 = 0$.

$s = 7$. A generic closed 4-form κ_6 on $\mathfrak{g}_7 = (0, 0, 12, 13, 14, 34 - 25, 0)$ has the following expression:

$$
\begin{aligned}
\kappa_7 = {}& c_{1234}e^{1234} + c_{1235}e^{1235} + c_{1236}e^{1236} + c_{1237}e^{1237} + c_{1245}e^{1245} + c_{1246}e^{1246} + c_{1247}e^{1247} \\
& + c_{1256}e^{1256} + c_{1257}e^{1257} + c_{1267}e^{1267} + c_{1345}e^{1345} + c_{1256}e^{1346} + c_{1347}e^{1347} + c_{1356}e^{1356} \\
& + c_{1357}e^{1357} + c_{1367}e^{1367} + c_{1456}e^{1456} + c_{1457}e^{1457} + c_{1467}e^{1467} + c_{2345}e^{2345} + c_{2346}e^{2346} \\
& + c_{2347}e^{2347} - c_{1267}e^{2357} + c_{2367}e^{2367} - c_{1367}e^{2457} - c_{1467}e^{3457},
\end{aligned}
$$

where c_{ijkl} are arbitrary real numbers. Thus, $(i_{e_5}(i_{e_7}\kappa_7))^2 = 0$.

$s = 8$. A generic closed 4-form κ_6 on $\mathfrak{g}_8 = (0, 0, 12, 13, 14 + 23, 34 - 25, 0)$ has the following expression:

$$
\begin{aligned}
\kappa_8 = {}& c_{1234}e^{1234} + c_{1235}e^{1235} + c_{1236}e^{1236} + c_{1237}e^{1237} + c_{1245}e^{1245} + c_{1246}e^{1246} + c_{1247}e^{1247} \\
& + c_{1256}e^{1256} + c_{1257}e^{1257} + c_{1267}e^{1267} + c_{1345}e^{1345} + c_{1256}e^{1346} + c_{1347}e^{1347} + c_{1356}e^{1356} \\
& + c_{1357}e^{1357} + c_{1367}e^{1367} + c_{1456}e^{1456} + c_{1457}e^{1457} + c_{1467}e^{1467} + c_{2345}e^{2345} + c_{2346}e^{2346} \\
& + c_{2347}e^{2347} - c_{1456}e^{2356} - (c_{1267} + c_{1457})e^{2357} + c_{2367}e^{2367} - c_{1367}e^{2457} - c_{1467}e^{3457},
\end{aligned}
$$

where c_{ijkl} are arbitrary real numbers. On can check that if the coefficient c_{1456} vanishes, then

$$
(\iota_{e_5}(\iota_{e_6}\kappa_8))^2 = 0,
$$

that is, $\mathfrak{g}_8$ satisfies the hypothesis of Lemma 3.3 for $X_8 = e_5$ and $Y_8 = e_6$. If $c_{2346} = 0$, then

$$
(\iota_{e_4}(\iota_{e_6}\kappa_8))^2 = 0.
$$

But if c_{1456} and c_{2346} are both non-zero, then for $X_8 = c_{2346}e_5 + c_{1456}e_4$ and $Y_8 = e_6$, we have

$$
(\iota_{c_{2346}e_5 + c_{1456}e_4}(\iota_{e_6}\kappa_8))^2 = 0.
$$

The proof is complete. $\qquad\square$

Moreover, using Lemma 3.4, we have the following proposition.

Proposition 4.2. *None of the following 7-dimensional nilpotent Lie algebras:*

$$
\begin{aligned}
\mathfrak{l}_1 &= (0, 0, 0, 12, 13 - 24, 14 + 23, 0), \\
\mathfrak{l}_2 &= (0, 0, 0, 12, 14, 13 - 24, 0), \\
\mathfrak{l}_3 &= (0, 0, 0, 12, 13 + 14, 24, 0)
\end{aligned}
$$

admits coclosed G_2-*structures.*

Proof. We will prove, by contradiction, that no closed 4-form τ_s defines a coclosed G_2-structure on the Lie algebra $\mathfrak{l}_s$ ($s = 1, 2, 3$). We proceed case by case.

$s = 1$. A generic closed 4-form τ_1 on $\mathfrak{l}_1 = (0, 0, 0, 12, 13 - 24, 14 + 23, 0)$ has the following expression:

$$
\begin{aligned}
\tau_1 = {}& c_{1234}e^{1234} + c_{1235}e^{1235} + c_{1236}e^{1236} + c_{1237}e^{1237} + c_{1245}e^{1245} + c_{1246}e^{1246} + c_{1247}e^{1247} \\
& + c_{1256}e^{1256} + c_{1257}e^{1257} + c_{1267}e^{1267} + c_{1345}e^{1345} + c_{1346}e^{1346} + c_{1347}e^{1347} + c_{1356}e^{1356} \\
& + c_{1357}e^{1357} + c_{1367}e^{1367} + c_{1456}e^{1456} + c_{1457}e^{1457} + c_{1467}e^{1467} + c_{2345}e^{2345} + c_{2346}e^{2346} \\
& + c_{2347}e^{2347} - c_{1456}e^{2356} + c_{2357}e^{2357} + c_{2367}e^{2367} - c_{1356}e^{2456} + (c_{1357} + c_{1467} + c_{2367})e^{2457} \\
& + c_{2467}e^{2467},
\end{aligned}
$$

where c_{ijkl} are arbitrary real numbers. Let us suppose that, for some real numbers c_{ijkl}, the 4-form τ_1 defines a coclosed G_2-structure on $\mathfrak{l}_1$. Since e_7 is in the center of $\mathfrak{l}_1$, Proposition 3.1 implies that

$$\nu_1 = -\pi_*(\iota_{e_7}\tau_1)$$

is a negative 3-form on the Lie algebra $\mathfrak{h}_1 = \mathfrak{l}_1/\operatorname{Span}\{e_7\}$, where $\pi\colon \mathfrak{l}_1 \to \mathfrak{h}_1$ is the projection. Thus,

$$\nu_1 = c_{1237}e^{123} + c_{1247}e^{124} + c_{1257}e^{125} + c_{1267}e^{126} + c_{1347}e^{134} + c_{1357}e^{135} + c_{1367}e^{136} + c_{1457}e^{145}$$
$$+ c_{1467}e^{146} + c_{2347}e^{234} + c_{2357}e^{235} + c_{2367}e^{236} + (c_{1357} + c_{1467} + c_{2367})e^{245} + c_{2467}e^{246}.$$

We claim that the map K_{ν_1}, defined by (2.2), has the following expression:

$$K_{\nu_1} = (K_{ab}^{\nu_1}) \otimes e^{123456} = \begin{pmatrix} K_{11}^{\nu_1} & K_{12}^{\nu_1} & 0 & 0 & 0 & 0 \\ K_{21}^{\nu_1} & K\nu_{122} & 0 & 0 & 0 & 0 \\ K_{31}^{\nu_1} & K_{32}^{\nu_1} & K_{33}^{\nu_1} & K_{34}^{\nu_1} & 0 & 0 \\ K_{41}^{\nu_1} & K_{42}^{\nu_1} & K_{43}^{\nu_1} & K_{44}^{\nu_1} & 0 & 0 \\ K_{51}^{\nu_1} & K_{52}^{\nu_1} & K_{53}^{\nu_1} & K_{54}^{\nu_1} & K_{55}^{\nu_1} & K_{56}^{\nu_1} \\ K_{61}^{\nu_1} & K_{62}^{\nu_1} & K_{63}^{\nu_1} & K_{64}^{\nu_1} & K_{65}^{\nu_1} & K_{66}^{\nu_1} \end{pmatrix} \otimes e^{123456},$$

where $a, b \in \{1, \ldots, 6\}$, and $K_{ab}^{\nu_1}$ is a polynomial function of the coefficients c_{ijkl} that appear in the expression of ν_1. In fact, by (2.2), it turns out that

$$(\iota_{e_b}\nu_1) \wedge \nu_1 = \sum_{1 \leq a \leq 6} K_{ab}^{\nu_1} e^{1\cdots\hat{a}\cdots 6}.$$

Therefore, $K_{ab}^{\nu_1} e^{1\cdots 6} = (\iota_{e_b}\nu_1) \wedge \nu_1 \wedge e^a$. Then, one can check that $K_{ab}^{\nu_1} = 0$ for $a = 1, 2$ and $b \geq 3$, and also $K_{ab}^{\nu_1} = 0$ for $a = 3, 4$ and $b = 5, 6$. Thus, the claim is true.

