Linear divisibility sequences and Salem numbers

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Abstract

We study linear divisibility sequences of order 4, providing a characterization by means of their characteristic polynomials and finding their factorization as a product of linear divisibility sequences of order 2. Moreover, we show a new interesting connection between linear divisibility sequences and Salem numbers. Specifically, we generate linear divisibility sequences of order 4 by means of Salem numbers modulo 1.

Mathematical Subject Classifications 2000: 11B37.

1 Introduction

A sequence $a = (a_n)_{n=0}^{\infty}$ is a divisibility sequence if $m|n$ implies $a_m|a_n$. Divisibility sequences that satisfy a linear recurrence relation are particularly studied. A classic example of linear divisibility sequence is the Fibonacci sequence. During the years linear divisibility sequences of order 2 have been deeply studied, see, e.g., [11] and [14]. Hall [10] studied divisibility sequences of order 3 and Bezivin et al. [3] have obtained more general results. Divisibility sequences are very interesting for their beautiful properties. For example, many studies can be found about their connection with elliptic curves [19], [12]. Further results on divisibility sequences can be found, e.g, in [8] where Cornelissen and Reynolds investigate matrix divisibility sequences, and in [22] where Horak and Skula characterize the second–order strong divisibility sequences.

Recently, linear divisibility sequences of order 4 have been deeply examined. In particular, Williams and Guy [20], [21] introduced and studied a class of linear divisibility sequences of order 4 that extends the Lehmer–Lucas theory for divisibility sequences of order 2. In section 2, we consider these sequences proving that all (non degenerate) divisibility sequences of order 4 have characteristic polynomial equals to the characteristic polynomial of sequences of Williams and Guy. Moreover, we provide all factorizations of divisibility sequences of order 4 into the product of divisibility sequences of order 2.

In section 3, we generate linear divisibility sequences of order 4 by means of powers of Salem numbers. This result is particularly intriguing, since connections between Salem numbers and divisibility sequences have been never highlighted. Moreover, the construction of divisibility sequences by means of powers of algebraic integers is an interesting research field that have been recently developed [18].

2 Standard linear divisibility sequences

Definition 1. *Given a ring $\mathcal{R}$, a sequence $a = (a_n)_{n=0}^{\infty}$ over $\mathcal{R}$ is a* divisibility sequence *if*

$$m|n \Rightarrow a_m|a_n.$$

Conventionally, we will consider $a_0 = 0$.

In the following, we will deal with *linear divisibility sequences* (LDSs), i.e., divisibility sequences that satisfy a linear recurrence. Classic LDSs are the Lucas sequences, i.e., the linear recurrence sequences with characteristic polynomial x^2-hx+k and initial conditions $0, 1$.

In [20] and [21], the authors introduced and studied some linear divisibility sequences of order 4. We recall these sequences in the following definition.

Definition 2. *Let us consider linear recurrence sequences of order 4 over* $\mathbb{Z}$ *with characteristic polynomial*

$$x^4 - px^3 + (q + 2r)x^2 - prx + r^2$$

and initial conditions

$$0, 1, p, p^2 - q - 3r.$$

We say that these sequences are standard *LDSs of order 4 and we call the previous polynomial as* standard polynomial.

In the next theorem, we prove that the product of two LDSs of order 2 is a standard LDS of order 4. First, we need the following lemma proved in [7].

Lemma 1. *Let* $a = (a_n)_{n=0}^{\infty}$ *and* $b = (b_n)_{n=0}^{\infty}$ *be linear recurrence sequences with characteristic polynomials* $f(x)$ *and* $g(x)$*, respectively. The sequence* $ab = (a_nb_n)_{n=0}^{\infty}$ *is a linear recurrence sequence that recurs with* $f(x) \otimes g(x)$*, the characteristic polynomial of the matrix* $F \otimes G$ *(Kronecker product of matrices), where* F *and* G *are the companion matrices of* $f(x)$ *and* $g(x)$*, respectively.*

Theorem 1. *Let* $a = (a_n)_{n=0}^{\infty}$ *and* $b = (b_n)_{n=0}^{\infty}$ *be LDSs of order 2 with characteristic polynomials* $x^2 - h_1x + k_1$*,* $x^2 - h_2x + k_2$*, respectively, and initial conditions* $0, 1$*. The sequence* $ab = (a_nb_n)_{n=0}^{\infty}$ *is a standard LDS of order 4 with initial conditions* $0, 1, h_1h_2, (h_1^2 - k_1)(h_2^2 - k_2)$*.*

