FORCING THE TRUTH OF A WEAK FORM OF SCHANUEL'S CONJECTURE

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Abstract. Schanuel's conjecture states that the transcendence degree over $\mathbb{Q}$ of the $2n$-tuple $(\lambda_1, \dots, \lambda_n, e^{\lambda_1}, \dots, e^{\lambda_n})$ is at least n for all $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ which are linearly independent over $\mathbb{Q}$; if true it would settle a great number of elementary open problems in number theory, among which the transcendence of e over π.

Wilkie [11], and Kirby [4, Theorem 1.2] have proved that there exists a smallest countable algebraically and exponentially closed subfield K of $\mathbb{C}$ such that Schanuel's conjecture holds relative to K (i.e. modulo the trivial counterexamples, $\mathbb{Q}$ can be replaced by K in the statement of Schanuel's conjecture). We prove a slightly weaker result (i.e. that there exists such a countable field K without specifying that there is a smallest such) using the forcing method and Shoenfield's absoluteness theorem.

This result suggests that forcing can be a useful tool to prove theorems (rather than independence results) and to tackle problems in domains which are apparently quite far apart from set theory.
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A brief introduction

We want to give an example of how we might use forcing to study a variety of expansions of the complex (or real) numbers enriched by *arbitrary* Borel predicates, still maintaining certain "tameness" properties of the theory of these expansions. We clarify what we intend by "tameness" as follows: in contrast with what happens for example with o-minimality in the case of real closed fields, we do not have to bother much with the complexity of the predicate P we wish to add to the real numbers (we can allow P to be an arbitrary Borel predicate), but we pay a price reducing significantly the variety of elementary superstructures (M, P_M) for which we are able to lift P to P_M so that $(\mathbb{R}, P) \prec (M, P_M)$ and for which we are able to use the forcing method to say something significant on the first order theory of (M, P_M). Nonetheless the family of superstructures M for which this is possible is still a large class, as we can combine (Woodin and) Shoenfield's absoluteness for the theory of projective sets of reals with a duality theorem relating certain spaces of functions to forcing constructions, to obtain the following[1]:

Theorem 1 (V. and Vaccaro [10])**.** *Let X be an extremally disconnected (i.e. such that the closure of open sets is open) compact Hausdorff space.*

Let $C^+(X)$ be the space of continuous functions $f : X \to \mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$ such that the preimage of ∞ is nowhere dense ($\mathbb{S}^2$ is the one point compactification of $\mathbb{C}$).

For any $p \in X$, let $C^+(X)/p$ be the ring of germs in p of functions in $C^+(X)$. Given any *Borel predicate R on $\mathbb{C}^n$, define a predicate $R_X/p \subseteq (C^+(X)/p)^n$ by*

[1]Theorem 1 generalizes results obtained by Jech [3] and Ozawa [8], we refer to [10] for further details on the relations between Theorem 1 and their works.

the rule $R_X/p([f_1],\dots[f_n])$ holds if there is an open neighboorhood U of p such that $R(f_1(x),\dots,f_n(x))$ holds for a comeager[2] *set of $x \in U$. Then*[3]

$$(\mathbb{C}, R) \prec_{\Sigma_2} (C^+(X)/p, R_X/p).$$

Moreover if we assume the existence of class many Woodin cardinals we get that

$$(\mathbb{C}, R) \prec (C^+(X)/p, R_X/p).$$

It turns out that the above spaces of functions are intrinsically intertwined with the forcing method: they provide an equivalent description of the forcing names for complex numbers for the notion of forcing given by the non-empty clopen subsets of X. Moreover these spaces are universal among the spaces of the form $C^+(Y)$ with Y compact Hausdorff, in the sense that for any such Y there is an isometric $*$-homomorphism of the unital C^*-algebra $C(Y)$ into a a C^*-algebra of the form $C(X)$ with X compact and extremely disconnected; this homomorphism extends to a $*$-monomorphism of the ring $C^+(Y)$ into the ring $C^+(X)$ (we refer the reader to [9, Chapter 4] for more details).

Playing with the choice of the compact space X and of the Borel predicate R we can cook up spaces in which it is possible to compute the solution of certain projective statements. Using the elementarity of these structures with respect to the standard complex numbers, we can conclude that the solution we computed in these expansions is the correct solution. This is exactly what we plan to do in the following for a weakening of the well known Schanuel's conjecture.

1. Main result

For a vector $\vec{v} = (v_1,\dots,v_n)$ and a function E we let $\vec{v}(c) = (v_1(c),\dots,v_n(c))$ if each v_i is a function and c is in the domain. $E(\vec{v}) = (E(v_1),\dots,E(v_n))$ if each v_i is in the domain of E. We also feel free (unless we feel this can generate misunderstandings) to confuse a vector $\vec{v} = (v_1,\dots,v_n)$ with the finite set of its elements $\{v_1,\dots,v_n\}$.

Definition 1.1. Given integral domains $Z \subseteq K \subseteq F$ with K, F fields of characteristic 0, let $\bar{Z}$ denote the field of fractions of Z. Fix $\{\lambda_1,\dots,\lambda_n\} \subseteq F$. Then:

- $\mathrm{Ldim}_{\bar{Z}}(\lambda_1,\dots,\lambda_n)$ denotes the $\bar{Z}$-linear dimension of the $\bar{Z}$-subspace V of F spanned by $\{\lambda_1,\dots,\lambda_n\}$.
- $\mathrm{Ldim}_{\bar{Z}}(\lambda_1,\dots,\lambda_n/K)$ is the $\bar{Z}$-codimension of K in the $\bar{Z}$-vector space $K+V$.
- $\mathrm{Trdg}_K(\lambda_1,\dots,\lambda_n)$ is the transcendence degree over K of the ring $K[\lambda_1,\dots,\lambda_n] \subset F$, i.e. the largest size of a subset A of $\{\lambda_1,\dots,\lambda_n\}$ such that no polynomial with coefficients in K and $|A|$-many variables vanishes on the elements of the subset.
- Let $E : F \to F^*$ be an homomorphism of the additive group $(F, +)$ on the multiplicative group $(F^*, \cdot)$. Let

  $$Z(F,E) = \{a \in F : \forall x(E(x) = 1 \to E(ax) = 1)\}.$$

  Then $Z(F,E) \subseteq F$ is a ring.

[2] $A \subseteq X$ is meager if it is the union of countably many nowhere dense sets. A is comeager in U if $U \setminus A$ is meager.

[3] Hidden in the conclusion of the theorem is the statement that R_X/p is a well defined relation for each $p \in X$.

- Assume $Z(F,E) \subseteq K \subseteq F$, and let $\bar{Z}$ denote the field of fractions of $Z(F,E)$. The *Ax character* of the pair (E,K) is the function:

$$\mathsf{AC}_{E,K}(\vec{\lambda}) = \mathrm{Trdg}_K(\vec{\lambda}, E(\vec{\lambda})) - \mathrm{Ldim}_{\bar{Z}}(\vec{\lambda}/K).$$

1.1. **Exponential fields.** We introduce axioms suitable to formulate our results on the exponential function relative to some algebraically closed field K. Since we will have to interplay between boolean valued semantics and standard Tarski semantics, and there are subtle points in the evaluation of function symbols in boolean valued models we do not want to address in the present paper, we overcome this problem assuming from now on that we are working always with relational first order languages. In particular when formally representing a function on a structure as the extension of a definable set, we will always assume that the formula defining the function does not contain any function symbol.

Definition 1.2. Consider a relational language for algebraically closed fields augmented by predicate symbols for an exponential map E, and for a special sub-field K.

$$(F, K, E, \cdot, +, 0, 1)$$

is a model of $T_{\mathsf{WSP}(K)}$ if it satisfies[4]:

(1) **AC FIELD:** F is an algebraically closed field of characteristic 0.
(2) **EXP FIELD:** The exponential map $E : F \to F^*$ is a surjective homomorphism of the additive group $(F,+)$ on the multiplicative group $(F^*, \cdot)$ with

$$\ker(E) = \omega \cdot Z(F,E) = \{\omega \cdot \lambda : \lambda \in Z(F,E)\}$$

for some $\omega \in F$ transcendental over $Z(F,E)$ ([5, Axioms $2'a$, $2'b$, Section 1.2]).
(3) **(K,E)-SP (Schanuel property for (K,E)):** $K \subseteq F$ is a field containing $Z(F,E)$ and $\mathsf{AC}_{E,K} : F^{<\omega} \to \mathbb{N}$ cannot get negative values and is 0 only on tuples contained in K.

$(\mathbb{Q}, \exp)$-SP is a strengthening of Schanuel's conjecture: *Assume $(\mathbb{Q}, \exp)$-SP holds and $\lambda_1, \dots, \lambda_n = \vec{\lambda}$ are $\mathbb{Q}$-linearly independent. Then either $\lambda_1, \dots, \lambda_{n-1}$ are $\mathbb{Q}$-linearly independent modulo $\mathbb{Q}$ or $\lambda_2, \dots, \lambda_n$ are $\mathbb{Q}$-linearly independent modulo $\mathbb{Q}$: Otherwise there are $s, r \in \mathbb{Q}$ and $s_1, \dots, s_{n-1}, r_2, \dots, r_n \in \mathbb{Q}$ such that $s = \sum_{i=1,\dots,n-1} s_i\lambda_i$ and $r = \sum_{i=2,\dots,n} r_i\lambda_i$. Then*

$$s \cdot r = \sum_{i=1,\dots,n-1} r \cdot s_i\lambda_i = \sum_{i=2,\dots,n} s \cdot r_i\lambda_i.$$

[4]The axioms we introduce are mostly taken from [5, Section 1.2], specifically axiom (2) corresponds to axioms $2'a$ and $2'b$ of [5, Section 1.2], we do not insist on the axiom $2'c$, while axiom (3) is a variation of the axiom $3'$ of [5, Section 1.2]. In order to be fully consistent with their axiomatization the Ax character in axiom (3) should be replaced by the "predimension" function $\Delta(\vec{\lambda}) = \mathrm{Trdg}_{Z(F,E)}(\vec{\lambda}, E(\vec{\lambda})/K) - \mathrm{Ldim}_{Z(F,E)}(\vec{\lambda}/K)$. Nonetheless the fields $K \subseteq F$ we will look at are such that $Z(F,E) \cup \ker(E) \subseteq K$ and it can be checked that for these fields $\Delta(\vec{\lambda}) \geq \mathsf{AC}_{E,K}(\vec{\lambda})$. In our analysis we will focus on the properties of the function $\mathsf{AC}_{E,K}$.

This shows that $\vec{\lambda}$ is not a vector of $\mathbb{Q}$-linearly independent numbers. Assume now that $\lambda_1, \dots, \lambda_{n-1}$ are $\mathbb{Q}$-linearly independent modulo $\mathbb{Q}$. By $(\mathbb{Q}, \exp)$-SP we get that

$$\begin{aligned}&\mathrm{Trdg}_{\mathbb{Q}}(\vec{\lambda}, e^{\vec{\lambda}}) \geq \\ &\geq \mathrm{Trdg}_{\mathbb{Q}}(\lambda_1, \dots, \lambda_{n-1}, e^{\lambda_1}, \dots, e^{\lambda_{n-1}}) = \\ &= \mathsf{AC}_{\mathbb{Q},\exp}(\lambda_1, \dots, \lambda_{n-1}) + \mathrm{Ldim}_{\mathbb{Q}}(\lambda_1, \dots, \lambda_{n-1}/\mathbb{Q}) > \\ &> \mathrm{Ldim}_{\mathbb{Q}}(\lambda_1, \dots, \lambda_{n-1}/\mathbb{Q}) = n-1,\end{aligned}$$

and we are done.

An exponential field is a pair (F, E) satisfying the field axioms and axiom (2).

