

## 2+1 quantum gravity for high genus

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## 2+1 quantum gravity for high genus

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I wish to review some recent results obtained in collaboration with T Regge [1] which concern quantum gravity in 2 + 1 (2 space, 1 time) dimensions.

In 2 + 1 dimensions the Einstein-Hilbert first order action with cosmological constant  $\Lambda$  [2]

$$-\frac{1}{2}(d\omega^{AB} - \omega_D^A \wedge \omega^{DB} + \Lambda e^A \wedge e^B) \wedge e^C \epsilon_{ABC} \quad A, B, C = 0, 1, 2$$

can be written as a Chern-Simons action

$$\alpha(d\omega^{ab} - \frac{2}{3}\omega_i^a \wedge \omega^{ib}) \wedge \omega^{cd} \epsilon_{abcd} / 8 \quad \alpha, b, c = 0, 1, 2, 3 \quad (1)$$

and is equal to the scalar curvature plus cosmological term: provided we identify the de Sitter spin connections  $\omega^{ab}$  with the variables  $e^A, \omega^{AB}$  in the following way.

For the triads  $e^A = \alpha \omega^{A3}$ ,  $\epsilon_{ABC3} = -\epsilon_{ABC}$  and the cosmological constant  $\Lambda = \frac{1}{3}k\alpha^{-2}$ . In the sequel  $\sqrt{k}$  means unambiguously +1 for  $k = 1$  and  $+i$  for  $k = -1$ , and the tangent Minkowski metric  $\eta_{ab}$  has signature  $(-1, 1, 1, k)$ ,  $k = +1$  for de Sitter,  $k = -1$  for anti-de Sitter space. The Riemannian curvature is:

$$R^{ab} = d\omega^{ab} - \omega^{at} \wedge \omega_t^b$$

and has components  $R^{AB} + \Lambda e^A \wedge e^B$ ,  $R^{A3} = \frac{1}{\alpha}R^A$ , with  $R^{AB}, R^A$  given by  $R^{AB} = d\omega^{AB} - \omega^{AT} \wedge \omega_T^B$ ,  $R^A = de^A - \omega^{AB} \wedge e - B$ .

The variational equations derived from (1) are  $R^{ab} = 0$  and imply that space-time is everywhere locally de Sitter.

The action (1) leads to the standard CCR:

$$[\omega_i^{ab}(x), \omega_j^{cd}(y)] = k\alpha^{-1} \epsilon_{ij} \epsilon^{abcd} \delta^2(x - y) \quad (2)$$

or equivalently:

$$\begin{aligned} [e_i^A(x), \omega_j^{BC}(y)] &= -\epsilon_{ij} \epsilon^{ABC} \delta^2(x - y) \\ [e_i^A(x), e_j^B(y)] &= [\omega_i^{AB}(x), \omega_j^{CD}(y)] = 0 \end{aligned}$$

† Speaker at the conference

where  $x, y \in \Sigma^2$  are generic points on the  $x^0 = \text{const.}$  surface  $\Sigma^2$ ,  $i, j = 1, 2$  are spatial vector indices on  $\Sigma^2$ ,  $\epsilon_{ij} = -\epsilon_{ji}$ ,  $\epsilon_{12} = 1$ .

Since we wish to use flat, gauge-invariant connections [3] we must satisfy the constraints  $R^{ab} = 0$ . This is done by considering an  $SO(3, 1)$  matrix  $\psi^{ab}$  which is a zero-form, given by

$$d\psi^{ab} = \omega^{ac} \psi_c^b. \tag{3}$$

If  $\Sigma^2$  is a Riemann surface [4] of genus  $g$ , and  $\gamma$  a path in  $\Sigma^2$  from a point  $P$  to a point  $Q$ , then a solution of (3)  $\psi^{ab}(P, Q, \{\gamma\})$  depends only on the end points and the homotopy class  $\{\gamma\}$  of  $\gamma$ . Setting  $P = Q$  we obtain the subset  $\psi^{ab}(P, \{\gamma\})$  which forms a representation of the fundamental group  $\pi_1(\Sigma^2)$  for  $\gamma \in \pi_1$ .