Proof. Since a and b are LDSs, it immediately follows that ab is a divisibility sequence and by Lemma 1, we know that it is a linear recurrence sequence of order 4 whose characteristic polynomial is

$$x^4 - h_1h_2x^3 + (k_1h_1^2 - k_2h_1^2 + 2k_1k_2)x^2 + h_1k_1h_2k_2x + k_1^2k_2^2.$$

By Definition 2, ab is a standard LDS for $p = h_1h_2$, $q = h_1^2k_2 + k_1(h_2^2 - 4k_2)$, $r = k_1k_2$. The initial conditions can be directly calculated. □

Moreover, we prove that all the LDSs of order 4 have characteristic polynomial equals to the characteristic polynomial of standard LDSs.

Theorem 2. *Let* $a = (a_n)_{n=0}^{\infty}$ *be a non degenerate LDS of order 4 with* $a_0 = 0$ *and* $a_1 = 1$*, then its characteristic polynomial is*

$$x^4 - px^3 + (q + 2r)x^2 - prx + r^2$$

for some p, q, r*.*

Proof. Let us suppose that the characteristic polynomial of a has distinct roots in order to avoid degenerate sequences, i.e., ratio of roots are not roots of unity. Since $a_0 = 0$, we have

$$a_n = c_1\alpha^n + c_2\beta^n + c_3\gamma^n - (c_1 + c_2 + c_3)\delta^n, \quad \forall n \geq 0.$$

Since

$$a_{2n} = c_1(\alpha^n - \delta^n)(\alpha^n + \delta^n) + c_2(\beta^n - \delta^n)(\beta^n + \delta^n) + c_3(\gamma^n - \delta^n)(\gamma^n + \delta^n),$$

we can write

$$a_{2n} = a_n(\alpha^n + \beta^n + \gamma^n + \delta^n) - R$$

where

$$R = (c_1 + c_2)((\alpha\beta)^n - (\gamma\delta)^n) + (c_1 + c_3)((\alpha\gamma)^n - (\beta\delta)^n) + (c_2 + c_3)((\beta\gamma)^n - (\alpha\delta)^n).$$

By hypothesis, $a_n|a_{2n}$, for all $n \geq 0$, thus $a_n|R$ for all $n \geq 0$. We will prove that $R = 0$. Indeed, let us pose $\delta^n = y$, $a_n|R$ means that polynomials

$$(c_1\alpha^n + c_2\beta^n + c_3\gamma^n) - (c_1 + c_2 + c_3)y$$

and

$$(c_1+c_2)(\alpha\beta)^n+(c_1+c_3)(\alpha\gamma)^n+(c_2+c_3)(\beta\gamma)^n-((c_1+c_2)\gamma^n+(c_1+c_3)\beta^n+(c_2+c_3)\alpha^n)y$$

have the common root $y = \frac{c_1\alpha^n+c_2\beta^n+c_3\gamma^n}{c_1+c_2+c_3}$. Let us observe that $c_1+c_2+c_3 \neq 0$, otherwise a has order 3. In this case, we should have

$$c_1c_2(\alpha^n - \beta^n)^2 + c_1c_3(\alpha^n - \gamma^n)^2 + c_2c_3(\beta^n - \gamma^n)^2 = 0 \tag{1}$$

for all $n \geq 0$, with $c_1c_2, c_1c_3, c_2c_3 \neq 0$ so that a has order 4 and $\alpha, \beta, \gamma, \delta$ distinct. We can write

$$c_1c_2 = -\frac{c_1c_3(\alpha^n - \gamma^n)^2 + c_2c_3(\beta^n - \gamma^n)^2}{(\alpha^n - \beta^n)^2}.$$

