A standard and internal calculus for Lewis counterfactual logics

A Standard Internal Calculus for Lewis' Counterfactual Logics

Nicola Olivetti[1] and Gian Luca Pozzato[2,⋆]

[1] Aix Marseille Université, CNRS, ENSAM, Université de Toulon, LSIS UMR 7296, 13397, Marseille, France
nicola.olivetti@univ-amu.fr
[2] Dip. di Informatica, Universitá di Torino, 10149 Torino, Italy
pozzato@di.unito.it

Abstract. The logic $\mathbb{V}$ is the basic logic of counterfactuals in the family of Lewis' systems. It is characterized by the whole class of so-called sphere models. We propose a new sequent calculus for this logic. Our calculus takes as primitive Lewis' connective of comparative plausibility $\preceq$: a formula $A \preceq B$ intuitively means that A is at least as plausible as B. Our calculus is standard in the sense that each connective is handled by a finite number of rules with a fixed and finite number of premises. Moreover our calculus is "internal", in the sense that each sequent can be directly translated into a formula of the language. We show that the calculus provides an optimal decision procedure for the logic $\mathbb{V}$.

1 Introduction

In the recent history of conditional logics the work by Lewis [15] has a prominent place (among others [5,18,12,10]). He proposed a formalization of conditional logics in order to represent a kind of hypothetical reasoning (if A were the case then B), that cannot be captured by classical logic with material implication. The original motivation by Lewis was to formalize *counterfactual sentences*, i.e. conditionals of the form "if A were the case then B would be the case", where A is false. But independently of counterfactual reasoning, conditional logics have found an interest also in several fields of artificial intelligence and knowledge representation. Just to mention a few: they have been used to reason about prototypical properties [7] and to model belief change [10,8]. Moreover, conditional logics can provide an axiomatic foundation of nonmonotonic reasoning [4,11], here a conditional $A \Rightarrow B$ is read as "in normal circumstances if A then B". Finally, a kind of (multi)-conditional logics [2,3] have been used to formalize epistemic change in a multi-agent setting and in some kind of epistemic "games", here each conditional operator expresses the "conditional beliefs" of an agent.

In this paper we concentrate on the logic $\mathbb{V}$ of counterfactual reasoning studied by Lewis. This logic is characterized by possible world models structured by a system of spheres. Intuitively, each world is equipped with a set of nested sets

⋆ Corresponding author.

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of worlds: inner sets represent "most plausible worlds" from the point of view of the given world and worlds belonging only to outer sets represent less plausible worlds. In other words, each sphere represent a degree of plausibility. The (rough) intuition involving the truth condition of a counterfactual $A \Rightarrow B$ at a world x is that B is true at the most plausible worlds where A is true, whenever there are worlds satisfying A. But Lewis is reluctant to assume that *most plausible worlds satisfying A exist* (whenever there are A-worlds), for philosophical reasons. He calls this assumption the Limit Assumption and he formulates his semantics in more general terms which do need this assumption (see below). The sphere semantics is the strongest semantics for conditional logics, in the sense that it characterizes only a subset of relatively strong systems; there are weaker (and more abstract) semantics such as the selection function semantics which characterize a wider range of systems [18].

From the point of view of proof-theory and automated deduction, conditional logics do not have a state of the art comparable with, say, the one of modal logics, where there are well-established alternative calculi, whose proof-theoretical and computational properties are well-understood. This is partially due to the lack of a unifying semantics. Similarly to modal logics and other extensions/alternatives to classical logics two types of calculi have been studied: *external* calculi which make use of labels and relations on them to import the semantics into the syntax, and *internal* calculi which stay within the language, so that a "configuration" (sequent, tableaux node...) can be directly interpreted as a formula of the language. Limiting our account to Lewis' counterfactual logics, some external calculi have been proposed in [9] which presents modular labeled calculi for preferential logic PCL and its extensions, including all counterfactual logics considered by Lewis. An external sequent calculus for Lewis' logic $\mathbb{VC}$ is also presented in [17]. Internal calculi have been proposed by Gent [6] and by de Swart [20] for $\mathbb{VC}$ and neighbours. These calculi manipulate sets of formulas and provide a decision procedure, although they comprise an *infinite set of rules and rules with a variable number of premises.* Finally in [14] the authors provide internal calculi for Lewis' conditional logic $\mathbb{V}$ and some extensions. Their calculi are formulated for a language comprising the comparative plausibility connective, the strong and the weak conditional operator. Both conditional operators can be defined in terms of the comparative plausibility connective. These calculi are actually an extension of Gent's and de Swart's ones and they comprise an infinite set of rules with a variable number of premises. We mention also a seminal work by Lamarre [12] who proposed a tableaux calculus for Lewis' logic, but it is actually a model building procedure rather than a calculus made of deductive rules.

In this paper we tackle the problem of providing a standard proof-theory for Lewis' logic $\mathbb{V}$ in the form of internal calculi. By "standard" we mean that we aim to obtain analytic sequent calculi where each connective is handled by a finite number of rules with a fixed and finite number of premises. As a first result, we propose a new internal calculus for Lewis' logic $\mathbb{V}$. This is the most general logic of Lewis' family and it is complete with respect to the whole class of sphere models. Our calculus takes as primitive Lewis' comparative plausibility connective $\preceq$: a formula $A \preceq B$ means, intuitively, that A is at least as plausible as B, so that a conditional $A \Rightarrow B$ can be defined as A is impossible or $A \wedge \neg B$ is less plausible than A[1]. In contrast to previous attempts, our calculus comprises *structured* sequents containing *blocks*, where a block is a new syntactic structure encoding a finite combination of $\preceq$. In other words, we introduce a new modal operator (but still definable in the logic) which encodes finite combinations of $\preceq$. This is the main ingredient to obtaining a standard and internal calculus for $\mathbb{V}$. We show a terminating strategy for proof search in the calculus, in particular that it provides an optimal decision procedure for the logic $\mathbb{V}$: indeed, we show that provability in $\mathcal{I}^{\mathbb{V}}$ is in PSPACE, matching the known complexity bound for the logic $\mathbb{V}$.

2 Lewis' Logic $\mathbb{V}$

We consider a propositional language $\mathcal{L}$ generated from a set of propositional variables *Varprop* and boolean connectives plus two special connectives $\preceq$ (comparative plausibility) and $\Rightarrow$ (conditional). A formula $A \preceq B$ is read as "A is at least as plausible as B". The semantics is defined in terms of sphere models, we take the definition by Lewis without the limit assumption.

Definition 1. *A model $\mathcal{M}$ has the form $\langle W, \$, [\ ]\rangle$, where W is a non-empty set whose elements are called worlds, $[\ ] : Varprop \longrightarrow Pow(W)$ is the propositional evaluation, and $\$: W \longrightarrow Pow(Pow(W))$. We write $\$_x$ for the value of the function $\$$ for $x \in W$, and we denote the elements of $\$_x$ by α, β.... Models have the following property:*

$$\forall \alpha, \beta \in \$_x \ \alpha \subseteq \beta \vee \beta \subseteq \alpha.$$

Truth definitions are the usual ones in the boolean cases; $[\ ]$ is extended to the other connectives as follows:

- $x \in [A \preceq B]$ *iff* $\forall \alpha \in \$_x$ *if* $\alpha \cap [B] \neq \emptyset$ *then* $\alpha \cap [A] \neq \emptyset$
- $x \in [A \Rightarrow B]$ *iff either* $\forall \alpha \in \$_x \ \alpha \cap [A] = \emptyset$ *or there is* $\alpha \in \$_x$*, such that* $\alpha \cap [A] \neq \emptyset$ *and* $\alpha \cap [A \wedge \neg B] = \emptyset$*.*

The semantic notions, satisfiability and validity are defined as usual. For the ease of reading we introduce the following conventions: we write $x \models A$, where the model is understood instead of $x \in [A]$. Moreover given $\alpha \in \$_x$, we use the following notations:

$\alpha \models^{\forall} A$ if $\alpha \subseteq [A]$, i.e. $\forall y \in \alpha \ y \models A$
$\alpha \models^{\exists} A$ if $\alpha \cap [A] \neq \emptyset$, i.e. $\exists y \in \alpha$ such that $y \models A$

Observe that with this notation, the truths conditions for $\preceq$ and $\Rightarrow$ become:

- $x \models A \preceq B$ iff $\forall \alpha \in \$_x$ either $\alpha \models^{\forall} \neg B$ or $\alpha \models^{\exists} A$
- $x \models A \Rightarrow B$ iff $\forall \alpha \in \$_x$ either $\alpha \models^{\forall} \neg A$ or there is $\beta \in \$_x$, such that $\beta \models^{\exists} A$ and $\beta \models^{\forall} A \rightarrow B$.