Recall that the fundamental group of a Riemann surface has  $2g$  generators  $U_1, V_1, \dots, U_g, V_g$ , satisfying

$$U_1 V_1 U_1^{-1} V_1^{-1} \dots U_g V_g U_g^{-1} V_g^{-1} = 1.$$

The brackets (2) can be integrated along intersecting paths  $\rho, \sigma$  in  $\Sigma^2$  to find, from (3), the brackets of the matrices  $\psi^{ab}$  [2].

It is actually more convenient to use the spinor groups. The spinor group of  $SO(2, 2)$  is  $SL(2, \mathbb{R}) \otimes SL(2, \mathbb{R})$ , that of  $SO(3, 1)$  is  $SL(2, \mathbb{C})$ , with 6 real parameters in both cases.

The  $SL(2, \mathbb{R})$  or  $SL(2, \mathbb{C})$  matrices  $S$  are related to the  $SO(3, 1)$  matrices  $\Psi$  by

$$\Psi^{ab} \gamma_b = S^{-1} \gamma^a S$$

The result is

$$[S^\pm(\rho)_\alpha^\beta, S^\pm(\sigma)_\gamma^r] = \pm i \left( \frac{s}{2\alpha\sqrt{k}} \right) (-S^\pm(\rho)_\alpha^\beta S^\pm(\sigma)_\gamma^r + 2S^\pm(\rho_3\sigma_1)_\alpha^r S^\pm(\sigma_3\rho_1)_\gamma^\beta) \tag{4}$$

$$[S^+(\rho)_\alpha^\beta, S^-(\sigma)_\gamma^r] = 0 \quad \alpha, \beta, \dots = 1, 2.$$

where the  $\pm$  refer to the upper/lower spinor components and  $s = s(\sigma, \rho)$  defines the orientation of the intersection of  $\rho$  and  $\sigma$ . A few comments are in order:

(i) Formula (4) is valid only when the paths  $\rho, \sigma$  have a single intersection. If they have no intersection then the brackets are trivially zero. If they have  $m > 1$  intersections then one can express one of the paths, say  $\rho$ , as  $\rho = \rho_1 \rho_2 \rho_3 \dots \rho_m$  where each  $\rho_i, i = 1 \dots m$ , has a single intersection and formula (4) can be applied. Alternatively one can give a direct geometrical derivation extending the one we have given here for  $m = 1$ .

(ii) Formula (4) is also valid for closed paths, but it is necessary to keep the base points separate to avoid false or ambiguous contributions from a common base point i.e. (4) is not gauge invariant since gauge invariance corresponds to a shift of base point (if it were gauge invariant we could always make the base points coincide which would lead to ambiguous contributions).

(iii) Formula (4) are Poisson brackets, not gauge invariant, and therefore do not respect the constraints  $R^{ab} = 0$  which generate gauge transformations. One should either use Dirac brackets or use gauge invariant quantities (observables) for which Dirac and Poisson brackets coincide.

However, for a generic closed path  $\sigma$  the traces:

$$c^\pm(\tau) = \frac{1}{2} S_\alpha^\pm(\tau)$$

are gauge invariant quantities since  $c^\pm(\tau) = c^\pm(\nu\tau\nu^{-1})$  for any open path  $\nu$ . This corresponds to a shift of base point along  $\nu$ . If  $\delta = \sigma_1^{-1}\rho_1$  denotes the open path from  $B$  to  $B'$ , then  $\sigma$  and  $\rho' = \delta\rho\delta^{-1}$  identify elements of the homotopy group  $\pi_1(\Sigma^2, B')$  based on the common base point  $B'$ . The algebra of these traces is then

$$\begin{aligned} [c^\pm(v), c^\pm(u)] &= \left( \pm is/2\alpha\sqrt{k} \right) (-c^\pm(v)c^\pm(u) + c^\pm(uv)) \\ [c^+(v), c^-(u)] &= 0 \end{aligned} \tag{5}$$

or alternatively<sup>1</sup>

$$[c^\pm(v), c^\pm(u)] = \left( \pm is/4\alpha\sqrt{k} \right) (c^\pm(uv) - c^\pm(uv^{-1}))$$

since for  $2 \times 2$  matrices the identity holds:

$$c^\pm(u)c^\pm(v) = \frac{1}{2}(c^\pm(vu) + c^\pm(vu^{-1})) \tag{6}$$

thus we have an infinite Lie algebra subject to non-linear constraints. Note that for  $k = -1$   $c^\pm$  are real and independent whereas for  $k = +1$  they are complex conjugates.