Comparing this identity for $n = 1, 2, 3$, we get

$$c_1c_3(\alpha - \gamma) = 0, \quad c_2c_3(\beta - \gamma) = 0$$

that contradict conditions on c_1, c_2, c_3 and $\alpha, \beta, \gamma, \delta$. Thus, Eq. (1) is not true under these hypothesis and consequently $R = 0$, i.e.,

$$(c_1 + c_2)((\alpha\beta)^n - (\gamma\delta)^n) + (c_1 + c_3)((\alpha\gamma)^n - (\beta\delta)^n) + (c_2 + c_3)((\beta\gamma)^n - (\alpha\delta)^n) = 0$$

and

$$c_1 + c_2 = -\frac{(c_1 + c_3)((\alpha\gamma)^n - (\beta\delta)^n) + (c_2 + c_3)((\beta\gamma)^n - (\alpha\delta)^n)}{(\alpha\beta)^n - (\gamma\delta)^n}$$

for all $n \geq 1$. Comparing this identity for $n = 1, 2, 3$, we get

$$(c_1 + c_3)(\alpha\gamma - \beta\delta) = 0, \quad (c_2 + c_3)(\beta\gamma - \alpha\delta) = 0, \quad c_1 + c_2 = 0.$$

These conditions are satisfied when

$$c_1 + c_3 = 0, \quad \beta\gamma - \alpha\delta = 0$$

or

$$c_2 + c_3 = 0, \quad \alpha\gamma - \beta\delta = 0.$$

We obtain

$$a_n = c_1(\alpha^n + \delta^n - \beta^n - \gamma^n), \quad \alpha\delta = \beta\gamma$$

or

$$a_n = c_1(\alpha^n + \gamma^n - \beta^n - \delta^n), \quad \alpha\gamma = \beta\delta.$$

Imposing $a_1 = 1$ we find

$$a_n = \frac{\alpha^n + \delta^n - \beta^n - \gamma^n}{\alpha + \delta - \beta - \gamma}, \quad \alpha\delta = \beta\gamma$$

or

$$a_n = \frac{\alpha^n + \gamma^n - \beta^n - \delta^n}{\alpha + \gamma - \beta - \delta}, \quad \alpha\gamma = \beta\delta$$

i.e., the characteristic polynomial of a is a standard polynomial. $\square$

Now, we see that any standard LDS can be factorized as a product of two LDS of order 2 over $\mathbb{C}$.

Definition 3. *Given the sequences $(u_n)_{n=0}^{+\infty}, (v_n)_{n=0}^{+\infty}, (s_n)_{n=0}^{+\infty}, (t_n)_{n=0}^{+\infty}$ over a ring $\mathcal{R}$, we say that the product sequences $(u_n v_n)_{n=0}^{+\infty}$ and $(s_n t_n)_{n=0}^{+\infty}$ are* equivalent *if*

$$u_n = \lambda^{n-1} s_n, \quad v_n = \lambda^{1-n} t_n$$

where $\lambda \in \mathcal{R}$ is a unit.

Theorem 3. *Let $a = (a_n)_{n=0}^{\infty}$ be a standard LDS over $\mathbb{Z}$, then $a_n = b_n c_n$, for all $n \geq 0$, where $b = (b_n)_{n=0}^{\infty}$ and $c = (c_n)_{n=0}^{\infty}$ are LDSs of order 2 over $\mathbb{C}$ with initial conditions $0, 1$ and characteristic polynomials*

$$\begin{cases} x^2 - \dfrac{\sqrt{q+4r+2p\sqrt{r}} \pm \sqrt{q+4r-2p\sqrt{r}}}{2\sqrt{r}} x + 1 \\ x^2 - \dfrac{\sqrt{q+4r+2p\sqrt{r}} \mp \sqrt{q+4r-2p\sqrt{r}}}{2} x + r \end{cases}$$

when $p \neq 0$. Moreover when $p = 0$ and $q + 4r \neq 0$, $q \neq 0$ (to avoid degenerate cases) we have the two possible families of characteristic polynomials for b and c given by

$$\begin{cases} x^2 + 1 \\ x^2 - \sqrt{q+4r}x + r \end{cases}, \quad \begin{cases} x^2 + 1 \\ x^2 - \sqrt{q}x - r \end{cases}$$

These are all the families of not equivalent factorizations of a over $\mathbb{C}$.

