HOLONOMY GROUPS OF G_2^*-MANIFOLDS

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Abstract. We classify the holonomy algebras of manifolds admitting an indecomposable torsion free G_2^*-structure, i.e. for which the holonomy representation does not leave invariant any proper non-degenerate subspace. We realize some of these Lie algebras as holonomy algebras of left-invariant metrics on Lie groups.

1. Introduction

Holonomy groups are a useful tool in the study of semi-Riemannian manifolds. They make it possible to apply algebraic methods to geometric problems such as the existence of special geometric structures or the decomposability of manifolds. So it is natural to ask which Lie groups can be the holonomy group of a semi-Riemannian manifold. Since we are mainly interested in connected holonomy groups we may equivalently ask which Lie subalgebras of $\mathfrak{so}(p,q)$ are holonomy algebras. For Riemannian manifolds, there is a complete answer to this question. Berger's list gives a classification of irreducible Riemannian holonomy algebras of non-locally symmetric spaces [Be, Br]. Moreover, holonomy algebras of locally symmetric Riemannian manifolds can be read off from Cartan's classification of Riemannian symmetric spaces. The pseudo-Riemannian situation is much more complicated. In general, a holonomy representation, i.e., the natural representation of a holonomy group on the tangent space can have isotropic invariant subspaces and is not necessarily completely reducible. Therefore it does not suffice to determine all irreducible holonomy groups. A complete classification is only known for Lorentzian manifolds, it is due to Leistner [Le]. For metrics of index greater than one only partial results are known. For instance, there are results for manifolds with special geometric structure. Galaev classified holonomy algebras of pseudo-Kählerian manifolds of index 2 [Ga]. Furthermore, holonomy groups of pseudo-quaternionic-Kählerian manifolds of non-zero scalar curvature were classified by Bezvitnaya [Bz].

In the present paper, we want to turn to another special geometry, namely the pseudo-Riemannian analogue of a torsion-free G_2-structure, which is well known from the holonomy theory of Riemannian manifolds since G_2 is one of the groups on Berger's list. While torsion-free G_2-structures exist on Riemannian 7-manifolds, their pseudo-Riemannian analogues are structures

Date: April 5, 2016.

on manifolds of signature $(4,3)$. They are characterised by the fact that their holonomy is contained in the non-compact subgroup $G_2^* \subset SO(4,3)$ of type G_2, which is defined as the stabiliser of a certain generic 3-form. There are other nice characterisations of this group, e.g., G_2^* is the stabiliser of a non-isotropic element of the real spinor representation of $\mathrm{Spin}(4,3)$ and it can also be understood as the stabiliser of a cross product on $\mathbb{R}^{4,3}$. Hence a torsion-free G_2^*-structure on a pseudo-Riemannian manifold M of signature $(4,3)$ can be understood as a parallel generic 3-form, a parallel non-isotropic spinor field or a parallel cross-product '$\times$' on M.

Our aim is to classify the holonomy algebras of manifolds admitting a torsion-free G_2^*-structure, where we want to assume that this G_2^*-structure is indecomposable, that is, its holonomy representation does not leave invariant any proper non-degenerate subspace. By a classification we mean a classification as subalgebras of $\mathfrak{g}_2^* \subset \mathfrak{so}(4,3)$ up to conjugation by elements of $\mathrm{SO}(4,3)$.

There are already some results in this direction. In [Ka2] indefinite symmetric spaces with G_2^*-structure are classified. Their holonomy algebras can be read off from this classification. It turns out that they are abelian and of dimension two or three. Furthermore, some results on left-invariant torsion-free G_2^*-structures on Lie groups (or, equivalently, G_2^*-structures on Lie algebras) are known. Examples of (decomposable) torsion-free G_2^*-structures with 1-dimensional and 2-dimensional holonomy have been found by M. Freibert [Fr] on almost abelian Lie algebras. In [FL] Fino and Lujan studied torsion-free G_2^*-structures with holonomy G_2^* on nilpotent Lie algebras, showing in particular that, up to isomorphism, there exists only one indecomposable nilpotent Lie algebra admitting a torsion-free G_2^*–structure such that the center is definite with respect to the induced inner product. In [FL] an example of an indecomposable torsion-free G_2^*-structure with 6-dimensional holonomy on a nilpotent Lie algebra is also given.

The first step in the classification of holonomy algebras is to get algebraic conditions for candidates for holonomy algebras $\mathfrak{h}$ strictly contained in $\mathfrak{g}_2^*$. These conditions can be derived from the following three facts. Firstly, since $\mathfrak{h}$ is a proper subalgebra of $\mathfrak{g}_2^*$, the natural representation of $\mathfrak{h} \subset \mathfrak{g}_2^* \subset \mathfrak{so}(4,3)$ on $\mathbb{R}^{4,3}$ has to leave invariant an isotropic subspace. Secondly, several restrictions come from the indecomposability of this representation. Thirdly, every holonomy algebra is a Berger algebra, i.e., it satisfies Berger's first criterion, which gives further conditions for $\mathfrak{h}$. In this paper we give a complete answer to this algebraic part of the classification problem. That is, we classify indecomposable Berger algebras strictly contained in $\mathfrak{g}_2^*$. The results can be summarised as follows. Let $\mathfrak{p}_1$, $\mathfrak{p}_2$ denote the two 9-dimensional parabolic subalgebras of $\mathfrak{g}_2^*$, which can be characterised by the action of G_2^* on isotropic subspaces of $\mathbb{R}^{4,3}$. The action of G_2^* on isotropic lines is transitive and $\mathfrak{p}_1$ is the Lie algebra of the stabiliser of an isotropic line. Furthermore, the action of G_2^* on 2-planes $E = \mathrm{span}\{b_1, b_2\}$ satisfying $b_1 \times b_2 = 0$ is transitive and $\mathfrak{p}_2$ is the Lie algebra of the stabiliser of such a 2-plane. We have $\mathfrak{p}_1 \cong \mathfrak{gl}(2,\mathbb{R}) \ltimes \mathfrak{m}$, where $\mathfrak{m}$ is three-step nilpotent and $\mathfrak{p}_2 \cong \mathfrak{gl}(2,\mathbb{R}) \ltimes \mathfrak{n}$, where $\mathfrak{n}$ is two-step nilpotent. Both $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are indecomposable Berger algebras. We will distinguish arbitrary indecomposable Berger algebras $\mathfrak{h} \subset \mathfrak{g}_2^* \subset \mathfrak{so}(4,3)$ by the dimension of the socle of their natural representation on $\mathbb{R}^{4,3}$. The socle is the maximal semisimple subrepresentation. By indecomposability, it is isotropic. We will say that $\mathfrak{h}$ is of Type I, II or III, if the dimension of the socle is one, two or three. In particular, $\mathfrak{p}_1$ is of Type I, $\mathfrak{p}_2$ is of Type II. We show that $\mathfrak{h} \subset \mathfrak{p}_1$ up to conjugation if $\mathfrak{h}$ is of Type I or III. If $\mathfrak{h}$ is of Type II, then $\mathfrak{h} \subset \mathfrak{p}_2$ up to conjugation. Let $\mathfrak{a}$ be the projection of $\mathfrak{h}$ to $\mathfrak{gl}(2,\mathbb{R}) \subset \mathfrak{p}_i$ for $i = 1, 2$, respectively. Then we may assume that $\mathfrak{a}$ is one of the representatives of conjugacy classes of subalgebras of $\mathfrak{gl}(2,\mathbb{R})$. Roughly speaking, for each of these representatives we classify the subalgebras $\hat{\mathfrak{m}} \subset \mathfrak{m}$ and $\hat{\mathfrak{n}} \subset \mathfrak{n}$ for which $\mathfrak{a} \ltimes \hat{\mathfrak{m}}$ and $\mathfrak{a} \ltimes \mathfrak{n}$ are indecomposable Berger algebras. Finally, for each type, we get a list of all (conjugacy classes of) indecomposable Berger algebras (Theorems 2.4, 2.10 and 2.12).

The second part of the classification consists in the realisation of the possible holonomy algebras by metrics. As we already mentioned above, left-invariant metrics on 7-dimensional nilpotent and solvable Lie groups may provide interesting examples of such metrics. In Section 4, we give new examples of left-invariant metrics with holonomy contained in $\mathfrak{g}_2^*$. In particular, we can provide examples for each of the Types I, II and III. As for Type I, we can realise $\mathfrak{m}$ and, furthermore, a 7-dimensional solvable Lie algebra and a 6-dimensional nilpotent one as a holonomy algebra. Besides $\mathfrak{n}$ and $\mathfrak{sl}(2,\mathbb{R}) \ltimes \mathfrak{n}$, we give a 3-dimensional abelian example of Type II. Finally, we can realise a three-dimensional abelian Lie algebra of Type III. Another special class of pseudo-Riemannian manifolds is that of symmetric spaces. As already mentioned, symmetric spaces with G_2^*-structure were determined in [Ka2]. In Section 3, we check how their holonomy algebras fit into the classification.

Acknoledgement We are very thankful for the opportunity to stay two weeks in the very stimulating atmosphere at Mathematisches Forschungsinstitut Oberwolfach, where we finished essential parts of this paper.

Notation If $b_1, \ldots, b_n$ is a basis of a vector space W, then we denote by $b^1, \ldots, b^n$ its dual basis of W^*. Furthermore, $b^{i_1 \cdots i_k} := b^{i_1} \wedge \cdots \wedge b^{i_k} \in \bigwedge^k W^*$, $b_{i_1 \ldots i_k} := b_{i_1} \wedge \cdots \wedge b_{i_k} \in \bigwedge^k W$ and $b_i^j := b_i \otimes b^j \in W \otimes W^* \cong \mathrm{End}(W)$.

2. Holonomy groups contained in G_2^*

2.1. **The group G_2^*.** Let M be a simply connected manifold of signature $(4,3)$. Suppose that M admits a G_2^*-structure, given by a generic three-form ω. Then ω defines an orientation and a metric g of signature $(4,3)$ on M. Here signature $(4,3)$ means that $g = \mathrm{diag}(-1,-1,-1,-1,1,1,1)$ with respect to a suitable basis. Let ω be parallel with respect to g. Then the holonomy group H of (M, g) is contained in G_2^*.

Equivalently, we could have started with a pseudo-Riemannian manifold (M, g) of signature $(4, 3)$ whose holonomy group H is contained in $\mathrm{G}_2^* \subset \mathrm{SO}(4, 3)$. Then G_2^* defines a parallel 3-form ω on (M, g) and g is induced by ω.

Let $\mathfrak{h}$ denote the Lie algebra of H. Suppose that the holonomy representation of $\mathfrak{h}$ on $V := T_x(M)$ is indecomposable but not irreducible.

A subspace $0 \neq U \subset V$ is called isotropic if $g(u, u) = 0$ for all $u \in U$.

Let us give explicit formulas. We choose a basis $e_1, \dots, e_7$ of V such that the 3-form ω equals

$$\omega_0 = \sqrt{2}(e^{167} + e^{235}) - e^4 \wedge (e^{15} - e^{26} - e^{37})\,.$$

Then the orientation of V is defined to be the orientation of $e_1, \dots, e_7$ and the induced metric equals $\langle \cdot\,, \cdot \rangle = 2(e^1 \cdot e^5 + e^2 \cdot e^6 + e^3 \cdot e^7) - (e^4)^2$. In particular, we can identify G_2^* with the subgroup of $\mathrm{GL}(7, \mathbb{R})$ that stabilises ω_0. Then $\mathrm{G}_2^* \subset SO(4, 3)$ with respect to $\langle \cdot\,, \cdot \rangle$. The Lie algebra $\mathfrak{g}_2^*$ consists of all matrices of the form

$$\begin{pmatrix} s_1 + s_4 & -s_{10} & s_9 & \sqrt{2}s_6 & 0 & -s_{11} & -s_{12} \\ -s_8 & s_1 & s_2 & \sqrt{2}s_9 & s_{11} & 0 & s_6 \\ s_7 & s_3 & s_4 & \sqrt{2}s_{10} & s_{12} & -s_6 & 0 \\ \sqrt{2}s_5 & \sqrt{2}s_7 & \sqrt{2}s_8 & 0 & \sqrt{2}s_6 & \sqrt{2}s_9 & \sqrt{2}s_{10} \\ 0 & s_{13} & s_{14} & \sqrt{2}s_5 & -s_1 - s_4 & s_8 & -s_7 \\ -s_{13} & 0 & -s_5 & \sqrt{2}s_7 & s_{10} & -s_1 & -s_3 \\ -s_{14} & s_5 & 0 & \sqrt{2}s_8 & -s_9 & -s_2 & -s_4 \end{pmatrix}, \qquad (1)$$

where $s_1, \dots, s_{14} \in \mathbb{R}$.

The 3-form ω defines a cross product on V by

$$\langle u \times v, w \rangle = \omega(u, v, w).$$

This cross product is antisymmetric and satisfies

$$\langle u \times v, u \rangle = 0, \quad u \times (u \times v) = -\langle u, u \rangle v + \langle u, v \rangle u,$$

for all $u, v \in V$.

The group G_2^* can also be understood as the stabiliser of a non-isotropic spinor ψ_0 in the real spinor representation Δ of $\mathrm{Spin}(V) \cong \mathrm{Spin}(4, 3)$. Indeed, the two-fold covering map $\lambda : \mathrm{Spin}(V) \to \mathrm{SO}(V)$ induces an isomorphism from this stabiliser to G_2^*. This is well known, see e.g. [Ka1] for details. We give also [Ka2] as a reference here since the above mentioned formulas for ω_0 and $\mathfrak{g}_2^*$ can be obtained from the description of the Clifford algebra of V in [Ka2] in replacing the basis $e_1, \dots, e_7$ used there by $e_7, e_5, e_6, e_4, e_3, e_1, e_2$. We want to recall the following well-known facts. The spinor module Δ admits an inner product $\langle \cdot\,, \cdot \rangle_\Delta$ of signature $(4,4)$ that is invariant with respect to the Clifford multiplication, i.e.,

$$\langle v \cdot \varphi, \psi \rangle_\Delta + \langle \varphi, v \cdot \psi \rangle_\Delta = 0,$$

for all $v \in V$. The Clifford multiplication of the non-isotropic spinor ψ_0 by a vector

$$V \longrightarrow \{\psi_0\}^\perp \subset \Delta, \quad v \longmapsto v \cdot \psi_0$$

is an isomorphism from V to $\{\psi_0\}^\perp$. The spinor ψ_0 defining G_2^* is related to the cross product by

$$u \cdot v \cdot \psi_0 + \langle u, v \rangle \psi_0 = (u \times v) \cdot \psi_0. \tag{2}$$

The map

$$\Delta \ni \varphi \longmapsto U(\varphi) := \{v \in V \mid v \cdot \varphi = 0\} \subset V$$

induces a bijection from the set of projective isotropic spinors to the set of 3-dimensional isotropic subspaces of V.

2.2. **The type of a holonomy algebra contained in $\mathfrak{g}_2^*$.** Since the holonomy representation of $\mathfrak{h}$ is indecomposable but not irreducible there exists at least one $\mathfrak{h}$-invariant isotropic subspace $\hat{E} \subset V$.

Lemma 2.1. *The following statements are true for any indecomposable subalgebra $\mathfrak{h}$ of $\mathfrak{g}_2^* \subset \mathfrak{so}(4,3)$. Let $E \subset V$ be an $\mathfrak{h}$-invariant isotropic subspace.*

1. *If $\dim E = 1$, then $\hat{E}(E) := \{v \in V \mid \forall e \in E : v \times e = 0\}$ is a three-dimensional isotropic $\mathfrak{h}$-invariant subspace of V containing E.*
2. *If $\dim E = 3$, then there exists a uniquely determined one-dimensional isotropic $\mathfrak{h}$-invariant subspace $E_0 \subset E$ such that $E = \hat{E}(E_0)$.*
3. *If $\dim E = 2$ and if $b_1 \times b_2 \neq 0$ for a basis b_1, b_2 of E, then there exists a one-dimensional $\mathfrak{h}$-invariant subspace $E_0 \subset V$ not contained in E such that $E \oplus E_0$ is isotropic.*

Proof. (1) Suppose $E = \mathbb{R}\cdot b$, $b \in V$. We want to show that $\hat{E}(E) = U(b\cdot\psi_0)$. Note first that $b \perp \hat{E}(E)$ since $0 = b \times (v \times b) = \langle b, b\rangle v - \langle b.v\rangle b = -\langle b, v\rangle b$ for all $v \in \hat{E}(E)$. Now (2) shows that $\hat{E}(E) \subset U(b \cdot \psi_0)$. Equation (2) also implies $\psi_0^\perp \ni (u\times b)\cdot\psi_0 = \langle u, b\rangle\psi_0$ for all $u \in U(b\cdot\psi_0)$, thus $U(b\cdot\psi_0) \subset \hat{E}(E)$. The subspace $U(b \cdot \psi_0)$ is three-dimensional and isotropic. Moreover, it is $\mathfrak{h}$-invariant since E is $\mathfrak{h}$-invariant and ψ_0 is annihilated by $\mathfrak{h}$. Hence the same is true for $\hat{E}(E)$.

(2) Suppose $\dim E = 3$. Then there exists an isotropic spinor φ_0 such that $E = U(\varphi_0)$. Since $\mathfrak{h}(E) \subset E$, we have $E \cdot (\mathfrak{h} \cdot \varphi_0) = \mathfrak{h} \cdot (E \cdot \varphi_0) = 0$, hence $E \cdot \varphi_0 \subset \mathbb{R} \cdot \varphi_0$. Assume that $\langle \varphi_0, \psi_0\rangle_\Delta \neq 0$. We define a vector X by $\psi_0 + X \cdot \psi_0 \in \mathbb{R} \cdot \varphi_0$. Then $\mathfrak{h}(X) = 0$ because of $\mathfrak{h} \cdot (\psi_0 + X \cdot \psi_0) = \mathfrak{h}(X) \cdot \psi_0 \in (\mathbb{R} \cdot \varphi_0) \cap \psi_0^\perp = 0$. On the other hand, X is not isotropic, which is a contradiction to indecomposability. Hence $\langle \psi_0, \varphi_0\rangle_\Delta = 0$. In particular, we can define $b \in V$ by $\varphi_0 = b\cdot\psi_0$. Then $E = U(\varphi_0) = \hat{E}(E_0)$ for $E_0 = \mathbb{R}\cdot b$.