Set  $c^\pm(u) = x^\pm$ ,  $c^\pm(v) = y^\pm$ ,  $c^\pm(uv) = c^\pm(vu) = z^\pm$  and consider (5).

$$[x^\pm, y^\pm] = -\frac{is}{2\alpha\sqrt{k}}(z^\pm - x^\pm y^\pm).$$

and cyclical permutations of  $x, y, z$ .

For simplicity we omit the  $+$  sign when there is no ambiguity.

We set  $(x, y) = xy - yx = i\hbar[x, y]$  and symmetrise the  $xy$  product to find

$$(x, y) = \frac{s\hbar}{2\alpha\sqrt{k}} \left( z - \frac{1}{2}(xy + yx) \right).$$

Set  $s\hbar/(2\alpha\sqrt{k}) = 2i \tan(\theta/2)$  to find

$$e^{(1/2)i\theta}xy - e^{-(1/2)i\theta}yx = i \sin \theta z \quad \text{and cyclical} \tag{7}$$

where in (7) we have replaced  $x, y, z$  with  $x/\cos(\theta/2)$  etc. This rescaling is irrelevant in the  $\hbar \rightarrow 0$  limit as it corresponds to changes of the order of  $\hbar^2$ .

The algebra (7) is trivially related to the Lie algebra of the quantum groups  $SU(2)$  [5].

It admits the central element:

$$F^2 = \cos^2 \frac{1}{2}\theta + 2e^{i\frac{1}{2}\theta}xyz - e^{-i\theta}(x^2 + z^2) - e^{-i\theta}y^2$$

<sup>1</sup> Compare with the contributions to this conference by A Ashtekar and C Rovelli

which is cyclically symmetric and whose classical limit

$$F^2 \rightarrow \Phi = 1 + 2xyz - x^2 - y^2 - z^2.$$

defines a cubic surface  $\Phi = \text{constant}$  whose values are related to the representations of  $SO(3, 1)$  or  $SO(2, 2)$ .

The situation for genus 2 is rather more complicated, and corresponds to a subalgebra of the algebra of a surface of arbitrary genus.

The subgroup of  $\pi_1$  generated by  $U_1, V_1, U_2, V_2$ , i.e. the paths around any 2 holes has traces of products of these paths of the form:

$$A_n = c(U_1^{n_0} V_1^{n_1} U_2^{n_2} V_2^{n_3}) \tag{8}$$

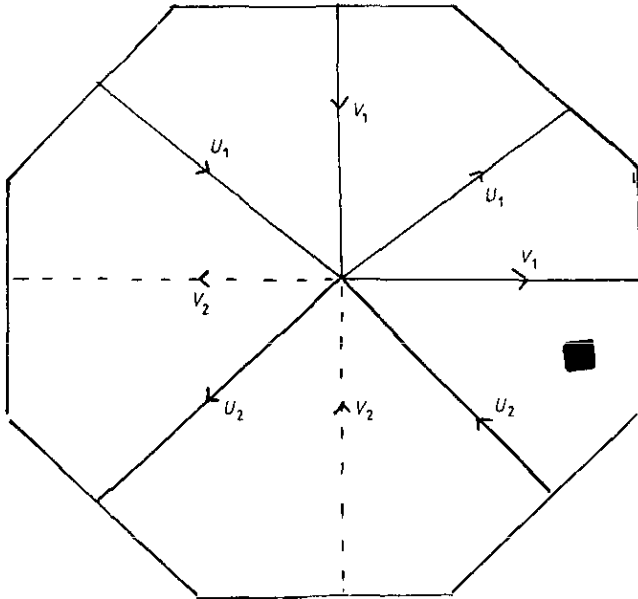
where  $n_0, n_1, n_2, n_3 = 0, 1$  and

$$\begin{aligned} n &= n_0 + 2n_1 + 4n_2 + 8n_3 \\ &= 1 \dots 15. \end{aligned}$$