Proof. We want to factorize a standard polynomial into the Kronecker product of two polynomials of degree 2, i.e., we want to find h_1, h_2, k_1, k_2 such that

$$(x^2 - h_1 x + k_1) \otimes (x^2 - h_2 x + k_2) = x^4 - px^3 + (q+2r)x^2 - px + r^2.$$

Let us observe that the characteristic polynomial of a must have distinct non zero roots in order to guarantee that a is a LDS of order 4 . Let γ_1, γ_2 and σ_1, σ_2 be the roots of $x^2 - h_1 x + k_1$ and $x^2 - h_2 x + k_2$, respectively. We have

$$\begin{cases} (\gamma_1 + \gamma_2)(\sigma_1 + \sigma_2) = p \\ (\gamma_1^2 + \gamma_2^2)\sigma_1\sigma_2 + \gamma_1\gamma_2(\sigma_1 + \sigma_2)^2 = q + 2r \\ \gamma_1\gamma_2\sigma_1\sigma_2(\gamma_1 + \gamma_2)(\sigma_1 + \sigma_2) = pr \\ (\gamma_1\gamma_2\sigma_1\sigma_2)^2 = r^2 \end{cases} \tag{2}$$

When $p \neq 0$ these conditions are equivalent to the system

$$\begin{cases} k_1 k_2 = r \\ h_1 h_2 = p \\ h_1^2 k_2 + h_2^2 k_1 = q + 4r \end{cases} \tag{3}$$

which is a particular case of

$$\begin{cases} k_1 k_2 = A \\ h_1 h_2 = B \\ h_1^2 k_2 + h_2^2 k_1 = C \end{cases}$$

where $A \neq 0$ since we suppose that the standard polynomial has not zero roots. Thus, we can obtain

$$A\left(\frac{h_1^2}{k_1}\right)^2 - C\left(\frac{h_1^2}{k_1}\right) + B^2 = 0$$

from which we have

$$h_1 = \pm\sqrt{k_1}\frac{\sqrt{C+2B\sqrt{A}} \pm \sqrt{C-2B\sqrt{A}}}{2\sqrt{A}}$$

and

$$h_2 = \pm\frac{\sqrt{C+2B\sqrt{A}} \mp \sqrt{C-2B\sqrt{A}}}{2\sqrt{k_1}}.$$

Thus solutions of system 3 are

$$\begin{cases} h_1 = \pm\sqrt{k_1}\dfrac{\sqrt{q+4r+2p\sqrt{r}} \pm \sqrt{q+4r-2p\sqrt{r}}}{2\sqrt{r}} \\ h_2 = \pm\dfrac{\sqrt{q+4r+2p\sqrt{r}} \mp \sqrt{q+4r-2p\sqrt{r}}}{2\sqrt{k_1}} \\ k_2 = \dfrac{r}{k_1} \end{cases}.$$

Let us pose

$$\lambda = \pm\sqrt{k_1},\ s = \frac{\sqrt{q+4r+2p\sqrt{r}} \pm \sqrt{q+4r-2p\sqrt{r}}}{2\sqrt{r}},,\ \bar{s} = \frac{\sqrt{q+4r+2p\sqrt{r}} \mp \sqrt{q+4r-2p\sqrt{r}}}{2}.$$

Thus, considering solutions of system 3, we have $x^2 - h_1x + k_1 = x^2 - s\lambda x + \lambda^2$ and $x^2 - h_2x + k_2 = x^2 - \frac{\bar{s}}{\lambda}x + \frac{r}{\lambda^2}$, whose roots are

$$\gamma_{1,2} = \lambda\left(\frac{s \pm \sqrt{s^2-4}}{2}\right), \quad \sigma_{1,2} = \frac{1}{\lambda}\left(\frac{\bar{s} \pm \sqrt{\bar{s}^2-4}}{2}\right).$$

In this case, we have $u_n = \lambda^{n-1}b_n$ and $v_n = \lambda^{1-n}c_n$, where b and c are Lucas sequences with characteristic polynomials $x^2 - sx + 1$ and $x^2 - \bar{s}x + r$, respectively. When $p = 0$ in conditions (2) we may suppose $\gamma_1 + \gamma_2 = h_1 = 0$ and find the two systems

$$\begin{cases} h_1 = 0 \\ k_1k_2 = r \\ h_2^2k_1 = p+4r \end{cases}, \quad \begin{cases} h_1 = 0 \\ k_1k_2 = -r \\ h_2^2k_1 = p \end{cases}$$

with respective solutions

$$\begin{cases} h_1 = 0 \\ h_2 = \pm\sqrt{\frac{p+4r}{k_1}} \\ k_2 = \frac{r}{k_1} \end{cases}, \quad \begin{cases} h_1 = 0 \\ h_2 = \pm\sqrt{\frac{p}{k_1}} \\ k_2 = -\frac{r}{k_1} \end{cases}$$

which give, with analogous considerations as in the case $p \neq 0$, with $\lambda = \pm\sqrt{k_1}$, the two families of characteristic polynomials for b and c related to this case. □

Remark 1. *It would be interesting to find when previous factorizations determine sequences in* $\mathbb{Z}$ *or* $\mathbb{Z}[i]$.

