

# Financial Leverage for Multi-Period Levered Investments

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## Abstract

Return on Equity (*ROE*) is one of the most popular performance metric related to equity involved in one-period investment. According to the Modigliani and Miller leverage formula, that applies to one-period levered investments, if the rate of Return on Investment (*ROI*) is not less than the Rate of Debt (*ROD*) then external financing increases *ROE*. The aim of this paper is to extend the Modigliani and Miller leverage formula to multi-period appraisals. First, we define the multi-period *ROE* as the Equity cash-flow Internal Rate of Return. Then, we achieve sufficient and necessary conditions to guaranteeing that external financing has beneficial effects on *ROE*. If projects are financed by up-front funds, as it is typical in long-term project financing, the Modigliani and Miller leverage formula can be extended.

**Keywords:** Capital Structure, Financial Leverage, Project finance, Multi-period Return on Equity

## 1 Introduction

The effects of debt on financial indices have long been a central question in numerous disciplines, including corporate finance, engineering economy and, in general, financial risk management (see [3] and [6] and the literature hereafter). In this paper we focus on the effects of financial leverage on one of the most popular measure of profitability of internal capital, the so called Return on Equity (*ROE*). Seminally [7] argue that if the Return on Investment (*ROI*) exceeds the Rate (or cost) of the Debt (*ROD*) then financial leverage has beneficial impact on *ROE* (see also [2]). Since typically long-term investments cover multi-period financing, we wonder whether the leverage rule still applies for multi-period appraisals. First, we extend the notion of *ROE* of multi-period appraisals as the Internal Rate of Return (*IRR*) of the Equity Cash-Flow. Sufficient and necessary conditions for leverage to increase *ROE* are obtained. In contrast to single-period projects, to detect the leverage impact on *ROE*, more complex conditions are to be checked. The augmented complexity stems from the presence of possible multiple *ROEs* at the same leverage level (for a seminal discussion on the *IRR* non-uniqueness, see [8]; and [5]). To guaranteeing the existence of a single *ROE* at any leverage level, we restrict our analysis to levered projects characterized by up-front financings. Typically that occurs in project financing when internal and external capitals are invested in one lump sum. In these circumstances the Modigliani and Miller leverage formula can be extended to multi-period valuations. Due the relevance of leverage in trade-off theory (see [10] and literature hereafter for trade-off theory) our results shed lights to further applications in the theory of capital structure.

The remainder of the paper is organized as follows. In Section 2 we introduce the basic notation. In Section 3 we derive sufficient and necessary conditions under which *ROE* increases with leverage. Section 4 concludes.

## 2 Notation setup

The impact of debt on *ROE* has been analysed extensively in the corporate finance literature. The *leverage formula* can be traced back to [7] seminal work and states that if  $ROI > ROD$ , an augmented financial leverage increases *ROE*. To extend the analysis in multi-period prospective, basic notation is set up.

Let consider an economic agent (i.e. a firm) facing the opportunity of investing in a financial project *A* that promises at time  $t_s$ , with  $s = 0, \dots, n$  the cash-flow  $a_s$ , with the usual convention that  $a_s < 0$  means that at time  $t_s$  there is a money outflow, while  $a_s > 0$  a money inflow and  $a_s = 0$  no cash movement.

For simplicity but without loss of generality, we assume to deal with a single project. Let  $t_0 = 0$ . The initial unitary borrowing  $f_0 = 1$  of a debt project  $D$  asks for payments  $f_s \leq 0$  at times  $t_s$  where  $s = 1, \dots, n$ . The initial outlay of project  $A$  is  $a_0 = -1$ . The project is  $\alpha \cdot 100\%$  debt financed and  $(1 - \alpha) \cdot 100\%$  equity financed with  $0 \leq \alpha \leq 1$ . If  $\alpha = 0$  project is all equity financed; whereas if  $\alpha = 1$  it is all debt financed.