[1] This definition avoids the Limit Assumption, in the sense that it works also for models where *at least* a sphere containing A worlds does not necessarily exist.

It can be observed that the two connectives $\preceq$ and $\Rightarrow$ are interdefinable, in particular:

$$A \Rightarrow B \equiv (\bot \preceq A) \vee \neg(A \wedge \neg B \preceq A)$$

Also the $\preceq$ connective can be defined in terms of the conditional $\Rightarrow$ as follows:

$$A \preceq B \equiv ((A \vee B) \Rightarrow \bot) \vee \neg((A \vee B) \Rightarrow \neg A)$$

The logic $\mathbb{V}$ can be axiomatized taking as primitive the conditional operator $\Rightarrow$ which gives the axiomatization here below [15]:

- classical axioms and rules
- if $A \leftrightarrow B$ then $(C \Rightarrow A) \leftrightarrow (C \Rightarrow B)$ (RCEC)
- if $A \rightarrow B$ then $(C \Rightarrow A) \rightarrow (C \Rightarrow B)$ (RCK)
- $((A \Rightarrow B) \wedge (A \Rightarrow C)) \rightarrow (A \Rightarrow B \wedge C)$ (AND)
- $A \Rightarrow A$ (ID)
- $((A \Rightarrow B) \wedge (A \Rightarrow C)) \rightarrow (A \wedge B \Rightarrow C)$ (CM)
- $(A \wedge B \Rightarrow C) \rightarrow ((A \Rightarrow B) \rightarrow (A \Rightarrow C))$ (RT) [2]
- $((A \Rightarrow B) \wedge \neg(A \Rightarrow \neg C)) \rightarrow ((A \wedge C) \Rightarrow B)$ (CV)
- $((A \Rightarrow C) \wedge (B \Rightarrow C)) \rightarrow (A \vee B \Rightarrow C)$ (OR)

together with the definition of $\preceq$ in terms of $\Rightarrow$ given above. The flat versions (i.e. without nested conditionals) of these axioms are part of KLM systems of nonmonotonic reasoning [11,13].

On the other hand, we can axiomatize $\mathbb{V}$ taking as primitive the connective $\preceq$ and the axioms are the following [15]:

- classical axioms and rules
- if $B \rightarrow (A_1 \vee \ldots \vee A_n)$ then $(A_1 \preceq B) \vee \ldots \vee (A_n \preceq B)$
- $(A \preceq B) \vee (B \preceq A)$
- $(A \preceq B) \wedge (B \preceq C) \rightarrow (A \preceq C)$
- $A \Rightarrow B \equiv (\bot \preceq A) \vee \neg(A \wedge \neg B \preceq A)$

3 An Internal Sequent Calculus for $\mathbb{V}$

We present $\mathcal{I}^{\mathbb{V}}$, a structured calculus for Lewis' conditional logic introduced in the previous section. The basic constituent of sequents are **blocks** of the form:

$$[A_1, \ldots, A_m \lhd B_1, \ldots, B_n]$$

where A_i, B_j are formulas. The interpretation is as follows: $x \models [A_1, \ldots, A_m \lhd B_1, \ldots, B_n]$ iff $\forall \alpha \in \$_x$ either $\alpha \models^{\forall} \neg B_j$ for some j, or $\alpha \models^{\exists} A_i$ for some i. Observe that

$$[A_1, \ldots, A_m \lhd B_1, \ldots, B_n] \;\leftrightarrow\; \bigvee_i^m \bigvee_j^n (A_i \preceq B_j)$$

[2] It is worth noticing that (CM) + (RT) are equivalent (in CK+ID) to the axiom known as (CSO):

$$((A \Rightarrow B) \;\wedge\; (B \Rightarrow A)) \rightarrow ((A \Rightarrow C) \leftrightarrow (B \Rightarrow C)) \qquad (CSO)$$

Therefore a block represents $n \times m$ disjunctions of $\preceq$ formulas.

We shall abbreviate multi-sets of formulas in blocks by Σ, Π, so that we shall write (since the order is irrelevant) $[\Sigma \lhd \Pi]$, $[\Sigma, A \lhd \Pi]$, $[\Sigma \lhd \Pi, B]$ and so on.

A **sequent** Γ is a multi-set $G_1, \dots G_k$, where each G_i is either a formula or a block. A sequent $\Gamma = G_1, \dots G_k$, is valid if for every model $\mathcal{M} = \langle W, \$, [\ ] \rangle$, for every world $x \in W$, it holds that $x \models G_1 \vee \dots \vee G_k$. The calculus $\mathcal{I}^{\mathbb{V}}$ comprises the following axiom and rules:

- Standard Axioms (given $P \in Varprop$): (i) $\Gamma, \top$ (ii) $\Gamma, \neg\bot$ (iii) $\Gamma, P, \neg P$
- Standard external rules of sequent calculi for boolean connectives
- Specific rules:

$$\frac{\Gamma, [A \lhd B]}{\Gamma, A \preceq B}\ (\preceq +) \qquad \frac{\Gamma, \neg(A \preceq B), [B, \Sigma \lhd \Pi] \quad \Gamma, \neg(A \preceq B), [\Sigma \lhd \Pi, A]}{\Gamma, \neg(A \preceq B), [\Sigma \lhd \Pi]}\ (\preceq -)$$

$$\frac{\Gamma, [\bot \lhd A], \neg(A \wedge \neg B \preceq A)}{\Gamma, A \Rightarrow B}\ (\Rightarrow +) \qquad \frac{\Gamma, \neg(\bot \preceq A) \qquad \Gamma, [A \wedge \neg B \lhd A]}{\Gamma, \neg(A \Rightarrow B)}\ (\Rightarrow -)$$

$$\frac{\Gamma, [\Sigma_1 \lhd \Pi_1, \Pi_2], [\Sigma_1, \Sigma_2 \lhd \Pi_2] \quad \Gamma, [\Sigma_2 \lhd \Pi_1, \Pi_2], [\Sigma_1, \Sigma_2 \lhd \Pi_1]}{\Gamma, [\Sigma_1 \lhd \Pi_1], [\Sigma_2 \lhd \Pi_2]}\ (Com)$$

$$\frac{\neg B_i, \Sigma}{\Gamma, [\Sigma \lhd B_1, \dots, B_n]}\ (Jump)$$

Some remarks on the rules: the rule $(\preceq^+)$ just introduces the block structure, showing that $\lhd$ is a generalization of $\preceq$; $(\preceq^-)$ prescribes case analysis and contributes to expanding the blocks; the rules $(\Rightarrow^+)$ and $(\Rightarrow^-)$ just apply the definition of $\Rightarrow$ in terms of $\preceq$. The *communication* rule (Com) is directly motivated by the *nesting* of spheres, which means a *linear order* on sphere inclusion; this rule is very similar to the homonymous one used in *hypersequent* calculi for handling truth in linearly ordered structures [1,16].