As for uniqueness, $\hat{E}(\mathbb{R}\cdot b)=\hat{E}(\mathbb{R}\cdot b')$ implies $U(b\cdot\psi_0)=U(b'\cdot\psi_0)$, thus $b'\in\mathbb{R}\cdot b$.

(3) Now assume that $E=\mathrm{span}\{b_1,b_2\}$ and that $b_1\times b_2=:b\neq 0$. If b were in E, then $b_1\times b_2=b_2$ without loss of generality. But this would imply $b_2=b_1\times(b_1\times b_2)=-\langle b_1,b_1\rangle b_2+\langle b_1,b_2\rangle b_1=0$, which contradicts $b_2\neq 0$. Thus $E_0:=\mathbb{R}\cdot b$ is not contained in E. Moreover, $E\oplus E_0$ is isotropic since $u\times v\perp u$ for all $u,v\in V$. □

Let S be the socle of the holonomy representation. Then S is isotropic. Indeed, $S\cap S^\perp$ is $\mathfrak{h}$-invariant. Since $\mathfrak{h}$ acts semisimply on S, there exists an $\mathfrak{h}$-invariant complement of $S\cap S^\perp$ in S. This complement is non-degenerate, hence trivial. Thus $S\cap S^\perp=S$.

Definition 2.2. *The holonomy representation is said to be of Type I, II or III if the dimension of S equals one, two or three, respectively.*

2.3. Berger algebras of Type I.

Let $\mathfrak{h}$ be of Type I .

Lemma 2.3. *If $\mathfrak{h}$ is of type I, then there exists a basis $b_1,\dots,b_7$ of V such that*

$$\begin{aligned}\langle\cdot,\cdot\rangle &= 2(b^1\cdot b^5+b^2\cdot b^6+b^3\cdot b^7)-(b^4)^2\\ \omega &= \sqrt{2}(b^{167}+b^{235})-b^4\wedge(b^{15}-b^{26}-b^{37})\end{aligned}$$

and $\mathfrak{h}$ is a subalgebra of

$$\mathfrak{h}^I:=\{h(A,v,u,y)\mid A\in\mathfrak{gl}(2,\mathbb{R}),\ v\in\mathbb{R},\ u,y\in\mathbb{R}^2\},$$

where

$$h(A,v,u,y)=\begin{pmatrix} \mathrm{tr}\,A & -u_2 & u_1 & \sqrt{2}v & 0 & -y_1 & -y_2\\ 0 & a_1 & a_2 & \sqrt{2}u_1 & y_1 & 0 & v\\ 0 & a_3 & a_4 & \sqrt{2}u_2 & y_2 & -v & 0\\ 0&0&0&0&\sqrt{2}v&\sqrt{2}u_1&\sqrt{2}u_2\\ 0&0&0&0&-\mathrm{tr}\,A&0&0\\ 0&0&0&0&u_2&-a_1&-a_3\\ 0&0&0&0&-u_1&-a_2&-a_4\end{pmatrix}$$

for $A=\begin{pmatrix}a_1&a_2\\a_3&a_4\end{pmatrix}$, $y=(y_1,y_2)^\top$, $u=(u_1,u_2)^\top$.

Proof. Let the socle be spanned by the isotropic vector b. Recall that G_2^* acts transitively on isotropic lines in $\mathbb{R}^{4,3}$. Hence we may assume $b=e_1$. Put $b_i:=e_i$, $i=1,\dots,7$. Then the assertion follows from (1). □

We define

$$\mathfrak{m}:=\{h(0,v,u,y)\mid v\in\mathbb{R},\ u,y\in\mathbb{R}^2\}\subset\mathfrak{h}^I$$

and identify $\mathfrak{gl}(2,\mathbb{R})$ with $\{h(A,0,0,0)\mid A\in\mathfrak{gl}(2,\mathbb{R})\}$. Then

$$\mathfrak{h}^I=\mathfrak{gl}(2,\mathbb{R})\ltimes\mathfrak{m}.$$

We define the matrices

$$C_a := \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix}, \ S := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ N := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};$$

and the following Lie subalgebras of $\mathfrak{gl}(2,\mathbb{R}) \subset \mathfrak{h}^I$:

$$\begin{array}{rcl} \mathfrak{d} & := & \{\mathrm{diag}(a,d) \mid a,d \in \mathbb{R}\}, \\ \mathfrak{u}(1) & := & \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \;\middle|\; a,b \in \mathbb{R} \right\}, \\ \hat{\mathfrak{b}}_2 & := & \mathrm{span}\{I, N\}, \\ \mathfrak{s}_\lambda & := & \mathrm{span}\{\mathrm{diag}(\lambda, \lambda-1),\ N\},\ \lambda \in \mathbb{R}, \\ \mathfrak{b}_2 & := & \text{Lie algebra of upper triangular matrices.} \end{array}$$

Furthermore, we define vector subspaces of $\mathfrak{m}$ by $\mathfrak{m}(0,0,0) := 0$ and

$$\begin{array}{rcl} \mathfrak{m}(1,0,0) & := & \{h(0,v,0,0) \mid v \in \mathbb{R}\}, \\ \mathfrak{m}(0,1,0) & := & \{h(0,0,(u_1,0)^\top,0) \mid u_1 \in \mathbb{R}\}, \\ \mathfrak{m}(0,0,1) & := & \{h(0,0,0,(y_1,0)^\top) \mid y_1 \in \mathbb{R}\}, \\ \mathfrak{m}(0,0,2) & := & \{h(0,0,0,y) \mid y \in \mathbb{R}^2\}. \end{array}$$

Now we put

$$\mathfrak{m}(i,j,k) = \mathfrak{m}(i,0,0) \oplus \mathfrak{m}(0,j,0) \oplus \mathfrak{m}(0,0,k)$$

for $i,j \in \{0,1\}$, $k \in \{0,1,2\}$.

Let $\mathfrak{a}$ be the projection of $\mathfrak{h}$ to $\mathfrak{gl}(2,\mathbb{R}) \subset \mathfrak{h}^I$.

Theorem 2.4. *If $\mathfrak{h}$ is of Type I, then there exists a basis of V such that we are in one of the following cases*

(1) $\mathfrak{a} \in \{0, \mathfrak{sl}(2,\mathbb{R}),\ \mathfrak{gl}(2,\mathbb{R}),\ \mathfrak{u}(1),\ \mathfrak{b}_2,\ \hat{\mathfrak{b}}_2,\ \mathfrak{d},\ \mathbb{R}\cdot C_a,\ \mathbb{R}\cdot S\}$ *and* $\mathfrak{h} = \mathfrak{a} \ltimes \mathfrak{m}$,
(2) $\mathfrak{a} = \mathfrak{s}_\lambda = \mathrm{span}\{X := \mathrm{diag}(\lambda,\lambda-1),\ N\}$ *and*
  (a) $\lambda \in \mathbb{R}$ *and* $\mathfrak{h} = \mathfrak{a} \ltimes \mathfrak{m}$,
  (b) $\lambda = 1$ *and* $\mathfrak{h} = \mathbb{R}\cdot h(X,0,(0,1)^\top,0) \ltimes (\mathbb{R}\cdot N \ltimes \mathfrak{m}(1,1,2))$,
  (c) $\lambda = 2$ *and* $\mathfrak{h} = \mathrm{span}\{X,\ h(N,0,(0,1)^\top,0)\} \ltimes \mathfrak{m}(i,j,2)$, *where* $i,j \in \{0,1\}$,
(3) $\mathfrak{a} = \mathbb{R}\cdot\mathrm{diag}(1,\mu)$ *and*
  (a) $\mu \in [-1,1]$ *and* $\mathfrak{h} = \mathfrak{a} \ltimes \mathfrak{m}$,
  (b) $\mu = 0$ *and* $\mathfrak{h} = \mathbb{R}\cdot h(\mathrm{diag}(1,0),0,(0,1)^\top,0) \ltimes \mathfrak{m}(1,1,2)$,
(4) $\mathfrak{a} = \mathbb{R}\cdot N$ *and*
  (a) $\mathfrak{h} = \mathfrak{a} \ltimes \mathfrak{m}$,
  (b) $\mathfrak{h} = \mathbb{R}\cdot h(N,0,(0,1)^\top,0) \ltimes \mathfrak{m}(1,j,2)$ *for* $j \in \{0,1\}$.

The remainder of this section is devoted to the proof of Theorem 2.4. Let us first have a closer look at the structure of $\mathfrak{h}^I = \mathfrak{gl}(2,\mathbb{R}) \ltimes \mathfrak{m}$. The element $A \in \mathfrak{gl}(2,\mathbb{R})$ acts on $\mathfrak{m}$ by

$$A \cdot h(0,v,u,y) = h(0,\mathrm{tr}(A)v, Au, (A + \mathrm{tr}\,A)y).$$

Furthermore, the Lie bracket on $\mathfrak{m}$ is given by

$$[h(0,v,u,y), h(0,\bar{v},\bar{u},\bar{y})] = h(0, 2\theta(u,\bar{u}), 0, 3(\bar{v}u - v\bar{u})), \tag{3}$$

where $\theta(u,\bar{u}) := u_1\bar{u}_2 - u_2\bar{u}_1$ for $u,\bar{u} \in \mathbb{R}^2$.

Similarly to $\mathfrak{gl}(2,\mathbb{R})$, we identify $\mathrm{GL}(2,\mathbb{R})$ with a subgroup of G_2^* consisting of block diagonal matrices:

$$\mathrm{GL}(2,\mathbb{R}) \ni g \longmapsto \mathrm{diag}(\det g, g, 1, (\det g)^{-1}, (g^\top)^{-1}) \in \mathrm{G}_2^*, \tag{4}$$

where G_2^* is considered with respect to the basis in Theorem 2.4. Then

$$\mathrm{Ad}(g)(h(A,v,u,y)) = h(gAg^{-1},\ \det(g)\cdot v,\ gu,\ \det(g)\cdot gy). \tag{5}$$

Lemma 2.5. *Either $\mathfrak{a} \in \{0,\ \mathfrak{sl}(2,\mathbb{R}),\ \mathfrak{gl}(2,\mathbb{R})\}$ or the basis $b_1,\dots,b_7$ in Lemma 2.3 can be chosen such that $\mathfrak{a}$ is equal to one of the following Lie algebras:*

(1) $\mathbb{R}\cdot A$, *where A is one of the matrices* C_a, S, N, $\mathrm{diag}(1,\mu)$, $\mu \in [-1,1]$;
(2) $\mathfrak{d}$, $\mathfrak{u}(1)$, $\hat{\mathfrak{b}}_2$, $\mathfrak{s}_\lambda$, $\lambda \in \mathbb{R}$;
(3) $\mathfrak{b}_2$.

Proof. We identify $\mathrm{GL}(2,\mathbb{R})$ with a subgroup of G_2^* as described above. The conjugation of $\mathfrak{h}$ by an element of $\mathrm{GL}(2,\mathbb{R})$ is given by (5). Hence the proof of the Lemma is just the well-known classification of subalgebras of $\mathfrak{gl}(2,\mathbb{R})$ up to conjugation by $\mathrm{GL}(2,\mathbb{R})$:

(1) Suppose that $\mathfrak{a}$ is generated by one matrix $A \neq 0$. If A has real eigenvalues, then we may assume that one of the eigenvalues equals 1. Otherwise, we may assume that the imaginary part of the eigenvalues equals ± 1. Conjugating by an element of $\mathrm{GL}(2,\mathbb{R})$ we can achieve that A has real Jordan normal form, which is $\mathrm{diag}(1,\mu)$, C_a, S or N.

(2) Now let $\mathfrak{a}$ be two-dimensional. If the natural representation of $\mathfrak{a}$ on $\mathbb{R}^2$ is semisimple, then $\mathfrak{a} = \mathfrak{d}$ or $\mathfrak{a} = \mathfrak{u}(1)$ after conjugation by an element of $\mathrm{GL}(2,\mathbb{R})$. If not, then $\mathfrak{a} = \hat{\mathfrak{b}}_2$ or $\mathfrak{a} = \mathfrak{s}_\lambda$ after conjugation depending on whether $\mathfrak{a}$ is abelian or not.

(3) If $\dim \mathfrak{a} = 3$ and $\mathfrak{a} \neq \mathfrak{sl}(2,\mathbb{R})$, then $\mathfrak{a}$ is solvable, thus conjugated to $\mathfrak{b}_2$. □

We define

$$\mathcal{K}(\mathfrak{h}) = \{R \in \textstyle\bigwedge^2 V^* \otimes \mathfrak{h} \mid \forall x,y,z \in V : R(x,y)z + R(y,z)x + R(z,x)y = 0\}$$

and

$$\underline{\mathfrak{h}} := \mathrm{span}\{R(x,y) \mid x,y \in V,\ R \in \mathcal{K}(\mathfrak{h})\}.$$

Berger's first criterion implies $\mathfrak{h} = \underline{\mathfrak{h}}$. Let $b_1,\dots,b_7$ be a basis as chosen in Lemma 2.3. If $R \in \mathcal{K}(\mathfrak{h})$, then

$$R_{ij} := R(b_i,b_j) = h(A^{ij}, v^{ij}, u^{ij}, y^{ij}).$$

Lemma 2.6. *If $R \in \mathcal{K}(\mathfrak{h})$, then*

(1) $R_{1j} = 0$ *for all* $j \neq 5$ *and* $R_{ij} = 0$ *for* $i, j \in \{2, 3, 4\}$,
(2) $\operatorname{tr} A^{ij} = 0$ *if* $i < j$ *and* $(i, j) \notin \{(5, 6), (5, 7)\}$,
(3) $R_{15} = h(0, 0, 0, (\operatorname{tr} A^{56}, \operatorname{tr} A^{57})^\top)$.

Proof. Let R be in $\mathcal{K}(\mathfrak{h})$. Since $\langle R_{ij}(b_k), b_l\rangle = \langle R_{kl}(b_i), b_j\rangle$ and $R_{kl} \in \mathfrak{h}$, assertion (1) follows.

We define $b(i, j, k) := R_{ij}(b_k) + R_{jk}(b_i) + R_{ki}(b_j)$. From $b(i, j, 5) = 0$ we get $\operatorname{tr} A^{ij} = 0$ for $i, j \neq 5$. Furthermore, $b(1, 5, 6) = b(1, 5, 7) = 0$ together with $R_{16} = R_{17} = 0$ implies $A^{15} = 0$, $u^{15} = 0$, $v^{15} = 0$ and $(\operatorname{tr} A^{56}, \operatorname{tr} A^{57})^\top = y^{15}$. Now $b(1, i, 5) = 0$ together with $u^{15} = 0$ and $v^{15} = 0$ gives $\operatorname{tr} A^{i5} = 0$ for $i = 2, 3, 4$. □

Corollary 2.7. *If* $\mathfrak{a}$ *contains an element* A *with* $\operatorname{tr} A \neq 0$*, then*

$$\mathfrak{y} := \{y \in \mathbb{R}^2 \mid h(0, 0, 0, y) \in \mathfrak{h}\} \neq 0.$$

Proposition 2.8. *The space* $\mathcal{K}(\mathfrak{h})$ *can be parametrised by* a_i, r_i, $x_i \in \mathbb{R}$ *(*$i = 1, 2, 3$*),* $b_k, c_k, , u_k, j_k \in \mathbb{R}$ *(*$k = 1, \dots, 4$*) and* $v_1, v_2, t \in \mathbb{R}$*, where* $R = h(A, v, u, y) \in \mathcal{K}(\mathfrak{h})$ *is given by the data in Table 1.*

Proof. Let R be in $\mathcal{K}(\mathfrak{h})$. As in the proof of Lemma 2.6, we use $\langle R_{ij}(b_k), b_l\rangle = \langle R_{kl}(b_i), b_j\rangle$ and $R_{kl} \in \mathfrak{h}$, which now gives

$$\sqrt{2}R_{25} = -R_{47},\ \sqrt{2}R_{35} = R_{46},\ \sqrt{2}R_{67} = -R_{45}$$

and

$$R_{37} = R_{15} - R_{26}.$$

Let us consider the equations $b(i, j, k) = 0$ for $i, j, k \neq 4$. These equations give, in particular,

$$v^{56} = -y_1^{67} =: v_1, \quad v^{56} = -y_1^{67} =: v_2, \quad y_1^{57} = y_2^{56} =: t.$$

Moreover, they imply the already proven properties of R stated in Lemma 2.6. The system of the remaining linear equations for the coefficients A^{ij}, v^{ij}, u^{ij} and y^{ij} of R_{ij} following from $b(i, j, k) = 0$ for $i, j, k \neq 4$ decomposes into five subsystems. Each of these subsystems is a system of equations in the elements of one of the following sets:

$$\begin{aligned}
M_1 :=&\{A^{26}, A^{27}, A^{36}, A^{37}\},\\
M_2 :=&\{A^{56}, A^{57}, y^{26}, y^{36}, y^{27}, y^{37}, v^{25}, v^{35}, u^{67}\},\\
M_3 :=&\{A^{25}, A^{35}, u^{27}, u^{37}, u^{26}, u^{36}\},\\
M_4 :=&\{A^{67}, u^{25}, u^{35}, v^{26}, v^{27}, v^{36}, v^{37}\},\\
M_5 :=&\{u^{56}, u^{57}, y^{25}, y^{35}, v^{67}\}.
\end{aligned}$$

$R(b_i, b_j)$	A	v	u	y
R_{15}	0	0	0	$(b_1 + b_4, c_1 + c_4)$
$-R_{25} = \frac{1}{\sqrt{2}} R_{47}$	$\begin{pmatrix} x_2 & -x_1 \\ x_3 & -x_2 \end{pmatrix}$	$c_1 - b_3$	(r_2, r_3)	(u_2, u_4)
R_{26}	$\begin{pmatrix} -a_1 & -a_2 \\ -a_3 & a_1 \end{pmatrix}$	$-r_2$	(x_1, x_2)	(b_1, c_1)
R_{27}	$\begin{pmatrix} -a_3 & a_1 \\ j_1 & a_3 \end{pmatrix}$	$-r_3$	(x_2, x_3)	(b_3, c_3)
$R_{35} = \frac{1}{\sqrt{2}} R_{46}$	$\begin{pmatrix} x_1 & x_4 \\ x_2 & -x_1 \end{pmatrix}$	$b_4 - c_2$	(r_1, r_2)	(u_1, u_3)
R_{36}	$\begin{pmatrix} -a_2 & j_2 \\ a_1 & a_2 \end{pmatrix}$	r_1	$(x_4, -x_1)$	(b_2, c_2)
$-R_{67} = \frac{1}{\sqrt{2}} R_{45}$	$\begin{pmatrix} -r_2 & r_1 \\ -r_3 & r_2 \end{pmatrix}$	$u_2 - u_3$	$(b_4 - c_2, c_1 - b_3)$	(v_1, v_2)
R_{56}	$\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$	v_1	(u_1, u_2)	(j_3, t)
R_{57}	$\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$	v_2	(u_3, u_4)	(t, j_4)
$R_{12} = R_{13} = R_{14} = R_{16} = R_{17} = R_{23} = R_{24} = R_{34} = 0$; $R_{37} = R_{15} - R_{26}$				

Table 1.

