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Twistor Geometry of Null Foliations in Complex Euclidean Space

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Abstract. We give a detailed account of the geometric correspondence between a smooth complex projective quadric hypersurface $\mathcal{Q}^n$ of dimension $n \geq 3$, and its twistor space $\mathbb{PT}$, defined to be the space of all linear subspaces of maximal dimension of $\mathcal{Q}^n$. Viewing complex Euclidean space $\mathbb{CE}^n$ as a dense open subset of $\mathcal{Q}^n$, we show how local foliations tangent to certain integrable holomorphic totally null distributions of maximal rank on $\mathbb{CE}^n$ can be constructed in terms of complex submanifolds of $\mathbb{PT}$. The construction is illustrated by means of two examples, one involving conformal Killing spinors, the other, conformal Killing–Yano 2-forms. We focus on the odd-dimensional case, and we treat the even-dimensional case only tangentially for comparison.

Key words: twistor geometry; complex variables; foliations; spinors

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1 Introduction

The *twistor space* $\mathbb{PT}$ of a smooth complex projective quadric hypersurface $\mathcal{Q}^n$ of dimension $n = 2m+1 \geq 3$, is defined to be the space of all *γ-planes*, i.e., m-dimensional linear subspaces of $\mathcal{Q}^n$. This is a complex projective variety of dimension $\frac{1}{2}(m+1)(m+2)$ equipped with a canonical holomorphic distribution D of rank $m+1$, and maximally non-integrable, i.e., $\mathrm{T}\mathbb{PT} = [\mathrm{D}, \mathrm{D}] + \mathrm{D}$. Here, $\mathrm{T}\mathbb{PT}$ denotes the holomorphic tangent bundle of $\mathbb{PT}$. Noting that a smooth quadric can be identified with a complexified n-sphere and is naturally equipped with a holomorphic conformal structure, we shall view complex Euclidean space $\mathbb{CE}^n$ as a dense open subset of $\mathcal{Q}^n$. In this context, we shall prove the following new results holding locally:

- totally geodetic integrable holomorphic γ-plane distributions on $\mathbb{CE}^n$ arise from $(m+1)$-dimensional complex submanifolds of $\mathbb{PT}$ – Theorem 3.5;
- totally geodetic integrable holomorphic γ-plane distributions on $\mathbb{CE}^n$ with integrable orthogonal complements arise from $(m+1)$-dimensional complex submanifolds of $\mathbb{PT}$ foliated by holomorphic curves tangent to D – Theorem 3.6;
- totally geodetic integrable holomorphic γ-plane distributions on $\mathbb{CE}^n$ with totally geodetic integrable orthogonal complements arise from m-dimensional complex submanifolds of a 1-dimensional reduction of a subset of $\mathbb{PT}$ known as *mini-twistor space* $\mathbb{MT}$ – Theorem 3.8.

Conversely, any such distributions arise in the ways thus described. These findings may be viewed as odd-dimensional counterparts of the work of [20], where it is shown that local foliations of a $2m$-dimensional smooth quadric $\mathcal{Q}^{2m}$ by *α-planes*, i.e., totally null self-dual m-planes, are in one-to-one correspondence with certain m-dimensional complex submanifolds of twistor space, here defined as the space of all α-planes in $\mathcal{Q}^{2m}$.

The first two of the above results are *conformally invariant*, and to arrive at them, we shall first describe the geometrical correspondence between $\mathcal{Q}^n$ and $\mathbb{PT}$ in a manifestly conformally invariant manner, by exploiting the vector and spinor representations of the complex conformal group $\mathrm{SO}(n+2,\mathbb{C})$ and of its double-covering $\mathrm{Spin}(n+2,\mathbb{C})$. Such a *tractor* or *twistor calculus*, as it is known, builds on Penrose's twistor calculus in four dimensions [29]. The more 'standard', local and Poincaré-invariant approach to twistor geometry will also be introduced to describe non-conformally invariant mini-twistor space $\mathbb{MT}$. In fact, a fairly detailed description of twistor geometry in odd dimensions will make up the bulk of this article, and should, we hope, have a wider range of applications than the one presented here. Once our calculus is all set up, our main results will follow almost immediately. The effectiveness of the tractor calculus will be exemplified by the construction of algebraic subvarieties of $\mathbb{PT}$, which describe the null foliations of $\mathcal{Q}^n$ arising from certain solutions of conformally invariant differential operators.

Another aim of the present article is to distil the *complex* geometry contained in a number of geometrical results on *real* Euclidean space and Minkowski space in dimensions three and four. In fact, our work is motivated by the findings of [27] and [2]. In the former reference, the author recasts the problem of finding pairs of analytic conjugate functions on $\mathbb{E}^n$ as a problem of finding closed null complex-valued 1-forms, and arrives at a description of the solutions in terms of real hypersurfaces of $\mathbb{C}^{n-1}$. The case $n=3$ is of particular interest, and is the focus of the article [2]: the kernel of a null complex 1-form on $\mathbb{E}^3$ consists of a complex line distribution $\mathrm{T}^{(1,0)}\mathbb{E}^3$ and the span of a real unit vector $\boldsymbol{u}$. This complex 2-plane distribution is in fact the orthogonal complement $\left(\mathrm{T}^{(1,0)}\mathbb{E}^3\right)^\perp$ of $\mathrm{T}^{(1,0)}\mathbb{E}^3$, and we can think of $\mathrm{T}^{(1,0)}\mathbb{E}^3$ as a CR-structure compatible with the conformal structure on $\mathbb{E}^3$ viewed as an open dense subset of S^3. The condition that $\left(\mathrm{T}^{(1,0)}\mathbb{E}^3\right)^\perp$ be integrable is equivalent to $\boldsymbol{u}$ being tangent to a *conformal foliation*, otherwise known as a *shearfree congruence* of curves. To find such congruences, the authors construct the S^2-bundle of unit vectors over S^3, which turns out to be a CR hypersurface in $\mathbb{CP}^3$. A section of this S^2-bundle defines a congruence of curves, and this congruence is shearfree if and only if the section is a 3-dimensional CR submanifold.

There are three antecedents for this result:

1) there is a one-to-one correspondence between local self-dual Hermitian structures on $\mathbb{E}^4\subset S^4$ and holomorphic sections of the S^2-bundle $\mathbb{CP}^3\to S^4$ known as the *twistor bundle* – this is a well-known result, see, e.g., [2, 4, 14, 20, 32];

2) there is a one-to-one correspondence between local analytic *shearfree congruences of null geodesics* in Minkowski space $\mathbb{M}$ and certain complex hypersurfaces of its twistor space, an auxilliary space isomorphic to $\mathbb{CP}^3$ – this is known as the *Kerr theorem* [11, 29, 31];

3) there is a one-to-one correspondence between local *shearfree congruences of geodesics* in $\mathbb{E}^3$ and certain holomorphic curves in its mini-twistor space, the holomorphic tangent bundle of $\mathbb{CP}^1\cong S^2$ – such congruences can also be equivalently described by *harmonic morphisms* [3, 36, 37].

Statements (1) and (2) are essentially the same result once they are cast in the complexification of $\mathbb{E}^4$ and $\mathbb{M}$.

The analogy between statement (1) and the result of [2] can be understood in the following terms: in the former case, the integrable complex null 2-plane distribution $\mathrm{T}^{(1,0)}\mathbb{E}^4$ defining the Hermitian structure is *totally geodetic*, i.e., $\nabla_{\boldsymbol{X}}\boldsymbol{Y}\in\Gamma\left(\mathrm{T}^{(1,0)}\mathbb{E}^4\right)$ for all $\boldsymbol{X},\boldsymbol{Y}\in\Gamma\left(\mathrm{T}^{(1,0)}\mathbb{E}^4\right)$. In the latter case, the condition that $\boldsymbol{u}$ be tangent to a shearfree congruence is also equivalent to the complex null line distribution $\mathrm{T}^{(1,0)}\mathbb{E}^3$ being (totally) geodetic. One could also think of the integrability of both $\mathrm{T}^{(1,0)}\mathbb{E}^3$ (trivially) and $\left(\mathrm{T}^{(1,0)}\mathbb{E}^3\right)^\perp$ as an analogue of the integrability of $\mathrm{T}^{(1,0)}\mathbb{E}^4$.

Finally, statement (3), unlike (1) and (2), breaks conformal invariance, and the additional data fixing a metric on $\mathbb{E}^3$ induces a reduction of the S^2-bundle constructed in [2] to mini-twistor space $\mathrm{T}S^2$ of (3). Correspondingly, for $\boldsymbol{u}$ to be tangent to a shearfree congruence of null geodesics, both $\mathrm{T}^{(1,0)}\mathbb{E}^3$ and $\left(\mathrm{T}^{(1,0)}\mathbb{E}^3\right)^\perp$ must be totally geodetic, which is not a conformally invariant condition.

The structure of the paper is as follows. Section 2 deals with the twistor geometry of a smooth quadric $\mathcal{Q}^n$ focussing mostly on the case $n = 2m+1$. In particular, we give an algebraic description of the canonical distribution on its twistor space. The geometric correspondence between $\mathcal{Q}^n$ and $\mathbb{PT}$ is made explicit. Propositions 2.12 and 2.13, and Corollary 2.14 give a twistorial articulation of incidence relations between γ-planes in $\mathcal{Q}^n$. The mini-twistor space $\mathbb{MT}$ of complex Euclidean space $\mathbb{CE}^n$ is introduced in Section 2.4. Points in $\mathbb{CE}^n$ correspond to embedded complex submanifolds of $\mathbb{PT}$ and $\mathbb{MT}$, and their normal bundles are described in Section 2.5. The main results, Theorems 3.5, 3.6 and 3.8, as outlined above, are given in Section 3. In each case, a purely geometrical explanation precedes a computational proof. In Section 4, we give two examples on how to relate null foliations in $\mathcal{Q}^n$ to complex varieties in $\mathbb{PT}$, based on certain solutions to the twistor equation, in Propositions 4.2 and 4.3, and the conformal Killing–Yano equation, in Proposition 4.7. We wrap up the article with Appendix A, which contains a description of standard open covers of twistor space and correspondence space.

2 Twistor geometry

We describe each of the three main protagonists involved in this article in turn: a smooth quadric hypersurface in projective space, its twistor space and a correspondence space fibered over them. The projective variety approach is very much along the line of [19, 31], while the reader should consult [5, 9] for the corresponding homogeneous space description.

Throughout $\mathbb{V}$ will denote an $(n+2)$-dimensional complex vector space. We shall make use of the following abstract index notation: elements of $\mathbb{V}$ and its dual $\mathbb{V}^*$ will carry upstairs and downstairs calligraphic upper case Roman indices respectively, i.e., $V^{\mathcal{A}} \in \mathbb{V}$ and $\alpha_{\mathcal{A}} \in \mathbb{V}^*$. Symmetrisation and skew-symmetrisation will be denoted by round and square brackets respectively, i.e., $\alpha_{(\mathcal{AB})} = \frac{1}{2}(\alpha_{\mathcal{AB}} + \alpha_{\mathcal{BA}})$ and $\alpha_{[\mathcal{AB}]} = \frac{1}{2}(\alpha_{\mathcal{AB}} - \alpha_{\mathcal{BA}})$. These conventions will apply to other types of indices used throughout this article. We shall also use Einstein's summation convention, e.g., $V^{\mathcal{A}}\alpha_{\mathcal{A}}$ will denote the natural pairing of elements of $\mathbb{V}$ and $\mathbb{V}^*$. We equip $\mathbb{V}$ with a non-degenerate symmetric bilinear form $h_{\mathcal{AB}}$, by means of which $\mathbb{V} \cong \mathbb{V}^*$: indices will be lowered and raised by $h_{\mathcal{AB}}$ and its inverse $h^{\mathcal{AB}}$ respectively. We also choose a complex orientation on $\mathbb{V}$, i.e., a complex volume element $\varepsilon_{\mathcal{A}_1\ldots\mathcal{A}_{n+2}}$ in $\wedge^{n+2}\mathbb{V}$. We shall denote by G the complex spin group $\mathrm{Spin}(n+2,\mathbb{C})$, the two-fold cover of the complex Lie group $\mathrm{SO}(n+2,\mathbb{C})$ preserving $h_{\mathcal{AB}}$ and $\varepsilon_{\mathcal{A}_1\ldots\mathcal{A}_{n+2}}$.

Turning now to the spinor representations of G, we distinguish the odd- and even-dimensional cases:

- $n = 2m+1$: denote by $\mathbb{S}$ the 2^{m+1}-dimensional irreducible spinor representation of G. Elements of $\mathbb{S}$ will carry upstairs bold lower case Greek indices, e.g., $S^{\boldsymbol{\alpha}} \in \mathbb{S}$, and dual elements, downstairs indices. The Clifford algebra $\mathcal{C}\ell(\mathbb{V}, h_{\mathcal{AB}})$ is linearly isomorphic to the exterior algebra $\wedge^\bullet\mathbb{V}$, and, identifying $\wedge^k\mathbb{V}$ with $\wedge^{2m+3-k}\mathbb{V}$ by Hodge duality for $k = 0, \ldots, m+1$, it is also isomorphic, as a matrix algebra, to the space $\mathrm{End}(\mathbb{S})$ of endomorphisms of $\mathbb{S}$. It is generated by matrices, denoted $\Gamma_{\mathcal{A}\boldsymbol{\alpha}}{}^{\boldsymbol{\gamma}}$, which satisfy the *Clifford identity*

$$\Gamma_{(\mathcal{A}\boldsymbol{\alpha}}{}^{\boldsymbol{\gamma}}\Gamma_{\mathcal{B})\boldsymbol{\gamma}}{}^{\boldsymbol{\beta}} = -h_{\mathcal{AB}}\delta_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}}. \tag{2.1}$$

Here $\delta_{\boldsymbol{\alpha}}^{\boldsymbol{\beta}}$ is the identity element on $\mathbb{S}$. There is a spin-invariant inner product on $\mathbb{S}$ denoted $\Gamma^{(0)}_{\boldsymbol{\delta\beta}} : \mathbb{S} \times \mathbb{S} \to \mathbb{C}$, yielding the isomorphism $\mathrm{End}(\mathbb{S}) \cong \mathbb{S} \otimes \mathbb{S}$. The resulting isomorphisms

$\mathcal{C}\ell(\mathbb{V}, h_{\mathcal{AB}}) \cong \wedge^\bullet \mathbb{V} \cong \mathbb{S} \otimes \mathbb{S}$ will be realised by means of the bilinear forms on $\mathbb{S}$ with values in $\wedge^k \mathbb{V}^*$, for $k = 1, \dots, n+2$:

$$\Gamma^{(k)}_{\mathcal{A}_1 \dots \mathcal{A}_k \boldsymbol{\alpha}\boldsymbol{\beta}} := \Gamma_{[\mathcal{A}_1 \boldsymbol{\alpha}}{}^{\boldsymbol{\gamma}_1} \cdots \Gamma_{\mathcal{A}_k]\boldsymbol{\gamma}_{k-1}}{}^{\boldsymbol{\delta}_k} \Gamma^{(0)}_{\boldsymbol{\delta}_k \boldsymbol{\beta}}. \tag{2.2}$$

These are symmetric in their spinor indices when $k \equiv m+1, m+2 \pmod 4$ and skew-symmetric otherwise.

- $n = 2m$: G has two 2^m-dimensional irreducible chiral spinor representations, which we shall denote $\mathbb{S}$ and $\mathbb{S}'$. Elements of $\mathbb{S}$ and $\mathbb{S}'$ will carry upstairs unprimed and primed lower case bold Greek indices respectively, i.e., $A^{\boldsymbol{\alpha}} \in \mathbb{S}$ and $B^{\boldsymbol{\alpha}'} \in \mathbb{S}'$. Dual elements will carry downstairs indices. The Clifford algebra $\mathcal{C}\ell(\mathbb{V}, h_{\mathcal{AB}})$ is isomorphic to $\mathrm{End}(\mathbb{S} \oplus \mathbb{S}')$ as a matrix algebra, and, linearly, to $\wedge^\bullet \mathbb{V}$. We can write its generators in terms of matrices $\Gamma_{\mathcal{A}\boldsymbol{\alpha}}{}^{\boldsymbol{\gamma}'}$ and $\Gamma_{\mathcal{A}\boldsymbol{\alpha}'}{}^{\boldsymbol{\gamma}}$ satisfying

$$\Gamma_{(\mathcal{A}\boldsymbol{\alpha}}{}^{\boldsymbol{\gamma}'} \Gamma_{\mathcal{B})\boldsymbol{\gamma}'}{}^{\boldsymbol{\beta}} = -h_{\mathcal{AB}} \delta^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}}, \qquad \Gamma_{(\mathcal{A}\boldsymbol{\alpha}'}{}^{\boldsymbol{\gamma}} \Gamma_{\mathcal{B})\boldsymbol{\gamma}}{}^{\boldsymbol{\beta}'} = -h_{\mathcal{AB}} \delta^{\boldsymbol{\beta}'}_{\boldsymbol{\alpha}'},$$

  where $\delta^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}}$ and $\delta^{\boldsymbol{\beta}'}_{\boldsymbol{\alpha}'}$ are the identity elements on $\mathbb{S}$ and $\mathbb{S}'$ respectively. There are spin-invariant bilinear forms on $\mathbb{S} \oplus \mathbb{S}'$ inducing isomorphisms $\mathbb{S}^* \cong \mathbb{S}'$, $(\mathbb{S}')^* \cong \mathbb{S}$ when m is even, and $\mathbb{S}^* \cong \mathbb{S}$ and $(\mathbb{S}')^* \cong \mathbb{S}'$ when m is odd, and denoted $\Gamma^{(0)}_{\boldsymbol{\alpha}\boldsymbol{\beta}'}$, $\Gamma^{(0)}_{\boldsymbol{\alpha}'\boldsymbol{\beta}}$, and $\Gamma^{(0)}_{\boldsymbol{\alpha}\boldsymbol{\beta}}$, $\Gamma^{(0)}_{\boldsymbol{\alpha}'\boldsymbol{\beta}'}$ respectively. The resulting isomorphisms $\mathcal{C}\ell(\mathbb{V}, h_{\mathcal{AB}}) \cong \wedge^\bullet \mathbb{V} \cong (\mathbb{S} \oplus \mathbb{S}') \otimes (\mathbb{S} \oplus \mathbb{S}')$ are realised by $\wedge^k \mathbb{V}$-valued bilinear forms $\Gamma^{(k)}_{\boldsymbol{\alpha}\boldsymbol{\beta}}$, for $k \equiv m+1 \pmod 2$, and $\Gamma^{(k)}_{\boldsymbol{\alpha}\boldsymbol{\beta}'}$, for $k \equiv m \pmod 2$ and so on.

We work in the holomorphic category throughout.

2.1 Smooth quadric hypersurface

Let us denote by $X^{\mathcal{A}}$ the position vector in $\mathbb{V}$, which can be viewed as standard Cartesian coordinates on $\mathbb{C}^{n+2}$. The equivalence class of non-zero vectors in $\mathbb{V}$ that projects down to the same point in the projective space $\mathbb{PV} \cong \mathbb{CP}^{n+1}$ will be denoted $[\cdot]$, and thus $[X^{\mathcal{A}}]$ will represent homogeneous coordinates on $\mathbb{PV}$.

The zero set of the quadratic form associated to $h_{\mathcal{AB}}$ on $\mathbb{V}$ defines a null cone $\mathcal{C}$ in $\mathbb{V}$, and the projectivisation of $\mathcal{C}$ defines a smooth quadric hypersurface $\mathcal{Q}^n$ in $\mathbb{PV}$, i.e.,

$$\mathcal{Q}^n = \left\{ \left[X^{\mathcal{A}}\right] \in \mathbb{PV} \colon h_{\mathcal{AB}} X^{\mathcal{A}} X^{\mathcal{B}} = 0 \right\}.$$

By taking a suitable cross-section of $\mathcal{C}$, one can identify $\mathcal{Q}^n$ with the complexification $\mathbb{C}S^n$ of the standard n-sphere S^n in Euclidean space $\mathbb{E}^{n+1}$. Using the affine structure on $\mathbb{V}$, $h_{\mathcal{AB}}$ can be viewed as a field of bilinear forms on $\mathbb{V}$ and thus on $\mathcal{C}$. We can then pull back $h_{\mathcal{AB}}$ to $\mathcal{Q}^n$ along any section of $\mathcal{C} \to \mathcal{Q}^n$ to a (holomorphic) metric on $\mathcal{Q}^n$. Different sections yield conformally related metrics on $\mathcal{Q}^n$, i.e., a (holomorphic) conformal structure on $\mathcal{Q}^n$. The *projective tangent space* at a point p of $\mathcal{Q}^n$ with homogeneous coordinate $\left[P^{\mathcal{A}}\right]$ is the linear subspace

$$\mathbf{T}_p \mathcal{Q}^n := \left\{ \left[X^{\mathcal{A}}\right] \in \mathcal{Q}^n \colon h_{\mathcal{AB}} X^{\mathcal{A}} P^{\mathcal{B}} = 0 \right\},$$

which can be seen to be the closure of the (holomorphic) tangent space $\mathrm{T}_p \mathcal{Q}^n$ at $p \in \mathcal{Q}^n$ in the usual sense. The intersection of $\mathbf{T}_p \mathcal{Q}^n$ and $\mathcal{Q}^n$ is a cone through p, and any point lying in this cone is connected to its vertex by a line that is null with respect to the conformal structure.

To obtain the Kleinian model of $\mathcal{Q}^n$, we fix a null vector $\mathring{X}^{\mathcal{A}}$ in $\mathbb{V}$, and denote by P the stabiliser of the line spanned by $\mathring{X}^{\mathcal{A}}$ in G. The transitive action of G on $\mathbb{V}$ descends to a transitive action on $\mathcal{Q}^n$, and since P stabilises a point in $\mathcal{Q}^n$, we obtain the identification $G/P \cong \mathcal{Q}^n$. The subgroup P is a parabolic subgroup of G, and its Lie algebra $\mathfrak{p}$ admits a Levi decomposition, that is, a splitting $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$, where $\mathfrak{p}_0$ is the reductive Lie algebra $\mathfrak{so}(n, \mathbb{C}) \oplus \mathbb{C}$, and $\mathfrak{p}_1$ is a nilpotent part, here isomorphic to $(\mathbb{C}^n)^*$. We choose a complement $\mathfrak{p}_{-1}$ of $\mathfrak{p}$ in $\mathfrak{g}$, dual to $\mathfrak{p}_1$ via the Killing form on $\mathfrak{g}$, so that $\mathfrak{g} = \mathfrak{p}_{-1} \oplus \mathfrak{p}$. There is a unique element spanning the centre $\mathfrak{z}(\mathfrak{p}_0) \cong \mathbb{C}$ of $\mathfrak{p}_0$, which acts diagonally on $\mathfrak{p}_0$, $\mathfrak{p}_1$ and $\mathfrak{p}_{-1}$ with eigenvalues 0, 1 and -1 respectively. For this reason, we refer to this element as the *grading element* of the splitting $\mathfrak{g} = \mathfrak{p}_{-1} \oplus \mathfrak{p}_0 \oplus \mathfrak{p}_1$. This splitting is compatible with the Lie bracket $[\cdot,\cdot] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ on $\mathfrak{g}$ in the sense that $[\mathfrak{p}_i, \mathfrak{p}_j] \subset \mathfrak{p}_{i+j}$, with the convention that $\mathfrak{p}_i = \{0\}$ for $|i| > 1$. In particular, it is invariant under $\mathfrak{p}_0$, but not under $\mathfrak{p}$. However, the filtration $\mathfrak{p}^1 \subset \mathfrak{p}^0 \subset \mathfrak{p}^{-1} := \mathfrak{g}$, where $\mathfrak{p}^1 := \mathfrak{p}_1$ and $\mathfrak{p}^0 := \mathfrak{p}_0 \oplus \mathfrak{p}_1$, *is* a filtration of $\mathfrak{p}$-modules on $\mathfrak{g}$, and each of the $\mathfrak{p}$-modules $\mathfrak{p}^{-1}/\mathfrak{p}^0$, $\mathfrak{p}^0/\mathfrak{p}^1$ and $\mathfrak{p}^1$ is linearly isomorphic to the $\mathfrak{p}_0$-modules $\mathfrak{p}_{-1}$, $\mathfrak{p}_0$ and $\mathfrak{p}_1$ respectively. These properties are most easily verified by realising $\mathfrak{g}$ in matrix form, i.e.,

$$\begin{pmatrix} \mathfrak{p}_0 & \mathfrak{p}_1 & \mathfrak{p}_1 & \mathfrak{p}_1 & 0 \\ \mathfrak{p}_{-1} & \mathfrak{p}_0 & \mathfrak{p}_0 & \mathfrak{p}_0 & \mathfrak{p}_1 \\ \mathfrak{p}_{-1} & \mathfrak{p}_0 & 0 & \mathfrak{p}_0 & \mathfrak{p}_1 \\ \mathfrak{p}_{-1} & \mathfrak{p}_0 & \mathfrak{p}_0 & \mathfrak{p}_0 & \mathfrak{p}_0 \\ 0 & \mathfrak{p}_{-1} & \mathfrak{p}_{-1} & \mathfrak{p}_0 & \mathfrak{p}_0 \end{pmatrix} \begin{matrix} \}_1 \\ \}_m \\ \}_1 \\ \}_m \\ \}_1 \end{matrix} \qquad \begin{pmatrix} \mathfrak{p}_0 & \mathfrak{p}_1 & \mathfrak{p}_1 & 0 \\ \mathfrak{p}_{-1} & \mathfrak{p}_0 & \mathfrak{p}_0 & \mathfrak{p}_1 \\ \mathfrak{p}_{-1} & \mathfrak{p}_0 & \mathfrak{p}_0 & \mathfrak{p}_0 \\ 0 & \mathfrak{p}_{-1} & \mathfrak{p}_0 & \mathfrak{p}_0 \end{pmatrix} \begin{matrix} \}_1 \\ \}_m \\ \}_m \\ \}_1 \end{matrix}$$

when $n = 2m + 1$ and $n = 2m$ respectively.

