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Surface approximation of basins of attraction through RBF interpolation schemes

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Abstract

In this paper we present the problem of determining the so-called *basins of attraction* of dynamical systems. They are found out so that there exist manifolds partitioning the phase space into different regions. The reconstruction of such surfaces is carried out by means of meshfree interpolation tools and specifically we apply the Partition of Unity (PU) method with local Radial Basis Function (RBF) interpolants.

Key words: scattered data approximation, partition of unity method, radial basis functions, dynamical systems, competition population models, basins of attraction
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1 Introduction

Over the last years the topic of numerical approximation of multivariate data has gained popularity in various disciplines, such as numerical solution of PDEs, image registration, neural networks, optimization, statistics and finance. In what follows we investigate an application to population dynamics [1].

Nowadays, mathematical modeling is commonly applied to major disciplines and by these models the prediction of the temporal evolution of the considered quantities, i.e. populations, cancer, divorces, is sought [9]. This is obtained in general via dynamical systems. Here we present a reliable algorithm for the reconstruction of unknown manifolds partitioning the phase state of dynamical systems into disjoint sets. Indeed, in an initial value problem, involving a set of ordinary differential equations, a particular solution of the system is completely determined by the Initial Condition (IC). Depending on the initial state of the system and on conditions involving the model parameters, the trajectories may in fact tend towards different equilibria.

The phase state of the dynamical system is thus partitioned into different regions, called the basins of attraction of each equilibrium, depending on where the trajectories originating in them will ultimately stabilize. In such cases, the final outcome of a mathematical model depends on the IC. If it lies in the basin of attraction of a certain equilibrium point, the system will finally settle to this specific steady state. To establish the ultimate system behavior, it is therefore important to assess for each possible attractor its domain of attraction.

Thus, we present a tool that allows to reconstruct the basin of attraction of each equilibrium, providing a graphical representation of the separatrix manifold [4, 5]. First, a suitable scheme is constructed for the generation of these manifolds. It provides points that, within a certain tolerance, lie on these sought manifolds. This is obtained via a suitable bisection-like routine that employs pairs of points belonging to two different sets of the partition. Then, since an attraction basin can be described by an implicit equation, we interpolate such points with the implicit PU method using local RBF approximants [6, 8, 10].

The model we consider throughout this paper, involving three populations P, Q and R, reads as follows

$$\begin{aligned} \frac{dP}{dt} &= p\left(1-\frac{P}{u}\right)P - aPQ - bPR, \\ \frac{dQ}{dt} &= q\left(1-\frac{Q}{v}\right)Q - cPQ - eQR, \\ \frac{dR}{dt} &= r\left(1-\frac{R}{w}\right)R - fPR - gQR, \end{aligned} \tag{1}$$

where p, q and r are the growth rates of P, Q and R, respectively, a, b, c, e, f and g are the competition rates, u, v and w are the carrying capacities of the three populations. The model describes the interaction of three competing populations within the same environment and has eight equilibria. For simplicity, we list here only those playing a role in this investigation, i.e. the origin $E_0 = (0, 0, 0)$, the points associated with the survival of only one population

$$E_1 = (u, 0, 0), \qquad E_2 = (0, v, 0), \qquad E_3 = (0, 0, w)$$

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and the coexistence equilibrium

$$E_* = \left(\frac{u[p(gvwe - qr) - avr(we - q) - bwq(vg - r)]}{p(gvwe - qr) + uva(rc - fwe) + uwb(fq - gcv)}, \frac{v[q(fuwb - pr) - rcu(wb - p) - pew(fu - r)]}{q(fuwb - pr) + cuv(ra - gwb) + evw(gp - afu)}, \frac{r[(cuva - pq) - gpv(cu - q) - ufq(va - p)]}{r(cuva - pq) + bwu(fq - vcg) + evw(gp - fua)} \right).$$

With the parameter setting $p = 1$, $q = 2$, $r = 2$, $a = 5$, $b = 4$, $c = 3$, $e = 7$, $f = 7$, $g = 10$, $u = 3$, $v = 2$, $w = 1$, the points associated with the survival of only one population are stable, the origin E_0 is an unstable equilibrium and the coexistence equilibrium E_* is a saddle point. The manifolds that partition the phase space into the different domains of attraction intersect only at the coexistence saddle point E_*.

2 Implicit RBF-PU approximation

In this section we present the method used to approximate the basins of attraction, i.e. the separatrix manifolds. Since they are often described by implicit surfaces, we consider the implicit PU approximation using locally radial kelnels or RBFs [2]. Such surfaces are defined by a point cloud data set $X_N = \{\boldsymbol{x}_i \in \mathbb{R}^3, \, i = 1, \dots, N\}$, which belongs to a surface in $\mathbb{R}^3$.