The *debt financing stream* reads  $[\alpha, \alpha f_1, \dots, \alpha f_n]$ . The *equity financing* at initial time 0 is  $e_0 = a_0 + \alpha f_0 = -1 + \alpha = -(1 - \alpha)$ . So, the *equity cash flow* generated by the project at time  $t_s$  is

$$e_s = a_s + \alpha f_s, \text{ for } s = 0, \dots, n.$$

By definition *ROI*, *ROE* and *ROD* are Internal Rate of Return (*IRR*) of the correspondent cash-flow. Denoting the *Discounted Cash-Flow* by *DCF*:

- *ROI* is an *IRR* of the *project cash-flow*  $A$ , so it is a solution of the equation
 
$$DCF_A(x) = a_0 + a_1(1+x)^{-t_1} + \dots + a_n(1+x)^{-t_n} = 0$$
- *ROD* is an *IRR* of the *unitary debt cash-flow*  $D$ , so it is a solution of the equation
 
$$DCF_D(x) = f_0 + f_1(1+x)^{-t_1} + \dots + f_n(1+x)^{-t_n} = 0.$$
- $ROE_\alpha$  is an *IRR* of the *equity cash-flow* generated by the  $\alpha \cdot 100\%$  debt financed project  $D$ , is a solution of the equation

$$\begin{aligned} DCF_{A+\alpha D}(x, \alpha) &= \\ &= (a_0 + \alpha f_0) + (a_1 + \alpha f_1)(1+x)^{-t_1} + \dots + (a_n + \alpha f_n)(1+x)^{-t_n} = 0 \end{aligned} \quad (1)$$

### 3 From Single vs. Multi-Period Project Appraisals

To get insights into the formula (1), we first analyze single-period projects. By definition  $ROE_\alpha$  is the solution of the equation (1), that takes for  $n = 1$  the form  $(a_0 + \alpha f_0) + (a_1 + \alpha f_1)(1+x)^{-t_1} = 0$  with  $0 \leq \alpha \leq 1$ . Due to the technical assumption  $-a_0 = f_0 = 1$ , we get

$$ROE_\alpha = \left( \frac{a_1 + \alpha f_1}{1 - \alpha} \right)^{\frac{1}{t_1}} - 1. \quad (2)$$

Plain remarks follow.  $ROE_\alpha$  exists and is positive if:

- $\alpha \neq 1$ , i.e. the project is equity financed in positive percentage  $(1-\alpha) \cdot 100\%$  ;
- $a_1 + \alpha f_1 > 1 - \alpha$ , with  $0 \leq \alpha < 1$ . That simply means that project must generate a positive return, i.e. the final revenue  $a_1 + \alpha f_1$  is not only positive but it must exceed the equity  $(1-\alpha)$  invested at time 0.

Substituting  $a_0 = -1$ ,  $a_1 = (1+ROI)^{t_1}$ ,  $f_0 = 1$  and  $f_1 = -(1+ROD)^{t_1}$  into (2), we get

$$ROE_\alpha = \left( \frac{(1+ROI)^{t_1} - \alpha(1+ROD)^{t_1}}{1-\alpha} \right)^{\frac{1}{t_1}} - 1, \text{ with } 0 \leq \alpha < 1. \quad (3)$$

We are ready now to provide sound foundations to the classical single-period leverage formula (see [7]). Let  $ROI = (a_1)^{\frac{1}{t_1}} - 1$ , with  $a_1 \geq 1$  and  $ROD = (-f_1)^{\frac{1}{t_1}} - 1$ , with  $-f_1 \geq 1$  so that project A does not destroy value (i.e.  $ROI \geq 0$ ) and debt is financially meaningful (i.e.  $ROD \geq 0$ ).

**Theorem 1.** *Let us consider a single-period project, with  $ROI \geq 0$ . The project is  $\alpha \cdot 100\%$  debt financed, with  $ROD \geq 0$ , and  $(1-\alpha) \cdot 100\%$  equity financed with  $0 \leq \alpha < 1$ . Then  $ROE_\alpha$  is nonnegative and increasing at  $\alpha$  with  $0 \leq \alpha < 1$  if and only if  $ROI \geq ROD$ .*

**Proof** (see the Appendix).

Let us now generalize previous results to multi-period levered projects. Multi-period ROE is referred to as an Internal Rate of Return of equity cash-flow (see [1]). Sufficient and necessary conditions for leverage to increase  $ROE$  are given.