Since $\nu_1 \in \Lambda_-(\mathfrak{h}_1^*)$, (2.5) and Theorem 2.4 imply that ν_1 defines the almost complex structure J_{ν_1} on $\mathfrak{h}_1$ given by

$$J_{\nu_1} = \frac{1}{\sqrt{|-\lambda(\nu_1)|}} \begin{pmatrix} K_{11}^{\nu_1} & K_{12}^{\nu_1} & 0 & 0 & 0 & 0 \\ K_{21}^{\nu_1} & K_{22}^{\nu_1} & 0 & 0 & 0 & 0 \\ K_{31}^{\nu_1} & K_{32}^{\nu_1} & K_{33}^{\nu_1} & K_{34}^{\nu_1} & 0 & 0 \\ K_{41}^{\nu_1} & K_{42}^{\nu_1} & K_{43}^{\nu_1} & K_{44}^{\nu_1} & 0 & 0 \\ K_{51}^{\nu_1} & K_{52}^{\nu_1} & K_{53}^{\nu_1} & K_{54}^{\nu_1} & K_{55}^{\nu_1} & K_{56}^{\nu_1} \\ K_{61}^{\nu_1} & K_{62}^{\nu_1} & K_{63}^{\nu_1} & K_{64}^{\nu_1} & K_{65}^{\nu_1} & K_{66}^{\nu_1} \end{pmatrix}.$$

Therefore, the subspace $W = \operatorname{Span}\{e_3, e_4, e_5, e_6\}$ is J_{ν_1}-invariant. Now, consider $\eta = \sum_{r=1}^{7} C_r e^r$, an arbitrary 1-form on $\mathfrak{l}_1$. According to (3.3), and taking into account the expressions of τ_1 and ν_1, we see that the 4-form $\sigma = \pi_*(\tau_1 - \pi^*\nu_1 \wedge \eta)$ on $\mathfrak{h}_1$, has zero component in e^{3456}. Hence, $\sigma|_W = \sigma(e_3, e_4, e_5, e_6) = 0$, contradicting Lemma 3.4. Thus, τ_1 never defines a coclosed G_2-structure.

$s = 2$. A generic closed four-form on $\mathfrak{l}_2 = (0, 0, 0, 12, 14, 13 - 24, 0)$ has the following expression:

$$\tau_2 = c_{1234}e^{1234} + c_{1235}e^{1235} + c_{1236}e^{1236} + c_{1237}e^{1237} + c_{1245}e^{1245} + c_{1246}e^{1246} + c_{1247}e^{1247}$$
$$+ c_{1256}e^{1256} + c_{1257}e^{1257} + c_{1267}e^{1267} + c_{1345}e^{1345} + c_{2456}e^{1346} + c_{1347}e^{1347} + c_{2456}e^{1356}$$
$$+ c_{1357}e^{1357} + (c_{2467} - c_{2357})e^{1367} + c_{1456}e^{1456} + c_{1457}e^{1457} + c_{1467}e^{1467} + c_{2345}e^{2345}$$
$$+ c_{2346}e^{2346} + c_{2347}e^{2347} + c_{2357}e^{2357} + c_{2367}e^{2367} + c_{2456}e^{2456} + c_{2457}e^{2457} + c_{2467}e^{2467},$$

where c_{ijkl} are arbitrary real numbers. Let us suppose that for some real numbers c_{ijkl}, the 4-form τ_2 defines a coclosed G_2-structure on $\mathfrak{l}_2$. Since e_7 is in the center of $\mathfrak{l}_2$, by Proposition 3.1, we have that $\nu_2 = -\pi_*(i_{e_7}\tau_2)$ is a negative 3-form on $\mathfrak{h}_2 = \mathfrak{l}_2/\operatorname{Span}\{e_7\}$, where $\pi\colon \mathfrak{l}_2 \to \mathfrak{h}_2$ is the natural projection. Thus, ν_2 is given by

$$\nu_2 = c_{1237}e^{123} + c_{1247}e^{124} + c_{1257}e^{125} + c_{1267}e^{126} + c_{1347}e^{134} + c_{1357}e^{135} + c_{1367}e^{136} + c_{1457}e^{145}$$
$$+ c_{1467}e^{146} + c_{2347}e^{234} + c_{2357}e^{235} + c_{2367}e^{236} + c_{2457}e^{245} + (c_{2357} + c_{1367})e^{246}.$$

We claim that the map K_{v_2}, defined by (2.2), has the following expression:

$$K_{v_2} = (K_{ab}^{v_2}) \otimes e^{123456} = \begin{pmatrix} K_{11}^{v_2} & K_{12}^{v_2} & 0 & 0 & 0 & 0 \\ K_{21}^{v_2} & K_{22}^{v_2} & 0 & 0 & 0 & 0 \\ K_{31}^{v_2} & K_{32}^{v_2} & K_{33}^{v_2} & K_{34}^{v_2} & 0 & 0 \\ K_{41}^{v_2} & K_{42}^{v_2} & K_{43}^{v_2} & K_{44}^{v_2} & 0 & 0 \\ K_{51}^{v_2} & K_{52}^{v_2} & K_{53}^{v_2} & K_{54}^{v_2} & K_{55}^{v_2} & K_{56}^{v_2} \\ K_{61}^{v_2} & K_{62}^{v_2} & K_{63}^{v_2} & K_{64}^{v_2} & K_{65}^{v_2} & K_{66}^{v_2} \end{pmatrix} \otimes e^{123456},$$

where $a, b \in \{1, \dots, 6\}$ and $K_{ab}^{v_2}$ is a polynomial function of the coefficients c_{ijkl} that appear in the expression of v_2. In fact, by (2.2) it turns out that

$$(\iota_{e_b} v_2) \wedge v_2 = \sum_{1 \leq a \leq 6} K_{ab}^{v_2} e^{1 \cdots \hat{a} \cdots 6}.$$

Therefore, $K_{ab}^{v_2} e^{1 \cdots 6} = (\iota_{e_b} v_2) \wedge v_2 \wedge e^a$. Then, one can check that $K_{ab}^{v_2} = 0$ for $a = 1, 2$ and $b \geq 3$, and also $K_{ab}^{v_2} = 0$ for $a = 3, 4$ and $b = 5, 6$. Thus, the claim is true.