Given a vector representation V of P, one can construct the holomorphic homogeneous vector bundle $G \times_P V$ over G/P: this is the orbit space of a point in $G \times V$ under the right action of G. In particular, the tangent bundle of $\mathcal{Q}^n$ can be described as $\mathrm{T}(G/P) \cong G \times_P (\mathfrak{g}/\mathfrak{p})$, and the tangent space at any point of $\mathcal{Q}^n$ is isomorphic to $\mathfrak{p}_{-1} \cong \mathfrak{g}/\mathfrak{p}$ – for a proof, see, e.g., [9]. Similarly, denoting by P_0 the reductive subgroup of P with Lie algebra $\mathfrak{p}_0$, we can construct holomorphic homogeneous vector bundles from representations of P_0.

2.1.1 The tractor bundle

An important homogeneous vector bundle over $\mathcal{Q}^n$ is the one constructed from the standard representation $\mathbb{V}$ of G. It leads to a conformal invariant calculus, known as *tractor calculus*. The reader should consult, e.g., [1, 12] for further details.

Definition 2.1. The (*complex*) *standard tractor bundle* over $\mathcal{Q}^n \cong G/P$ is the rank-$(n+2)$ vector bundle $\mathcal{T} := G \times_P \mathbb{V} \cong G/P \times \mathbb{V}$.

The symmetric bilinear form $h_{\mathcal{AB}}$ on $\mathbb{V}$ induces a non-degenerate holomorphic section of $\odot^2 \mathcal{T}^* \to \mathcal{Q}^n$ on $\mathcal{T}$, called the *tractor metric*, also denoted by $h_{\mathcal{AB}}$. Further, the affine structure on $\mathbb{V}$ induces a unique *tractor connection* on $\mathcal{T}$ preserving $h_{\mathcal{AB}}$.

The vector space $\mathbb{V}$ admits a filtration of P-modules $\mathbb{V} =: \mathbb{V}^{-1} \supset \mathbb{V}^0 \supset \mathbb{V}^1$, where $\mathbb{V}^1 = \langle \mathring{X}^{\mathcal{A}} \rangle$ and $\mathbb{V}^0$ is the orthogonal complement of $\mathbb{V}^1$. These P-modules and their quotients $\mathbb{V}^i/\mathbb{V}^{i+1}$ give rise to P-invariant vector bundles as explained above. For convenience, we choose a splitting

$$\mathbb{V} = \mathbb{V}_{-1} \oplus \mathbb{V}_0 \oplus \mathbb{V}_1, \tag{2.3}$$

where $\mathbb{V}_1 := \mathbb{V}^{-1}$, $\mathbb{V}_{-1}$ is a null line in $\mathbb{V}$ complementary to $\mathbb{V}^0 \subset \mathbb{V}^{-1}$, and $\mathbb{V}_0$ is the n-dimensional vector subspace orthogonal to both $\mathbb{V}_{-1}$ and $\mathbb{V}_1$. We note the linear isomorphisms $\mathbb{V}_{-1} \cong \mathbb{V}^{-1}/\mathbb{V}^0$ and $\mathbb{V}_0 \cong \mathbb{V}^0/\mathbb{V}^1$.

Let us introduce some abstract index notation. Elements of $\mathbb{V}_0$ and its dual $(\mathbb{V}_0)^*$ will be adorned with upstairs and downstairs lower-case Roman indices respectively, e.g., $V^a \in \mathbb{V}_0$ and $\alpha_a \in (\mathbb{V}_0)^*$. We fix a null vector $\mathring{Y}^{\mathcal{A}}$ spanning $\mathbb{V}_{-1}$ such that $\mathring{X}^{\mathcal{A}} \mathring{Y}_{\mathcal{A}} = 1$. We also introduce the injector $\mathring{Z}_a^{\mathcal{A}} \colon \mathbb{V}_0 \to \mathbb{V}$. Then, $h_{\mathcal{AB}}$ restricts to a non-degenerate symmetric bilinear form $g_{ab} := \mathring{Z}_a^{\mathcal{A}} \mathring{Z}_b^{\mathcal{B}} h_{\mathcal{AB}}$ on $\mathbb{V}_0$. Indices can be raised or lowered by means of $h_{\mathcal{AB}}$, g_{ab} and their inverses.

A geometric interpretation of $\mathcal{T} \to G/P$ can be found in [12] in a real setting. Here, we note that the line subbundle $G \times_P \mathbb{V}^1$ of $\mathcal{T}$ can be identified with the pull-back $\mathcal{O}[-1]$ of the tautological line bundle $\mathcal{O}(-1)$ on $\mathbb{PV}$ to $\mathcal{Q}^n$. The bundle $G \times_P \left(\mathbb{V}^{-1}/\mathbb{V}^0\right)$ is isomorphic to the dual of $\mathcal{O}[-1]$, i.e., to the pullback $\mathcal{O}[1]$ of the hyperplane bundle $\mathcal{O}(1)$ on $\mathbb{PV}$. Finally, since $\mathfrak{p}_{-1} \otimes \mathbb{V}_1 \cong \mathbb{V}_0$, we have the identification $G \times_P \left(\mathbb{V}^0/\mathbb{V}^1\right) \cong \mathrm{T}\mathcal{Q}^n \otimes \mathcal{O}[-1]$.

The structure sheaf of $\mathcal{Q}^n$ will be denoted $\mathcal{O}$, and the sheaf of germs of holomorphic functions on $\mathcal{Q}^n$ homogeneous of degree w by $\mathcal{O}[w]$. We shall write $\mathcal{O}^a$ for the sheaf of germs of holomorphic sections of $\mathrm{T}\mathcal{Q}^n$, and extend this notation in the obvious way to tensor products, e.g., $\mathcal{O}_{ab}^{\mathcal{A}}[w] := \mathcal{O}^{\mathcal{A}} \otimes \mathcal{O}_{ab} \otimes \mathcal{O}[w]$, and so on. In particular, the sheaf of germs of holomorphic sections of the tractor bundle $\mathcal{T}$ reads

$$\mathcal{O}^{\mathcal{A}} = \mathcal{O}[1] + \mathcal{O}^a[-1] + \mathcal{O}[-1]. \tag{2.4}$$

The line bundle $\mathcal{O}[1]$ has the geometric interpretation of the bundle of *conformal scales*, and the conformal structure on $\mathcal{Q}^n$ can be equivalently encoded in terms of a distinguished global section $\mathbf{g}_{ab}$ of $\mathcal{O}_{(ab)}[2]$ called the *conformal metric*. For any non-vanishing local section σ of $\mathcal{O}[1]$, $g_{ab} = \sigma^{-2}\mathbf{g}_{ab}$ is a metric in the conformal class. A choice of metric in the conformal class is essentially equivalently to a splitting of (2.4), i.e., a choice of section $Y^{\mathcal{A}}$ of $\mathcal{O}^{\mathcal{A}}[-1]$ such that $Y^{\mathcal{A}}Y_{\mathcal{A}} = 0$ and $X^{\mathcal{A}}Y_{\mathcal{A}} = 1$, where we view $X^{\mathcal{A}} \in \mathcal{O}^{\mathcal{A}}[1]$ as the Euler vector field on $\mathcal{C} \subset \mathbb{V}$. We can then choose a section $Z_a^{\mathcal{A}}$ of $\mathcal{O}_a^{\mathcal{A}}[1]$ satisfying $Z_a^{\mathcal{A}} Z_{b\mathcal{A}} = \mathbf{g}_{ab}$ and $Z_a^{\mathcal{A}} X_{\mathcal{A}} = Z_a^{\mathcal{A}} Y_{\mathcal{A}} = 0$, so that the tractor metric takes the form $h_{\mathcal{AB}} = 2X_{(\mathcal{A}}Y_{\mathcal{B})} + Z_{\mathcal{A}}^a Z_{\mathcal{B}}^b \mathbf{g}_{ab}$ – see, e.g., [15]. A section $\Sigma^{\mathcal{A}}$ of $\mathcal{O}^{\mathcal{A}}$ can be expressed as

$$\Sigma^{\mathcal{A}} = \sigma Y^{\mathcal{A}} + \varphi^a Z_a^{\mathcal{A}} + \rho X^{\mathcal{A}}, \qquad \text{where} \quad (\sigma, \varphi^a, \rho) \in \mathcal{O}[1] \oplus \mathcal{O}^a[-1] \oplus \mathcal{O}[-1]. \tag{2.5}$$

We shall denote both the tractor connection and the Levi-Civita connection of a metric in the conformal class by ∇_a. The explicit formula for the tractor connection on a section (2.5) of $\mathcal{O}^{\mathcal{A}}$ in terms of a splitting of (2.4) can then be recovered from the Leibniz rule and the formulae

$$\nabla_a X^{\mathcal{A}} = Z_a^{\mathcal{A}}, \qquad \nabla_a Z_b^{\mathcal{A}} = -\mathrm{P}_{ab} X^{\mathcal{A}} - \mathbf{g}_{ab} Y^{\mathcal{A}}, \qquad \nabla_a Y^{\mathcal{A}} = \mathrm{P}_a{}^b Z_b^{\mathcal{A}}, \tag{2.6}$$

where P_{ab} is the Schouten tensor of ∇_a defined by the relation $2\nabla_{[a}\nabla_{b]}V^c = 2\mathrm{P}^c{}_{[a}V_{b]} - 2V^d\mathrm{P}_{d[a}\delta_{b]}^c$.

Complex Euclidean space. Most of this paper will be concerned with the geometry on n-dimensional complex Euclidean space $\mathbb{CE}^n$ viewed as a dense open subset of $\mathcal{Q}^n$, i.e., $\mathbb{CE}^n = \mathcal{Q}^n \setminus \{\infty\}$ where ∞ is a point at 'infinity' on $\mathbb{C}S^n \cong \mathcal{Q}^n$. We choose a conformal scale $\sigma \in \mathcal{O}[1]$ so that g_{ab} is the flat metric, i.e., $\mathrm{P}_{ab} = 0$. To realise σ geometrically, we use the splitting (2.3). Then, $\mathbb{CE}^n$ arises as the intersection of the affine hyperplane $\mathcal{H} := \{X^{\mathcal{A}} \in \mathbb{V} \colon X^{\mathcal{A}}\mathring{Y}_{\mathcal{A}} = 1\}$ with $\mathcal{C}$: $\mathbb{V}_1 = \langle \mathring{X}^{\mathcal{A}} \rangle$ descends to the origin on $\mathbb{CE}^n$, and $\mathbb{V}_{-1} = \langle \mathring{Y}^{\mathcal{A}} \rangle$ represents ∞ on $\mathcal{Q}^n$. The flat metric g_{ab} is obtained by pulling back $h_{\mathcal{AB}}$ along the local section $\mathcal{C} \cap \mathcal{H}$ of $\mathcal{C} \to \mathcal{Q}^n$. Letting $\{x^a\}$ be flat coordinates on $\mathbb{CE}^n$ so that $\nabla_a = \frac{\partial}{\partial x^a}$, we can integrate (2.6) explicitly to get

$$Y^{\mathcal{A}} = \mathring{Y}^{\mathcal{A}}, \qquad Z_a^{\mathcal{A}} = \mathring{Z}_a^{\mathcal{A}} - g_{ab}x^b\mathring{Y}^{\mathcal{A}}, \qquad X^{\mathcal{A}} = \mathring{X}^{\mathcal{A}} + x^a\mathring{Z}_a^{\mathcal{A}} - \tfrac{1}{2}g_{ab}x^a x^b \mathring{Y}^{\mathcal{A}}. \tag{2.7}$$

This description is also consistent with the identification of $\mathbb{CE}^n$ with the tangent space at the 'origin' of $\mathcal{Q}^n$. In this case, the coordinates $\{x^a\}$ arise from $\mathfrak{p}_{-1} \cong \mathbb{V}_{-1} \otimes \mathbb{V}_0$ via the exponential map, which provides an embedding of $\mathbb{CE}^n$ into $\mathcal{Q}^n$, $x^a \mapsto [X^{\mathcal{A}}]$ where $X^{\mathcal{A}}$ is given by (2.7). The embedding can in fact be extended to a *conformal* embedding

$$\begin{aligned} &\mathbb{CE}^n \to \mathcal{C} \to \mathcal{Q}^n, \\ &x^a \mapsto \Omega X^{\mathcal{A}} = \Omega\mathring{X}^{\mathcal{A}} + x^a\Omega\mathring{Z}_a^{\mathcal{A}} - \tfrac{1}{2}\left(\Omega^2 g_{ab}x^a x^b\right)\Omega^{-1}\mathring{Y}^{\mathcal{A}} \mapsto \left[X^{\mathcal{A}}\right], \end{aligned}$$

obtained by intersecting $\mathcal{C}$ with the affine hypersurface $\mathcal{H}_\Omega := \{X^{\mathcal{A}} \in \mathbb{V} \colon X^{\mathcal{A}} \mathring{Y}_{\mathcal{A}} = \Omega\}$, where Ω is a non-vanishing holomorphic function on $\mathbb{V}$.

2.1.2 The tractor spinor bundle

We can play the same game by considering bundles over $\mathcal{Q}^n$ arising from the spinor representations of $G = \mathrm{Spin}(n+2,\mathbb{C})$. Again, we distinguish the odd- and even-dimensional cases.

Odd dimensions. Assume $n = 2m+1$.

Definition 2.2. The *tractor spinor bundle* and *dual tractor spinor bundle* over $\mathcal{Q}^n \cong G/P$ are the holomorphic homogeneous vector bundles $\mathcal{S} := G \times_P \mathbb{S}$ and $\mathcal{S}^* := G \times_P \mathbb{S}^*$ respectively.

The generators ${\Gamma_{\mathcal{A}\boldsymbol{\alpha}}}^{\boldsymbol{\beta}}$ of the Clifford algebra $(\mathbb{V}, h_{\mathcal{AB}})$ induce holomorphic sections of $\mathcal{T}^* \otimes \mathcal{S}^* \otimes \mathcal{S}$ on $\mathcal{Q}^n$, which we shall also denote by ${\Gamma_{\mathcal{A}\boldsymbol{\alpha}}}^{\boldsymbol{\beta}}$. The tractor connection on $\mathcal{Q}^n$ extends to a *tractor spinor connection* on $\mathcal{S}$ preserving ${\Gamma_{\mathcal{A}\boldsymbol{\alpha}}}^{\boldsymbol{\beta}}$, and thus $h_{\mathcal{AB}}$.

There is a filtration of P-submodules $\mathbb{S} =\colon \mathbb{S}^{-\frac{1}{2}} \supset \mathbb{S}^{\frac{1}{2}}$. These P-modules and their quotients give rise to P-invariant vector bundles on $\mathcal{Q}^n$ in the standard way. The splitting (2.3) of $\mathbb{V}$ induces a splitting

$$\mathbb{S} \cong \mathbb{S}_{-\frac{1}{2}} \oplus \mathbb{S}_{\frac{1}{2}}, \tag{2.8}$$

where $\mathbb{S}_{\frac{1}{2}} \cong \mathbb{V}_1 \otimes \mathbb{S}_{-\frac{1}{2}}$, and we can identify $\mathbb{S}_{-\frac{1}{2}}$, and thus $\mathbb{S}_{\frac{1}{2}}$, as the spinor representation for $(\mathbb{V}_0, g_{ab})$. Similar considerations apply to $\mathbb{S}^*$. See, e.g., [16, 17] for details.

Elements of $\mathbb{S}_{\pm\frac{1}{2}}$ will carry bold upper case Roman indices, e.g., $\xi^{\mathbf{A}} \in \mathbb{S}_{\pm\frac{1}{2}}$. The Clifford algebra generators ${\gamma_{a\mathbf{A}}}^{\mathbf{B}}$ satisfy ${\gamma_{(a\mathbf{A}}}^{\mathbf{C}} {\gamma_{b)\mathbf{C}}}^{\mathbf{B}} = -g_{ab}\delta_{\mathbf{A}}^{\mathbf{B}}$, where $\delta_{\mathbf{A}}^{\mathbf{B}}$ is the identity on $\mathbb{S}_{\pm\frac{1}{2}}$. There is a spin-invariant bilinear form $\gamma^{(0)}_{\mathbf{AB}}$ on $\mathbb{S}_{\pm\frac{1}{2}}$, by means of which we can define bilinear forms

$$\gamma^{(k)}_{a_1\dots a_k\mathbf{AB}} := {\gamma_{[a_1\mathbf{A}}}^{\mathbf{C_1}} \cdots {\gamma_{a_k]\mathbf{C_{k-1}}}}^{\mathbf{C_k}} \gamma^{(0)}_{\mathbf{C_k B}},$$

from $\mathbb{S}_{\pm\frac{1}{2}} \times \mathbb{S}_{\pm\frac{1}{2}}$ to $\wedge^k \mathbb{V}_0$ for $k = 1,\dots,n$. We introduce projectors $\mathring{O}^{\mathbf{A}}_{\boldsymbol{\alpha}} \colon \mathbb{S} \to \mathbb{S}_{-\frac{1}{2}}$ and $\mathring{I}^{\mathbf{A}}_{\boldsymbol{\alpha}} \colon \mathbb{S} \to \mathbb{S}_{\frac{1}{2}}$, and injectors $\mathring{I}^{\boldsymbol{\alpha}}_{\mathbf{A}} \colon \mathbb{S}_{-\frac{1}{2}} \to \mathbb{S}$ and $\mathring{O}^{\boldsymbol{\alpha}}_{\mathbf{A}} \colon \mathbb{S}_{\frac{1}{2}} \to \mathbb{S}$, which satisfy $\mathring{O}^{\mathbf{B}}_{\boldsymbol{\alpha}} \mathring{I}^{\boldsymbol{\alpha}}_{\mathbf{A}} = \delta^{\mathbf{B}}_{\mathbf{A}}$ and $\mathring{O}^{\mathbf{A}}_{\boldsymbol{\alpha}} \mathring{I}^{\boldsymbol{\beta}}_{\mathbf{A}} + \mathring{I}^{\mathbf{A}}_{\boldsymbol{\alpha}} \mathring{O}^{\boldsymbol{\beta}}_{\mathbf{A}} = \delta^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}}$. Then one can check that the relation between ${\Gamma_{\mathcal{A}\boldsymbol{\alpha}}}^{\boldsymbol{\beta}}$ and ${\gamma_{a\mathbf{A}}}^{\mathbf{B}}$ is given by

$${\Gamma_{\mathcal{A}\boldsymbol{\alpha}}}^{\boldsymbol{\beta}} = \mathring{Z}^a_{\mathcal{A}} \big(\mathring{O}^{\mathbf{A}}_{\boldsymbol{\alpha}} \mathring{I}^{\boldsymbol{\beta}}_{\mathbf{B}} {\gamma_{a\mathbf{A}}}^{\mathbf{B}} - \mathring{I}^{\mathbf{A}}_{\boldsymbol{\alpha}} \mathring{O}^{\boldsymbol{\beta}}_{\mathbf{B}} {\gamma_{a\mathbf{A}}}^{\mathbf{B}} \big) + \sqrt{2} \mathring{Y}_{\mathcal{A}} \mathring{O}^{\mathbf{A}}_{\boldsymbol{\alpha}} \mathring{O}^{\boldsymbol{\beta}}_{\mathbf{A}} - \sqrt{2} \mathring{X}_{\mathcal{A}} \mathring{I}^{\mathbf{A}}_{\boldsymbol{\alpha}} \mathring{I}^{\boldsymbol{\beta}}_{\mathbf{A}}. \tag{2.9}$$

Sheaves of germs of holomorphic sections of $G \times_P \left(\mathbb{S}^{-\frac{1}{2}}/\mathbb{S}^{-\frac{1}{2}}\right)$ will be denoted $\mathcal{O}^{\mathbf{A}}$, and we shall write $\mathcal{O}^{\mathbf{A}}[-1] := \mathcal{O}^{\mathbf{A}} \otimes \mathcal{O}[-1]$, and similarly for dual bundles in the obvious way. In particular, the sheaves of germs of holomorphic sections of $\mathcal{S}$ and its dual are given by

$$\mathcal{O}^{\boldsymbol{\alpha}} = \mathcal{O}^{\mathbf{A}} + \mathcal{O}^{\mathbf{A}}[-1], \qquad \mathcal{O}_{\boldsymbol{\alpha}} = \mathcal{O}_{\mathbf{A}}[1] + \mathcal{O}_{\mathbf{A}}, \tag{2.10}$$

respectively. The splitting of (2.10) can be realised by means of injectors/projectors $O^{\mathbf{A}}_{\boldsymbol{\alpha}} \in \mathcal{O}^{\mathbf{A}}_{\boldsymbol{\alpha}}$, $I^{\mathbf{A}}_{\boldsymbol{\alpha}} \in \mathcal{O}^{\mathbf{A}}_{\boldsymbol{\alpha}}[-1]$, $O^{\boldsymbol{\alpha}}_{\mathbf{A}} \in \mathcal{O}^{\boldsymbol{\alpha}}_{\mathbf{A}}[1]$ and $I^{\boldsymbol{\alpha}}_{\mathbf{A}} \in \mathcal{O}^{\boldsymbol{\alpha}}_{\mathbf{A}}$, such that $O^{\mathbf{A}}_{\boldsymbol{\alpha}} I^{\boldsymbol{\alpha}}_{\mathbf{B}} = \delta^{\mathbf{A}}_{\mathbf{B}}$, $I^{\mathbf{A}}_{\boldsymbol{\alpha}} O^{\boldsymbol{\alpha}}_{\mathbf{B}} = \delta^{\mathbf{A}}_{\mathbf{B}}$, and $O^{\mathbf{A}}_{\boldsymbol{\alpha}} I^{\boldsymbol{\beta}}_{\mathbf{A}} + I^{\mathbf{A}}_{\boldsymbol{\alpha}} O^{\boldsymbol{\beta}}_{\mathbf{A}} = \delta^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}}$, while all the other pairings are zero. In particular, we shall express a section of $\mathcal{O}^{\boldsymbol{\alpha}}$ as

$$\Xi^{\boldsymbol{\alpha}} = I^{\boldsymbol{\alpha}}_{\mathbf{A}} \xi^{\mathbf{A}} + O^{\boldsymbol{\alpha}}_{\mathbf{A}} \zeta^{\mathbf{A}}, \qquad \text{where} \quad \left(\xi^{\mathbf{A}}, \zeta^{\mathbf{A}}\right) \in \mathcal{O}^{\mathbf{A}} + \mathcal{O}^{\mathbf{A}}[-1],$$

and similarly for dual tractor spinors.