Zilber [12] showed that there is a natural axiom system T_{Zilber} expanding $T_{\mathsf{WSP}(\mathbb{Q})}$ and axiomatizable in the logic $L_{\omega_1,\omega}(Q)$ (where Q stands for the quantifier for uncountably many elements) such that for each uncountable cardinal κ there is exactly one field $\mathbb{B}$ and one exponential function $E : \mathbb{B} \to \mathbb{B}^*$ with $\ker(E) = \omega \cdot \mathbb{Z}$ for some $\omega \in \mathbb{B}$ transcendental over $\mathbb{Q}$ and such that $(\mathbb{B}, \mathbb{Q}, E, +, 0, 1)$ is a standard model of T_{Zilber}. Roughly T_{Zilber} extends $T_{\mathsf{WSP}(\mathbb{Q})}$ requiring that Axiom (3) is replaced by the full Schanuel's conjecture[5] stating that

$$\mathrm{Trdg}_{\mathbb{Q}}(\vec{\lambda}, E(\vec{\lambda})) \geq \mathrm{Ldim}_{\mathbb{Q}}(\vec{\lambda}) \text{ for all } \vec{\lambda}.$$

Furthermore T_{Zilber} requires two other sorts of axioms requiring the existence of generic points for certain kind of irreducible varieties (the so called normal or rotund varieties) and specifying further properties of these generic points (see MR2102856 (2006a:03051) for a short account of the axiom system). However in the present paper we are not interested in this other part of Zilber's axiomatization of the theory of exponential fields. Zilber conjectures that $(\mathbb{C}, \mathbb{Q}, e^x, +, \cdot)$ is a model of T_{Zilber}.

We give a proof based on forcing and generic absoluteness of the following theorem:

Theorem 1.3 (Kirby [4], Wilkie [11])**.** *There exists a* countable *(algebraically and exponentially closed) field $K_0 \subseteq \mathbb{C}$ such that $(\mathbb{C}, K_0, e^x, +, \cdot)$ is a model of $T_{\mathsf{WSP}(K_0)}$.*

Essentially what we have to prove is the following:

> There exists a *countable* (algebraically and exponentially closed) field $K_0 \subseteq \mathbb{C}$ such that
> $$\mathsf{AC}_{K_0,\exp}(\vec{\lambda}) \geq 0$$
> for all $\vec{\lambda} \in \mathbb{C}^{<\mathbb{N}}$ (where $\exp(\lambda) = e^{\lambda}$), with equality holding only if $\vec{\lambda} \subseteq K_0$.

The proof is articulated in three steps and runs as follows:

(1) The above statement is expressible by the lightface Σ^1_2-formula

$$\mathsf{WSP} \equiv \exists f \in \mathbb{C}^{\mathbb{N}} (\mathrm{ran}(f) = K_0 \text{ is a field } \wedge \forall \vec{\lambda} \in \mathbb{C}^{<\mathbb{N}}\, \mathrm{Trdg}_{K_0}(\vec{\lambda}, e^{\vec{\lambda}}) \geq \mathrm{Ldim}_{\mathbb{Q}}(\vec{\lambda}/K_0)),$$

[5]Whenever a field $K \subseteq \mathbb{C}$ is closed with respect to the graph of the exponential function, the formal analogue of Schanuel's conjecture obtained replacing all occurrences of $\mathbb{Q}$ by K in its statement is false (i.e. the statement asserting that $\mathrm{Trdg}_K(\vec{\lambda}, e^{\vec{\lambda}}) \geq \mathrm{Ldim}_K(\vec{\lambda})$ for all $\vec{\lambda}$). A counterexample is given by $\mathrm{Trdg}_K(1, e) = 0 < 1 = \mathrm{Ldim}_K(1)$. The content of Theorem 1.3 below is that in essence this is the unique relevant countexample for certain subfields K of $\mathbb{C}$. This is the reason why we chose to formulate Schanuel's property for fields K in the form of axiom 3.

since it is a rather straightforward calculation to check that the formulae

$$\phi(f) \equiv (f \in \mathbb{C}^{\mathbb{N}} \wedge \operatorname{ran}(f) = K_0 \text{ is a field })$$

and

$$\mathsf{WSP}(\vec{\lambda}, f) \equiv \phi(f) \wedge (\vec{\lambda} \in \mathbb{C}^{<\mathbb{N}} \wedge \operatorname{Trdg}_{K_0}(\vec{\lambda}, e^{\vec{\lambda}}) \geq \operatorname{Ldim}_{\mathbb{Q}}(\vec{\lambda}/K_0))$$

are Borel statements definable over the parameters $f, \vec{\lambda}$ which require only to quantify over the countable sets f, $\mathbb{N}$, $\mathbb{Q}$. It is a classical result of set theory (known as Shoenfield's absoluteness) that any Σ^1_2-property known to hold in some forcing extension is actually true. So in order to establish the theorem it is enough to prove the above formula consistent by means of forcing i.e. to prove that $[\![\mathsf{WSP}]\!]_{\mathsf{B}} = 1_{\mathsf{B}}$ in the boolean valued model for set theory V^{B} for some complete boolean algebra B.

(2) The second step relies on the following observation: whenever B is any complete boolean algebra and V is the universe of sets (i.e. the standard model of ZFC), the family of B-names for complex numbers in the boolean valued model V^{B} (which we denote by $\dot{\mathbb{C}}$) "corresponds" to the space of continuous functions

$$C^+(\operatorname{St}(\mathsf{B})) = \{f : \operatorname{St}(\mathsf{B}) \to \mathbb{S}^2 : f \text{ is continuous and } f^{-1}[\{\infty\}] \text{ is nowhere dense}\},$$

where $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$ is the one point compactification of $\mathbb{C}$ with the euclidean topology, and $\operatorname{St}(\mathsf{B})$ is the space of ultrafilters on $\operatorname{St}(\mathsf{B})$ (equivalently of ring homomorphisms of the boolean ring B onto the ring $\mathbb{Z}_2$). More precisely there is a natural embedding of the structure $C^+(\operatorname{St}(\mathsf{B}))$ into the boolean valued model V^{B} which identifies $C^+(\operatorname{St}(\mathsf{B}))$ with

$$\dot{\mathbb{C}} = \{\tau \in V^{\mathsf{B}} : [\![\tau \text{ is a complex number}]\!]_{\mathsf{B}} = 1_{\mathsf{B}}\}.$$

Various facets of this identification are common knowledge for the set theory scholars, and this isomorphism has been proved in full details by Jech [3] and Ozawa [8] for $C^+(\operatorname{St}(\mathsf{B}))$. An account of this correspondence which is closer to the approach taken in the current papers and is more general than what is outlined in Jech's and Ozawa's works can be found in [10].

The reader should be aware that these spaces of functions may not be exotic: for example if MALG is the complete boolean algebra given by Lebesgue-measurable sets modulo Lebesgue null sets, $C(\operatorname{St}(\mathsf{MALG}))$ is isometric to $L^\infty(\mathbb{R})$ via the Gelfand-transform of the C^*-algebra $L^\infty(\mathbb{R})$ and consequently $\operatorname{St}(\mathsf{MALG})$ is homeomorphic to the space of characters of $L^\infty(\mathbb{R})$ endowed with the *weak-$*$ topology* inherited from the dual of $L^\infty(\mathbb{R})$.

What is more important to us is that for all complete boolean algebras B and for all $G \in \operatorname{St}(\mathsf{B})$ the space of germs given by $C^+(\operatorname{St}(\mathsf{B}))/G$ is an algebraically closed field to which any "natural" (i.e. for example Borel) relation defined on $\mathbb{C}^n$ can be extended: for example the exponential function can be extended to $C^+(\operatorname{St}(\mathsf{B}))/G$ by the map $[f]_G \mapsto [e^f]_G$. Moreover we can identify $\mathbb{C}$ inside $C^+(\operatorname{St}(\mathsf{B}))/G$ as the subfield given by germs of constant functions. We invite the reader to skim through [10] to get a thorough presentation of the properties of the spaces $C^+(\operatorname{St}(\mathsf{B}))$ seen as B-valued extensions of the complex numbers.

In this paper we are also interested in canonical subfields of $C^+(\operatorname{St}(\mathsf{B}))/G$ which give the correct lift to $C^+(\operatorname{St}(\mathsf{B}))/G$ of $\mathbb{Q}, \mathbb{C}$, these are respectively:

- The field $\check{\mathbb{C}}/G$ given by germs of locally constant functions, i.e. functions f in $C^+(\mathrm{St}(\mathsf{B}))$ such that
$$\bigcup\{f^{-1}[\{\lambda\}] : \lambda \in \mathbb{C}, f^{-1}[\{\lambda\}] \text{ is clopen}\}$$
is an open dense subset of $\mathrm{St}(\mathsf{B})$.
- The subfield $\check{\mathbb{Q}}/G$ (respectively the subring $\check{\mathbb{Z}}/G$) of $\check{\mathbb{C}}/G$ given by germs of locally constant functions with range contained in $\mathbb{Q}$ (respectively in $\mathbb{Z}$).

These rings corresponds in the forcing terminology of set theory respectively: to the B-names for complex numbers of the ground model, to the B-names for rational numbers of the ground model, to the B-names for integer numbers of the ground model. This characterization will play an important role in our proof.

The second step of our proof will show that if $G \in \mathrm{St}(\mathsf{B})$ and B is a complete boolean algebra, the structure
$$(C^+(\mathrm{St}(\mathsf{B}))/G, \check{\mathbb{C}}/G, [f]_G/[g]_G \mapsto [e^{f/g}]_G, \ldots, [0]_G, [1]_G)$$
is a model of $T_{\mathsf{WSP}(\check{\mathbb{C}}/G)}$ for any $G \in \mathrm{St}(\mathsf{B})$.

The key arguments in this second step do not require any specific training in set theory and needs just a certain amount of familiarity with first order logic, the basic properties of algebraic varieties, and with the combinatorics of forcing as expressed in terms of complete atomless boolean algebras. In particular there is no need to be acquainted with forcing or set theory to follow the proof of the above results (such a familiarity will nonetheless be of great help to follow the arguments).

The basic ideas for the proof are the following:

(A) For any $[\vec{f}]_G = ([f_1]_G, \ldots, [f_n]_G) \in (C^+(\mathrm{St}(\mathsf{B}))/G)^n$, the variety
$$V(\bar{I}_G(\vec{f}, e^{\vec{f}}), \check{\mathbb{C}}/G)$$
given by the 0-set of polynomials in $\check{\mathbb{C}}/G[\vec{x}, \vec{y}]$ vanishing at $[\vec{f}, e^{\vec{f}}]_G$ in $(C^+(\mathrm{St}(\mathsf{B}))/G)^{2n}$ has dimension equal to the transcendence degree of the tuple $[\vec{f}, e^{\vec{f}}]_G$ over $\check{\mathbb{C}}/G$. To compute the Ax Character for $[\vec{f}]_G$ over $\check{\mathbb{C}}/G$ it is enough to study the algebraic dimension of this variety in $(\check{\mathbb{C}}/G)^{2n}$.

(B) For a dense open set of G, the ideal $\bar{I}_G(\vec{f}, e^{\vec{f}})$ is generated by polynomials $p_1, \ldots, p_k$ with *complex coefficients*, consequently the algebraic dimension of $V(\bar{I}_G(\vec{f}, e^{\vec{f}}), \check{\mathbb{C}}/G))$ is equal to the algebraic dimension of the complex variety $V(p_1, \ldots, p_k, \mathbb{C})$ given by points in $\mathbb{C}^{2n}$ on which all the p_j vanish.

(C) Let $[\vec{f}]_G = ([f_1]_G, \ldots, [f_n]_G)$ be given by nowhere locally constant functions which are $\check{\mathbb{Q}}/G$-linearly independent modulo $\check{\mathbb{C}}/G$, by (B) above the transcendence degree of the $2n$-tuple $[\vec{f}, e^{\vec{f}}]_G$ over $\check{\mathbb{C}}/G$ is equal to the transcendence degree of the same $2n$-tuple over $\mathbb{C}$ (seen as a subfield of $C^+(\mathrm{St}(\mathsf{B}))/G$).

