TIME-FREQUENCY ANALYSIS OF BORN-JORDAN PSEUDODIFFERENTIAL OPERATORS

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Abstract. Born-Jordan operators are a class of pseudodifferential operators arising as a generalization of the quantization rule for polynomials on the phase space introduced by Born and Jordan in 1925. The weak definition of such operators involves the Born-Jordan distribution, first introduced by Cohen in 1966 as a member of the Cohen class. We perform a time-frequency analysis of the Cohen kernel of the Born-Jordan distribution, using modulation and Wiener amalgam spaces. We then provide sufficient and necessary conditions for Born-Jordan operators to be bounded on modulation spaces. We use modulation spaces as appropriate symbols classes.

1. Introduction

In 1925 Born and Jordan [2] introduced for the first time a rigorous mathematical explanation of the notion of "quantization". This rule was initially restricted to polynomials as symbol classes but was later extended to the class of tempered distribution $\mathcal{S}'(\mathbb{R}^{2d})$ [1, 6]. Roughly speaking, a quantization is a rule which assigns an operator to a function (called symbol) on the phase space $\mathbb{R}^{2d}$. The Born-Jordan quantization was soon superseded by the most famous Weyl quantization rule proposed by Weyl in [38], giving rise to the well-known Weyl operators (transforms) (see, e.g. [39]).

Recently there has been a regain in interest in the Born-Jordan quantization, both in Quantum Physics and Time-frequency Analysis [17]. The second of us has proved that it is the correct rule if one wants matrix and wave mechanics to be equivalent quantum theories [16]. Moreover, as a time-frequency representation, the Born-Jordan distribution has been proved to be better than the Wigner distribution since it damps very well the unwanted "ghost frequencies", as shown in [1, 37]. For a throughout and rigorous mathematical explanation of these phenomena we refer to [9] whereas [25, Chapter 5] contains the relevant engineering literature about the geometry of interferences and kernel design.

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To be more specific, the (cross-)Wigner distribution of signals f, g in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ is defined by

$$W(f,g)(x,\omega) = \int_{\mathbb{R}^d} e^{-2\pi i y\omega} f(x+\frac{y}{2})\overline{g(x-\frac{y}{2})}\, dy. \tag{1}$$

The Weyl operator $\mathrm{Op_W}(a)$ with symbol $a \in \mathcal{S}'(\mathbb{R}^{2d})$ can be defined in terms of the Wigner distribution by the formula

$$\langle \mathrm{Op_W}(a) f, g\rangle = \langle a, W(g,f)\rangle.$$

For $z = (x,\omega)$, consider the Cohen kernel

$$\Theta(z) := \mathrm{sinc}(x\omega) = \begin{cases} \dfrac{\sin(\pi x\omega)}{\pi x\omega} & \text{for } \omega x \neq 0 \\ 1 & \text{for } \omega x = 0. \end{cases} \tag{2}$$

The (cross-)Born-Jordan distribution $Q(f,g)$ is then defined by

$$Q(f,g) = W(f,g) * \Theta_\sigma, \quad f,g \in \mathcal{S}(\mathbb{R}^d), \tag{3}$$

where Θ_σ is the symplectic Fourier transform recalled in (22) below. Likewise the Weyl operator, a Born-Jordan operator with symbol $a \in \mathcal{S}'(\mathbb{R}^{2d})$ can be defined as

$$\langle \mathrm{Op_{BJ}}(a) f, g\rangle = \langle a, Q(g,f)\rangle \quad f,g \in \mathcal{S}(\mathbb{R}^d). \tag{4}$$

Any pseudodifferential operator admits a representation in the Born-Jordan form $\mathrm{Op_{BJ}}(a)$, as stated in [8].

Now, a first relevant feature of this work is to have computed the Cohen kernel Θ_σ explicitly (cf. the subsequent Proposition 3.4). Namely

$$\Theta_\sigma(\zeta_1,\zeta_2) = \begin{cases} -2\,\mathrm{Ci}(4\pi|\zeta_1\zeta_2|), & (\zeta_1,\zeta_2) \in \mathbb{R}^2,\ d=1 \\ \mathcal{F}(\chi_{\{|s|\geq 2\}}|s|^{d-2})(\zeta_1\zeta_2), & (\zeta_1\zeta_2) \in \mathbb{R}^{2d},\ d \geq 2, \end{cases}$$

where $\chi_{\{|s|\geq 2\}}$ is the characteristic function of the set $\{s \in \mathbb{R} : |s| \geq 2\}$ and where

$$\mathrm{Ci}(t) = -\int_t^{+\infty} \frac{\cos s}{s}\, ds, \quad t \in \mathbb{R}. \tag{5}$$

is the cosine integral function.

This expression of Θ_σ shows that this kernel behaves badly in general: it does not even belong to L^∞_{loc} (see Corollary 3.5) and has no decay at infinity (see Corollary 3.6). In spite of these facts, it was proved in [9] that some directional smoothing effect is still present, but the analysis carried on there also shows the necessity of a systematic and general study of the boundedness of such operators $\mathrm{Op_{BJ}}(a)$ on modulation spaces, in dependence of the Born-Jordan symbol space. Modulation spaces, introduced by Feichtinger in [19], have been widely employed in the literature to investigate properties of pseudodifferential operators, in particular we highlight the contributions [3, 4, 14, 24, 28, 31, 32, 33, 34, 35, 36]. For their definition and main properties we refer to the successive section.

The main result concerning the sufficient boundedness conditions of Born-Jordan operators on modulation spaces shows that they behave similarly to Weyl pseudodifferential operators or any other τ-form of pseudodifferential operators. For comparison, see [12, Theorem 5.2, Proposition 5.3], [13, Theorem 1.1] and [35, Theorem 4.3]. The necessary boundedness conditions still contain some open problems, as shown in the following result. We denote q' the conjugate exponent of $q \in [1, \infty]$; it is defined by $1/q + 1/q' = 1$.

Theorem 1.1. *Consider* $1 \leq p, q, r_1, r_2 \leq \infty$*, such that*

$$p \leq q' \tag{6}$$

and

$$q \leq \min\{r_1, r_2, r_1', r_2'\}. \tag{7}$$

Then the Born-Jordan operator $\mathrm{Op}_{\mathrm{BJ}}(a)$*, from* $\mathcal{S}(\mathbb{R}^d)$ *to* $\mathcal{S}'(\mathbb{R}^d)$*, having symbol* $a \in M^{p,q}(\mathbb{R}^{2d})$*, extends uniquely to a bounded operator on* $\mathcal{M}^{r_1,r_2}(\mathbb{R}^d)$*, with the estimate*

$$\| \mathrm{Op}_{\mathrm{BJ}}(a) f\|_{\mathcal{M}^{r_1,r_2}} \lesssim \|a\|_{M^{p,q}} \|f\|_{\mathcal{M}^{r_1,r_2}} \quad f \in \mathcal{M}^{r_1,r_2}. \tag{8}$$

Vice-versa, if this conclusion holds true, the constraints (6) *is satisfied and it must hold*

$$\max\left\{\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_1'}, \frac{1}{r_2'}\right\} \leq \frac{1}{q} + \frac{1}{p} \tag{9}$$

which is (7) *when* $p = \infty$.

Notice that the condition (9) is weaker than (7) when $p < \infty$. The condition (9) is obtained by working with rescaled Gaussians which provide the best localization in terms of Wigner distribution (cf. [29]). On the Fourier side, the Born-Jordan distribution is the point-wise multiplication of the Wigner distribution with the kernel Θ. This reasoning conduces to conjecture that the condition (9) should be the optimal one so that the sufficient boundedness conditions for Born-Jordan operators might be weaker than the corresponding ones for Weyl and τ-pseudodifferential operators. But the matter is really subtle and requires a new and most refined analysis of the kernel Θ. In particular the zeroes of the Θ function should play a key for a thorough understanding of such operators, which certainly deserve further study.

The paper is organized as follows. Section 2 is devoted to some preliminary results from Time-frequency Analysis. In Section 3 we perform an analysis of the kernel Θ and we prove the above formula for Θ_σ. In Sections 4 and 5 we study the

Cohen kernels and the boundedness of Born-Jordan operators in the framework of modulation spaces.

2. Preliminaries

In this section we recall the definition of the spaces involved in our study and present the main time-frequency tools used.

Modulation and Wiener amalgam spaces. The modulation and Wiener amalgam space norms are a measure of the joint time-frequency distribution of $f \in \mathcal{S}'$. For their basic properties we refer to the original literature [18, 19, 20] and the textbooks [15, 23].