Since $v_2 \in \Lambda_-(\mathfrak{h}_2^*)$, (2.5) and Theorem 2.4 imply that v_2 defines the almost complex structure J_{v_2} on $\mathfrak{h}_2$ given by

$$J_{v_2} = \frac{1}{|\sqrt{-\lambda(v_2)}|} \begin{pmatrix} K_{11}^{v_2} & K_{12}^{v_2} & 0 & 0 & 0 & 0 \\ K_{21}^{v_2} & K_{22}^{v_2} & 0 & 0 & 0 & 0 \\ K_{31}^{v_2} & K_{32}^{v_2} & K_{33}^{v_2} & K_{34}^{v_2} & 0 & 0 \\ K_{41}^{v_2} & K_{42}^{v_2} & K_{43}^{v_2} & K_{44}^{v_2} & 0 & 0 \\ K_{51}^{v_2} & K_{52}^{v_2} & K_{53}^{v_2} & K_{54}^{v_2} & K_{55}^{v_2} & K_{56}^{v_2} \\ K_{61}^{v_2} & K_{62}^{v_2} & K_{63}^{v_2} & K_{64}^{v_2} & K_{65}^{v_2} & K_{66}^{v_2} \end{pmatrix}.$$

Therefore, the subspace $W = \mathrm{Span}\{e_3, e_4, e_5, e_6\}$ is J_{v_2}-invariant. Now, consider $\eta = \sum_{r=1}^{7} C_r e^r$, an arbitrary 1-form on $\mathfrak{l}_2$. According to (3.3), and taking into account the expressions of τ_2 and v_2, we see that the 4-form $\sigma = \pi_*(\tau_2 - \pi^* v_2 \wedge \eta)$ on $\mathfrak{h}_2$ has zero component in e^{3456}. Hence, $\sigma|_W = \sigma(e_3, e_4, e_5, e_6) = 0$, contradicting Lemma 3.4. Thus, τ_2 never defines a coclosed G_2-structure.

$s = 3$. A generic closed four-form on $\mathfrak{l}_3 = (0, 0, 0, 12, 13 + 14, 24, 0)$ has the following expression:

$$\begin{aligned} \tau_2 = {} & c_{1234} e^{1234} + c_{1235} e^{1235} + c_{1236} e^{1236} + c_{1237} e^{1237} + c_{1245} e^{1245} + c_{1246} e^{1246} + c_{1247} e^{1247} \\ & + c_{1256} e^{1256} + c_{1257} e^{1257} + c_{1267} e^{1267} + c_{1345} e^{1345} + c_{1346} e^{1346} + c_{1347} e^{1347} + c_{1357} e^{1357} \\ & + c_{1367} e^{1367} + c_{1456} e^{1456} + c_{1457} e^{1457} + c_{1467} e^{1467} + c_{2345} e^{2345} + c_{2346} e^{2346} + c_{2347} e^{2347} \\ & + c_{2357} e^{2357} + c_{2367} e^{2367} + c_{2356} e^{2456} + (c_{1367} + c_{2357}) e^{2457} + c_{2467} e^{2467}, \end{aligned}$$

where c_{ijkl} are arbitrary real numbers. Let us suppose that for some real numbers c_{ijkl}, the 4-form τ_3 defines a coclosed G_2-structure on $\mathfrak{l}_2$. Since e_7 is in the center of $\mathfrak{l}_1$, by Proposition 3.1, we have that $v_3 = -\pi_*(i_{e_7} \tau_3)$ is a negative 3-form on $\mathfrak{h}_3 = \mathfrak{l}_3 / \mathrm{Span}\{e_7\}$, where $\pi \colon \mathfrak{l}_3 \to \mathfrak{h}_3$ is the natural projection. Thus, v_3 is given by

$$\begin{aligned} v_3 = {} & c_{1237} e^{123} + c_{1247} e^{124} + c_{1257} e^{125} + c_{1267} e^{126} + c_{1347} e^{134} + c1357 e^{135} + c_{1367} e^{136} + c_{1457} e^{145} \\ & + c_{1467} e^{146} + c_{2347} e^{234} + c_{2357} e^{235} + c_{2367} e^{236} + (c_{2357} - c_{1367}) e^{245} + c_{2467} e^{246}. \end{aligned}$$

We claim that the map K_{v_3}, defined by (2.2), has the following expression:

$$K_{v_3} = (K_{ab}^{v_3}) \otimes e^{123456} = \begin{pmatrix} K_{11}^{v_3} & K_{12}^{v_3} & 0 & 0 & 0 & 0 \\ K_{21}^{v_3} & K_{22}^{v_3} & 0 & 0 & 0 & 0 \\ K_{31}^{v_3} & K_{32}^{v_3} & K_{33}^{v_3} & K_{34}^{v_3} & 0 & 0 \\ K_{41}^{v_3} & K_{42}^{v_3} & K_{43}^{v_3} & K_{44}^{v_3} & 0 & 0 \\ K_{51}^{v_3} & K_{52}^{v_3} & K_{53}^{v_3} & K_{54}^{v_3} & K_{55}^{v_3} & K_{56}^{v_3} \\ K_{61}^{v_3} & K_{62}^{v_3} & K_{63}^{v_3} & K_{64}^{v_3} & K_{65}^{v_3} & K_{66}^{v_3} \end{pmatrix} \otimes e^{123456},$$

where $a, b \in \{1, \ldots, 6\}$, and $K_{ab}^{v_3}$ is a polynomial function of the coefficients c_{ijkl} that appear in the expression of v_3. In fact, by (2.2), it turns out that

$$(\iota_{e_b} v_3) \wedge v_3 = \sum_{1 \leq a \leq 6} K_{ab}^{v_3} e^{1 \cdots \hat{a} \cdots 6}.$$

Therefore, $K_{ab}^{v_3} e^{1 \cdots 6} = (\iota_{e_b} v_3) \wedge v_3 \wedge e^a$. Then one can check that $K_{ab}^{v_3} = 0$ for $a = 1, 2$ and $b \geq 3$, and also $K_{ab}^{v_3} = 0$ for $a = 3, 4$ and $b = 5, 6$. Thus, the claim is true.

Since $v_3 \in \Lambda_-(\mathfrak{h}_3{}^*)$, (2.5) and Theorem 2.4 imply that v_3 defines the almost complex structure $J_{v_3 2}$ on $\mathfrak{h}_3$ given by

$$J_{v_3} = \frac{1}{|\sqrt{-\lambda(v_3)}|} \begin{pmatrix} K_{11}^{v_3} & K_{12}^{v_3} & 0 & 0 & 0 & 0 \\ K_{21}^{v_3} & K_{22}^{v_3} & 0 & 0 & 0 & 0 \\ K_{31}^{v_3} & K_{32}^{v_3} & K_{33}^{v_3} & K_{34}^{v_3} & 0 & 0 \\ K_{41}^{v_3} & K_{42}^{v_3} & K_{43}^{v_3} & K_{44}^{v_3} & 0 & 0 \\ K_{51}^{v_3} & K_{52}^{v_3} & K_{53}^{v_3} & K_{54}^{v_3} & K_{55}^{v_3} & K_{56}^{v_3} \\ K_{61}^{v_3} & K_{62}^{v_3} & K_{63}^{v_3} & K_{64}^{v_3} & K_{65}^{v_3} & K_{66}^{v_3} \end{pmatrix}.$$

Therefore, the subspace $W = \mathrm{Span}\{e_3, e_4, e_5, e_6\}$ is J_{v_3}-invariant. Now, consider $\eta = \sum_{r=1}^{7} C_r e^r$, an arbitrary 1-form on $\mathfrak{l}_3$. According to (3.3), and taking into account the expressions of τ_3 and v_3, we see that the 4-form $\sigma = \pi_*(\tau_3 - \pi^* v_3 \wedge \eta)$ on $\mathfrak{h}_3$ has zero component in e^{3456}. Hence, $\sigma|_W = \sigma(e_3, e_4, e_5, e_6) = 0$, contradicting Lemma 3.4. Thus, τ_3 never defines a coclosed G$_2$-structure. □

By Gong's classification, there exist, up to isomorphism, 35 decomposable 7-dimensional nilpotent Lie algebras. We will show that 24 of these Lie algebras admit a coclosed G$_2$-structure.