Any other trace can be reduced to a polynomial in the  $A_n$  by repeated use of  $SL(2, \mathbb{R})$  identities such as (6). There are only two exceptions to this rule:

$A_3 = -c(U_1 V_1) + 2c(U_1)c(V_1) = c(U_1 V_1^{-1})$ ,  $A_{12} = -c(U_2 V_2) + 2c(U_2)c(V_2)$ . Some examples of the general rule are  $A_4 = c(U_2)$ ,  $A_9 = c(U_1 V_2)$ ,  $A_{11} = c(U_1 V_1 V_2)$ .

The 105 brackets of these 15 elements were calculated by direct geometrical methods. We do not report all the details but try to explain our reasoning and method. We made ample use of (4) and of the representation of a compact surface of genus  $g$  by means of a polygon of  $4g$  sides suitably identified (see Fig. 1).



**Figure 1.** Octagon with identified sides showing two holes of a surface of arbitrary genus. The black square is an obstruction leading into the remainder of the surface.

After calculation of each bracket, using (4), the paths are reassembled, then we trace and simplify using the trace properties of  $SL(2, \mathbb{R})$  matrices. The final result is best displayed by the complete hexagon appearing in Fig. 2. Also we omit the factor  $-1/2a$ . To each element  $A_i$  we associate the  $i$ th line of the hexagon. If the lines  $i, j$  have no point in common then the corresponding paths are homotopic to non-intersecting paths and:

$$[A_i, A_j] = 0 \quad (30 \text{ brackets}) \quad (9)$$

for example

$$[A_1, A_5] = [A_3, A_{12}] = 0$$

If the sequence of lines  $i, k, j$  forms a triangle and runs clockwise around its perimeter, then the corresponding paths intersect once and:

$$[A_i, A_k] = A_i A_k - A_j \quad (10)$$

There are 20 of such triangles, for example

$$[A_1, A_2] = A_1 A_2 - A_3.$$

Finally there are pairs of diagonal lines, say  $n, p$ , which intersect at one point  $P$  inside the hexagon. These correspond to traces of paths which have 2 intersections. Let  $n$  have end points  $P_1, P_3$  and  $p$  have end points  $P_2, P_4$  and let  $i_{ab} = i_{ba}$  the line connecting the points  $P_a$  and  $P_b$ . If we connect the points  $P_a, a = 1 \dots 4$  in all possible ways we obtain a quadrilateral with diagonals  $n, p$ . We may always choose a convention such that the triples  $i_{12}, i_{23}, i_{31} = n$  and  $i_{42} = p, i_{23}, i_{34}$  run clockwise (see Fig. 3). We have then:

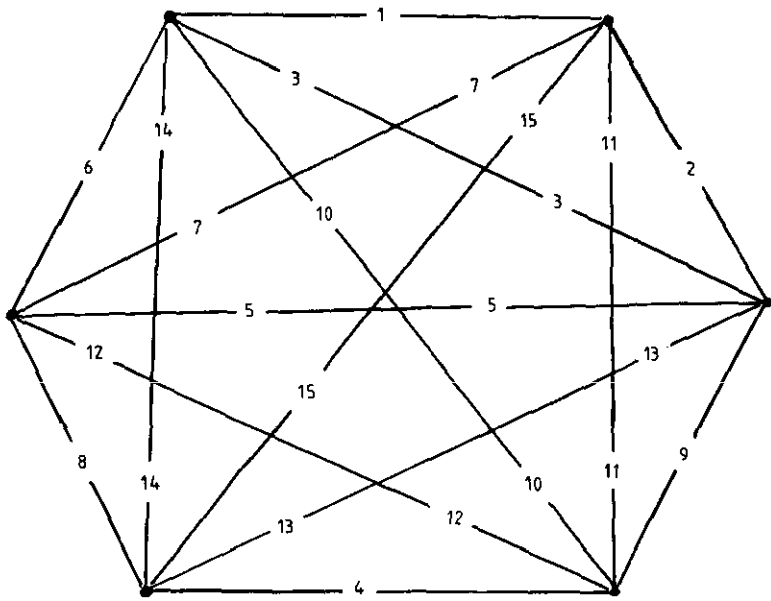
$$[A_n, A_p] = 2A_{i_{12}} A_{i_{34}} - 2A_{i_{23}} A_{i_{14}} \quad (11)$$