In the next section, we see a new connection between LDS of order 4 and Salem numbers.

3 Construction of linear divisibility sequences by means of Salem numbers of order 4

The Salem numbers have been introduced in 1944 by Raphael Salem [17] and they are closely related to the Pisot numbers [16]. There are several results regarding Pisot numbers and recurrence sequences [4], [5], [6]. In the following, we relate Salem numbers and LDS.

There are many equivalent definitions of Salem numbers, here we report the following one.

Definition 4. *A* Salem number *is an algebraic integer $\tau > 1$ of degree $d \geq 4$ such that all the conjugate elements belong to the unitary circle, unless τ and τ^{-1}.*

In the following, we work on Salem numbers of degree 4, which can be characterized as follows (see [2], pag. 81).

Proposition 1. *The Salem numbers of degree 4 are all the real roots $\tau > 1$, of the following polynomials with integer coefficients*

$$x^4 + tx^3 + cx^2 - tx + 1$$

where

$$2(t-1) < c < -2(t+1).$$

It is immediate to see that previous polynomials are standard polynomials for $p = -t$, $q = -2 + c$, $r = 1$.

Definition 5. *We call* Salem standard polynomials *the polynomials*

$$x^4 - px^3 + (q+2)x^2 - px + 1$$

with

$$2(-p-1) < 2 + q < -2(-p+1).$$

The study of the distribution modulo 1 of the powers of a given real number greater than 1 is a rich and classic research field (see, e.g, [13]). In the following, we use the same notation of [2] (pag. 61).

Definition 6. *Given a real number α, let $E(\alpha)$ be the nearest integer to α, i.e., $\alpha = E(\alpha) + \epsilon(\alpha)$ where $\epsilon(\alpha) \in [-\frac{1}{2}, \frac{1}{2}]$ is called α* modulo 1.

In the original work of Salem [17], he proved that if α is a Pisot number, then α^n modulo 1 tends to zero and if α is a Salem number, then α^n modulo 1 is dense in the unit interval. Further results on the distribution modulo 1 of the Salem numbers can be found, e.g., in [1] and [23]. Moreover, integer and fractional parts of Pisot and Salem numbers have been studied, e.g., in [9] and [24].

Let $\mathcal{R} \subseteq \mathbb{C}$ be a ring and $\alpha \in \mathcal{R}$ with $\alpha \notin \mathcal{R}^*$, then the sequence $(\alpha^n)_{n=0}^{\infty}$ is clearly a LDS. Given a couple of irrational numbers λ and α, it is interesting to study when the sequence $(E(\lambda\alpha^n))_{n=0}^{\infty}$ is a LDS.

Example 1. If we consider $\frac{1}{\sqrt{5}}$ and the golden mean ϕ, it is well–known that

$$E\left(\frac{1}{\sqrt{5}}\phi^n\right) = F_n,$$

where F_n is the n–th Fibonacci number, consequently we get a LDS.

Let $g(x)$ be a Salem standard polynomial, $g(x)$ has real roots $\alpha > 1$, α^{-1} and complex roots γ, γ^{-1} with norm 1. Let $(u_n)_{n=0}^{\infty}$ be a standard LDS with characteristic polynomial $g(x)$. By the Binet formula, there exist $\lambda, \lambda_1, \lambda_2, \lambda_3$ such that

$$u_n = \lambda\alpha^n + \lambda_1\alpha^{-n} + \lambda_2\gamma^n + \lambda_3\gamma^{-n}.$$