Let $f \in \mathcal{S}'(\mathbb{R}^d)$. We define the short-time Fourier transform of f as

$$V_g f(z) = \langle f, \pi(z)g\rangle = \mathcal{F}[fT_x g](\omega) = \int_{\mathbb{R}^d} f(y)\, \overline{g(y-x)}\, e^{-2\pi i y\omega}\, dy \tag{10}$$

for $z = (x,\omega) \in \mathbb{R}^d \times \mathbb{R}^d$.

Given a non-zero window $g \in \mathcal{S}(\mathbb{R}^d)$, $1 \le p, q \le \infty$, the *modulation space* $M^{p,q}(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $V_g f \in L^{p,q}(\mathbb{R}^{2d})$ (weighted mixed-norm spaces). The norm on $M^{p,q}$ is

$$\|f\|_{M^{p,q}} = \|V_g f\|_{L^{p,q}} = \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p (x,\omega)^p\, dx\right)^{q/p} d\omega\right)^{1/p}$$

(with natural modifications when $p = \infty$ or $q = \infty$). If $p = q$, we write M^p instead of $M^{p,p}$.

The space $M^{p,q}(\mathbb{R}^d)$ is a Banach space whose definition is independent of the choice of the window g, in the sense that different nonzero window functions yield equivalent norms. The modulation space $M^{\infty,1}$ is also called Sjöstrand's class [31].

The closure of $\mathcal{S}(\mathbb{R}^d)$ in the $M^{p,q}$-norm is denoted $\mathcal{M}^{p,q}(\mathbb{R}^d)$. Then

$$\mathcal{M}^{p,q}(\mathbb{R}^d) \subseteq M^{p,q}(\mathbb{R}^d), \quad \text{and } \mathcal{M}^{p,q}(\mathbb{R}^d) = M^{p,q}(\mathbb{R}^d),$$

provided $p < \infty$ and $q < \infty$.

Recalling that the conjugate exponent p' of $p \in [1,\infty]$ is defined by $1/p + 1/p' = 1$, for any $p, q \in [1,\infty]$ the inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ extends to a continuous sesquilinear map $M^{p,q}(\mathbb{R}^d) \times M^{p',q'}(\mathbb{R}^d) \to \mathbb{C}$.

Modulation spaces enjoy the following inclusion properties:

$$\mathcal{S}(\mathbb{R}^d) \subseteq M^{p_1,q_1}(\mathbb{R}^d) \subseteq M^{p_2,q_2}(\mathbb{R}^d) \subseteq \mathcal{S}'(\mathbb{R}^d), \quad p_1 \le p_2,\ q_1 \le q_2. \tag{11}$$

The Wiener amalgam spaces $W(\mathcal{F}L^p, L^q)(\mathbb{R}^d)$ are given by the distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{W(\mathcal{F}L^p,L^q)(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|V_gf(x,\omega)|^p\,d\omega\right)^{q/p}dx\right)^{1/q} < \infty$$

(with obvious changes for $p = \infty$ or $q = \infty$). Using Parseval identity in (10), we can write the so-called fundamental identity of time-frequency analysis $V_gf(x,\omega) = e^{-2\pi i x\omega}V_{\hat{g}}\hat{f}(\omega,-x)$, hence

$$|V_gf(x,\omega)| = |V_{\hat{g}}\hat{f}(\omega,-x)| = |\mathcal{F}(\hat{f}\,T_\omega\overline{\hat{g}})(-x)|$$

so that

$$\|f\|_{M^{p,q}} = \left(\int_{\mathbb{R}^d}\|\hat{f}\,T_\omega\overline{\hat{g}}\|^q_{\mathcal{F}L^p}(\omega)\,d\omega\right)^{1/q} = \|\hat{f}\|_{W(\mathcal{F}L^p,L^q)}.$$

This means that these Wiener amalgam spaces are simply the image under Fourier transform of modulation spaces:

$$\mathcal{F}(M^{p,q}) = W(\mathcal{F}L^p, L^q). \tag{12}$$

We will often use the following product property of Wiener amalgam spaces ([18, Theorem 1 (v)]):

$$f \in W(\mathcal{F}L^1, L^\infty) \text{ and } g \in W(\mathcal{F}L^p, L^q) \Longrightarrow fg \in W(\mathcal{F}L^p, L^q). \tag{13}$$

In order to prove the necessary boundedness conditions for Born-Jordan operators we shall use the dilation properties for Gaussian functions. Precisely, consider $\varphi(x) = e^{-\pi|x|^2}$ and define

$$\varphi_\lambda(x) = \varphi(\sqrt{\lambda}x) = e^{-\pi\lambda|x|^2}, \quad \lambda > 0. \tag{14}$$

The dilation properties for the Gaussian φ_λ in modulation spaces were proved in [35, Lemma 1.8] (see also [7, Lemma 3.2]).

Lemma 2.1. *For $1 \leq p, q \leq \infty$, we have*

$$\|\varphi_\lambda\|_{M^{p,q}} \asymp \lambda^{-\frac{d}{2q'}} \quad \text{as } \lambda \to +\infty \tag{15}$$

$$\|\varphi_\lambda\|_{M^{p,q}} \asymp \lambda^{-\frac{d}{2p}} \quad \text{as } \lambda \to 0^+. \tag{16}$$

The following dilation properties are a straightforward generalization of [9, Lemma 2.3].

Lemma 2.2. *Consider $1 \leq p, q \leq \infty$, $\psi \in C_c^\infty(\mathbb{R}^d) \setminus \{0\}$ and $\lambda > 0$. Then*

$$\|\psi(\sqrt{\lambda}\,\cdot)\|_{W(\mathcal{F}L^p,L^q)} \asymp \lambda^{-\frac{d}{2p'}} \quad \text{as } \lambda \to +\infty \tag{17}$$

$$\|\psi(\sqrt{\lambda}\,\cdot)\|_{W(\mathcal{F}L^p,L^q)} \asymp \lambda^{-\frac{d}{2q}} \quad \text{as } \lambda \to 0^+. \tag{18}$$

The same conclusion holds uniformly with respect to λ if ψ varies in bounded subsets of $C_c^\infty(\mathbb{R}^d)$.

Another tool for obtaining the optimality of our results is the cross-Wigner distribution of rescaled Gaussian functions. The proof is a straightforward computation (see Prop. 244 in [15]):

Lemma 2.3. *Consider $\varphi(x) = e^{-\pi|x|^2}$ and φ_λ as in (14). Then*

$$W(\varphi,\varphi_\lambda)(x,\omega) = \frac{2^d}{(\lambda+1)^{\frac{d}{2}}} e^{-\frac{4\pi\lambda}{\lambda+1}|x|^2} e^{-\frac{4\pi}{\lambda+1}|\omega|^2} e^{-4\pi i \frac{\lambda-1}{\lambda+1} x\omega}. \tag{19}$$

It follows that:

Corollary 2.4. *Consider φ and φ_λ as in the assumptions of Lemma 2.3. Then*

$$\mathcal{F}W(\varphi,\varphi_\lambda)(\zeta_1,\zeta_2) = \frac{1}{(\lambda+1)^{\frac{d}{2}}} e^{-\frac{\pi}{\lambda+1}\zeta_1^2} e^{-\frac{\pi\lambda}{\lambda+1}\zeta_2^2} e^{-\pi i \frac{\lambda-1}{\lambda+1}\zeta_1\zeta_2}. \tag{20}$$

Proof. Formula (20) is easily obtained from (19) using well-known Gaussian integral formulas; it can also be painlessly obtained from (19) by observing that for any functions $\psi,\phi \in L^2(\mathbb{R}^d)$ the following relation between the cross-Wigner distribution and its Fourier transform holds:

$$\mathcal{F}W(\psi,\phi)(x,\omega) = 2^{-d} W(\psi,\phi^\vee)(\tfrac{1}{2}\omega, \tfrac{1}{2}x)$$

where $\phi^\vee(x) = \phi(-x)$ (see formula (9.27) in [15], or formula (1.90) in Folland [22]). $\square$

We denote by σ the symplectic form on the phase space $\mathbb{R}^{2d} \equiv \mathbb{R}^d \times \mathbb{R}^d$; the phase space variable is denoted $z = (x,\omega)$ and the dual variable by $\zeta = (\zeta_1,\zeta_2)$. By definition $\sigma(z,\zeta) = Jz\cdot\zeta = \omega\cdot\zeta_1 - x\cdot\zeta_2$, where

$$J = \begin{pmatrix} 0_{d\times d} & I_{d\times d} \\ -I_{d\times d} & 0_{d\times d} \end{pmatrix}. \tag{21}$$