The subsystem for M_1 is

$$\begin{aligned} &-a_1^{26} = a_4^{26} = a_2^{27} = a_1^{37} = -a_4^{37} =: a_1, \quad a_4^{26} = a_3^{36}, \\ &-a_2^{26} = -a_1^{36} = a_4^{36} = a_2^{37} =: a_2, \\ &-a_3^{26} = -a_1^{27} = a_4^{27} = a_3^{37} =: a_3. \end{aligned}$$

Together with $a_3^{27} =: j_1$, $a_2^{36} =: j_2$ this gives the parametrisation of M_1 claimed in the proposition.

For M_2, we have

$$\begin{aligned}
&a_1^{56} = y_1^{26} =: b_1, \quad a_2^{56} = y_1^{36} =: b_2,\\
&a_3^{57} = y_2^{27} =: c_3, \quad a_4^{57} = y_2^{37} =: c_4,\\
&v^{25} + y_2^{26} = a_3^{56} = a_1^{57} + u_2^{67} =: b_3,\\
&v^{35} + y_2^{36} = a_4^{56} = a_2^{57} - u_1^{67} =: b_4,\\
&v^{25} + a_1^{57} = y_1^{27} = y_2^{26} + u_2^{67},\\
&v^{35} + a_2^{57} = y_1^{37} = y_2^{36} - u_1^{67}
\end{aligned}$$

and for M_3

$$\begin{aligned}
&a_1^{35} = a_2^{25} = u_1^{26} = -u_2^{36} = -u_1^{37} = -a_4^{35} =: x_1,\\
&a_4^{25} = a_3^{25} = -u_2^{37} = u_1^{27} = u_2^{26} = -a_1^{25} =: x_2,\\
&u_2^{27} = -a_3^{25} =: x_3, \quad u_1^{36} = a_2^{35} =: x_4.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&a_1^{67} = -v^{26} = -u_1^{25} = u_2^{35} = v^{37} = -a_4^{67} =: r_2,\\
&-a_2^{67} = v^{36} = u_1^{35} =: r_1,\\
&a_3^{67} = -v^{27} = -u_2^{25} =: r_3
\end{aligned}$$

is the system for M_4 and

$$\begin{aligned}
&y_1^{35} = u_1^{56} =: u_1, \quad -y_1^{25} = u_2^{56} =: u_2,\\
&y_2^{35} = u_1^{57} =: u_3, \quad -y_2^{25} = u_2^{57} =: u_4,\\
&v^{67} = u_1^{57} - u_2^{56}
\end{aligned}$$

that for M_5. This shows that $R \in \mathcal{K}(\mathfrak{h})$ is given as in Table 1. Moreover, the above considered systems of linear equations imply that each R that is defined as in Table 1 for an arbitrary choice of the parameters $a_i, r_i, x_i, b_k, c_k, u_k, j_k, v_1, v_2, t$ satisfies $b(i,j,k) = 0$ for $i,j,k \neq 4$. It is easy to show by a direct calculation that it also satisfies $b(i,j,4) = 0$ for $1 \leq i,j \leq 7$. □

Proof of Theorem 2.4. We define

$$\mathfrak{u} := \{u \in \mathbb{R}^2 \mid \exists v \in \mathbb{R}, \exists y \in \mathbb{R}^2 : h(0,v,u,y) \in \mathfrak{h}\}, \tag{6}$$

$$\mathfrak{v} := \{v \in \mathbb{R} \mid \exists y \in \mathbb{R}^2 : h(0,v,0,y) \in \mathfrak{h}\} \tag{7}$$

Below, we will several times conjugate by elements of G_2^*. In particular, we will use the following formulas, which can be obtained from (3):

$$\mathrm{Ad}(\exp h(0,\bar v,0,0))(h(A,v,u,y)) = h(A, v - \mathrm{tr}(A)\bar v, u, y - 3\bar v u), \tag{8}$$

$$\begin{aligned}
&\mathrm{Ad}(\exp h(0,0,\bar u,0))(h(A,v,u,y)) = \\
&\quad h(A, v - 2\theta(u,\bar u) - \theta(\bar u, A\bar u), u - A\bar u, y + (3v - 3\theta(u,\bar u) - \theta(\bar u, A\bar u))\bar u),
\end{aligned} \tag{9}$$

$$\mathrm{Ad}(\exp h(0,0,0,\bar y))(h(A,v,u,y)) = h(A,v,u,y - (A + \mathrm{tr}\,A)\cdot \bar y). \tag{10}$$

(1) If $\mathfrak{a} = 0$, then $\dim \mathfrak{u} = 2$ since otherwise the kernel of $\mathfrak{h}$ would be at least two-dimensional. Thus there exist elements $v_1, v_2 \in \mathbb{R}$, $y_1, y_2 \in \mathbb{R}^2$ such that

$$h_1 := h(0, v_1, (1,0)^\top, y_1),\ \ h_2 := h(0, v_2, (0,1)^\top, y_2) \in \mathfrak{h}.$$

Then also $h_3 := [h_1, h_2] = h(0, 2, 0, 3(v_2, -v_1)^\top)$, $[h_1, h_3] = h(0, 0, 0, (6,0)^\top)$ and $[h_2, h_3] = h(0, 0, 0, (0,6)^\top)$ are elements of $\mathfrak{h}$, hence $\mathfrak{h} = \mathfrak{m}$.

Suppose $\mathfrak{a} = \mathfrak{sl}(2,\mathbb{R})$. Assume that $\mathfrak{u} = 0$. Then for each $A \in \mathfrak{a}$ there exists a unique element $u_A \in \mathbb{R}^2$ such that $h(A, v, u_A, y) \in \mathfrak{h}$ for some $v \in \mathbb{R}, y \in \mathbb{R}^2$. The map $A \mapsto u_A$ is a cocycle with respect to the standard representation of $\mathfrak{a}$ on $\mathbb{R}^2$. Since $H^1(\mathfrak{a}, \mathbb{R}^2) = 0$ by Whitehead's lemma, it is a coboundary, i.e., $u_A = A\hat{u}$ for some $\hat{u} \in \mathbb{R}^2$. After conjugation of $\mathfrak{h}$ by $\exp h(0, 0, \hat{u}, 0)$ according to (9), the projection of $\mathfrak{h}$ to $\{h(0, 0, u, 0) \mid u \in \mathbb{R}^2\}$ is trivial. But then $\mathfrak{h}$ is of Type III. Hence $\mathfrak{u} \neq 0$. Since $\mathfrak{u}$ is invariant under $\mathfrak{a}$, we obtain $\dim \mathfrak{u} = 2$ and as above we conclude $\mathfrak{m} \subset \mathfrak{h}$.

For $\mathfrak{a} = \mathbb{R} \cdot C_a$ we can argue similarly. Indeed, $h(C_a, v, u, y)$ for some v, u, y. Since C_a is non-singular, we have $u = C_a(\hat{u})$ for some $\hat{u} \in \mathbb{R}^2$ and we can proceed as above.

Now suppose $\mathfrak{a} = \mathbb{R} \cdot S$. Since S defines a bijective map we may assume that (after conjugation) $h(S, v, 0, y) \in \mathfrak{h}$ for some $v \in \mathbb{R}$ and $y \in \mathbb{R}^2$. Hence $\mathfrak{u} \neq 0$ since otherwise $\mathfrak{h}$ would be of Type II. Assume that $\dim \mathfrak{u} = 1$. Then $\mathfrak{u} = \mathbb{R} \cdot (1,0)^\top$ since $\mathfrak{u}$ is invariant under $\mathfrak{a}$. But then $\mathfrak{h}$ again would be of Type II. Hence $\dim \mathfrak{u} = 2$, which implies $\mathfrak{m} \subset \mathfrak{h}$.

Before we continue with the remaining cases let us make the following remark. Suppose $I \in \mathfrak{a}$ and $\mathfrak{u} = 0$. Then, as above, $h(A, v, u_A, y) \in \mathfrak{h}$ for a cocycle $A \mapsto u_A$ and some $v \in \mathbb{R}, y \in \mathbb{R}^2$. Since $I \in \mathfrak{a}$, we have, in particular, $0 = u_{[I,A]} = I(u_A) - A(u_I)$. Hence $u_A = A(u_I)$ for all $A \in \mathfrak{a}$. As in the case of $\mathfrak{a} = \mathfrak{sl}(2,\mathbb{R})$, after conjugation of $\mathfrak{h}$ by $\exp h(0, 0, \hat{u}, 0)$ the projection of $\mathfrak{h}$ to $\{h(0, 0, u, 0) \mid u \in \mathbb{R}^2\}$ is trivial and $\mathfrak{h}$ is not of Type I.

Suppose $\mathfrak{a} \in \{\mathfrak{gl}(2,\mathbb{R}), \mathfrak{u}(1)\}$. Then $\mathfrak{u} \neq 0$ by the above remark. Since $\mathfrak{u}$ is $\mathfrak{a}$-invariant we get $\dim \mathfrak{u} = 2$, thus $\mathfrak{m} \subset \mathfrak{h}$.

Now let $\mathfrak{a}$ be one of the Lie algebras $\mathfrak{b}_2$, $\hat{\mathfrak{b}}_2$. Then $\mathfrak{u} \neq 0$. If $\dim \mathfrak{u} = 2$, then $\mathfrak{m} \subset \mathfrak{h}$. Thus we have to discuss the case $\dim \mathfrak{u} = 1$. Because of the $\mathfrak{a}$-invariance of $\mathfrak{u}$ we have $\mathfrak{u} = \mathbb{R} \cdot (1,0)^\top$. Since $\operatorname{tr} I \neq 0$ and I and $I + \operatorname{tr} I$ act bijectively on $\mathbb{R}^2$, we may assume that $h_1 := h(I, 0, 0, 0)$ is in $\mathfrak{h}$ (after conjugation according to (8) – (10)). Let $h(A, v_A, (0,u')^\top, y_A)$ be in $\mathfrak{h}$. Then also

$$[h_1, h(A, v_A, (0,u')^\top, y_A)] = h(0, 2v_A, (0,u')^\top, 3y_A)$$

is in $\mathfrak{h}$, which implies $u' = 0$. But then $\mathfrak{h}$ would be of Type II.

Finally, we consider the case $\mathfrak{a} = \mathfrak{d}$. Then again $\mathfrak{u} \neq 0$. If $\dim \mathfrak{u} = 2$, we are done. Assume $\dim \mathfrak{u} = 1$. Because of the invariance of $\mathfrak{u}$ under $\mathfrak{a}$ we have $\mathfrak{u} = \mathbb{R} \cdot (1,0)^\top$ or $\mathfrak{u} = \mathbb{R} \cdot (0,1)^\top$. Since $\operatorname{tr} I \neq 0$ and I and $I + \operatorname{tr} I$ acts bijectively on $\mathbb{R}^2$, we may assume that $h_1 := h(I, 0, 0, 0)$ is in $\mathfrak{h}$ (after conjugation). For $H = \mathrm{diag}(1, -1)$, there are elements $v \in \mathbb{R}$, $u \in \mathbb{R}^2$, $y \in \mathbb{R}^2$ such that $h(H, v, u, y) \in \mathfrak{h}$. Since

$$[h_1, h(H, v, u, y)] = h(0, 2v, u, 3y) \in \mathfrak{h},$$

$u \in \mathfrak{u}$ by (6). Thus we may assume that $h(H, v, 0, y) \in \mathfrak{h}$ for some $v \in \mathbb{R}$, $y \in \mathbb{R}^2$. But then $\mathfrak{h}$ is of Type II.

(2) Now let $\mathfrak{a}$ be equal to $\mathfrak{s}_\lambda$. First we want to show that $\mathfrak{u} = 0$ implies $\lambda = 2$. If $\mathfrak{u} = 0$ we can define a cocycle $A \mapsto u_A$ by $h(A, v, u_A, y) \in \mathfrak{h}$ for some v, y. Since $[X, N] = N$ we obtain $X \cdot u_N - N \cdot u_X = u_N$, which yields

$$(\lambda - 1)u_N^1 = u_X^2, \quad (\lambda - 2)u_N^2 = 0 \tag{11}$$

for the components u_X^1, u_X^2 of u_X and u_N^1, u_N^2 of u_N. We may assume that $u_X^2 = 0$. Indeed, if $\lambda \neq 1$, then $\{(0, u_2)^\top \mid u_2 \in \mathbb{R}\}$ is in the image of X, thus we can find a suitable conjugation of $\mathfrak{h}$. If $\lambda = 1$, then $u_X^2 = 0$ follows from (11). If $\lambda \neq 2$, then (11) gives $u_N^2 = 0$ and, consequently, the projection of $\mathfrak{h}$ to $\{h(0, 0, (0, u_2)^\top, 0) \mid u_2 \in \mathbb{R}\}$ vanishes. Hence $\mathfrak{h}$ is of Type II. Thus $\mathfrak{u} = 0$ can hold only if $\lambda = 2$.

Now take, $\lambda \in \mathbb{R}$, $\lambda \notin \{1, 2\}$. We have already seen that $\mathfrak{u} \neq 0$. If $\dim \mathfrak{u} = 2$, we are done. Assume that $\dim \mathfrak{u} = 1$. Then $\mathfrak{u} = \mathbb{R} \cdot (1, 0)^\top$ by $\mathfrak{a}$-invariance of $\mathfrak{u}$. Since $\lambda \neq 1$, we see as above that we may assume that $h(X, v_X, u_X, y_X)$ is in $\mathfrak{h}$ for some v_X, u_X, y_X with $u_X = (u', 0)$. Hence $h_X := h(X, v_X, 0, y_X) \in \mathfrak{h}$. Furthermore, $h_N := h(N, v_N, u_N, y_N) \in \mathfrak{h}$ for some v_N, u_N, y_N with $u_N = (0, u'')^\top$. Then

$$[h_X, h_N] = h(N, (2\lambda - 1)v_N, (\lambda - 1)u_N, y)$$

for some $y \in \mathbb{R}^2$. Since $\lambda \neq 2$, we obtain $u'' = 0$. Hence the projection of $\mathfrak{h}$ to $\{h(0, 0, (0, u_2)^\top, 0) \mid u_2 \in \mathbb{R}\}$ vanishes and $\mathfrak{h}$ is of Type II, which contradicts our assumption.

Suppose now $\lambda = 1$. We already know that $\mathfrak{u} \neq 0$. If $\dim \mathfrak{u} = 2$, then we are in case (2)(a). Let us consider the case $\dim \mathfrak{u} = 1$. Then $\mathfrak{u} = \mathbb{R} \cdot (1, 0)^\top$. Choose $v_X, u_X, y_X, v_N, u_N, y_N$ such that $h_X := h(X, v_X, u_X, y_X)$ and $h_N := (N, v_N, u_N, y_N)$ are in $\mathfrak{h}$. Since $\mathrm{tr}\, X \neq 0$ and $X + \mathrm{tr}\, X$ acts bijectively, we may assume $v_X = 0$ and $y_X = 0$. Then $[h_X, h_N] = h(N, v_N, Xu_N, (X + \mathrm{tr}\, X)y_N)$, which implies $u_N - Xu_N \in \mathfrak{u}$. Since also Xu_N is in $\mathfrak{u}$, we see that $u_N \in \mathfrak{u}$. Thus we can choose $u_N = 0$. To summarise, we get

$$h_X = h(X, 0, (0, u')^\top, 0), \quad h_N = h(N, v_N, 0, y_N),$$

where $u' \neq 0$ since otherwise $\mathfrak{h}$ would not be of Type I. We choose v_0, y_0 such that $h_0 := h(0, v_0, (1, 0)^\top, y_0) \in \mathfrak{h}$. Then $[h_X, h_0] = h(0, v_0 - 2u', (1, 0)^\top, y)$ for some $y \in \mathbb{R}^2$. Since $u' \neq 0$, this implies $\mathfrak{v} \neq 0$. Hence, there exists $\hat{y} = (\hat{y}_1, \hat{y}_2) \in \mathbb{R}^2$ such that $h_v := h(0, 1, 0, \hat{y}) \in \mathfrak{h}$. We have

$$[h_X, h_v] = h(0, 1, 0, (2\hat{y}_1, \hat{y}_2 + 3u')^\top).$$

Hence $(\hat{y}_1, 3u') \in \mathfrak{y}$. Since $\mathfrak{y}$ is $\mathfrak{a}$-invariant and $u' \neq 0$ we obtain $\mathfrak{y} = \mathbb{R}^2$. Consequently, $\mathfrak{h} = \mathbb{R} \cdot h_X \ltimes (\mathbb{R} \cdot N \ltimes \mathfrak{m}(1,1,2))$. Conjugating by $\mathrm{diag}(u', (u')^{-1}) \in \mathrm{SL}(2,\mathbb{R}) \subset \exp \mathfrak{h} \subset \mathrm{SO}(4,3)$ we get $u' = 1$ and we are in case (2)(b).