(D) For an n-tuple $[\vec{f}]_G$ as above we can show that the transcendence degree over $\mathbb{C}$ of the $2n$-tuple $[\vec{f}, e^{\vec{f}}]_G$ is at least $n+1$ as follows: we can find $\phi_1, \ldots, \phi_n$ analytic functions from $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ to $\mathbb{C}$ linearly independent over $\mathbb{Q}$ modulo $\mathbb{C}$ with the following property: Let $[\phi]$ denote the germ of ϕ at 0. Then the map $[\phi_i] \mapsto [f_i]_G$, $[e^{\phi_i}] \mapsto [e^{f_i}]_G$ extends to an isomorphism of the corresponding finitely generated subfields. The desired conclusion follows, since the field of germs at 0 of analytic functions from some open neighborhood U of 0 to $\mathbb{C}$ is a field to which Ax's theorem on Schanuel's property for functions fields apply (i.e. Theorem 2.1 below).

(3) The third step of our paper combines steps (1) and (2) as follows: We choose a boolean algebra B such that in the boolean valued model V^{B}, $[\![\check{\mathbb{C}} \text{ is countable }]\!]_{\mathsf{B}} = 1_{\mathsf{B}}$ (for example we can choose B to be the boolean algebra of regular open subsets of $\mathbb{C}^{\mathbb{N}}$ where $\mathbb{C}$ is endowed with the discrete topology). In particular in V^{B} we will have that

$$\left[\!\!\left[\check{\mathbb{C}} \text{ is countable as witnessed by } \dot{f} \wedge \mathsf{WSP}(\dot{f}, \dot{\mathbb{C}})\right]\!\!\right]_{\mathsf{B}} = 1_{\mathsf{B}},$$

i.e. $[\![\mathsf{WSP}]\!]_{\mathsf{B}} = 1_{\mathsf{B}}$ holds in V^{B}. By the results of step (1), we thus get that WSP holds in V concluding the proof of Theorem 1.3.

We will not expand any further on step (1), the core of the paper concerns the proof of the results in step (2), we add some more comments in the last part regarding step (3). We try (as much as possible) to make the arguments in step (2) accessible to persons which are not acquainted with the forcing techniques and more generally with logic. For this reason we shall limit the use of techniques which are specific of set theory just to the last step.

2. Step (2)

2.1. **Results from complex analysis and algebraic geometry.** We need just classical results in the field and we use as a general reference text [7], though some of the results we need may not be covered in that textbook. We will use the following definitions and theorems:

(1) **Ax's theorem on Schanuel's property.** The following corollary of Ax's theorem [1, Theorem 3]:

Theorem 2.1. *Assume (F, E) is an exponential field which is algebraically closed. Let $D : F \to F$ be a derivation (i.e. $D(f + g) = D(f) + D(g)$ and $D(fg) = D(f)g + fD(g)$ for all $f, g \in F$) such that $D(E(f)) = D(f) \cdot E(f)$ for all $f \in F$.*

Then for all $\vec{f} = (f_1, \dots, f_n) \in F^n$ which are $\mathbb{Q}$-linearly independent modulo $\ker(D)$ *we have that*

$$\mathrm{Trdg}_{\ker(D)}(f_1, \dots, f_n, E(f_1), \dots, E(f_n)) \geq n + 1.$$

(2) The field of fractions $\mathcal{O}^\Omega$ of germs at 0 of analytic functions (i.e. defined on some open neighborhood $U \subseteq \mathbb{C}$ of 0 by a convergent power series) $f : U \to \mathbb{C}$ with differential $D([f]/[g]) = \frac{[f'g - g'f]}{[g^2]}$ satisfies the assumptions of Ax's theorem with $\ker(D) = \mathbb{C}$ (identifying $\mathbb{C}$ as the subfield of $\mathcal{O}^\Omega$ given by germs of constant functions).

(3) **Algebraic dimension of affine irreducible algebraic varieties.** Any ideal $I \subseteq K[x_1, \dots, x_m]$ with K a field is finitely generated.

Given L a field containing all the coefficients of a set of generators $p_1, \dots, p_m$ for I, we let I^L denote the ideal generated by $p_1, \dots, p_m$ in $L[x_1, \dots, x_m]$.

- Any irreducible affine algebraic variety in K^n with K algebraically closed field is of the form
$$V(I,K) = \left\{\vec{\lambda} \in K^n : p(\vec{\lambda}) = 0\, \forall p(\vec{x}) \in I\right\}$$
with I a finitely generated prime ideal in $K[x_1, \dots, x_n]$.
- Given an ideal $I \subseteq K[x_1, \dots, x_n]$, and $L \subseteq K$ field containing all the coefficients of the polynomials in a set of generators for I,
  $\vec{\lambda} \in V(I,K)$ is an L-generic point for $V(I,K)$ if any polynomial in $L[x_1, \dots, x_n]$ is in I^L if and only if it vanishes on $\vec{\lambda}$.
- The algebraic dimension of the irreducible variety $V(I,K)$ can be computed as follows: fix some countable field $L \subseteq K$ finitely generated and containing the coefficients of a set of generators for I. Fix $\vec{\lambda} \in V(I,K)$ an L-generic point ($\vec{\lambda}$ exists since L is countable, by a simple Baire's category argument). The algebraic dimension of $V(I,K)$ is the number $\mathrm{Trdg}_L(\vec{\lambda})$ and depends neither on the choice of L nor on that of K in the following sense: Assume $L_1 \subseteq K_1$ are any other fields (L_1 need not be countable) such that L_1 contains the coefficients of a set of generators for I and K_1 is algebraically closed, and $\vec{\lambda}_1 \in K_1^n$ is an L_1-generic point for I^{L_1}, then $\mathrm{Trdg}_L(\vec{\lambda}) = \mathrm{Trdg}_{L_1}(\vec{\lambda}_1)$.

(4) A quasi-affine variety is the intersection of an affine variety with a Zariski-open set. The set of regular (or smooth) points of an irreducible quasi-affine variety on $\mathbb{C}^n$ is an open non-empty Zariski subset of the variety, and any generic point of the variety is smooth (recall that for a 0-set of a finite family of differentiable functions defined on $U \subseteq \mathbb{C}^n$, $\vec{a} \in U$ is a smooth point for this 0-set if the rank of the Jacobian of the finite set of functions defining U attains its maximum in $\vec{a}$).

(5) **Relations between algebraic affine varieties and analytic manifolds.** Any quasi-affine and smooth irreducible variety contained in $\mathbb{C}^n$ (i.e. a Zariski open set of an irreducible algebraic variety in $\mathbb{C}^n$ contained in the non-singular points of the variety) is also an analytic manifold and its algebraic dimension is equal to its dimension as an analytic manifold (i.e. the unique n such that some open neighborhood of the manifold is homeomorphic to $\mathbb{C}^n$).

(6) **Analytic implicit function theorem.** Assume U is the zero-set of a finite family of analytic functions defined on some open subset of $\mathbb{C}^k$ in the Euclidean topology. Let $\vec{a} \in U$ be a smooth point of U. Then for some unique n, there is an analytic, open (in the Euclidean topology on U) map $\phi : \mathbb{C}^n \to U$ with $\vec{a}$ in the target of ϕ.

(7) **Analytic paths inside an analytic manifold of positive dimension.** Let $V \subseteq \mathbb{C}^k$ be the injective image of an analytic map $\phi : \mathbb{C}^n \to \mathbb{C}^k$. Then any family of m distinct points $\{q_0, \dots, q_{m-1}\}$ in V can be connected by an analytic path i.e. an analytic map $\vec{\phi} : \Delta \to V$ such that $\{q_0, \dots, q_{m-1}\} \subseteq \mathrm{ran}(\vec{\phi})$, $\vec{\phi}(0) = q_0$.We sketch a proof since we are not able to give a proper reference other than wikipedia:Polynomial-interpolation: Let $\phi : \mathbb{C}^n \to V$

be analytic and injective. Let $\phi(\vec{x}_j) = q_j$ for all $j = 0, \ldots, m-1$. Let $\vec{x}_j = (x^j_0, \ldots, x^j_{n-1})$ By the interpolation theorem, we can find unique polynomials $p_l(x) \in \mathbb{C}[x]$ of degree m such that $p_l(\frac{j}{m}) = x^j_l$ for all $l = 0, \ldots, n-1$ and $j = 0, \ldots, m-1$. Then $\psi : \Delta \to V$ mapping $a \mapsto \phi(p_0(a), \ldots p_{n-1}(a))$ is analytic and maps $\frac{j}{m}$ to q_j for all $j = 0, \ldots, m-1$.

2.2. **Forcing on $C^+(\mathrm{St}(\mathsf{B}))$.** We refer the reader to [9, Chapters 2, 3, 4] for a detailed account on the material presented here.

Given a topological space (X, τ), an open set $A \in \tau$ is regular open if $A = \mathsf{Int}\,(\mathsf{Cl}\,(A))$, where for any $B \subseteq X$, $\mathsf{Int}\,(B)$ is the largest open set contained in B and $\mathsf{Cl}\,(B)$ the smallest closed set containing B. We define $\mathsf{Reg}\,(A) = \mathsf{Int}\,(\mathsf{Cl}\,(A))$. Recall that the algebra of regular open sets of a topological space (X, τ) consists of those $A \subseteq X$ such that $A = \mathsf{Reg}\,(A)$ and is always a complete boolean algebra with operations:

- $\bigvee\{A_i : i \in I\} = \mathsf{Reg}\,(\bigcup\{A_i : i \in I\})$,
- $\neg A = \mathsf{Int}\,(X \setminus A_i)$,
- $A \wedge B = A \cap B$.

- A topological space (X, τ) is 0-dimensional if its clopen sets form a base for τ.
- A compact topological space (X, τ) is extremally (extremely) disconnected if its algebra of clopen sets $\mathsf{CLOP}(X)$ is equal to its algebra of regular open sets $\mathsf{RO}(X)$.

For a boolean algebra B we let $\mathrm{St}(\mathsf{B})$ be the Stone space of its ultrafilters with topology generated by the clopen sets

$$N_b = \{G \in \mathrm{St}(\mathsf{B}) : b \in G\}.$$

We remark the following:

- $\mathrm{St}(\mathsf{B})$ is a compact 0-dimensional Haussdorf space and any 0-dimensional compact space (X, τ) is isomorphic to $\mathrm{St}(\mathsf{CLOP}(X))$,
- A compact Hausdorff space (X, τ) is extremely disconnected if and only if its algebra of clopen sets is a complete boolean algebra. In particular $\mathrm{St}(\mathsf{B})$ is extremely disconnected if and only if $\mathsf{B} = \mathsf{CLOP}(\mathrm{St}(\mathsf{B}))$ is complete.

An antichain on a boolean algebra B is a subset A such that $a \wedge b = 0_{\mathsf{B}}$ for all $a, b \in A$, $\mathsf{B}^+ = \mathsf{B} \setminus \{0_{\mathsf{B}}\}$ is the family of positive elements of B and a dense subset of B^+ is a subset D such that for all $b \in \mathsf{B}^+$ there is $a \in D$ such that $a \leq_{\mathsf{B}} b$. In a complete boolean algebra B any dense subset D of B^+ contains an antichain A such that $\bigvee A = \bigvee D = 1_{\mathsf{B}}$.

Another key observation on Stone spaces of complete boolean algebras we often need is the following:

Fact 2.2. *Assume B is a complete atomless boolean algebra, then on its Stone space $\mathrm{St}(\mathsf{B})$:*

- $N_{\bigvee_{\mathsf{B}} A} = \mathit{Cl}\left(\bigcup_{a \in A} N_a\right)$ *for all $A \subseteq \mathsf{B}$.*
- $N_{\bigvee_{\mathsf{B}} A} = \bigcup_{a \in A} N_a$ *for all finite sets $A \subseteq \mathsf{B}$.*
- *For any infinite antichain $A \subseteq \mathsf{B}^+$, $\bigcup_{a \in A} N_a$ is properly contained in $N_{\bigvee_{\mathsf{B}} A}$ as a dense open subset.*

Given a compact extremely disconnected topological space X, we let $C^+(X)$ be the space of continuous functions

$$f : X \to \mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$$

(where $\mathbb{S}^2$ is seen as the one point compactification of $\mathbb{C}$) with the property that $f^{-1}[\{\infty\}]$ is a closed nowhere dense subset of X. In this manner we can endow $C^+(X)$ of the structure of a commutative ring of functions with involution letting the operations be defined pointwise on all points whose image is in $\mathbb{C}$ and be undefined on the preimage of ∞. More precisely $f + g$ is the unique continuous function

$$h : X \to \mathbb{S}^2$$

such that $h(x) = f(x) + g(x)$ whenever this makes sense (it makes sense on an open dense subset of X, since the preimage of the point at infinity under f, g is closed nowhere dense) and is extended by continuity on the points on which $f(x) + g(x)$ is undefined: since X is extremely disconnected and compact, any h which is continuous on a dense open subset of X admits a unique continuous extension to the whole of X by the rule $h(p) = a$, where a is the unique element of

$$\bigcap\{U : U \text{ is closed and } p \in \mathsf{Int}\left(f^{-1}[U]\right)\}.$$

Thus $f + g \in C^+(X)$ if $f, g \in C^+(X)$. Similarly we define the other operations. We take the convention that constant functions are always denoted by their constant value, and that $0 = 1/\infty$.