Theorem 4.3. *Among the 35 decomposable nilpotent Lie algebras, of dimension 7, those that have a coclosed G$_2$-structure are*

$$\mathfrak{n}_1 = (0, 0, 0, 0, 0, 0, 0), \qquad \mathfrak{n}_2 = (0, 0, 0, 0, 0, 12, 0),$$

$$\mathfrak{n}_3 = (0, 0, 0, 0, 0, 12 + 34, 0), \qquad \mathfrak{n}_4 = (0, 0, 0, 0, 12, 13, 0),$$

$$\mathfrak{n}_5 = (0, 0, 0, 0, 12, 34, 0), \qquad \mathfrak{n}_6 = (0, 0, 0, 0, 13 - 24, 14 + 23, 0),$$

$$\mathfrak{n}_7 = (0, 0, 0, 0, 12, 14 + 23, 0), \qquad \mathfrak{n}_8 = (0, 0, 0, 0, 12, 14 + 25, 0),$$

$$\mathfrak{n}_9 = (0, 0, 0, 0, 12, 15 + 34, 0), \qquad \mathfrak{n}_{10} = (0, 0, 0, 12, 13, 23, 0),$$

$$\mathfrak{n}_{11} = (0, 0, 0, 12, 13, 24, 0), \qquad \mathfrak{n}_{12} = (0, 0, 0, 12, 13, 14 + 23, 0),$$

$$\mathfrak{n}_{13} = (0, 0, 0, 12, 23, 14 + 35, 0), \qquad \mathfrak{n}_{14} = (0, 0, 0, 12, 23, 14 - 35, 0),$$

$$\mathfrak{n}_{15} = (0, 0, 0, 12, 13, 14 + 35, 0), \qquad \mathfrak{n}_{16} = (0, 0, 0, 12, 14, 15, 0),$$

$$\mathfrak{n}_{17} = (0, 0, 0, 12, 14, 15 + 24, 0), \qquad \mathfrak{n}_{18} = (0, 0, 0, 12, 14, 15 + 24 + 23, 0),$$

$$\mathfrak{n}_{19} = (0, 0, 0, 12, 14, 15 + 23, 0), \qquad \mathfrak{n}_{20} = (0, 0, 0, 12, 14 - 23, 15 + 34, 0),$$

$$\mathfrak{n}_{21} = (0, 0, 12, 13, 23, 14 + 25, 0), \qquad \mathfrak{n}_{22} = (0, 0, 12, 13, 23, 14 - 25, 0),$$

$$\mathfrak{n}_{23} = (0, 0, 12, 13, 23, 14, 0), \qquad \mathfrak{n}_{24} = (0, 0, 12, 13, 14 + 23, 15 + 24, 0).$$

Proof. By Propositions 4.1 and 4.2, we know that there are 11 decomposable nilpotent Lie algebras not admitting coclosed G$_2$-structures. This implies that there are at most 24 decomposable nilpotent Lie algebras having a coclosed G$_2$-structure. But all the 24 Lie algebras listed in the statement have such a G$_2$-structure. This is clear on the abelian Lie algebra $\mathfrak{n}_1 = (0, 0, 0, 0, 0, 0, 0)$. Moreover, every non-abelian Lie algebra $\mathfrak{n}_s$, $s \in \{2, \ldots, 24\}$, listed in the statement, is a direct sum of two ideals $\mathfrak{h}_s$ and $\mathbb{R}$. In fact, $\mathfrak{n}_s$ is an abelian extension of $\mathfrak{h}_s$,

$$\mathfrak{n}_s = \mathfrak{h}_s \oplus \mathbb{R} e_7,$$

where $\mathfrak{h}_s$ is a 6-dimensional Lie algebra, and $e_7 = \frac{\partial}{\partial t}$ with t a coordinate on $\mathbb{R}$.

By [4], we know that $\mathfrak{h}_s$ has a half-flat SU(3)-structure (ω_s, ψ_s^-). Thus, the 4-form ϕ_s on $\mathfrak{n}_s$, given by

$$\phi_s = \frac{1}{2}\omega_s^2 + \psi_s^- \wedge dt,$$

defines a coclosed G_2-structure on $\mathfrak{n}_s$, which completes the proof. □

Corollary 4.4. *Any decomposable 2-step nilpotent Lie algebra, of dimension seven, has a coclosed G_2-structure.*

Proof. Among the 35 decomposable nilpotent Lie algebras, of dimension seven, those that are 2-step nilpotent are the 7 Lie algebras $\mathfrak{n}_s$, where $s \in \{2, 3, 4, 5, 6, 7, 10\}$, defined in Theorem 4.3. Thus, all of them carry coclosed G_2-structures. □

5 Indecomposable 2-step nilpotent case

In this section we complete the classification of 7-dimensional 2-step nilpotent Lie algebras which admit a coclosed G_2-structure. We have seen that there are exactly 7 decomposable Lie algebras of this type. In order to discuss the indecomposable 2-step nilpotent Lie algebras, we refer to Gong's classification [14] (see also [5]). This list consists of 131 Lie algebras and 9 1-parameter families. None of the 1-parameter families defines a 2-step nilpotent Lie algebra. Among the indecomposable Lie algebras (with no parameters), there are only nine which are 2-step nilpotent Lie algebras. They are the following:

$$17 = (0, 0, 0, 0, 0, 0, 12 + 34 + 56),$$
$$27A = (0, 0, 0, 0, 0, 12, 14 + 35),$$
$$27B = (0, 0, 0, 0, 0, 12 + 34, 15 + 23),$$
$$37A = (0, 0, 0, 0, 12, 23, 24),$$
$$37B = (0, 0, 0, 0, 12, 23, 34),$$
$$37B1 = (0, 0, 0, 0, 12 - 34, 13 + 24, 14),$$
$$37C = (0, 0, 0, 0, 12 + 34, 23, 24),$$
$$37D = (0, 0, 0, 0, 12 + 34, 13, 24),$$
$$37D1 = (0, 0, 0, 0, 12 - 34, 13 + 24, 14 - 23).$$

Theorem 5.1. *Up to isomorphism, the unique indecomposable 2-step nilpotent Lie algebras carrying a coclosed G_2-structure are the following: $17, 37A, 37B, 37B1, 37C, 37D$ and $37D1$.*

Proof. We know that a G_2-structure on a Lie algebra $\mathfrak{g}$ can be defined either by a 3-form or, equivalently, by a 4-form, which have the expression given by (3.1) or (3.2), respectively, with respect to some orthonormal coframe $\{e^1, \ldots, e^7\}$ of $\mathfrak{g}^*$. For each Lie algebra appearing in the statement, an appropriate coframe and the corresponding 4-form are given as follows:

- 17: $\phi_1 = e^{1234} + e^{1256} + e^{1367} + e^{1457} + e^{2357} - e^{2467} + e^{3456}$, $\{e^1, e^2, e^3, e^4, e^5, e^6, e^7\}$,
- $37A$: $\phi_2 = e^{1234} + e^{1257} - e^{1356} - e^{1467} - e^{2367} + e^{2456} + e^{3457}$, $\{e^3, e^1, e^2, e^4, e^5, e^6, e^7\}$,
- $37B$: $\phi_3 = e^{1234} + e^{1457} + e^{2357} + \frac{\sqrt{2}}{2}(e^{1256} - e^{1356} + e^{1267} + e^{1367} + e^{2456}) + \frac{\sqrt{2}}{2}(e^{3456} - e^{2467} + e^{3467})$, $\{e^1, \frac{\sqrt{2}}{2}(e^2 - e^3), \frac{\sqrt{2}}{2}(e^2 + e^3), e^4, e^5, e^6, e^7\}$,
- $37B1$: $\phi_4 = e^{1234} + e^{1267} + e^{1357} + e^{1456} + e^{2356} - e^{2457} + e^{3467}$, $\{e^1, e^4, e^2, e^3, e^5, e^6, e^7\}$,
- $37C$: $\phi_5 = -e^{1234} - e^{1267} + e^{1357} + e^{1456} - e^{2356} + e^{2457} + e^{3467}$, $\{e^2, e^3, e^4, e^1, e^6, e^5, e^7\}$,
- $37D$: $\phi_6 = e^{1234} + e^{1267} + e^{3467} + \frac{\sqrt{2}}{2}(e^{1356} + e^{1357} + e^{1456} - e^{1457} + e^{2356}) - \frac{\sqrt{2}}{2}(e^{2357} - e^{2456} - e^{2457})$, $\{e^1, \frac{\sqrt{2}}{2}(e^3 + e^4), e^2, \frac{\sqrt{2}}{2}(e^3 - e^4), e^5, e^6, e^7\}$,
- $37D1$: $\phi_7 = -e^{1234} - e^{1267} - e^{1356} + e^{1457} - e^{2357} - e^{2456} + e^{3467}$, $\{e^1, e^3, e^2, e^4, e^5, -e^6, e^7\}$.

It is straightforward to verify that each ϕ_i is closed on the corresponding Lie algebra.

It remains to prove that the Lie algebras $27A$ and $27B$ do not admit any coclosed G_2-structure. To this end, we show that for these Lie algebras the hypothesis of Lemma 3.3 is satisfied for $X = e_6$ and $Y = e_7$. In

fact, let α be a generic closed 4-form on $27A$. Then, one can check that α has the following expression:

$$
\begin{aligned}
\alpha = {} & c_{1234}e^{1234} + c_{1235}e^{1235} + c_{1236}e^{1236} + c_{1237}e^{1237} + c_{1245}e^{1245} + c_{1246}e^{1246} + c_{1247}e^{1247} \\
& + c_{1256}e^{1256} + c_{1257}e^{1257} + c_{1345}e^{1345} + c_{1346}e^{1346} + c_{1347}e^{1347} + c_{1356}e^{1356} \\
& + c_{1357}e^{1357} + c_{1367}e^{1367} + c_{1456}e^{1456} + c_{1457}e^{1457} + c_{1567}e^{1567} + c_{2345}e^{2345} \\
& + c_{2346}e^{2346} + c_{2347}e^{2347} + c_{2356}e^{2356} + c_{2357}e^{2357} + c_{2456}e^{2456} + c_{2457}e^{2457} \\
& + (c_{1247} - c_{2357})e^{3456} + c_{3457}e^{3457},
\end{aligned}
$$

where c_{ijkl} are arbitrary real numbers. Thus, $(\iota_{e_6}(\iota_{e_7}\alpha))^2 = 0$. A generic closed 4-form β on the Lie algebra $27B$ is expressed as

$$
\begin{aligned}
\beta = {} & c_{1234}e^{1234} + c_{1235}e^{1235} + c_{1236}e^{1236} + c_{1237}e^{1237} + c_{1245}e^{1245} + c_{1246}e^{1246} + c_{1247}e^{1247} \\
& + c_{1256}e^{1256} + c_{1257}e^{1257} + c_{1345}e^{1345} + c_{1346}e^{1346} + c_{1347}e^{1347} + c_{1356}e^{1356} \\
& + c_{1357}e^{1357} + c_{1367}e^{1367} + c_{1456}e^{1456} + c_{1457}e^{1457} + c_{2345}e^{2345} \\
& + c_{2346}e^{2346} + c_{2347}e^{2347} + c_{2356}e^{2356} + c_{2357}e^{2357} + c_{2456}e^{2456} + c_{2457}e^{2457} \\
& + (c_{1256} + c_{1457} - c_{2347})e^{3456} + c_{3457}e^{3457},
\end{aligned}
$$

where c_{ijkl} are arbitrary real numbers. Hence, $(\iota_{e_6}(\iota_{e_7}\beta))^2 = 0$. □

6 Coclosed G$_2$-structures inducing nilsolitons

In this section we prove that any 2-step nilpotent Lie algebra, admitting a coclosed G$_2$-structure, also has a nilsoliton inner product determined by a coclosed G$_2$-structure. This result is not true for higher steps. Indeed, we give an example of a 3-step nilpotent Lie algebra supporting a nilsoliton and coclosed G$_2$-structures but none coclosed G$_2$-structure induces a nilsoliton.

Let $\mathfrak{n}$ be a nilpotent Lie algebra. According to Lauret [21], an inner product g on $\mathfrak{n}$ is called *nilsoliton* if its Ricci endomorphism $\mathrm{Ric}(g)$ differs from a derivation D of $\mathfrak{n}$ by a scalar multiple of the identity map I, that is, if there exists a real number λ such that

$$\mathrm{Ric}(g) = \lambda I + D.$$

Not all nilpotent Lie algebras admit nilsoliton inner products, but if a nilsoliton exists, then it is unique up to automorphism and scaling [21].

Theorem 6.1. *Any indecomposable 2-step nilpotent Lie algebra admitting a coclosed G$_2$-structure also has a coclosed G$_2$-structure inducing a nilsoliton.*

Proof. By Theorem 5.1, we know that, up to isomorphism, the Lie algebras $17, 37A, 37B, 37B1, 37C, 37D$ and $37D1$ are the unique indecomposable 2-step nilpotent admitting coclosed G$_2$-structures. For each of these Lie algebras, defined in Section 5 in terms of a basis $\{e^1, \ldots, e^7\}$ of the dual space, we consider a new basis $\{f^1, \ldots, f^7\}$ and a coclosed G$_2$-structure defined by a 3-form φ_i ($1 \leq i \leq 7$) which determines the inner product so that the basis $\{f^1, \ldots, f^7\}$ is orthonormal. The basis $\{f^j\}$, the G$_2$-structure φ_i and the appropriate coframe defining φ_i are given as follows:

- $17 \colon f^i = e^i$, $1 \leq i \leq 6$, $f^7 = \frac{\sqrt{6}}{6}e^7$. Then

$$17 = \left(0, 0, 0, 0, 0, 0, \tfrac{\sqrt{6}}{6}f^{12} + \tfrac{\sqrt{6}}{6}f^{34} + \tfrac{\sqrt{6}}{6}f^{56}\right)$$

  and

$$\varphi_1 = f^{127} + f^{347} + f^{567} + f^{135} - f^{146} - f^{236} - f^{245}, \quad \{f^1, f^2, f^3, f^4, f^5, f^6, f^7\}.$$

- $37A \colon f^i = e^i$, $1 \leq i \leq 4$, $f^j = \frac{\sqrt{6}}{6}e^j$, $5 \leq j \leq 7$. Then,

$$37A = \left(0, 0, 0, 0, \tfrac{\sqrt{6}}{6}f^{12}, \tfrac{\sqrt{6}}{6}f^{23}, \tfrac{\sqrt{6}}{6}f^{24}\right)$$

and

$$\varphi_2 = -f^{137} + f^{247} + f^{567} - f^{126} - f^{145} - f^{235} - f^{346}, \quad \{f^3, f^1, f^2, f^4, f^5, f^6, f^7\}.$$