By abuse of notation, the connection on $\mathcal{S}$ and the spin connection associated to a metric in the conformal class will both be denoted ∇_a. They satisfy

$$\nabla_a O^{\mathbf{A}}_{\boldsymbol{\alpha}} = -\frac{1}{\sqrt{2}}\boldsymbol{\gamma}_{a\mathbf{B}}{}^{\mathbf{A}} I^{\mathbf{B}}_{\boldsymbol{\alpha}}, \qquad \nabla_a I^{\mathbf{A}}_{\boldsymbol{\alpha}} = -\frac{1}{\sqrt{2}}\mathrm{P}_{ab}\boldsymbol{\gamma}^b{}_{\mathbf{B}}{}^{\mathbf{A}} O^{\mathbf{B}}_{\boldsymbol{\alpha}},$$
$$\nabla_a O^{\boldsymbol{\alpha}}_{\mathbf{A}} = \frac{1}{\sqrt{2}}\boldsymbol{\gamma}_{a\mathbf{A}}{}^{\mathbf{B}} I^{\boldsymbol{\alpha}}_{\mathbf{B}}, \qquad \nabla_a I^{\boldsymbol{\alpha}}_{\mathbf{A}} = \frac{1}{\sqrt{2}}\mathrm{P}_{ab}\boldsymbol{\gamma}^b{}_{\mathbf{A}}{}^{\mathbf{B}} O^{\boldsymbol{\alpha}}_{\mathbf{B}}, \tag{2.11}$$

where $\boldsymbol{\gamma}_{a\mathbf{A}}{}^{\mathbf{B}} \in \mathcal{O}_{a\mathbf{A}}{}^{\mathbf{B}}[1]$ satisfy $\boldsymbol{\gamma}_{(a\mathbf{A}}{}^{\mathbf{C}}\boldsymbol{\gamma}_{b)\mathbf{C}}{}^{\mathbf{B}} = -\mathbf{g}_{ab}\delta^{\mathbf{B}}_{\mathbf{A}}$. The bundle analogue of (2.9) is

$$\Gamma_{\mathcal{A}\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}} = Z^a_{\mathcal{A}}\big(O^{\mathbf{A}}_{\boldsymbol{\alpha}} I^{\boldsymbol{\beta}}_{\mathbf{B}}\boldsymbol{\gamma}_{a\mathbf{A}}{}^{\mathbf{B}} - I^{\mathbf{A}}_{\boldsymbol{\alpha}} O^{\boldsymbol{\beta}}_{\mathbf{B}}\boldsymbol{\gamma}_{a\mathbf{A}}{}^{\mathbf{B}}\big) + \sqrt{2}Y_{\mathcal{A}} O^{\mathbf{A}}_{\boldsymbol{\alpha}} O^{\boldsymbol{\beta}}_{\mathbf{A}} - \sqrt{2}X_{\mathcal{A}} I^{\mathbf{A}}_{\boldsymbol{\alpha}} I^{\boldsymbol{\beta}}_{\mathbf{A}}.$$

With a choice of conformal scale $\sigma \in \mathcal{O}[1]$ for which $g_{ab} = \sigma^{-2}\mathbf{g}_{ab}$ is flat, i.e., $\mathrm{P}_{ab} = 0$, equations (2.11) can be integrated explicitly to give

$$I^{\boldsymbol{\alpha}}_{\mathbf{A}} = \mathring{I}^{\boldsymbol{\alpha}}_{\mathbf{A}}, \qquad O^{\boldsymbol{\alpha}}_{\mathbf{A}} = \mathring{O}^{\boldsymbol{\alpha}}_{\mathbf{A}} + \frac{1}{\sqrt{2}}x^a\gamma_{a\mathbf{A}}{}^{\mathbf{B}}\mathring{I}^{\boldsymbol{\alpha}}_{\mathbf{B}}, \qquad I^{\mathbf{A}}_{\boldsymbol{\alpha}} = \mathring{I}^{\mathbf{A}}_{\boldsymbol{\alpha}}, \qquad O^{\mathbf{A}}_{\boldsymbol{\alpha}} = \mathring{O}^{\mathbf{A}}_{\boldsymbol{\alpha}} - \frac{1}{\sqrt{2}}x^a\gamma_{a\mathbf{A}}{}^{\mathbf{B}}\mathring{I}^{\mathbf{B}}_{\boldsymbol{\alpha}},$$

where $\gamma_{a\mathbf{A}}{}^{\mathbf{B}} = \sigma^{-1}\boldsymbol{\gamma}_{a\mathbf{A}}{}^{\mathbf{B}}$.

Even dimensions. When $n = 2m$, the story is similar, except that, by virtue of the two chiral spinor representations, we have an unprimed tractor spinor bundle and a primed tractor spinor bundle, defined as $\mathcal{S} := G \times_P \mathbb{S}$ and $\mathcal{S}' := G \times_P \mathbb{S}'$ respectively. We shall view the generators $\Gamma_{\mathcal{A}\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}'}$ and $\Gamma_{\mathcal{A}\boldsymbol{\alpha}'}{}^{\boldsymbol{\beta}}$ as holomorphic sections of $\mathcal{T}^* \otimes \mathcal{S}^* \otimes \mathcal{S}'$ and $\mathcal{T}^* \otimes (\mathcal{S}')^* \otimes \mathcal{S}$ respectively on $\mathcal{Q}^n$, both of which are preserved by the extension of the tractor connection to $\mathcal{S} \oplus \mathcal{S}'$.

The spinor spaces $\mathbb{S}$ and $\mathbb{S}'$ admit filtrations of P-submodules $\mathbb{S} =: \mathbb{S}^{-\frac{1}{2}} \supset \mathbb{S}^{\frac{1}{2}}$ and $\mathbb{S}' =: \mathbb{S}'^{-\frac{1}{2}} \supset \mathbb{S}'^{\frac{1}{2}}$. These P-modules and their quotients give rise to P-invariant vector bundles on $\mathcal{Q}^n$ in the standard way. The splitting (2.3) on $\mathbb{V}$ induces a splitting of these filtrations

$$\mathbb{S} \cong \mathbb{S}_{-\frac{1}{2}} \oplus \mathbb{S}_{\frac{1}{2}}, \qquad \mathbb{S}' \cong \mathbb{S}'_{-\frac{1}{2}} \oplus \mathbb{S}'_{\frac{1}{2}}, \tag{2.12}$$

where $\mathbb{S}'_{\frac{1}{2}} \cong \mathbb{V}_1 \otimes \mathbb{S}_{-\frac{1}{2}}$ and $\mathbb{S}_{\frac{1}{2}} \cong \mathbb{V}_1 \otimes \mathbb{S}'_{-\frac{1}{2}}$, and we can identify $\mathbb{S}_{-\frac{1}{2}}$ and $\mathbb{S}'_{-\frac{1}{2}}$, and thus $\mathbb{S}'_{\frac{1}{2}}$ and $\mathbb{S}_{\frac{1}{2}}$, as the chiral spinor representations of $(\mathbb{V}_0, g_{ab})$. Elements of $\mathbb{S}_{-\frac{1}{2}}$ and $\mathbb{S}_{\frac{1}{2}}$ will carry unprimed and primed upper case Roman indices respectively, e.g., $\eta^{\mathbf{A}} \in \mathbb{S}_{\frac{1}{2}}$ and $\xi^{\mathbf{A}'} \in \mathbb{S}_{-\frac{1}{2}}$. The generators of the Clifford algebra are matrices denoted $\gamma_{a\mathbf{A}}{}^{\mathbf{B}'}$ and $\gamma_{a\mathbf{B}'}{}^{\mathbf{A}}$, satisfying the Clifford identities $\gamma_{(a\mathbf{A}}{}^{\mathbf{C}'}\gamma_{b)\mathbf{C}'}{}^{\mathbf{B}} = -g_{ab}\delta^{\mathbf{B}}_{\mathbf{A}}$ and $\gamma_{(a\mathbf{A}'}{}^{\mathbf{C}}\gamma_{b)\mathbf{C}}{}^{\mathbf{B}'} = -g_{ab}\delta^{\mathbf{B}'}_{\mathbf{A}'}$, where $\delta^{\mathbf{B}'}_{\mathbf{A}'}$ and $\delta^{\mathbf{B}}_{\mathbf{A}}$ are the identity elements on $\mathbb{S}_{-\frac{1}{2}}$ and $\mathbb{S}_{\frac{1}{2}}$ respectively. We also obtain spin invariant bilinear forms $\gamma^{(k)}_{\mathbf{A}'\mathbf{B}'}$, $\gamma^{(k)}_{\mathbf{A}\mathbf{B}}$ and $\gamma^{(k)}_{\mathbf{A}\mathbf{B}'}$. The story for $\mathbb{S}'$ is similar.

We introduce projectors $\mathring{O}^{\mathbf{A}}_{\boldsymbol{\alpha}}$, $\mathring{I}^{\mathbf{A}'}_{\boldsymbol{\alpha}}$ and injectors $\mathring{I}^{\boldsymbol{\alpha}}_{\mathbf{A}}$ and $\mathring{O}^{\boldsymbol{\alpha}}_{\mathbf{A}'}$ for the splitting (2.12), normalised in the obvious way. The relation between the generators of the Clifford algebra $\mathcal{C}\ell(\mathbb{V}, h_{\mathcal{AB}})$ and those of $\mathcal{C}\ell(\mathbb{V}_0, g_{ab})$ is then given by

$$\Gamma_{\mathcal{A}\boldsymbol{\alpha}}{}^{\boldsymbol{\beta}'} = \mathring{Z}^a_{\mathcal{A}}\big(\mathring{O}^{\mathbf{A}}_{\boldsymbol{\alpha}}\mathring{I}^{\boldsymbol{\beta}'}_{\mathbf{B}'}\gamma_{a\mathbf{A}}{}^{\mathbf{B}'} - \mathring{I}^{\mathbf{A}'}_{\boldsymbol{\alpha}}\mathring{O}^{\boldsymbol{\beta}'}_{\mathbf{B}}\gamma_{a\mathbf{A}'}{}^{\mathbf{B}}\big) + \sqrt{2}\mathring{Y}_{\mathcal{A}}\mathring{O}^{\mathbf{A}}_{\boldsymbol{\alpha}}\mathring{O}^{\boldsymbol{\beta}'}_{\mathbf{A}} - \sqrt{2}\mathring{X}_{\mathcal{A}}\mathring{I}^{\mathbf{A}'}_{\boldsymbol{\alpha}}\mathring{I}^{\boldsymbol{\beta}'}_{\mathbf{A}'},$$

and similar for $\Gamma_{\mathcal{A}\boldsymbol{\alpha}'}{}^{\boldsymbol{\beta}}$ by interchanging primed and unprimed indices.

These algebraic objects extend to weighted tensor or spinor fields just as in odd dimensions in the obvious way and notation. In particular, we have composition series of the unprimed and primed tractor spinor bundles:

$$\mathcal{O}^{\boldsymbol{\alpha}} = \mathcal{O}^{\mathbf{A}} + \mathcal{O}^{\mathbf{A}'}[-1], \qquad \mathcal{O}^{\boldsymbol{\alpha}'} = \mathcal{O}^{\mathbf{A}'} + \mathcal{O}^{\mathbf{A}}[-1],$$
$$\mathcal{O}_{\boldsymbol{\alpha}} = \mathcal{O}_{\mathbf{A}'}[1] + \mathcal{O}_{\mathbf{A}}, \qquad \mathcal{O}_{\boldsymbol{\alpha}'} = \mathcal{O}_{\mathbf{A}}[1] + \mathcal{O}_{\mathbf{A}'}.$$

2.2 Twistor space

The linear subspaces of $\mathcal{Q}^n$ can be described in terms of representations of $G = \mathrm{Spin}(n+2,\mathbb{C})$. We shall be interested in those of maximal dimension, arising from maximal totally null vector subspaces of $(\mathbb{V}, h_{\mathcal{AB}})$. In even dimensions, the complex orientation on $\mathbb{V}$ determines the duality of the corresponding linear subspaces, via Hodge duality, which are then described as either self-dual or anti-self-dual.

Definition 2.3. An m-dimensional linear subspace of $\mathcal{Q}^{2m+1}$ is called a *γ-plane*. A self-dual, respectively, anti-self-dual, m-dimensional linear subspace of $\mathcal{Q}^{2m}$ is called an *α-plane*, respectively, a *β-plane*.

We call the space of all γ-planes in $\mathcal{Q}^{2m+1}$ the *twistor space* of $\mathcal{Q}^{2m+1}$, and denote it by $\mathbb{PT}_{(2m+1)}$. The space of all α-planes, respectively, β-planes in $\mathcal{Q}^{2m}$ will be called the *twistor space* $\mathbb{PT}_{(2m)}$, respectively, the *primed twistor space* $\mathbb{PT}'_{(2m)}$.

A point in $\mathbb{PT}$ will be referred to as a *twistor*.

We shall often write $\mathbb{PT}$ and $\mathbb{PT}'$ for $\mathbb{PT}_{(2m+1)}$ or $\mathbb{PT}_{(2m)}$, and $\mathbb{PT}'_{(2m)}$ respectively. We now distinguish the odd- and even-dimensional cases.

2.2.1 Odd dimensions

Assume $n = 2m+1$. Let $Z^{\boldsymbol{\alpha}}$ be a non-zero spinor in $\mathbb{S}$, and define the linear map

$$Z^{\boldsymbol{\alpha}}_{\mathcal{A}} := \Gamma_{\mathcal{A}\boldsymbol{\beta}}{}^{\boldsymbol{\alpha}} Z^{\boldsymbol{\beta}} : \ \mathbb{V} \to \mathbb{S}. \tag{2.13}$$

By (2.1), the kernel of (2.13) is a totally null vector subspace of $\mathbb{V}$, and if it is non-trivial, descends to a linear subspace of $\mathcal{Q}^n$.

Definition 2.4. We say that a non-zero spinor $Z^{\boldsymbol{\alpha}}$ in $\mathbb{S}$ is *pure* if the kernel of $Z^{\boldsymbol{\alpha}}_{\mathcal{A}} := \Gamma_{\mathcal{A}\boldsymbol{\beta}}{}^{\boldsymbol{\alpha}} Z^{\boldsymbol{\beta}}$ has maximal dimension $m+1$.

The $(m+1)$-dimensional totally null subspace of $\mathbb{V}$ associated in this way to a pure spinor descends to a γ-plane in $\mathcal{Q}^n$. Clearly, any two pure spinors differing by a factor give rise to the same γ-plane. Further, one can show that any γ-plane in $\mathcal{Q}^n$ arises from a pure spinor up to scale. Hence,

Proposition 2.5 ([10])**.** *The twistor space $\mathbb{PT}$ of $\mathcal{Q}^{2m+1}$ is isomorphic to the projectivisation of the space of all pure spinors in $\mathbb{S}$.*

Every non-zero spinor in $\mathbb{S}$ is pure when $m = 1$. Let us recall that the $\Gamma^{(k)}_{\boldsymbol{\alpha\beta}}$ in the next theorem denote the spin bilinear forms defined by (2.2).

Theorem 2.6 ([10])**.** *When $m > 1$, a non-zero spinor $Z^{\boldsymbol{\alpha}}$ in $\mathbb{S}$ is pure if and only if it satisfies*

$$\Gamma^{(k)}_{\boldsymbol{\alpha\beta}} Z^{\boldsymbol{\alpha}} Z^{\boldsymbol{\beta}} = 0, \qquad \textit{for all} \quad k < m+1, \quad k \equiv m+2, m+1 \pmod 4, \tag{2.14}$$

and $\Gamma^{(m+1)}_{\boldsymbol{\alpha\beta}} Z^{\boldsymbol{\alpha}} Z^{\boldsymbol{\beta}} \neq 0$.

Alternatively, the quadratic relations (2.14) can be expressed more succinctly by [35]

$$Z^{\mathcal{A}\boldsymbol{\alpha}} Z^{\boldsymbol{\beta}}_{\mathcal{A}} + Z^{\boldsymbol{\alpha}} Z^{\boldsymbol{\beta}} = 0. \tag{2.15}$$

In analogy with the description of the quadric, we shall view $Z^{\boldsymbol{\alpha}}$ as a position vector or coordinates on $\mathbb{S}$. The twistor space of $\mathcal{Q}^n$ can then be described as a complex projective variety of the projectivisation $\mathbb{PS}$ of $\mathbb{S}$ with homogeneous coordinates $[Z^{\boldsymbol{\alpha}}]$ satisfying (2.14) or (2.15) when $m > 1$. For $\mathcal{Q}^3$, we have $\mathbb{PT}_{(3)} \cong \mathbb{CP}^3$.

We shall adopt the following notation: if Z is a point in $\mathbb{PT}$, with homogeneous coordinates $[Z^{\boldsymbol{\alpha}}]$, then the corresponding γ-plane in $\mathcal{Q}^n$ will be denoted $\check{Z}$, i.e.,

$$\check{Z} := \left\{ \left[X^{\mathcal{A}}\right] \in \mathcal{Q}^n \colon X^{\mathcal{A}} Z_{\mathcal{A}}^{\boldsymbol{\alpha}} = 0 \right\}.$$

Let Ξ be a twistor with homogeneous coordinates $[\Xi^{\boldsymbol{\alpha}}]$ and associated γ-plane $\check{\Xi}$ in $\mathcal{Q}^n$. The projective tangent space of $\mathbb{PT}$ at Ξ is the linear subspace of $\mathbb{PS}$ defined by

$$\mathbf{T}_{\Xi}\mathbb{PT} := \left\{ [Z^{\boldsymbol{\alpha}}] \in \mathbb{PS} \colon \Gamma^{(k)}_{\boldsymbol{\alpha\beta}} Z^{\boldsymbol{\alpha}} \Xi^{\boldsymbol{\beta}} = 0, \text{ for all } k < m-1 \right\}. \tag{2.16}$$

This is the closure of the holomorphic tangent space $\mathrm{T}_{\Xi}\mathbb{PT}$ at Ξ, and contains the linear subspace

$$\mathbf{D}_{\Xi} := \left\{ [Z^{\boldsymbol{\alpha}}] \in \mathbb{PS} \colon \Gamma^{(k)}_{\boldsymbol{\alpha\beta}} Z^{\boldsymbol{\alpha}} \Xi^{\boldsymbol{\beta}} = 0, \text{ for all } k < m \right\}. \tag{2.17}$$

This is the closure of a subspace D_{Ξ} of $\mathrm{T}_{\Xi}\mathbb{PT}$. The smooth assignment of every point Ξ of $\mathbb{PT}$ of D_{Ξ} yields a distribution that we shall denote D. Another convenient way of expressing the locus in (2.17) is [35]

$$0 = Z^{\mathcal{A}\boldsymbol{\alpha}} \Xi_{\mathcal{A}}^{\boldsymbol{\beta}} + 2 Z^{\boldsymbol{\beta}} \Xi^{\boldsymbol{\alpha}} - Z^{\boldsymbol{\alpha}} \Xi^{\boldsymbol{\beta}}, \tag{2.18}$$

where $Z_{\mathcal{A}}^{\boldsymbol{\alpha}} := \Gamma_{\mathcal{A}\boldsymbol{\beta}}{}^{\boldsymbol{\alpha}} Z^{\boldsymbol{\beta}}$ and $\Xi_{\mathcal{A}}^{\boldsymbol{\alpha}} := \Gamma_{\mathcal{A}\boldsymbol{\beta}}{}^{\boldsymbol{\alpha}} \Xi^{\boldsymbol{\beta}}$.

To understand $\mathbb{PT}$ more fully, we realise it as a Kleinian geometry. Let us fix a pure spinor $\Xi^{\boldsymbol{\alpha}}$, and denote by R the stabiliser of its span in G. This is a parabolic subgroup of G. Then, $\mathbb{PT}$ is isomorphic to G/R. One could equivalently realise $\mathbb{PT}$ as the quotient of $\mathrm{SO}(n+2,\mathbb{C})$ by the stabiliser of the corresponding γ-plane $\check{\Xi}$ in $\mathcal{Q}^n$. The Lie algebra $\mathfrak{r}$ of R induces a $|2|$-grading on $\mathfrak{g}$, i.e., $\mathfrak{g} = \mathfrak{r}_{-2} \oplus \mathfrak{r}_{-1} \oplus \mathfrak{r}_0 \oplus \mathfrak{r}_1 \oplus \mathfrak{r}_2$, where $\mathfrak{r} = \mathfrak{r}_0 \oplus \mathfrak{r}_1 \oplus \mathfrak{r}_2$, with $\mathfrak{r}_0 \cong \mathfrak{gl}(m+1,\mathbb{C})$, $\mathfrak{r}_{-1} \cong \mathbb{C}^{m+1}$ and $\mathfrak{r}_{-2} \cong \wedge^2 \mathbb{C}^{m+1}$, and $\mathfrak{r}_{-1} \cong (\mathfrak{r}_1)^*$, $\mathfrak{r}_{-2} \cong (\mathfrak{r}_2)^*$. In matrix form, this reads as

$$\left(\begin{array}{c:c:c:c:c} \mathfrak{r}_0 & \mathfrak{r}_0 & \mathfrak{r}_1 & \mathfrak{r}_2 & 0 \\ \hdashline \mathfrak{r}_0 & \mathfrak{r}_0 & \mathfrak{r}_1 & \mathfrak{r}_2 & \mathfrak{r}_2 \\ \hdashline \mathfrak{r}_{-1} & \mathfrak{r}_{-1} & 0 & \mathfrak{r}_1 & \mathfrak{r}_1 \\ \hdashline \mathfrak{r}_{-2} & \mathfrak{r}_{-2} & \mathfrak{r}_{-1} & \mathfrak{r}_0 & \mathfrak{r}_0 \\ \hdashline 0 & \mathfrak{r}_{-2} & \mathfrak{r}_{-1} & \mathfrak{r}_0 & \mathfrak{r}_0 \end{array} \right) \begin{array}{l} \}_1 \\ \}_m \\ \}_1 \\ \}_m \\ \}_1 \end{array}$$

These $\mathfrak{r}_0$-modules satisfy the commutation relations $[\mathfrak{r}_i, \mathfrak{r}_j] \subset \mathfrak{r}_{i+j}$ where $\mathfrak{r}_i = \{0\}$ for $|i| > 2$. Further, $\mathfrak{g}$ is equipped with a filtration of $\mathfrak{r}$-modules $\mathfrak{g} := \mathfrak{r}^{-2} \supset \mathfrak{r}^{-1} \supset \mathfrak{r}^0 \supset \mathfrak{r}^1 \supset \mathfrak{r}^2$ where $\mathfrak{r}^i := \mathfrak{r}_i \oplus \mathfrak{r}^{i+1}$ satisfy $[\mathfrak{r}^i, \mathfrak{r}^j] \subset \mathfrak{r}^{i+j}$. In particular, $\mathfrak{g}/\mathfrak{r}$ is not an irreducible $\mathfrak{r}$-module, but admits a splitting into irreducible $\mathfrak{r}$-submodules $\mathfrak{r}^{-1}/\mathfrak{r}$ and $\mathfrak{r}^{-2}/\mathfrak{r}^{-1}$. Since the tangent space at any point of G/R can be identified with the quotient $\mathfrak{g}/\mathfrak{r}$, i.e., $\mathrm{T}(G/R) \cong G \times_R (\mathfrak{g}/\mathfrak{r})$, the tangent bundle of $\mathbb{PT}$ admits a filtration of R-invariant subbundles $\mathrm{T}\mathbb{PT} = \mathrm{T}^{-2}\mathbb{PT} \supset \mathrm{T}^{-1}\mathbb{PT}$, where the rank-$(m+1)$ distribution

$$\mathrm{T}^{-1}\mathbb{PT} := G \times_R \left(\mathfrak{r}^{-1}/\mathfrak{r}\right) \tag{2.19}$$

is *maximally non-integrable* by virtue of the commutation relations among the various graded pieces of $\mathfrak{g}$, i.e., at every point $Z \in \mathbb{PT}$, $\mathrm{T}^{-1}_Z\mathbb{PT} \cong \mathfrak{r}_{-1}$ and $[\mathrm{T}^{-1}_Z\mathbb{PT}, \mathrm{T}^{-1}_Z\mathbb{PT}] + \mathrm{T}^{-1}_Z\mathbb{PT} \cong \mathfrak{r}_{-1} \oplus \mathfrak{r}_{-2}$.

We shall presently show that the distributions D defined in terms of (2.17) and $\mathrm{T}^{-1}\mathbb{PT}$ defined by (2.19) are the same. We first note that any spinor $Z^{\boldsymbol{\alpha}} \in \mathbb{S}$ can be expressed as

$$\begin{aligned} Z^{\boldsymbol{\alpha}} = Z_{(0)} \Xi^{\boldsymbol{\alpha}} &+ \sum_{k=1}^{[(m+1)/2]} \left(-\frac{1}{4}\right)^k \frac{1}{k!} \left(Z_{(-2k)} \cdot \Xi\right)^{\boldsymbol{\alpha}} \\ &+ \frac{\mathrm{i}}{2} \sum_{k=0}^{[(m+1)/2]} \left(-\frac{1}{4}\right)^k \frac{1}{k!} \left(Z_{(-2k-1)} \cdot \Xi\right)^{\boldsymbol{\alpha}}, \end{aligned} \tag{2.20}$$

where $Z_{(-i)} \in \wedge^i \mathfrak{r}_{-1} \cong \wedge^i \mathbb{C}^{m+1}$, and $\left[\frac{m+1}{2}\right]$ is $\frac{m+1}{2}$ when $m+1$ is even, $\frac{m}{2}$ when $m+1$ is odd. Here, the $\cdot$ denotes the Clifford action, i.e., $(\Phi \cdot \Xi)^{\boldsymbol{\alpha}} = \Phi^{\mathcal{A}} \Gamma_{\mathcal{A}\boldsymbol{\beta}}{}^{\boldsymbol{\alpha}} \Xi^{\boldsymbol{\beta}}$, and so on as extended to the action of $\wedge^\bullet \mathbb{V}$ on $\mathbb{S}$. The factors have been chosen for convenience. The representation (2.20) is sometimes referred to as the *Fock representation* [7], and is already used implicitly in Cartan's work [10], where the $Z_{(-i)}$ are viewed as the components of a spinor.