Finally, suppose $\lambda = 2$. Since $X = \mathrm{diag}(2,1)$, $\mathrm{tr}\, X \neq 0$ and $X + \mathrm{tr}\, X$ acts bijectively, we have $X \in \mathfrak{h}$ after a suitable conjugation. Cor. 2.7 together with the $\mathfrak{a}$-invariance of $\mathfrak{y}$ implies that $\mathfrak{y}$ contains $\mathbb{R} \cdot (1,0)^\top$. In particular, $h_N := h(N, v_N, u_N, y_N) \in \mathfrak{h}$ for some $v_N, u_N = (u_N^1, u_N^2)^\top$ and $y_N = (0, y')^\top$. Then

$$[X, h_N] = h(N, 3v_N, (2u_N^1, u_N^2)^\top, (0, 4y')^\top). \tag{12}$$

Hence $(u_N^1, 0) \in \mathfrak{u}$, thus we can choose $u_N^1 = 0$. Let us first consider the case $\mathfrak{v} = 0$. Then $h_N = h(N, 0, (0,u')^\top, (0,y')^\top)$ and $h_v = h(0,1,0,y_v)$ are in $\mathfrak{h}$ for some $y_v \in \mathbb{R}^2$, thus also $[h_N, h_v] = h(0,0,0,(y', 3u')^\top) \in \mathfrak{h}$. If now $u' \neq 0$, this shows $\mathfrak{y} = \mathbb{R}^2$. Conjugation by $(u')^{-1} \cdot I \in \mathrm{GL}(2,\mathbb{R}) \subset \mathrm{SO}(4,3)$ shows that we may assume $u' = 1$. Hence we are in case (2)(c) with $i = 1$ or $\mathfrak{h} = \mathfrak{a} \ltimes \mathfrak{m}$. If $u' = 0$, then $\mathfrak{u} = \mathbb{R}^2$ since $\mathfrak{h}$ is of Type I and $\mathfrak{u}$ is $\mathfrak{a}$-invariant. Hence $\mathfrak{h} = \mathfrak{a} \ltimes \mathfrak{m}$. If $\mathfrak{v} = 0$, then (12) implies $v_N = 0$. Assume $\mathfrak{y} = \mathbb{R} \cdot (1,0)^\top$. Then (12) would imply $y' = 0$. Hence the projection of $\mathfrak{h}$ to $\{h(0, v, 0, (0, y_2)^\top) \mid v, y_2 \in \mathbb{R}\}$ would be trivial. Thus Prop. 2.8 would imply $c_1 = \ldots = c_4 = 0$ and $b_4 = 0$. But then $X \notin \underline{\mathfrak{h}}$, which would mean that $\mathfrak{h}$ is not a Berger algebra. Hence $\mathfrak{y} = \mathbb{R}^2$. If $u' = 0$, then $\mathfrak{u} = \mathbb{R}^2$ since $\mathfrak{h}$ is of Type I and $\mathfrak{u}$ is $\mathfrak{a}$-invariant. Thus $\mathfrak{h} = \mathfrak{a} \ltimes \mathfrak{m}$. If $u' \neq 0$ we again may assume $u' = 1$. Then $\mathfrak{h} = \mathfrak{a} \ltimes \mathfrak{m}$ or we are in case (2)(c) with $i = 0$.

(3) Let $\mathfrak{a}$ be spanned by $\mathrm{diag}(1, \mu)$. Assume first that $\mu \neq 0$. Then, possibly after a further conjugation, $h(\mathrm{diag}(1,\mu), \hat{v}, 0, \hat{y}) \in \mathfrak{h}$ for some $\hat{v} \in \mathbb{R}$, $\hat{y} \in \mathbb{R}^2$. If $\mu \neq 1$, then $\mathbb{R} \cdot (1,0)^\top$ and $\mathbb{R} \cdot (0,1)^\top$ are the only proper $\mathfrak{a}$-invariant subspaces of $\mathfrak{u}$. Since $\mathfrak{h}$ is of Type I, we obtain $\mathfrak{u} = \mathbb{R}^2$, thus $\mathfrak{h} = \mathfrak{a} \ltimes \mathfrak{m}$. If $\mu = 1$, then the operators $h(0, v, u, y)$ with $u \in \mathfrak{u}$ do not have a non-trivial common kernel since $\mathfrak{h}$ is of Type I. Hence $\dim \mathfrak{u} = 2$, which implies $\mathfrak{h} = \mathfrak{a} \ltimes \mathfrak{m}$. Now we consider $\mu = 0$. Then we may assume that $\hat{h} := h(\mathrm{diag}(1,\mu), 0, (0,u')^\top, 0) \in \mathfrak{h}$ for some $u' \in \mathbb{R}$. Since $\mathfrak{h}$ is of Type I, we have $\mathbb{R} \cdot (1,0)^\top \subset \mathfrak{u}$. If $\mathfrak{u} = \mathbb{R}^2$, then $\mathfrak{h} = \mathfrak{a} \times \mathfrak{m}$. If $\mathfrak{u} = \mathbb{R} \cdot (1,0)^\top$, then $u' \neq 0$ since $\mathfrak{h}$ is of Type I. Thus we may assume $u' = 1$. Furthermore, $h_0 := h(0, v_0, (1,0)^\top, y_0) \in \mathfrak{h}$ for some v, y_0. Since $[\hat{h}, h_0] = h(0, v_0 - 2, (1,0)^\top, y)$ for some $y \in \mathbb{R}^2$, we see that $\mathfrak{v} \neq 0$. Hence, $\hat{h} = h(\mathrm{diag}(1,\mu), 0, (0,1)^\top, 0)$, $h_0 = h(0, 0, (1,0)^\top, y_0)$, $h_v := h(0,1,0,y_v)$ are in $\mathfrak{h}$ for some $y_0, y_v = (y_v^1, y_v^2)^\top \in \mathbb{R}^2$. Since $[\hat{h}, h_v] = h(0,1,0,(2y_v^1, y_v^2)^\top)$, we get $\mathbb{R} \cdot (0,1)^\top \subset \mathfrak{y}$ and $[h_0, h_v] = h(0,0,0,(3,0)^\top)$ implies $\mathbb{R} \cdot (1,0)^\top \subset \mathfrak{y}$. Thus $\mathfrak{y} = \mathbb{R}^2$ and we are in case (3)(b).

(4) Finally, we consider $\mathfrak{a} = \mathbb{R} \cdot N$. If $\mathfrak{u} = \mathbb{R}^2$, then $\mathfrak{h} = \mathfrak{a} \ltimes \mathfrak{m}$. Suppose that $\mathfrak{u} \neq \mathbb{R}^2$. Then $\mathfrak{u} = 0$ or $\mathfrak{u} = \mathbb{R} \cdot (1,0)^\top$. Possibly after conjugation, $h_N := h(N, v_N, (0,u')^\top, (0,y')^\top) \in \mathfrak{h}$ for suitable $v_N, u', y' \in \mathbb{R}$. Then $u' \neq 0$ since $\mathfrak{h}$ is of Type I. We may assume $u' = 1$. Then, again

after conjugation, $v_N = y' = 0$. Let us first consider the case $\mathfrak{v} \neq 0$. Then $h_v := h(0,1,0,y_v) \in \mathfrak{h}$ for some $y_v \in \mathbb{R}^2$. Because of $[h_N, h_v] = [h(N,0,(0,1)^\top,0), h_v] = h(0,0,0,(y_1,3)^\top)$ for some $y_1 \in \mathbb{R}$. Since $\mathfrak{y}$ is $\mathfrak{a}$-invariant, we obtain $\mathfrak{y} = \mathbb{R}^2$. Hence $\mathfrak{h}$ is as claimed in case (4)(b). Now suppose $\mathfrak{v} = 0$. If $h_0 := h(0, v_0, (1,0)^\top, y_0)$ would be in $\mathfrak{h}$ for some v_0, y_0, then also $[h_N, h_0] = [h(N,0,(0,1)^\top,0), h_0] = h(0,-2,0,\hat{y}) \in \mathfrak{h}$ for some $\hat{y} \in \mathbb{R}^2$, which would contradict $\mathfrak{v} = 0$. Thus $\mathfrak{u} = \mathfrak{v} = 0$. This implies that all parameters describing $\mathcal{K}(\mathfrak{h})$ are zero except of j_3, j_4, t, see Prop. 2.8. But then $h_N = h(N,0,(0,1)^\top,0)$ would not be in $\underline{\mathfrak{h}}$, which would contradict Berger's first criterion. □

2.4. **Berger algebras of Type III.**

Lemma 2.9. *If $\mathfrak{h}$ is of Type III, then there exists a basis $b_1, \dots, b_7$ of V such that the metric on V equals $2\sigma^1 \cdot \sigma^5 + 2\sigma^2 \cdot \sigma^6 + 2\sigma^3 \cdot \sigma^7 - (\sigma^4)^2$ with respect to the dual basis and $\mathfrak{h}$ is a subalgebra of*

$$\mathfrak{h}^s = \{h^s(A,v,y) := h(A,v,0,y) \mid A \in \mathfrak{gl}(2,\mathbb{R}),\ v \in \mathbb{R},\ y \in \mathbb{R}^2\} \subset \mathfrak{h}^I.$$

Let $\mathfrak{a}$ and $\mathfrak{m}(i,j,k)$ be defined as in Section 2.3.

Theorem 2.10. *If $\mathfrak{h}$ is of Type III, then there exists a basis such that we are in one of the following cases*

(1) $\mathfrak{a} \in \{\mathfrak{sl}(2,\mathbb{R}),\ \mathfrak{gl}(2,\mathbb{R}),\ \mathfrak{u}(1), \mathfrak{d}\}$ *and* $\mathfrak{h} = \mathfrak{a} \ltimes \mathfrak{m}(1,0,2)$
(2) $\mathfrak{a} \in \{0,\ \mathbb{R} \cdot \mathrm{diag}(1,0)\}$ *and* $\mathfrak{h} = \mathfrak{a} \ltimes \mathfrak{m}(1,0,k)$ *for* $k \in \{1,2\}$.

Proof. If the socle S is three-dimensional, it defines a one-dimensional invariant subspace in a natural way, see Lemma 2.1. Since G_2^* acts transitively on isotropic lines we may assume that this space is spanned by e_1. Then $S = \mathrm{span}\{e_1, e_2, e_3\}$, see Lemma 2.1. Now we take the same basis as in Lemma 2.3., i.e., $b_i = e_i$, $i = 1, \dots, 7$. Then $\mathfrak{h} \subset \mathfrak{h}^I$. Since the representation of $\mathfrak{h}$ on S is semisimple, $\mathbb{R} \cdot b_1$ has an invariant complement $\hat{S}$ in S. Hence $\mathfrak{u} = \{u \in \mathbb{R}^2 \mid \exists v \in \mathbb{R}, \exists y \in \mathbb{R}^2 : h(0,v,u,y) \in \mathfrak{h}\} = 0$. Since $\mathfrak{h}$ acts semisimply on $\hat{S}$, $\mathfrak{a} \cong \mathrm{GL}(2,\mathbb{R})$ acts semisimply on $\mathbb{R}^2$. Thus $\mathfrak{a} \in \{0, \mathfrak{gl}(2,\mathbb{R}), \mathfrak{sl}(2,\mathbb{R})\}$ or $\mathfrak{a}$ is conjugated to one of the Lie algebras $\mathfrak{u}(1), \mathfrak{d}, \mathbb{R} \cdot \mathrm{diag}(1,\mu)$ or $\mathbb{R} \cdot C_a$. So we may assume that $\mathfrak{a}$ is one of these Lie algebras. In the proof of Theorem 2.4, we have seen that if $\mathfrak{a} \in \{\mathfrak{sl}(2,\mathbb{R}), \mathbb{R} \cdot C_a\}$ or if $I \in \mathfrak{a}$, then $\mathfrak{u} = 0$ implies that, after a suitable conjugation, the projection of $\mathfrak{h}$ to $\{h(0,0,u,0) \mid u \in \mathbb{R}^2\}$ is trivial, which means that $\mathfrak{h} \subset \mathfrak{h}^s$. Furthermore, if $\mathfrak{a} = \mathbb{R} \cdot \mathrm{diag}(1,\mu)$, $\mu \neq 0$, then we also have $\mathfrak{h} \subset \mathfrak{h}^s$ after a suitable conjugation according to (9) since $\mathrm{diag}(1,\mu)$ acts bijectively. If $\mu = 0$, then $h(\mathrm{diag}(1,\mu), v_0, (0,u'), y_0) \in \mathfrak{h}$ after conjugation according to (9) since $\mathbb{R} \cdot (1,0)^\top$ is in the image of $\mathrm{diag}(1,0)$. Since $\mathfrak{h}$ acts semisimply on $S = \mathrm{span}\{b_1, b_2, b_3\}$, it follows that $u' = 0$.

Now we are using Berger's criterion in order to determine $\mathfrak{h}$. Since $\mathfrak{h} \subset \mathfrak{h}^s$, Prop. 2.8 implies $0 = r_1 = r_2 = r_3 = x_1 = \cdots = x_4 = u_1 = \cdots = u_4$ and

$b_4 = c_2$, $c_1 = b_3$ for the parameters of $\mathcal{K}(\mathfrak{h})$. Since $\mathfrak{h}$ is indecomposable, we have $\mathfrak{v} \neq 0$ since otherwise the non-isotropic vector b_4 would be in the kernel of $\mathfrak{h}$. Hence $v_1 \neq 0$ or $v_2 \neq 0$, which implies $\mathfrak{y} \neq 0$. For $\mathfrak{a} \in \{\mathfrak{gl}(1,\mathbb{R}), \mathfrak{sl}(2,\mathbb{R}), \mathfrak{u}(1)\}$ we get $\mathfrak{y} = \mathbb{R}^2$ since there is no one-dimensional invariant subspace of $\mathbb{R}^2$. Thus $\mathfrak{a} \subset \mathfrak{h}$ and $\mathfrak{h} \cap \mathfrak{m} = \mathfrak{m}(1,0,2)$. If $\mathfrak{a} \subset \mathfrak{d}$, then the parameters of $\mathcal{K}(\mathfrak{h})$ satisfy in addition $0 = a_1 = a_2 = a_3 = j_1 = j_2 = 0$ and $b_2 = b_3 = b_4 = c_1 = c_2 = c_3 = 0$. Hence all parameters appearing in some A^{ij} in Table 1 are zero except of b_1 and c_4. For $\mathfrak{a} = \mathfrak{d}$ this immediately implies $\mathfrak{y} = \mathbb{R}^2$. For $\mathfrak{a} = \mathbb{R} \cdot \mathrm{diag}(1,\mu)$ we obtain $\mu = 0$ and $b_1 \neq 0$. Since $\mathrm{diag}(1,0) + \mathrm{tr}\,\mathrm{diag}(1,0)$ act bijectively on $\mathbb{R}^2$ we may have $\mathfrak{a} \subset \mathfrak{h}$ after conjugation according to (10). Furthermore, $b_1 \neq 0$ yields $\mathfrak{h} \cap \mathfrak{m} = \mathfrak{m}(1,0,k)$ for $k = 1,2$, see Table 1. If $\mathfrak{a} = 0$, then $\mathfrak{y} = \mathbb{R}^2$ or $\mathfrak{y} = \mathbb{R} \cdot (1,0)^\top$ after conjugation, which proves the assertion for this case.

It remains to exclude $\mathfrak{a} = \mathbb{R} \cdot C_a$. For $\mathfrak{a} = \mathbb{R} \cdot C_a$, Prop. 2.8 implies the system

$$b_1 = b_4 = c_2 = -c_3,\ -b_2 = b_3 = c_1 = c_4,\ b_1 = ab_3,\ c_1 = ac_3$$

of linear equations, which has only the trivial solution. This gives a contradiction to $h(C_a, 0, 0, y_0) \in \underline{\mathfrak{h}}$ for some $y_0 \in \mathbb{R}^2$.

□

2.5. **Berger algebras of Type II.** Let $\mathfrak{h}$ be of Type II.