As usual, given a formula $G \in \mathcal{L}$, in order to check whether G is valid we look for a derivation of G in the calculus $\mathcal{I}^{\mathbb{V}}$. Given a sequent Γ, we say that Γ is derivable in $\mathcal{I}^{\mathbb{V}}$ if it admits a *derivation*. A derivation of Γ is a tree where:

- the root is Γ;
- every leaf is an instance of standard axioms;
- every non-leaf node is (an instance of) the conclusion of a rule having (an instance of) the premises of the rule as children.

Here below we show some examples of derivations in $\mathcal{I}^{\mathbb{V}}$.

Example 1. A derivation of $(P \preceq Q) \vee (Q \preceq P)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg P, P}{[P \lhd Q, P], [P, Q \lhd P]}\,(Jump) \qquad \dfrac{\neg Q, Q}{[Q \lhd Q, P], [P, Q \lhd Q]}\,(Jump)}{[P \lhd Q], [Q \lhd P]}\,(Com)}{[P \lhd Q], Q \preceq P}\,(\preceq^+)}{P \preceq Q, Q \preceq P}\,(\preceq^+)}{(P \preceq Q) \vee (Q \preceq P)}\,(\vee^+)$$

Example 2. A derivation of an instance of Lewis' axiom **CV**.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg P, P, \bot}{\ldots, [P, \bot \lhd P]}\,(Jump) \qquad \dfrac{\bot, \neg\bot}{\ldots, [\bot \lhd P, \bot]}\,(Jump)}{(P \wedge Q) \Rightarrow R, \neg(\bot \preceq P), [\bot \lhd P], \neg(P \wedge \neg\neg Q \preceq P)}\,(\preceq^-)}{(P \wedge Q) \Rightarrow R, P \Rightarrow \neg Q, \neg(\bot \preceq P)}\,(\Rightarrow^+) \qquad \overset{\clubsuit}{P \Rightarrow \neg Q, (P \wedge Q) \Rightarrow R, [P \wedge \neg R \lhd P]}}{\neg(P \Rightarrow R), P \Rightarrow \neg Q, (P \wedge Q) \Rightarrow R}\,(\Rightarrow^-)}{\neg(P \Rightarrow R), \neg\neg(P \Rightarrow \neg Q), (P \wedge Q) \Rightarrow R}\,(\neg)}{\neg((P \Rightarrow R) \wedge \neg(P \Rightarrow \neg Q)), (P \wedge Q) \Rightarrow R}\,(\wedge^-)}{((P \Rightarrow R) \wedge \neg(P \Rightarrow \neg Q)) \rightarrow ((P \wedge Q) \Rightarrow R)}\,(\rightarrow^+)$$

where $\clubsuit$ is the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{P, P \wedge \neg R, \neg P}{\ldots, [P, P \wedge \neg R \lhd P]}\,(Jump) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg P, \neg Q, P, P \wedge \neg R \qquad \neg P, \neg Q, Q, P \wedge \neg R}{\neg P, \neg Q, P \wedge Q, P \wedge \neg R}\,(\wedge^+)}{\neg P, \neg\neg(\neg Q), P \wedge Q, P \wedge \neg R}\,(\neg)}{\neg(P \wedge \neg\neg Q), P \wedge Q, P \wedge \neg R}\,(\wedge^-)}{\ldots, [P \wedge Q, P \wedge \neg R \lhd P, P \wedge \neg\neg Q]}\,(Jump) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P, \neg P, \neg Q, R \qquad \neg R, \neg P, \neg Q, R}{P \wedge \neg R, \neg P, \neg Q, R}\,(\wedge^+)}{P \wedge \neg R, \neg(P \wedge Q), R}\,(\wedge^-)}{P \wedge \neg R, \neg(P \wedge Q), \neg\neg R}\,(\neg)}{P \wedge \neg R, \neg(P \wedge Q \wedge \neg R)}\,(\wedge^-)}{\ldots, [P \wedge \neg R \lhd P, P \wedge \neg\neg Q, P \wedge Q \wedge \neg R]}\,(Jump)}{[\bot \lhd P], [\bot \lhd P \wedge Q], [P \wedge \neg R \lhd P, P \wedge \neg\neg Q], \neg(P \wedge Q \wedge \neg R \preceq P \wedge Q), \neg(P \wedge \neg\neg Q \preceq P)}\,(\preceq^-)}{[\bot \lhd P], [P \wedge \neg R \lhd P], [\bot \lhd P \wedge Q], \neg(P \wedge Q \wedge \neg R \preceq P \wedge Q), \neg(P \wedge \neg\neg Q \preceq P)}\,(\preceq^-)}{[\bot \lhd P], \neg(P \wedge \neg\neg Q \preceq P), (P \wedge Q) \Rightarrow R, [P \wedge \neg R \lhd P]}\,(\Rightarrow^+)}{P \Rightarrow \neg Q, (P \wedge Q) \Rightarrow R, [P \wedge \neg R \lhd P]}\,(\Rightarrow^+)$$

We terminate this section by proving the soundness of the calculus $\mathcal{I}^{\mathbb{V}}$ and by stating some standard structural properties of it[3].

Theorem 1 (Soundness). *Given a sequent Γ, if Γ is derivable then it is valid.*

Proof. By induction on the height of derivation. For the base case, we have to consider sequents that are instances of standard axioms. The proof is easy and left to the reader. For the inductive step, we have to consider all the possible

[3] To save space, detailed proofs are given in the accompanying report [19].

rules ending a derivation. We only show the most interesting cases of $(\preceq^-)$ and (Com).
$(\preceq^-)$: the derivation of Γ is ended by an application of $(\preceq^-)$ as follows:

$$\dfrac{(i)\ \Gamma', \neg(A \preceq B), [B, \Sigma \lhd \Pi] \qquad\qquad (ii)\ \Gamma', \neg(A \preceq B), [\Sigma \lhd \Pi, A]}{\Gamma', \neg(A \preceq B), [\Sigma \lhd \Pi]}\ (\preceq^-)$$

By inductive hypothesis, (i) and (ii) are valid sequents. By absurd, suppose that the conclusion is not, that is to say there is a model $\mathcal{M} = \langle W, \$, [\,]\rangle$ and a world $x \in W$ such that (1) $x \not\models G_i$, for all $G_i \in \Gamma'$, (2) $x \not\models \neg(A \preceq B)$ and (3) $x \not\models [\Sigma \lhd \Pi]$. From (1), (2) and the fact that (i) is valid, we conclude that (a) $x \models [B, \Sigma \lhd \Pi]$. Reasoning in the same way, from (1), (2) and the validity of (ii), we conclude that (b) $x \models [\Sigma \lhd \Pi, A]$. By the interpretation of a block, for all $\alpha \in \$_x$, from (a) we have that either $\alpha \models^\forall \neg B_j$ for some $B_j \in \Pi$ or $\alpha \models^\exists A_i$ for some $A_i \in \Sigma$ or $(*)$ $\alpha \models^\exists B$. Similarly, from (b) we have that either $\alpha \models^\forall \neg B_j$ for some $B_j \in \Pi$ or $(**)$ $\alpha \models^\forall \neg A$ or $\alpha \models^\exists A_i$ for some $A_i \in \Sigma$. If $\alpha \models^\forall \neg B_j$ for some $B_j \in \Pi$, then, by the interpretation of a block, we have that $x \models [\Sigma \lhd \Pi]$, and this contradicts (3). For the same reason, it cannot be also the case that $\alpha \models^\exists A_i$ for some $A_i \in \Sigma$. The only case left is when $(*)$ $\alpha \models^\exists B$ and $(**)$ $\alpha \models^\forall \neg A$. This contradicts (2). Indeed, (2) $x \not\models \neg(A \preceq B)$ means that $x \models A \preceq B$, namely, by the truth condition of $\preceq$, for all $\alpha \in \$_x$ we have that either $\alpha \models^\forall \neg B$, and this contradicts (*), or $\alpha \models^\exists A$, and this contradicts (**);
(Com): the derivation of Γ is ended by an application of (Com) as follows:

$$\dfrac{(i)\ \Gamma', [\Sigma_1 \lhd \Pi_1, \Pi_2], [\Sigma_1, \Sigma_2 \lhd \Pi_2] \qquad\qquad (ii)\ \Gamma', [\Sigma_2 \lhd \Pi_1, \Pi_2], [\Sigma_1, \Sigma_2 \lhd \Pi_1]}{\Gamma', [\Sigma_1 \lhd \Pi_1], [\Sigma_2 \lhd \Pi_2]}\ (Com)$$