The six elements  $A_{12} A_{23} A_{13} A_{14} A_{24} A_{34}$  generate a subalgebra of the full algebra. There are 15 such subalgebras corresponding to the 15 quadrilaterals contained in Fig. 2. For example:

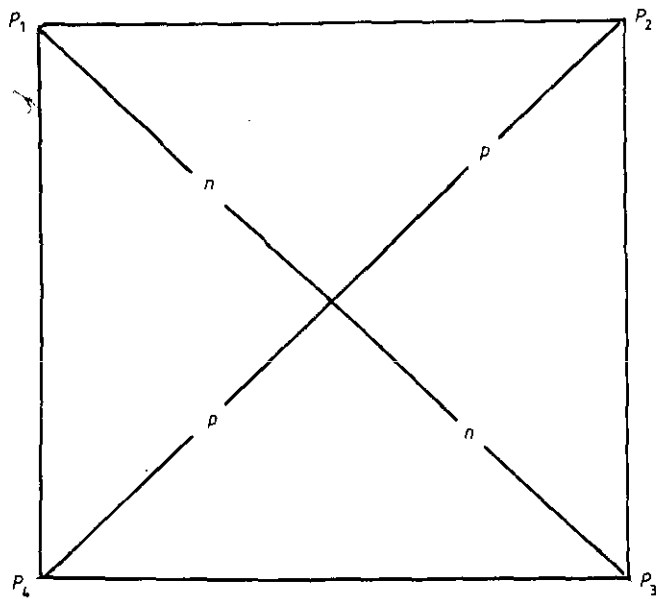
$$[A_5, A_{10}] = 2A_9 A_6 - 2A_3 A_{12}$$

One can show that it is impossible to eliminate completely the double intersections by considering traces of the form (8) and reversing the sign of any  $n_i$ , that is, one can only change the intersection number by  $\pm 2$ .

The algebra of these 15 elements  $A_n$  has a hierarchy of nested subalgebras with an interesting geometrical interpretation. If any clockwise triangle (3 oriented lines) of the hexagon can instead be thought of as 3 oriented points on one line, then any 4 triangles, namely a quadrilateral like Fig. 3, correspond to 6 oriented points on 4 lines (see e.g. Fig. 4) and so on. Any 10 triangles (pentagon) with 5 points and 10 lines corresponds to a Desargues diagram with 10 lines and 10 points (Fig. 5). The full algebra (hexagon, Fig. 2) corresponds to 4 lines through each point, in total 20



**Figure 2.** Diagram showing the combinatorial rules for the Poisson brackets used in the text.



**Figure 3.** Diagram used for double intersections.

lines and 15 points. In general, if there are  $n$  lines through each point, there are  $n(n + 1)(n + 2)/6$  lines and  $(n + 1)(n + 2)/2$  points of the generalised Desargues diagram, alternately represented by a complete  $(n + 2)$ -gon.

Now, the full algebra (hexagon) is highly symmetric.

For instance, each element  $A_i, i = 1 \dots 15$  generates an infinitesimal canonical transformation which can be exponentiated to give an automorphism of the algebra.