For $z = (z_1, \ldots, z_4) \in \mathbb{R}^4$, we define

$$\sigma(z) := \begin{pmatrix} z_2 & \sqrt{2}z_3 & z_4 \\ z_1 & \sqrt{2}z_2 & z_3 \end{pmatrix}, \qquad \sigma(z)^* = \begin{pmatrix} -z_4 & -z_3 \\ \sqrt{2}z_3 & \sqrt{2}z_2 \\ -z_2 & -z_1 \end{pmatrix}$$

and, for $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in \mathfrak{gl}(2,\mathbb{R})$ and $c \in \mathbb{R}$, we put

$$\rho(A) := \begin{pmatrix} a_1 - a_4 & -\sqrt{2}a_2 & 0 \\ -\sqrt{2}a_3 & 0 & -\sqrt{2}a_2 \\ 0 & -\sqrt{2}a_3 & -a_1 + a_4 \end{pmatrix}, \qquad U(c) := \begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix}.$$

Lemma 2.11. *If $\mathfrak{h}$ is of type II, then there exists a basis $b_1, \ldots, b_7$ of V such that*

$$\begin{aligned} \langle \cdot\,, \cdot \rangle &= 2(b^1 \cdot b^6 + b^2 \cdot b^7 + b^3 \cdot b^5) - (b^4)^2, \\ \omega &= \sqrt{2}(-b^{157} + b^{236}) - b^4 \wedge (b^{16} - b^{27} - b^{35}) \end{aligned}$$

and $\mathfrak{h}$ is a subalgebra of

$$\mathfrak{h}^{II} := \{h(A,z,c) \mid A \in \mathfrak{gl}(2,\mathbb{R}),\ z \in \mathbb{R}^4,\ c \in \mathbb{R}\},$$

where

$$h(A,z,c) = \begin{pmatrix} A & \sigma(z) & U(c) \\ 0 & \rho(A) & \sigma(z)^* \\ 0 & 0 & -A^\top \end{pmatrix}.$$

Proof. Let b_1, b_2 be a basis of the socle S. Then $b_1 \times b_2 = 0$ since otherwise S would be contained in a 3-dimensional isotropic subspace on which $\mathfrak{h}$ acts semisimply, see Lemma 2.1 (3). Since G_2^* acts transitively on isotropic vectors, we may assume $b_1 = e_1$. Because of $b_1 \times b_2 = 0$, the vector b_2 is in $\mathrm{span}\{e_1, e_2, e_3\}$. Furthermore, the subgroup $\mathrm{GL}(2,\mathbb{R}) \subset \mathrm{G}_2^*$ defined by (4) acts as $\mathrm{GL}(2,\mathbb{R})$ on $\mathrm{span}\{e_2, e_3\}$, thus we may assume $b_2 = e_2$. Moreover, we put $b_3 := e_3$, $b_4 := e_4$, $b_5 := e_7$, $b_6 := e_5$, $b_7 := e_6$. □

Note that ω and $\langle\cdot\,,\cdot\rangle$ with respect to the chosen basis differ from those that we considered in the section on Type I. We will also consider another embedding of $\mathfrak{gl}(2,\mathbb{R})$ into $\mathfrak{g}_2^*$. In this subsection we identify $\mathfrak{gl}(2,\mathbb{R})$ with

$$\{h(A,0,0) \mid A \in \mathfrak{gl}(2,\mathbb{R})\} \cong \mathfrak{gl}(2,\mathbb{R}) \tag{13}$$

and define $\mathfrak{a}$ to be the projection of $\mathfrak{h}$ to $\mathfrak{gl}(2,\mathbb{R}) \subset \mathfrak{h}^{II}$. We set

$$\mathfrak{n} := \{h(0,z,c) \mid z \in \mathbb{R}^4,\ c \in \mathbb{R}\}$$

and, for $i,j,k \in \{1,2,3,4\}$, we define

$$\begin{aligned} \mathfrak{n}(i,j) &:= \{h(0,z,c) \mid z \in \mathbb{R}^4,\ z_l = 0 \text{ if } l \notin \{i,j\},\ c \in \mathbb{R}\}, \\ \mathfrak{n}(i,j,k) &:= \{h(0,z,c) \mid z \in \mathbb{R}^4,\ z_l = 0 \text{ if } l \notin \{i,j,k\},\ c \in \mathbb{R}\}. \end{aligned}$$

Theorem 2.12. *If $\mathfrak{h}$ is of Type II, then there exists a basis of V such that we are in one of the following cases*

(1) $\mathfrak{a} \in \{\mathfrak{sl}(2,\mathbb{R}),\ \mathfrak{gl}(2,\mathbb{R})\}$ *and* $\mathfrak{h} = \mathfrak{a} \ltimes \mathfrak{n}$,

(2) $\mathfrak{a} \in \{\mathfrak{u}(1),\ \mathbb{R}\cdot C_a\}$ *and* $\mathfrak{h} = \mathfrak{a} \ltimes \mathfrak{n}$ *or*
$$\mathfrak{h} = \mathfrak{a} \ltimes \{h(0,(3r,s,r,3s),c) \mid r,s,c \in \mathbb{R}\}.$$

(3) $\mathfrak{a} = \mathfrak{d}$ *and* $\mathfrak{h} = \mathfrak{a} \ltimes \mathfrak{n}_1$, *where*
$$\mathfrak{n}_1 \in \{\mathfrak{n},\ \mathfrak{n}(1,3),\ \mathfrak{n}(2,3),\ \mathfrak{n}(1,2,3),\ \mathfrak{n}(1,2,4)\},$$

(4) $\mathfrak{a} = \mathbb{R}\cdot \mathrm{diag}(1,\mu)$, $\mu \in [-1,1)$, *and*
   - (a) $\mu \in [-1,1)$ *and* $\mathfrak{h} = \mathfrak{a} \ltimes \mathfrak{n}_1$, *where*
     $\mathfrak{n}_1 \in \{\mathfrak{n},\ \mathfrak{n}(2,3),\ \mathfrak{n}(1,2,3),\ \mathfrak{n}(1,2,4),\ \mathfrak{n}(1,3,4),\ \mathfrak{n}(2,3,4)\}$,
   - (b) $\mu = 1/2$ *and* $\mathfrak{h} = \mathbb{R}\cdot h(\mathrm{diag}(1,1/2),(1,0,0,0),0) \ltimes \mathfrak{n}_1$, *where* $\mathfrak{n}_1 \in \{\mathfrak{n}(2,3), \mathfrak{n}(2,3,4)\}$,
   - (c) $\mu = 0$ *and* $\mathfrak{h} = \mathfrak{a} \ltimes \mathfrak{n}(2,4)$ *or*
     $$\mathfrak{h} = \mathbb{R}\cdot h(\mathrm{diag}(1,0),(0,1,0,0),0) \ltimes \mathfrak{n}_1,$$
     where $\mathfrak{n}_1 \in \{\mathfrak{n}(1,4),\ \mathfrak{n}(3,4),\ \mathfrak{n}(1,3,4)\}$,

(5) $\mathfrak{a} \in \{0,\ \mathbb{R}\cdot I\}$ *and* $\mathfrak{h} = \mathfrak{a} \ltimes \mathfrak{n}_1$, *where*
$$\mathfrak{n}_1 \in \{\mathfrak{n},\ \mathfrak{n}(1,3),\ \mathfrak{n}(2,3),\ \mathfrak{n}(1,3,4),\ \mathfrak{n}(2,3,4)\},$$
or $\mathfrak{n}_1$ *is one of the Lie algebras* $\{h(0,z,c) \mid z \in Z,\ c \in \mathbb{R}\}$ *for*
   - (a) $Z = \{(z_1,0,z_1,z_4) \mid z_1,z_4 \in \mathbb{R}\}$,
   - (b) $Z = \{(0,z_2,z_3,-z_2) \mid z_1,z_4 \in \mathbb{R}\}$,
   - (c) $Z = \{(z_1,\alpha z_1,\alpha z_4,z_4) \mid z_1,z_4 \in \mathbb{R}\}$, $\alpha \in \left[\frac{\sqrt{3}-1}{\sqrt{6}}, \frac{\sqrt{3}+1}{\sqrt{6}}\right]$,
   - (d) $Z = \{(sz_1,\alpha z_2,-\alpha z_1,-z_2) \mid z_1,z_2 \in \mathbb{R}\}$, $s \in (0,1]$, $\alpha \in \mathbb{R}$ *such that* $3\alpha^2 - (s+1)\alpha - s = 0$,
   - (e) $Z = \{(z_1,z_2,\kappa z_1,z_4) \mid z_1,z_2,z_4 \in \mathbb{R}\}$, $\kappa = \pm 1$.

The remainder of this section is concerned with the proof of Theorem 2.12. Let us first describe the structure of $\mathfrak{h}^{II} = \mathfrak{gl}(2,\mathbb{R}) \ltimes \mathfrak{n}$. The Lie bracket on $\mathfrak{n}$ is given by

$$[h(0,z,c), h(0,\hat{z},\hat{c})] = h(0,0,\eta(z,\hat{z})),$$

where

$$\eta(z,\hat{z}) = -z_1\hat{z}_4 + z_4\hat{z}_1 + 3z_2\hat{z}_3 - 3z_3\hat{z}_2.$$

Moreover, $\mathfrak{h}^{II} = \mathfrak{gl}(2,\mathbb{R}) \ltimes \mathfrak{n}$, where $A \in \mathfrak{gl}(2,\mathbb{R})$ acts on $\mathfrak{n}$ by

$$A \cdot h(0,z,c) = h(0, A\cdot z, \mathrm{tr}(A)\cdot c),$$

where the representation of $\mathfrak{gl}(2,\mathbb{R})$ on $\mathbb{R}^4$ is given by the equation

$$\sigma(A\cdot z) = A \circ \sigma(z) - \sigma(z) \circ \rho(A).$$

In particular, the basis vectors $I \in \mathfrak{gl}(2,\mathbb{R})$ and

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

act by $I\cdot z = z$, $H \cdot z = (-3z_1, -z_2, z_3, 3z_4)$, $X\cdot z = (0, z_1, 2z_2, 3z_3)$ and $Y \cdot z = (3z_2, 2z_3, z_4, 0)$. This representation integrates to a representation of $\mathrm{GL}^+(2,\mathbb{R})$. Putting $\mathrm{diag}(1,-1)\cdot z = (z_1, -z_2, z_3, -z_4)$, we finally get a representation of $\mathrm{GL}(2,\mathbb{R})$ on $\mathbb{R}^4$, which we will consider in the following.

Let P_d denote the space of homogeneous polynomials of degree d in x, y. On P_d, we consider the representation of $\mathrm{GL}(2,\mathbb{R})$ given by $(A\cdot p)(x,y) := p((x,y)A)$. Then

$$\begin{array}{rcl} \phi_1 :\ \mathbb{R}^4 & \longrightarrow & P_3 \\ z = (z_1,\dots,z_4) & \longmapsto & z_1y^3 + 3z_2xy^2 + 3z_3x^2y + z_4x^3 \end{array}$$

is an isomorphism satisfying

$$\phi_1(A\cdot z) = \lambda_1^{-1}(A)\cdot \phi_1(z),$$

where λ_1 denotes the automorphism

$$\lambda_1 : \mathrm{GL}(2,\mathbb{R}) \longrightarrow \mathrm{GL}(2,\mathbb{R}), \quad A \longmapsto \det(A)A.$$

In particular, ϕ_1 maps $\mathrm{GL}(2,\mathbb{R})$-orbits in $\mathbb{R}^4$ to $\mathrm{GL}(2,\mathbb{R})$-orbits in P_3.

The representation of $\mathrm{GL}(2,\mathbb{R})$ on $\mathbb{R}^4$ that we considered above induces representations of $\mathrm{GL}(2,\mathbb{R})$ on $\bigwedge^n \mathbb{R}^4$. We consider these representations for $n = 2, 3$. Let $e_1, \dots, e_4$ be the standard basis of $\mathbb{R}^4$. Let us start with $n = 2$. The complementary subspaces $W_0 := \mathrm{span}\{w_0 := e_{23} - 3e_{14}\}$ and $W' := \mathrm{span}\{e_{12}, e_{13}, w' := e_{23} + 3e_{14}, e_{24}, e_{34}\}$ of $\bigwedge^2 \mathbb{R}^4$ are invariant under $\mathrm{GL}(2,\mathbb{R})$. The isomorphism $\phi_2 : W' \to P_4$ defined by

$$\begin{array}{rrcl} \phi_2 : & e_{12} & \longmapsto & y^4 \\ & e_{13} & \longmapsto & 2xy^3 \\ & w' & \longmapsto & 6x^2y^2 \\ & e_{24} & \longmapsto & 2x^3y \\ & e_{34} & \longmapsto & x^4 \end{array}$$

satisfies

$$\phi_2(A\cdot u) = \lambda_2^{-1}(A)\cdot\phi_2(u),$$

where λ_2 denotes the automorphism

$$\lambda_2 : \mathrm{GL}(2,\mathbb{R}) \longrightarrow \mathrm{GL}(2,\mathbb{R}), \quad A \longmapsto \mathrm{sgn}(\det A)\sqrt{|\det(A)|}A.$$

For $n = 3$ the situation is even simpler. The representation $\bigwedge^3\mathbb{R}^4$ is equivalent to P_3. An equivalence is given by

$$\Phi_3 : \; e_{123} \longmapsto y^3, \;\; e_{124} \longmapsto xy^2, \;\; e_{134} \longmapsto x^2y, \;\; e_{234} \mapsto x^3. \qquad (14)$$

Next we determine the orbits of the $\mathrm{GL}(2,\mathbb{R})$-action on the projective spaces $\mathbb{P}(P_3)$ and $\mathbb{P}(P_4)$. The line spanned by a polynomial p is denoted by $[p]$.

Lemma 2.13. *The elements* $[x^3]$, $[x^2y]$ *and* $[x(x^2\pm y^2)]$ *constitute a complete system of representatives of the orbit space* $\mathrm{GL}(2,\mathbb{R})\setminus\mathbb{P}(P_3)$.

The following elements constitute a complete system of representatives of $\mathrm{GL}(2,\mathbb{R})\setminus\mathbb{P}(P_4)$*:* $[x^4]$, $[x^3y]$, $[x^2y^2]$, $[xy(x^2+rxy+y^2)]$ *for* $r\in[0, 3/\sqrt{2}]$, *and* $[(x^2+y^2)(x^2+sy^2)]$ *for* $s\in[0,1]$.

Proof. The first assertion is well known and easy to prove. Let us check the second one. We denote by $\langle p\rangle$ the orbit of $[p]$. By a zero of $p = p(x,y)$ we mean a (real) zero of p on $\mathbb{R}P^1$. If p has a zero of multiplicity three or four, then $\langle p\rangle = \langle x^4\rangle$ or $\langle p\rangle = \langle x^3y\rangle$. If p has two zeroes of multiplicity two, then $\langle p\rangle = \langle x^2y^2\rangle$. Let us now consider the remaining cases.

(1) Suppose that p has exactly two or four zeroes and all zeroes are simple. Then

$$\langle p\rangle = \langle xy(x^2+bxy+cy^2)\rangle = \langle xy(x^2+rxy+\kappa y^2)\rangle, \quad \kappa=\pm 1,$$

by rescaling x and y. We want to show that we may assume $\kappa = 1$. If $\kappa = -1$, then $x^2+rxy+\kappa y^2 = x^2+rxy-y^2$ has two zeroes, hence

$$x^2+rxy-y^2 = (x+qy)(x-\tfrac{1}{q}y), \quad |q|\geq 1.$$

We choose $\hat{q}\in\mathbb{R}$ such that $\hat{q}^2 = q^2+1$ and put $x = \hat{x}+\hat{q}\hat{y}$, $y = q\hat{x}$. Then

$$\begin{aligned}[xy(x+qy)(x-y/q)] &= [(\hat{x}+\hat{q}\hat{y})\hat{x}(\hat{x}+\hat{q}\hat{y}+q^2\hat{x})\hat{y}]\\ &= [(\hat{x}+\hat{q}\hat{y})\hat{x}(\hat{q}^2\hat{x}+\hat{q}\hat{y})\hat{y}] \;=\; [\hat{x}\hat{y}(\hat{x}+\hat{q}\hat{y})(\hat{x}+\hat{y}/\hat{q})]\\ &= [\hat{x}\hat{y}(\hat{x}^2+\hat{r}\hat{x}\hat{y}+\hat{y}^2)]\,.\end{aligned}$$

Thus $\langle p\rangle = \langle xy(x^2+rxy+y^2)\rangle$, where $r\geq 0$ (otherwise replace x by $-x$) and $r\neq 2$. We distinguish two cases.

(1)(a) If $r\in[0,2)$, then p has exactly two zeroes. Polynomials with different values for r belong to different $\mathrm{GL}(2,\mathbb{R})$-orbits since any transformation that maps $[xy(x^2+rxy+y^2)]$ to $[xy(x^2+\hat{r}xy+y^2)]$ leaves invariant the set $\{[1:0],\,[0:1]\}\subset\mathbb{R}P^1$ of zeroes.

(1)(b) If $r > 2$, then p has four different zeroes. Thus $\langle p\rangle = \langle xy(x+qy)(x+y/q)]$, where $0 < q < 1$. If $q < 1/\sqrt{2}$, then we choose $\hat{q} \in (1/\sqrt{2}, 1)$ such that $q^2 + \hat{q}^2 = 1$. For $x = \hat{x} + \hat{q}\hat{y}$ and $y = -q\hat{x}$, we get

$$\begin{aligned}[xy(x+qy)(x+y/q)] &= [(\hat{x}+\hat{q}\hat{y})\hat{x}(\hat{x}+\hat{q}\hat{y}-q^2\hat{x})\hat{y}] \\ &= [(\hat{x}+\hat{q}\hat{y})\hat{x}(\hat{q}^2\hat{x}+\hat{q}\hat{y})\hat{y}] \;=\; [\hat{x}\hat{y}(\hat{x}+\hat{q}\hat{y})(\hat{x}+\hat{y}/\hat{q})]\,.\end{aligned}$$

Hence $\langle p\rangle = \langle xy(x+qy)(x+y/q)\rangle$ for some $q \in [1/\sqrt{2}, 1)$. Next we show that polynomials of the form $xy(x+qy)(x+y/q)$ with different values of $q \in [1/\sqrt{2}, 1)$ belong to different orbits. For $A \in \mathrm{GL}(2,\mathbb{R})$, the projective transformation of $\mathbb{R}P^1$ induced by $(A^{-1})^\top$ maps the set of zeroes of p to the set of zeroes of $A \cdot p$. Each projective transformation of $\mathbb{R}P^1$ preserves the cross-ratio of four points. Hence the set of all cross-ratios of the zeroes of a polynomial is an invariant of the $\mathrm{GL}(2,\mathbb{R})$-action. That is, if $z_1, \ldots, z_4 \in \mathbb{R}P^1 \cong \mathbb{R} \cup \{\infty\}$ are the four different zeroes of $p \in P_4$, then

$$C(\langle p\rangle) := \left\{ \frac{z_i - z_k}{z_j - z_k} : \frac{z_i - z_l}{z_j - z_l} \;\middle|\; \{i,j,k,l\} = \{1,2,3,4\} \right\}$$

is well defined. The zeroes of $xy(x+qy)(x+y/q)$ are $0, \infty, -q, -1/q$. Thus

$$C(\langle p\rangle) = \{q^{\pm 2}, (1-q^2)^{\pm 1}, (1-1/q^2)^{\pm 1}\}.$$

For $q \in (1/\sqrt{2}, 1)$, we have $C(\langle p\rangle) \cap (1/2, 1) = q^2$. If $q = 1/\sqrt{2}$, then $C(\langle p\rangle) \cap (1/2, 1) = \emptyset$. Thus different values of $q \in [1/\sqrt{2}, 1)$ give different orbits. Since $xy(x+qy)(x+y/q) = xy(x^2+rxy+y^2)$, where $r = q + 1/q$, we see that $\langle p\rangle = \langle xy(x^2+rxy+y^2)\rangle$ for exactly one $r \in (2,\, 3/\sqrt{2}]$.

(2) If p has no zero, then $\langle p\rangle = \langle p_1 p_2\rangle$, where $p_1(x,y) = x^2+y^2$ and $p_2(x,y) = x^2 + 2bxy + cy^2$, $b^2 < c$. The positive definite quadratic forms p_1 and p_2 are simultaneous diagonalisable, thus $\langle p\rangle = \langle (x^2+y^2)(x^2+sy^2)\rangle$. Obviously, we can choose $s \in (0,1]$.