By inductive hypothesis, (i) and (ii) are valid. Suppose the conclusion $\Gamma', [\Sigma_1 \lhd \Pi_1], [\Sigma_2 \lhd \Pi_2]$ is not, namely there is a model $\mathcal{M} = \langle W, \$, [\,]\rangle$ and a world $x \in W$ such that (1) $x \not\models G_k$ for all $G_k \in \Gamma'$, (2) $x \not\models [\Sigma_1 \lhd \Pi_1]$ and (3) $x \not\models [\Sigma_2 \lhd \Pi_2]$. By the interpretation of blocks, from (2) it follows that there is $\alpha \in \$_x$ such that $\alpha \not\models^\exists A_i$, for all $A_i \in \Sigma_1$, and $\alpha \not\models^\forall \neg B_j$ for all $B_j \in \Pi_1$. Similarly, from (3) it follows that there is $\beta \in \$_x$ such that $\beta \not\models^\exists C_k$, for all $C_k \in \Sigma_2$, and $\beta \not\models^\forall \neg D_l$ for all $D_l \in \Pi_2$. By Definition 1, either (*) $\beta \subseteq \alpha$ or (**) $\alpha \subseteq \beta$. (*) If $\beta \subseteq \alpha$, we have also that $\beta \not\models^\exists A_i$, for all $A_i \in \Sigma_1$, and $\beta \not\models^\forall \neg B_j$ for all $B_j \in \Pi_1$. Let us consider (ii): we have that $\beta \not\models^\exists C_k$, for all $C_k \in \Sigma_2$, as well as $\beta \not\models^\forall \neg B_j$ for all $B_j \in \Pi_1$ and $\beta \not\models^\forall \neg D_l$ for all $D_l \in \Pi_2$: by the definition of interpretation of a block, we have that (4) $x \not\models [\Sigma_2 \lhd \Pi_1, \Pi_2]$. Furthermore, since $\beta \not\models^\exists A_i$, for all $A_i \in \Sigma_1$, $\beta \not\models^\exists C_k$, for all $C_k \in \Sigma_2$ and $\beta \not\models^\forall \neg B_j$ for all $B_j \in \Pi_1$, then we have that (5) $x \not\models [\Sigma_1, \Sigma_2 \lhd \Pi_1]$. However, from (1), (4) and (5) we obtain that (ii) is not valid, against the inductive hypothesis. (**) If $\alpha \subseteq \beta$, we reason analogously. We can observe that also $\alpha \not\models^\exists C_k$, for all $C_k \in \Sigma_2$, and $\alpha \not\models^\forall \neg D_l$ for all $D_l \in \Pi_2$. Therefore, we have that (6) $x \not\models [\Sigma_1 \lhd \Pi_1, \Pi_2]$, since $\alpha \not\models^\exists A_i$, for all $A_i \in \Sigma_1$, $\alpha \not\models^\forall \neg B_j$ for all $B_j \in \Pi_1$ and $\alpha \not\models^\forall \neg D_l$ for all $D_l \in \Pi_2$. Furthermore, (7) $x \not\models [\Sigma_1, \Sigma_2 \lhd \Pi_2]$ since $\alpha \not\models^\exists A_i$, for all $A_i \in \Sigma_1$, $\alpha \not\models^\exists C_k$, for all $C_k \in \Sigma_2$, and $\alpha \not\models^\forall \neg D_l$ for all $D_l \in \Pi_2$. From (1), (6) and (7) we have that (ii) is not valid, again against the inductive hypothesis. ∎

Proposition 1 (Weakening). *Weakening is height-preserving admissible in the following cases: (1) if Γ is derivable, then Γ, F is derivable where F is a formula or a block; (2) if $\Gamma, [\Sigma \lhd \Pi]$ is derivable, so are $\Gamma, [\Sigma, A \lhd \Pi]$ and $\Gamma, [\Sigma \lhd \Pi, B]$.*

Proposition 2 (Contraction). *Contraction is height-preserving admissible in the following cases: (1) if $\Gamma, [A, A, \Sigma \lhd \Pi]$ is derivable then $\Gamma, [A, \Sigma \lhd \Pi]$ is derivable too. (2) if $\Gamma, [\Sigma \lhd \Pi, B, B]$ is derivable then $\Gamma, [\Sigma \lhd \Pi, B]$ is derivable too. (3) if Γ, F, F is derivable then Γ, F is derivable too, where F is either a formula or a block.*

4 Termination and Completeness

In this section we prove both the termination and the completeness of the calculus. Both results make use of the notion of *saturated sequent*: intuitively any sequent that is obtained by backwards applying the rules "as much as possible". To get termination we show that any derivation without *redundant* applications of the rules is finite and its leaves are axioms or saturated sequents. Completeness is proved by induction on the modal degree of a sequent (defined next), by taking advantage of the fact that backward applications of the rules do not increase the modal degree of a sequent and eventually reduce it (the (*Jump*) rule).

Definition 2. *The modal degree md of a formula/sequent is defined as follows:*

$$
\begin{aligned}
&md(P) = 0\\
&md(A * B) = max(md(A), md(B)),\ \textit{for}\ * \in \{\wedge, \vee, \rightarrow\}\\
&md(\neg A) = md(A)\\
&md(A \preceq B) = md(A \Rightarrow B) = max(md(A), md(B)) + 1\\
&md(\Delta) = max\{md(A) \mid A \in \Delta\}\ \textit{for a multi-set}\ \Delta\\
&md([\Sigma \lhd \Pi]) = max(md(\Sigma), md(\Gamma)) + 1
\end{aligned}
$$

We can prove the following propositions:

Proposition 3. *All rules preserve the modal degree, i.e. the premises of rules have a modal degree no greater than the one of the respective conclusion.*

Proposition 4 (Invertibility). *All rules, except (Jump), are height-preserving invertible: if the conclusion is derivable then the premises must be derivable with a derivation of no greater height.*

Definition 3. *A sequent Γ is saturated if it has the form $\Gamma_N, \Lambda, [\Sigma_1 \lhd \Pi_1], \ldots, [\Sigma_n \lhd \Pi_n]$ where Γ_N, Λ are possible empty, $n \geq 0$ and:*

1. *Γ_N is a multi-set of negative $\preceq$-formulas,*
2. *Λ is a multi-set of literals,*
3. *for every $\neg(A \preceq B) \in \Gamma_N$ and every $[\Sigma_i \lhd \Pi_i]$ either $B \in \Sigma_i$ or $A \in \Pi_i$*
4. *for every $[\Sigma_i \lhd \Pi_i]$ and $[\Sigma_j \lhd \Pi_j]$: either $\Sigma_i \subseteq \Sigma_j$ or $\Sigma_j \subseteq \Sigma_i$ and either $\Pi_i \subseteq \Pi_j$ or $\Pi_j \subseteq \Pi_i$.*

We want to prove now that $\mathcal{I}^{\mathbb{V}}$ terminates, provided we restrict attention to *non-redundant* derivations, a notion that we define next. An application of a rule (R) is redundant if the conclusion can be obtained from one of its premises by *contraction* or *weakening*.