Now, using (2.1) and (2.2), together with (2.14) applied to $\Xi^{\boldsymbol{\alpha}}$, we compute $\Gamma^{(m+2)}_{\boldsymbol{\alpha\beta}} Z^{\boldsymbol{\alpha}} \Xi^{\boldsymbol{\alpha}} = Z_{(0)} \Gamma^{(m+2)}_{\boldsymbol{\alpha\beta}} \Xi^{\boldsymbol{\alpha}} \Xi^{\boldsymbol{\alpha}}$ and, for $k \geq 0$,

$$\begin{aligned}
\Gamma^{(m-2k+1)}_{\mathcal{A}_1 \ldots \mathcal{A}_{m-2k+1} \boldsymbol{\alpha\beta}} Z^{\boldsymbol{\alpha}} \Xi^{\boldsymbol{\alpha}} &= \left(-\frac{1}{4}\right)^k \frac{1}{k!} Z^{\mathcal{B}_1 \ldots \mathcal{B}_{2k}}_{(-2k)} \Gamma^{(m+1)}_{\mathcal{B}_1 \ldots \mathcal{B}_{2k} \mathcal{A}_1 \ldots \mathcal{A}_{m-2k+1} \boldsymbol{\alpha\beta}} \Xi^{\boldsymbol{\alpha}} \Xi^{\boldsymbol{\alpha}} \\
&\quad + \frac{\mathrm{i}}{2} \left(-\frac{1}{4}\right)^k \frac{1}{k!} Z^{\mathcal{B}_1 \ldots \mathcal{B}_{2k+1}}_{(-2k-1)} \Gamma^{(m+2)}_{\mathcal{B}_1 \ldots \mathcal{B}_{2k+1} \mathcal{A}_1 \ldots \mathcal{A}_{m-2k+1} \boldsymbol{\alpha\beta}} \Xi^{\boldsymbol{\alpha}} \Xi^{\boldsymbol{\alpha}}, \\
\Gamma^{(m-2k)}_{\mathcal{A}_1 \ldots \mathcal{A}_{m-2k} \boldsymbol{\alpha\beta}} Z^{\boldsymbol{\alpha}} \Xi^{\boldsymbol{\alpha}} &= \frac{\mathrm{i}}{2} \left(-\frac{1}{4}\right)^k \frac{1}{k!} Z^{\mathcal{B}_1 \ldots \mathcal{B}_{2k+1}}_{(-2k-1)} \Gamma^{(m+1)}_{\mathcal{B}_1 \ldots \mathcal{B}_{2k+1} \mathcal{A}_1 \ldots \mathcal{A}_{m-2k} \boldsymbol{\alpha\beta}} \Xi^{\boldsymbol{\alpha}} \Xi^{\boldsymbol{\alpha}} \\
&\quad + \left(-\frac{1}{4}\right)^{k+1} \frac{1}{(k+1)!} Z^{\mathcal{B}_1 \ldots \mathcal{B}_{2k+2}}_{(-2k-2)} \Gamma^{(m+2)}_{\mathcal{B}_1 \ldots \mathcal{B}_{2k+2} \mathcal{A}_1 \ldots \mathcal{A}_{m-2k} \boldsymbol{\alpha\beta}} \Xi^{\boldsymbol{\alpha}} \Xi^{\boldsymbol{\alpha}}. \quad (2.21)
\end{aligned}$$

Here, we have added tractor indices to the $Z_{(-i)}$. We can immediately conclude

Lemma 2.7. *The conditions that $[Z^{\boldsymbol{\alpha}}] \in \mathbb{PS}$ lies in $\mathbf{T}_\Xi \mathbb{PT}$ and $\mathbf{D}_\Xi$ respectively are equivalent to*

$$Z^{\boldsymbol{\alpha}} = Z_{(0)} \Xi^{\boldsymbol{\alpha}} + \tfrac{\mathrm{i}}{2} \big(Z_{(-1)} \cdot \Xi\big)^{\boldsymbol{\alpha}} - \tfrac{1}{4} \big(Z_{(-2)} \cdot \Xi\big)^{\boldsymbol{\alpha}}, \quad (2.22)$$

$$Z^{\boldsymbol{\alpha}} = Z_{(0)} \Xi^{\boldsymbol{\alpha}} + \tfrac{\mathrm{i}}{2} \big(Z_{(-1)} \cdot \Xi\big)^{\boldsymbol{\alpha}}, \quad (2.23)$$

respectively, up to overall factors. When $Z_{(0)}$ is non-zero, $[Z^{\boldsymbol{\alpha}}]$ given by (2.22) and (2.23) lies in $\mathrm{T}_\Xi \mathbb{PT}$ and D_Ξ respectively. In particular, $\mathrm{D} \cong \mathrm{T}^{-1} \mathbb{PT}$.

Proof. Equations (2.22) and (2.23) follow from definitions (2.16) and (2.17) using (2.21). Equation (2.23) with $Z_{(0)} = 1$ coincides with the exponential of an element of $\mathfrak{r}_{-1}$ and thus describes a point in $\mathrm{T}^{-1}_\Xi \mathbb{PT}$. The story for (2.22) is similar. ■

On the other hand, using (2.14) or referring to [10], the condition that $Z^{\boldsymbol{\alpha}}$ be pure is that

$$\begin{aligned}
&Z_{(0)} Z_{(-2k-1)} = Z_{(-1)} \wedge Z_{(-2k)}, \\
&Z_{(0)} Z_{(-2k)} = Z_{(-2)} \wedge Z_{(-2k+2)}, \qquad k = 1, \ldots, [(m+1)/2]. \quad (2.24)
\end{aligned}$$

A dense open subset of $\mathbb{PT}$ containing $[\Xi^{\boldsymbol{\alpha}}]$ can be obtained by intersecting the locus (2.24) with the affine subspace $Z_{(0)} = 1$ in $\mathbb{S}$. Summarising,

Proposition 2.8. *The twistor space $\mathbb{PT}$ of a $(2m+1)$-dimensional smooth quadric $\mathcal{Q}^{2m+1}$ has dimension $\frac{1}{2}(m+1)(m+2)$, and is equipped with a maximally non-integrable distribution D of rank $m+1$, i.e., $\mathrm{T}\mathbb{PT} = \mathrm{D} + [\mathrm{D}, \mathrm{D}]$, where, for any $\Xi \in \mathbb{PT}$, D_Ξ is a dense open subset of $\mathbf{D}_\Xi$ as defined by (2.17).*

Further, for any $\Xi \in \mathbb{PT}$, the projective tangent space $\mathbf{T}_\Xi \mathbb{PT}$ intersects $\mathbb{PT}$ in a $(2m+1)$-dimensional linear subspace of $\mathbb{PT}$, and $\mathbf{D}_\Xi$ is an $(m+1)$-dimensional linear subspace of $\mathbb{PT}$.

Proof. The first part has already been explained and stems from the general theory of [9].

For the second part, we fix a pure spinor $\Xi^{\boldsymbol{\alpha}}$, and let $[Z^{\boldsymbol{\alpha}}]$ be an element of the projective tangent space $\mathbf{T}_\Xi \mathbb{PT}$ so that $Z^{\boldsymbol{\alpha}}$ takes the form (2.22). If $[Z^{\boldsymbol{\alpha}}]$ also lies in $\mathbb{PT}$, then, with reference to (2.24), $Z_{(-1)} \wedge Z_{(-2)} = 0$ and $Z_{(-2)} \wedge Z_{(-2)} = 0$. Generically, $Z_{(-1)}$ is non-zero, so $Z_{(-2)} = Z_{(-1)} \wedge \Phi_{(-1)}$ for some $\Phi_{(-1)} \in \mathfrak{r}_{-1}$. The form of $Z_{(-2)}$ remains invariant under the transformation $\Phi_{(-1)} \mapsto \Phi_{(-1)} + aZ_{(-1)}$ for any $a \in \mathbb{C}$. The choice of $Z_{(0)}$ is cancelled out by the freedom in the choice of scale of (2.22). Thus, $\dim(\mathbf{T}_\Xi\mathbb{PT} \cap \mathbb{PT}) = 2 \times (m+1) - 1 = 2m+1$. If $[Z^{\alpha}]$ lies in $\mathbf{D}_\Xi\mathbb{PT}$, then it takes the form (2.23). In this case, the purity conditions (2.24) do not yield any further constraints, and thus $[Z^{\alpha}]$ must also lie in $\mathbb{PT}$. ∎

Definition 2.9. The rank-$(m+1)$ distribution D will be referred to as the *canonical distribution* of $\mathbb{PT}$.

When $m = 1$, the twistor space of $\mathcal{Q}^3$ is simply $\mathbb{CP}^3$ and the canonical distribution D is the rank-2 contact distribution annihilated by the contact 1-form $\boldsymbol{\alpha} := \Gamma^{(0)}_{\alpha\beta} Z^{\alpha} \mathrm{d} Z^{\beta}$. The appropriate generalisation of this contact 1-form to dimension $2m+1$ is then the set of 1-forms

$$\boldsymbol{\alpha}^{\alpha\beta} := Z^{\mathcal{A}\alpha} \mathrm{d} Z^{\beta}_{\mathcal{A}} + 2 Z^{\beta} \mathrm{d} Z^{\alpha} - Z^{\alpha} \mathrm{d} Z^{\beta}, \tag{2.25}$$

annihilating the canonical distribution D. Here, the homogeneous coordinates $[Z^{\alpha}]$ are assumed to satisfy (2.14) or (2.15).

The following lemma follows directly from the exponential map from a given complement of $\mathfrak{r}$ in $\mathfrak{g}$ to a dense open subset of $\mathbb{PT}$.

Lemma 2.10. *Let Ξ be a point in $\mathbb{PT}$, and let $\mathfrak{r}$ be its stabiliser in $\mathfrak{g}$. Then $\mathbf{D}_\Xi$ is foliated by a family of distinguished curves passing through Ξ parametrised by the points of the $(m+1)$-dimensional module $\mathfrak{r}_{-1}$, for any decomposition $\mathfrak{r} = \mathfrak{r}_{-2} \oplus \mathfrak{r}_{-1} \oplus \mathfrak{r}_0 \oplus \mathfrak{r}_1 \oplus \mathfrak{r}_2$.*

Geometric correspondences. The bilinear forms (2.2) can also be used to characterise the intersections of γ-planes in terms of their corresponding pure spinors.

Theorem 2.11 ([10, 17])**.** *Let Z and W be two twistors with homogeneous coordinates $[Z^{\alpha}]$ and $[W^{\alpha}]$, and corresponding γ-planes $\check{Z}$ and $\check{W}$ in $\mathcal{Q}^n$ respectively. Then*

$$\dim\left(\check{Z} \cap \check{W}\right) \geq k \quad \Longleftrightarrow \quad \Gamma^{(\ell)}_{\alpha\beta} Z^{\alpha} W^{\beta} = 0, \qquad \text{for all} \quad \ell \leq k.$$

Further, $\dim(\check{Z} \cap \check{W}) = k$ *if and only if in addition* $\Gamma^{(k+1)}_{\alpha\beta} Z^{\alpha} W^{\beta} \neq 0$.

A direct application leads to

Proposition 2.12. *Let Ξ and Z be two twistors with corresponding γ-planes $\check{\Xi}$ and $\check{Z}$ respectively. Then*

1. $\dim(\check{\Xi} \cap \check{Z}) \geq m-3$ *if and only if there exists* $W \in \mathbb{PT}$ *such that* $W \in \mathbf{D}_\Xi \cap \mathbf{T}_Z\mathbb{PT}$ *or* $W \in \mathbf{D}_Z \cap \mathbf{T}_\Xi\mathbb{PT}$.
2. $\dim(\check{\Xi} \cap \check{Z}) \geq m-2$ *if and only if* $\Xi \in \mathbf{T}_Z\mathbb{PT}$ *if and only if* $Z \in \mathbf{T}_\Xi\mathbb{PT}$ *if and only if there exists* $W \in \mathbb{PT}$ *such that* $Z, \Xi \in \mathbf{D}_W$*, or equivalenly* $W \in \mathbf{D}_Z \cap \mathbf{D}_\Xi$.
3. $\dim(\check{\Xi} \cap \check{Z}) \geq m-1$ *if and only if* $Z \in \mathbf{D}_\Xi$ *if and only if* $\Xi \in \mathbf{D}_Z$.

Proof. We fix Ξ^{α} and we assume that Z^{α} is given by (2.20) with components $Z_{(-i)}$ satisfying (2.24). In each case, we apply Theorem 2.11 and compute $\Gamma^{(\ell)}_{\alpha\beta} Z^{\alpha} W^{\beta} = 0$ to derive conditions on $Z_{(-i)}$. With no loss of generality, we may assume $Z_{(0)} = 1$.

1. We have $Z_{(-i)}$ for all $i \geq 4$ and $Z_{(-2)} \wedge Z_{(-2)} = 0$, i.e., $Z_{(-2)} = \Phi_{(-1)} \wedge \Psi_{(-1)}$ for some $\Phi_{(-1)}, \Psi_{(-1)} \in \mathfrak{r}_{-1}$, and $Z_{(-3)} = Z_{(-1)} \wedge Z_{(-2)} = Z_{(-1)} \wedge \Phi_{(-1)} \wedge \Psi_{(-1)}$. A suitable $W \in \mathbf{D}_\Xi \cap \mathbf{T}_Z\mathbb{PT}$ is given by $W^{\alpha} = \Xi^{\alpha} + \frac{\mathrm{i}}{2}(Z_{(-1)} \cdot \Xi)^{\alpha}$ and $W^{\alpha} = Z^{\alpha} + \frac{1}{4}((\Phi_{(-1)} \wedge \Psi_{(-1)}) \cdot Z)^{\alpha}$, and similarly for a suitable $W \in \mathbf{D}_Z \cap \mathbf{T}_\Xi\mathbb{PT}$.

2. The first two equivalences follow immediately from Proposition 2.8 and Theorem 2.11. For the last equivalence, we have $Z_{(-i)}$ for all $i \geq 3$, so that $Z_{(-2)} \wedge Z_{(-2)} = 0$ and $Z_{(-1)} \wedge Z_{(-2)} = 0$, i.e., $Z_{(-2)} = Z_{(-1)} \wedge \Phi_{(-1)}$ for some $\Phi_{(-1)} \in \mathfrak{r}_{-1}$. A suitable $W \in \mathbf{D}_Z \cap \mathbf{D}_\Xi$ is given by $W^{\alpha} = \Xi^{\alpha} + \frac{\mathrm{i}}{2}(Z_{(-1)} \cdot \Xi)^{\alpha}$ and $W^{\alpha} = Z^{\alpha} - \frac{\mathrm{i}}{2}(\Phi_{(-1)} \cdot Z)^{\alpha}$.

3. This follows immediately from Proposition 2.8 and Theorem 2.11. ∎

In a similar vein, we obtain

Proposition 2.13. *Fix a twistor Ξ in $\mathbb{PT}$ and let $\check{\Xi}$ be its corresponding γ-plane in $\mathcal{Q}^n$. Let Z and W be two twistors in $\mathbf{T}_\Xi \mathbb{PT}$, corresponding to γ-planes $\check{Z}$ and $\check{W}$. Then $\dim(\check{Z} \cap \check{W}) \geq m-4$.*

Further, if Z and W take the respective forms

$$Z^{\boldsymbol{\alpha}} = Z_{(0)}\Xi^{\boldsymbol{\alpha}} + \tfrac{\mathrm{i}}{2}\big(Z_{(-1)} \cdot \Xi\big)^{\boldsymbol{\alpha}} - \tfrac{1}{4}\big(Z_{(-2)} \cdot \Xi\big)^{\boldsymbol{\alpha}},$$
$$W^{\boldsymbol{\alpha}} = W_{(0)}\Xi^{\boldsymbol{\alpha}} + \tfrac{\mathrm{i}}{2}\big(W_{(-1)} \cdot \Xi\big)^{\boldsymbol{\alpha}} - \tfrac{1}{4}\big(W_{(-2)} \cdot \Xi\big)^{\boldsymbol{\alpha}},$$

where $Z_{(0)}Z_{(-2)} = Z_{(-1)} \wedge Z_{(-1)}$, $Z_{(-2)} \wedge Z_{(-2)} = 0$, $W_{(0)}W_{(-2)} = W_{(-1)} \wedge W_{(-1)}$ and $W_{(-2)} \wedge W_{(-2)} = 0$, then

$$\dim\big(\check{Z} \cap \check{W}\big) \geq m-3 \quad \Longleftrightarrow \quad Z_{(-2)} \wedge W_{(-2)} = 0, \tag{2.26a}$$
$$\dim\big(\check{Z} \cap \check{W}\big) \geq m-2 \quad \Longleftrightarrow \quad Z_{(-1)} \wedge W_{(-2)} + W_{(-1)} \wedge Z_{(-2)} = 0, \tag{2.26b}$$
$$\dim\big(\check{Z} \cap \check{W}\big) \geq m-1 \quad \Longleftrightarrow \quad W_{(-2)} - Z_{(-2)} - W_{(-1)} \wedge Z_{(-1)} = 0. \tag{2.26c}$$

Proof. Let us rewrite

$$W^{\boldsymbol{\alpha}} = Z^{\boldsymbol{\alpha}} + \tfrac{\mathrm{i}}{2}\big(\Phi_{(-1)} \cdot Z\big)^{\boldsymbol{\alpha}} - \tfrac{1}{4}\big(\Phi_{(-2)} \cdot Z\big)^{\boldsymbol{\alpha}}$$
$$\qquad - \tfrac{\mathrm{i}}{8}\big(\big(\Phi_{(-1)} \wedge \Phi_{(-2)}\big) \cdot Z\big)^{\boldsymbol{\alpha}} + \tfrac{1}{32}\big(\big(\Phi_{(-2)} \wedge \Phi_{(-2)}\big) \cdot Z\big)^{\boldsymbol{\alpha}},$$

where $\Phi_{-1} := W_{-1} - Z_{-1}$ and $\Phi_{-2} := W_{-2} - Z_{-2} - W_{-1} \wedge Z_{-1}$. It suffices to compute $\Gamma^{(m-k)}_{\boldsymbol{\alpha\beta}} Z^{\boldsymbol{\alpha}} W^{\boldsymbol{\beta}} = 0$ for all $k \geq 4$, and

1) $\Gamma^{(m-k)}_{\boldsymbol{\alpha\beta}} Z^{\boldsymbol{\alpha}} W^{\boldsymbol{\beta}} = 0$ for all $k \geq 3$ if and only if $\Phi_{(-2)} \wedge \Phi_{(-2)} = 0$;

2) $\Gamma^{(m-k)}_{\boldsymbol{\alpha\beta}} Z^{\boldsymbol{\alpha}} W^{\boldsymbol{\beta}} = 0$ for all $k \geq 2$ if and only if $\Phi_{(-1)} \wedge \Phi_{(-2)} = 0$;

3) $\Gamma^{(m-k)}_{\boldsymbol{\alpha\beta}} Z^{\boldsymbol{\alpha}} W^{\boldsymbol{\beta}} = 0$ for all $k \geq 1$ if and only if $\Phi_{(-2)} = 0$.

Equivalences (2.26a), (2.26b) and (2.26c) now follow from the definitions of $\Phi_{(-1)}$ and $\Phi_{(-2)}$. ■

A special case of this proposition is given below.

Corollary 2.14. *Fix a twistor Ξ in $\mathbb{PT}$ and let $\check{\Xi}$ be its corresponding γ-plane in $\mathcal{Q}^n$. Let Z and W be two twistors in $\mathbf{D}_\Xi$, corresponding to γ-planes $\check{Z}$ and $\check{W}$. Then $\dim(\check{Z} \cap \check{W}) \geq m-2$.*

Further, Z and W belong to the same distinguished curve in $\mathbf{D}_\Xi$, as defined in Lemma 2.10, if and only if $\dim(\check{Z} \cap \check{W}) \geq m-1$.

Proof. This is a direct consequence of Proposition 2.13 with $Z_{(-2)} = W_{(-2)} = 0$, and Lemma 2.10. ■

2.2.2 Even dimensions

Assume $n = 2m$. Any non-zero chiral spinor $Z^{\boldsymbol{\alpha}}$ defines a linear map $Z^{\boldsymbol{\alpha}'}_{\mathcal{A}} := \Gamma_{\mathcal{A}\boldsymbol{\beta}}{}^{\boldsymbol{\alpha}'} Z^{\boldsymbol{\beta}} : \mathbb{V} \to \mathbb{S}$, and similarly for primed spinors. Again, any non-trivial kernel of this map descends to a linear subspace of $\mathcal{Q}^n$. A non-zero chiral spinor $Z^{\boldsymbol{\alpha}}$ is *pure* if the kernel of $Z^{\boldsymbol{\alpha}}_{\mathcal{A}}$ has maximal dimension $m+1$, and similarly for primed spinors.

Proposition 2.15 ([10])**.** *The twistor space $\mathbb{PT}$ and the primed twistor space $\mathbb{PT}'$ of $\mathcal{Q}^{2m}$ are isomorphic to the projectivations of the spaces of all pure spinors in $\mathbb{S}$ and $\mathbb{S}'$ respectively.*

When $m = 2$, all spinors in $\mathbb{S}$ and $\mathbb{S}'$ are pure. When $m > 2$, the analogue of the purity condition (2.14) is now [10]

$$\Gamma^{(k)}_{\alpha\beta} Z^{\alpha} Z^{\beta} = 0, \qquad \text{for all} \quad k < m+1, \quad k \equiv m+1 \pmod 4, \tag{2.27}$$

or alternatively, [20, 34], $Z^{\mathcal{A}\alpha'} Z^{\beta'}_{\mathcal{A}} = 0$. Again, we will think of $\mathbb{PT}$ and $\mathbb{PT}'$ as complex projective varieties of $\mathbb{PS}$ and $\mathbb{PS}'$ respectively, when $m > 2$, while for $\mathcal{Q}^4$, we have $\mathbb{PT}_{(4)} \cong \mathbb{CP}^3$.

The Kleinian model is again a homogeneous space G/R, where R is parabolic. But its parabolic Lie algebra $\mathfrak{r}$ this time induces a $|1|$-grading $\mathfrak{g} = \mathfrak{r}_{-1} \oplus \mathfrak{r}_0 \oplus \mathfrak{r}_1$ on $\mathfrak{g}$, where $\mathfrak{r}_0 \cong \mathfrak{gl}(m+1, \mathbb{C})$, $\mathfrak{r}_{-1} \cong \wedge^2 \mathbb{C}^{m+1}$ and $\mathfrak{r}_1 \cong \wedge^2 (\mathbb{C}^{m+1})^*$, and $\mathfrak{r} = \mathfrak{r}_0 \oplus \mathfrak{r}_1$, as given in matrix form by

$$\left(\begin{array}{c:c:c:c} \mathfrak{r}_0 & \mathfrak{r}_0 & \mathfrak{r}_1 & 0 \\ \hdashline \mathfrak{r}_0 & \mathfrak{r}_0 & \mathfrak{r}_1 & \mathfrak{r}_1 \\ \hdashline \mathfrak{r}_{-1} & \mathfrak{r}_{-1} & \mathfrak{r}_0 & \mathfrak{r}_0 \\ \hdashline 0 & \mathfrak{r}_{-1} & \mathfrak{r}_0 & \mathfrak{r}_0 \end{array} \right) \begin{array}{l} \}_1 \\ \}_m \\ \}_m \\ \}_1 \end{array}$$

Again, the one-dimensional center of $\mathfrak{r}_0$ is spanned by a unique grading element with eigenvalues i on $\mathfrak{r}_i$. In this case, the tangent space of any point of G/R is irreducible and linearly isomorphic to $\mathfrak{r}_{-1}$.

Unlike in odd dimensions, the twistor space of $\mathcal{Q}^{2m}$ is not equipped with any canonical rank-m distribution. As we shall see in Section 2.2.3, one requires an additional structure to endow $\mathbb{PT}_{(2m)}$ with one.

Proposition 2.16. *The twistor space $\mathbb{PT}$ of a $2m$-dimensional smooth quadric $\mathcal{Q}^{2m}$ has dimension $\frac{1}{2}m(m+1)$. Further, for any Z of $\mathbb{PT}$, the projective tangent space $\mathbf{T}_Z\mathbb{PT}$ intersects $\mathbb{PT}$ in a $(2m-1)$-dimensional linear subspace of $\mathbb{PT}$.*

Arguments similar to those used in odd dimensions lead to the following proposition.

Proposition 2.17. *Let Z and W be two twistors corresponding to α-planes $\check{Z}$ and $\check{W}$. Then $\dim(\check{Z} \cap \check{W}) \in \{m-2, m\}$ if and only if $Z \in \mathbf{T}_W\mathbb{PT}$, or equivalently, $W \in \mathbf{T}_Z\mathbb{PT}$. Further, if Z and W lie in $\mathbf{T}_\Xi\mathbb{PT}$ for some twistor Ξ in $\mathbb{PT}$, then $\dim(\check{Z} \cap \check{W}) \in \{m-4, m-2, m\}$.*

2.2.3 From even to odd dimensions

We note that as $\frac{1}{2}(m+1)(m+2)$-dimensional projective complex varieties of $\mathbb{CP}^{2^{m+1}-1}$, the respective twistor spaces $\mathbb{PT} := \mathbb{PT}_{(2m+1)}$ and $\widetilde{\mathbb{PT}} := \mathbb{PT}_{(2m+2)}$ of $\mathcal{Q}^{2m+1}$ and $\mathcal{Q}^{2m+2}$ are isomorphic. The only geometric structure that distinguishes the former from the latter is the rank-$(m+1)$ canonical distribution. It is shown in [13] how $\widetilde{\mathbb{PT}}$ can be viewed as a 'Fefferman space' over $\mathbb{PT}$ – in fact, this reference deals with a more general, curved, setting. Here, we explain how the canonical distribution on $\mathbb{PT}$ arises as one 'descends' from $\widetilde{\mathbb{PT}}$ to $\mathbb{PT}$.