The Fourier transform of a function f on $\mathbb{R}^d$ is normalized as

$$\mathcal{F}f(\omega) = \int_{\mathbb{R}^d} e^{-2\pi i x\omega} f(x)\,dx,$$

and the symplectic Fourier transform of a function F on the phase space $\mathbb{R}^{2d}$ is

$$\mathcal{F}_\sigma F(\zeta) = \int_{\mathbb{R}^{2d}} e^{-2\pi i \sigma(\zeta,z)} F(z)\,dz. \tag{22}$$

Observe that $\mathcal{F}_\sigma F(\zeta) = \mathcal{F}F(J\zeta)$. Hence the symplectic Fourier transform of the Wigner distribution (19) is given by

$$\mathcal{F}_\sigma W(\varphi,\varphi_\lambda)(\zeta_1,\zeta_2) = \frac{1}{(\lambda+1)^{\frac{d}{2}}} e^{-\frac{\pi\lambda}{\lambda+1}\zeta_1^2} e^{-\frac{\pi}{\lambda+1}\zeta_2^2} e^{\pi i \frac{\lambda-1}{\lambda+1}\zeta_1\zeta_2}. \tag{23}$$

We will also use the important relation

$$\mathcal{F}_\sigma[F*G]=\mathcal{F}_\sigma F\,\mathcal{F}_\sigma G. \tag{24}$$

The convolution relations for modulation spaces are essential in the proof of the boundedness of a Born-Jordan operator on these spaces and were proved in [10, Proposition 2.4]:

Proposition 2.1. *Let* $1\leq p,q,r,s,t\leq\infty$. *If*

$$\frac{1}{p}+\frac{1}{q}-1=\frac{1}{r},\qquad \frac{1}{t}+\frac{1}{t'}=1\,,$$

then

$$M^{p,st}(\mathbb{R}^d)*M^{q,st'}(\mathbb{R}^d)\hookrightarrow M^{r,s}(\mathbb{R}^d) \tag{25}$$

with $\|f*h\|_{M^{r,s}}\lesssim\|f\|_{M^{p,st}}\|h\|_{M^{q,st'}}$.

We also recall the useful result proved in [9, Lemma 5.1].

Lemma 2.5. *Let* $\chi\in C_c^\infty(\mathbb{R})$. *Then, for* $\zeta_1,\zeta_2\in\mathbb{R}^d$, *the function* $\chi(\zeta_1\zeta_2)$ *belongs to* $W(\mathcal{F}L^1,L^\infty)(\mathbb{R}^{2d})$.

3. Analysis of the Cohen kernel Θ

Consider the Cohen kernel Θ defined in (2). Obviously $\Theta\in\mathcal{C}^\infty(\mathbb{R}^{2d})\cap L^\infty(\mathbb{R}^{2d})$ but displays a vary bad decay at infinity, as clarified in what follows.

Proposition 3.1. *For* $1\leq p<\infty$, *the function* $\Theta\notin L^p(\mathbb{R}^{2d})$.

Proof. Observe that, for $t\in\mathbb{R}$, $|t|\leq 1/2$, the function $\mathrm{sinc}t$ satisfies $|\mathrm{sinc}\,t|\geq 2/\pi$. Hence, for any $1\leq p<\infty$,

$$\begin{aligned}\int_{\mathbb{R}^{2d}}|\Theta(x,\omega)|^p\,dxd\omega&=\int_{\mathbb{R}^{2d}}|\mathrm{sinc}(x\omega)|^p\,dxd\omega\\&\geq\int_{|x\omega|\leq 1/2}|\mathrm{sinc}(x\omega)|^p\,dxd\omega\\&\geq\left(\frac{2}{\pi}\right)^p\int_{|x\omega|\leq 1/2}dxd\omega\\&=\left(\frac{2}{\pi}\right)^p meas\{(x,\omega)\,:\,|x\omega|\leq 1/2\}=+\infty.\end{aligned}$$

This concludes the proof. ∎

We continue our investigation of the function Θ by looking for the right Wiener amalgam and modulation spaces containing this function. For this reason, we first reckon explicitly the STFT of the Θ function, with respect to the Gaussian window $g(x,\omega)=e^{-\pi x^2}e^{-\pi\omega^2}\in\mathcal{S}(\mathbb{R}^{2d})$.

Proposition 3.2. *For* $z_1, z_2, \zeta_1, \zeta_2 \in \mathbb{R}^d$,

$$\begin{aligned} &V_g\Theta(z_1, z_2, \zeta_1, \zeta_2) \\ &= \int_{-1/2}^{1/2} \frac{1}{(t^2+1)^{d/2}} e^{-2\pi i[\frac{1}{t}\zeta_1\zeta_2 + \frac{t}{t^2+1}(z_1-\frac{1}{t}\zeta_2)(z_2-\frac{1}{t}\zeta_1)]} e^{-\pi\frac{t^2}{t^2+1}[(z_1-\frac{1}{t}\zeta_2)^2+(z_2-\frac{1}{t}\zeta_1)^2]}\, dt. \end{aligned} \tag{26}$$

Proof. We write $\Theta(z_1, z_2) = F_1(z_1, z_2) + F_2(z_1, z_2)$, where $F_1(z_1, z_2) = \int_0^{1/2} e^{2\pi i z_1 z_2 t} dt$ and $F_2(z) = F_1(Jz)$, $z = (z_1, z_2)$. Let us first reckon $V_g F_1(z, \zeta)$, $z = (z_1, z_2)$, $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$, where g is the Gaussian function above. For $t > 0$ we define the function $H_t(z_1, z_2) = e^{2\pi i t z_1 z_2}$ and observe that

$$\mathcal{F}H_t(\zeta_1, \zeta_2) = \frac{1}{t^d} e^{-2\pi i \frac{1}{t}\zeta_1\zeta_2} \tag{27}$$

(cf. [22, Appendix A, Theorem 2]). By the Dominated Convergence Theorem,

$$
\begin{aligned}
V_gF_1(z,\zeta) &= \int_0^{1/2} \mathcal{F}(H_tT_zg)(\zeta)dt = \int_0^{1/2} (\mathcal{F}(H_t)*M_{-z}\hat{g})(\zeta_1,\zeta_2)dt \\
&= \int_0^{1/2} \frac{1}{t^d}\int_{\mathbb{R}^{2d}} e^{-2\pi i\frac{1}{t}(\zeta_1-y_1)\cdot(\zeta_2-y_2)}e^{-2\pi i(z_1,z_2)\cdot(y_1,y_2)}e^{-\pi y_1^2}e^{-\pi y_2^2}\,dy_1dy_2dt \\
&= \int_0^{1/2} \frac{1}{t^d}e^{-2\pi i\frac{1}{t}\zeta_1\zeta_2}\int_{\mathbb{R}^{2d}} e^{-2\pi i\frac{1}{t}y_1y_2+2\pi i\frac{1}{t}(\zeta_2y_1+\zeta_1y_2)-2\pi i(z_1y_1+z_2y_2)}e^{-\pi(y_1^2+y_2^2)}\,dy_1dy_2dt \\
&= \int_0^{1/2} \frac{1}{t^d}e^{-2\pi i\frac{1}{t}\zeta_1\zeta_2}\int_{\mathbb{R}^d} e^{2\pi i(\frac{1}{t}\zeta_1y_2-z_2y_2)}e^{-\pi y_2^2} \\
&\qquad\cdot\left(\int_{\mathbb{R}^d} e^{-2\pi iy_1\cdot(\frac{1}{t}y_2-\frac{1}{t}\zeta_2+z_1)}e^{-\pi y_1^2}\,dy_1\right)dy_2dt \\
&= \int_0^{1/2} \frac{1}{t^d}e^{-2\pi i\frac{1}{t}\zeta_1\zeta_2}\int_{\mathbb{R}^d} e^{-2\pi iy_2\cdot(z_2-\frac{1}{t}\zeta_1)}e^{-\pi y_2^2}e^{-\pi(\frac{1}{t}y_2-\frac{1}{t}\zeta_2+z_1)^2}\,dy_2dt \\
&= \int_0^{1/2} \frac{1}{t^d}e^{-2\pi i\frac{1}{t}\zeta_1\zeta_2}e^{-\pi(z_1-\frac{1}{t}\zeta_2)^2}\int_{\mathbb{R}^d} e^{-2\pi iy_2\cdot(z_2-\frac{1}{t}\zeta_1)}e^{-\pi((1+\frac{1}{t^2})y_2^2-2(z_1-\frac{1}{t}\zeta_2)\cdot\frac{1}{t}y_2)}dy_2dt \\
&= \int_0^{1/2} \frac{1}{t^d}e^{-2\pi i\frac{1}{t}\zeta_1\zeta_2}e^{-\pi\frac{t^2}{t^2+1}(u_1-\frac{1}{t}\zeta_2)^2}\int_{\mathbb{R}^d} e^{-2\pi iy_2\cdot(u_2-\frac{1}{t}\zeta_1)} \\
&\qquad\cdot\ e^{-\pi(\frac{\sqrt{t^2+1}}{t}y_2-\frac{t}{\sqrt{t^2+1}}(z_1-\frac{1}{t}\zeta_2))^2}dy_2dt \\
&= \int_0^{1/2} \frac{1}{(t^2+1)^{d/2}}e^{-2\pi i\frac{1}{t}\zeta_1\zeta_2}e^{-\pi\frac{t^2}{t^2+1}(z_1-\frac{1}{t}\zeta_2)^2} \\
&\qquad\cdot\int_{\mathbb{R}^d} e^{-2\pi i(\frac{t}{\sqrt{t^2+1}}w+\frac{t}{t^2+1}(z_1-\frac{1}{t}\zeta_2))(z_2-\frac{1}{t}\zeta_1)}e^{-\pi w^2}dwdt \\
&= \int_0^{1/2} \frac{1}{(t^2+1)^{d/2}}e^{-2\pi i\frac{1}{t}\zeta_1\zeta_2}e^{-\pi\frac{t^2}{t^2+1}(z_1-\frac{1}{t}\zeta_2)^2}e^{-\pi\frac{t^2}{t^2+1}(z_2-\frac{1}{t}\zeta_1)^2} \\
&\qquad\cdot\ e^{-2\pi i\frac{t}{t^2+1}(z_1-\frac{1}{t}\zeta_2)(z_2-\frac{1}{t}\zeta_1)}dt.
\end{aligned}
$$