(3) Suppose that p has a zero of multiplicity two and that all other zeroes (if further ones exist) are simple.

(3)(a) If p has two further zeroes, then $\langle p\rangle = \langle xy(ax+by)^2\rangle$, $a \neq 0, b \neq 0$. Rescaling x and y we get $\langle p\rangle = \langle xy(x+y)^2\rangle = \langle xy(x^2+rxy+y^2)\rangle$ for $r = 2$.

(3)(b) If p has no further zero, then $\langle p\rangle = \langle x^2(x^2+bxy+cy^2)\rangle = \langle x^2(x^2+y^2)\rangle$ by completing the square (with respect to y) and rescaling x afterwards. In particular, $\langle p\rangle = \langle (x^2+y^2)(x^2+sy^2)\rangle$ for $s = 0$. □

Let $b_1, \ldots, b_7$ be a basis as chosen in Lemma 2.11. If $R \in \mathcal{K}(\mathfrak{h})$, then

$$R_{ij} := R(b_i, b_j) = h(A^{ij}, z^{ij}, c^{ij}).$$

Proposition 2.14. *The space $\mathcal{K}(\mathfrak{h})$ can be parametrised by real numbers $x_1, \ldots, x_5, y_1, \ldots, y_5, r_1, \ldots, r_4, t, t_1, \ldots, t_6, s_1, s_2, j_1, j_2$, where $R = h(A,z,c) \in \mathcal{K}(\mathfrak{h})$ is given by the data in Table 2.*

$R(b_i, b_j)$	A	z	c
R_{16}	0	(x_4, x_3, x_2, x_1)	$t_1 + t$
$R_{17} = \frac{1}{\sqrt{2}} R_{34}$	0	(x_5, x_4, x_3, x_2)	$t_4 - t_5$
$R_{26} = -\frac{1}{\sqrt{2}} R_{45}$	0	(y_4, y_3, y_2, y_1)	$t_2 - t_3$
R_{27}	0	(y_5, y_4, y_3, y_2)	$t_6 + t$
R_{56}	$\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$	(t_6, t_2, s_2, j_2)	r_1
$R_{57} = \frac{1}{\sqrt{2}} R_{46}$	$\begin{pmatrix} x_2 & y_2 \\ x_3 & y_3 \end{pmatrix}$	(t_5, t_1, t_3, s_2)	r_2
$R_{36} = \frac{1}{\sqrt{2}} R_{47}$	$\begin{pmatrix} x_3 & y_3 \\ x_4 & y_4 \end{pmatrix}$	(s_1, t_4, t_1, t_2)	r_3
R_{37}	$\begin{pmatrix} x_4 & y_4 \\ x_5 & y_5 \end{pmatrix}$	(j_1, s_1, t_5, t_6)	r_4
R_{67}	$\begin{pmatrix} t_1 + t & t_2 - t_3 \\ t_4 - t_5 & t_6 + t \end{pmatrix}$	(r_4, r_3, r_2, r_1)	0
$R_{12} = R_{13} = R_{14} = R_{15} = R_{23} = R_{24} = R_{25} = 0$			
$R_{35} = R_{16} - R_{27}$			

Table 2.

Proof. Let R be in $\mathcal{K}(\mathfrak{h})$. Since $\langle R_{ij}(b_k), b_l \rangle = \langle R_{kl}(b_i), b_j \rangle$ and $R_{kl} \in \mathfrak{h}$, we have

$$R_{ij} = 0, \quad i \leq 2 \leq j \leq 5. \tag{15}$$

The same argument gives

$$\sqrt{2} R_{17} = R_{34}, \ \sqrt{2} R_{26} = -R_{45}, \ \sqrt{2} R_{57} = R_{46}, \ \sqrt{2} R_{36} = R_{47}$$

and

$$R_{35} = R_{16} - R_{27}. \tag{16}$$

As above, we use the notation $b(i,j,k) := R_{ij}(b_k) + R_{jk}(b_i) + R_{ki}(b_j)$. From $b(i,j,6) = b(i,j,7) = 0$, we conclude $A^{ij} = 0$ for $i < j \leq 5$. By (15) and $b(i,4,6) = 0$, we get $a_2^{i6} = 0$ and $0 = a_3^{i6} = a_1^{i7} = a_4^{i7}$, where the last identity follows from $b(i,3,7) = 0$. Together with (15) and (16) this implies

$$A^{16} = A^{17} = A^{26} = A^{27} = 0.$$

Let us consider the equations $b(i,j,k)=0$ for $i,j,k\in\{1,2,3,5,6,7\}$. These equations give, in particular,

$$\begin{array}{ll} z_4^{67}=c^{56}=:r_1, & z_2^{37}=z_1^{36}=:s_1,\\ c^{57}=z_3^{67}=:r_2, & z_3^{56}=z_4^{57}=:s_2,\\ c^{36}=z_2^{67}=:r_3, & z_1^{17}=a_3^{37}=:x_5,\\ z_1^{67}=c^{37}=:r_4, & z_4^{26}=a_2^{56}=:y_1. \end{array}$$

The system of the remaining linear equations for the coefficients $A^{ij},\ z^{ij},\ c^{ij}$ that follow from $b(i,j,k)=0$ for $i,j,k\in\{1,2,3,5,6,7\}$ decomposes into six subsystems each of which is a system of equations in the elements of one of the following sets:

$$\begin{aligned} M_1 :=&\{a_2^{67},z_4^{35},z_2^{56},z_3^{57},c^{26}\},\\ M_2 :=&\{a_3^{67},z_2^{36},z_3^{37},z_1^{57},c^{17}\},\\ M_3 :=&\{a_1^{56},a_4^{56},a_2^{57},z_4^{16},z_3^{26},z_4^{27},z_4^{35},c^{25}\},\\ M_4 :=&\{a_1^{67},a_4^{67},z_3^{36},z_4^{37},z_1^{56},z_2^{57},c^{16},c^{25},c^{35}\},\\ M_5 :=&\{a_2^{36},a_3^{56},a_1^{57},a_4^{57},z_3^{16},z_4^{17},z_2^{26},z_3^{27},z_3^{35},c^{15}\},\\ M_6 :=&\{a_1^{36},a_3^{36},a_4^{36},a_1^{37},a_2^{37},a_4^{37},a_3^{57},\\ &\quad z_1^{16},z_2^{16},z_2^{17},z_3^{17},z_1^{26},z_1^{27},z_2^{27},z_1^{35},z_2^{35},c^{13},c^{23}\}. \end{aligned}$$

The subsystem for M_1 is

$$z_4^{36}=z_2^{56}=:t_2,\quad z_2^{56}-a_2^{67}=z_3^{57}=:t_3,\quad c^{26}=a_2^{67}.$$

Similarly, M_2 is parametrised by t_4 and t_5 as claimed in the proposition. For M_3, we have

$$\begin{aligned} &a_1^{56}=z_4^{16}=:x_1,\quad a_2^{57}=a_4^{56}=:y_2,\quad z_3^{26}=z_4^{27},\\ &a_2^{57}-z_4^{27}=c^{25}=-a_4^{56}+z_3^{26},\quad z_4^{35}=a_1^{56}-a_4^{56}, \end{aligned}$$

and for M_4,

$$\begin{aligned} &c^{27}=a_4^{67}=:t+t_6,\quad c^{16}=a_1^{67}=:t_1+t\\ &z_3^{36}-z_4^{37}=a_1^{67}-a_4^{67}=-z_1^{56}+z_2^{57},\\ &z_3^{36}-z_1^{56}=c^{35}=z_2^{57}-z_4^{37}, \end{aligned}$$

where we put $t_1:=z_3^{36}$. The equations containing elements of M_5 are

$$\begin{aligned} &a_3^{56}=a_1^{57}=:x_2,\quad a_2^{36}=z_2^{26}=z_3^{27}=a_4^{57}=:y_3,\quad z_3^{16}=z_4^{17},\\ &a_1^{57}-z_4^{17}=c^{15}=-a_3^{56}+z_3^{16},\\ &a_1^{57}-a_4^{57}=z_3^{35}=-a_2^{36}+a_3^{56}. \end{aligned}$$

Finally, for M_6, we get

$$
\begin{aligned}
&a_1^{36} = z_2^{16} = z_3^{17} = a_3^{57} =: x_3, \quad a_3^{36} = a_1^{37} =: x_4,\\
&z_1^{26} = z_2^{27} =: y_4, \quad a_4^{37} = z_1^{27} =: y_5\\
&z_1^{16} = z_2^{17}, \quad a_4^{36} = a_2^{37},\\
&a_1^{37} - z_2^{17} = c^{13} = -a_3^{36} + z_1^{16}, \quad a_2^{37} - z_2^{27} = c^{23} = -a_4^{36} + z_1^{26},\\
&a_1^{36} - a_4^{36} = z_2^{35} = -a_2^{37} + a_3^{57}, \quad z_1^{35} = a_1^{37} - a_4^{37}.
\end{aligned}
$$

This shows that $R \in \mathcal{K}(\mathfrak{h})$ is given as in Table 2. Moreover, the above considered systems of linear equations imply that each R that is defined as in Table 2 for an arbitrary choice of the parameters $x_1, x_2, \dots, j_1, j_2$ satisfies $b(i,j,k) = 0$ for $i,j,k \in \{1,2,3,5,6,7\}$. It is easy to show by a direct calculation that it also satisfies $b(i,j,4) = 0$ for $1 \le i,j \le 7$. $\square$

The embedding of $\mathfrak{gl}(2,\mathbb{R})$ into $\mathfrak{g}_2^*$ defined by (13) gives us an embedding of $\mathrm{GL}^+(2,\mathbb{R})$ into G_2^*. If we send, moreover, $\mathrm{diag}(1,-1) \in \mathrm{GL}(2,\mathbb{R})$ to $\mathrm{diag}(1,-1,-1,1,-1,1,-1) \in \mathrm{G}_2^*$, we obtain an embedding of $\mathrm{GL}(2,\mathbb{R})$ into G_2^*, which we want to consider in this section. Note that this embedding is different from that defined by (4). With this identification we have

$$\mathrm{Ad}(g)(h(A,z,c)) = h(gAg^{-1},\, g\cdot z,\, \det(g)c). \tag{17}$$

Lemma 2.15. *Either $\mathfrak{a} \in \{0,\ \mathfrak{sl}(2,\mathbb{R}),\ \mathfrak{gl}(2,\mathbb{R})\}$ or the basis $b_1, \dots, b_7$ in Lemma 2.11 can be chosen such that $\mathfrak{a}$ is equal to one of the following Lie algebras:*

1. *$\mathbb{R} \cdot C_a$, $\mathbb{R} \cdot \mathrm{diag}(1,\mu)$, $\mu \in [-1,1]$;*
2. *$\mathfrak{d}$, $\mathfrak{u}(1)$.*

Proof. In a similar way as for Berger algebras of Type I we may conjugate $\mathfrak{a}$ by elements of $GL(2,\mathbb{R})$, now according to (17). Hence $\mathfrak{a}$ is one of the Lie algebras listed in Lemma 2.5. Since $\mathfrak{h}$ acts semisimply on S, $\mathfrak{a} \cong \mathfrak{gl}(2,\mathbb{R})$ acts semisimply on $\mathbb{R}^2$. This gives the assertion of the Lemma. $\square$

Proof of Theorem 2.12. Below, we will use the conjugation

$$\mathrm{Ad}(\exp h(0,\bar z,0))(h(A,z,c)) = h(A, z - A\cdot\bar z, c - \eta(z,\bar z) - \tfrac{1}{2}\eta(\bar z, A\cdot \bar z)) \tag{18}$$

several times. We define

$$Z := \{z \in \mathbb{R}^4 \mid \exists\, c \in \mathbb{R} :\ h(0,z,c) \in \mathfrak{h}\}.$$

Since $\mathfrak{h}$ is of Type II, we have $Z \ne 0$. Obviously, Z is invariant under $\mathfrak{a}$. Let $e_1, \dots, e_4$ be the standard basis of $\mathbb{R}^4$ and denote by $Z(j_1,\dots,j_k) \subset \mathbb{R}^4$, $k = 1,2,3$, the span of $e_{j_1}, \dots, e_{j_k}$.

(1) If $\mathfrak{a} \in \{\mathfrak{sl}(2,\mathbb{R}), \mathfrak{gl}(2,\mathbb{R})\}$, then $Z = \mathbb{R}^4$, since the action of $\mathfrak{sl}(2,\mathbb{R})$ on P_d is irreducible.

(2) Suppose $\mathfrak{a} \in \{\mathbb{R}\cdot C_a, \mathfrak{u}(1)\}$. If $\mathfrak{a} = \mathbb{R}\cdot C_a$, then $h(C_a,0,c_0) \in \mathfrak{h}$ for some $c_0 \in \mathbb{R}$ after a suitable conjugation of $\mathfrak{h}$ according to (18) since C_a

is bijective. If $\mathfrak{a} = \mathfrak{u}(1)$, then, possibly after conjugation, for all $U \in \mathfrak{u}(1)$, there is a real number c such that $h(U, 0, c)$ is in $\mathfrak{h}$. Indeed, $h(I, 0, c_I) \in \mathfrak{h}$ after a suitable conjugation according to (18) since I is bijective. Since

$$[h(I, 0, c_I), h(U, z_U, c_U)] = h(0, z_U, 2c_U),$$

we have $h(U, 0, -c_U) \in \mathfrak{h}$. The restrictions of the representation of $\mathfrak{gl}(2,\mathbb{R})$ on $\mathbb{R}^4$ to $\mathfrak{u}(1)$ and $\mathbb{R}\cdot C_a$ decomposes into the two irreducible representations

$$Z_1 := \{(r, s, -r, -s) \mid r, s \in \mathbb{R}\}, \quad Z_2 := \{(3r, s, r, 3s) \mid r, s \in \mathbb{R}\}.$$

Indeed, both subspaces are invariant under $\mathfrak{u}(1)$ and they are irreducible since C_a has eigenvalues $a \pm 3i$ on Z_1 and $a \pm i$ on Z_2. If Z were equal to Z_1, then the non-isotropic vector $b_3 + b_5$ would be in the kernel of $\mathfrak{h}$. Hence $Z = Z_2$ or $Z = \mathbb{R}^4$, which gives the assertion.

(3) If $\mathfrak{a} = \mathfrak{d}$, then we can again conjugate $\mathfrak{h}$ according to (18) such that for all $D \in \mathfrak{d}$ there exists a $c \in \mathbb{R}$ such that $h(D, 0, c) \in \mathfrak{h}$. Indeed, as above $h(I, 0, c_I) \in \mathfrak{h}$ after a suitable conjugation. Thus $[h(I, 0, c_I), h(D, z_D, c_D)] = h(0, z_D, 2c_D) \in \mathfrak{h}$ and $h(D, 0, -c_D) \in \mathfrak{h}$ follows. The subspace $Z \subset \mathbb{R}^4$ is invariant under $\mathfrak{d}$ if and only if it is invariant under H. Thus Z is a direct sum of eigenspaces of H. Since $\mathfrak{d}$ considered as a subspace of $\mathfrak{gl}(2,\mathbb{R})$ is invariant under conjugation by $U := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, we may conjugate $\mathfrak{h}$ by $U \in \mathrm{GL}(2,\mathbb{R}) \subset \mathrm{G}_2^*$. U acts on Z by $(z_1, z_2, z_3, z_4) \mapsto (z_4, -z_3, z_2, -z_1)$. Thus we may assume that Z is one of the subspaces $\mathfrak{n}, Z(1), Z(2), Z(1,2), Z(1,3), Z(1,4), Z(2,3), Z(1,2,3), Z(1,2,4)$. For $Z = Z(1)$ and $Z = Z(1,4)$, $\mathfrak{h}$ is decomposable since b_4 is in the kernel. Moreover, we can exclude $Z = Z(2)$ and $Z = Z(1,2)$ since for these Z the Lie algebra $\mathfrak{h}$ would be of Type III. For $Z(2,3), Z(1,2,3)$ and $Z(1,2,4)$ we get immediately $\mathfrak{n}(2,3) \subset \mathfrak{h}, \mathfrak{n}(1,2,3) \subset \mathfrak{h}$ and $\mathfrak{n}(1,2,4) \subset \mathfrak{h}$, respectively. Hence these three cases give Lie algebras that are on the list. For $Z = Z(1,3)$ we have to use Berger's criterion to show that $h(0,0,1)$ is in $\mathfrak{h}$. The parameters of $\mathcal{K}(\mathfrak{h})$ satisfy $x_1 = \cdots = x_5 = y_1 = \cdots = y_4 = t_1 = \cdots = t_6 = s_1 = s_2 = 0$. Hence $r_2 \neq 0$, which implies $h(0,0,1) \in \mathfrak{h}$. Thus $\mathfrak{n}(1,3) \subset \mathfrak{h}$, which gives the remaining Lie algebra in case (3).

(4) Let $\mathfrak{a}$ be spanned by $\mathrm{diag}(1,\mu)$. The action of $\mathrm{diag}(1,\mu)$ on $\mathbb{R}^4$ equals the multiplication by the matrix $D_0 := \mathrm{diag}(-1+2\mu, \mu, 1, 2-\mu)$. Since $\mu \in [-1,1)$, all eigenvalues of D_0 are different. Hence every invariant subspace is spanned by elements of the standard basis of $\mathbb{R}^4$.

(4)(a) Suppose first that $\mu \notin \{0, 1/2\}$. Then D_0 is invertible. Hence we may assume that $h_0 := h(\mathrm{diag}(1,\mu), 0, c_0) \in \mathfrak{h}$ for some $c_0 \in \mathbb{R}$. Consequently, $\dim Z > 1$ since otherwise $\mathfrak{h}$ would be decomposable. Furthermore, by indecomposability, Z cannot be one of the spaces $Z(1,2)$ or $Z(1,4)$. Moreover, $Z \neq Z(3,4)$ since $\mathfrak{h}$ is of Type II. The remaining spaces $Z(1,3)$ and $Z(2,4)$ can be excluded by Berger's criterion. Note first that the parameters x_i and y_i $(i = 1, \ldots, 5)$ of $\mathcal{K}(\mathfrak{h})$ vanish since $\mathfrak{a}$ is spanned by $\mathrm{diag}(1,\mu)$ with $\mu \neq 0$.