Definition 2.3. Let G be an ultrafilter on B. For $f, g \in C^+(\mathrm{St}(\mathsf{B}))$ let $[f]_G = [h]_G$ iff for some $a \in G$, $f \restriction N_a = g \restriction N_a$.

$C^+(\mathrm{St}(\mathsf{B}))/G$ is the quotient ring of $C^+(\mathrm{St}(\mathsf{B}))$ by G given by the equivalence classes $[f]_G$ for $f \in C^+(\mathrm{St}(\mathsf{B}))$.

In the sequel given a vector $\vec{f} = (f_1, \dots, f_n) \in C^+(\mathrm{St}(\mathsf{B}))^n$, $b \in \mathsf{B}$, $G \in \mathrm{St}(\mathsf{B})$:

- $[\vec{f}]_G$ is a shorthand for $([f_1]_G, \dots, [f_n]_G)$,
- $\vec{f}(G)$ is a shorthand for $(f_1(G), \dots, f_n(G))$,
- $\vec{f} \restriction N_b$ is a shorthand for $(f_1 \restriction N_b, \dots, f_n \restriction N_b)$,
- For $g : \mathbb{C} \to \mathbb{C}$, $g(\vec{f})$ is a shorthand for $(g \circ f_1, \dots, g \circ f_n)$.

We also define the following family of rings indexed by positive elements of a complete boolean algebra:

Definition 2.4. Let B be a complete boolean algebra and $b \in \mathsf{B}^+$.

- $\check{\mathbb{C}}_c \subseteq C^+(N_c)$ is the ring of functions $f \in C^+(N_c)$ which are locally constant i.e. such that
$$\bigcup \left\{f^{-1}[\{\lambda\}] : f^{-1}[\{\lambda\}] \text{ is clopen}\right\}$$
is open dense in N_c. $\check{\mathbb{C}}$ stands for $\check{\mathbb{C}}_{1_{\mathsf{B}}}$.
- Let K be a structure among $\mathbb{Q}, \mathbb{Z}, \mathbb{N}$, we define $\check{K}_c$ to be the family of functions $f \in \check{\mathbb{C}}_c$ such that $\operatorname{ran} f \subseteq K$. $\check{K}$ stands for $\check{K}_{1_{\mathsf{B}}}$.

As a warm-up for the sequel we can already prove the following:

Fact 2.5. *Assume B is a complete boolean algebra. Then:*

(1) $(C^+(\mathrm{St}(\mathsf{B}))/G, [f]_G \mapsto [e^f]_G)$ and $(\check{\mathbb{C}}/G, [f]_G \mapsto [e^f]_G)$ are exponential fields with kernel $2\pi \cdot (\check{\mathbb{Z}}/G)$ for all $G \in \mathrm{St}(\mathsf{B})$.

(2) $\check{\mathbb{Q}}/G$ *is a field for all* $G \in \mathrm{St}(\mathsf{B})$.

Proof. Left to the reader. For what concerns the field structure of $C^+(\mathrm{St}(\mathsf{B}))/G$, it is not hard to check that for a non-zero $[f]_G \in C^+(\mathrm{St}(\mathsf{B}))/G$, we can find some N_b with $b \in G$ so that $g \in C^+(N_b)$ and $g \cdot (f \restriction N_b) = 1$ in $C^+(N_b)$. We can then extend g arbitrarily to a continuous function in $C^+(\mathrm{St}(\mathsf{B}))$ out of N_b. The rest is similar or easier. □

Germs of continuous functions on Stone spaces and forcing. We need to consider $C^+(\mathrm{St}(\mathsf{B}))$ as a B-boolean valued model. This is done as follows:

Definition 2.6. We identify a cba B with the complete boolean algebra of clopen (regular open) sets of $\mathrm{St}(\mathsf{B})$. The equality relation on $C^+(\mathrm{St}(\mathsf{B}))$ is the map

$$\begin{aligned} =: & C^+(\mathrm{St}(\mathsf{B}))^2 \to \mathsf{B} \\ & (f,g) \mapsto \mathsf{Reg}\left(\{H : f(H) = g(H)\}\right) \end{aligned}$$

We denote $=(f,g)$ by $[\![f=g]\!]$.

This equality boolean relation satisfies:

$$[\![f=g]\!] \wedge [\![h=g]\!] \leq [\![f=h]\!]$$

and

$$[\![f=g]\!] = [\![g=f]\!]$$

for all f,g,h.

A forcing relation on $C^+(\mathrm{St}(\mathsf{B}))$ is a map

$$R : C^+(\mathrm{St}(\mathsf{B}))^n \to \mathsf{B}$$

such that

$$R(f_1,\dots,f_n) \wedge [\![f_i = h]\!] \leq R(f_1,\dots,f_{i-1},h,f_{i+1},\dots,f_n)$$

for all $f_1,\dots,f_n,h$.

Let $R_1,\dots,R_m$ be forcing relations on $C^+(\mathrm{St}(\mathsf{B}))$ with $R_i : C^+(\mathrm{St}(\mathsf{B}))^{n_i} \to \mathsf{B}$. Consider the language $\{R_1,\dots,R_n\}$ with R_i relation of arity n_i. We define for ϕ,ψ formulae of this language:

- $\left[\!\!\left[R_i(\vec{f})\right]\!\!\right] = R_i(\vec{f})$ for all $i \leq n$,
- $[\![\phi \wedge \psi]\!] = [\![\phi]\!] \wedge [\![\psi]\!]$,
- $[\![\phi \vee \psi]\!] = [\![\phi]\!] \vee [\![\psi]\!]$,
- $[\![\neg\phi]\!] = \neg\,[\![\phi]\!]$,
- $$\left[\!\!\left[\exists x \phi(x,\vec{f})\right]\!\!\right] = \bigvee\{\left[\!\!\left[\phi(g,\vec{f})\right]\!\!\right] : g \in C^+(\mathrm{St}(\mathsf{B}))\}.$$

Given G ultrafilter on B we make $C^+(\mathrm{St}(\mathsf{B}))/G$ a Tarski structure for the language $\{R_1,\dots,R_n\}$ letting

$$C^+(\mathrm{St}(\mathsf{B}))/G \models R_i/G([\vec{f}]_G)$$

if and only if $R_i(\vec{f}) \in G$.

We have the following Theorems:

Lemma 2.7 (Mixing Lemma)**.** *Assume* B *is a complete boolean algebra and* $A \subseteq \mathsf{B}$ *is an antichain. Then for all family* $\{f_a : a \in A\} \subseteq C^+(\mathrm{St}(\mathsf{B}))$*, there exists* $f \in C^+(\mathrm{St}(\mathsf{B}))$ *such that*

$$a \leq [\![f = f_a]\!]$$

for all $a \in A$*.*

Proof. Sketch: Let $f \in C^+(\mathrm{St}(\mathsf{B}))$ be the unique function such that $f \restriction N_{(\neg \bigvee A)} = 0$ and $f \restriction N_a = f_a \restriction N_a$ for all $a \in A$. Check that f is well defined and works. □

Lemma 2.8 (Fullness Lemma)**.** *Let* $R_1, \ldots, R_n$ *be forcing relations on* $C^+(\mathrm{St}(\mathsf{B}))^{<\omega}$*. Then for all formulae* $\phi(x, \vec{y})$ *in the language* $\{R_1, \ldots, R_n\}$ *and all* $\vec{f} \in C^+(\mathrm{St}(\mathsf{B}))^n$ *there exists* $g \in C^+(\mathrm{St}(\mathsf{B}))$ *such that*

$$\left[\!\!\left[\exists x \phi(x, \vec{f})\right]\!\!\right] = \left[\!\!\left[\phi(g, \vec{f})\right]\!\!\right].$$

Proof. Sketch: Find A maximal antichain among the b such that $\left[\!\!\left[\phi(g_b, \vec{f})\right]\!\!\right] \geq b > 0_{\mathsf{B}}$ for some g_b. Now apply the Mixing Lemma to patch together all the g_a for $a \in A$ in a g. Check that

$$\left[\!\!\left[\exists x \phi(x, \vec{f})\right]\!\!\right] = \left[\!\!\left[\phi(g, \vec{f})\right]\!\!\right].$$

□

Theorem 2.9 (Cohen's forcing Theorem)**.** *Let* $R_1, \ldots, R_n$ *be forcing relations on* $C^+(\mathrm{St}(\mathsf{B}))$*. Then for all* $\vec{f} \in C^+(\mathrm{St}(\mathsf{B}))^n$ *and all formulae* $\phi(\vec{x})$ *in the language* $\{R_1, \ldots, R_n\}$*:*

(1) $C^+(\mathrm{St}(\mathsf{B}))/G \models \phi([\vec{f}]_G)$ *if and only if* $\left[\!\!\left[\phi(\vec{f})\right]\!\!\right] \in G$,

(2) *for all* $a \in \mathsf{B}$ *the following are equivalent:*
   (a) $[\![\phi(f_1, \ldots, f_n)]\!] \geq a$,
   (b) $C^+(\mathrm{St}(\mathsf{B}))/G \models \phi([\vec{f}]_G)$ *for* all $G \in N_a$,
   (c) $C^+(\mathrm{St}(\mathsf{B}))/G \models \phi([\vec{f}]_G)$ *for* densely many $G \in N_a$.

Proof. Sketch: Proceed by induction on the complexity of ϕ using the Mixing Lemma and the Fullness Lemma to handle the quantifier's cases. □

2.3. $T_{\mathsf{WSP}(\check{\mathbb{C}}/G)}$ holds in $C^+(\mathrm{St}(\mathsf{B}))/G$.

Theorem 2.10. *Assume* B *is a cba and* $G \in \mathrm{St}(\mathsf{B})$*. Then*

$$\mathsf{AC}_{\check{\mathbb{C}}/G, \exp/_G}([\vec{f}]_G) \geq 0$$

for all $[\vec{f}]_G \in (C^+(\mathrm{St}(\mathsf{B}))/G)^n$ *(where* $\exp/_G([f]_G) = [e^f]_G$*), with equality holding only if* $[\vec{f}]_G \subseteq (\check{\mathbb{C}}/G)^n$*.*

Before embarking on the proof of the above Theorem, let us show how the forcing theorem simplifies our task and let us also outline some caveat.

For any $b \in \mathsf{B}$ we can consider $C^+(N_b)$ both as a ring of functions in the usual sense, or as a boolean valued model on the boolean algebra $\mathsf{B} \restriction b$ in which we consider the sum and product operations as forcing relations, imposing for example for the sum:

$$[\![f + g = h]\!] = \mathsf{Reg}\left(\{H \in N_b : f(H) + g(H) = h(H)\}\right)$$

and similarly for the other field operations. By the forcing theorem, we will get that $[\![\phi]\!] = 1_{\mathsf{B}}$ for all field axioms ϕ expressed in the language with ternary relation symbols to code the operations, since each $C^+(\mathrm{St}(\mathsf{B}))/G$ is a field for all $G \in \mathrm{St}(\mathsf{B})$. Notice in sharp contrast that $C^+(\mathrm{St}(\mathsf{B}))$ is *not* a field when we consider it as an algebraic ring. This outlines a serious distinction between the theory of $C^+(\mathrm{St}(\mathsf{B}))$ seen as a boolean valued model and its theory seen as an algebraic ring.