- 37B: $f^i = e^i$, $1 \leq i \leq 4$, $f^5 = \frac{\sqrt{5}}{5} e^5$, $f^6 = \frac{\sqrt{6}}{6} e^6$, $f^7 = \frac{\sqrt{5}}{5} e^7$. Thus,

$$37B = (0, 0, 0, 0, \tfrac{\sqrt{5}}{5} f^{12}, \tfrac{\sqrt{10}}{10} f^{23}, \tfrac{\sqrt{5}}{5} f^{34})$$

and

$$\varphi_3 = -f^{146} - f^{236} + f^{567} + \tfrac{\sqrt{2}}{2}(f^{123} + f^{125} - f^{135} - f^{137} - f^{245}) + \tfrac{\sqrt{2}}{2}(f^{247} + f^{345} + f^{347}),$$
$$\{f^1, \tfrac{\sqrt{2}}{2}(f^2 - f^3), \tfrac{\sqrt{2}}{2}(f^2 + f^3), f^4, f^5, f^6, f^7\}.$$

- 37B1: $f^i = e^i$, $1 \leq i \leq 4$, $f^j = \frac{\sqrt{10}}{10} e^j$, $5 \leq j \leq 7$. So,

$$37B1 = (0, 0, 0, 0, \tfrac{\sqrt{10}}{10}(f^{12} - f^{34}), \tfrac{\sqrt{10}}{10}(f^{13} + f^{24}), \tfrac{\sqrt{10}}{10} f^{14})$$

and

$$\varphi_4 = f^{125} + f^{345} + f^{567} + \tfrac{\sqrt{2}}{2}(-f^{136} + f^{137} + f^{146} + f^{147} + f^{236}) + \tfrac{\sqrt{2}}{2}(f^{237} + f^{246} - f^{247}),$$
$$\{f^1, \tfrac{\sqrt{2}}{2}(f^3 + f^4), f^2, \tfrac{\sqrt{2}}{2}(f^3 - f^4), f^5, f^6, f^7\}.$$

- 37C: $f^i = e^i$, $1 \leq i \leq 4$, $f^j = \frac{\sqrt{2}}{4} e^j$, $5 \leq j \leq 7$. Thus,

$$37C = (0, 0, 0, 0, \tfrac{\sqrt{2}}{4}(f^{12} + f^{34}), \tfrac{\sqrt{2}}{4} f^{23}, \tfrac{\sqrt{2}}{4} f^{24})$$

and

$$\varphi_5 = f^{147} + f^{237} + f^{567} + f^{125} - f^{136} + f^{246} + f^{345}, \quad \{f^1, f^4, f^2, f^3, f^5, f^6, f^7\}.$$

- 37D: $f^i = e^i$, $1 \leq i \leq 4$, $f^j = \frac{\sqrt{3}}{6} e^j$, $5 \leq j \leq 7$. Then,

$$37D = (0, 0, 0, 0, \tfrac{\sqrt{3}}{6}(f^{12} + f^{34}, \tfrac{\sqrt{6}}{6} f^{13}, \tfrac{\sqrt{6}}{6} f^{24})$$

and

$$\varphi_6 = -f^{147} + f^{237} - f^{567} + f^{125} + f^{136} + f^{246} - f^{345}, \quad \{f^2, f^3, f^4, f^1, f^6, f^5, f^7\}.$$

- 37D1: $f^i = e^i$, $1 \leq i \leq 4$, $f^j = \frac{\sqrt{3}}{6} e^j$, $5 \leq j \leq 7$. Hence,

$$37D1 = (0, 0, 0, 0, \tfrac{\sqrt{3}}{6}(f^{12} - f^{34}, \tfrac{\sqrt{6}}{6}(f^{13} + f^{24}), \tfrac{\sqrt{6}}{6}(f^{14} - f^{23})),$$

and

$$\varphi_7 = f^{137} + f^{247} - f^{567} + f^{125} + f^{146} - f^{236} - f^{345}, \quad \{f^1, f^3, f^2, f^4, f^5, -f^6, f^7\}.$$

It is straightforward to check that each φ_i is coclosed and induces the inner product $g_i = \sum_{j=1}^{7} (f^j)^2$. By [8], each g_i is a nilsoliton on the corresponding Lie algebra, which completes the proof.   □

Remark 6.2. Note that for each of the Lie algebras $17-37D1$, considered in the previous theorem, the change of basis from $\{e^i\}$ to $\{f^i\}$, given in the proof of Theorem 6.1, defines an automorphism of the Lie algebra, but it is not an isomorphism between the G_2-structure defined in the proof of Theorem 6.1 and the G_2-structure defined in the proof of Theorem 5.1. In fact, these structures define different metrics.

Theorem 6.3. *Any decomposable 2-step nilpotent Lie algebra has a coclosed G_2-structure inducing a nilsoliton.*

Proof. By Corollary 4.4, any decomposable 2-step nilpotent Lie algebra admitting a coclosed G_2-structure is isomorphic to one of the following seven Lie algebras $\mathfrak{n}_s$, $s \in \{2, 3, 4, 5, 6, 7, 10\}$, defined in Theorem 4.3. For each of these Lie algebras, we consider the G_2 form φ, and the appropriate coframe defining φ, given as follows:

- $\mathfrak{n}_2$: $\varphi = -e^{137} - e^{247} - e^{567} + e^{126} + e^{145} - e^{235} - e^{346}$, $\{e^1, e^3, e^2, e^4, e^6, -e^5, -e^7\}$,
- $\mathfrak{n}_3$: $\varphi = -e^{127} + e^{347} - e^{567} + e^{135} + e^{146} - e^{236} + e^{245}$, $\{e^1, e^2, e^3, -e^4, e^5, e^6, -e^7\}$,

- $\mathfrak{n}_4$: $\varphi = e^{147} + e^{237} + e^{567} - e^{126} - e^{135} + e^{245} - e^{346}$, $\{e^1, e^4, e^2, e^3, -e^6, e^5, e^7\}$,
- $\mathfrak{n}_5$: $\varphi = \frac{\sqrt{2}}{2}(e^{125} + e^{126} + e^{135} - e^{136} - e^{245} + e^{345} + e^{346} + e^{246}) + e^{147} + e^{237} + e^{567}$,
  $\{e^1, e^2, \frac{\sqrt{2}}{2}(e^5 + e^6), e^3, \frac{\sqrt{2}}{2}(e^5 - e^6), e^4, e^7\}$,
- $\mathfrak{n}_6$: $\varphi = e^{127} + e^{347} + e^{567} - e^{235} + e^{246} - e^{136} - e^{145}$, $\{e^1, e^2, e^6, -e^5, e^3, e^4, e^7\}$,
- $\mathfrak{n}_7$: $\varphi = e^{137} + e^{247} - e^{567} - e^{125} - e^{146} + e^{236} + e^{345}$, $\{e^1, e^3, e^2, e^4, -e^5, e^6, e^7\}$,
- $\mathfrak{n}_{10}$: $\varphi = -e^{147} - e^{257} - e^{367} + e^{126} - e^{135} + e^{234} - e^{456}$, $\{e^1, e^4, e^3, e^6, -e^5, e^2, -e^7\}$.