Now, an easy computation shows

$$V_gF_2(z,\zeta) = V_gF_1(Jz,J\zeta)$$

so that $V_g\Theta = V_gF_1 + V_gF_2$ and we obtain (26). ◻

Proposition 3.3. *The function* Θ *in* (2) *belongs to* $W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$.

Proof. We simply have to calculate

$$\sup_{z\in\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} |V_g\Theta(z,\zeta)|d\zeta.$$

From (26) we observe that

$$\begin{aligned}\|V_g\Theta(z,\cdot)\|_1 &\leq \int_{-1/2}^{1/2}\int_{\mathbb{R}^{2d}} \frac{1}{(t^2+1)^{d/2}} e^{-\pi\frac{t^2}{t^2+1}(z_1-\frac{1}{t}\zeta_2)^2} e^{-\pi\frac{t^2}{t^2+1}(z_2-\frac{1}{t}\zeta_1)^2} d\zeta_1 d\zeta_2 dt\\ &= \int_{-1/2}^{1/2}\int_{\mathbb{R}^{2d}} \frac{1}{(t^2+1)^{d/2}} e^{-\pi\frac{1}{t^2+1}(tz_1-\zeta_2)^2} e^{-\pi\frac{1}{t^2+1}(tz_2-\zeta_1)^2} d\zeta_1 d\zeta_2 dt\\ &= \int_{-1/2}^{1/2}\int_{\mathbb{R}^{2d}} (t^2+1)^{d/2} e^{-\pi(v_1^2+v_2^2)} dv_1 dv_2 dt = C < \infty,\end{aligned}$$

from which the claim follows. □

Using the STFT of the function Θ in (26) we observe that

$$\|V_g\Theta(\cdot,\zeta)\|_1 \leq \int_{-1/2}^{1/2}\int_{\mathbb{R}^{2d}} \frac{1}{(t^2+1)^{d/2}} e^{-\pi\frac{t^2}{t^2+1}(u_1-\frac{1}{t}\zeta_2)^2} e^{-\pi\frac{t^2}{t^2+1}(u_2-\frac{1}{t}\zeta_1)^2} du_1 du_2 dt = +\infty$$

so that we conjecture that $\Theta \notin M^{1,\infty}(\mathbb{R}^{2d})$. The previous claim will follow if we prove that $\Theta_\sigma \notin W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$.

Note that $\Theta_\sigma(\zeta) = \mathcal{F}\Theta(J\zeta) = \mathcal{F}\Theta(\zeta)$. Furthermore, the distributional Fourier transform of Θ can be computed explicitly as follows. First, recall the definition of the cosine integral function (5).

Proposition 3.4. *For $d \geq 1$ the distribution symplectic Fourier transform Θ_σ of the function Θ is provided by*

$$\Theta_\sigma(\zeta_1,\zeta_2) = \begin{cases} -2\,\mathrm{Ci}(4\pi|\zeta_1\zeta_2|), & (\zeta_1,\zeta_2)\in\mathbb{R}^2,\ d=1\\ \mathcal{F}(\chi_{\{|s|\geq 2\}}|s|^{d-2})(\zeta_1\zeta_2), & (\zeta_1\zeta_2)\in\mathbb{R}^{2d},\ d\geq 2,\end{cases} \tag{28}$$

where $\chi_{\{|s|\geq 2\}}$ is the characteristic function of the set $\{s\in\mathbb{R} : |s|\geq 2\}$. The case $d=1$ can be recaptured by the case $d\geq 2$ using (5).

Proof. We carry out the computations of Θ_σ by studying first the case in dimension $d=1$ and secondly, inspired by the former case, $d>1$.

First step: $d=1$. By Proposition 3.1, the function Θ is in

$$L^\infty(\mathbb{R}^2)\setminus L^p(\mathbb{R}^2)\subset \mathcal{S}'(\mathbb{R}^2), \quad 1\leq p<\infty,$$

so that the Fourier transform is meant in $\mathcal{S}'(\mathbb{R}^2)$. Observe that

$$\mathcal{F}\Theta(\zeta_1,\zeta_2) = \mathcal{F}_2\mathcal{F}_1\Theta(\zeta_1,\zeta_2),$$

where $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) is the partial Fourier transform with respect to the first (resp. second) variable. Indeed, for every test function $\varphi\in\mathcal{S}(\mathbb{R}^2)$,

$$\langle\mathcal{F}\Theta,\varphi\rangle = \langle\Theta,\mathcal{F}^{-1}\varphi\rangle$$

and $\mathcal{F}^{-1}\varphi(x,\omega) = \mathcal{F}_1^{-1}\mathcal{F}_2^{-1}\varphi(x,\omega) = \mathcal{F}_2^{-1}\mathcal{F}_1^{-1}\varphi(x,\omega)$, by Fubini's Theorem.

Using

$$\mathcal{F}_1 \mathrm{sinc}(y_2\cdot)(\zeta_1) = \frac{1}{|y_2|} p_{1/2}(\zeta_1/y_2), \quad y_2 \neq 0,$$

where $p_{1/2}(t)$ is the box function defined by $p_{1/2}(t) = 1$ for $|t| \leq 1/2$ and $p_{1/2}(t) = 0$ otherwise, we obtain, for $\zeta_1\zeta_2 \neq 0$ (hence in particular $|\zeta_1| > 0$),

$$\begin{aligned}
\mathcal{F}\Theta(\zeta_1, \zeta_2) &= \int_{\mathbb{R}} e^{-2\pi i \zeta_2 y_2} \frac{1}{|y_2|} p_{1/2}(\zeta_1/y_2)\, dy_2 = \int_{|y_2|\geq 2|\zeta_1|} e^{-2\pi i \zeta_2 y_2} \frac{1}{|y_2|} dy_2 \\
&= \int_{|s|\geq 2|\zeta_1\zeta_2|} e^{-2\pi i s} \frac{1}{|s|}\, ds \\
&= \int_{|s|\geq 2|\zeta_1\zeta_2|} \frac{\cos(2\pi s) - i\sin(2\pi s)}{|s|}\, ds \\
&= \int_{|s|\geq 2|\zeta_1\zeta_2|} \frac{\cos 2\pi s}{|s|}\, ds \\
&= 2\int_{2|\zeta_1\zeta_2|}^{+\infty} \frac{\cos 2\pi s}{s}\, ds = -2\mathrm{Ci}(4\pi|\zeta_1\zeta_2|),
\end{aligned}$$

by (5), so that, since $\zeta_1\zeta_2 = 0$ is a set of Lebesgue measure equal zero on $\mathbb{R}^2$, we can write

$$\Theta_\sigma(\zeta_1, \zeta_2) = -2\mathrm{Ci}(4\pi|\zeta_1\zeta_2|), \quad (\zeta_1, \zeta_2) \in \mathbb{R}^2. \tag{29}$$