As an example, the transformation generated by  $A_1 = \text{ch}\theta$  on  $A_2$  and  $A_3$  can be displayed as follows:

$$\begin{aligned} \begin{pmatrix} A_2(t) \\ A_3(t) \end{pmatrix} &= \frac{1}{\text{sh}\theta} \begin{pmatrix} \text{sh}(\theta(1-t)) & \text{sh}(\theta t) \\ -\text{sh}(\theta t) & \text{sh}(\theta(1+t)) \end{pmatrix} \begin{pmatrix} A_2 \\ A_3 \end{pmatrix} \\ &= \Omega(t) \begin{pmatrix} A_2 \\ A_3 \end{pmatrix} \end{aligned}$$

where for  $t$  integer, the transformation is polynomial (otherwise a transcendental function). In fact  $\Omega(n) = \Omega(1)^n$  so it is sufficient to take  $t = 1$ .

The transformations from double intersections are more complicated but are reported in full in [1], and are still polynomial in the  $A$ 's. Define  $D(n)$  to be the transformation generated by  $A_n$  with  $t = 1$ . The algebra is invariant under all the  $D(n), n = 1 \dots 15$ . These Dehn transformations form an infinite discrete group  $D$  isomorphic to the original algebra.

The same hexagon appearing in Fig. 2 can be used to classify the identities satisfied by the  $D(n)$  just as we did for the  $A_n$ . In particular if the lines  $i, j$  do not intersect we have:

$$[D(i), D(j)] = 0 \tag{12}$$

If the lines  $i, k, j$  run clockwise around a triangle of Fig. 2 we have:

$$\begin{aligned} D(i)D(j)D(i) &= D(j)D(i)D(j) \quad \text{and cyclical} \\ D(i)D(j) &= D(j)D(k) = D(k)D(i) \end{aligned} \tag{13}$$

For doubly intersecting paths we find more complicated relations which follow directly from the ones quoted above and will not be quoted here. By using these identities we can express all  $D(n)$  in terms of a subset of 5 elements only, say  $D(8), D(6), D(1), D(2), D(9)$ , i.e. the sides of the hexagon with the 6th missing. The exclusion of  $D(4)$  is purely conventional and does not reflect any breaking of the hexagonal symmetry.

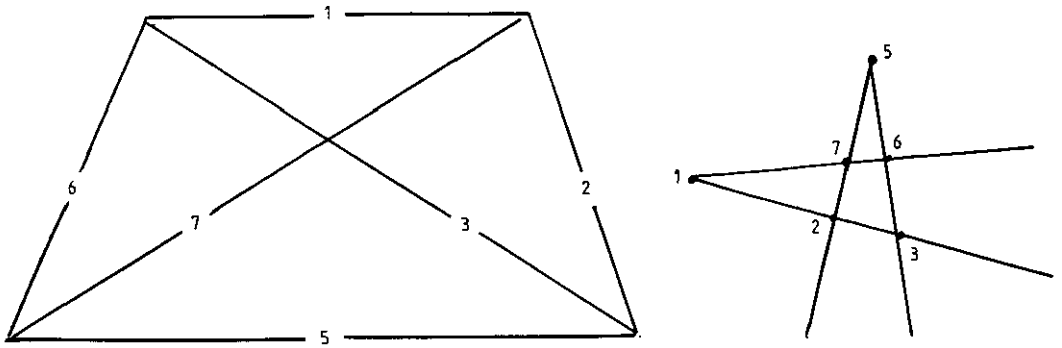
We set  $\zeta_1 = D(8), \zeta_2 = D(6), \zeta_3 = D(1), \zeta_4 = D(2), \zeta_5 = D(9)$  and verify from (12), (13) that:

$$\begin{aligned} \zeta_i \zeta_j &= \zeta_j \zeta_i \quad \text{if } |i - j| \geq 2 \quad 1 \leq i, j \leq 5 \\ \zeta_i \zeta_{i+1} \zeta_i &= \zeta_{i+1} \zeta_i \zeta_{i+1} \quad 1 \leq i \leq 4 \end{aligned}$$