The assumption $Z = Z(1,3)$ of $Z = Z(2,4)$ would imply $t_1 = t_6 = 0$, Thus $h_0 \notin \underline{\mathfrak{h}}$, which is a contradiction.

(4)(b) Now suppose $\mu = 1/2$. Then $(1,0,0,0)$ spans the kernel of D_0. Hence we may assume that $h(\mathrm{diag}(1,\mu),(t,0,0,0),c_0) \in \mathfrak{h}$ for certain $t, c_0 \in \mathbb{R}$. If Z contains $Z(1)$, we choose $t = 0$. If $t = 0$, then we proceed as in (4)(a). Take now $t \neq 0$. Then we can achieve $t = 1$ conjugating by a suitable multiple of $I \in \mathrm{GL}(2,\mathbb{R})$. Again, $Z = Z(2)$ and $Z = Z(4)$ cannot occur because of indecomposability. For the remaining possibilities for Z we want to use Berger's criterion. In the same way as in (4)(a) we see that x_i and y_i vanish for $i = 1,\dots,5$. Assume that $Z \subset Z(3,4)$ or $Z = Z(2,4)$. Then $t_1 = t_6 = 0$, which as above leads to a contradiction.

(4)(c) Finally, take $\mu = 0$. The kernel of D_0 is spanned by $(0,1,0,0)$. Hence we may assume that $h(\mathrm{diag}(1,0),(0,t,0,0),c_0) \in \mathfrak{h}$ for certain $t, c_0 \in \mathbb{R}$. If Z contains $Z(1)$, we choose $t = 0$. Suppose first that $t = 0$. As in (4)(a), $\dim Z > 1$ and $Z \notin \{Z(1,2), Z(1,4), Z(3,4)\}$. Moreover, $Z = Z(1,3)$ can be excluded by Berger's criterion. However, in contrast to the case $\mu \neq 0$ we cannot rule out $Z = Z(2,4)$. Indeed, if $Z = Z(2,4)$, then $\mathfrak{h}$ satisfies Berger's criterion if and only if $h(0,0,1) \in \mathfrak{h}$. Now we consider the case $t \neq 0$. We may assume $t = 1$. If Z were in $\{Z(1), Z(2)\}$, then $\mathfrak{h}$ would be decomposable. For $Z \subset Z(1,3)$, $\mathfrak{h}$ would not satisfy Berger's criterion, see Proposition 2.14.

(5) Now let $\mathfrak{a}$ be either trivial or equal to $\mathbb{R}\cdot I$. Then Z can be an arbitrary subspace of $\mathbb{R}^4$. We may conjugate $\mathfrak{h}$ by $\mathrm{GL}(2,\mathbb{R}) \subset \mathrm{G}_2^*$ without changing $\mathfrak{a}$. We want to use that in order to find a certain normal form for Z. Let $\langle Z\rangle$ denote the $\mathrm{GL}(2,\mathbb{R})$-orbit of Z.

Let us first consider the case $\dim Z = 1$. By Lemma 2.13, Z is in one of the orbits $\langle[\Phi_1^{-1}(x^3)]\rangle$, $\langle[\Phi_1^{-1}(x^2y)]\rangle$ or $\langle[\Phi_1^{-1}(x(x^2 \pm y^2))]\rangle$, where for $[z]$ denotes the line spanned by $z \in \mathbb{R}^4$. For $[\Phi_1^{-1}(x^3)] = Z(4)$, the corresponding Lie algebra $\mathfrak{h}$ is decomposable since b_4 is in the kernel of $\mathfrak{h}$. For $[\Phi_1^{-1}(x^2y)] = Z(3)$, we get a Lie algebra $\mathfrak{h}$ of Type III. For $[\Phi_1^{-1}(x(x^2 \pm y^2))] = \{(0,z/3,0,z) \mid z \in \mathbb{R}\}$, the corresponding Lie algebra $\mathfrak{h}$ is decomposable since $3b_3 \mp b_5$ is in the kernel.

Now we turn to the case $\dim Z = 2$. Let us consider the Plücker embedding of the Grassmannian of 2-planes in $\mathbb{R}^4$ into $\mathbb{P}(\bigwedge^2\mathbb{R}^4)$. The line spanned by

$$t_0w_0 + z_{12}e_{12} + z_{13}e_{13} + t'w' + z_{24}e_{24} + z_{34}e_{34} \in \bigwedge\nolimits^2\mathbb{R}^4$$

is in the image of this embedding if and only if

$$z_{12}z_{34} - z_{13}z_{24} = 3(t_0^2 - t'^2)\,. \tag{19}$$

Let us denote by $\hat{Z}$ the image of Z under the Plücker embedding and by $\langle\hat{Z}\rangle$ the $\mathrm{GL}(2,\mathbb{R})$-orbit of $\hat{Z}$. Furthermore, for $\alpha \in \bigwedge^2\mathbb{R}^4$ denote by $[\alpha]$ the element of $\mathbb{P}(\bigwedge^2\mathbb{R}^4)$ represented by α. Then $\langle\hat{Z}\rangle = \langle[\Phi_2^{-1}(p) + t_0w_0]\rangle$ for some polynomial $p \in P_4$ and some $t_0 \in \mathbb{R}$. By Lemma 2.13 we may assume that p is one of the polynomials x^4, x^3y, x^2y^2, $xy(x^2+rxy+y^2)$ for $r\in[0,\,3/\sqrt{2}]$ or $(x^2+y^2)(x^2+sy^2)$ for $s\in[0,1]$. Let us start with $p=x^4$. Since $\Phi_2^{-1}(x^4)+t_0w_0=e_{34}+t_0w_0$ is in the image of the Plücker embedding if and only if $t_0=0$, we see that $\hat{Z}$ is in the $\mathrm{GL}(2,\mathbb{R})$-orbit of e_{34}. Since the Plücker embedding is $\mathrm{GL}(2,\mathbb{R})$-equivariant, Z is in the $\mathrm{GL}(2,\mathbb{R})$-orbit of $Z(3,4)$. But then $\mathfrak{h}$ is of Type III. Now we consider $p=x^3y$. We have $\langle\hat{Z}\rangle=\langle[\Phi_2^{-1}(x^3y)+t_0w_0]\rangle=\langle[\Phi_2^{-1}(y^3x)+t_0'w_0]\rangle=\langle[e_{13}+t_0''w_0]\rangle$. Since $e_{13}+t_0''w_0$ has to be in the image of the Plücker imbedding, we get $t_0''=0$. Thus Z is in the $\mathrm{GL}(2,\mathbb{R})$-orbit of $Z(1,3)$. In the same way as in case (3) we can show that $h(0,0,1)$ is in $\mathfrak{h}$ using Berger's Criterion. Hence $\mathfrak{h}=\mathfrak{a}\ltimes\mathfrak{n}(1,3)$ up to conjugation. For $p=x^2y^2$, we get $\langle\hat{Z}\rangle=\langle[\Phi_2^{-1}(x^2y^2)+t_0w_0]\rangle=\langle[w'+t_0'w_0]\rangle$. By (19), $t_0'=\pm1$. If $t_0'=1$, then $\langle\hat{Z}\rangle=\langle w'+w_0\rangle=\langle e_{23}\rangle$. Hence Z is in the orbit of $Z(2,3)$, which implies $\mathfrak{h}=\mathfrak{a}\ltimes\mathfrak{n}(2,3)$. For $t_0'=-1$ we obtain $\langle\hat{Z}\rangle=\langle w'-w_0\rangle=\langle e_{14}\rangle$. Thus Z is in the orbit of $Z(1,4)$. But then $\mathfrak{h}$ is decomposable since b_4 is in the kernel of $\mathfrak{h}$. Now take $p=xy(x^2+rxy+y^2)$, $r\in[0,\,3/\sqrt{2}]$. Then

$$\begin{aligned}\langle\hat{Z}\rangle&=\langle[\Phi_2^{-1}(p)+t_0'w_0]\rangle=\langle[\tfrac{1}{2}e_{24}+\tfrac{r}{6}w'+\tfrac{1}{2}e_{13}+t_0'w_0]\rangle\\&=\langle[e_{24}+\tfrac{r}{3}w'+e_{13}+t_0w_0]\rangle\\&=\langle[e_{13}+(\tfrac{r}{3}+t_0)e_{23}+(r-3t_0)e_{14}+e_{24}]\rangle.\end{aligned}\tag{20}$$

By (19), we have $9t_0^2=r^2-3$, which is possible only for $r\geq\sqrt{3}$. The map

$$\begin{aligned}M:=\big\{(r,t_0)\bigm| 9t_0^2=r^2-3,\ r\in[\sqrt{3},\,3/\sqrt{2}]\,\big\}&\longrightarrow\Big[\tfrac{\sqrt{3}-1}{\sqrt{6}},\tfrac{\sqrt{3}+1}{\sqrt{6}}\Big]\\(r,t_0)&\longmapsto\alpha(r,t_0):=\tfrac{r}{3}+t_0\end{aligned}$$

is a bijection. Note that $\alpha(r,t_0)\cdot(r-3t_0)=1$. Now (20) implies

$$\begin{aligned}\langle\hat{Z}\rangle&=\langle[e_{13}+\alpha e_{23}+(1/\alpha)e_{14}+e_{24}]\rangle=\langle[\alpha e_{13}+\alpha^2e_{23}+e_{14}+\alpha e_{24}]\rangle\\&=\langle(e_1+\alpha e_2)\wedge(\alpha e_3+e_4)]\rangle\end{aligned}$$

for $\alpha:=\alpha(r,t_0)$. Hence Z and $\{(z_1,\alpha z_1,\alpha z_4,z_4)\mid z_1,z_4\in\mathbb{R}\}$ are in the same orbit. Thus we are in case (5c) of the theorem. At last, we consider $p=(x^2+y^2)(x^2+sy^2)$, $s\in[0,1]$. Then

$$\begin{aligned}\langle\hat{Z}\rangle&=\langle[\Phi_2^{-1}(p)+t_0w_0]\rangle=\langle[e_{34}+\tfrac{s+1}{6}w'+se_{13}+t_0w_0]\rangle\\&=\langle[e_{24}+\tfrac{r}{3}w'+e_{13}+t_0w_0]\rangle\\&=\langle[se_{12}+3(\tfrac{s+1}{6}-t_0)e_{14}+(\tfrac{s+1}{6}+t_0)e_{23}+e_{34}]\rangle.\end{aligned}\tag{21}$$

By (19), we have $s=3\Big(t_0^2-\big(\tfrac{s+1}{6}\big)^2\Big)$. Thus, for a given parameter s, the coefficients $\frac{s+1}{6}-t_0$ and $\frac{s+1}{6}+t_0$ appearing in (21) are exactly the roots $\alpha_{1,2}$ of $3\alpha^2-(s+1)\alpha-s$. In particular, $\alpha_1\alpha_2=-s/3$. Now suppose in addition that $s \neq 0$. Then the latter equation implies $3\alpha_1 = -s/\alpha_2$. Consequently,

$$\begin{aligned}\langle \hat{Z} \rangle &= \langle [se_{12} - (s/\alpha)e_{14} + \alpha e_{23} + e_{34}] \rangle = \langle [s\alpha e_{12} - se_{14} + \alpha^2 e_{23} + \alpha e_{34}] \rangle \\ &= \langle [\,(se_1 - \alpha e_3) \wedge (\alpha e_2 - e_4)\,] \rangle\end{aligned}$$

where $\alpha \in \{\alpha_1, \alpha_2\}$. Hence, if $s \neq 0$, then Z is in the $\mathrm{GL}(2,\mathbb{R})$-orbit of $\{(sz_1, \alpha z_2, -\alpha z_1, -z_2) \mid z_1, z_2 \in \mathbb{R}\}$, which implies that $\mathfrak{h}$ is conjugated to the Lie algebra in (5d). If $s = 0$, then (21) implies that $\langle \hat{Z} \rangle = \langle [e_{14} + e_{34}] \rangle = \langle [(e_1 + e_3) \wedge e_4] \rangle$ or

$$\langle \hat{Z} \rangle = \langle [\tfrac{1}{6}(w' + w_0) + e_{34}] \rangle = \langle [-\tfrac{1}{2}(w' + w_0) - e_{34}] \rangle = \langle [e_3 \wedge (e_2 - e_4)] \rangle.$$

Thus we are in case (5a) or (5b), respectively.

Finally, we study the case $\dim Z = 3$. Recall that the action of $\mathrm{GL}(2,\mathbb{R})$ on $\bigwedge^3 \mathbb{R}^4$ is equivalent to the representation on P_3, see (14). By Lemma 2.13, the image of Z under the Plücker embedding is in the same $\mathrm{GL}(2,\mathbb{R})$-orbit as either $[\Phi_3^{-1}(x^3)]$, $[\Phi_3^{-1}(x^2y)]$ or $[\Phi_3^{-1}(x(x^2 \pm y^2))]$. Because of $\Phi_3^{-1}(x^3) = e_{234}$, $\Phi_3^{-1}(x^2y) = e_{134}$ and $[\Phi_3^{-1}(x(x^2 \pm y^2))] = [e_{234} \pm e_{124}] = [(e_1 \pm e_3) \wedge e_2 \wedge e_4]$ we see that $\mathfrak{h}$ is conjugated to $\mathfrak{a} \ltimes \mathfrak{n}_1$, where $\mathfrak{n}_1 = \mathfrak{n}(2,3,4)$ or $\mathfrak{n}_1 = \mathfrak{n}(1,3,4)$ or $\mathfrak{h}$ is conjugated to the Lie algebra in item (5e). □

3. Holonomy of symmetric spaces admitting a G_2^*-structure

Indefinite symmetric spaces of signature $(4,3)$ with a G_2^*-structure were classified in [Ka2]. Their holonomy algebras can be easily read off from this classification. They are abelian and two- or three-dimensional (unfortunately, there is an obvious mistake in the formulation of Cor. 6.9 in [Ka2]). Let us check how they fit into the classification of holonomy algebras in Section 2.

Let $X = G/G_+$ be a (pseudo-)Riemannian symmetric space, where G is the transvection group of X. The reflection at the base point $x_0 := eG_+ \in G/G_+$ defines an involution θ on the Lie algebra $\mathfrak{g}$ of G. Let $\mathfrak{g}_+$ and $\mathfrak{g}_-$ denote the eigenspaces of θ with eigenvalue 1 and -1, respectively. Then $\mathfrak{g}_-$ can be identified with the tangent space of X at x_0 and $\mathfrak{g}_+$ can be identified with the holonomy algebra (as an abstract Lie algebra). The holonomy representation is given by the adjoint representation of $\mathfrak{g}_+$ on $\mathfrak{g}_-$.

The classification of symmetric spaces with G_2^*-structure is given by the list in Theorem 6.8 in [Ka2]. Item 1 of this list contains a one-parameter family of symmetric spaces and each of the items 2 (a) and (b) contains a single space.

Let us first consider the family in item 1. In the notation of the theorem, the holonomy algebra $\mathfrak{g}_+$ is spanned by Z_B, A_1, B and the tangent space $\mathfrak{g}_-$ by $Z_1, Z_2, Z_3, A, L_1, L_2, L_3$. The adjoint representation of $\mathfrak{g}_+$ on $\mathfrak{g}_-$ is defined by the Lie bracket given by Equation (8) in [Ka2]. It is easy to verify that the holonomy algebra as a subalgebra of $\mathfrak{so}(4,3)$ is of Type II and equals $\mathfrak{n}(1,3)$ with respect to the basis $Z_2, Z_3, L_1, A, Z_1, L_2, L_3$.

The space in item 2 (a) has a holonomy algebra $\mathfrak{g}_+ =$ spanned by Z_B, A_1, B. As a subalgebra of $\mathfrak{so}(4,3)$ the holonomy algebra is of Type III and equals $\mathfrak{m}(1,0,2)$ with respect to the basis $Z_3, Z_2, Z_1, A, L_3, L_2, L_1$. For the symmetric space in item 1 (b) we get a similar result. Its holonomy algebra is spanned by Z_B, B and using the adjoint representation it can be identified with $\mathfrak{m}(1,0,1)$ with respect to the same basis of $\mathfrak{g}_-$. In particular, it is also of Type III.

4. Left-invariant metrics with holonomy in G_2^*

In this section we construct examples of Lie groups G admitting left-invariant torsion free G_2^*-structures whose holonomy algebra is one of the list in Theorems 2.4, 2.10 and 2.12. The G_2^*-structure on G is induced by a three-form ω on the Lie algebra $\mathfrak{g}$ of G. If we denote by $\langle\cdot\,,\cdot\rangle$ the induced inner product, the Levi-Civita connection ∇ is determined by

$$2\langle\nabla_u v, w\rangle = \langle[x,y],z\rangle - \langle[y,z],x\rangle + \langle[z,x],y\rangle, \quad x,y,z \in \mathfrak{g}.$$

If $b_1,\dots,b_7$ is a basis of $\mathfrak{g}$ we will denote by Λ_j the endomorphism corresponding to ∇_{b_j} and by R_{kl} the curvature operator $R(b_k,b_l)$. We compute the holonomy algebra using the Ambrose-Singer holonomy theorem [AS], which states that the holonomy algebra is spanned by the curvature operators $R_{xy}, x, y \in \mathfrak{g}$, together with their covariant derivatives.