2.1 Radial kernels

Generally, to recover a function $f : \Omega \to \mathbb{R}$ on a bounded domain $\Omega \subset \mathbb{R}^3$, we use a set of samples of f on N pairwise distinct data points or nodes $X_N \subset \Omega$, i.e. $\boldsymbol{f} = [f_1, \dots, f_N]^T$, with $f_i = f(\boldsymbol{x}_i)$, $\boldsymbol{x}_i \in X_N$. For this aim, we define a positive definite and symmetric kernel $\Phi : \Omega \times \Omega \to \mathbb{R}$, obtaining the interpolant expressed as follows

$$u(\boldsymbol{x}) = \sum_{j=1}^{N} c_j \Phi(\boldsymbol{x}, \boldsymbol{x}_j), \qquad \boldsymbol{x} \in \Omega. \tag{2}$$

Here Φ is a radial kernel depending on a positive *shape parameter* ε for all $\boldsymbol{x}, \boldsymbol{z} \in \Omega$, i.e.

$$\Phi(\boldsymbol{x}, \boldsymbol{z}) = \phi_\varepsilon(||\boldsymbol{x} - \boldsymbol{z}||_2) = \phi(\varepsilon ||\boldsymbol{x} - \boldsymbol{z}||_2),$$

where $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ defines a radial basis function. Table 1 shows a list of a few strictly positive definite radial kernels with their smoothness degrees [10].

The coefficients $\boldsymbol{c} = [c_1, \dots, c_N]^T$ in (2) are found by solving the system of linear equations

$$A\boldsymbol{c} = \boldsymbol{f}, \tag{3}$$

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RBF	$\phi_\varepsilon(r)$
Inverse MultiQuadric C^∞ (IMQ)	$(1+\varepsilon^2 r^2)^{-1/2}$
Matérn C^6 (M6)	$\mathrm{e}^{-\varepsilon r}(\varepsilon^3 r^3 + 6\varepsilon^2 r^2 + 15\varepsilon r + 15)$
Matérn C^4 (M4)	$\mathrm{e}^{-\varepsilon r}(\varepsilon^2 r^2 + 3\varepsilon r + 3)$
Wendland C^6 (W6)	$(1-\varepsilon r)_+^8\,(32\varepsilon^3 r^3 + 25\varepsilon^2 r^2 + 8\varepsilon r + 1)$
Wendland C^4 (W4)	$(1-\varepsilon r)_+^6\,(35\varepsilon^2 r^2 + 18\varepsilon r + 3)$

Table 1: Some examples of strictly positive definite radial kernels in $\mathbb{R}^3$.

where entries of the interpolation matrix are given by

$$A_{ij} = \Phi(\boldsymbol{x}_i, \boldsymbol{x}_j), \quad i, j = 1, \ldots, N.$$

The resulting solution u is a function of the *native Hilbert space* $\mathcal{N}_\Phi(\Omega)$ uniquely associated with the kernel, and, if $f \in \mathcal{N}_\Phi(\Omega)$, it is in particular the $\mathcal{N}_\Phi(\Omega)$-projection of f into the subspace

$$\mathcal{N}_\Phi(X_N) = \mathrm{span}\{\Phi(\boldsymbol{x}, \boldsymbol{x}_j), \boldsymbol{x}_j \in X_N\}$$

spanned by the standard basis of translates $\mathcal{T}_{X_N} = \{\Phi(\boldsymbol{x}, \boldsymbol{x}_j), 1 \leq j \leq N\}$, see [6, Ch. 14].

However, since in some situations the matrix A in (3) might turn out to be very ill-conditioned and, accordingly, the interpolant (2) unstable, many efforts have recently been made to derive more stable bases (see [7] for an overview).

2.2 Implicit PU interpolation

In order to determine the implicit PU interpolant, we consider additional interpolation conditions taking an extra set of off-surface points. Hence we generate the extra off-surface points referring to a small step away along the surface normals $\boldsymbol{n}_i$. This enables us to obtain for each node $\boldsymbol{x}_i$ two additional off-surface points: the former defined as $\boldsymbol{x}_{N+i} = \boldsymbol{x}_i + \delta \boldsymbol{n}_i$ is located outside the surface, the latter expressed in the form $\boldsymbol{x}_{2N+i} = \boldsymbol{x}_i - \delta \boldsymbol{n}_i$ is set inside, δ being the stepsize [6, Ch. 30].

After creating the data point set, we can construct the PU interpolant, whose zero contour or iso-surface interpolates three sets, i.e., X_N, $X_\delta^+ = \{\boldsymbol{x}_{N+1}, \ldots, \boldsymbol{x}_{2N}\}$ and $X_\delta^- = \{\boldsymbol{x}_{2N+1}, \ldots, \boldsymbol{x}_{3N}\}$.