Moreover in the sequel we do not work simply with the boolean valued model $C^+(\mathrm{St}(\mathsf{B}))$ in the language for fields. We will consider it as a boolean valued model in the language with predicate symbols for the relations and operations $\check{\mathbb{C}}, \mathsf{exp}, +, \cdot$, we will also add a predicate symbol for the ring $\check{\mathbb{Q}}$ ($\check{\mathbb{Z}}$) given by the locally constant $\mathbb{Q}$-valued ($\mathbb{Z}$-valued) functions and for the forcing relations expressing $\check{\mathbb{Z}}$-linear independence over $\check{\mathbb{C}}$ and the $\check{\mathbb{C}}$-transcendence degree forcing relation.

Definition 2.11. Let B be a complete boolean algebra. For all $c \in \mathsf{B}$:

- $\check{\mathbb{C}}_c \subsetneq C^+(N_c)$ is the ring of functions which are locally constant and $\check{\mathbb{C}}$ stands for $\check{\mathbb{C}}_{1_{\mathsf{B}}}$.
- Let K be a structure among $\mathbb{Q}, \mathbb{Z}, \mathbb{N}$, we define $\check{K}_c$ to be the family of functions given by $f \in \check{\mathbb{C}}_c$ such that $\operatorname{ran} f \subseteq K$ and $\check{K}$ stands for $\check{K}_{1_{\mathsf{B}}}$.

Given $\vec{f} = (f_1, \ldots, f_n) \in C^+(\mathrm{St}(\mathsf{B}))^n$ and $c \in \mathsf{B}$, let:

- $[\![\mathrm{Trdg}_{\check{\mathbb{C}}}(\vec{f}) = m]\!] =$
$$= \bigvee_{\mathsf{B}} \{b \in \mathsf{B} : \forall G \in N_b(\mathrm{Trdg}_{\check{\mathbb{C}}/G}([\vec{f}]_G) = m)\},$$
- $[\![\mathrm{Ldim}_{\check{\mathbb{Q}}}(\vec{f}/\check{\mathbb{C}}) = m]\!] =$
$$= \bigvee_{\mathsf{B}} \{b \in \mathsf{B} : \forall G \in N_b(\mathrm{Ldim}_{\check{\mathbb{Q}}/G}([\vec{f}]_G/_{\check{\mathbb{C}}/G}) = m)\}.$$

Fact 2.12. *The above relations are forcing relation for $C^+(\mathrm{St}(\mathsf{B}))$.*

Proof. Left to the reader. □

On the face of the definitions we get that

$$[\![\mathrm{Ldim}_{\check{\mathbb{Q}}}(\vec{f}/\check{\mathbb{C}}) = m]\!] = \bigvee_{\mathsf{B}} \{b \in \mathsf{B} : \forall G \in N_b \, \mathrm{Ldim}_{\check{\mathbb{Q}}/G}([\vec{f}]_G/\check{\mathbb{C}}/_G) = m\}$$

entails that

$$\mathrm{Ldim}_{\check{\mathbb{Q}}/H}([\vec{f}]_H/\check{\mathbb{C}}/H) = m$$

only on an open dense subset of

$$H \in N_{[\![\mathrm{Ldim}_{\check{\mathbb{Q}}}(\vec{f}/\check{\mathbb{C}}) = m]\!]}.$$

Similarly for the boolean predicate $[\![\mathrm{Trdg}_{\check{\mathbb{C}}}(\vec{f}) = m]\!]$.

First of all we observe that for these two boolean predicates this open dense subset is the whole of $N_{[\![\mathrm{Ldim}_{\check{\mathbb{C}}}(\vec{f}) = m]\!]}$ ($N_{[\![\mathrm{Trdg}_{\check{\mathbb{C}}}(\vec{f}) = m]\!]}$):

Fact 2.13. *Let B be a complete boolean algebra and $\vec{f} = (f_1, \ldots, f_n) \in C^+(\mathrm{St}(\mathsf{B}))^n$. Then for all $G \in \mathrm{St}(\mathsf{B})$:*

(1) $[\![\mathrm{Trdg}_{\check{\mathbb{C}}}(\vec{f}) = m]\!] \in G$ *if and only if*
$$\mathrm{Trdg}_{\check{\mathbb{C}}/G}([\vec{f}]_G) = m.$$

(2) $[\![\mathrm{Ldim}_{\check{\mathbb{Q}}}(\vec{f}/\check{\mathbb{C}}) = m]\!] \in G$ *if and only if*

$$\mathrm{Ldim}_{\check{\mathbb{Q}}/G}([\vec{f}]_G/(\check{\mathbb{C}}/_G)) = m.$$

Proof. The proof is a standard application of the forcing method. To get the reader acquainted with what we shall be doing in the remainder we give some of its parts. Let $\vec{f} = (f_0, \dots, f_n)$ be a tuple of $C^+(\mathrm{St}(\mathsf{B}))$-functions.

Assume that

$$\mathrm{Trdg}_{\check{\mathbb{C}}/G}([\vec{f}]_G) < m.$$

Then there is a polynomial $p(x_0, \dots, x_{m-1})$ in $\check{\mathbb{C}}/G[x_0, \dots, x_{m-1}]$ such that

$$p([\vec{f}]_G) = [0]_G.$$

By the forcing theorem we get that $[\![p(\vec{f}) = 0]\!] \in G$. Let

$$p(\vec{x}) = \sum_\alpha f_\alpha \vec{x}^\alpha,$$

where α ranges over the appropriate multiindexes and each $f_\alpha \in C^+(\mathrm{St}(\mathsf{B}))$. Then we also get that $(f_\alpha \restriction N_b) \in \check{\mathbb{C}}_b$ for all α for some $b \in G$ refining $[\![p(\vec{f}) = 0]\!]$.

This gives that $\mathrm{Trdg}_{\check{\mathbb{C}}/_H}([\vec{f}]_H) < m$ as witnessed by

$$\sum_\alpha ([f_\alpha]_H)\vec{x}^\alpha$$

for all $H \in N_b$.

On the other hand, assume towards a contradiction that $d = [\![\mathrm{Trdg}_{\check{\mathbb{C}}}(\vec{f}) = m]\!] \in G$. This means that for an open dense subset A of N_d we have that $\mathrm{Trdg}_{\check{\mathbb{C}}/_H}([\vec{f}]_H) = m$. Since $G \in N_d \cap N_b$, and A is dense in N_d, we also get that $A \cap N_b$ is non-empty. Any H in $A \cap N_b$ witnesses that

$$m = \mathrm{Trdg}_{\check{\mathbb{C}}/_H}([\vec{f}]_H) < m,$$

a contradiction.

The converse direction for Trdg and the proof for the other predicate are left to the reader. □

We leave to the reader to check that:

$$\forall G \in N_b\, \mathrm{Trdg}_{\check{\mathbb{C}}/_G}([\vec{f}]_G) = m$$

if and only if

$$\forall c \leq b\, \mathrm{Trdg}_{\check{\mathbb{C}}_c}(\vec{f} \restriction N_c) = m,$$

and also that:

$$\forall G \in N_b\, \mathrm{Ldim}_{\check{\mathbb{Q}}/_G}([\vec{f}]_G/\check{\mathbb{C}}/_G) = m$$

if and only if

$$\forall c \leq b\, \mathrm{Ldim}_{\check{\mathbb{C}}_c}(\vec{f} \restriction N_c) = m,$$

2.3.1. *Key Lemmas.* Let $b \in \mathsf{B}$, and $\vec{f} = (f_1, \dots, f_n)$ be a tuple of $C^+(\mathrm{St}(\mathsf{B}))$-functions.

- $I_b(\vec{f})$ is the ideal in $\mathbb{C}[\vec{x}]$ given by polynomials $p(\vec{x})$ with coefficients in $\mathbb{C}$ such that
$$p(\vec{f}(H)) = 0 \text{ for all } H \in N_b.$$
- $I_G(\vec{f})$ is the ideal in $\mathbb{C}[\vec{x}]$ of polynomials $p(\vec{x})$ with coefficients in $\mathbb{C}$ such that
$$p([\vec{f}]_G) = 0.$$
- $\bar{I}_b(\vec{f})$ is the ideal in $\check{\mathbb{C}}_b[\vec{x}]$ given by polynomials $p(\vec{x})$ with coefficients in $\check{\mathbb{C}}_b$ such that
$$p(\vec{f} \restriction N_b) = 0.$$
- $\bar{I}_G(\vec{f})$ is the ideal in $\check{\mathbb{C}}/_G[\vec{x}]$ of polynomials $p(\vec{x})$ with coefficients in $\check{\mathbb{C}}_b$ for some $b \in G$ such that
$$[p]_G([\vec{f}]_G) = 0.$$

If no confusion can arise we let I_b denote $I_b(\vec{f})$ and similarly for all the other ideals defined above.

Notice the following:

- $I_b \subseteq I_G$ for all $G \in N_b$,
- $I_b \subseteq \bar{I}_b$,
- $I_G \subseteq \bar{I}_G$ for all $G \in N_b$,
- $[p]_G \in \bar{I}_G$ for all $p \in \bar{I}_b$ and for all $G \in N_b$, where
$$[p]_G = \sum_\alpha [f_\alpha]_G x^\alpha \text{ if } p = \sum_\alpha f_\alpha x^\alpha.$$

Fact 2.14. *$V(I_G, \mathbb{C})$ and $V(\bar{I}_G, \check{\mathbb{C}}/G)$ are irreducible algebraic varieties.*

Proof. Assume $p(\vec{x})q(\vec{x}) \in I_G(\vec{f})$. Then $[p \circ \vec{f}]_G [q \circ \vec{f}]_G = 0$ in $C^+(\mathrm{St}(\mathsf{B}))/G$. Since the latter is a field we get that $[p \circ \vec{f}]_G$ or $[q \circ \vec{f}]_G$ must be 0, which yields the desired conclusion. The proof for $V(\bar{I}_G, \check{\mathbb{C}}/G)$ is identical. □

Lemma 2.15. *Assume B is a complete boolean algebra. For each $b \in \mathsf{B}^+$ and $\vec{f} = (f_1, \dots, f_n)$ tuple of $C^+(\mathrm{St}(\mathsf{B}))$-functions, there exists $c \leq_{\mathsf{B}} b$ in B^+ such that for all $G \in N_c$:*

- *$I_c(\vec{f}) = I_G(\vec{f})$,*
- *$[\vec{f}]_G$ is a generic point for $V(I_G(\vec{f}), C^+(\mathrm{St}(\mathsf{B}))/G)$.*

Proof. Assume the first conclusion of the Lemma fails for b and $\vec{f}$. Let $b_0 = b$ and $I_0 = I_b(\vec{f})$ and build by induction a strictly increasing chain of ideals I_n on C and a decreasing chain of elements $b_n >_{\mathsf{B}} 0_{\mathsf{B}}$ as follows:

Given $I_n = I_{b_n}(\vec{f})$, find —if possible— some $p(\vec{x}) \in \mathbb{C}[\vec{x}]$ which is not in I_n and vanishes on $[\vec{f}]_G$ for some $G \in N_{b_n}$. Then
$$p([\vec{f}]_G) = [0]_G$$
if and only if
$$[\![p(\vec{f}) = 0]\!] \in G.$$
If we can proceed for all n,
$$\{I_n : n \in \mathbb{N}\}$$

is a stricly increasing chain of ideals on the Noetherian ring $\mathbb{C}[x_1, \dots, x_n]$. This is impossible, so we can find $b_n = c$ such that $I_G(\vec{f}) = I_c(\vec{f})$ for any $G \in N_c$.