A derivation is non-redundant if (a) it does not contain redundant applications of the rules, (b) if a sequent is an axiom then it is a leaf of the derivation. As a consequence of the height-preserving admissibility of contraction (Proposition 2) and of weakening (Proposition 1), if a sequent is derivable then it has a non-redundant derivation. Thus we can safely restrict proof search to non-redundant derivations.

In the search of a non-redundant derivation we can assume that the rule:

$$\frac{\Gamma, [\Sigma_1 \lhd \Pi_1, \Pi_2], [\Sigma_1, \Sigma_2 \lhd \Pi_2] \quad \Gamma, [\Sigma_2 \lhd \Pi_1, \Pi_2], [\Sigma_1, \Sigma_2 \lhd \Pi_1]}{\Gamma, [\Sigma_1 \lhd \Pi_1], [\Sigma_2 \lhd \Pi_2]} \ (Com)$$

is applied provided it satisfies the following restriction, where inclusions are intended as set inclusions:

(RestCom) $\quad (\Sigma_1 \not\subseteq \Sigma_2 \text{ and } \Sigma_2 \not\subseteq \Sigma_1)$ or $(\Pi_1 \not\subseteq \Pi_2 \text{ and } \Pi_2 \not\subseteq \Pi_1)$.

Fact 1 *If an application of* (Com) *is non-redundant, then it must respect the restriction (RestCom).*

Proof. We must check that the 4 cases of violation of (RestCom):

1. $\Sigma_1 \subseteq \Sigma_2$ and $\Pi_1 \subseteq \Pi_2$ 2. $\Sigma_1 \subseteq \Sigma_2$ and $\Pi_2 \subseteq \Pi_1$
3. $\Sigma_2 \subseteq \Sigma_1$ and $\Pi_1 \subseteq \Pi_2$ 4. $\Sigma_2 \subseteq \Sigma_1$ and $\Pi_2 \subseteq \Pi_1$

produce a *redundant* application of (Com).

In cases 2 and 3 the conclusion corresponds to one of the premises. Let us consider case 2 as an example. Assume that $\Sigma_1 \subseteq \Sigma_2$ and $\Pi_2 \subseteq \Pi_1$: the leftmost premise of (Com) is therefore $\Gamma, [\Sigma_1 \lhd \Pi_1, \Pi_2], [\Sigma_1, \Sigma_2 \lhd \Pi_2] = \Gamma, [\Sigma_1 \lhd \Pi_1], [\Sigma_2 \lhd \Pi_2]$ and corresponds to the conclusion. The case 3 is similar and left to the reader.

In cases 1 and 4 both the premises are different from the conclusion, however we observe that the conclusion can be obtained by weakening from one of the premises of an application of (Com), which is therefore redundant. Let us consider the case 1, i.e. $\Sigma_1 \subseteq \Sigma_2$ and $\Pi_1 \subseteq \Pi_2$. Consider also the rightmost premise of (Com), namely $\Gamma, [\Sigma_2 \lhd \Pi_1, \Pi_2], [\Sigma_1, \Sigma_2 \lhd \Pi_1] = (*)\ \Gamma, [\Sigma_2 \lhd \Pi_2], [\Sigma_2 \lhd \Pi_1]$. Since $\Pi_1 \subseteq \Pi_2$, from $(*)$ we obtain that also $(**)\ \Gamma, [\Sigma_2 \lhd \Pi_2], [\Sigma_2 \lhd \Pi_2]$ is derivable by weakening (Proposition 1). Since contraction is admissible, from $(**)$ we obtain a proof of $\Gamma, [\Sigma_2 \lhd \Pi_2]$, from which the conclusion of (Com), namely $\Gamma, [\Sigma_1 \lhd \Pi_1], [\Sigma_2 \lhd \Pi_2]$, can be obtained by weakening. Therefore, an application of (Com) would be redundant, since its rightmost premise allows to obtain the conclusion by weakening and contraction and without (Com). Case 4 is similar and left to the reader. ∎

The proposition below states that for any sequent Γ (derivable or not in the calculus), there is a (non-redundant) derivation tree whose leaves (no matter whether they are derivable or not in the calculus) are saturated sequents with no greater modal degree. In order to prove it, we introduce some complexity measure of sequents. The aim will be to show that each application of a rule decreases this measure. Let Γ be of the form:

$$\Delta, [\Sigma_1 \lhd \Pi_1], \ldots, [\Sigma_n \lhd \Pi_n]$$

- First we define a complexity measure of formulas:
  $Cp(A) = 0$ if A is either a literal or it has the form $\neg(C \preceq D)$,
  $Cp(A) = 1$ if A has one of the forms $C \preceq D, C \Rightarrow D, \neg(C \Rightarrow D)$
  $Cp(\neg\neg A) = Cp(A) + 1$
  $Cp(A * B) = Cp(A) + Cp(B) + 1$, where $*$ is a boolean connective.
  Next we let
  $$CP(\Gamma) = \text{ multi-set } \{Cp(A) \mid A \in \Gamma\}$$
- To take care of the application of $(\preceq^-)$, we define:
  $$CN(\Gamma) = Card(\{(\neg(A \preceq B), [\Sigma \lhd \Pi]) \mid \neg(A \preceq B), [\Sigma \lhd \Pi] \in \Gamma, B \notin \Sigma, A \notin \Pi\})$$
- To take care of the application of (Com), we proceed as follows. First, for a multi-set Λ, we still denote by $Card(\Lambda)$ the cardinality of Λ as a set (or, in other words, of its *support*). Next, given $\Gamma = \Delta, [\Sigma_1 \lhd \Pi_1], \ldots, [\Sigma_n \lhd \Pi_n]$, we let $\Sigma_\Gamma = \bigcup_i \Sigma_i$ and $\Pi_\Gamma = \bigcup_i \Pi_i$ (set-union), we define:
  $$CC(\Gamma) = n * (Card(\Sigma_\Gamma) + Card(\Pi_\Gamma)) - \sum_{i=1}^{n} (Card(\Sigma_i) + Card(\Pi_i))$$
- We finally define the rank of a sequent Γ, $rank(\Gamma)$ as the triple
  $$rank(\Gamma) = \langle CP(\Gamma), CN(\Gamma), CC(\Gamma) \rangle$$
  taken in *lexicographic* order, where we consider the *multi-set ordering* for $CP(\Gamma)$.

Observe that a *minimal* rank has the form $\langle 0^*, 0, m \rangle$, where $m \geq 0$. We are ready to prove the following proposition.

Proposition 5. *Given a sequent Γ, every branch of any derivation-tree starting with Γ eventually ends with a* saturated *sequent with* no greater *modal degree than that of Γ. Moreover the set of such saturated sequents for a given derivation tree is finite.*

Proof. By Proposition 3, no rule applied backward augments the modal degree of a sequent. It can be shown that every (non-redundant) application of a rule (R) with premises Γ_i and conclusion Γ reduces the rank of Γ in the sense that $rank(\Gamma_i) < rank(\Gamma)$. In order to see this, we note:
- the application of classical propositional rule reduces $CP(\Gamma)$
- the application of $(\preceq^+), (\Rightarrow^+), (\Rightarrow^-)$ rules reduces $CP(\Gamma)$