Let $\widetilde{\mathbb{V}}$ be a $(2m+4)$-dimensional oriented complex vector space equipped with a non-degenerate symmetric bilinear form $\tilde{h}_{\mathcal{AB}}$. Denote by $X^{\mathcal{A}}$ the standard coordinates on $\widetilde{\mathbb{V}}$. As before, we realise $\mathcal{Q}^{2m+2}$ as a smooth quadric of $\mathbb{P}\widetilde{\mathbb{V}}$ with twistor spaces $\widetilde{\mathbb{PT}}$ and $\widetilde{\mathbb{PT}}'$ induced from the irreducible spinor representations $\widetilde{\mathbb{S}}$ and $\widetilde{\mathbb{S}}'$ of $(\widetilde{\mathbb{V}}, \tilde{h}_{\mathcal{AB}})$. Now, fix a unit vector $U^{\mathcal{A}}$ in $\widetilde{\mathbb{V}}$, so that $\widetilde{\mathbb{V}} = \mathbb{U} \oplus \mathbb{V}$, where $\mathbb{U} := \langle U^{\mathcal{A}} \rangle$, and $\mathbb{V} := \mathbb{U}^{\perp}$ is its orthogonal complement in $\widetilde{\mathbb{V}}$. Then $\mathbb{V}$ is equipped with a non-degenerate symmetric bilinear form $h_{\mathcal{AB}} := \tilde{h}_{\mathcal{AB}} - U_{\mathcal{A}} U_{\mathcal{B}}$, and we can realise $\mathcal{Q}^{2m+1}$ as a smooth quadric of $\mathbb{PV}$ with twistor space $\mathbb{PT}$ induced from the irreducible spinor representation $\mathbb{S}$ of $(\mathbb{V}, h_{\mathcal{AB}})$.

Observe that $U^{\mathcal{A}}$ defines two invertible linear maps,

$$U_{\alpha'}^{\beta} := U^{\mathcal{A}} \widetilde{\Gamma}_{\mathcal{A}\alpha'}{}^{\beta} : \ \widetilde{\mathbb{S}}' \to \widetilde{\mathbb{S}}, \qquad U_{\alpha}^{\beta'} := U^{\mathcal{A}} \widetilde{\Gamma}_{\mathcal{A}\alpha}{}^{\beta'} : \ \widetilde{\mathbb{S}} \to \widetilde{\mathbb{S}}',$$

where $\widetilde{\Gamma}_{\mathcal{A}\alpha'}{}^{\beta}$ and $\widetilde{\Gamma}_{\mathcal{A}\alpha}{}^{\beta'}$ generate the Clifford algebra $\mathcal{C}\ell(\widetilde{\mathbb{V}}, \tilde{h}_{\mathcal{AB}})$. These maps allow us to identify $\widetilde{\mathbb{S}}$ with $\widetilde{\mathbb{S}}'$, and thus $\widetilde{\mathbb{PT}}$ with $\widetilde{\mathbb{PT}}'$. Further, using the Clifford property, it is straightforward to check that $\Gamma_{\mathcal{A}\alpha}{}^{\beta} := h_{\mathcal{A}}^{\mathcal{B}} \widetilde{\Gamma}_{\mathcal{B}\alpha}{}^{\gamma'} U_{\gamma'}^{\beta} = -h_{\mathcal{A}}^{\mathcal{B}} U_{\alpha}^{\gamma'} \widetilde{\Gamma}_{\mathcal{B}\gamma'}{}^{\beta} = U^{\mathcal{B}} \widetilde{\Gamma}_{\mathcal{AB}\alpha}{}^{\beta}$ generate the Clifford algebra $\mathcal{C}\ell(\mathbb{V}, h_{\mathcal{AB}})$. More generally, the relation between the spanning elements of $\mathcal{C}\ell(\mathbb{V}, h_{\mathcal{AB}})$ and those of $\mathcal{C}\ell(\widetilde{\mathbb{V}}, \widetilde{h}_{\mathcal{AB}})$ is given by

$$\begin{aligned}
&\Gamma^{(k)}_{\mathcal{A}_1 \ldots \mathcal{A}_k \alpha\beta} = h_{\mathcal{A}_1}^{\mathcal{B}_1} \cdots h_{\mathcal{A}_k}^{\mathcal{B}_k} \widetilde{\Gamma}^{(k)}_{\mathcal{B}_1 \ldots \mathcal{B}_k \alpha\beta}, \qquad k \equiv m+2 \pmod 2, \\
&\Gamma^{(k)}_{\mathcal{A}_1 \ldots \mathcal{A}_k \alpha\beta} = U^{\mathcal{B}} \widetilde{\Gamma}^{(k)}_{\mathcal{A}_1 \cdots \mathcal{A}_k \mathcal{B} \alpha\beta} = (-1)^k h_{\mathcal{A}_1}^{\mathcal{B}_1} \cdots h_{\mathcal{A}_k}^{\mathcal{B}_k} U_{\alpha}^{\gamma'} \widetilde{\Gamma}^{(k)}_{\mathcal{B}_1 \cdots \mathcal{B}_k \gamma' \beta}, \qquad k \equiv m+1 \pmod 2.
\end{aligned} \tag{2.28}$$

If we now introduce homogeneous coordinates $[Z^{\alpha}]$ on $\mathbb{P}\widetilde{\mathbb{S}}$, we can identify the twistor space $\mathbb{PT}$ equipped with its canonical distribution with the twistor space $\widetilde{\mathbb{PT}}$, as can be seen by inspection of (2.14) and (2.27). Note that we could have played the same game with $\widetilde{\mathbb{PT}}'$.

Let us interpret this more geometrically. Clearly, the embedding of $\mathcal{Q}^{2m+1}$ into $\mathcal{Q}^{2m+2}$ arises as the intersection of the hyperplane $U_{\mathcal{A}} X^{\mathcal{A}} = 0$ in $\mathbb{P}\widetilde{\mathbb{V}}$ with the cone over $\mathcal{Q}^{2m+2}$. A γ-plane of $\mathcal{Q}^{2m+1}$ then arises as the intersection of an α-plane of $\mathcal{Q}^{2m+2}$ with $\mathcal{Q}^{2m+1}$, and similarly for β-planes. An α-plane $\check{Z}$ and a β-plane $\check{W}$ define the same γ-plane if and only if their corresponding twistors satisfy $Z^{\alpha} = U_{\beta'}^{\alpha} W^{\beta'}$. In particular, such a pair must intersect maximally, i.e., in an m-plane in $\mathcal{Q}^{2m+2}$. This much is already outlined in the appendix of [31].

Finally, we can see how the canonical distribution D on $\mathbb{PT}$ arises geometrically from $\widetilde{\mathbb{PT}}$ and $\widetilde{\mathbb{PT}}'$. Fix a point $[\Xi^{\alpha}]$ in $\widetilde{\mathbb{PT}}$. This represents an α-plane $\check{\Xi}$ in $\mathcal{Q}^{2m+2}$, and so a γ-plane in $\mathcal{Q}^{2m+1}$, which also corresponds to the unique β-plane with associated primed twistor $[U_{\beta}^{\alpha'} \Xi^{\beta}]$ in $\widetilde{\mathbb{PT}}'$. We claim that the β-planes intersecting $\check{\Xi}$ maximally are in one-to-one correspondence with the points of $\mathbf{D}_{\Xi}$. To see this, let $[Z^{\alpha}]$ be a point in $\mathbf{T}_{\Xi} \widetilde{\mathbb{PT}} \subset \mathbb{P}\widetilde{\mathbb{S}}$ so that

$$\widetilde{\Gamma}^{(k)}_{\alpha\beta} Z^{\alpha} \Xi^{\beta} = 0, \qquad \text{for all} \quad k < m, \quad k \equiv m \pmod 2.$$

We can then conclude $[Z^{\alpha}] \in \mathbf{T}_{\Xi} \mathbb{PT}$ by virtue of (2.16) and (2.28) as expected. Now, consider the set of all β-planes intersecting $\check{\Xi}$ maximally: these correspond to all primed twistors $[W^{\alpha'}] \in \widetilde{\mathbb{PT}}'$ satisfying

$$\widetilde{\Gamma}^{(k)}_{\alpha'\beta} W^{\alpha'} \Xi^{\beta} = 0, \qquad \text{for all} \quad k < m+1, \quad k \equiv m+1 \pmod 2.$$

Identifying β-planes and α-planes on $\mathcal{Q}^{2m+1}$, i.e., setting $Z^{\alpha} = U_{\beta'}^{\alpha} W^{\beta'}$, and using (2.28) again precisely yield that $[Z^{\alpha}] \in \mathbf{D}_{\Xi}$ by virtue of (2.17).

2.3 Correspondence space

We now formalise the correspondence between $\mathcal{Q}^n$ and $\mathbb{PT}$.

2.3.1 Odd dimensions

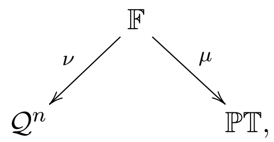
Assume $n = 2m+1$.

Definition 2.18. The *correspondence space* $\mathbb{F}$ of $\mathcal{Q}^n$ and $\mathbb{PT}$ is the projective complex subvariety of $\mathcal{Q}^n \times \mathbb{PT}$ defined as the set of points $([X^{\mathcal{A}}], [Z^{\alpha}])$ satisfying *the incidence relation*

$$X^{\mathcal{A}} Z_{\mathcal{A}}^{\beta} = 0, \tag{2.29}$$

where $Z_{\mathcal{A}}^{\beta} := \Gamma_{\mathcal{A}\alpha}{}^{\beta} Z^{\alpha}$.

The usual way of understanding the twistor correspondence is by means of the double fibration



where μ and ν denote the usual projections of maximal rank.

Clearly, since, by definition, a twistor $[Z^{\boldsymbol{\alpha}}]$ in $\mathbb{PT}$ corresponds to a γ-plane of $\mathcal{Q}^n$, namely the set of points $[X^{\mathcal{A}}]$ in $\mathcal{Q}^n$ satisfying (2.29), we see that each fiber of μ is isomorphic to $\mathbb{CP}^m$.

Now, a point x of $\mathcal{Q}^n$ is sent to a compact complex submanifold $\hat{x}$ of $\mathbb{PT}$ isomorphic to the fiber $\mathbb{F}_x$ of $\mathbb{F}$ over x, and similarly, a subset $\mathcal{U}$ of $\mathcal{Q}^n$ will correspond to a subset $\widehat{\mathcal{U}}$ of $\mathbb{PT}$ swept out by those complex submanifolds $\{\hat{x}\}$ parametrised by the points $x \in \mathcal{U}$, i.e.,

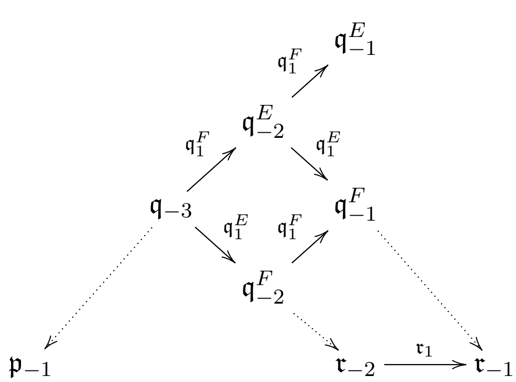
$$x \in \mathcal{Q}^n \mapsto \mathbb{F}_x := \nu^{-1}(x) \mapsto \hat{x} := \mu(\mathbb{F}_x), \qquad \mathcal{U} \subset \mathcal{Q}^n \mapsto \mathbb{F}_{\mathcal{U}} := \bigcup_{x\in\mathcal{U}} \nu^{-1}(x) \mapsto \widehat{\mathcal{U}} := \bigcup_{x\in\mathcal{U}} \mu(\mathbb{F}_x).$$

To describe $\hat{x}$, it is enough to describe the fiber $\mathbb{F}_x$. By definition, this is the set of all γ-planes incident on x. If $\check{Z}$ is a γ-plane incident on x, the intersection $\check{Z} \cap \mathrm{T}_x\mathcal{Q}^n$ is an m-dimensional subspace totally null with respect to the bilinear form on $\mathrm{T}_x\mathcal{Q}^n \cong \mathbb{CE}^n$, which we shall also refer to a γ-plane. This descends to a γ-plane in $\mathcal{Q}^{2m-1}$ viewed as the projectivisation of the null cone through x. Thus, $\hat{x} \cong \mathbb{F}_x$ is isomorphic to the $\frac{1}{2}m(m+1)$-dimensional twistor space $\mathbb{PT}_{(2m-1)}$ of $\mathcal{Q}^{2m-1}$.

We can get a little more information about $\mathbb{F}$ by viewing it as the homogeneous space G/Q where $Q := P \cap R$ is the intersection of P, the stabiliser of a null line in $\mathbb{V}$, and R the stabiliser of a totally null $(m+1)$-plane containing that line. The Lie algebra $\mathfrak{q}$ of Q induces a $|3|$-grading on $\mathfrak{g}$, i.e., $\mathfrak{g} = \mathfrak{q}_{-3} \oplus \mathfrak{q}_{-2} \oplus \mathfrak{q}_{-1} \oplus \mathfrak{q}_0 \oplus \mathfrak{q}_1 \oplus \mathfrak{q}_2 \oplus \mathfrak{q}_3$, where $\mathfrak{q} = \mathfrak{q}_0 \oplus \mathfrak{q}_1 \oplus \mathfrak{q}_2 \oplus \mathfrak{q}_3$. For convenience, we split $\mathfrak{q}_{\pm 1}$ and $\mathfrak{q}_{\pm 2}$ further as $\mathfrak{q}_{\pm 1} = \mathfrak{q}^E_{\pm 1} \oplus \mathfrak{q}^F_{\pm 1}$ and $\mathfrak{q}_{\pm 2} = \mathfrak{q}^E_{\pm 2} \oplus \mathfrak{q}^F_{\pm 2}$. Also, $\mathfrak{q}_0 \cong \mathfrak{gl}(m,\mathbb{C}) \oplus \mathbb{C}$, $\mathfrak{q}^E_{-1} \cong \mathbb{C}^m$, $\mathfrak{q}^F_{-1} \cong (\mathbb{C}^m)^*$, $\mathfrak{q}^E_{-2} \cong \mathbb{C}$, $\mathfrak{q}^F_{-2} \cong \wedge^2\mathbb{C}^m$ and $\mathfrak{q}_{-3} \cong (\mathbb{C}^m)^*$ with $(\mathfrak{q}_i)^* \cong \mathfrak{q}_{-i}$. In matrix form, $\mathfrak{g}$ reads as

$$\left(\begin{array}{c:c:c:c:c} \mathfrak{q}_0 & \mathfrak{q}^E_1 & \mathfrak{q}^E_2 & \mathfrak{q}_3 & 0 \\ \hdashline \mathfrak{q}^E_{-1} & \mathfrak{q}_0 & \mathfrak{q}^F_1 & \mathfrak{q}^F_2 & \mathfrak{q}_3 \\ \hdashline \mathfrak{q}^E_{-2} & \mathfrak{q}^F_{-1} & 0 & \mathfrak{q}^F_1 & \mathfrak{q}^E_2 \\ \hdashline \mathfrak{q}_{-3} & \mathfrak{q}^F_{-2} & \mathfrak{q}^F_{-1} & \mathfrak{q}_0 & \mathfrak{q}^E_1 \\ \hdashline 0 & \mathfrak{q}_{-3} & \mathfrak{q}^E_{-2} & \mathfrak{q}^E_{-1} & \mathfrak{q}_0 \end{array}\right) \begin{array}{l} \}_1 \\ \}_m \\ \}_1 \\ \}_m \\ \}_1 \end{array}$$

These modules satisfy the commutation relations $[\mathfrak{q}_i, \mathfrak{q}_j] \subset \mathfrak{q}_{i+j}$ where $\mathfrak{q}_i = \{0\}$ for $|i| > 3$. More precisely, the action of $\mathfrak{q}_1$ on these modules, carefully distinguishing $\mathfrak{q}^E_1$ and $\mathfrak{q}^F_1$, can be recorded in the form of a diagram:



where the dotted arrows give the relations between $\mathfrak{q}_0$-modules, and $\mathfrak{p}_0$- and $\mathfrak{r}_0$-modules. Invariance follows from the inclusions $\mathfrak{q}_1^E \subset \mathfrak{r}_0$, $\mathfrak{q}_1^F \subset \mathfrak{p}_0$, $\mathfrak{q}_1^E \subset \mathfrak{p}_1$ and $\mathfrak{q}_1^F \subset \mathfrak{r}_1$.

Beside the filtration of vector subbundles of $\mathrm{T}\mathbb{F}$ determined by the grading on $\mathfrak{g}$, we distinguish three Q-invariant distributions of interest on $\mathbb{F}$:

- the rank-$\frac{1}{2}m(m+1)$ distribution $\mathrm{T}^{-2}_F\mathbb{F}$ corresponding to $\mathfrak{q}^F_{-2} \oplus \mathfrak{q}^F_{-1}$. It is integrable and tangent to the fibers of $\nu : G/Q \to G/P$, each isomorphic to the homogeneous space P/Q. This follows from the relations $[\mathfrak{q}^F_{-1}, \mathfrak{q}^F_{-1}] \subset \mathfrak{q}^F_{-2}$, $[\mathfrak{q}^F_{-1}, \mathfrak{q}^F_{-2}] = 0$, and $[\mathfrak{q}^F_{-2}, \mathfrak{q}^F_{-2}] = 0$, and the fact that the kernel of the projection $\mathfrak{g}/\mathfrak{q} \to \mathfrak{g}/\mathfrak{p}$ is precisely $\mathfrak{q}^F_{-2} \oplus \mathfrak{q}^F_{-1} \cong \mathfrak{p}/\mathfrak{q}$. In fact, since $[\mathfrak{q}^F_{-1}, \mathfrak{q}^F_{-1}] \subset \mathfrak{q}^F_{-2}$, each fiber is itself equipped with the canonical distribution on $\mathbb{PT}_{(2m-1)}$.
- the rank-m distribution $\mathrm{T}^{-1}_E\mathbb{F}$ corresponding to $\mathfrak{q}^E_{-1}$. It is integrable and tangent to the fibers of $\mu : G/Q \to G/R$, each isomorphic to the homogeneous space R/Q. This follows from the relations $[\mathfrak{q}^E_{-1}, \mathfrak{q}^E_{-1}] = 0$ and the fact that the kernel of the projection $\mathfrak{g}/\mathfrak{q} \to \mathfrak{g}/\mathfrak{r}$ is precisely $\mathfrak{q}^E_{-1} \cong \mathfrak{r}/\mathfrak{q}$.
- the rank-$(2m+1)$ distribution $\mathrm{T}^{-2}_E\mathbb{F}$ corresponding to $\mathfrak{q}^E_{-2} \oplus \mathfrak{q}^F_{-1} \oplus \mathfrak{q}^E_{-1}$. It is non-integrable and bracket generates $\mathrm{T}\mathbb{F}$ since we have $[\mathfrak{q}^E_{-1}, \mathfrak{q}^F_{-1}] \subset \mathfrak{q}^E_{-2}$, $[\mathfrak{q}^E_{-1}, \mathfrak{q}^E_{-2}] = 0$, $[\mathfrak{q}^E_{-1}, \mathfrak{q}^F_{-2}] \subset \mathfrak{q}_{-3}$, $[\mathfrak{q}^F_{-1}, \mathfrak{q}^E_{-2}] \subset \mathfrak{q}_{-3}$. Further, the quotient $\mathrm{T}^{-2}_E\mathbb{F}/\mathrm{T}^{-1}_E\mathbb{F}$ descends to the canonical distribution $\mathrm{T}^{-1}\mathbb{PT}$.

The twistor space and correspondence space of $\mathbb{CE}^{2m+1}$. At this stage, we introduce a splitting (2.3) of $\mathbb{V}$, and as before denote by $\mathring{X}^{\mathcal{A}}$, $\mathring{Y}^{\mathcal{A}}$ and $\mathring{Z}^{\mathcal{A}}_a$ vectors in $\mathbb{V}_1$, $\mathbb{V}_{-1}$ and $\mathbb{V}_0$ respectively. There is an induced splitting (2.8) of $\mathbb{S}$, and we shall accordingly split the homogeneous twistor coordinates as $Z^{\boldsymbol{\alpha}} = (\omega^{\mathbf{A}}, \pi^{\mathbf{A}})$, or, using the injectors, as

$$Z^{\boldsymbol{\alpha}} = \mathring{I}^{\boldsymbol{\alpha}}_{\mathbf{A}}\omega^{\mathbf{A}} + \mathring{O}^{\boldsymbol{\alpha}}_{\mathbf{A}}\pi^{\mathbf{A}}. \tag{2.30}$$

Needless to say that Cartan's theory of spinors applies to $\mathbb{S}_{-\frac{1}{2}}$ and $\mathbb{S}_{\frac{1}{2}}$ in the obvious way and notation, as we have done in Section 2.2. In particular, a spinor $\pi^{\mathbf{A}}$ is pure if and only if the kernel of the map $\pi^{\mathbf{A}}_a := \pi^{\mathbf{B}}\gamma_{a\mathbf{B}}{}^{\mathbf{A}}$ is of maximal dimension m, and so on. The purity condition on $Z^{\boldsymbol{\alpha}}$ can then be re-expressed as follows.

Lemma 2.19. *Let $Z^{\boldsymbol{\alpha}} = (\omega^{\mathbf{A}}, \pi^{\mathbf{A}})$ be a non-zero spinor in $\mathbb{S} \cong \mathbb{S}_{-\frac{1}{2}} \oplus \mathbb{S}_{\frac{1}{2}}$. Then $Z^{\boldsymbol{\alpha}}$ is pure, i.e., satisfies (2.14), if and only if $\omega^{\mathbf{A}}$ and $\pi^{\mathbf{A}}$ satisfy*

$$\gamma^{(k)}_{\mathbf{AB}}\pi^{\mathbf{A}}\pi^{\mathbf{B}} = 0, \qquad \text{for all} \quad k < m, \quad k \equiv m+1, m \pmod 4, \tag{2.31a}$$
$$\gamma^{(k)}_{\mathbf{AB}}\omega^{\mathbf{A}}\omega^{\mathbf{B}} = 0, \qquad \text{for all} \quad k < m, \quad k \equiv m+1, m \pmod 4, \tag{2.31b}$$
$$\gamma^{(k)}_{\mathbf{AB}}\omega^{\mathbf{A}}\pi^{\mathbf{B}} = 0, \qquad \text{for all} \quad k < m-1. \tag{2.31c}$$

Proof. This is a direct computation using (2.15), (2.9) and (2.30). Writing $\pi^{\mathbf{A}}_a := \pi^{\mathbf{B}}\gamma_{a\mathbf{B}}{}^{\mathbf{A}}$ and $\omega^{\mathbf{A}}_a := \omega^{\mathbf{B}}\gamma_{a\mathbf{B}}{}^{\mathbf{A}}$, we find

$$\pi^{a\mathbf{A}}\pi^{\mathbf{B}}_a + \pi^{\mathbf{A}}\pi^{\mathbf{B}} = 0, \qquad \omega^{a\mathbf{A}}\omega^{\mathbf{B}}_a + \omega^{\mathbf{A}}\omega^{\mathbf{B}} = 0, \qquad \pi^{a\mathbf{A}}\omega^{\mathbf{B}}_a - \pi^{\mathbf{A}}\omega^{\mathbf{B}} + 2\omega^{\mathbf{A}}\pi^{\mathbf{B}} = 0,$$

which are equivalent to (2.31a), (2.31b) and (2.31c) respectively [35]. ■

By Cartan's theory of spinors, condition (2.31a) is equivalent to $\pi^{\mathbf{A}}$ being pure provided it is non-zero, and similarly for condition (2.31b) and $\omega^{\mathbf{A}}$. Condition (2.31c) is equivalent to the γ-planes of $\pi^{\mathbf{A}}$ and $\omega^{\mathbf{A}}$ intersecting in an m- or $(m-1)$-plane in $\mathbb{V}_0$ provided these are non-zero.