We are left to prove that $[\vec{f}]_G \in (C^+(\mathrm{St}(\mathsf{B}))/G)^n$ is a generic point for $V(I_c(\vec{f}), C^+(\mathrm{St}(\mathsf{B}))/G)$ for any $G \in N_c$. This is immediate for all $G \in N_c$, since:

$$p([\vec{f}]_G) = 0 \text{ iff } p(\vec{x}) \in I_G(\vec{f}) = I_c(\vec{f}).$$

The proof of the Lemma is completed. □

Lemma 2.16. *Assume* B *is a complete boolean algebra. Let* $\vec{f} = (f_1, \dots, f_n)$ *be a tuple of* $C^+(\mathrm{St}(\mathsf{B}))$*-functions, and* $c \in \mathsf{B}$ *be such that* $I_c(\vec{f}) = I_G(\vec{f})$ *for all* $G \in N_c$. *Then* $I_d(\vec{f})$ *is a set of generators for* $\bar{I}_d(\vec{f} \upharpoonright N_d)$ *in* $\check{\mathbb{C}}_d[\vec{x}]$ *for all* $d \leq_{\mathsf{B}} c$ *and* $I_G(\vec{f})$ *is a set of generators for* $\bar{I}_G(\vec{f})$ *for all* $G \in N_c$. *In particular*

$$V(I_G(\vec{f}), C^+(\mathrm{St}(\mathsf{B}))/G) = V(\bar{I}_G(\vec{f}), C^+(\mathrm{St}(\mathsf{B}))/G)$$

for all $G \in N_c$.

Proof. Let $p_1, \dots, p_k \in \mathbb{C}[\vec{x}]$ be a set of generators for $I_c(\vec{f})$. We claim that $p_1, \dots, p_k$ is also a set of generators for $\bar{I}_c(\vec{f})$ in $\check{\mathbb{C}}_c[\vec{x}]$: Pick some $p \in \check{\mathbb{C}}_c[\vec{x}]$ such that $p \in \bar{I}_c(\vec{f})$. Since the coefficients of p are locally constant functions defined on N_c, we can find a maximal antichain $\{d_j : j \in J\}$ such that each d_j refines c and is such that

$$p \upharpoonright N_{d_j} = \sum_\alpha f_\alpha \upharpoonright N_{d_j} x^\alpha \in \mathbb{C}[\vec{x}].$$

This gives that

$$p \upharpoonright N_{d_j}(\vec{f}) \in I_{d_j}(\vec{f}) = I_c(\vec{f})$$

for all $j \in J$. Find thus $q_j^1, \dots q_j^k \in \mathbb{C}[\vec{x}]$ such that

$$p \upharpoonright N_{d_j} = \sum_{l=1,\dots,k} q_j^l p_l.$$

Define for each $l = 1, \dots, k$ $q_l \in C^+(N_c)$ by the requirement that

$$q_l \upharpoonright N_{d_j} = q_j^l$$

for all $j \in J$.

Then $q_l \in \check{\mathbb{C}}_c[\vec{x}]$ for all $l = 1, \dots, k$ and

$$p = \sum_{l=1,\dots,k} q_l \cdot p_l \in \bar{I}_c(\vec{f}).$$

Since $p \in I_c(\vec{f})$ was chosen arbitrarily, we conclude that $p_1, \dots, p_k$ are a set of generators for $\bar{I}_c(\vec{f})$ in $\check{\mathbb{C}}_c[\vec{x}]$. This proves the first part of the Lemma.

For the second part observe that $p_1, \dots, p_k$ are a set of generators for $I_G(\vec{f})$ for all $G \in N_c$.

Now pick $[p]_G \in \bar{I}_G(\vec{f})$ for $G \in N_c$. Then for some $d \leq_{\mathsf{B}} c$ in G $p \upharpoonright N_d \in \bar{I}_d(\vec{f})$. But since $c \geq_{\mathsf{B}} d$ it is immediate to check that $p_1, \dots, p_k$ are generators also for $\bar{I}_d(\vec{f})$. We conclude that $p \upharpoonright N_d$ can be obtained as a linear combination of $p_1, \dots, p_k$ with coefficients in $\check{\mathbb{C}}_d[\vec{x}]$. Thus this occurs as well for $[p]_G$ taking the germs of these coefficients in $C^+(\mathrm{St}(\mathsf{B}))/G$. The proof of the Lemma is completed. □

Lemma 2.17. *Let $b \in \mathsf{B}$ and $\vec{f} = (f_1, \ldots, f_n)$ be a tuple of $C^+(\mathrm{St}(\mathsf{B}))$-functions. Assume that*

$$[\![\mathrm{Ldim}_{\check{\mathbb{Q}}}(f_1, \ldots, f_n/\check{\mathbb{C}}) = n]\!] \geq_{\mathsf{B}} b$$

(i.e. $[f_1]_H, \ldots, [f_n]_H$ are $\check{\mathbb{Q}}/H$-linearly independent modulo $\check{\mathbb{C}}/H$ for all $H \in N_b$).

Then there exists an ultrafilter $G \in N_b$ such that

$$\mathrm{Trdg}_{\check{\mathbb{C}}/G}([\vec{f}]_G, [e^{\vec{f}}]_G) \geq n + 1.$$

Clearly the proof of this Lemma concludes the proof of Theorem 2.10 since it shows that the statement

$$\mathsf{AC}_{\check{\mathbb{Q}}/_H, \exp/_H}([\vec{f}]_H) > 0$$

holds for a dense set of H for any fixed $\vec{f} \in (C^+(\mathrm{St}(\mathsf{B}))^{<\mathbb{N}}$ such that $[\vec{f}]_H$ is not contained in $\check{\mathbb{C}}/_H$. In particular we get that for all $\vec{f} \in (C^+(\mathrm{St}(\mathsf{B}))^n$ and for all $n \in \mathbb{N}$

$$[\![\mathsf{AC}_{\check{\mathbb{Q}}, \exp}(\vec{f}) \geq 0]\!]_{\mathsf{B}} = 1_{\mathsf{B}}$$

and

$$[\![\mathsf{AC}_{\check{\mathbb{Q}}, \exp}(\vec{f}) = 0]\!]_{\mathsf{B}} = [\![\vec{f} \subseteq \check{\mathbb{C}}^n]\!]_{\mathsf{B}}.$$

Using the observations regarding the properties of the forcing predicates $[\![\mathrm{Ldim}_{\check{\mathbb{Q}}}(\vec{f}/\check{\mathbb{C}})]\!]_{\mathsf{B}}$ and $[\![\mathrm{Trdg}_{\check{\mathbb{C}}}(\vec{f})]\!]_{\mathsf{B}}$, and once again the forcing theorem, we get that

$$\mathsf{AC}_{\check{\mathbb{Q}}/H, \exp/_H}([\vec{f}]_H) > 0$$

holds for *all* H and for any fixed $\vec{f} \in (C^+(\mathrm{St}(\mathsf{B}))^{<\mathbb{N}}$ not contained in $(\check{\mathbb{C}}/_H)^{<\mathbb{N}}$, which is the desired conclusion.

We now prove the Lemma:

Proof. First of all we choose $c \leq b$ such that

$$I_c(\vec{f}, e^{\vec{f}}) = I_G(\vec{f}, e^{\vec{f}})$$

for all $G \in N_c$, which is possible by Lemma 2.15. We let $I = I_c = I_G$ in what follows, and $p_1, \ldots, p_m \in \mathbb{C}[\vec{x}, \vec{y}]$ be a set of generators of minimal size for I. Then $\bar{I}_G$ is also generated by $p_1, \ldots, p_m$.

Now we have that for all algebraically closed fields K containing all the coefficients of $p_1, \ldots, p_m$, the algebraic dimension of $V(I, K)$ is the same and is given by $\mathrm{Trdg}_L(\vec{\lambda})$ with $\vec{\lambda} \in K^{2n}$ an L-generic point and L a field containing all the coefficients of $p_1, \ldots, p_m$.

This gives that the dimension of $V(\bar{I}_G, C^+(\mathrm{St}(\mathsf{B}))/G)$ as a variety over $(C^+(\mathrm{St}(\mathsf{B}))/G)^{2n}$ is equal to the transcendence degree of $([\vec{f}, e^{\vec{f}}]_G)$ over $\mathbb{C}$ as well as over $\check{\mathbb{C}}/G$, since —by Lemma 2.16— the latter is a generic point of the variety

$$V(I, C^+(\mathrm{St}(\mathsf{B}))/G) = V(\bar{I}_G, C^+(\mathrm{St}(\mathsf{B}))/G)$$

for any $G \in N_c$ for the field $\check{\mathbb{C}}/G$.

So in order to prove the Lemma we can also study the algebraic dimension of $V(I, C^+(\mathrm{St}(\mathsf{B}))/G)$ as a subvariety of $(C^+(\mathrm{St}(\mathsf{B}))/G)^{2n}$ and prove that it is at least $n + 1$ for some $G \in N_c$.

To prove this we argue as follows:

(1) First of all we use classical arguments rooted in the equality of the notion of algebraic dimension of an irreducible smooth quasi-affine variety contained in $\mathbb{C}^{2n}$ and of the notion of topological (or analytic) dimension of the same variety seen as an analytic manifold, to argue that the analytic dimension of $V(I,\mathbb{C})$ (which is equal to its algebraic dimension, and thus also to the algebraic dimension of $V(I,C^+(\mathrm{St}(\mathsf{B}))/G))$ is positive.
(2) Next we argue that we can find n distinct analytic paths with the same origin inside $V(I,\mathbb{C})$ which are $\mathbb{Q}$-linearly independent modulo $\mathbb{C}$ to conclude that that the algebraic dimension of $V(I,C^+(\mathrm{St}(\mathsf{B}))/G)$ is at least $n+1$ by means of Ax's theorem 2.1.

Let

$$\mathsf{Exp}_n = \{(\vec{\lambda}, e^{\vec{\lambda}}) : \lambda \in \mathbb{C}^n\}.$$

Remark that $V(I,\mathbb{C})\cap\mathsf{Exp}_n$ is the zero-set of the finite set of analytic functions

$$\{p_1,\dots,p_m,y_1-e^{x_1},\dots,y_n-e^{x_n}\},$$

where $p_1,\dots,p_m \in \mathbb{C}[\vec{x},\vec{y}]$ is a set of generators for I. Consider the Jacobian $J : \mathbb{C}^{2n} \to \mathbb{C}^{(n+m)^2}$ of this finite set of functions and the map

$$\begin{aligned}\phi :& N_c \to n+m+1\\ & G \mapsto \mathrm{rank}(J(\vec{f}(G), e^{\vec{f}(G)})).\end{aligned}$$

ϕ is continuous with range on a discrete space, so it must be constant on a clopen non-empty subset of N_c. By refining furher c if necessary, we can assume that ϕ is constant on N_c with value k. Pick $G \in N_c$. Then $\mathrm{rank}(J(\vec{f}(G), e^{\vec{f}(G)})) = k$ entails that on an open neighborhood $U_G \subseteq \mathbb{C}^{2n}$ of $(\vec{f}(G), e^{\vec{f}(G)})$, $\mathrm{rank}(J(\vec{f}(G), e^{\vec{f}(G)})) = k$. This gives that for any $G \in N_c$ the analytic dimension of $V(I,\mathbb{C})\cap\mathsf{Exp}_n$ around $(\vec{f}(G), e^{\vec{f}(G)})$ is k since the rank of the Jacobian of the functions

$$\{p_1,\dots,p_m,y_1-e^{x_1},\dots,y_n-e^{x_n}\}$$

of which $V(I,\mathbb{C})\cap\mathsf{Exp}_n$ is the 0-set attains its maximum k on all points of $U_G \cap V(I,\mathbb{C})\cap\mathsf{Exp}_n$. Therefore $(\vec{f}(G), e^{\vec{f}(G)})$ is a smooth point of $V(I,\mathbb{C})\cap\mathsf{Exp}_n$ for all $G \in N_c$.

Fix now some $G \in N_c$. By the implicit function theorem applied to the point $(\vec{f}(G), e^{\vec{f}(G)}) \in V(I,\mathbb{C})\cap\mathsf{Exp}_n$, there is an open (in the euclidean topology on $V(I,\mathbb{C})$) analytic map from some $\mathbb{C}^k$ to $V(I,\mathbb{C})\cap\mathsf{Exp}_n \subseteq \mathbb{C}^{2n}$ which is an homeomorphism with its target and has $(\vec{f}(G), e^{\vec{f}(G)})$ in its range. Let $V' \subseteq V(I,\mathbb{C})\cap\mathsf{Exp}_n$ be the target of this map. Then V' is an open subset of $V(I,\mathbb{C})\cap\mathsf{Exp}_n$ in the euclidean topology.

Claim 1. $\dim(V') > 0$.