Second step: $d > 1$. This is a simple generalization on the former step. For $(z_1, z_2), (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$, $d > 1$, we write

$$z_i = (z_i', z_{i,d}),\ \zeta_i = (\zeta_i', \zeta_{i,d}), \quad z_i', \zeta_i' \in \mathbb{R}^{d-1},\ z_{i,d}, \zeta_{i,d} \in \mathbb{R}, \quad i = 1, 2. \tag{30}$$

We decompose $\mathcal{F}\Theta = \mathcal{F}_{2d}\mathcal{F}'\mathcal{F}_1\Theta$ where, for $\Theta = \Theta(z_1, z_2)$, $\mathcal{F}_1$ is the partial Fourier transform with respect to the variable $z_{1,d}$, $\mathcal{F}'$ is the partial Fourier transform with respect the $2d-2$ variables $(z_1', z_2') \in \mathbb{R}^{2d-2}$ and $\mathcal{F}_{2d}$ is the partial Fourier transform with respect to the last variable $z_{2,d}$. We start with computing the partial Fourier transform $\mathcal{F}_1$:

$$\begin{aligned}
\mathcal{F}_1\Theta(z_1', \cdot, z_2', z_{2,d})(\zeta_{1,d}) &= \mathcal{F}_1(T_{\frac{-z_1' z_2'}{z_{2,d}}} \mathrm{sinc}(z_{2,d}\cdot))(\zeta_{1,d}) \\
&= e^{2\pi i \frac{\zeta_{1,d}}{z_{2,d}} z_1' z_2'} \frac{1}{|z_{2,d}|} \mathcal{F}_1(\mathrm{sinc}) \left(\frac{\zeta_{1,d}}{z_{2,d}}\right) \\
&= e^{2\pi i \frac{\zeta_{1,d}}{z_{2,d}} z_1' z_2'} \frac{1}{|z_{2,d}|} p_{1/2} \left(\frac{\zeta_{1,d}}{z_{2,d}}\right).
\end{aligned}$$

Using the Gaussian integrals in [22, Appendix A, Theorem 2]) we calculate

$$\mathcal{F}'(e^{2\pi i \frac{\zeta_{1,d}}{z_{2,d}} z_1' z_2'})(\zeta_1', \zeta_2') = \left|\frac{z_{2,d}}{\zeta_{1,d}}\right|^{d-1} e^{-2\pi i \frac{z_{2,d}}{\zeta_{1,d}} \zeta_1' \zeta_2'},$$

so that

$$\begin{aligned}
\mathcal{F}\Theta(\zeta_1, \zeta_2) &= \mathcal{F}_{2d}\left(e^{-2\pi i \frac{z_{2,d}}{\zeta_{1,d}} \zeta_1' \zeta_2'} \left|\frac{z_{2,d}}{\zeta_{1,d}}\right|^{d-1} \frac{1}{|z_{2,d}|} p_{1/2}\left(\frac{\zeta_{1,d}}{z_{2,d}}\right)\right)(\zeta_{2,d}) \\
&= \int_{\left|\frac{\zeta_{1,d}}{z_{2,d}}\right| \le \frac{1}{2}} \left|\frac{z_{2,d}}{\zeta_{1,d}}\right|^{d-1} \frac{1}{|z_{2,d}|} e^{-2\pi i \frac{z_{2,d}}{\zeta_{1,d}} \zeta_1 \zeta_2} \, dz_{2,d} \\
&= \int_{|s| \ge 2} e^{-2\pi i s(\zeta_1 \zeta_2)} |s|^{d-2} \, ds,
\end{aligned}$$

as claimed. ∎

Notice that the second equation (28) can be written

$$\Theta_\sigma(\zeta_1, \zeta_2) = \int_{|s| \ge 2} e^{-2\pi i s(\zeta_1 \zeta_2)} |s|^{d-2} \, ds.$$

Corollary 3.5. *We have*

$$\Theta_\sigma \notin L^\infty_{loc}(\mathbb{R}^{2d}).$$

Proof. For the case $d = 1$, recall that the cosine integral $\mathrm{Ci}(x)$ has the series expansion

$$\mathrm{Ci}(x) = \gamma + \log x + \sum_{k=1}^{+\infty} \frac{(-x^2)^k}{2k(2k)!}, \quad x > 0$$

where γ is the Euler–Mascheroni constant, from which our claim easily follows.

For $d \ge 2$, Θ_σ is only defined as a tempered distribution. ∎

Corollary 3.6. *The function $\Theta_\sigma \notin L^p(\mathbb{R}^{2d})$, for any $1 \le p \le \infty$.*

Proof. The case $p = \infty$ is already treated in Corollary 3.5. For $d \ge 2$ again we observe that Θ_σ is not defined as function but only as a tempered distribution. For $d = 1$, $1 \le p < \infty$, the claim follows by the expression (29). Indeed, choose $0 < \epsilon < \pi/2$, then $|\mathrm{Ci}(x)| \ge |\mathrm{Ci}(\epsilon)|$, for $0 < x < \epsilon$, so that

$$\begin{aligned}
\int_{\mathbb{R}^2} |\Theta_\sigma(\zeta_1, \zeta_2)|^p \, d\zeta_1 d\zeta_2 &\ge 2 \int_{|\zeta_1 \zeta_2| < \frac{\epsilon}{4\pi}} |\mathrm{Ci}(4\pi |\zeta_1 \zeta_2|)|^p \, d\zeta d\zeta_2 \\
&\ge C_p meas\{(\zeta_1, \zeta_2) : |\zeta_1 \zeta_2| < \frac{\epsilon}{4\pi}\} = +\infty,
\end{aligned}$$

for a suitable constant $C_p > 0$. ∎

Since $\mathcal{F}L^1 \subset L^\infty$, the Wiener amalgam space $W(\mathcal{F}L^1, L^\infty)$ is included in L^∞_{loc}. This proves our claim:

Corollary 3.7. *The function $\Theta_\sigma \notin W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$ or, equivalently, $\Theta \notin M^{1,\infty}(\mathbb{R}^{2d})$.*

4. Cohen Kernels in modulation and Wiener spaces

In this section we focus on other members of the Cohen class, introduced by Cohen in [5], which define, for $\tau \in [0,1]$, the (cross-)τ-Wigner distributions

$$W_\tau(f,g)(x,\omega) = \int_{\mathbb{R}^d} e^{-2\pi i y\zeta} f(x+\tau y)\overline{g(x-(1-\tau)y)}\,dy \quad f,g \in \mathcal{S}(\mathbb{R}^d). \tag{31}$$

Such distributions enter in the definition of the $\tau-$pseudodifferential operators as follows

$$\langle \mathrm{Op}_\tau(a)f, g\rangle = \langle a, W_\tau(g,f)\rangle \quad f,g \in \mathcal{S}(\mathbb{R}^d). \tag{32}$$

It is then natural to investigate the time-frequency properties of such kernels and compare to the corresponding Weyl and Born-Jordan ones. The Cohen class consists of elements of the type

$$M(f,f)(x,\omega) = W(f,f) * \sigma$$

where $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ is called the Cohen kernel. When $\sigma = \delta$, then $M(f,f) = W(f,f)$ and we come back to the Wigner distribution. When $\sigma = \Theta_\sigma$, then $M(f,f) = Q(f,f)$, that is the Born-Jordan distribution. The τ-Wigner function $W_\tau(f,f)$ belongs to the Cohen class for every $\tau \in [0,1]$, as proved in [1, Proposition 5.6]:

$$W_\tau(f,f) = W(f,f) * \sigma_\tau,$$

where

$$\sigma_\tau(x,\omega) = \frac{2^d}{|2\tau-1|^d} e^{2\pi i \frac{2}{2\tau-1} x\omega}, \quad \tau \neq \frac{1}{2}$$

and $\sigma_{1/2} = \delta$ (the case of the Wigner distribution, as already observed).

In what follows we study the properties of the Cohen kernels σ_τ in the realm of modulation and Wiener amalgam spaces. As we shall see, the Born-Jordan kernel and the Wigner one display similar time-frequency properties and are locally worse than the kernels σ_τ, $\tau \neq 1/2$.