- the application of $(\preceq^-)$ reduces $CN(\Gamma)$, *without increasing* $CP(\Gamma)$
- the application of (Com) reduces $CC(\Gamma)$, *without increasing* neither $CP(\Gamma)$, nor $CN(\Gamma)$. We first show that an application of (Com) rule reduces $CC(\Gamma)$. Let $\Gamma = \Delta, [\Sigma_1 \lhd \Pi_1], [\Sigma_2 \lhd \Pi_2], \ldots, [\Sigma_n \lhd \Pi_n]$. To simplify indexing (since the order does not matter) suppose that the application of (Com) concerns the blocks $[\Sigma_1 \lhd \Pi_1]$, $[\Sigma_2 \lhd \Pi_2]$, so that the premises of the application of (Com) leading to Γ will be:

$$\Gamma_1 = \Delta, [\Sigma_1 \lhd \Pi_1, \Pi_2], [\Sigma_1, \Sigma_2 \lhd \Pi_2], [\Sigma_3 \lhd \Pi_3], \ldots, [\Sigma_n \lhd \Pi_n]$$
$$\Gamma_2 = \Delta, [\Sigma_2 \lhd \Pi_1, \Pi_2], [\Sigma_1, \Sigma_2 \lhd \Pi_1], [\Sigma_3 \lhd \Pi_3], \ldots, [\Sigma_n \lhd \Pi_n]$$

  Observe that the overall set of formulas in blocks does not change so that, referring to the above notation:

$$\Sigma_{\Gamma_i} = \Sigma_\Gamma \text{ and } \Pi_{\Gamma_i} = \Pi_\Gamma, \qquad \text{for } i = 1, 2$$

  Let us abbreviate $a = n*(Card(\Sigma_\Gamma)+Card(\Pi_\Gamma))$ and $c = \sum_{i=3}^{n}(Card(\Sigma_i)+Card(\Pi_i))$, so that we have:

$$CC(\Gamma) = a - ((Card(\Sigma_1) + Card(\Pi_1)) + (Card(\Sigma_2) + Card(\Pi_2)) + c)$$
$$CC(\Gamma_1) = a - ((Card(\Sigma_1) + Card(\Pi_1 \cup \Pi_2)) + (Card(\Sigma_1 \cup \Sigma_2) + Card(\Pi_2)) + c)$$
$$CC(\Gamma_2) = a - ((Card(\Sigma_2) + Card(\Pi_1 \cup \Pi_2)) + (Card(\Sigma_1 \cup \Sigma_2) + Card(\Pi_1)) + c)$$

  Obviously $CC(\Gamma_1) \leq CC(\Gamma)$ and $CC(\Gamma_2) \leq CC(\Gamma)$, since $Card(\Sigma_1 \cup \Sigma_2) \geq Card(\Sigma_i)$ and $Card(\Pi_1 \cup \Pi_2) \geq Card(\Pi_i)$, $i = 1, 2$. But since the application of (Com) is non-redundant, it respects the restriction (RestCom) and therefore either (a) $\Sigma_1 \not\subseteq \Sigma_2$ and $\Sigma_2 \not\subseteq \Sigma_1$ or (b) $\Pi_1 \not\subseteq \Pi_2$ and $\Pi_2 \not\subseteq \Pi_1$. Thus some of the inequalities are strict. In case (a) we get that $Card(\Sigma_1 \cup \Sigma_2) > Card(\Sigma_i)$, $i = 1, 2$, thus $(Card(\Sigma_1 \cup \Sigma_2) + Card(\Pi_2)) > (Card(\Sigma_2) + Card(\Pi_2))$ whence $CC(\Gamma_1) < CC(\Gamma)$ and $(Card(\Sigma_1 \cup \Sigma_2) + Card(\Pi_1)) > (Card(\Sigma_1) + Card(\Pi_1))$, whence $CC(\Gamma_2) < CC(\Gamma)$. In case (b) we get that $Card(\Pi_1 \cup \Pi_2) > Card(\Pi_i)$, $i = 1, 2$, thus $(Card(\Sigma_1) + Card(\Pi_1 \cup \Pi_2)) > (Card(\Sigma_1) + Card(\Pi_1))$, whence $CC(\Gamma_1) < CC(\Gamma)$ and $(Card(\Sigma_2) + Card(\Pi_1 \cup \Pi_2)) > (Card(\Sigma_2) + Card(\Pi_2))$, whence $CC(\Gamma_2) < CC(\Gamma)$.

  We now show the second claim, that an application of (Com) does not increase $CN(\Gamma)$: let Γ_1 be the leftmost premise of (Com), and $\neg(A \preceq B) \in \Gamma_1$ and consider for instance $[\Sigma_1 \lhd \Pi_1, \Pi_2]$. If $B \not\in \Pi_1, \Pi_2$ and $A \not\in \Sigma_1$, obviously also $B \not\in \Pi_1$ and since $\neg(A \preceq B) \in \Gamma$ the pair $(\neg(A \preceq B), [\Sigma_1 \lhd \Pi_1])$ will contribute to $CN(\Gamma)$; a similar reasoning applies to the block $[\Sigma_1, \Sigma_2 \lhd \Pi_2]$. Hence we get that $CN(\Gamma_1) \leq CN(\Gamma)$. The same argument applies to the rightmost premise.

Thus each branch of every derivation with root Γ has a finite length and ends with a saturated sequent. Since the derivation is *finitary* (each rule has at most two premises) it is also finite, thus the set of saturated sequents as leaves is finite. This ends the proof. ■

The following theorem shows that the calculus is terminating, whence it provides a decision procedure for $\mathbb{V}$, assuming restriction to non-redundant derivations.

Proposition 6. *Given a sequent Γ, any non-redundant derivation-tree of Γ is finite.*

Proof. By induction on the modal degree m of Γ. If $m = 0$ then we rely on the corresponding property of classical sequent calculus. If $m > 0$, by the previous Proposition 5, Γ has a finite derivation tree ending with a set of saturated sequents Γ_i. For each Γ_i either it is an axiom and Γ_i will be a leaf of the derivation, or the only applicable rule (by non-redundancy restriction) is (*Jump*), but the premise of (*Jump*) has a smaller modal degree and we apply the induction hypothesis to the premise of (*Jump*). ∎

The above proposition means that for any sequent Γ (derivable or not in the calculus), there is a derivation tree whose leaves (no matter whether they are derivable or not in the calculus) are saturated sequents with no greater modal degree.

The termination result can be strengthened in order to show that the calculus $\mathcal{I}^{\mathbb{V}}$ can be used to describe an optimal decision procedure for $\mathbb{V}$, provided we adopt a specific strategy on the application of the rules. The strategy is the following:

1. apply propositional rules and $(\preceq^+)$, $(\Rightarrow^+)$ and $(\Rightarrow^-)$ as much as possible;
2. apply $(\preceq^-)$ as much as possible;
3. apply (*Com*) as much as possible with the restriction (RestCom).

If the last sequent so obtained is not an instance of standard axioms, then it is saturated: we can then apply the rule (*Jump*) and then restart from 1. The completeness of the strategy is justified by the following proposition:

Proposition 7. *The rule (Com) permutes over all the other rules, except (Jump).*

We are now ready to prove the following theorem.

Theorem 2. *Provability in $\mathcal{I}^{\mathbb{V}}$ is in* PSPACE.

Proof. Let n be the length of the string representing a sequent Δ. Given any derivation tree built starting with Δ, we show that the length of each branch is polynomial in n, and that the size of each sequent occurring in it is polynomial in n. We proceed by induction on the modal degree of Δ. For the base case, $md(\Delta) = 0$, that is to say all formulas in Δ are propositional formulas. In this case we immediately conclude since the above claims hold for the propositional calculus. For the inductive step, we apply the rules of the calculus $\mathcal{I}^{\mathbb{V}}$ to build any branch **B** until the last sequent of **B** is an axiom or a saturated sequent. According to the above strategy, **B** is built as follows:

- first, propositional rules and $(\preceq^+)$, $(\Rightarrow^+)$, and $(\Rightarrow^-)$ are applied as much as possible: since the number of connectives in F is bounded by n, the number

of applications of these rules is $O(n)$. Since all the rules are analytic, the size of each sequent is $O(n)$ (see comments below concerning the application of the $(\Rightarrow^-)$ rule);
- then, the rule $(\preceq^-)$ is applied as much as possible, by considering all combinations of blocks and formulas $\neg(A \preceq B)$: since all possible blocks are $O(n)$ and all possible formulas $\neg(A \preceq B)$ are $O(n)$, the number of applications of the rule $(\preceq^-)$ is $O(n^2)$ and, again, the size of each sequent is polynomial in n;
- the rule (Com) is applied as much as possible with the restriction (RestCom): as already shown in the proof of Proposition 5, the number of applications of (Com) is bounded by the measure $CC(\Gamma) = n * (Card(\Sigma_\Gamma) + Card(\Pi_\Gamma)) - \sum_{i=1}^{n}(Card(\Sigma_i) + Card(\Pi_i))$, and is therefore $O(n^2)$.