Remark 2.20. The annihilator (2.25) of the canonical distribution of $\mathbb{PT}$ can be re-expressed as

$$\begin{aligned}
\boldsymbol{\alpha}^{\mathbf{AB}}_{(\omega,\omega)} &:= \omega^{a\mathbf{A}}\mathrm{d}\omega_a^{\mathbf{B}} + 2\omega^{\mathbf{B}}\mathrm{d}\omega^{\mathbf{A}} - \omega^{\mathbf{A}}\mathrm{d}\omega^{\mathbf{B}}, \\
\boldsymbol{\alpha}^{\mathbf{AB}}_{(\pi,\pi)} &:= \pi^{a\mathbf{A}}\mathrm{d}\pi_a^{\mathbf{B}} + 2\pi^{\mathbf{B}}\mathrm{d}\pi^{\mathbf{A}} - \pi^{\mathbf{A}}\mathrm{d}\pi^{\mathbf{B}}, \\
\boldsymbol{\alpha}^{\mathbf{AB}}_{(\omega,\pi)} &:= \omega^{a\mathbf{A}}\mathrm{d}\pi_a^{\mathbf{B}} + \omega^{\mathbf{A}}\mathrm{d}\pi^{\mathbf{B}} + 4\pi^{[\mathbf{A}}\mathrm{d}\omega^{\mathbf{B}]}, \\
\boldsymbol{\alpha}^{\mathbf{AB}}_{(\pi,\omega)} &:= \pi^{a\mathbf{A}}\mathrm{d}\omega_a^{\mathbf{B}} + \pi^{\mathbf{A}}\mathrm{d}\omega^{\mathbf{B}} + 4\omega^{[\mathbf{A}}\mathrm{d}\pi^{\mathbf{B}]},
\end{aligned} \tag{2.32}$$

where we have used (2.30) and (2.9), and it is understood that $\omega^{\mathbf{A}}$ and $\pi^{\mathbf{A}}$ satisfy (2.31).

The twistor correspondence associates to the point ∞ in $\mathcal{Q}^n$, with coordinates $[\mathring{Y}^{\mathcal{A}}]$, a complex submanifold $\widehat{\infty}$ of $\mathbb{PT}$ defined by the locus $\mathring{Y}^{\mathcal{A}} Z^{\boldsymbol{\alpha}}_{\mathcal{A}} = 0$ in $\mathbb{PT}$, i.e.,

$$\infty \in \mathcal{Q}^n \mapsto \mathbb{F}_\infty := \nu^{-1}(\infty) \mapsto \widehat{\infty} := \mu(\mathbb{F}_\infty) = \mu \circ \nu^{-1}(\infty).$$

Points of $\widehat{\infty}$ are parametrised by $[\omega^{\mathbf{A}}, 0]$. Since removing ∞ from $\mathcal{Q}^n$ yields complex Euclidean space $\mathbb{CE}^n$, we accordingly remove $\widehat{\infty}$ to obtain the twistor space $\mathbb{PT}\backslash\{\widehat{\infty}\} = \mu\circ\nu^{-1}(\mathbb{CE}^n)$ of $\mathbb{CE}^n$. This will be denoted by $\mathbb{PT}_{\backslash\widehat{\infty}}$. This region of twistor space is parametrised by $\{[\omega^{\mathbf{A}}, \pi^{\mathbf{A}}]\colon \pi^{\mathbf{A}} \neq 0\}$.

The correspondence space of $\mathbb{CE}^n$ will be denoted $\mathbb{F}_{\mathbb{CE}^n}$, and is parametrised by the coordinates $(x^a, [\pi^{\mathbf{A}}])$, where $\{x^a\}$ are the flat standard coordinates on $\mathbb{CE}^n$ and $[\pi^{\mathbf{A}}]$ are homogeneous pure spinor coordinates on the fibers of $\mathbb{F}$. These parametrise the γ-planes of the tangent space $\mathrm{T}_x\mathbb{CE}^n$ at a point x in $\mathbb{CE}^n$, and are related to $[\omega^{\mathbf{A}}, \pi^{\mathbf{B}}]$ by means of the incidence relation (2.29)

$$\omega^{\mathbf{A}} = \tfrac{1}{\sqrt{2}} x^a \pi_a^{\mathbf{A}}, \tag{2.33}$$

which can be obtained from (2.7), (2.9) and (2.30). Indeed, the γ-plane defined by $[\pi^{\mathbf{A}}]$ through the origin is given by the locus $\frac{1}{\sqrt{2}} x^a \pi_a^{\mathbf{A}}$, so that the γ-plane defined by $[\pi^{\mathbf{A}}]$ through any other point $\mathring{x}^a$ is given by (2.33) with $\omega^{\mathbf{A}} = \frac{1}{\sqrt{2}}\mathring{x}^a \pi_a^{\mathbf{A}}$.

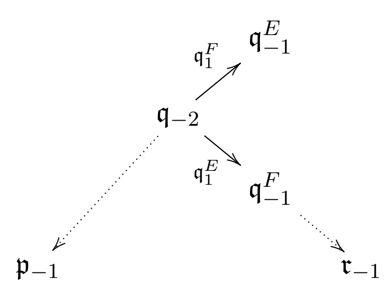
Remark 2.21. By (2.33) and (2.31a), for a holomorphic function f on $\mathbb{F}$ to descend to $\mathbb{PT}$, it must be annihilated by the differential operator $\pi^{[\mathbf{A}}\pi^{a\mathbf{B}]}\nabla_a$.

2.3.2 Even dimensions

The double fibration picture in dimension $n = 2m$ is very similar to the odd-dimensional case, and we only summarise the discussion here.

We realise $\mathbb{F}$ as a homogeneous space G/Q. Here, the Lie algebra $\mathfrak{q}$ of Q induces a $|2|$-grading $\mathfrak{g} = \mathfrak{q}_{-2} \oplus \mathfrak{q}_{-1} \oplus \mathfrak{q}_0 \oplus \mathfrak{q}_1 \oplus \mathfrak{q}_2$ on $\mathfrak{g}$, where $\mathfrak{q} = \mathfrak{q}_0 \oplus \mathfrak{q}_1 \oplus \mathfrak{q}_2$. We split $\mathfrak{q}_{\pm 1}$ further as $\mathfrak{q}_{\pm 1} = \mathfrak{q}^E_{\pm 1} \oplus \mathfrak{q}^F_{\pm 1}$, and we have $\mathfrak{q}_0 \cong \mathfrak{gl}(m, \mathbb{C}) \oplus \mathbb{C}$, $\mathfrak{q}^E_{-1} \cong \mathbb{C}^m$, $\mathfrak{q}^F_{-1} \cong \wedge^2\mathbb{C}^m$ and $\mathfrak{q}_{-2} \cong (\mathbb{C}^m)^*$ with $(\mathfrak{q}_i)^* \cong \mathfrak{q}_{-i}$. The action of $\mathfrak{q}_1$ on these $\mathfrak{q}_0$-modules is recorded below together with the matrix form of the splitting:

$$\begin{pmatrix} \mathfrak{q}_0 & \mathfrak{q}_1^E & \mathfrak{q}_2 & 0 \\ \mathfrak{q}_{-1}^E & \mathfrak{q}_0 & \mathfrak{q}_1^F & \mathfrak{q}_2 \\ \mathfrak{q}_{-2} & \mathfrak{q}_{-1}^F & \mathfrak{q}_0 & \mathfrak{q}_1^E \\ 0 & \mathfrak{q}_{-2} & \mathfrak{q}_{-1}^E & \mathfrak{q}_0 \end{pmatrix} \begin{matrix} \}_1 \\ \}_m \\ \}_m \\ \}_1 \end{matrix}$$



The modules $\mathfrak{q}^F_{-1}$ and $\mathfrak{q}^E_{-1}$ give rise to two integrable Q-invariant distributions $\mathrm{T}^{-1}_F\mathbb{F}$ and $\mathrm{T}^{-1}_E\mathbb{F}$ on $\mathbb{F}$ of rank $\frac{1}{2}m(m-1)$ and m respectively, and tangent to the fibers of $G/Q \to G/P$ and $G/Q \to G/R$ respectively.

The twistor space and correspondence space of $\mathbb{CE}^{2m}$. The even-dimensional analogue of Lemma 2.19 is recorded below.

Lemma 2.22. *Let $Z^{\boldsymbol{\alpha}} = (\omega^{\mathbf{A}}, \pi^{\mathbf{A}'})$ be a spinor in $\mathbb{S} \cong \mathbb{S}_{-\frac{1}{2}} \oplus \mathbb{S}'_{\frac{1}{2}}$. Then $Z^{\boldsymbol{\alpha}}$ is pure if and only if $\omega^{\mathbf{A}}$ and $\pi^{\mathbf{A}'}$ satisfy*

$$\gamma^{(k)}_{\mathbf{A}'\mathbf{B}'}\pi^{\mathbf{A}'}\pi^{\mathbf{B}'} = 0, \qquad \textit{for all} \quad k < m, \quad k \equiv m \pmod 4, \tag{2.34a}$$
$$\gamma^{(k)}_{\mathbf{A}\mathbf{B}}\omega^{\mathbf{A}}\omega^{\mathbf{B}} = 0, \qquad \textit{for all} \quad k < m, \quad k \equiv m \pmod 4, \tag{2.34b}$$
$$\gamma^{(k)}_{\mathbf{A}\mathbf{B}'}\omega^{\mathbf{A}}\pi^{\mathbf{B}'} = 0, \qquad \textit{for all} \quad k < m-1, \quad k \equiv m-1 \pmod 2. \tag{2.34c}$$

Conditions (2.34a), (2.34b) and (2.34c) can equivalently be expressed as

$$\pi^{a\mathbf{A}}\pi_a^{\mathbf{B}} = 0, \qquad \omega^{a\mathbf{A}'}\omega_a^{\mathbf{B}'} = 0, \qquad \pi^{a\mathbf{A}}\omega_a^{\mathbf{B}'} + 2\omega^{\mathbf{A}}\pi^{\mathbf{B}'} = 0,$$

respectively. By Cartan's theory of spinors, condition (2.34a) is equivalent to $\pi^{\mathbf{A}'}$ being pure provided it is non-zero, and similarly for conditions (2.34b) for $\omega^{\mathbf{A}}$. Condition (2.34c) is equivalent to the α-plane of $\pi^{\mathbf{A}'}$ and the β-plane of $\omega^{\mathbf{A}}$ intersecting in an $(m-1)$-plane in $\mathbb{V}_0$ provided these are non-zero.

Just as in the odd-dimensional case, the twistor space of $\mathbb{CE}^{2m}$ is obtained by removing the $\frac{1}{2}m(m-1)$-dimensional complex submanifold $\widehat{\infty}$ corresponding to ∞ on $\mathcal{Q}^{2m}$ from $\mathbb{PT}$. We can use $[\pi^{\mathbf{A}'}]$ as homogeneous coordinates on the fibers of $\mathbb{F}_{\mathbb{CE}^n}$, and the incidence relation (2.29) can be expressed as $\omega^{\boldsymbol{A}} = \frac{1}{\sqrt{2}}x^a\pi_a^{\mathbf{A}}$.

2.4 Co-γ-planes and mini-twistor space

In odd dimensions, there is an additional geometric object of interest.

Definition 2.23. A *co-γ-plane* is an $(m+1)$-dimensional affine subspace of $\mathbb{CE}^{2m+1}$ with the property that the orthogonal complement of its tangent space at any of its point is totally null with respect to the metric.

The space of all co-γ-planes in $\mathbb{CE}^{2m+1}$ is called the *mini-twistor space* of $\mathbb{CE}^{2m+1}$, and is denoted $\mathbb{MT}$.

Viewed as a vector subspace of $\mathrm{T}_x\mathbb{CE}^n \cong \mathbb{CE}^n$, a co-$\gamma$-plane through a point x in $\mathbb{CE}^n$ is the orthogonal complement of a γ-plane through x. Consider a co-γ-plane through the origin, and let $[\pi^{\mathbf{A}}]$ be a projective pure spinor associated to the γ-plane orthogonal to it. Then, it is easy to check that this co-γ-plane consists of the set of points x^a satisfying $t\pi^{\mathbf{A}} = \frac{1}{\sqrt{2}}x^a\pi_a^{\mathbf{A}}$ where $t \in \mathbb{C}$ with $x^a x_a = -2t^2$. Shifting the origin to $\mathring{x}^a$ say, a point in a co-γ-plane containing $\mathring{x}^a$ now satisfies $\omega^{\mathbf{A}} + \pi^{\mathbf{A}}t = \frac{1}{\sqrt{2}}x^a\pi_a^{\mathbf{A}}$ for some $t \in \mathbb{C}$, and where $\omega^{\mathbf{A}} := \frac{1}{\sqrt{2}}\mathring{x}^a\pi_a^{\mathbf{A}}$. Thus, a co-$\gamma$-plane through $\mathring{x}^a$ consists of the set of points satisfying the incidence relation

$$\omega^{[\mathbf{A}}\pi^{\mathbf{B}]} = \tfrac{1}{\sqrt{2}}x^a\pi_a^{[\mathbf{A}}\pi^{\mathbf{B}]}, \tag{2.35}$$

where $[\pi^{\mathbf{C}}]$ is a projective pure spinor and $\omega^{\mathbf{A}} := \frac{1}{\sqrt{2}}\mathring{x}^a\pi_a^{\mathbf{A}}$. In particular, a co-$\gamma$-plane consists of a 1-parameter family of γ-planes, and thus corresponds to the curve

$$\mathbb{C} \ni t \mapsto \left[\omega^{\mathbf{A}} + \pi^{\mathbf{A}}t, \pi^{\mathbf{A}}\right] \in \mathbb{PT}_{\backslash\widehat{\infty}}. \tag{2.36}$$

The relation between $\mathbb{MT}$ and $\mathbb{PT}_{\setminus\widehat{\infty}}$ can be made precise by involving our choice of 'infinity' $[\mathring{Y}^{\mathcal{A}}]$ to define $\mathbb{CE}^n$. Let us write $(\mathring{Y} \cdot Z)^{\boldsymbol{\alpha}} := \mathring{Y}^{\mathcal{A}} \Gamma_{\mathcal{A}\boldsymbol{\beta}}{}^{\boldsymbol{\alpha}} Z^{\boldsymbol{\beta}}$. We can then define the vector field

$$\boldsymbol{Y} := -\frac{\mathrm{i}}{2} \big(\mathring{Y} \cdot Z\big)^{\boldsymbol{\alpha}} \frac{\partial}{\partial Z^{\boldsymbol{\alpha}}} = \frac{\mathrm{i}}{\sqrt{2}} \pi^{\mathbf{A}} \frac{\partial}{\partial \omega^{\mathbf{A}}}, \tag{2.37}$$

on $\mathbb{PT}_{\setminus\widehat{\infty}}$, the factors having been added for later convenience. It is now pretty clear that the curve (2.36) is an integral curve of the vector field (2.37) passing through the point $[\omega^{\mathbf{A}}, \pi^{\mathbf{A}}]$. We therefore conclude

Lemma 2.24. *Mini-twistor space $\mathbb{MT}$ is the quotient of $\mathbb{PT}_{\setminus\widehat{\infty}}$ by the flow of $\boldsymbol{Y}$ defined by* (2.37).

An alternative geometric interpretation can be obtained by introducing weighted homogeneous coordinates on $\mathbb{MT}$ as follows. Since $\pi^{\mathbf{A}}$ is pure, we can view $[\pi^{\mathbf{A}}]$ as homogeneous coordinates on $\mathbb{PT}_{(2m-1)}$. Let $\underline{\omega}_{a_1 \dots a_{m-1}}$ be an $(m-1)$-form satisfying

$$\pi^{a_1 \mathbf{A}} \underline{\omega}_{a_1 a_2 \dots a_{m-1}} = 0, \qquad m > 1. \tag{2.38}$$

Write $[\underline{\omega}_{a_1 \dots a_{m-1}}, \pi^{\mathbf{A}}]_{2,1}$ for the equivalence class of pairs $(\underline{\omega}_{a_1 \dots a_{m-1}}, \pi^{\mathbf{A}})$ defined by the relation

$$\big(\underline{\omega}_{a_1 \dots a_{m-1}}, \pi^{\mathbf{A}}\big) \sim \big(\lambda^2 \underline{\omega}_{a_1 \dots a_{m-1}}, \lambda \pi^{\mathbf{A}}\big) \qquad \text{for some} \quad \lambda \in \mathbb{C}^*.$$

Then $[\underline{\omega}_{a_1 \dots a_{m-1}}, \pi^{\mathbf{A}}]_{2,1}$ constitute weighted homogeneous coordinates on $\mathbb{MT}$. To see this, we note that for any choice of representative, the condition (2.38) is equivalent to

$$\underline{\omega}_{a_1 \dots a_{m-1}} = \gamma^{(m-1)}_{a_1 \dots a_{m-1} \mathbf{AB}} \pi^{\mathbf{A}} \omega^{\mathbf{B}} \tag{2.39}$$

for some pure spinor $\omega^{\mathbf{A}}$ satisfying (2.31b). Then, the projection of any $[\omega^{\mathbf{A}}, \pi^{\mathbf{A}}]$ in $\mathbb{PT}_{\setminus\widehat{\infty}}$ to $[\underline{\omega}_{a_1 \dots a_{m-1}}, \pi^{\mathbf{A}}]_{2,1}$ is independent of the choice of representative of $[\omega^{\mathbf{A}}, \pi^{\mathbf{A}}]$, and further, since $\pi^{\mathbf{A}}$ is pure, i.e., satisfies (2.31a), sending $\omega^{\mathbf{A}}$ to $\omega^{\mathbf{A}} + t\pi^{\mathbf{A}}$ for any $t \in \mathbb{C}$ leaves (2.39) unchanged.

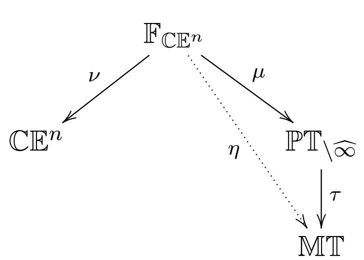
With these coordinates, we can rewrite the incidence relation (2.35) as

$$\underline{\omega}_{a_1 \dots a_{m-1}} = \tfrac{1}{\sqrt{2}} x^a \gamma^{(m)}_{a a_1 \dots a_{m-1} \mathbf{AB}} \pi^{\mathbf{A}} \pi^{\mathbf{B}}. \tag{2.40}$$

Now, turning to the geometrical interpretation, we fix a point π in $\mathbb{PT}_{(2m-1)}$ with a choice of pure spinor $\pi^{\mathbf{A}}$. Since $\mathrm{T}^{-1}_{\pi} \mathbb{PT}_{(2m-1)}$ is a dense open subset of an m-dimensional linear subspace of $\mathbb{PT}_{(2m-1)}$ containing π, we can identify a vector in $\mathrm{T}^{-1}_{\pi} \mathbb{PT}_{(2m-1)}$ with a point in this subspace, which can be represented by a pure spinor $\omega^{\mathbf{A}}$ satisfying (2.31b). At this stage, this identification is valid provided the scale of $\pi^{\mathbf{A}}$ is *fixed*. Clearly the origin in $\mathrm{T}^{-1}_{\pi} \mathbb{PT}_{(2m-1)}$ is $\pi^{\mathbf{A}}$ itself, so that $(\omega^{\mathbf{A}}, \pi^{\mathbf{A}})$ maps injectively to $(\underline{\omega}_{a_1 a_2 \dots a_{m-1}}, \pi^{\mathbf{A}})$. That this map is also surjective follows immediately from (2.39). Hence, we can conclude

Proposition 2.25. *The mini-twistor space $\mathbb{MT}$ of $\mathbb{CE}^{2m+1}$ is a $\frac{1}{2}m(m+3)$-dimensional complex manifold isomorphic to the total space of the canonical rank-m distribution $\mathrm{T}^{-1}\mathbb{PT}_{(2m-1)}$ of the twistor space $\mathbb{PT}_{(2m-1)}$ of $\mathcal{Q}^{2m-1}$.*

For clarity, we represent $\mathbb{MT}$ by means of an extended double fibration



where μ, ν, τ and η are the usual projections. We shall introduce the following notation for submanifolds of $\mathbb{MT}$ corresponding to points in $\mathbb{CE}^n$:

$$x \in \mathbb{CE}^n \mapsto \mathbb{F}_x := \nu^{-1}(x) \mapsto \underline{\hat{x}} := \tau(\hat{x}) = \eta(\mathbb{F}_x),$$
$$\mathcal{U} \subset \mathbb{CE}^n \mapsto \mathbb{F}_{\mathcal{U}} := \bigcup_{x\in\mathcal{U}} \nu^{-1}(x) \mapsto \underline{\widehat{\mathcal{U}}} := \tau(\widehat{\mathcal{U}}) = \eta(\mathbb{F}_{\mathcal{U}}).$$

Remark 2.26. For a holomorphic function on $\mathbb{F}$ to descend to $\mathbb{MT}$, it must be annihilated by the differential operator $\pi^{a\mathbf{A}}\nabla_a$.

2.5 Normal bundles

It will also be convenient to think of the correspondence space as an analytic family $\{\hat{x}\}$ of compact complex submanifolds of twistor space parametrised by the points x of $\mathcal{Q}^n$. The way each $\hat{x}$ is embedded in $\mathbb{PT}$ is described by its (holomorphic) *normal bundle* $\mathrm{N}\hat{x}$ in $\mathbb{PT}$, which is the rank-$(m+1)$ vector bundle defined by the short exact sequence

$$0 \to \mathrm{T}\hat{x} \to \mathrm{T}\,\mathbb{PT}|_{\hat{x}} \to \mathrm{N}\hat{x} \to 0.$$

As we shall see there are some crucial difference between the odd- and even-dimensional cases.

2.5.1 Odd dimensions

Assume $n = 2m+1$. We first note that the canonical distribution D on $\mathbb{PT}$ defines a subbundle $\mathrm{D}|_{\hat{x}} + \mathrm{T}\hat{x}$ of $\mathrm{T}\mathbb{PT}|_{\hat{x}}$ containing $\mathrm{T}\hat{x}$. How much of this subbundle descends to $\mathrm{N}\hat{x}$ is answered by the following lemma.

Lemma 2.27. *Let x be a point in $\mathcal{Q}^{2m+1}$. Then, for any $Z \in \hat{x} \subset \mathbb{PT}$, the intersection of D_Z and $\mathrm{T}_Z\hat{x}$ has dimension m. In particular, $\hat{x}$ is equipped with a maximally non-integrable rank-m distribution $\mathrm{T}^{-1}\hat{x} := \mathrm{D}|_{\hat{x}} \cap \mathrm{T}\hat{x}$. Further, there is a distinguished line subbundle of the normal bundle $\mathrm{N}\hat{x}$ of $\hat{x}$ given by $\mathrm{N}^{-1}\hat{x} := (\mathrm{D}|_{\hat{x}} + \mathrm{T}\hat{x})/\mathrm{T}\hat{x}$.*

Proof. Denote by $[X^{\mathcal{A}}]$ the homogeneous coordinates of $x \in \mathcal{Q}^{2m+1}$, and let $\Xi \in \hat{x} \subset \mathbb{PT}$ so that $X^{\mathcal{A}}\Xi_{\mathcal{A}}^{\boldsymbol{\alpha}} = 0$. Then, by Lemma 2.7 and Proposition 2.8, a vector tangent to D_Ξ can be identified with a point $Z^{\boldsymbol{\alpha}} = \Xi^{\boldsymbol{\alpha}} + \frac{\mathrm{i}}{2}(Z_{(-1)} \cdot \Xi)^{\boldsymbol{\alpha}}$ of a dense open subset of $\mathbf{D}_\Xi \subset \mathbb{PT}$. Here, $Z_{(-1)} \in \mathfrak{r}_{-1} \cong \mathbb{C}^{m+1}$ lies in a complement of the stabiliser $\mathfrak{r}$ of Ξ as explained in Section 2.2. The condition that this vector is also tangent to $\hat{x}$ is equivalent to $0 = X^{\mathcal{A}}Z_{\mathcal{A}}^{\boldsymbol{\alpha}} = -2X_{\mathcal{A}}Z_{(-1)}^{\mathcal{A}}\Xi^{\boldsymbol{\alpha}}$, by (2.1), i.e., $X_{\mathcal{A}}Z_{(-1)}^{\mathcal{A}} = 0$. This gives a single additional algebraic condition on $Z_{(-1)}^{\mathcal{A}}$, and thus the intersection of D_Ξ and $\mathrm{T}_\Xi\hat{x}$ is m-dimensional (for a description in affine coordinates, see the end of Appendix A.1). This defines a rank-m distribution $\mathrm{T}^{-1}\hat{x} := \mathrm{D}|_{\hat{x}} \cap \mathrm{T}\hat{x}$ on $\hat{x}$. Since D is maximally non-integrable, so must be $\mathrm{T}^{-1}\hat{x}$. That the subbundle $\mathrm{N}^{-1}\hat{x} := (\mathrm{D}|_{\hat{x}} + \mathrm{T}\hat{x})/\mathrm{T}\hat{x}$ of $\mathrm{N}\hat{x}$ is of rank 1 follows from the isomorphism $\mathrm{D}|_{\hat{x}}/(\mathrm{D}|_{\hat{x}} \cap \mathrm{T}\hat{x}) \cong (\mathrm{D}|_{\hat{x}} + \mathrm{T}\hat{x})/\mathrm{T}\hat{x}$. ■

That $\hat{x}$ is endowed with a canonical rank-m distribution comes as no surprise since each $\hat{x}$ is isomorphic to the generalised flag manifold $P/Q \cong \mathbb{PT}_{(2m-1)}$.