Since

$$|u_n - \lambda\alpha^n| \geq |\lambda_1\alpha^{-n}| + |\lambda_2| + |\lambda_3|,$$

for all $\epsilon > 0$, with n sufficiently large, we have

$$|u_n - \lambda\alpha^n| \geq \epsilon + |\lambda_2| + |\lambda_3|.$$

Thus, if $|\lambda_2| + |\lambda_3| < \frac{1}{2}$, there exists n_0 such that

$$u_n = E(\lambda\alpha^n), \quad \forall n > n_0$$

and if $|\lambda_1\alpha^{-1}| + |\lambda_2| + |\lambda_3| < \frac{1}{2}$, then

$$u_n = E(\lambda\alpha^n), \quad \forall n \geq 1.$$

An interesting case is given by the Salem standard polynomial

$$x^4 - tx^3 + tx^2 - tx + 1$$

for $t \geq 6$. In this case, we have the Salem numbers

$$\alpha = \frac{1}{4}\left(t + \sqrt{(t-4)t+8} + \sqrt{2}\sqrt{t(t+\sqrt{(t-4)t+8}-2)-4}\right)$$

and

$$\lambda = \frac{1}{\sqrt{(t-4)t+8}}.$$

Thus, we can determine infinitely many LDSs generated by powers of a Salem number, specifically the sequences

$$(\theta_n(t))_{n=1}^{\infty} = E(\lambda\alpha^n), \quad \forall t \geq 6 \in \mathbb{Z}$$

For example, when $t = 6$ we have the LDS

$$1, 6, 29, 144, 725, 3654, 18409, ...$$

when $t = 7$, we have

$$1, 7, 41, 245, 8897, 53621, ...$$

These sequences appear to be new, since they are not listed in OEIS [15]. Moreover, as a consequence, we have the following property on Salem numbers, i.e.,

$$d|n \Rightarrow E(\lambda\alpha^d)|E(\lambda\alpha^n).$$

Finally, in the following proposition we characterize all the Salem standard polynomials that produces LDSs of this kind.

Proposition 2. *With the above notation, if* $|\lambda_1\alpha^{-1}| + |\lambda_2| + |\lambda_3| < \frac{1}{2}$, *then the integer coefficients* p, q *of* $g(x)$ *must satisfy the following inequalities*

$$\begin{cases} 2 \leq p \leq 8, & -4 - 2p < q < \dfrac{p^4 + 8p^3 - 160p - 400}{4p^2 + 32p + 64} \\ p > 8, & -4 - 2p < q < -4 + 2p \end{cases}$$

Proof. The real root $\alpha > 1$ of $g(x)$ can be written as

$$\alpha = \frac{1}{4\left(p+\sqrt{p^2-4q}+\sqrt{(p+\sqrt{p^2-4q})^2-16}\right)}.$$

Moreover, by the Binet formula

$$\lambda = \lambda_1 = \frac{\alpha\gamma}{(\alpha-\gamma)(\alpha\gamma-1)}, \quad \lambda_2 = \lambda_3 = -\frac{\alpha\gamma}{(\alpha-\gamma)(\alpha\gamma-1)}.$$

Thus, from $|\lambda_1\alpha^{-1}| + |\lambda_2| + |\lambda_3| < \frac{1}{2}$ we get

$$|(\alpha-\gamma)(\alpha\gamma-1)| > 2\alpha+2.$$

Posing $\gamma = a+ib$, with some calculations we find

$$\alpha^4 - 4a\alpha^3 + 2(2a^2-7)\alpha^2 - 4(a+4)\alpha - 3 > 0$$

from which we have

$$\alpha > 2 + a + \sqrt{(a+2)^2+1}$$

since $-1 < a < 1$ and $\alpha > 1$. Using the explicit expression of α and that $a = \frac{p-\sqrt{p^2-4q}}{4}$, we finally obtain

$$\frac{1}{4}(p+\sqrt{-16+(-p-\sqrt{p^2-4q})^2}+\sqrt{p^2-4q}) > 2+\frac{p}{4}+\sqrt{1+\frac{1}{16}(8+p-\sqrt{p^2-4q})^2}-\frac{1}{4}\sqrt{p^2-4q},$$

whose solutions are

$$\begin{cases} 2 \le p \le 8, \quad -4-2p < q < \dfrac{p^4+8p^3-160p-400}{4p^2+32p+64} \\ p > 8, \quad -4-2p < q < -4+2p \end{cases}.$$

□

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