We conclude that **B** has length polynomial in n and contains sequents whose sizes are polynomial in n. The last sequent of **B** is either (i) an instance of a standard axiom or (ii) saturated. In case (i), we are done. In case (ii), the rule $(Jump)$ is the only applicable one: let Γ be the sequent of **B** to which we apply (backward) the rule $(Jump)$, and let Γ' be its premise. Since $md(\Gamma') < md(\Gamma)$, we can apply the inductive hypothesis, to conclude that any branch **B'** built in the derivation starting with Γ' is polynomial in n and that each sequent in it has a polynomial size in n: this immediately follows from the facts that Γ belongs to the derivation tree having Δ as a root (therefore, its size is polynomial in n) and that Γ' is a subsequent of Γ, then its size is polynomial in n, too. It is worth noticing that this also holds when the rule $(\Rightarrow^-)$ is considered: let Δ contain $\neg(A_1 \Rightarrow B_1), \neg(A_2 \Rightarrow B_2), \ldots, \neg(A_k \Rightarrow B_k)$, in the worst case a branch contains a block of the form $[A_1 \wedge \neg B_1, A_2 \wedge \neg B_2, \ldots, A_k \wedge \neg B_k, \Sigma \lhd A_1, A_2, \ldots, A_k, \Pi]$, whose size could be higher than the one of Δ. However, an application of $(Jump)$ would lead to a premise, in the worst case, of the form $A_1 \wedge \neg B_1, A_2 \wedge \neg B_2, \ldots, A_k \wedge \neg B_k, \Sigma, \neg A_i$, and backward applications of $(\wedge^-)$ to formulas $A_1 \wedge \neg B_1, A_2 \wedge \neg B_2, \ldots, A_k \wedge \neg B_k$ would obviously lead to sequents whose size is strictly lower than the one of Δ.

We can conclude that the length of the branch obtained by concatenating **B** and **B'** is polynomial in n and each sequent in it has a polynomial size in n, and we are done.

In order to prove that a formula F is valid, we try to build a derivation in $\mathcal{I}^{\mathbb{V}}$ having F as a root. Let n be the length of the string representing F. By the argument shown above, given any derivation tree built starting with F, we have that the length of each branch is polynomial in n, and the size of each sequent occurring in it is polynomial in n, and this concludes the proof. ∎

The following proposition is the last ingredient we need for the completeness proof.

Proposition 8 (Semantic Invertibility). *All rules, except* $(Jump)$ *are semantically invertible: if the conclusion is valid then the premises are also valid.*

Theorem 3 (Completeness of the Calculus $\mathcal{I}^{\mathbb{V}}$). *If* Γ *is valid then it is derivable.*

Proof. By induction on the modal degree of Γ. If $md(\Gamma) = 0$ then Γ is just a multi-set of propositional formulas, and we rely on the completeness of sequent calculus for classical logic. Suppose now that $md(\Gamma) > 0$, by Proposition 5, Γ can be derived from a set of saturated sequents Γ_i of no greater modal degree. But by Proposition 8 (semantic invertibility) since Γ is valid then also each Γ_i is valid. We are left to prove that any *saturated and valid* sequent Γ_i is derivable. To this purpose we prove that if Γ_i is valid then either (i) it is an axiom or (ii) there must exist a *valid* sequent Δ such that Γ_i is obtained by (*Jump*) from Δ. In the first case (i) the result is obvious. In case (ii) we reason as follows: since $md(\Delta) < md(\Gamma_i)$ by inductive hypothesis, Δ is derivable in $\mathcal{I}^{\mathbb{V}}$, and so is Γ_i indeed by the (*Jump*) rule.

Let us prove that if Γ_i is valid and saturated and it is not an axiom, then there exists a *valid* sequent Δ such that Γ_i is obtained by (*Jump*) from Δ. Suppose that Γ_i is valid and it is not an axiom. We let $\Gamma_i = \Gamma_N, \Lambda, [\Sigma_1 \triangleleft \Pi_1], \dots, [\Sigma_n \triangleleft \Pi_n]$ as in Definition 3. Observe that Λ does not contain axioms. By saturation (and weakening and contraction) we can assume that the blocks in the sequence are ordered as follows:

- $\Sigma_1 \supseteq \Sigma_2 \supseteq \dots \Sigma_n$
- $\Pi_1 \subseteq \Pi_2 \subseteq \dots \subseteq \Pi_n$

A quick argument: by saturation blocks are ordered with respect to set-inclusion for both components Σ and Π, consider them ordered first by decreasing Σ: let two blocks in the sequence: $[\Sigma \triangleleft \Pi], [\Sigma' \triangleleft \Pi']$ with $\Sigma' \subseteq \Sigma$, we can assume that $\Pi \subseteq \Pi'$ otherwise it would be $\Pi' \subset \Pi$, but then any sequent containing **both** $[\Sigma \triangleleft \Pi]$ and $[\Sigma' \triangleleft \Pi']$ is semantically equivalent to a sequent containing **only** $[\Sigma \triangleleft \Pi]$ (syntactically we get rid of $[\Sigma' \triangleleft \Pi']$ by weakening and contraction)[4]. Thus we let:

$$\begin{aligned}
&\Pi_1 = B_{1,1}, \dots, B_{1,k_1}\\
&\Pi_2 = B_{1,1}, \dots, B_{1,k_1}, B_{2,1}, \dots, B_{2,k_2}\\
&\dots\\
&\Pi_n = B_{1,1}, \dots, B_{1,k_1}, \dots, B_{2,k_2}, \dots, B_{n,k_n}
\end{aligned}$$

Suppose now for a contradiction that no application of (*Jump*) leads to a valid sequent. Thus for each $l = 1, \dots, n$, and $t = 1, \dots, k_l$, the sequent $\neg B_{l,t}, \Sigma_l$ is not valid. Starting from $l = 1$ up to n, there are increasing sequences of models:

$$\begin{aligned}
&\mathcal{M}_{1,1}, \dots, \mathcal{M}_{1,k_1},\\
&\mathcal{M}_{1,1}, \dots, \mathcal{M}_{1,k_1}, \mathcal{M}_{2,1}, \dots, \mathcal{M}_{2,k_2}\\
&\mathcal{M}_{1,1}, \dots, \mathcal{M}_{1,k_1}, \dots, \mathcal{M}_{2,k_2}, \dots, \mathcal{M}_{n,k_n}
\end{aligned}$$

where $\mathcal{M}_{l,t} = (W_{l,t}, \$^{l,t}, [\ \ ]_{l,t})$ for $l = 1, \dots, n$, and $t = 1, \dots, k_l$ and some elements $x_{l,t} \in W_{l,t}$ such that $\mathcal{M}_{l,t}, x_{l,t} \models B_{l,t}$ and $\mathcal{M}_{l,t}, x_{l,t} \not\models C$ for all $C \in \Sigma_l$. Observe that if $\mathcal{M}_{l,t}, x_{l,t} \not\models C$ for all $C \in \Sigma_s$ and $s < t$ then $\mathcal{M}_{l,t}, x_{l,t} \not\models$

[4] An alternative argument: Γ_i must contain a *valid* subsequent Γ_i' where the blocks satisfy the above ordering conditions. Then the proof carry on considering Γ_i'.