4.1. Examples of Type I.

In all examples of this subsection, $b_1,\dots,b_7$ will be a basis of a Lie algebra $\mathfrak{g}$ such that ω equals

$$\sqrt{2}(b^{127}+b^{356}) - e^4 \wedge (b^{15}+b^{26}-b^{37})$$

and the induced inner product on $\mathfrak{g}$ equals

$$2(b^1\cdot e^5 + b^2\cdot e^6 + b^3\cdot e^7) - (b^4)^2.$$

Example 4.1. *(5-dimensional holonomy) Let $\mathfrak{g}$ be the Lie algebra with structure equations*

$$\begin{aligned}
db^1 &= -2b^{15} - b^{56},\\
db^2 &= -2b^{25} - b^{35} - b^{56},\\
db^3 &= -b^{35},\\
db^4 &= b^{45} - \sqrt{2}b^{67},\\
db^5 &= 0,\\
db^6 &= -b^{56},\\
db^7 &= b^{36} - b^{56}.
\end{aligned}$$

Then

$$
\begin{aligned}
\Lambda_1 &= 0\\
\Lambda_2 &= \tfrac{1}{2}(b_2^5 - b_1^6)\\
\Lambda_3 &= b_2^3 - b_7^6 + \tfrac{1}{2}(b_3^5 - b_1^7),\\
\Lambda_4 &= -b_1^4 - b_4^5 + \tfrac{1}{\sqrt{2}}(b_3^6 - b_2^7),\\
\Lambda_5 &= -2(b_1^1 - b_5^5) - \tfrac{3}{2}(b_2^2 - b_6^6) - \tfrac{1}{2}(b_3^3 - b_7^7) - b_2^3 + b_7^6 - b_2^5 + b_1^6,\\
\Lambda_6 &= -\tfrac{1}{2}(b_1^2 - b_6^5) + \tfrac{1}{\sqrt{2}}(b_3^4 + b_4^7) - b_2^5 + b_1^6,\\
\Lambda_7 &= -\tfrac{1}{2}(b_1^3 - b_7^5) - \tfrac{1}{\sqrt{2}}(b_2^4 + b_4^6).
\end{aligned}
$$

For the holonomy algebra $\mathfrak{h}$*, we obtain*

$$\mathfrak{h} = \operatorname{span}\{R_{25}, R_{35}, R_{45}, R_{56}, R_{57}\} = \mathfrak{m}.$$

Example 4.2. *(6-dimensional holonomy) Let* $\mathfrak{g}$ *be the Lie algebra with structure equations*

$$
\begin{aligned}
db^1 &= db^5 = db^6 = 0,\\
db^2 &= \sqrt{2}(b^{25} + b^{35} - b^{56} - b^{57}),\\
db^3 &= -db^7 = -\tfrac{1}{\sqrt{2}}b^{35} - b^{46} - \sqrt{2}b^{56} + \tfrac{1}{\sqrt{2}}b^{57},\\
db^4 &= -b^{36} + b^{67}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\Lambda_1 &= 0\\
\Lambda_2 &= -\tfrac{1}{\sqrt{2}}(b_2^5 - b_1^6),\\
\Lambda_3 &= \tfrac{1}{\sqrt{2}}(b_1^3 - b_7^5) + b_2^4 + b_4^6,\\
\Lambda_4 &= b_2^3 - b_7^6,\\
\Lambda_5 &= \tfrac{1}{\sqrt{2}}(b_2^2 - b_6^6 - b_3^3 + b_7^7) + \sqrt{2}(b_2^3 - b_7^6),\\
\Lambda_6 &= \tfrac{1}{\sqrt{2}}(b_1^2 - b_6^5) - b_3^4 - b_4^7 - \sqrt{2}(b_2^5 - b_1^6 + b_3^5 - b_1^7),\\
\Lambda_7 &= -\sqrt{2}(b_2^5 - b_1^6) + \tfrac{1}{\sqrt{2}}(b_3^5 - b_1^7).
\end{aligned}
$$

For the holonomy algebra $\mathfrak{h}$*, we obtain*

$$\mathfrak{h} = \operatorname{span}\{R_{25}, R_{35}, R_{36}, R_{56}, R_{67}, \nabla_{e_6}R_{36}\} = \mathbb{R}\cdot N \ltimes \mathfrak{m}.$$

Example 4.3. *(7-dimensional holonomy) Let $\mathfrak{g}$ be the Lie algebra with structure equations*

$$\begin{aligned}
db^1 &= -b^{16} - \tfrac{1}{\sqrt{2}}b^{26} + \tfrac{1}{2}b^{36} + (1+\sqrt{2})b^{46} + b^{56} - b^{67},\\
db^2 &= -\tfrac{3}{2}b^{26} + (\tfrac{\sqrt{2}}{4} + \tfrac{1}{2})b^{36} + (3 + \tfrac{1}{\sqrt{2}})b^{46} + (\tfrac{1}{\sqrt{2}} - 1)b^{56} - (\tfrac{1}{\sqrt{2}} + 1)b^{67},\\
db^3 &= \tfrac{\sqrt{2}}{4}b^{35} + \tfrac{1}{2}b^{36} + \tfrac{1}{2}b^{46} - b^{56} - \tfrac{1}{\sqrt{2}}b^{57},\\
de^4 &= \tfrac{1}{4}b^{36} + (1+\sqrt{2})b^{56} - \tfrac{1}{2}b^{67},\\
db^5 &= b^{56},\\
db^6 &= 0,\\
db^7 &= -\tfrac{\sqrt{2}}{8}b^{35} - \tfrac{1}{2}b^{36} - \tfrac{1}{4}b^{46} - (\tfrac{1}{2} + 3\sqrt{2})b^{56} + \tfrac{\sqrt{2}}{4}b^{57} + \tfrac{1}{2}b^{67}.
\end{aligned}$$

Then

$$\begin{aligned}
\Lambda_1 &= 0\\
\Lambda_2 &= -\tfrac{\sqrt{2}}{4}(b_2^5 - b_1^6),\\
\Lambda_3 &= -\tfrac{\sqrt{2}}{8}(b_1^3 - b_7^5) - \tfrac{1}{2}(b_2^3 - b_7^6) - \tfrac{1}{4}(b_2^4 + b_4^6) - \tfrac{3}{\sqrt{2}}(b_2^5 - b_1^6),\\
\Lambda_4 &= -\frac{1}{4}(b_2^3 - b_7^6),\\
\Lambda_5 &= \tfrac{\sqrt{2}}{4}(-b_2^2 + b_6^6 + b_3^3 - b_7^7) - \tfrac{3}{\sqrt{2}}(b_2^3 - b_7^6) + b_2^5 - b_1^6,\\
\Lambda_6 &= -b_1^1 + b_5^5 - \tfrac{3}{2}(b_2^2 - b_6^6) + \tfrac{1}{2}(b_3^3 - b_7^7) - \tfrac{\sqrt{2}}{4}(b_1^2 - b_6^5) + (\tfrac{1}{2} + \tfrac{3}{\sqrt{2}})(b_1^3 - b_7^5)\\
&\quad + (\tfrac{\sqrt{2}}{4} + \tfrac{1}{2})(b_2^3 - b_7^6) + (1+\sqrt{2})(b_1^4 + b_4^5) + (3 + \tfrac{1}{\sqrt{2}})(b_2^4 + b_4^6)\\
&\quad + \tfrac{1}{2}(b_3^4 + b_4^7) + (\tfrac{1}{\sqrt{2}} - 1)(b_2^5 - b_1^6) - b_3^5 + b_1^7) - (\tfrac{1}{\sqrt{2}} + 1)(b_3^6 - b_2^7),\\
\Lambda_7 &= -\tfrac{1}{\sqrt{2}}(b_3^5 - b_1^7).
\end{aligned}$$

For the holonomy algebra $\mathfrak{h}$, we obtain

$$\mathfrak{h} = \operatorname{span}\{R_{25}, R_{35}, R_{36}, R_{45}, R_{56}, R_{57}, \nabla_{b_5}R_{56}\} = \mathfrak{s}_{1/2} \ltimes \mathfrak{m}.$$

4.2. **Examples of Type II.** In all examples of this subsection, $b_1, \dots, b_7$ will be a basis of a Lie algebra $\mathfrak{g}$ such that ω equals

$$\sqrt{2}(-b^{157} + b^{236}) - b^4 \wedge (b^{16} - b^{27} - b^{35})$$

and the induced inner product on $\mathfrak{g}$ equals

$$2(b^1 \cdot b^6 + b^2 \cdot b^7 + b^3 \cdot b^5) - (b^4)^2.$$

Example 4.4. *(3-dimensional holonomy) Let $\mathfrak{g}$ be the Lie algebra with structure equations*

$$\begin{aligned}
db^1 &= -b^{17} + \tfrac{1}{\sqrt{2}}b^{34} + (1-\sqrt{2})b^{37} - b^{46} - b^{57} + b^{67},\\
db^2 &= -b^{27} + b^{37} - b^{47} - b^{67}\\
db^3 &= (1-\sqrt{2})b^{67},\\
db^4 &= -b^{47},\\
db^5 &= -b^{37} + \tfrac{1}{\sqrt{2}}b^{46} + b^{67},\\
db^6 &= b^{67},\\
db^7 &= 0.
\end{aligned}$$

Then

$$\begin{aligned}
\Lambda_1 &= \Lambda_2 = 0,\\
\Lambda_3 &= -b_2^3 + b_5^7 + (1-\tfrac{1}{\sqrt{2}})(b_2^6 - b_1^7),\\
\Lambda_4 &= \tfrac{1}{\sqrt{2}}(b_1^3 - b_5^6) + (b_2^4 + e_4^7),\\
\Lambda_5 &= \tfrac{1}{\sqrt{2}}(b_1^7 - b_2^6),\\
\Lambda_6 &= (1-\tfrac{1}{\sqrt{2}})(b_2^3 - b_5^7) - \tfrac{1}{\sqrt{2}}(b_2^5 - b_3^7) + b_2^6 - b_1^7 - b_1^4 - b_4^6,\\
\Lambda_7 &= b_2^3 - b_5^7 - \tfrac{1}{\sqrt{2}}(b_1^3 - b_5^6) - b_2^2 + b_7^7 - b_1^1 + b_6^6 - b_2^4 - b_4^7\\
&\quad - (1-\tfrac{1}{\sqrt{2}})(b_1^5 - b_3^6) - b_2^6 + b_1^7.
\end{aligned}$$

For the holonomy algebra $\mathfrak{h}$, we obtain

$$\mathfrak{h} = \operatorname{span}\{R_{37}, R_{46}, R_{67}\} = \mathfrak{n}(1,3).$$

Example 4.5. *(5-dimensional holonomy) Let $\mathfrak{g}$ be the Lie algebra with structure equations*

$$\begin{aligned}
db^1 &= 2b^{15} + 4b^{56} + b^{57},\\
db^2 &= b^{17} + b^{25} + \sqrt{2}b^{34} - \sqrt{2}b^{46} - e^{56} - b^{57} - b^{67},\\
db^3 &= b^{35} - 3b^{56},\\
db^4 &= -\sqrt{2}b^{37} - \sqrt{2}b^{67},\\
db^5 &= 0,\\
db^6 &= 2b^{56},\\
db^7 &= b^{57}.
\end{aligned}$$

Then

$$\begin{aligned}
\Lambda_1 &= \Lambda_2 = \Lambda_3 = 0,\\
\Lambda_4 &= \sqrt{2}(b_2^3 - b_5^7 + b_2^6 - b_1^7),\\
\Lambda_5 &= 2(b_1^1 - b_6^6) + b_2^2 - b_7^7 + b_3^3 - b_5^5 - 3(b_1^5 - b_3^6) + b_2^6 - b_1^7,\\
\Lambda_6 &= 4(b_1^5 - b_3^6),\\
\Lambda_7 &= b_2^1 - b_6^7 - \sqrt{2}(b_4^3 + b_1^4 + b_5^4 + b_4^6) - b_2^5 + b_3^7 - b_2^6 + b_1^7.
\end{aligned}$$

For the holonomy algebra $\mathfrak{h}$*, we obtain*

$$\mathfrak{h} = \mathrm{span}\{R_{37}, R_{46}, R_{56}, R_{67}, \nabla_{b_7} R_{67}\} = \mathfrak{n}.$$

Example 4.6. *(8-dimensional holonomy) Let* $\mathfrak{g}$ *be the Lie algebra with structure equations*

$$\begin{aligned}
db^1 &= -b^{17} + b^{26} - b^{36} - b^{37} - b^{46} + e^{56} - b^{57} - b^{67}, \\
db^2 &= \tfrac{5}{3}b^{16} - b^{27} + \tfrac{13}{3}b^{36} + \tfrac{11}{9}b^{37} - \tfrac{4}{3}\sqrt{2}b^{46} + (\tfrac{8}{3}\sqrt{2} + \tfrac{5}{3})b^{47} \\
&\quad -9b^{56} - \tfrac{1}{3}b^{57} - b^{67} \\
db^3 &= b^{37} - 2\sqrt{2}b^{46} - b^{57} - (\sqrt{2} - 10)b^{67}, \\
db^4 &= \tfrac{2}{3}b^{47} + \tfrac{2}{3}\sqrt{2}b^{67}, \\
db^5 &= \tfrac{7}{9}b^{37} - \tfrac{2}{3}\sqrt{2}b^{46} - \tfrac{5}{3}b^{57} - (2 + \tfrac{5}{3}\sqrt{2})b^{67}, \\
db^6 &= db^7 = 0.
\end{aligned}$$

Then

$$\begin{aligned}
\Lambda_1 &= -\tfrac{4}{3}(b_2^6 - b_1^7) \\
\Lambda_2 &= 0 \\
\Lambda_3 &= \tfrac{7}{9}(b_2^3 - b_5^7) - \tfrac{\sqrt{2}}{3}(b_1^4 + b_4^6) - \tfrac{1}{3}(b_2^5 - b_3^7) - (\tfrac{11}{3} + \tfrac{5}{6}\sqrt{2})(b_2^6 - b_1^7), \\
\Lambda_4 &= -\tfrac{\sqrt{2}}{3}(b_1^3 - b_5^6) - \tfrac{2}{3}(b_2^4 + b_4^7) - \sqrt{2}(b_1^5 - b_3^6) + \tfrac{\sqrt{2}}{3}(b_2^6 - b_1^7), \\
\Lambda_5 &= -\tfrac{1}{3}(b_2^3 - b_5^7) - \sqrt{2}(b_1^4 + b_4^6) - b_2^5 + b_3^7 + (9 - \tfrac{1}{\sqrt{2}})(b_2^6 - b_1^7) \\
\Lambda_6 &= \tfrac{1}{3}(b_2^1 - b_6^7) + b_1^2 - b_7^6 - b_1^3 + b_5^6 + (\tfrac{2}{3} - \tfrac{5}{6}\sqrt{2})(b_2^3 - b_5^7) - \tfrac{\sqrt{2}}{3}(b_4^3 + b_5^4) \\
&\quad -b_1^4 - b_4^6 - \sqrt{2}(b_2^4 + b_4^7 + b_3^4 + b_4^5) + b_1^5 - b_3^6 - \tfrac{1}{\sqrt{2}}(b_2^5 - b_3^7) - b_2^6 + b_1^7, \\
\Lambda_7 &= \tfrac{1}{3}(b_1^1 - b_6^6) - b_2^2 + b_7^7 + \tfrac{4}{3}(b_3^3 - b_5^5) + (\tfrac{8}{3} + \tfrac{5}{6}\sqrt{2})(b_1^3 - b_5^6) + \tfrac{11}{9}(b_2^3 - b_5^7) \\
&\quad -\tfrac{1}{3}\sqrt{2}(b_1^4 + b_4^6) + (\tfrac{8}{3}\sqrt{2} + \tfrac{5}{3})(b_2^4 + b_4^7) - (10 - \tfrac{1}{\sqrt{2}})(b_1^5 - b_3^6) \\
&\quad -\tfrac{1}{3}(b_2^5 - b_3^7) - b_2^6 + b_7^1.
\end{aligned}$$

For the holonomy algebra $\mathfrak{h}$*, we obtain*

$$\mathfrak{h} = \mathrm{span}\{R_{17}, R_{36}, R_{37}, R_{56}, R_{57}, R_{67}, \nabla_{b_6} R_{67}, \nabla_{b_7} R_{67}\} = \mathfrak{sl}(2, \mathbb{R}) \ltimes \mathfrak{n}.$$

4.3. **Example of Type III.** In the example of this subsection, $b_1, \ldots, b_7$ will be a basis of a Lie algebra $\mathfrak{g}$ such that ω equals

$$\sqrt{2}(b^{127} + b^{356}) - e^4 \wedge (b^{15} + b^{26} - b^{37})$$

and the induced inner product on $\mathfrak{g}$ equals

$$2(b^1 \cdot e^5 + b^2 \cdot e^6 + b^3 \cdot e^7) - (b^4)^2.$$

Example 4.7. *(3-dimensional holonomy) Let $\mathfrak{g}$ be the Lie algebra with structure equations*

$$\begin{aligned} db^1 &= -b^{15} - b^{45} - b^{56}, \\ db^2 &= -\tfrac{1}{3}b^{25} - b^{35} - b^{36} - \sqrt{2}b^{45} - b^{56} + (\sqrt{2}-1)b^{57} + b^{67}, \\ db^3 &= -\tfrac{2}{3}b^{35} - b^{56} + \tfrac{4}{3}b^{57}, \\ db^4 &= -\sqrt{2}b^{56}, \\ db^5 &= 0, \\ db^6 &= -\tfrac{1}{3}b^{56}, \\ db^7 &= -b^{56} - \tfrac{2}{3}b^{57}. \end{aligned}$$

Then

$$\begin{aligned} \Lambda_1 &= \Lambda_2 = \Lambda_3 = 0 \\ \Lambda_4 &= \sqrt{2}(b_2^5 - b_1^6) \\ \Lambda_5 &= -b_1^1 + b_5^5 - \tfrac{1}{3}(b_2^2 - b_6^6) - \tfrac{2}{3}(b_3^3 - b_7^7) - b_2^3 + b_7^6 - b_1^4 - b_4^5 - b_2^5 + b_1^6 \\ &\quad + \tfrac{1}{\sqrt{2}}(b_3^6 - b_2^7) \\ \Lambda_6 &= -b_2^3 + b_7^6 - \sqrt{2}(b_1^4 + b_4^5) - b_2^5 + b_1^6 - (1 - \tfrac{1}{\sqrt{2}})(b_3^5 - b_1^7) + b_3^6 - b_2^7 \\ \Lambda_7 &= -(1 - \tfrac{1}{\sqrt{2}})(b_2^5 - b_1^6) + \tfrac{4}{3}(b_3^5 - b_1^7). \end{aligned}$$

For the holonomy algebra $\mathfrak{h}$, we obtain

$$\mathfrak{h} = \mathrm{span}\{R_{45}, R_{56}, R_{57}\} = \mathfrak{m}(1,0,2).$$

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