It is straightforward to check that each form φ is coclosed on the corresponding Lie algebra, and it defines the metric $g = \sum_{i=1}^{7}(e^i)^2$, which is a nilsoliton as it is proved in [27]. □

Finally, we show an example of a 3-step nilpotent Lie algebra admitting coclosed G_2-structures and a nilsoliton which is not determined by any coclosed G_2-structure.

Example 6.4. Consider the 3-step nilpotent decomposable Lie algebra $\mathfrak{n}_8$ defined in Theorem 4.3 as

$$\mathfrak{n}_8 = (0, 0, 0, 0, 12, 14 + 25, 0),$$

with respect to a basis $\{e^1, \ldots, e^7\}$ of the dual space $\mathfrak{n}_8^*$. Clearly, in the basis

$$\{f^1 = e^1, f^2 = \tfrac{\sqrt{2}}{2}e^2, f^3 = e^3, \ldots, f^7 = e^7\},$$

the Lie algebra $\mathfrak{n}_8$ is defined by the structure equations

$$(0, 0, 0, 0, \sqrt{2}f^{12}, f^{14} + \sqrt{2}f^{25}, 0).$$

By [13], the inner product

$$g = \sum_{i=1}^{7} (f^i)^2$$

is a nilsoliton on $\mathfrak{n}_8$, and so it is unique up to scaling and automorphism of the Lie algebra [21].

On the other hand, by Theorem 4.3, $\mathfrak{n}_8$ admits a coclosed G_2-structure. However, we will show that the Lie algebra $\mathfrak{n}_8$ does not carry any coclosed G_2-structure inducing the nilsoliton g. For this, we consider the G_2-structure defined by the 3-form

$$\varphi_0 = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}.$$

Denote by $\star$ the Hodge operator defined by φ_0. By using [3, formula (3.6)], we know that every positive 3-form φ on $\mathfrak{n}_8$, compatible with the previous metric g and orientation, can be written as

$$\varphi = (a^2 - |\alpha|^2)\varphi_0 + 2a * (\alpha \wedge \varphi_0) + i(\alpha \circ \alpha),$$

where a is a constant, α is a 1-form satisfying $a^2 + |\alpha|^2 = 1$, and $i(\alpha \circ \alpha)$ is a 3-form depending on α and φ_0. Consequently,

$$*\varphi = (a^2 - |\alpha|^2) * \varphi_0 + 2a(\alpha \wedge \varphi_0) + *i(\alpha \circ \alpha).$$

Consider a generic 1-form $\alpha = \sum_{j=1}^{7} \alpha_j e^j$, with $\alpha_j \in \mathbb{R}$. Since

$$i(\alpha \circ \alpha) = \sum_{1 \le j, k \le 7} \alpha_j \alpha_k e^j \wedge (\iota_{e_k} \varphi_0),$$

we obtain the following explicit expression for $*\varphi$:

$$
\begin{aligned}
*\varphi = {} & (a^2 - |\alpha|^2)e^{3456} + (2a\alpha_3 + \alpha_1\alpha_5 - \alpha_2\alpha_6 - \alpha_4\alpha_7)e^{3567} \\
& + (2a\alpha_1 - \alpha_2\alpha_7 + \alpha_3\alpha_5 - \alpha_4\alpha_6)e^{1347} + (a^2 - |\alpha|^2)e^{1256} + (a^2 - |\alpha|^2)e^{2357} \\
& + (2a\alpha_2 + \alpha_1\alpha_7 - \alpha_3\alpha_6 - \alpha_4\alpha_5)e^{2347} + (2a\alpha_2 + \alpha_1\alpha_7 + \alpha_3\alpha_6 + \alpha_4\alpha_5)e^{2567} \\
& + (a^2 - |\alpha|^2)e^{1367} + (2a\alpha_4 - \alpha_1\alpha_6 - \alpha_2\alpha_5 + \alpha_3\alpha_7)e^{4567} + (a^2 - |\alpha|^2)e^{1457} \\
& + (2a\alpha_1 - \alpha_2\alpha_7 - \alpha_3\alpha_5 + \alpha_4\alpha_6)e^{1567} + (-a^2 + |\alpha|^2)e^{2467} + (a^2 - |\alpha|^2)e^{1234}
\end{aligned}
$$

$$
\begin{aligned}
&+ (2 - a\alpha_1 - \alpha_2\alpha_7 - \alpha_3\alpha_5 - \alpha_4\alpha_6)e^{1236} + (2 - a\alpha_1 - \alpha_2\alpha_7 + \alpha_3\alpha_5 + \alpha_4\alpha_6)e^{1246} \\
&+ (2 - a\alpha_2 + \alpha_1\alpha_7 + \alpha_3\alpha_6 - \alpha_4\alpha_5)e^{1235} + (2a\alpha_3 - \alpha_1\alpha_5 + \alpha_2\alpha_6 - \alpha_4\alpha_7)e^{1237} \\
&+ (2a\alpha_2 - \alpha_1\alpha_7 + \alpha_3\alpha_6 - \alpha_4\alpha_5 e^{1246} + (2a\alpha_4 + \alpha_1\alpha_6 + \alpha_2\alpha_5 + \alpha_3\alpha_7)e^{1247} \\
&+ (2a\alpha_5 + \alpha_1\alpha_3 - \alpha_2\alpha_4 - \alpha_6\alpha_7)e^{1257} + (2a\alpha_6 - \alpha_1\alpha_4 - \alpha_2\alpha_3 + \alpha_5\alpha_7)e^{1267} \\
&+ (2a\alpha_4 + \alpha_1\alpha_6 - \alpha_2\alpha_5 - \alpha_3\alpha_7 e^{1345} + (2a\alpha_3 - \alpha_1\alpha_5 - \alpha_2\alpha_6 + \alpha_4\alpha_7)e^{1346} \\
&+ (-2a\alpha_6 + \alpha_1\alpha_4 - \alpha_2\alpha_3 + \alpha_5\alpha_7)e^{1356}(-2a\alpha_7 - \alpha_1\alpha_2 - \alpha_3\alpha_4 - \alpha_5\alpha_6)e^{1357} \\
&+ (-2a\alpha_5 - \alpha_1\alpha_3 - \alpha_2\alpha_4 - \alpha_6\alpha_7)e^{1456} + (2a\alpha_7 + \alpha_1\alpha_2 - \alpha_3\alpha_4 - \alpha_5\alpha_6)e^{1467} \\
&+ (2a\alpha_3 + \alpha_1\alpha_5 + \alpha_2\alpha_6 + \alpha_4\alpha_7)e^{2345} + (-2a\alpha_4 + \alpha_1\alpha_6 - \alpha_2\alpha_5 + \alpha_3\alpha_7)e^{2346} \\
&+ (-2a\alpha_5 + \alpha_1\alpha_3 + \alpha_2\alpha_4 - \alpha_6\alpha_7)e^{2356} + (2a\alpha_7 - \alpha_1\alpha_2 + \alpha_3\alpha_4 - \alpha_5\alpha_6)e^{2367} \\
&+ (2a\alpha_6 + \alpha_1\alpha_4 - \alpha_2\alpha_3 - \alpha_5\alpha_7)e^{2456} + (2a\alpha_7 - \alpha_1\alpha_2 - \alpha_3\alpha_4 + \alpha_5\alpha_6)e^{2457} \\
&+ (2a\alpha_5 - \alpha_1\alpha_3 + \alpha_2\alpha_4 - \alpha_6\alpha_7)e^{3457} + (2a\alpha_6 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_5\alpha_7)e^{3467}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
d(*\varphi) = &-(a^2 - |\alpha|^2)\sqrt{2}e^{12357} - (2a\alpha_4 - \alpha_1\alpha_6 - \alpha_2\alpha_5 + \alpha_3\alpha_7)\sqrt{2}e^{12467} \\
&+ (-2a\alpha_7\sqrt{2} - \sqrt{2}\alpha_1\alpha_2 + \sqrt{2}\alpha_3\alpha_4 + \sqrt{2}\alpha_5\alpha_6 - 2a\alpha_2 - \alpha_1\alpha_7 - \alpha_3\alpha_6 - \alpha_4\alpha_5)e^{12457} \\
&+ (-2a\alpha_3\sqrt{2} + \sqrt{2}\alpha_1\alpha_5 + \sqrt{2}\alpha_2\alpha_6 - \sqrt{2}\alpha_4\alpha_7 + 2a\alpha_5 - \alpha_1\alpha_3 - \alpha_2\alpha_4 + \alpha_6\alpha_7)e^{12345} \\
&- (2a\alpha_3 + \alpha_1\alpha_5 - \alpha_2\alpha_6 - \alpha_4\alpha_7)\sqrt{2}e^{12367} + (a^2 - |\alpha|^2)\sqrt{2}e^{12346} \\
&+ (2a\alpha_5\sqrt{2} - \sqrt{2}\alpha_1\alpha_3 + \sqrt{2}\alpha_2\alpha_4 - \sqrt{2}\alpha_6\alpha_7 + 2a\alpha_7 - \alpha_1\alpha_2 + \alpha_3\alpha_4 - \alpha_5\alpha_6)e^{12347} \\
&+ (2a\alpha_6\sqrt{2} + \sqrt{2}\alpha_1\alpha_4 + \sqrt{2}\alpha_2\alpha_3 + \sqrt{2}\alpha_5\alpha_7)e^{23457} + (-2a\alpha_3 - \alpha_1\alpha_5 + \alpha_2\alpha_6 + \alpha_4\alpha_7)e^{13457}.
\end{aligned}
$$