C for all $C \in \Sigma_t$, as $\Sigma_t \subseteq \Sigma_s$. We suppose that all models are disjoint and we define a new model $\mathcal{M} = \langle \mathcal{W}, \$, [\ \ ] \rangle$ as follows:

$\mathcal{W} = (\bigcup_l \bigcup_t (\mathcal{W}_{l,t})) \cup \{x\}$ for a new element x $\qquad [P] = \bigcup_l \bigcup_t [P]_{l,t}$ if $\neg P \notin \Lambda$
$\$_z = \$_z^{l,t}$ if $z \in \mathcal{W}_{l,t}$ for some l, t $\qquad [P] = \bigcup_l \bigcup_t [P]_{l,t} \cup \{x\}$ if $\neg P \in \Lambda$

In order to define the evaluation function [ ] we let:

$$\begin{aligned}
&\alpha_1 = \{x_{1,1}, \ldots, x_{1,k_1}\} \\
&\alpha_2 = \{x_{1,1}, \ldots, x_{1,k_1}, x_{2,1}, \ldots, x_{2,k_2}\} \\
&\ldots \\
&\alpha_n = \{x_1, \ldots, x_{1,k_1}, x_{2,1}, \ldots, x_{2,k_2}, \ldots, x_{n,k_n}\}
\end{aligned}$$

We finally let $\$_x = \{\alpha_1, \ldots, \alpha_n\}$. Observe that the "spheres" α_l are nested. To complete the proof we must show that x falsifies Γ_i in $\mathcal{M}$. In particular we have to show that:

(1) $\mathcal{M}, x \not\models L$ for every $L \in \Lambda$
(2) $\mathcal{M}, x \not\models [\Sigma_l \lhd \Pi_l]$ for $l = 1, \ldots, n$
(3) $\mathcal{M}, x \not\models \neg(A \preceq B)$ for every $\neg(A \preceq B) \in \Gamma_N$

(1) is obvious by definition: if $P \in \Lambda$, then $\neg P \notin \Lambda$ (otherwise Γ_i would be an axiom) and $x \notin [P]$, if $\neg P \in \Lambda$, then $x \in [P]$.

To prove (2), first observe that for $z \in \mathcal{W}_{l,t}$ and every formula F, we have $z \in [F]$ if and only if $z \in [F]_{l,t}$. This is proved by a straightforward induction on F. Then we prove (2) by induction on l. For $l = 1$, we have that, for $x_{1,l} \in \alpha_1$, it holds $\mathcal{M}, x_{1,l} \models B_{1,l}$, whence $\alpha_1 \not\models^\forall \neg B_{1,t}$ for $t = 1, \ldots, k_1$. On the other hand, putting $\Sigma_1 = C_{1,1}, \ldots, C_{1,r_1}$, we have, for every $u = 1, \ldots, r_1$ and $x_{1,t}$, $t = 1, \ldots, k_1$, that $\mathcal{M}, x_{1,t} \not\models C_{1,u}$, but this means that $\alpha_1 \not\models^\exists C_{1,u}$ for $u = 1, \ldots, r_1$. Thus we get $\mathcal{M}, x \not\models [\Sigma_1 \lhd \Pi_1]$. For $l > 1$, since $\Sigma_l \supseteq \Sigma_{l-1}$ and $\Pi_{l-1} \subseteq \Pi_l$, the argument is the same (using possibly the induction hypothesis).

We consider now (3): let $\neg(A \preceq B) \in \Gamma_N$ and let $\alpha_l \in \$_x$. Let us consider $[\Sigma_l \lhd \Pi_l]$, by saturation either $A \in \Pi_l$ or $B \in \Sigma_l$. For what we have just shown, in the former case we have $\alpha_l \models^\exists A$ and in the latter case we have $\alpha_l \models^\forall \neg B$. Thus, for any $\alpha_l \in \$_x$, either $\alpha_l \models^\exists A$ or $\alpha_l \models^\forall \neg B$, whence $\mathcal{M}, x \models A \preceq B$. ■

5 Further Research

In future research, we aim at extending our approach to all the other conditional logics of the Lewis' family, in particular we aim at focusing on the logics $\mathbb{VN}$, $\mathbb{VT}$, $\mathbb{VW}$ and $\mathbb{VC}$. Actually, for $\mathbb{VN}$, whose sphere models are known as *normal* ($\$_x \neq \emptyset$), the extension is straightforward: it is sufficient to add to the calculus $\mathcal{I}^{\mathbb{V}}$ the following rule:

$$\frac{\Gamma, [\bot \lhd \top]}{\Gamma}\ (N)$$

Observe that the flat version (i.e. without nested conditionals) of $\mathbb{VN}$ is exactly rational logic **R** presented in [13]. Thus, as far as we know, our calculus provides the first internal calculus for **R**. The other cases are currently under investigation.

In [14], ingenious and optimal sequent calculi for the whole family of Lewis' logics are proposed. The calculus for $\mathbb{V}$ contains an infinite set of rules $R_{n,m}$ (with $n \geq 1, m \geq 0$) with a variable number of premises:

$$\frac{\{\neg B_k, A_1, \ldots, A_n, D_1, \ldots, D_m \mid k \leq n\} \ \cup \ \{\neg C_k, A_1, \ldots, A_n, D_1, \ldots, D_{k-1} \mid k \leq m\}}{\Gamma, \neg(C_1 \preceq D_1), \ldots, \neg(C_m \preceq D_m), A_1 \preceq B_1, \ldots, A_n \preceq B_n} R_{n,m}$$

We wish to study the precise relation between our calculus $\mathcal{I}^{\mathbb{V}}$ and the one introduced in [14]. As an example, we show that, in the case $n = 1$ and $m = 1$, the rule

$$\frac{\neg B_1, A_1, D_1 \qquad\qquad \neg C_1, A_1}{\Gamma, \neg(C_1 \preceq D_1), A_1 \preceq B_1} R_{1,1}$$

is derivable in $\mathcal{I}^{\mathbb{V}}$ as follows:

$$\dfrac{\dfrac{\dfrac{\neg B_1, A_1, D_1}{\Gamma, \neg(C_1 \preceq D_1), [A_1, D_1 \lhd B_1]}\,(Jump) \qquad \dfrac{\neg C_1, A_1}{\Gamma, \neg(C_1 \preceq D_1), [A_1 \lhd B_1, C_1]}\,(Jump)}{\Gamma, \neg(C_1 \preceq D_1), [A_1 \lhd B_1]}\,(\preceq^-)}{\Gamma, \neg(C_1 \preceq D_1), A_1 \preceq B_1}\,(\preceq^+)$$

We conjecture that all instances $R_{n,m}$, $(n \geq 1, m \geq 0)$, are derivable in $\mathcal{I}^{\mathbb{V}}$: this will be subject of further investigation.

Last, in future research we shall provide an efficient implementation of $\mathcal{I}^{\mathbb{V}}$.

6 Conclusions

In this paper we begin a proof-theoretical investigation of Lewis' logics of counterfactuals characterized by the sphere-model semantics. We have presented a simple, analytic calculus $\mathcal{I}^{\mathbb{V}}$ for logic $\mathbb{V}$, the most general logic characterized by the sphere-model semantics. The calculus $\mathcal{I}^{\mathbb{V}}$ is *standard*, namely it contains a finite a number of rules with a fixed number of premises, and *internal*, in the sense that each sequent denotes a formula of $\mathbb{V}$. The novel ingredient of $\mathcal{I}^{\mathbb{V}}$ is that sequents are structured objects containing blocks, where a block is a structure or a sort of n-ary modality encoding a finite combination of formulas with the connective $\preceq$. $\mathcal{I}^{\mathbb{V}}$ ensures termination, in particular we have shown that provability is in PSpace, therefore it provides an optimal decision procedure for $\mathbb{V}$.

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