As explained in [24], the tangent space at a point x of $\mathcal{Q}^{2m+1}$ injects into $H^0(\hat{x}, \mathcal{O}(\mathrm{N}\hat{x}))$, the space of global holomorphic sections of $\mathrm{N}\hat{x}$. If V^a is a vector in $\mathrm{T}_x\mathcal{Q}^{2m+1}$ and y the point infinitesimally separated from x by V^a, then the corresponding section of $H^0(\hat{x}, \mathcal{O}(\mathrm{N}\hat{x}))$ can be identified with $\hat{y}$. Let us fix x to be the origin in $\mathbb{CE}^{2m+1} \subset \mathcal{Q}^{2m+1}$. Then V^a can be identified with y^a. We view $\pi^{\mathbf{A}}$ as coordinates on $\hat{x}$ given by the locus $\omega^{\mathbf{A}} = 0$. The infinitesimal displacement of $\hat{x}$ along V^a at the origin is $V^{\mathbf{A}} := V^a\nabla_a\omega^{\mathbf{A}}$, i.e., $V^{\mathbf{A}} = \frac{1}{\sqrt{2}}V^a\pi_a^{\mathbf{A}}$. This represents a global holomorphic section $\widehat{V}_{\hat{x}}$ of $\mathrm{N}\hat{x}$, and can be identified with the complex submanifold $\hat{y}$ given by $\omega^{\mathbf{A}} = \frac{1}{\sqrt{2}}y^a\pi_a^{\mathbf{A}}$.

Before describing such sections, we shall need the following two lemmata.

Lemma 2.28. *Let V^a be a non-zero vector in $\mathbb{CE}^{2m+1}$, and let $V_{\mathbf{A}}^{\mathbf{B}} := V^a \gamma_{a\mathbf{A}}{}^{\mathbf{B}}$ be the corresponding spin endomorphism. Then V^a is null if and only if $V_{\mathbf{A}}^{\mathbf{B}}$ has a zero eigenvalue. Further,*

- *if V^a is null, $V_{\mathbf{A}}^{\mathbf{B}}$ has a single zero eigenvalue of algebraic multiplicity 2^m, and its eigenspace is isomorphic to the 2^{m-1}-dimensional spinor space of $\mathbb{CE}^{2m-1}$,*
- *if V^a is non-null, $V_{\mathbf{A}}^{\mathbf{B}}$ has a pair of eigenvalues $\pm \mathrm{i}\sqrt{V^a V_a}$, each of algebraic multiplicity 2^{m-1}, and their respective eigenspaces are isomorphic to the 2^{m-1}-dimensional chiral spinor spaces of $\mathbb{CE}^{2m}$.*

Proof. By the Clifford property, we have $V_{\mathbf{A}}^{\mathbf{C}} V_{\mathbf{C}}^{\mathbf{B}} = -V^a V_a \delta_{\mathbf{A}}^{\mathbf{B}}$, and it follows that any eigenvalue of $V_{\mathbf{A}}^{\mathbf{B}}$ must be equal to $\pm \mathrm{i}\sqrt{V^a V_a}$. Hence V^a is null if and only if it has a zero eigenvalue. This zero eigenvalue must be of algebraic multiplicity 2^m since in this case $V_{\mathbf{A}}^{\mathbf{B}}$ is nilpotent. One can check that the kernel of $V_{\mathbf{A}}^{\mathbf{B}}$ can be identified with the 2^{m-1}-dimensional spinor space of $\mathbb{CE}^{2m-1}$ as the orthogonal complement of V^a in $\mathbb{CE}^{2m+1}$ quotiented by $\langle V^a \rangle$.

If V^a is non-null, the square of $V_{\mathbf{A}}^{\mathbf{B}}$ is proportional to the identity, and thus, each of the eigenvalues $\pm \mathrm{i}\sqrt{V^a V_a}$ must have algebraic multiplicity 2^{m-1}. Each of the eigenspaces can be identified with each of the chiral spinor spaces of $\mathbb{CE}^{2m}$ as the orthogonal complement of $\langle V^a \rangle$ in $\mathbb{CE}^{2m+1}$ – see, e.g., [31]. ■

Lemma 2.29. *Let x and y be two points in $\mathcal{Q}^{2m+1}$ infinitesimally separated by a non-null vector V^a. Then, for every $Z \in \hat{x} \subset \mathbb{PT}$ such that V^a is tangent to the co-γ-plane $\check{Z}^{\perp} \subset \mathrm{T}_x \mathcal{Q}^{2m+1}$, D_Z intersects $\hat{y}$ in a unique point W, say, such that the corresponding γ-planes $\check{Z}$ and $\check{W}$ intersect maximally.*

Proof. With no loss of generality, we may assume that x is the origin in $\mathbb{CE}^{2m+1} \subset \mathcal{Q}^{2m+1}$. We then have $V^a = y^a$. Since V^a is non-null, it must lie on some co-γ-plane of some twistor Z. Following the discussion of Section 2.4, it can be represented by a 1-parameter family of γ-planes. In particular, y must lie on one such γ-plane. If Z is a point on $\hat{x}$, then $[Z^{\boldsymbol{\alpha}}] = [0, \pi^{\mathbf{A}}]$ for some $\pi^{\mathbf{A}}$. The condition that y lies on the co-γ-plane $Z^{\perp}$ is that $\pi^{\mathbf{A}}$ is an eigenspinor of V^a with eigenvalue t or $-t$ where $t := \mathrm{i}\sqrt{V^a V_a}$. For definiteness, let us assume that the eigenvalue is t. With reference to (2.36), the point y^a lies in the γ-plane W given by $[W^{\boldsymbol{\alpha}}] = [t\pi^{\mathbf{A}}, \pi^{\mathbf{A}}]$. Re-expressing this twistor as $W^{\boldsymbol{\alpha}} = \Xi^{\boldsymbol{\alpha}} - \frac{t}{2} \mathring{Y}^{\mathcal{A}} \Gamma_{\mathcal{A}\boldsymbol{\beta}}{}^{\boldsymbol{\alpha}} \Xi^{\boldsymbol{\alpha}}$, we see, by Lemma 2.7, that W lies in the intersection of D_Z and $\hat{y}$. In fact, one can see that the connecting vector from Z to W is given by $\sqrt{V^a V_a}\boldsymbol{Y}$, where $\boldsymbol{Y}$ is given by (2.37). Finally, by Proposition 2.12, Z and W must intersect maximally. ■

Proposition 2.30. *Let x be a point in $\mathcal{Q}^{2m+1}$ with corresponding submanifold $\hat{x}$ in $\mathbb{PT}$. Let V be a tangent vector at x, and $\widehat{V}_{\hat{x}}$ its corresponding global holomorphic section of $\mathrm{N}\hat{x}$.*

- *Suppose V is null. When $m = 1$, $\widehat{V}_{\hat{x}}$ vanishes at a single point on $\hat{x}$, which corresponds to the unique γ-plane (i.e., null line) to which V is tangent. When $m > 1$, there is a $\frac{1}{2}m(m-1)$-dimensional algebraic subset of $\hat{x}$ biholomorphic to $\mathbb{PT}_{(2m-3)}$ on which $\widehat{V}_{\hat{x}}$ vanishes. Each point of this subset corresponds to a γ-plane to which V is tangent.*
- *Suppose V is non-null. When $m = 1$, there are precisely two points, $Z_{\pm}$ say, on $\hat{x}$, at which $\widehat{V}_{\hat{x}}(Z_{\pm}) \in \mathrm{N}_{Z_{\pm}}^{-1}\hat{x}$. Further, V is tangent to the two co-γ-planes determined by $Z_{\pm}$. When $m > 1$, there are two disjoint $\frac{1}{2}m(m-1)$-dimensional algebraic subsets of $\hat{x}$, biholomorphic to $\mathbb{PT}_{(2m-2)}$ and $\mathbb{PT}'_{(2m-2)}$, over which $\widehat{V}_{\hat{x}}$ is a section of $\mathrm{N}^{-1}\hat{x}$. Each point of these subsets corresponds to a co-γ-plane to which V is tangent.*

Conversely, if $\widehat{V}_{\hat{x}}$ vanishes at a point, then V must be null, and if $\widehat{V}_{\hat{x}}(Z) \in \mathrm{N}_Z^{-1}\hat{x}$ for some $Z \in \hat{x}$, then V must be non-null.

Proof. Again, let us assume that x is the origin in $\mathbb{CE}^{2m+1} \subset \mathcal{Q}^{2m+1}$, and set $V_{\mathbf{A}}^{\mathbf{B}} := V^a \gamma_{a\mathbf{A}}{}^{\mathbf{B}}$.

If V^a is null, the vanishing of $\widehat{V}_{\hat{x}}$ of a point $\pi^{\mathbf{A}}$ of $\hat{x}$ is simply equivalent to $V_{\mathbf{A}}^{\mathbf{B}} \pi^{\mathbf{A}} = 0$, i.e., $\pi^{\mathbf{A}}$ is a pure eigenspinor of $V_{\mathbf{A}}^{\mathbf{B}}$. By Lemma 2.28, we can immediately conclude that $\widehat{V}_{\hat{x}}$ vanishes at a point when $m = 1$, and on a subset of $\hat{x}$ biholomorphic to $\mathbb{PT}_{(2m-3)}$ when $m > 1$. Clearly, each point of this subset corresponds to a γ-plane to which V^a is tangent.

If V^a is non-null, we know by Lemma 2.28 that $V_{\mathbf{A}}^{\mathbf{B}}$ has eigenvalues $\pm \mathrm{i}\sqrt{V^a V_a}$. In particular, the pure eigenspinors up to scale determine two distinct points on $\hat{x}$ when $m = 1$, and two disjoint subsets of $\hat{x}$ biholomorphic to the twistor spaces $\mathbb{PT}_{(2m-2)}$ and $\mathbb{PT}'_{(2m-2)}$ when $m > 1$. A point Z on any of these sets corresponds to a co-γ-plane $\tilde{Z}^{\perp}$ to which V^a is tangent. By Lemma 2.29, the corresponding submanifold $\hat{y}$ intersects D_Z at a point W. The connecting vector from Z to W clearly lies in D_Z, but is not tangent to $\hat{x}$. In particular, it descends to an element of $\mathrm{N}_Z^{-1}\hat{x}$. Thus, the restriction of $\widehat{V}_{\hat{x}}$ to these subsets is a section of $\mathrm{N}^{-1}\hat{x}$.

Finally, if $\widehat{V}_{\hat{x}}$ vanishes at a point Z say, then V is tangent to the γ-plane $\check{Z}$, and so must be null. The non-null case is similar. ■

2.5.2 Mini-twistor space

For any point x of $\mathcal{Q}^n$, the normal bundle $\mathrm{N}\underline{\hat{x}}$ of $\underline{\hat{x}}$ in $\mathbb{MT}$ is given by $0 \to \mathrm{T}\underline{\hat{x}} \to \mathrm{T}\,\mathbb{MT}|_{\underline{\hat{x}}} \to \mathrm{N}\underline{\hat{x}} \to 0$. In this case, $\mathrm{N}\underline{\hat{x}}$ can be identified with $\mathrm{T}^{-1}\hat{x}$, i.e., mini-twistor space itself, as follows from the description of Section 2.4: taking x in $\mathbb{CE}^n$ to be the origin, then the complex submanifold $\underline{\hat{x}}$ in $\mathbb{MT}$ is defined by $\underline{\omega}_{a_1 \dots a_{m-1}} = 0$, $\pi^{\mathbf{A}}$ will be coordinates on $\underline{\hat{x}}$, and we shall view $\underline{\omega}_{a_1\dots a_{m-1}}$ as coordinates off $\underline{\hat{x}}$.

Again, for any $x \in \mathbb{CE}^n$, $\mathrm{T}_x\mathbb{CE}^n$ injects into $H^0(\underline{\hat{x}}, \mathcal{O}(\mathrm{N}\underline{\hat{x}}))$. If x is the origin and $V \in \mathrm{T}_x\mathbb{CE}^n$ be the vector connecting x to a point y, we can identify the global holomorphic section $\widehat{V}_{\underline{\hat{x}}}$ of $\mathrm{N}\underline{\hat{x}}$ as in the previous section. If $\pi^{\mathbf{A}}$ are coordinates on $\hat{x}$ given by the locus $\omega^{\mathbf{A}} = 0$, $\widehat{V}_{\underline{\hat{x}}}$ can be identified with the complex submanifold $\underline{\hat{y}}$ given by $\underline{\omega}_{a_1\dots a_{m-1}} = \frac{1}{\sqrt{2}} y^a \gamma^{(m)}_{a a_1 \dots a_{m-1}\mathbf{AB}} \pi^{\mathbf{A}} \pi^{\mathbf{B}}$, where $y^a = V^a$.

Proposition 2.31. *Let x be a point in $\mathbb{CE}^{2m+1}$ with corresponding submanifold $\underline{\hat{x}}$ in $\mathbb{MT}$. Let V be a tangent vector at x, and $\widehat{V}_{\underline{\hat{x}}}$ its corresponding global holomorphic section of $\mathrm{N}\underline{\hat{x}}$.*

- *Suppose V is null. When $m = 1$, $\widehat{V}_{\underline{\hat{x}}}$ has a double zero, which corresponds to the γ-plane to which V is tangent. When $m > 1$, $\widehat{V}_{\underline{\hat{x}}}$ vanishes on a $\frac{1}{2}m(m-1)$-dimensional algebraic subset of $\underline{\hat{x}}$ biholomorphic to $\mathbb{PT}_{(2m-1)}$ of multiplicity 2^m. Each point of this subset corresponds to a γ-plane to which V is tangent.*
- *Suppose V is non-null. When $m = 1$, $\widehat{V}_{\underline{\hat{x}}}$ has two simple zeros, each of which determines a co-γ-plane to which V is tangent. When $m > 1$, $\widehat{V}_{\underline{\hat{x}}}$ vanishes on two disjoint $\frac{1}{2}m(m-1)$-dimensional algebraic subsets of $\underline{\hat{x}}$ bihomolomorphic to $\mathbb{PT}_{(2m-2)}$ and $\mathbb{PT}'_{(2m-2)}$, each of multiplicity 2^{m-1}. Each point of these subsets corresponds to a co-γ-plane to which V is tangent.*

Proof. With no loss, we assume that x is the origin in $\mathbb{CE}^{2m+1} \subset \mathcal{Q}^{2m+1}$, and set $V_{\mathbf{A}}^{\mathbf{B}} := V^a \gamma_{a\mathbf{A}}{}^{\mathbf{B}}$. To determine the zero set of $\underline{\hat{V}}_{\underline{\hat{x}}}$, we simply remark that $V^a \gamma^{(m)}_{a a_1 \dots a_{m-1}\mathbf{AB}} \pi^{\mathbf{A}} \pi^{\mathbf{B}} = 0$ is equivalent to the eigenspinor equation $\pi^{\mathbf{C}} V_{\mathbf{C}}^{[\mathbf{A}} \pi^{\mathbf{B}]} = 0$. We can then proceed as in the proof of Proposition 2.30 according to whether V^a is null or non-null, and obtain the required zero sets of the section $\widehat{V}_{\underline{\hat{x}}}$ in each case, the multiplicities being given by the algebraic multiplicities of the eigenvalues of $V_{\mathbf{C}}^{\mathbf{A}}$. In particular, when $m = 1$, the solution set is defined by the vanishing of a single homogeneous polynomial of degree 2, which has two distinct roots generically, but a single root of multiplicity two when V^a is null – see, e.g., [21]. ■

2.5.3 Even dimensions

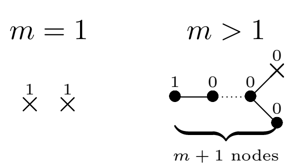
The analysis when $n = 2m$ is very similar to the odd-dimensional case without the added complication of the canonical distribution. Again, for any x of $\mathcal{Q}^{2m}$, $\mathrm{T}_x\mathbb{CE}^{2m}$ injects into $H^0(\hat{x}, \mathcal{O}(\mathrm{N}\hat{x}))$. A null vector in V^a is $\mathrm{T}_x\mathbb{CE}^{2m}$ defines a global section $\widehat{V}_{\hat{x}}$ of $\mathrm{N}\hat{x}$, which vanishes at a single point when $m = 2$, and on a $\frac{1}{2}(m-1)(m-2)$-dimensional algebraic subset of $\hat{x}$, isomorphic to $\mathbb{PT}_{(2m-2)}$, when $m > 2$. Each point of this subset corresponds to an α-plane to which V^a is tangent.

2.5.4 Kodaira's theorem and completeness

Let us now turn to the question of whether $\mathrm{T}_x\mathcal{Q}^n$ maps to $H^0(\hat{x}, \mathcal{O}(\mathrm{N}\hat{x}))$ bijectively, and not merely injectively, for any $x \in \mathcal{Q}^n$. By Kodaira's theorem [24], $\mathrm{T}_x\mathcal{Q}^n \cong H^0(\hat{x}, \mathcal{O}(\mathrm{N}\hat{x})) \cong \mathbb{C}^n$ if and only if the family $\{\hat{x}\}$ in $\mathbb{PT}$ is *complete*, i.e., *any* infinitesimal deformation of $\hat{x}$ arises from an element of $\mathrm{T}_x\mathcal{Q}^n$. As we have seen in Section 2.2.3, the twistor space $\mathbb{PT}$ of $\mathcal{Q}^{2m+1}$ and the twistor space $\widetilde{\mathbb{PT}}$ of $\mathcal{Q}^{2m+2}$ are both $\frac{1}{2}(m+1)(m+2)$-dimensional complex projective varieties in $\mathbb{CP}^{2^{m+1}-1}$, and it is the embedding $\mathcal{Q}^{2m+1} \subset \mathcal{Q}^{2m+2}$ that induces the canonical distribution D on $\mathbb{PT}$. The issue here is that Kodaira's theorem is only concerned with the holomorphic structure of the underlying manifolds, and does not depend on the additional distribution on $\mathbb{PT}$.

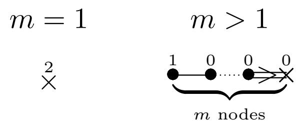
Now, by the twistor correspondences, any point x in $\mathcal{Q}^{2m+1}$ and $\mathcal{Q}^{2m+2}$ gives rise to a $\frac{1}{2}m(m+1)$-dimensional complex submanifold $\hat{x}$ of $\mathbb{PT}$ and $\widetilde{\mathbb{PT}}$ respectively. This means that the analytic family $\{\hat{x}\}$ parametrised by the points $\{x\}$ of $\mathcal{Q}^{2m+1}$ can be completed to a larger family parametrised by the points $\{x\}$ of $\mathcal{Q}^{2m+2}$ via the embedding $\mathcal{Q}^{2m+1} \subset \mathcal{Q}^{2m+2}$. Further, a complex submanifold $\hat{x}$ corresponds to a point x in $\mathcal{Q}^{2m+1}$ if and only if $\hat{x}$ is tangent to an m-dimensional subspace of D_Z at every point $Z \in \hat{x}$.

We also need to check whether the family of $\hat{x}$ is complete when $x \in \mathcal{Q}^{2m+2}$. If it were not, one would be able to find a group of biholomorphic automorphisms of $\mathbb{PT}$ larger than $\mathrm{Spin}(2m+4, \mathbb{C})$ and a parabolic subgroup such that the quotient models $\mathbb{PT}$. But the work of [13, 28] tells us that there is no such group. The same applies to each $\hat{x}$, and since these are biholomorphic to flag varieties, the normal bundle $\mathrm{N}\hat{x}$ can be identified with a rank-$(m+1)$ holomorphic homogeneous vector bundle over $\hat{x}$. In the notation of [5], we find that for a point x in $\mathcal{Q}^{2m+1}$ or $\mathcal{Q}^{2m+2}$, the normal bundle $\mathrm{N}\hat{x}$ in $\mathbb{PT} \cong \widetilde{\mathbb{PT}}$ is given by



Here, the mutilated Dynkin diagram corresponds to the parabolic subalgebra underlying the flag variety $\hat{x}$, and the coefficients over the nodes to the irreducible representation that determines the vector bundle. When $m = 1$, i.e., for $\mathcal{Q}^3$ and $\mathcal{Q}^4$, we recover the well-known result $\mathrm{N}_{\hat{x}} \cong \mathcal{O}_{\hat{x}}(1) \oplus \mathcal{O}_{\hat{x}}(1)$, where $\mathcal{O}_{\hat{x}}(1)$ is the hyperplane bundle over $\hat{x} \cong \mathbb{CP}^1$. We can compute the cohomology using the Bott–Borel–Weil theorem, and verify that indeed $H^0(\hat{x}, \mathcal{O}(\mathrm{N}\hat{x})) \cong \mathbb{C}^{2m+2}$ and $H^1(\hat{x}, \mathcal{O}(\mathrm{N}\hat{x})) = 0$ – this latter condition tells us that there is no obstruction for the existence of our family.

We can play the same game with the family of compact complex submanifolds $\{\underline{\hat{x}}\}$ in $\mathbb{MT}$ parametrised by the points x of $\mathbb{CE}^{2m+1}$. But in this case, for any x of $\mathbb{CE}^{2m+1}$, the normal bundle $\mathrm{N}\underline{\hat{x}}$ is essentially the total space of $\mathrm{T}^{-1}\hat{x} \to \hat{x}$, and is described, in the notation of [5], as the rank-m holomorphic homogeneous vector bundle



When $m = 1$, i.e., $\mathcal{Q}^3$, $\underline{\hat{x}} \cong \mathbb{CP}^1$, and we recover the well-known result $\mathcal{O}(\mathrm{N}\underline{\hat{x}}) \cong \mathcal{O}_{\underline{\hat{x}}}(2) := \otimes^2 \mathcal{O}_{\underline{\hat{x}}}(1)$. Again, the Bott–Borel–Weil theorem confirms that $H^0(\underline{\hat{x}}, \mathcal{O}(\mathrm{N}\underline{\hat{x}})) \cong \mathbb{C}^{2m+1}$ and $H^1(\underline{\hat{x}}, \mathcal{O}(\mathrm{N}\underline{\hat{x}})) = 0$.

Remark 2.32. When $n = 3$, this analysis was already exploited in [25] in the curved setting, where the twistor space of a three-dimensional holomorphic conformal structure is identified with the space of null geodesics. See also [18].

3 Null foliations

As before, we work in the holomorphic category throughout, i.e., vector fields and distributions will be assumed to be holomorphic.

Definition 3.1. An *almost null structure* is a holomorphic totally null m-plane distribution on $\mathcal{Q}^n$, where $n = 2m$ or $2m + 1$.

In other words, an almost null structure is a γ-plane, α-plane or β-plane distribution. From the discussion of Section 2.3, an almost null structure, self-dual when $n = 2m$, can be viewed as a holomorphic section of $\mathbb{F} \to \mathcal{Q}^n$, or equivalently as a projective pure spinor field on $\mathcal{Q}^n$, that is a spinor field defined up to scale, and which is pure at every point. The geometric properties of an almost null structure on a general spin complex Riemannian manifold can be expressed in terms of the differential properties of its corresponding projective pure spinor field as described in [34, 35].

The question we now wish to address is the following one: given an almost null structure, how can we encode its geometric properties in twistor space $\mathbb{PT}$?

3.1 Odd dimensions

When $n = 2m + 1$, an almost null structure is more adequately expressed as an inclusion of distributions $N \subset N^\perp$ where N is a holomorphic totally null m-plane distribution and $N^\perp$ is its orthogonal complement. One can then investigate the geometric properties of N and $N^\perp$ independently. In the following, $\Gamma(\mathcal{U}, \mathcal{O}(N))$ denotes the space of holomorphic sections of N over an open subset $\mathcal{U}$ of $\mathcal{Q}^n$, and similarly for $N^\perp$.

Definition 3.2. Let $N \subset N^\perp$ be an almost null structure on some open subset $\mathcal{U}$ of $\mathcal{Q}^n$. We say that N is

- *integrable* if $[\boldsymbol{X}, \boldsymbol{Y}] \in \Gamma(\mathcal{U}, \mathcal{O}(N))$ for all $\boldsymbol{X}, \boldsymbol{Y} \in \Gamma(\mathcal{U}, \mathcal{O}(N))$,
- *totally geodetic* if $\nabla_{\boldsymbol{Y}} \boldsymbol{X} \in \Gamma(\mathcal{U}, \mathcal{O}(N))$ for all $\boldsymbol{X}, \boldsymbol{Y} \in \Gamma(\mathcal{U}, \mathcal{O}(N))$,
- *co-integrable* if $[\boldsymbol{X}, \boldsymbol{Y}] \in \Gamma(\mathcal{U}, \mathcal{O}(N^\perp))$ for all $\boldsymbol{X}, \boldsymbol{Y} \in \Gamma(\mathcal{U}, \mathcal{O}(N^\perp))$,
- *totally co-geodetic* if $\nabla_{\boldsymbol{Y}} \boldsymbol{X} \in \Gamma(\mathcal{U}, \mathcal{O}(N^\perp))$ for all $\boldsymbol{X}, \boldsymbol{Y} \in \Gamma(\mathcal{U}, \mathcal{O}(N^\perp))$.

An integrable almost null structure will be referred to as a *null structure*.

There is however some dependency regarding the geometric properties of N and $N^\perp$.