Proposition 4.1. *We have, for every $\tau \in [0,1] \setminus \{1/2\}$,*

$$\sigma_\tau \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d}) \cap M^{1,\infty}(\mathbb{R}^{2d}).$$

Proof. We exploit the dilation properties for Wiener spaces (cf. [33, Lemma 3.2] and its generalization in [7, Corollary 3.2]): for $A = \lambda I$, $\lambda > 0$,

$$\|f(A\,\cdot)\|_{W(\mathcal{F}L^p, L^q)} \leq C\lambda^{d\left(\frac{1}{p}-\frac{1}{q}-1\right)}(\lambda^2+1)^{d/2}\|f\|_{W(\mathcal{F}L^p, L^q)}. \tag{33}$$

Using the dilation relations for Wiener amalgam spaces (33) for $\lambda = \sqrt{t}$, $0 < t < 1/2$, $p = 1$, $q = \infty$, we obtain

$$\|e^{\pm 2\pi i \zeta_1 \zeta_2 t}\|_{W(\mathcal{F}L^1, L^\infty)} \leq C \|e^{\pm 2\pi i \zeta_1 \zeta_2}\|_{W(\mathcal{F}L^1, L^\infty)}$$

with constant $C > 0$ independent on the parameter t. For $t = \frac{2}{2\tau - 1}$, when $\tau > 1/2$ and $t = -\frac{2}{2\tau-1}$, when $0 \leq \tau < 1/2$, we obtain that $\sigma_\tau \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$. Now, an easy computation gives

$$\mathcal{F}\sigma_\tau(\zeta_1, \zeta_2) = e^{-\pi i (2\tau - 1)\zeta_1 \zeta_2},$$

so that, using $\mathcal{F}M^{1,\infty}(\mathbb{R}^{2d}) = W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$ and repeating the previous argument we obtain $\sigma_\tau \in M^{1,\infty}(\mathbb{R}^{2d})$ for every $\tau \in [0,1] \setminus \{1/2\}$. ∎

The case $\tau = 1/2$ is different. Indeed, $\sigma_{1/2} = \delta$ and for any fixed $g \in \mathcal{S}(\mathbb{R}^{2d}) \setminus \{0\}$ the STFT $V_g \delta$ is given by

$$V_g \delta(z, \zeta) = \langle \delta, M_\zeta T_z g \rangle = \overline{g(-z)},$$

that yields $\delta \in M^{1,\infty}(\mathbb{R}^{2d}) \setminus W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$.

The Born-Jordan kernel Θ_σ behaves analogously. Indeed, using Proposition 3.3 and Corollary 3.7, we obtain

$$\Theta_\sigma \in M^{1,\infty}(\mathbb{R}^{2d}) \setminus W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d}).$$

Those distributions can be used in the definition of the τ-pseudodifferential operators

5. Symbols in modulation spaces

This section is focused on the proof of Theorem 1.1. We first demonstrate the sufficient boundedness conditions.

Theorem 5.1. *Assume that* $1 \leq p, q, r_1, r_2 \leq \infty$. *Then the pseudodifferential operator* $\mathrm{Op}_{\mathrm{BJ}}(a)$, *from* $\mathcal{S}(\mathbb{R}^d)$ *to* $\mathcal{S}'(\mathbb{R}^d)$, *having symbol* $a \in M^{p,q}(\mathbb{R}^{2d})$, *extends uniquely to a bounded operator on* $\mathcal{M}^{r_1,r_2}(\mathbb{R}^d)$, *with the estimate* (8) *and the indices' conditions* (6) *and* (7).

The result relies on a thorough understanding of the action of the mapping

$$A: \ a \longmapsto a * \Theta_\sigma, \tag{34}$$

which gives the Weyl symbol of an operator with Born-Jordan symbol a, on modulation spaces.

Proposition 5.1. *For every* $1 \leq p, q \leq \infty$, *the mapping* A *in* (34), *defined initially on* $\mathcal{S}'(\mathbb{R}^{2d})$, *restricts to a linear continuous map on* $M^{p,q}(\mathbb{R}^{2d})$, *i.e., there exists a constant* $C > 0$ *such that*

$$\|Aa\|_{M^{p,q}} \leq C \|a\|_{M^{p,q}}. \tag{35}$$

Proof. By Proposition 3.3, the function $\Theta \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d})$. Its symplectic Fourier transform Θ_σ belongs to $\mathcal{F}_\sigma W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d}) = M^{1,\infty}(\mathbb{R}^{2d})$. Now, for every $1 \leq p, q \leq \infty$, the convolution relations for modulation space s (25) give

$$M^{p,q}(\mathbb{R}^{2d}) * M^{1,\infty}(\mathbb{R}^{2d}) \hookrightarrow M^{p,q}(\mathbb{R}^{2d})$$

and this shows the claim (35). ∎

Proof of Theorem 5.1. Assume $a \in M^{p,q}(\mathbb{R}^{2d})$, then Proposition 5.1 proves that $Aa = a * \Theta_\sigma \in M^{p,q}(\mathbb{R}^{2d})$ as well. We next write $\mathrm{Op_{BJ}}(a) = \mathrm{Op_W}(Aa)$ and use the sufficient conditions for Weyl operators in [12, Theorem 5.2]: if the Weyl symbol Aa is in $M^{p,q}(\mathbb{R}^{2d})$, then $\mathrm{Op_W}(Aa)$ extends to a bounded operator on $\mathcal{M}^{r_1,r_2}(\mathbb{R}^d)$, with

$$\|\mathrm{Op_{BJ}}(a)f\|_{\mathcal{M}^{r_1,r_2}} = \|\mathrm{Op_W}(Aa)f\|_{\mathcal{M}^{r_1,r_2}} \lesssim \|Aa\|_{M^{p,q}}\|f\|_{\mathcal{M}^{r_1,r_2}}$$

where the indices r_1, r_2, p, q satisfy (6) and (7). The inequality (35) then provides the claim. ∎

The necessary conditions of Theorem 1.1 require some preliminaries.

We reckon the adjoint operator $\mathrm{Op_{BJ}}(a)^*$ of a Born-Jordan operator $\mathrm{Op_{BJ}}(a)$ using the connection with Weyl operators. Recall that $\mathrm{Op_W}(b)^* = \mathrm{Op_W}(\bar{b})$ [26], so that

$$\mathrm{Op_{BJ}}(a)^* = \mathrm{Op_W}(a*\Theta_\sigma)^* = \mathrm{Op_W}(\overline{a*\Theta_\sigma}) = \mathrm{Op_W}(\bar{a}*\bar{\Theta}_\sigma) = \mathrm{Op_W}(\bar{a}*\Theta_\sigma) = \mathrm{Op_{BJ}}(\bar{a})$$

because Θ is an even real-valued function. Hence the adjoint of a Born-Jordan operator $\mathrm{Op_{BJ}}(a)$ with symbol a is the Born-Jordan operator having symbol $\bar{a}$ (the complex-conjugate of a). This nice property is the key argument for the following auxiliary result, already obtained for the case of Weyl operators in [12, Lemma 5.1]. The proof uses the same pattern as the former result and hence is omitted.

Lemma 5.2. *Suppose that, for some $1 \leq p, q, r_1, r_2 \leq \infty$, the following estimate holds:*

$$\|\mathrm{Op_{BJ}}(a)f\|_{M^{r_1,r_2}} \leq C\|a\|_{M^{p,q}}\|f\|_{M^{r_1,r_2}}, \quad \forall a \in \mathcal{S}(\mathbb{R}^{2d}),\ \forall f \in \mathcal{S}(\mathbb{R}^d).$$

Then the same estimate is satisfied with r_1, r_2 replaced by r_1', r_2' (even if $r_1 = \infty$ or $r_2 = \infty$).

The above instruments let us show the necessity of (6) and (9).

Theorem 5.2. *Suppose that, for some $1 \leq p, q, r_1, r_2 \leq \infty$, $C > 0$ the estimate*

$$\|\mathrm{Op_{BJ}}(a)f\|_{M^{r_1,r_2}} \leq C\|a\|_{M^{p,q}}\|f\|_{M^{r_1,r_2}} \qquad \forall a \in \mathcal{S}(\mathbb{R}^{2d}),\ f \in \mathcal{S}(\mathbb{R}^d) \tag{36}$$

holds. Then the constraints in (6) *and* (9) *must hold.*

Proof. The estimate (36) can be written as

$$|\langle a, Q(f,g)\rangle| \leq C\|a\|_{M^{p,q}}\|f\|_{M^{r_1,r_2}}\|g\|_{M^{r_1',r_2'}} \qquad \forall a \in \mathcal{S}(\mathbb{R}^{2d}),\ f,g \in \mathcal{S}(\mathbb{R}^d)$$

which is equivalent to

$$\|Q(f,g)\|_{M^{p',q'}} \leq C\|f\|_{M^{r_1,r_2}}\|g\|_{M^{r_1',r_2'}} \qquad \forall f,g \in \mathcal{S}(\mathbb{R}^d).$$

Now, one should test this estimate on families of functions f_λ, g_λ such that $Q(f_\lambda, g_\lambda)$ is concentrated inside the hyperbola $|x \cdot \omega| < 1$ (say), see Figure 1, where $\theta \asymp 1$, so that the left-hand side is comparable to $\|W(f_\lambda, g_\lambda)\|_{M^{p',q'}}$ and can be estimated from below.