Proof. Assume $\dim V' = 0$. Then we get that V' is the homeomorphic image of $\mathbb{C}^0$, which is a space consisting of a single point. Thus V' consists of a single point and is an open subset of a connected component of the analytic manifold $V(I,\mathbb{C})\cap\mathsf{Exp}_n$ in the euclidean topology on $V(I,\mathbb{C})\cap\mathsf{Exp}_n$. This gives that V' is a clopen subset of this connected component in this topology, and thus must be equal to this connected component of $V(I,\mathbb{C})\cap\mathsf{Exp}_n$.

Hence we can find an open neighboorhood $B \subseteq \mathbb{C}^{2n}$ in the Euclidean topology on $\mathbb{C}^{2n}$, such that

$$B \cap V(I,\mathbb{C}) \cap \mathsf{Exp}_n = V' = \{(\vec{f}(G), e^{\vec{f}(G)})\}.$$

This gives that $(\vec{f}(H), e^{\vec{f}(H)}) = (\vec{f}(G), e^{\vec{f}(G)})$ for all H such that $(\vec{f}(H), e^{\vec{f}(H)}) \in B \cap V(I,\mathbb{C})$. However

$$I = I_H(\vec{f}, e^{\vec{f}}) = I_G(\vec{f}, e^{\vec{f}}) = I_c(\vec{f}, e^{\vec{f}})$$

for all $H \in N_c$. In particular $p(\vec{f}(H), e^{\vec{f}(H)}) = 0$ for all $p \in I$ and all $H \in N_c$, i.e $(\vec{f}(H), e^{\vec{f}(H)}) \in V(I,\mathbb{C})$ for all $H \in N_c$. Hence $(\vec{f}(H), e^{\vec{f}(H)}) = (\vec{f}(G), e^{\vec{f}(G)})$ for all $H \in N_c$ with $(\vec{f}(H), e^{\vec{f}(H)}) \in B$. We conclude that $\vec{f}$ is constant with value $\vec{f}(G)$ on an open subset of N_c, contradicting our assumptions that $\vec{f}$ is nowhere locally constant on $N_b \supseteq N_c$. □

By the Claim we get that for any $G \in N_c$, the algebraic dimension of $V(I,\mathbb{C})$ around $(\vec{f}(G), e^{\vec{f}(G)})$ is positive since $V(I,\mathbb{C})$ contains the analytic variety of positive dimension V'.

We now come to the heart of the proof of this Lemma:

Claim 2. *For some $G \in N_c$*

$$\mathrm{Trdg}_{\mathbb{C}}([f_1]_G, \dots, [f_n]_G, [e^{f_1}]_G, \dots, [e^{f_n}]_G) \geq n+1.$$

Proof. Let $c_1 \leq c$ be such that $(\vec{f}(H), e^{\vec{f}(H)}) \in V'$ for all $H \in N_{c_1}$. Our assumptions give that

$$(f_1(H), \dots, f_n(H), e^{f_1(H)}, \dots, e^{f_n(H)}) \in V'$$

for all $H \in N_{c_1}$ and that V' is a connected analytic manifold of positive dimension.

Let $C^{\Omega}(V')$ denote the vector valued paths $\phi : \Delta \to \mathbb{C}^{2n}$ which are analytic and with range contained in $V' \subseteq \mathbb{C}^{2n}$.

We will use the following standard fact (Observation 7 on analytic manifolds):

Fact 2.18. *For any distinct $H_1, \dots, H_k$ with $\vec{f}(H_i) \neq \vec{f}(H_j)$ for all $0 < i \neq j \leq k$ in N_{c_1} there is a path in $C^{\Omega}(V')$ passing through*

$$(f_1(H_j), \dots, f_n(H_j), e^{f_1(H_j)}, \dots, e^{f_n(H_j)})$$

for all $0 < j \leq k$.

For each $H \in N_{c_1}$ consider the family Path_H of $C^{\Omega}(V')$-paths

$$\vec{\phi} : \Delta \to V' \subseteq \mathbb{C}^{2n}$$

with

$$\vec{\phi}(0) = (f_1(H), \dots, f_n(H), e^{f_1(H)}, \dots, e^{f_n(H)})$$

Let $\mathcal{H}$ be the family of hypersurfaces (relative to the analytic manifold given by $\mathsf{Exp}_n \subseteq \mathbb{C}^{2n}$) given by points $(\vec{x}, \vec{y}) \in \mathsf{Exp}_n$ satisfying

$$\sum_{i=1,\dots,n} m_i x_i = a; \prod_{i=1,\dots,n} y_i^{m_i} = e^a$$

for some $a \in \mathbb{C}$ and some vector $(m_1, \dots, m_n) \in \mathbb{N}^n$.

Subclaim 1. *For all $G \in N_{c_1}$ the set D_G of $H \in N_{c_1}$ such that any $C^{\Omega}(V')$-path in Path_G passing through*

$$(f_1(H), \dots, f_n(H), e^{f_1(H)}, \dots, e^{f_n(H)})$$

is contained in some hypersurface in $\mathcal{H}$ is nowhere dense.

Proof. Assume not for some G. Let $d \in \mathsf{B}$ be such that $D = D_G \cap N_d$ is dense in N_d.

By our assumptions, any $C^{\Omega}(V')$-path contained in V' starting in the point

$$(f_1(G), \dots, f_n(G), e^{f_1(G)}, \dots, e^{f_n(G)})$$

and passing through

$$(f_1(H), \dots, f_n(H), e^{f_1(H)}, \dots, e^{f_n(H)})$$

for some $H \in D$ is contained in an hypersurface in $\mathcal{H}$. Since V' is connected, for any $G_1, \dots, G_k \in D$ there is a $C^{\Omega}(V')$-path in Path_G passing through

$$(f_1(G_j), \dots, f_n(G_j), e^{f_1(G_j)}, \dots, e^{f_n(G_j)}).$$

By our assumptions this path is contained in some hypersurface of the form

$$\sum_{i=1,\dots,n} m_i x_i = a; \prod_{i=1,\dots,n} y_i^{m_i} = e^a$$

belonging to $\mathcal{H}$. Now select for as long as it is possible for each $0 \leq j \leq n$ some $G_j \in D$ so that $G_0 = G$ and for all $j < n$

$$(f_1(G_{j+1}), \dots, f_n(G_{j+1}), e^{f_1(G_{j+1})}, \dots, e^{f_n(G_{j+1})}).$$

does not belong to the unique j-dimensional hypersurface E_j determined as follows: Let A_j be the unique j-dimensional hyperplane in $\mathbb{C}^n$ passing for the points

$$(f_1(G_k), \dots, f_n(G_k))$$

with $k \leq j$. Let E_j consists of the points of the form $(\vec{\lambda}, e^{\vec{\lambda}})$ with $\vec{\lambda} \in A_j$. E_j is an hypersuperface contained in some element of $\mathcal{H}$ for each $0 \leq j < n$. To proceed in the construction notice that E_j is a closed subset of $\mathbb{C}^{2n}$ for all $j < n$, thus

$$U_j = \{H \in N_d : (f_1(H), \dots, f_n(H), e^{f_1(H)}, \dots, e^{f_n(H)}) \in E_j\}$$

is a closed subset of N_d. So either the latter set is equal to N_d, or its complement has open and non-empty intersection with N_d, in which case we can find $G_{j+1} \in D_G \setminus U_j$ since D_G is dense in N_d. Continue this way for all $0 \leq j \leq n$ for which this is possible until $j = n$, if possible.

We show that this j cannot exist, reaching a contradiction.

- If we stop at stage $j < n$, this occurs only if for all $H \in D_G \setminus \{G_0, \dots, G_j\}$

  $$(f_1(H), \dots, f_n(H), e^{f_1(H)}, \dots, e^{f_n(H)}) \in E_j.$$

  However $E_j \subseteq M$ for some hypersurface $M \in \mathcal{H}$. This M is therefore the 0-set of equations of the form

  $$\sum_{i=1,\dots,n} m_i x_i = a, \prod_{i=1,\dots,n} y_i^{m_i} = e^a.$$

  In particular we get that for a dense set of $H \in N_d$

  $$(f_1(H), \dots, f_n(H), e^{f_1(H)}, \dots, e^{f_n(H)}) \in E_j.$$

Since belonging to E_j is a closed property of $\mathbb{C}^{2n}$, and the map $H \mapsto (f_1(H), \dots, f_n(H), e^{f_1(H)}, \dots, e^{f_n(H)})$ is continuous on N_d, we get that for all $H \in N_d$

$$(f_1(H), \dots, f_n(H), e^{f_1(H)}, \dots, e^{f_n(H)}) \in E_j.$$

Then in $C^+(N_d)$

$$\sum_{i=1,\dots,n} m_i f_i \restriction N_d = a,$$

This contradicts the $\mathbb{Q}$-linear independence modulo $\mathbb{C}$ of the vector $f_1 \restriction N_d, \dots, f_n \restriction N_d$ on N_d for a $d \leq b$, which was an assumption of the Lemma.

- Otherwise we can continue up to stage $j = n$. This gives that

$$\{(f_1(G_k), \dots, f_n(G_k)) : 0 \leq k \leq n\}$$

are points in $\mathbb{C}^n$ in general position, i.e. they are not contained in any proper affine subspace of $\mathbb{C}^n$. Since $(f_1(G_k), \dots, f_n(G_k), e^{f_1(G_k)}, \dots, e^{f_n(G_k)})$ are all in V' for all $k \leq n$, and V' is an analytic variety homeomorphic to $\mathbb{C}^k$ (with $k > 0$) via an analytic map, there is a $C^\Omega(V')$-path $(\phi_1, \dots, \phi_{2n})$ connecting all of these points and starting in $(\vec{f}(G_0), e^{\vec{f}(G_0)})$. Now observe that $\vec{\phi} = (\phi_1, \dots, \phi_n)$ is an analytic path passing through $n+1$-points in $\mathbb{C}^n$ in general position. Thus it cannot be contained in any hyperplane of $\mathbb{C}^n$. In particular $(\vec{\phi}, e^{\vec{\phi}}) \in \mathsf{Path}_G$ cannot be contained in any hypersurface belonging to $\mathcal{H}$, which is a contradiction.

The subclaim is proved. □

By the above subclaim, we can fix $G \in N_{c_1}$ and find $H \in N_{c_1} \setminus D_G$ (since this latter set contains a dense open subset of N_{c_1}). Then we can pick an analytic path $(\vec{\phi}, e^{\vec{\phi}})$ in Path_G passing through $(\vec{f}(H), e^{\vec{f}(H)})$ and not contained in any hyperplane in $\mathcal{H}$.

Consider finally the field of fractions of germs $[f]$ of analytic functions $f : U \to \mathbb{C}$ for some $U \subseteq \mathbb{C}$ open neighborhood of 0. around the point 0, where $[f] = [g]$ are equivalent germs if f and g agree on U for some open $U \subseteq \Delta$. This is a differential field $\mathcal{O}^\Omega$ with differential

$$D : \mathcal{O}^\Omega \to \mathcal{O}^\Omega$$

mapping

$$[f]/[g] \to [f'g - g'f]/[g^2]$$

and $\ker(D) = \mathbb{C}$ given by the germs of constant functions.

Since we chose $\vec{\phi}$ so that $(\vec{\phi}, e^{\vec{\phi}})$ is not contained in E for any hypersurface $E \in \mathcal{H}$, we get that $[\vec{\phi}]$ is a vector of elements of the differential field $\mathcal{O}^\Omega$ which are $\mathbb{Q}$-linearly independent modulo $\mathbb{C}$, so that the hypothesis of Ax's theorem apply to these elements. By Ax's result 2.1, we get that

$$\mathrm{Trdg}_{\mathbb{C}}([\vec{\phi}, e^{\vec{\phi}}]) \geq n + 1.$$

Now let

$$J = \{p \in \mathbb{C}[\vec{x}, \vec{y}] : p([\vec{\phi}, e^{\vec{\phi}}]) = 0\},$$

we get that $I = I_{N_{c_1}} \subseteq J$ since $(\vec{\phi}, e^{\vec{\phi}})$ has range contained in $V(I, \mathbb{C})$. In particular

$$\mathrm{Trdg}_{\check{\mathbb{C}}/G}([\vec{f}]_G, [e^{\vec{f}}]_G) = \dim(V(I, C^+(\mathrm{St}(\mathsf{B}))/_G)) =$$
$$= \dim(V(I, \mathbb{C}) \geq \dim(V(J, \mathbb{C})) = \mathrm{Trdg}_{\mathbb{C}}([\vec{\phi}, e^{\vec{\phi}}]) \geq n + 1.$$

This concludes the proof of the Claim and of the Lemma[6]. □

The proof of the Lemma is completed. □

3. Step (3)

From now on we shall assume the reader has some familiarity with the boolean valued model approach to forcing in set theory. Standard references for the material of this section can be [2] or [6], and a detailed account of the results we sketch here can be found in [9]. We briefly sketch the general picture of the forcing theory in the next subsection.