Then we see that $d(*\varphi) = 0$ is equivalent to the following system:

$$
\begin{cases}
a^2 - |\alpha|^2 = 0, \\
a\alpha_4 - \alpha_1\alpha_6 - \alpha_2\alpha_5 + \alpha_3\alpha_7 = 0, \\
2a\alpha_3 + \alpha_1\alpha_5 - \alpha_2\alpha_6 - \alpha_4\alpha_7 = 0, \\
2a\alpha_6\sqrt{2} + \sqrt{2}\alpha_1\alpha_4 + \sqrt{2}\alpha_2\alpha_3 + \sqrt{2}\alpha_5\alpha_7 = 0, \\
-2a\alpha_3\sqrt{2} + \sqrt{2}\alpha_1\alpha_5 + \sqrt{2}\alpha_2\alpha_6 - \sqrt{2}\alpha_4\alpha_7 + 2a\alpha_5 - \alpha_1\alpha_3 - \alpha_2\alpha_4 + \alpha_6\alpha_7 = 0, \\
2a\alpha_5\sqrt{2} - \sqrt{2}\alpha_1\alpha_3 + \sqrt{2}\alpha_2\alpha_4 - \sqrt{2}\alpha_6\alpha_7 + 2a\alpha_7 - \alpha_1\alpha_2 + \alpha_3\alpha_4 - \alpha_5\alpha_6 = 0, \\
-2a\alpha_7\sqrt{2} - \sqrt{2}\alpha_1\alpha_2 + \sqrt{2}\alpha_3\alpha_4 + \sqrt{2}\alpha_5\alpha_6 - 2a\alpha_2 - \alpha_1\alpha_7 - \alpha_3\alpha_6 - \alpha_4\alpha_5 = 0,
\end{cases}
$$

in the variables $a, \alpha_j, j = 1, \ldots, 7$. We can show that the above system has no solution. In fact, from the first equation it follows that $a^2 = |\alpha|^2$, so it is not restrictive to suppose that $2a = 1$. By the second, third and fourth equations, we obtain

$$
\alpha_4 = \alpha_1\alpha_6 + \alpha_2\alpha_5 - \alpha_3\alpha_7, \quad \alpha_3 = \frac{-\alpha_1\alpha_6\alpha_7 - \alpha_2\alpha_5\alpha_7 + \alpha_1\alpha_5 - \alpha_2\alpha_6}{1 + \alpha_7^2}, \quad \alpha_6 = -\alpha_5\alpha_7.
$$

By the last equation $(d(*\varphi))_{12457} = 0$, we get $\alpha_5^2 = -1$, and so $\mathfrak{n}_8$ does not admit any coclosed G_2-structure inducing the nilsoliton.

7 Contact metric structures induced by coclosed G_2-structures

In general, the existence of a coclosed G_2-structure is independent of the existence of a contact form. We recall that a contact metric structure on a Riemannian manifold (M, g) is a unit length vector field ξ such that the endomorphism ϕ defined by $g(\phi \cdot, \cdot) = \frac{1}{2}d\eta(\cdot, \cdot)$ and the 1-form $\eta = \langle \xi, \cdot \rangle$ are related by

$$
\phi^2 = -\mathrm{Id} + \eta \otimes \xi.
$$

A contact metric structure (M, g, ξ, ϕ) is called K-contact if the Reeb vector field ξ is Killing, and it is said to be normal if the induced almost complex structure J on the product manifold $M \times \mathbb{R}$ is integrable. A Sasakian manifold is a normal contact metric manifold.

A natural question is whether on a 7-manifold there exists a contact metric structure such that the metric is induced by a coclosed G_2-structure. For 2-step nilmanifolds we can show the following.

Proposition 7.1. *If a 2-step nilmanifold $\Gamma \setminus G$ admits an invariant contact metric structure (g, ξ, ϕ) such that the metric g is induced by a coclosed G_2-structure, then G is isomorphic to the 7-dimensional Heisenberg Lie group. Moreover, $\Gamma \setminus G$ has an invariant Sasakian structure.*

Proof. Since the involved structures are invariant, we can work at the level of the Lie algebra $\mathfrak{g}$ of G. In [20], Kutsak showed that a decomposable 7-dimensional nilpotent Lie algebra does not admit any contact structure, and the only indecomposable 2-step nilpotent Lie algebra admitting a contact form is the Lie algebra 17, which is the Lie algebra of the 7-dimensional Heisenberg Lie group. Therefore, the result follows by Theorem 5.1. □

Note that, by [1], the Heisenberg Lie algebra 17 is indeed the only 7-dimensional nilpotent Lie algebra admitting a Sasakian structure. So a natural problem is whether on a (not Sasakian) 7-dimensional nilpotent Lie algebra there exists a K-contact metric structure such that the metric is induced by a coclosed structure.

Example 7.2. Consider the following indecomposable 3-step nilpotent Lie algebra

$$\mathfrak{n} = (0, 0, 0, 12, 13, 0, 16 + 25 + 34).$$

By [15], we know that $\mathfrak{n}$ admits a K-contact metric structure defined by $g = \sum_{i=1}^{7} (e^i)^2$ and the Reeb vector $\xi = e_7$. We can show that $\mathfrak{g}$ admits a coclosed G_2-structure φ, whose underlying metric coincides with g. Indeed, consider the 3-form

$$\varphi = -e^{167} - e^{237} + e^{457} - e^{124} - e^{135} - e^{256} + e^{346}.$$

One can check that $g_\varphi = g$ and that the Hodge dual $\star\varphi$ of φ, given by

$$\star\varphi = e^{1236} - e^{1456} - e^{1257} + e^{1347} + e^{2467} + e^{2357},$$

is closed.

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