The idea of the PU method is to decompose a (usually) large problem or domain $\Omega \subseteq \mathbb{R}^3$ into d small problems or subdomains Ω_j such that $\Omega \subseteq \bigcup_{j=1}^d \Omega_j$ with some mild overlap among the subdomains. Associated with these subdomains we construct a partition of

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unity, i.e. a family of compactly supported, non-negative, continuous functions w_j with $\mathrm{supp}(w_j) \subseteq \Omega_j$ such that

$$\sum_{j=1}^{d} w_j(\boldsymbol{x}) = 1, \qquad \boldsymbol{x} \in \Omega.$$

Therefore the global approximant may be expressed as follows

$$P(\boldsymbol{x}) = \sum_{j=1}^{d} u_j(\boldsymbol{x}) w_j(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega, \tag{4}$$

where $w_j : \Omega_j \to \mathbb{R}$ defines the *Shepard weight*

$$w_j(\boldsymbol{x}) = \frac{\varphi_j(\boldsymbol{x})}{\sum_{k=1}^{d} \varphi_k(\boldsymbol{x})},$$

φ_j being the compactly supported Wendland C^{2k}, $k \geq 1$ functions [10]. The local RBF interpolant $u_j : \Omega_j \to \mathbb{R}$ in (4) assumes the form

$$u_j(\boldsymbol{x}) = \sum_{i=1}^{N_j} c_i^j \phi(||\boldsymbol{x} - \boldsymbol{x}_i^j||_2),$$

where N_j indicates the number of data points in Ω_j, i.e. the points $\boldsymbol{x}_i^j \in X_j = X_N \cap \Omega_j$.

The PU approach is thus a simple and effective computational method that allows us to decompose a large problem into many small subproblems, ensuring that the accuracy obtained for the local fits is carried over to the global one (for further details see [3, 10]).

3 Numerical experiments

In this numerical section we show how the implicit PU method can be applied to reconstruct the domains of attraction of dynamical systems presenting three stable equilibria.

Specifically, using the routine given in [5], we can approximate the basins of attraction of the system (1). In fact, considering n equispaced points on each edge of the cube $[0, \gamma]^3$, with $\gamma \in \mathbb{R}^+$, we can define a set of ICs, i.e.

$$\begin{array}{lll} P^1_{i_1,i_2} = (x_{i_1}, y_{i_2}, 0) \quad \text{and} \quad P^2_{i_1,i_2} = (x_{i_1}, y_{i_2}, \gamma), & i_1, i_2 = 1, \ldots, n, \\ P^3_{i_1,i_2} = (x_{i_1}, 0, z_{i_2}) \quad \text{and} \quad P^4_{i_1,i_2} = (x_{i_1}, \gamma, z_{i_2}), & i_1, i_2 = 1, \ldots, n, \\ P^5_{i_1,i_2} = (0, y_{i_1}, z_{i_2}) \quad \text{and} \quad P^6_{i_1,i_2} = (\gamma, y_{i_1}, z_{i_2}), & i_1, i_2 = 1, \ldots, n. \end{array}$$

Taking then such points in pairs and applying a bisection algorithm, we find a certain number of separatrix points that lie on the basins of attraction. As an example, the points

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lying on the separatrix manifolds depicted in Figure 1 have been found assuming $n = 15$ and $\gamma = 6$.

In Figure 1 we report the plot of the three surfaces showing the domains of attraction. The latter have been approximated by using the method (4) described in Section 2. Such surfaces have been obtained by taking $d = 4$ subdomains and the W6 function in Table 1 with $\varepsilon = 0.1$. In this case the parameters used are $p = 1$, $q = 2$, $r = 2$, $a = 5$, $b = 4$, $c = 3$, $e = 7$, $f = 7$, $g = 10$, $u = 3$, $v = 2$ and $w = 1$.

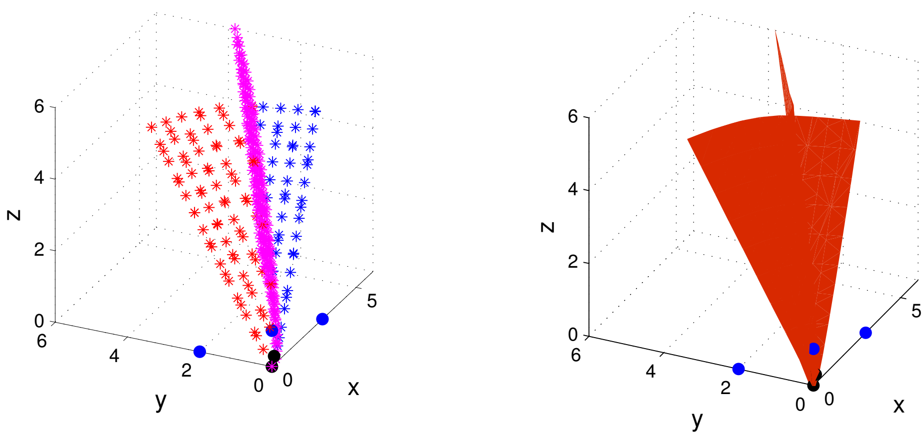

Figure 1: Detection of points on the surfaces determining the basins of attraction (left) and approximation of the domains of E_1, E_2 and E_3 (right). Stable equilibria are marked by a blue dot, unstable saddle points E_0 and E_* are represented by a black dot.

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