3.1. **A brief outline of forcing over the standard model of set theory.** Recall that for $(V, \in)$ the standard model of ZFC for the first order language $\{\in, =\}$ and B a complete boolean algebra in V we can define (by transfinite recursion) the class of B-names V^{B} given by $\tau \in V$ if τ is a function with domain contained in V^{B} and range contained in B. We can also define forcing relations

$$\begin{aligned} \in_{\mathsf{B}} : (V^{\mathsf{B}})^2 &\to \mathsf{B} \\ (\tau, \sigma) &\mapsto [\![\tau \in \sigma]\!] \end{aligned}$$

$$\begin{aligned} =_{\mathsf{B}} : (V^{\mathsf{B}})^2 &\to \mathsf{B} \\ (\tau, \sigma) &\mapsto [\![\tau = \sigma]\!] \end{aligned}$$

such that $(V^{\mathsf{B}}, \in_{\mathsf{B}}, =_{\mathsf{B}})$ is a full B-valued model for the language of set theory and $[\![\phi]\!] = 1_{\mathsf{B}}$ for all axioms ϕ of ZFC.

Letting

$$[\tau]_G = \{\sigma : [\![\tau = \sigma]\!] \in G\}$$

and

$$[\tau]_G \in [\sigma]_G \text{ if and only if } [\![\tau \in \sigma]\!] \in G$$

We also have that

$$[\![\phi(\tau_1, \ldots, \tau_n)]\!] \in G \text{ if and only if } V^{\mathsf{B}}/G \models \phi([\tau_1]_G, \ldots, [\tau_n]_G)$$

for all formulae $\phi(x_1, \ldots, x_n)$ in this language and all $G \in \mathrm{St}(\mathsf{B})$.

Finally we recall that G is V-generic for a cba B if $G \cap D$ is nonempty for all D dense subset of B^+ and that for such a G and all $\tau \in V^{\mathsf{B}}$ we can define:

$$\tau_G = \{\sigma_G : \tau(\sigma) \in G\}$$

and let

$$V[G] = \{\tau_G : \tau \in V^{\mathsf{B}}\}.$$

With this choice of G we have that the map $[\tau]_G \mapsto \tau_G$ define an isomorphism of $(V^{\mathsf{B}}/G, \in_G)$ with $(V[G], \in)$.

[6]With some extra work one can check that $J = I$ for an open dense set of $H \in N_{c_1}$.

Moreover any element $u \in V$ has a canonical name $\check{u} \in V^{\mathsf{B}}$ such that $\check{u}_G = u$ whenever G is V-generic for B.

It s well known that V-generic filters cannot exist for an atomless complete boolean algebra, nonetheless there is a wide spectrum of solutions to overcome this issue, and work under the assumption that for any such B, V-generic filters can be found.

3.2. **The relation between $C^+(\mathrm{St}(\mathsf{B}))$ and V^{B}.** We have the following theorem linking the boolean valued model $C^+(\mathrm{St}(\mathsf{B}))$ to the set theoretic boolean valued model V^{B} (see [9, Theorem 4.3.5]):

Theorem 3.1. *Let* B *be a cba,* $b \in \mathsf{B}$*, and* $\{U_n : n \in \omega\}$ *be a countable base for the euclidean topology on* $\mathbb{C}$*. Given* $f \in C^+(N_b)$ *for some* $b \in \mathsf{B}$*, let* $\tau_f \in V^{\mathsf{B}}$ *be a* B*-name for the unique object in* V^{B} *satisfying in* V^{B}*:*

$$[\![\tau_f \in U_n]\!] = \mathsf{Reg}\left(f^{-1}[U_n]\right).$$

Given R *a forcing relation on* $C^+(N_b)^n$ *let* $\bar{R} \in V^{\mathsf{B}}$ *be a* B*-name for a* n*-ary relation on the* n*-tuples of complex numbers* $\mathbb{C}^n$ *as computed in* V^{B} *such that*

$$\left[\!\left[\bar{R}(\tau_{f_1}, \dots, \tau_{f_n})\right]\!\right]^{V^{\mathsf{B}}} = R(f_1, \dots, f_n).$$

Then the assignment $f \mapsto \tau_f$*,* $R \mapsto \bar{R}$ *is an embedding of the boolean valued models* $C^+(\mathrm{St}(\mathsf{B}))$ *and* $C^+(N_b)$ *for* $b \in \mathsf{B}$ *in the boolean valued model* V^{B} *such that:*

- *the equality forcing relation on* $C^+(\mathrm{St}(\mathsf{B}))$ *is mapped to the equality relation on* V^{B}*;*
- *for all* $\tau \in V^{\mathsf{B}}$ *such that*

  $$[\![\tau \text{ is a complex number }]\!]^{V^{\mathsf{B}}} = b,$$

  there exists $f \in C^+(N_b)$ *such that*

  $$[\![\tau = \tau_f]\!]^{V^{\mathsf{B}}} = b;$$

- *for all forcing relations* R *on* $C^+(\mathrm{St}(\mathsf{B}))^n$ *and all* $f_1, \dots, f_n \in C^+(\mathrm{St}(\mathsf{B}))$

  $$\left[\!\left[\bar{R}(\tau_{f_1}, \dots, \tau_{f_n})\right]\!\right]^{V^{\mathsf{B}}} = R(f_1, \dots, f_n).$$

3.3. **Shoenfield's absoluteness.** We say that $A \subseteq \mathbb{C}^m$ is a Σ^1_2-property if there is a Borel predicate $R \subseteq \mathbb{C}^{<\omega}$ and $\vec{a} \in \mathbb{C}^{<\omega}$ such that $A(\vec{a})$ holds if and only if

$$\exists x \forall y R(x, y, \vec{a}).$$

Given a Borel predicate $R \subseteq \mathbb{C}^n$ and a complete boolean algebra B, we let

$$\begin{aligned} R_{\mathsf{B}} :& C^+(\mathrm{St}(\mathsf{B}))^n \to \mathsf{B} \\ & (f_1, \dots, f_n) \mapsto \mathsf{Reg}\left(\{H : R(f_1(H), \dots, f_n(H))\}\right) \end{aligned}$$

and

$$\begin{aligned} \bar{R}_{\mathsf{B}} :& (V^{\mathsf{B}})^n \to \mathsf{B} \\ & (\tau_1, \dots, \tau_n) \mapsto \bigwedge_{j=1,\dots,n} [\![\tau_j \text{ is a complex number}]\!] \wedge R_{\mathsf{B}}(f_{\tau_1}, \dots, f_{\tau_n}) \end{aligned}$$

Theorem 3.2 (Shoenfield's absoluteness)**.** *Assume A is a Σ^1_2-property defined by the Borel predicate R as $\exists y \forall x R(x, y, \vec{a})$. Then $A(a_1, \dots, a_n)$ holds in V for complex numbers $a_1, \dots, a_n$ if and only if*

$$\left[\!\left[\exists x \forall y \bar{R}_{\mathsf{B}}(x, y, \check{a}_1, \dots, \check{a}_n)\right]\!\right]_{\mathsf{B}} = 1_{\mathsf{B}}$$

for some complete boolean algebra B*.*

3.4. WSP **holds for $\mathbb{C}$ relative to a countable subfield.** We can now prove Theorem 1.3: Shoenfield's absoluteness gives a simple proof of the following:

Corollary 3.3. *$C^+(\mathrm{St}(\mathsf{B}))/G$ is an algebraically closed field for any $G \in \mathrm{St}(\mathsf{B})$ and for any complete boolean algebra* B*.*

Proof. The graph of the multiplication and of the addition are Borel relations on $\mathbb{C}^3$, and the field axioms and the algebraic closure axioms are expressible as Σ_2-properties of these operations. □

Now let B be the complete boolean algebra of regular sets in $\mathbb{C}^{\mathbb{N}}$ where $\mathbb{C}$ is endowed with the discrete topology. In $V[G]$ there is a new bijection f of $\mathbb{C}^V = \mathbb{C}$ with $\mathbb{N}$ given by $f(n) = a$ if and only if

$$\{g \in \mathbb{C}^{\mathbb{N}} : g(n) = a\}$$

is in G. Moreover

$$V[G] \models \phi((\tau_1)_G, \dots, (\tau_n)_G) \text{ if and only if } [\![\phi(\tau_1, \dots, \tau_n)]\!] \in G.$$

Now we observe that the following holds in $V[G]$ if G is V-generic for B:

- $C^+(\mathrm{St}(\mathsf{B}), \mathbb{C})/G$ is isomorphic to the complex numbers of $V[G]$ via the map
$$[f]_G = (\tau_f)_G$$
- $e^{V[G]}$ is the unique analytic function on the field
$$\mathbb{C}^{V[G]} = \{\tau_G : [\![\tau \text{ is a complex number}]\!] \in G\}$$
whose power series expansion is
$$\sum_{n=0,\infty} x^n/n!.$$
Moreover $e^{V[G]}$ is the graph of $[f]_G \mapsto [e^f]_G$ modulo the isomorphism of $C^+(\mathrm{St}(\mathsf{B}), \mathbb{C})/G$ with $\mathbb{C}^{V[G]}$,
- $\mathbb{C} \cap V = \mathbb{C}^V = \check{\mathbb{C}}_G$ is identified with $\check{\mathbb{C}}/G$ modulo the above isomorphism and $\mathbb{N}^{V[G]} \cap V = \mathbb{N}^V = \mathbb{N}^{V[G]} = \check{\mathbb{N}}_G$ are the natural numbers both in V and in $V[G]$.
- The Key Lemmas for $\vec{f}$ give that
$$\mathrm{Trdg}_{\mathbb{C}^V}([\vec{f}]_G, e^{[\vec{f}]_G}) \geq n$$
whenever $[\vec{f}]_G$ is a family of $\mathbb{Q}$-linearly independent vectors modulo $\mathbb{C}^V$, since the boolean value of this statement is 1_{B} (notice that such vectors are identified to complex numbers of $V[G] \setminus V$, since the complex numbers of V are represented by the locally constant functions).
- $V[G]$ models that $\mathbb{C}^V$ is a countable exponentially and algebraically closed subfield of $\mathbb{C}^{V[G]}$ and the latter is the field of complex numbers in $V[G]$.

In particular $V[G]$ models that:

There exists $\mathbb{C}^V$*, countable algebraically and exponentially closed subfield of* $\mathbb{C}^{V[G]}$*, such that for all* $\vec{f} \in (\mathbb{C}^{V[G]})^n$

$$\mathrm{Trdg}_{\mathbb{C}^V}([\vec{f}]_G, e^{[\vec{f}]_G}) \geq \mathrm{Ldim}_{\mathbb{Q}}(\vec{f}/\mathbb{C}^V),$$

with equality holding only if $\vec{f} \subseteq \mathbb{C}^V$.

This is a Σ^1_2-statement in no parameters and a few (lightface definable) Borel predicates which holds in

$$(\mathbb{C}^{V[G]}, \mathbb{C}^V, \mathbb{N}^V, e^{V[G]}, \mathrm{Trdg}_{\mathbb{C}^V}, \mathrm{Ldim}_{\mathbb{C}^V}).$$

By Shoenfield's absoluteness it holds in V, since all of the above predicates are Borel.

More precisely the forcing theorem gives that V^{B} models the above statement with boolean value 1_{B} and Shoenfield's absoluteness shows that it also holds in V.

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