The choice $f_\lambda(x) = g_\lambda(x) = e^{-\pi\lambda|x|^2}$, provides the estimate (6) when $\lambda \to +\infty$. Indeed in this case we argue exactly as in the proof of [9, Theorem 1.4]. We recall this pattern, useful also for other cases. Remind that $\varphi(x) = e^{-\pi|x|^2}$ and φ_λ is defined in (14). By (15) we obtain the estimate

$$\|\varphi_\lambda\|_{M^{r_1,r_2}}\|\varphi_\lambda\|_{M^{r_1',r_2'}} \lesssim \lambda^{-\frac{d}{2r_2'}}\lambda^{-\frac{d}{2r_2}}. \tag{37}$$

We gauge from below the norm $\|Q(\varphi_\lambda, \varphi_\lambda)\|_{M^{p',q'}}$ as follows. By taking the symplectic Fourier transform and using Lemma 2.5 and the product property (13) we have

$$\begin{aligned}
\|Q(\varphi_\lambda, \varphi_\lambda)\|_{M^{p',q'}} &= \|\Theta_\sigma * W(\varphi_\lambda, \varphi_\lambda)\|_{M^{p',q'}} \\
&\asymp \|\Theta\mathcal{F}_\sigma[W(\varphi_\lambda, \varphi_\lambda)]\|_{W(\mathcal{F}L^{p'},L^{q'})} \\
&\gtrsim \|\Theta(\zeta_1,\zeta_2)\chi(\zeta_1\zeta_2)\mathcal{F}_\sigma[W(\varphi_\lambda, \varphi_\lambda)]\|_{W(\mathcal{F}L^{p'},L^{q'})}
\end{aligned}$$

for any $\chi \in C_c^\infty(\mathbb{R})$. Choosing χ supported in the interval $[-1/4, 1/4]$ and $= 1$ in the interval $[-1/8, 1/8]$, we write

$$\chi(\zeta_1\zeta_2) = \chi(\zeta_1\zeta_2)\Theta(\zeta_1,\zeta_2)\Theta^{-1}(\zeta_1,\zeta_2)\tilde{\chi}(\zeta_1\zeta_2),$$

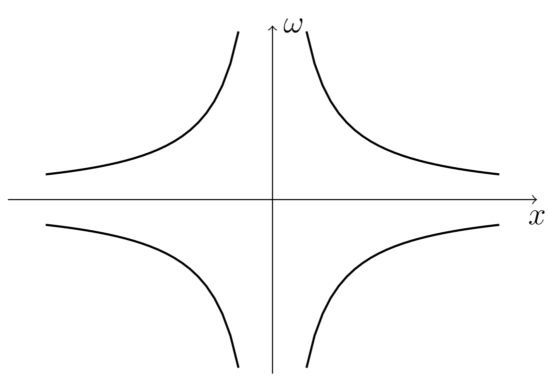

FIGURE 1. The region $|x \cdot \omega| < 1$ $(d = 1)$.

with $\tilde{\chi}\in C_c^\infty(\mathbb{R})$ supported in $[-1/2,1/2]$ and $\tilde{\chi}=1$ on $[-1/4,1/4]$, therefore on the support of χ. Since by Lemma 2.5 the function $\Theta^{-1}(\zeta_1,\zeta_2)\tilde{\chi}(\zeta_1\zeta_2)$ belongs to $W(\mathcal{F}L^1,L^\infty)$, again by the product property the last expression is estimated from below as

$$\gtrsim \|\chi(\zeta_1\zeta_2)\mathcal{F}_\sigma[W(\varphi_\lambda,\varphi_\lambda)]\|_{W(\mathcal{F}L^{p'},L^{q'})}.$$

Consider a function $\psi\in C_c^\infty(\mathbb{R}^d)\setminus\{0\}$, supported in $[-1/4,1/4]$. Using

$$|\zeta_1\zeta_2|\leq\frac{1}{2}(|\sqrt{\lambda}\zeta_1|^2+|\sqrt{\lambda}^{-1}\zeta_2|^2)$$

we see that $\chi(\zeta_1\zeta_2)=1$ on the support of $\psi(\sqrt{\lambda}\zeta_1)\psi(\sqrt{\lambda}^{-1}\zeta_2)$, for every $\lambda>0$.

Then, we can write

$$\psi(\sqrt{\lambda}\zeta_1)\psi(\sqrt{\lambda}^{-1}\zeta_2)=\chi(\zeta_1\zeta_2)\psi(\sqrt{\lambda}\zeta_1)\psi(\sqrt{\lambda}^{-1}\zeta_2)$$

and by Lemma 2.2 we also infer

$$\|\psi(\sqrt{\lambda}\zeta_1)\psi(\sqrt{\lambda}^{-1}\zeta_2)\|_{W(\mathcal{F}L^1,L^\infty)}\lesssim 1$$

so that we can continue the above estimate as

$$\gtrsim \|\psi(\sqrt{\lambda}\zeta_1)\psi(\sqrt{\lambda}^{-1}\zeta_2)\mathcal{F}_\sigma[W(\varphi_\lambda,\varphi_\lambda)]\|_{W(\mathcal{F}L^{p'},L^{q'})}.$$

Using (see e.g. [23, Formula (4.20)])

$$W(\varphi_\lambda,\varphi_\lambda)(x,\omega)=2^{\frac{d}{2}}\lambda^{-\frac{d}{2}}\varphi(\sqrt{2\lambda}\,x)\varphi(\sqrt{\frac{2}{\lambda}}\,\omega), \tag{38}$$

we calculate

$$\mathcal{F}_\sigma[W(\varphi_\lambda,\varphi_\lambda)](\zeta_1,\zeta_2)=2^{\frac{d}{2}}\lambda^{-\frac{d}{2}}\varphi((\sqrt{2\lambda})^{-1}\,\zeta_2)\varphi(\sqrt{\frac{\lambda}{2}}\,\zeta_1),$$

so that

$$\|\psi(\sqrt{\lambda}\zeta_1)\psi(\sqrt{\lambda}^{-1}\zeta_2)\mathcal{F}_\sigma[W(\varphi_\lambda,\varphi_\lambda)]\|_{W(\mathcal{F}L^{p'},L^{q'})}$$
$$=2^{d/2}\lambda^{-\frac{d}{2}}\|\psi(\sqrt{\lambda}\zeta_1)\varphi((1/\sqrt{2})\sqrt{\lambda}\,\zeta_1)\|_{W(\mathcal{F}L^{p'},L^{q'})}\|\psi(\sqrt{\lambda}^{-1}\zeta_2)\varphi((\sqrt{2\lambda})^{-1}\,\zeta_2)\|_{W(\mathcal{F}L^{p'},L^{q'})}.$$

By Lemma 2.2 we can estimate the last expression so that

$$\|Q(\varphi_\lambda,\varphi_\lambda)\|_{M^{p',q'}}\gtrsim\lambda^{-d+\frac{d}{2p'}+\frac{d}{2q'}}\quad\text{as }\lambda\to+\infty.$$

Finally, using the estimate (37) we infer (6).