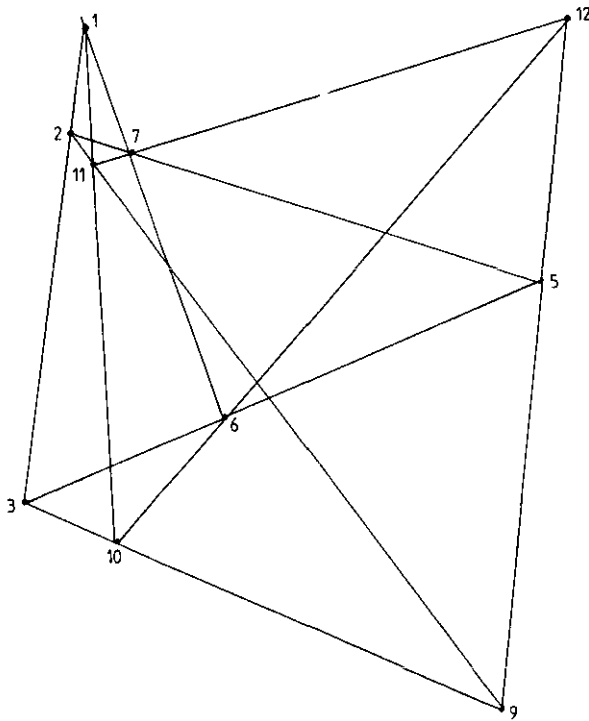
which are satisfied by the elements of  $B(6)$ , the braid group of order 6. In particular the element  $\zeta_i$  corresponds to the element of  $B(6)$  which exchanges the braids  $i, i + 1$ . It follows that  $D$  yields a representation of  $B(6)$  [6].

From this identification the symmetries of the braid group  $B(6)$  can be described by symmetries of the hexagon (Fig. 2). For example, a clockwise rotation of the





**Figure 4.** Quadrilateral Subalgebra



**Figure 5.** Desargues Diagram

hexagon by  $\pi/3$  corresponds, at the level of the traces  $A_n$ , to a transformation whose cube simply exchanges the labelling of the holes 1 and 2. At the level of  $B(6)$  it is the transformation

$$\zeta_i \rightarrow \zeta_{i+1} \quad i = 1 \cdots 5, \text{ mod } 5.$$

Another example is a reflection of the hexagon around the dashed line. Clearly its square is the identity; it corresponds to replacing  $\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5$  by  $\zeta_5, \zeta_4, \zeta_3, \zeta_2, \zeta_1$  in order (think of looking at the six strings of  $B(6)$  from behind, or wrapping them around a cylinder and looking from inside).

Both of these transformations can be generated by the group  $D$  which, recall, acts on the traces  $A_n$  defined by (8).

The Dehn group  $D$  can be lifted to the level of the homotopy group  $\pi_1(\Sigma^2)$ . As an example, define.

$$H(1) = \{V_1 \rightarrow U_1^{-1}V_1\}$$

$$H(2) = \{U_1 \rightarrow U_1V_1\}$$

These maps  $H(i), i = 1, 2, 6, 8, 9$  satisfy the same identities (12), (13) as the  $D(n)$  and generate a group  $H$  of homomorphisms of  $\pi_1$ , which is induced by the mapping class group. They reduce to the  $D(n)$  on the  $A_n$ . We thus have a minimal representation of the mapping class group  $\Gamma_{g,o}$  for  $g \geq 2$  [7].

The extension to  $g > 2$  is very promising. Consider the algebras  $A(n)$  associated to complete  $n$ -gons formed by  $n$  points joined by  $n(n-1)/2$  lines and including  $n(n-1)(n-2)/6$  triangles and where the brackets are defined as a straightforward generalisation of (9)–(11). For  $n < 6$  these algebras are isomorphic to subalgebras of the full algebra and in particular  $A(3)$  contains the triple associated with a triangle (genus 1).

The quantisation of this classical theory is highly non-trivial. We have found a quantum ordering which reproduces the classical algebra. There are many quantum Casimirs (polynomials in the  $A_n$ ) which should annihilate the physical states. The Dehn maps can be implemented as unitary operators on the physical states. This will appear elsewhere [8].

## Acknowledgments

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