**Theorem 2.** Let  $a_s$  be the cash-flow at time  $t_s$ , for  $s=0, \dots, n$ , of a multi-period project. The project is initially  $\alpha \cdot 100\%$  debt financed and  $(1-\alpha) \cdot 100\%$  equity financed, with  $0 \leq \alpha < 1$ . Let  $\alpha f_s$ ,  $s=0, \dots, n$  denote the debt cash-flow. Then,  $x = ROE_\alpha$  is increasing with respect to  $\alpha$  if and only if

$$\frac{1 + f_1(1+x)^{-t_1} + \dots + f_n(1+x)^{-t_n}}{t_1(a_1 + \alpha f_1)(1+x)^{-t_1} + \dots + t_n(a_n + \alpha f_n)(1+x)^{-t_n}}(1+x) \geq 0, \quad (4)$$

where  $x$  is a solution of the equation

$$g(x, \alpha) = -1 + \alpha + (a_1 + \alpha f_1)(1+x)^{-t_1} + \dots + (a_n + \alpha f_n)(1+x)^{-t_n} = 0.$$

**Proof** (see the Appendix).

In contrast to single-period context, the impact of leverage on  $ROE$  no longer depends on the  $ROI$  and  $ROD$  spread only. The augmented complexity is imputable to the definition of  $IRR$  as a solution of an equation of  $n$ -th order.  $IRR$  drawbacks in this respect have been seminally pinpointed by [8] and long debated in academia. Recently, [5] sums up “eighteen fallacies” of  $IRR$ ; the most relevant in our context concerns the presence of multiple  $IRR$  (for a critical analysis for a practitioner audience, see [9])

### 3.1 Norström (1972)’s conditions

To eliminate possible  $IRR$  inconsistencies and guarantee the existence of a single  $ROI, ROE_\alpha, ROD$  at any leverage level  $\alpha$ , we confine our analysis to levered projects that apply [8] conditions. So, we restrict our analysis to levered projects for which the sequence of non-discounted cash flow changes sign only once. To state it differently, we consider levered projects such that outflows come before inflows, i.e.  $e_0 = -1 + \alpha < 0$ ,  $e_s = a_s + \alpha f_s \geq 0$ , for  $s = 1, \dots, n-1$  and  $e_n = a_n + \alpha f_n > 0$ . In addition, to deal with meaningful financial indices, we concern unlevered projects with  $ROI \geq 0$  and financings with  $ROD \geq 0$ .

The single-period leverage rule described in Theorem 1 is plainly extended to multi-period appraisals.

**Theorem 3.** *Let  $a_s$  be the cash-flow at time  $t_s$ ,  $s = 0, \dots, n$ , of a multi-period project. The initial outflow  $a_0 = -1$  is  $\alpha \cdot 100\%$  debt financed and  $(1-\alpha) \cdot 100\%$  equity financed, with  $0 \leq \alpha < 1$ . Let exist a single  $ROI$  and  $ROD$  with  $ROI, ROD \geq 0$  and  $e_0 = -1 + \alpha < 0$ ,  $e_s = a_s + \alpha f_s \geq 0$ , for  $s = 1, \dots, n-1$  and  $e_n = a_n + \alpha f_n > 0$ . Then  $ROE_\alpha$  is nonnegative and increasing with respect to  $\alpha$  if and only if  $ROE_\alpha \geq ROD$  for any  $0 \leq \alpha < 1$ .*

**Proof** (see the Appendix).

**Corollary 1.** If  $\alpha = 0$ , then  $ROI = ROE_{\alpha=0} \geq ROD$ .

Corollary 1 highlights that whenever investment is fully equity financed (i.e.  $\alpha = 0$ ) the classical Modigliani and Miller condition  $ROI \geq ROD$  comes out.

## 4 Conclusion

In this short note we extend the Modigliani and Miller financial leverage rule to multi-period appraisals. Sufficient and necessary conditions guaranteeing that financial leverage increases  $ROE$ , referred as equity IRR, are stated. To avoid the undesirable occurrence of multiple  $ROEs$ , we restrict our focus on levered projects characterized by one lump internal and external capitals as it is typical to occur in long-term project financing. In these circumstances, the Modigliani and Miller single-period leverage rule – if  $ROI$  is not less than  $ROD$ , leverage arises  $ROE$  – is extended to multi-period appraisals.