We now prove that $\max\{1/r_1,1/r_1'\}\leq 1/q+1/p$. If we show the estimate $1/r_1\leq 1/q+1/p$, then the constraint $1/r_1'\leq 1/q+1/p$ follows by the duality argument of Lemma 5.2. To reach this goal, we consider $f_\lambda=\varphi$ (independent of the parameter λ) and $g=\varphi_\lambda$ as before and use the previous pattern for these families of functions, in the case $\lambda\to 0^+$. By (15) the upper estimate becomes

$$\|\varphi\|_{M^{r_1,r_2}}\|\varphi_\lambda\|_{M^{r_1',r_2'}}\lesssim \lambda^{-\frac{d}{2r_1'}}. \tag{39}$$

The same arguments as before let us write

$$\|Q(\varphi,\varphi_\lambda)\|_{M^{p',q'}}\gtrsim \|\psi(\sqrt{\lambda}\zeta_1)\psi(\sqrt{\lambda}^{-1}\zeta_2)\mathcal{F}_\sigma[W(\varphi,\varphi_\lambda)]\|_{W(\mathcal{F}L^{p'},L^{q'})},$$

where $\mathcal{F}_\sigma[W(\varphi,\varphi_\lambda)]$ is computed in (23). Observe that, given any $F\in W(\mathcal{F}L^{p'},L^{q'})$,

$$\begin{aligned}\|e^{\pi i\frac{\lambda-1}{\lambda+1}\zeta_1\zeta_2}F(\zeta_1,\zeta_2)\|_{W(\mathcal{F}L^{p'},L^{q'})}&\gtrsim \|e^{-\pi i\frac{\lambda-1}{\lambda+1}\zeta_1\zeta_2}e^{\pi i\frac{\lambda-1}{\lambda+1}\zeta_1\zeta_2}F(\zeta_1,\zeta_2)\|_{W(\mathcal{F}L^{p'},L^{q'})}\\&=\|F(\zeta_1,\zeta_2)\|_{W(\mathcal{F}L^{p'},L^{q'})},\end{aligned}$$

because $\|e^{-\pi i\frac{\lambda-1}{\lambda+1}\zeta_1\zeta_2}\|_{W(\mathcal{F}L^1,L^\infty)}\le C$, for every $\lambda>0$ by [9, Proposition 3.2]. So, writing

$$c_\lambda=\frac{1}{(\lambda+1)^{\frac{d}{2}}}$$

(notice $c_\lambda\to 1$ for $\lambda\to 0^+$) we are reduced to

$$\|Q(\varphi,\varphi_\lambda)\|_{M^{p',q'}}\gtrsim c_\lambda\|\psi(\sqrt{\lambda}\zeta_1)e^{-\frac{\pi\lambda}{\lambda+1}\zeta_1^2}\|_{W(\mathcal{F}L^{p'},L^{q'})}\|\psi(\sqrt{\lambda}^{-1}\zeta_2)e^{-\frac{\pi}{\lambda+1}\zeta_2^2}\|_{W(\mathcal{F}L^{p'},L^{q'})}.$$

By Lemma 2.2 we can estimate, for $\lambda\to 0^+$,

$$\|\psi(\sqrt{\lambda}\zeta_1)e^{-\frac{\pi\lambda}{\lambda+1}\zeta_1^2}\|_{W(\mathcal{F}L^{p'},L^{q'})}=\|\psi(\sqrt{\lambda}\zeta_1)e^{-\frac{\pi}{\lambda+1}(\sqrt{\lambda}\zeta_1)^2}\|_{W(\mathcal{F}L^{p'},L^{q'})}\asymp\lambda^{-\frac{d}{2q'}},$$

whereas

$$\begin{aligned}\|\psi(\sqrt{\lambda}^{-1}\zeta_2))e^{-\frac{\pi}{\lambda+1}\zeta_2^2}\|_{W(\mathcal{F}L^{p'},L^{q'})}&\gtrsim\lambda^{\frac{d}{2}}(\lambda+1)^{\frac{d}{2}}\|\int\hat{\psi}(\sqrt{\lambda}(\zeta_2-\eta))e^{-\pi(\lambda+1)|\eta|^2}\,d\eta\|_{L^{p'}}\\&=\lambda^{\frac{d}{2}}(\lambda+1)^{\frac{d}{2}}\lambda^{-\frac{d}{2p'}}\|\int\hat{\psi}(\zeta_2-\sqrt{\lambda}\eta))e^{-\pi(\lambda+1)|\eta|^2}\,d\eta\|_{L^{p'}}\\&=(\lambda+1)^{\frac{d}{2}}\lambda^{-\frac{d}{2p'}}\|\int\hat{\psi}(\zeta_2-t)e^{-\pi\frac{\lambda+1}{\lambda}|t|^2}dt\|_{L^{p'}}\\&=\lambda^{\frac{d}{2}-\frac{d}{2p'}}\|\hat{\psi}*K_{1/\sqrt{\lambda}}\|_{L^{p'}}\\&\sim\lambda^{\frac{d}{2}-\frac{d}{2p'}}\|\hat{\psi}\|_{p'},\ \text{as }\lambda\to 0^+\end{aligned}$$

where $K_{1/\sqrt{\lambda}}(\zeta_2)=\lambda^{-\frac{d}{2}}(\lambda+1)^{\frac{d}{2}}e^{-\frac{\pi(\lambda+1)}{\lambda}|\zeta_2|^2}$, $\lambda\to 0^+$, is an approximate identity. So that

$$\lambda^{-\frac{d}{2r_1'}}\gtrsim\lambda^{-\frac{d}{2q'}}\lambda^{\frac{d}{2p}}$$

and, for $\lambda\to 0^+$, we obtain

$$\frac{1}{r_1}\le\frac{1}{q}+\frac{1}{p},$$

as desired.

It remains to prove that $\max\{1/r_2, 1/r_2'\} \leq 1/q + 1/p$. Again, it is enough to show that $1/r_2 \leq 1/q + 1/p$ and invoke Lemma 5.2 for $1/r_2' \leq 1/q + 1/p$.

An explicit computation (see [12, Proposition 5.3]) shows that

$$\mathcal{F}^{-1}\operatorname{Op_W}(\sigma)\mathcal{F} = \operatorname{Op_W}(\sigma \circ J), \tag{40}$$

where $J(x, \omega) = (\omega, -x)$ as defined in (21) (this is also a consequence of the intertwining property of the metaplectic operator $\mathcal{F}$ with the Weyl operator $\operatorname{Op_W}(\sigma)$ [15, Corollary 221]).

Now, observing that $\Theta_\sigma \circ J = \Theta_\sigma$, we obtain

$$\begin{aligned}(a * \Theta_\sigma)(Jz) &= \int_{\mathbb{R}^{2d}} a(u)\Theta_\sigma(Jz - u)\, du = \int_{\mathbb{R}^{2d}} a(u)\Theta_\sigma(J(z - J^{-1}u))du \\ &= \int_{\mathbb{R}^{2d}} a(u)\Theta_\sigma(z - J^{-1}u)du = \int_{\mathbb{R}^{2d}} a(Ju)\Theta_\sigma(z-u)\, du \\ &= (a \circ J) * \Theta_\sigma(z).\end{aligned}$$

The previous computations together with (40) gives

$$\mathcal{F}^{-1}\operatorname{Op_{BJ}}(a)\mathcal{F} = \mathcal{F}^{-1}\operatorname{Op_{BJ}}(a \circ J)\mathcal{F}.$$

On the other hand, the map $a \longmapsto a \circ J$ is an isomorphism of $M^{p,q}$, so that (36) is in fact equivalent to

$$\|\operatorname{Op_{BJ}}(a)f\|_{W(\mathcal{F}L^{r_1},L^{r_2})} \lesssim \|a\|_{M^{p,q}}\|f\|_{W(\mathcal{F}L^{r_1},L^{r_2})} \qquad \forall a \in \mathcal{S}(\mathbb{R}^{2d}),\ f \in \mathcal{S}(\mathbb{R}^d). \tag{41}$$

The estimate (41) can be written as

$$|\langle a, Q(f,g)\rangle| \leq C\|a\|_{M^{p,q}}\|f\|_{W(\mathcal{F}L^{r_1},L^{r_2})}\|g\|_{W(\mathcal{F}L^{r_1'},L^{r_2'})} \qquad \forall a \in \mathcal{S}(\mathbb{R}^{2d}),\ f, g \in \mathcal{S}(\mathbb{R}^d)$$

which is equivalent to

$$\|Q(f,g)\|_{M^{p',q'}} \leq C\|f\|_{W(\mathcal{F}L^{r_1},L^{r_2})}\|g\|_{W(\mathcal{F}L^{r_1'},L^{r_2'})} \qquad \forall f, g \in \mathcal{S}(\mathbb{R}^d).$$

Now, taking $f = \varphi$ and $g = \varphi_\lambda$ as before, we observe that, for $\lambda \to 0^+$, by (15),

$$\|\varphi_\lambda\|_{W(\mathcal{F}L^{r_1'},L^{r_2'})} \asymp \lambda^{-\frac{d}{2}}\|\varphi_{1/\lambda}\|_{M^{r_1',r_2'}} \asymp \lambda^{-\frac{d}{2}+\frac{d}{2r_2}} = \lambda^{-\frac{d}{2r_2'}}.$$

Arguing as in the previous case we obtain $1/r_2 \leq 1/q + 1/p$. This concludes the proof. ◻

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