## Appendix

**Proof of Theorem 1** Let assume that  $ROI \geq ROD \geq 0$ , by (3) it results that  $ROE_\alpha \geq 0$ . To detect  $ROE_\alpha$  monotonicity, let calculate

$$\frac{dROE_\alpha}{d\alpha} = \frac{1}{t_1} \left( \frac{(1+ROI)^{t_1} - \alpha(1+ROD)^{t_1}}{1-\alpha} \right)^{\frac{1}{t_1}-1} \frac{(1+ROI)^{t_1} - (1+ROD)^{t_1}}{(1-\alpha)^2}$$

with  $0 \leq \alpha < 1$ . Since  $ROI \geq ROD \geq 0$ , it follows that  $dROE_\alpha/d\alpha \geq 0$ . Therefore  $ROE_\alpha$  is nonnegative and increasing at any leverage level  $\alpha$ , with  $0 \leq \alpha < 1$ .

Now, let  $ROE_\alpha$  be nonnegative and increasing at any leverage level  $\alpha$ , with  $0 \leq \alpha < 1$ . Then  $dROE_\alpha/d\alpha \geq 0$  and consequently,  $ROI \geq ROD$ . Q.E.D.

**Proof of Theorem 2** The sketch of the proof follows the seminal path suggested in [4]. By definition,  $ROE_\alpha$  is a solution of the equation  $g(x, \alpha) = DCF_{A+\alpha D}(x, \alpha) = DCF_A(x) + \alpha DCF_D(x) = 0$ . Since  $g$  has continuous partial derivatives, Dini's Implicit Function Theorem provides the derivative of  $x = ROE_\alpha$  with respect to  $\alpha$ :

$$\begin{aligned} \frac{dx}{d\alpha} &= -\frac{\partial g / \partial \alpha}{\partial g / \partial x} = \frac{1 + f_1(1+x)^{-t_1} + \dots + f_n(1+x)^{-t_n}}{t_1(a_1 + \alpha f_1)(1+x)^{-t_1-1} + \dots + t_n(a_n + \alpha f_n)(1+x)^{-t_n-1}} = \\ &= \frac{1 + f_1(1+x)^{-t_1} + \dots + f_n(1+x)^{-t_n}}{t_1(a_1 + \alpha f_1)(1+x)^{-t_1} + \dots + t_n(a_n + \alpha f_n)(1+x)^{-t_n}} (1+x) \end{aligned}$$

It follows that univariate function  $x = ROE_\alpha$  has derivative  $dx/d\alpha \geq 0$ , if and only if  $x = ROE_\alpha$  is increasing with respect to  $\alpha$ . Q.E.D.

### Proof of Theorem 3

Let  $ROE_\alpha$  is nonnegative and increasing with respect to  $\alpha$ .

By Theorem 2, it results

$$\frac{dROE_\alpha}{d\alpha} = \frac{DCF_D(ROE_\alpha)}{\sum_{s=1}^n t_s (a_1 + \alpha f_s)(1 + ROE_\alpha)^{-t_s}} (1 + ROE_\alpha) \geq 0 \quad (5)$$

for all  $0 \leq \alpha < 1$ .

Due to Norström (1972)'s conditions, if a cash-flow displays one change in sign, the equation  $DCF(x) = 0$  admits a unique solution  $x \geq -1$ . So, for any  $0 \leq \alpha < 1$  there exists a unique  $ROE_\alpha$ . As a consequence,  $(1 + ROE_\alpha) > 0$  in (5). Since also the denominator of (5) is positive, it follows that  $DCF_D(ROE_\alpha) \geq 0$ . It is easy to prove that  $DCF_D(x) = 1 + f_1(1+x)^{-t_1} + \dots + f_n(1+x)^{-t_n}$  is a strictly increasing function with  $DCF_D(ROD) = 0$ . Since  $DCF_D(ROE_\alpha) \geq 0$ , it follows that  $ROE_\alpha \geq ROD$  for any  $0 \leq \alpha < 1$ .

Now, let  $ROE_\alpha \geq ROD$  with  $ROD \geq 0$ , for all  $0 \leq \alpha < 1$ .

Due to the assumptions,  $ROE_\alpha \geq 0$ . Since  $DCF_D$  is a strictly increasing function with  $DCF_D(ROD) = 0$ , it follows that  $dROE_\alpha/d\alpha \geq 0$ . The function  $ROE_\alpha$  results nonnegative and increasing with respect to  $\alpha$ . Q.E.D.

**Acknowledgements.** We would like to thank Lorenzo Peccati for his helpful comments. The usual disclaimer applies.

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**Received: April 11, 2017; Published: May 17, 2017**