Lemma 3.3 ([35]). *Let N be an almost null structure. Then*

- *if N is totally co-geodetic, it is also integrable and co-integrable,*
- *if N is integrable and co-integrable, it is also geodetic,*
- *if N is totally geodetic, it is also integrable.*

Another important point is the conformal invariance of the above properties. All with the exception of the totally co-geodetic property are conformal invariant – see [35].

3.1.1 Local description

The next theorems will be local in nature. That means that we shall work on $\mathbb{CE}^n$ viewed as a dense open subset of $\mathcal{Q}^n$. For their proofs, we shall make use of the local coordinates on $\mathbb{CE}^n$, $\mathbb{F}_{\mathbb{CE}^n}$ and $\mathbb{PT}_{\backslash\infty}$ given in Appendix A.1. Let N be an almost null structure on some open subset $\mathcal{U}$ of $\mathbb{CE}^n = \{z^A, z_A, u\}$, and view N as a local holomorphic section of $\mathbb{F} \to \mathbb{CE}^n$, i.e., a holomorphic projective pure spinor field $[\xi^{\mathbf{A}}]$. We may assume that locally, $[\xi^{\mathbf{A}}]$ defines a complex submanifold of $\mathcal{U} \times \mathcal{U}_0$, where $(\mathcal{U}_0, (\pi^A, \pi^{AB}))$ is a coordinate chart on the fibers of $\mathbb{F}_{\mathcal{U}}$, given by the graph

$$\Gamma_\xi := \big\{(x,\pi) \in \mathcal{U} \times \mathcal{U}_0 \colon \pi^{AB} = \xi^{AB}(x),\, \pi^A = \xi^A(x)\big\}, \tag{3.1}$$

for some $\frac{1}{2}m(m-1)$ and m holomorphic functions $\xi^{AB} = \xi^{[AB]}$ and ξ^A respectively on $\mathcal{U}$. In this case, the distribution N is spanned by the m holomorphic vector fields

$$\boldsymbol{Z}^A = \partial^A + \big(\xi^{AD} - \tfrac{1}{2}\xi^A\xi^D\big)\,\partial_D + \xi^A\partial, \tag{3.2}$$

while its orthogonal complement $N^\perp$ by the $m+1$ holomorphic vector fields

$$\boldsymbol{Z}^A = \partial^A + \big(\xi^{AD} - \tfrac{1}{2}\xi^A\xi^D\big)\,\partial_D + \xi^A\partial, \qquad \boldsymbol{U} = \partial - \xi^D\partial_D, \tag{3.3}$$

where $\partial^A := \frac{\partial}{\partial z_A}$, $\partial_A := \frac{\partial}{\partial z^A}$ and $\partial := \frac{\partial}{\partial u}$. Here, we shall make a slight abuse of notation by denoting the vector fields spanning N and $N^\perp$, and their lifts to $\mathbb{F}_{\mathcal{U}}$, both by (3.2) and (3.3).

Remark 3.4. It will be understood that when $m = 1$ there are no coordinates π^{AB}. This does not affect the veracity of the following results in this case – see however Remark 3.7.

3.1.2 Totally geodetic null structures

Let $\mathcal{W}$ be an $(m+1)$-dimensional complex submanifold of $\mathbb{PT}$ and let $\mathcal{U}$ be an open subset of $\mathcal{Q}^{2m+1}$. Suppose that for every point x of $\mathcal{U}$, $\hat{x} \in \widehat{\mathcal{U}}$ intersects $\mathcal{W}$ transversely in a point. Then each point of $\mathcal{W} \cap \hat{x}$ determines a point in the fiber $\mathbb{F}_x$, and thus a γ-plane through x. Smooth variations of the point x in $\mathcal{U}$ thus define a holomorphic section of $\mathbb{F}_{\mathcal{U}} \to \mathcal{U}$ and an $(m+1)$-dimensional analytic family of γ-planes, each of which being the totally geodetic leaf of an integrable almost null structure. Conversely, consider a local foliation by totally null and totally geodetic m-dimensional leaves. Then, each leaf must be some affine subset of a γ-plane. The $(m+1)$-dimensional leaf space of the foliation constitutes an $(m+1)$-dimensional analytic family of γ-planes, and thus defines an $(m+1)$-dimensional complex submanifold of $\mathbb{PT}$.

Theorem 3.5. *A totally geodetic null structure on some open subset $\mathcal{U}$ of $\mathcal{Q}^{2m+1}$ gives rise to an $(m+1)$-dimensional complex submanifold of $\widehat{\mathcal{U}} \subset \mathbb{PT}$ intersecting $\hat{x} \subset \widehat{\mathcal{U}}$ transversely for each $x \in \mathcal{U}$. Conversely, any totally geodetic null structure locally arises in this way.*

Proof. Let N be an almost null structure as described in Section 3.1.1. The condition that N be totally geodetic is $\boldsymbol{g}(\nabla_{\boldsymbol{Z}^A}\boldsymbol{Z}^B, \boldsymbol{Z}^C) = \boldsymbol{g}(\nabla_{\boldsymbol{Z}^A}\boldsymbol{Z}^B, \boldsymbol{U}) = 0$, i.e.,

$$\begin{aligned} &\left(\partial^A + \left(\xi^{AD} - \tfrac{1}{2}\xi^A\xi^D\right)\partial_D + \xi^A\partial\right)\xi^{BC} = 0,\\ &\left(\partial^A + \left(\xi^{AD} - \tfrac{1}{2}\xi^A\xi^D\right)\partial_D + \xi^A\partial\right)\xi^{B} = 0. \end{aligned} \tag{3.4}$$

We re-express the system (3.4) of holomorphic partial differential equations as

$$\begin{aligned} &\rho^{ABC} + \left(\pi^{AD} - \tfrac{1}{2}\pi^A\pi^D\right)\rho^{BC}_D + \pi^A\rho^{BC} = 0,\\ &\sigma^{AB} + \left(\pi^{AD} - \tfrac{1}{2}\pi^A\pi^D\right)\sigma^{B}_D + \pi^A\sigma^{B} = 0, \end{aligned} \tag{3.5}$$

where $\rho^{ABC} := \partial^A\pi^{BC}$, $\rho^{BC}_A := \partial_A\pi^{BC}$, $\rho^{AB} := \partial\pi^{AB}$, $\sigma^{AB} := \partial^A\pi^B$, $\sigma^B_A := \partial_A\pi^B$, $\sigma^A := \partial\pi^A$. In the language of jets, the locus (3.5) defines a complex submanifold of the first jet space $\mathcal{J}^1(\mathbb{CE}^n, \mathcal{U}_0)$, of which the prolongation of the section Γ_ξ is a submanifold. Now, the distribution $\mathrm{T}^{-1}_E\mathbb{F} = \langle \boldsymbol{Z}^A\rangle$ tangent to the fibers of $\mathbb{F} \to \mathbb{PT}$ is annihilated by the 1-forms $\mathrm{d}\pi^A$, $\mathrm{d}\pi^{AB}$, $\boldsymbol{\theta}^A$ and $\boldsymbol{\theta}^0$ as defined in Appendix A.1, which can be pulled back to $\mathcal{J}^1(\mathbb{CE}^n, \mathcal{U}_0)$. The 1-forms defined by

$$\begin{aligned} &\boldsymbol{\phi}^A := \mathrm{d}\pi^A - \sigma^A_C\boldsymbol{\theta}^C - \left(\sigma^A - \sigma^A_C\pi^C\right)\boldsymbol{\theta}^0,\\ &\boldsymbol{\phi}^{AB} := \mathrm{d}\pi^{AB} - \rho^{AB}_C\boldsymbol{\theta}^C - \left(\rho^{AB} - \rho^{AB}_C\pi^C\right)\boldsymbol{\theta}^0, \end{aligned} \tag{3.6}$$

vanish on the locus (3.5), and this implies in particular that, for generic ρ^{AB}_C, ρ^{BC}, ρ^A_C, ρ^C, the section Γ_ξ must be constant along the fibers of $\mathrm{T}^{-1}_E\mathbb{F}$, i.e., the functions (ξ^A, ξ^{AB}) depend only on the coordinates $(\omega^0, \omega^A, \pi^A, \pi^{AB})$ of the chart $\mathcal{V}_0$ of $\mathbb{PT}$. Thus, quotienting Γ_ξ along the fibers of $\mathbb{F} \to \mathbb{PT}$ yields an $(m+1)$-dimensional complex submanifold of $\mathbb{PT}$ intersecting each $\hat{x}$ transversely in a point.

The converse is also true: we start with an $(m+1)$-dimensional complex submanifold $\mathcal{W}$, say, of $\mathbb{PT}$, which can be locally represented by the vanishing of $\frac{1}{2}m(m+1)$ holomorphic functions (F^{AB}, F^A) on the chart $(\mathcal{V}_0, (\omega^0, \omega^A, \pi^A, \pi^{AB}))$. Then $(\mathrm{d}F^{AB}, \mathrm{d}F^A)$ are a set of 1-forms vanishing on $\mathcal{W}$. We shall assume that for each $x \in \mathcal{U}$, the submanifold $\hat{x} \subset \widehat{\mathcal{U}}$ intersects $\mathcal{W}$ transversely in a point. This singles out a local holomorphic section $[\xi^{\mathbf{A}}]$ of $\mathcal{U}\times\mathcal{U}_0 \subset \mathbb{F} \to \mathcal{U}$. By the implicit function theorem, we may assume with no loss of generality that this is the graph Γ_ξ given by (3.1). The pullbacks of $(\mathrm{d}F^{AB}, \mathrm{d}F^A)$ to $\mathbb{F}$ vanish on Γ_ξ and give the restriction

$$\begin{pmatrix} Q^A_C & Q^A_{CD} \\ Q^{AB}_C & Q^{AB}_{CD}\end{pmatrix}\begin{pmatrix}\mathrm{d}\pi^C \\ \mathrm{d}\pi^{CD}\end{pmatrix} + \begin{pmatrix}\boldsymbol{Y}F^A & \boldsymbol{X}_C F^A \\ \boldsymbol{Y}F^{AB} & \boldsymbol{X}_C F^{AB}\end{pmatrix}\begin{pmatrix}\boldsymbol{\theta}^0 \\ \boldsymbol{\theta}^C\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}, \tag{3.7}$$

where

$$\begin{pmatrix} Q^A_C & Q^A_{CD} \\ Q^{AB}_C & Q^{AB}_{CD}\end{pmatrix} := \begin{pmatrix} \left(\dfrac{\partial}{\partial\pi^C} + \dfrac{1}{2}u\dfrac{\partial}{\partial\omega^C} - z_C\dfrac{\partial}{\partial\omega^0}\right)F^A & \left(\dfrac{\partial}{\partial\pi^{CD}} + z_{[C}\dfrac{\partial}{\partial\omega^{D]}}\right)F^A \\ \left(\dfrac{\partial}{\partial\pi^C} + \dfrac{1}{2}u\dfrac{\partial}{\partial\omega^C} - z_C\dfrac{\partial}{\partial\omega^0}\right)F^{AB} & \left(\dfrac{\partial}{\partial\pi^{CD}} + z_{[C}\dfrac{\partial}{\partial\omega^{D]}}\right)F^{AB}\end{pmatrix}. \tag{3.8}$$

At generic points, the matrix (3.8) is invertible, and equations (3.7) can immediately be seen to be equivalent to the vanishing of the forms (3.6). In particular, $\pi^{AB} = \xi^{AB}(x)$ and $\pi^A = \xi^A(x)$ satisfy (3.4), i.e., the distribution associated to the graph Γ_ξ is integrable and totally geodetic. ■

3.1.3 Co-integrable null structures

Let us now suppose that our almost null structure N is integrable and co-integrable on $\mathcal{U}$. We then have two foliations of $\mathcal{U}$, one for N and the other for $N^\perp$. By Lemma 3.3, we know that each leaf of N is totally geodetic and therefore a γ-plane. Since $N \subset N^\perp$, each $(m+1)$-dimensional leaf of $N^\perp$ contains a one-parameter holomorphic family $\{\check{Z}_t\}$ of γ-planes, i.e., of leaves of N. Thus each leaf of $N^\perp$ descends to a holomorphic curve on the leaf space of N. In particular, by Theorem 3.5, we can identify the leaf space of N with an $(m+1)$-dimensional complex submanifold $\mathcal{W}$ of $\mathbb{PT}$ foliated by curves, each of which being a one-parameter of twistors $\{Z_t\}$ and, as we shall show, tangent to the canonical distribution D of $\mathbb{PT}$.

We start by the remark that at any point Z of $\mathcal{W}$, any submanifold $\hat{x}$ intersects $\mathcal{W}$ transversely, i.e., $\mathrm{T}_Z\mathbb{PT} = \mathrm{T}_Z\hat{x} \oplus \mathrm{T}_Z\mathcal{W}$. Hence, by Lemma 2.27 the intersection of D_Z with $\mathrm{T}_Z\mathcal{W}$ can only be at most one-dimensional. Now, let Z_0 and Z_t be two points on $\mathcal{W}$ corresponding to two infinitesimally separated γ-planes, $\check{Z}_0$ and $\check{Z}_t$ in $\{\check{Z}_t\}$, contained in the co-γ-plane $\check{Z}_0^\perp$. Let x and y be points on $\check{Z}_0$ and $\check{Z}_t$ respectively, so that their corresponding complex submanifolds $\hat{x}$ and $\hat{y}$ of $\widehat{\mathcal{U}}$ intersect $\mathcal{W}$ in Z_0 and Z_t respectively. The vector V^a in $\mathrm{T}_x\mathcal{U}$ tangent to $\check{Z}_0^\perp$ connecting x to y is non-null, and we know by Lemma 2.29 that the vector connecting Z_0 to Z_t must lie in D_{Z_0}. This is clearly independent of the choice of points x and y on $\check{Z}_0$ and $\check{Z}_t$. Assigning a vector tangent to D_{Z_t} at every point of $\{Z_t\}$ yields a curve corresponding to a leaf of $N^\perp$. Proceeding in this way for each leaf of $N^\perp$ gives rise to a foliation by holomorphic curves tangent to D on $\mathcal{W}$. Conversely, any such foliation by curves on a given $(m+1)$-submanifold of $\mathbb{PT}$ gives rise to an integrable and co-integrable almost null structure.

Theorem 3.6. *An integrable and co-integrable almost null structure on some open subset $\mathcal{U}$ of $\mathcal{Q}^{2m+1}$ gives rise to an $(m+1)$-dimensional complex submanifold of $\widehat{\mathcal{U}} \subset \mathbb{PT}$ foliated by holomorphic curves tangent to D and intersecting $\hat{x} \subset \widehat{\mathcal{U}}$ transversely for each $x \in \mathcal{U}$. Conversely, any integrable and co-integrable almost null structure locally arises in this way.*

Proof. We recycle the setting and notation of the proof of Theorem 3.5. In particular, we take N and $N^\perp$ to be spanned by the vector fields (3.2) and (3.3). The assumption that N be integrable and co-integrable, i.e., $\boldsymbol{g}(\nabla_{\boldsymbol{Z}^A}\boldsymbol{Z}^B, \boldsymbol{Z}^C) = \boldsymbol{g}(\nabla_{\boldsymbol{Z}^A}\boldsymbol{Z}^B, \boldsymbol{U}) = \boldsymbol{g}(\nabla_{\boldsymbol{U}}\boldsymbol{Z}^B, \boldsymbol{Z}^C) = 0$, gives (3.4) and in addition,

$$(\partial - \xi^D\partial_D)\xi^{BC} + \big((\partial - \xi^D\partial_D)\xi^{[B}\big)\xi^{C]} = 0. \tag{3.9}$$

Thus, the system $\{(3.4), (3.9)\}$ can be encoded as the complex submanifold of $\mathcal{J}^1(\mathbb{CE}^n, \mathcal{U}_0)$ arising from the intersection of the locus (3.5) and the locus

$$\rho^{BC} - \pi^D\rho_D^{BC} + \sigma^{[B}\pi^{C]} - \pi^D\sigma_D^{[B}\pi^{C]} = 0, \tag{3.10}$$

and the prolongation of Γ_ξ must lie in this intersection. Now, let us define $\boldsymbol{\psi}^{AB} := \boldsymbol{\phi}^{AB} - \pi^{[A}\boldsymbol{\phi}^{B]}$, where $\boldsymbol{\phi}^A$ and $\boldsymbol{\phi}^{AB}$ are the 1-forms (3.6). From the proof of Theorem 3.5, the 1-forms $\boldsymbol{\psi}^{AB}$ and $\boldsymbol{\phi}^A$ vanish on the locus (3.5). On the other hand, on restriction to the locus (3.10), we have $\boldsymbol{\psi}^{AB} = \boldsymbol{\alpha}^{AB} - \left(\rho_C^{AB} - \pi^{[A}\sigma_C^{B]}\right)\boldsymbol{\theta}^C$, where $\langle \boldsymbol{\alpha}^{AB}, \boldsymbol{\theta}^A \rangle$ annihilate the rank-$(2m+1)$ distribution $\mathrm{T}_E^{-2}\mathbb{F} = \langle \boldsymbol{U}, \boldsymbol{W}_A, \boldsymbol{Z}^A \rangle$. One can further check that $\langle \boldsymbol{\psi}^{AB}, \boldsymbol{\phi}^A \rangle$ annihilate the $m+1$ vector fields $\boldsymbol{U} + \left(\sigma^A - \sigma_B^A\pi^B\right)\boldsymbol{W}_A$ and $\boldsymbol{Z}^A$. These span a rank-$(m+1)$ subdistribution L, say, of $\mathrm{T}_E^{-2}\mathbb{F}$ tangent to Γ_ξ. By Theorem 3.5, Γ_ξ descends to an $(m+1)$-dimensional complex submanifold $\mathcal{W}$ of $\mathbb{PT}$. The quotient $\mathrm{L}/\mathrm{T}_E^{-1}\mathbb{F}$ is a rank-1 subbundle of $\mathrm{T}_E^{-2}\mathbb{F}/\mathrm{T}_E^{-1}\mathbb{F}$, which also descends to a rank-1 subdistribution of $\mathrm{D} = \mathrm{T}^{-1}\mathbb{PT}$ tangent to $\mathcal{W}$. This proves the first part of the theorem.

Conversely, consider a complex submanifold $\mathcal{W}$ of $\mathbb{PT}$, transverse to every $\hat{x}$ in $\widehat{\mathcal{U}}$, given by the vanishing of holomorphic functions (F^{AB}, F^A) on the chart $(\mathcal{V}_0, (\omega^0, \omega^A, \pi^A, \pi^{AB}))$. By Theorem 3.5, we can associate to $\mathcal{W}$ a local section $[\xi^{\mathbf{A}}]$ of $\mathcal{U} \times \mathcal{U}_0 \subset \mathbb{F}$ with graph Γ_ξ, so that equations (3.5) hold. Assume further that the intersection of $\mathrm{T}\mathcal{W}$ and $\mathrm{D}|_{\mathcal{W}}$ is one-dimensional at every point. Then the pullbacks of $(\mathrm{d}F^{AB}, \mathrm{d}F^{A})$ to $\mathcal{U} \times \mathcal{U}_0 \subset \mathbb{F}$ must vanish on Γ_ξ and annihilate both $\mathrm{T}^{-1}_E\mathbb{F}$ and a rank-$(m+1)$ subbundle of $\mathrm{T}^{-2}_E\mathbb{F} \supset \mathrm{T}^{-1}_E\mathbb{F}$. Thus, there exists a vector field $\boldsymbol{V} = \boldsymbol{U} + V^A\boldsymbol{W}_A$, for some holomorphic functions V^A on Γ_ξ, annihilating the 1-forms (3.6). It is then straightforward to check that this gives us precisely the additional restrictions (3.10). In particular, $\pi^{AB} = \xi^{AB}(x)$ and $\pi^A = \xi^A(x)$ satisfy (3.4) and (3.9), i.e., the distribution associated to the graph Γ_ξ is integrable and co-integrable. ■

Remark 3.7. When $n = 3$, Theorems 3.5 and 3.6 are equivalent: since $\mathbb{PT}$ is 3-dimensional and D has rank 2, any 2-dimensional complex submanifold of $\mathbb{PT}$ satisfying the transversality property of the theorems must have non-trivial intersection with D.

3.1.4 Totally co-geodetic null structures

Finally, we consider a totally co-geodetic null structure N. The key point here is that this stronger requirement is not conformally invariant, and for this reason, the appropriate arena is the mini-twistor space $\mathbb{MT}$ of $\mathbb{CE}^{2m+1}$. In this case, each leaf of the foliation of $N^\perp$ is totally geodetic, and must therefore be a co-γ-plane. The m-dimensional leaf space can then be identified as an m-dimensional complex submanifold $\underline{\mathcal{W}}$ of $\mathbb{MT}$.

Alternatively, we can recycle the setting of Theorems 3.5 and 3.6: since N is in particular integrable and co-integrable, its leaf space is an $(m+1)$-dimensional complex submanifold $\mathcal{W}$ of $\mathbb{PT}_{\backslash\widehat{\infty}}$ foliated by curves. However, these curves are very particular since they correspond to totally geodetic leaves of $N^\perp$. Breaking of the conformal invariance can be translated into these curves being the integral curves of the vector field $\boldsymbol{Y}$ induced by the point ∞ on $\mathcal{Q}^n$. Quotienting the submanifold $\mathcal{W}$ by the flow of $\boldsymbol{Y}$ thus yields an m-dimensional complex submanifold $\underline{\mathcal{W}}$ of $\mathbb{MT}$.

Theorem 3.8. *A totally co-geodetic null structure on some open subset $\mathcal{U}$ of $\mathbb{CE}^{2m+1}$ gives rise to an m-dimensional complex submanifold of $\underline{\widehat{\mathcal{U}}} \subset \mathbb{MT}$ intersecting each $\hat{x} \subset \underline{\widehat{\mathcal{U}}}$ transversely for each $x \in \mathcal{U}$. Conversely, any totally co-geodetic null structure locally arises in this way.*

Proof. Suppose N and $N^\perp$ are both integrable as in the previous section. As already pointed out the integral manifolds of N are totally geodetic. We now impose the further assumption that the integral manifolds of $N^\perp$ are also totally geodetic on $\mathcal{U}$, i.e., $\boldsymbol{g}(\nabla_{\boldsymbol{Z}^A}\boldsymbol{Z}^B, \boldsymbol{Z}^C) = \boldsymbol{g}(\nabla_{\boldsymbol{Z}^A}\boldsymbol{Z}^B, \boldsymbol{U}) = \boldsymbol{g}(\nabla_{\boldsymbol{U}}\boldsymbol{Z}^B, \boldsymbol{Z}^C) = \boldsymbol{g}(\nabla_{\boldsymbol{U}}\boldsymbol{Z}^A, \boldsymbol{U}) = 0$. Then, in addition to (3.4), we have

$$(\partial - \xi^D\partial_D)\xi^{AB} = 0, \qquad (\partial - \xi^D\partial_D)\xi^{A} = 0, \tag{3.11}$$

which can be seen to imply (3.9). As before, using the same notation as in the proof of Theorem 3.5, we express the system (3.4), (3.11) as a complex submanifold of $\mathcal{J}^1(\mathbb{CE}^n, \mathcal{U}_0)$ defined by (3.5) and

$$\rho^{AB} - \pi^D\rho^{AB}_D = 0, \qquad \sigma^A - \pi^D\sigma^A_D = 0. \tag{3.12}$$

In particular, the 1-forms $\mathrm{d}\pi^{AB} - \rho^{AB}_C\boldsymbol{\theta}^C$ and $\mathrm{d}\pi^A - \sigma^A_C\boldsymbol{\theta}^C$ vanish on the locus (3.5) and (3.12), and this implies in particular that, for generic ρ^{AB}_C, ρ^{BC}, ρ^A_C, ρ^C, the section Γ_ξ must be constant along the fibers of $\mathbb{F} \to \mathbb{MT}$, i.e., the functions (ξ^A, ξ^{AB}) depend only on the coordinates $(\underline{\omega}^A, \pi^A, \pi^{AB})$ on the chart $\underline{\mathcal{V}}_0$ of $\mathbb{MT}$. Thus, quotienting Γ_ξ along the fibers of $\mathbb{F} \to \mathbb{MT}$ yields an m-dimensional complex submanifold of $\mathbb{MT}$.

For the converse, we simply run the argument backwards as in the proof of Theorem 3.5. ■

3.2 Even dimensions

The even-dimensional case is somewhat more tractable than the odd-dimensional case. For one, the orthogonal complement of an α-plane or β-plane distribution N *is* N itself, i.e., $N^\perp = N$. Definition 3.2 still applies albeit with much redundancy. In particular, N is integrable if and only if it is co-integrable. The question now reduces to whether N is integrable or not, and if so, whether it is totally geodetic. But it turns out that these two questions are equivalent.

Lemma 3.9. *An almost null structure is integrable if and only if it is totally geodetic.*

For a proof, see for instance [33, 34]. The argument leading up to Theorem 3.5 equally applies to the even-dimensional case – simply substitute γ-plane for α-plane. For the sake of completeness, we restate the theorem, which was first used in four dimensions in [23], reformulated in twistor language in [29], and generalised to higher even dimensions in [20]. The proof of Theorem 3.5 can be recycled entirely by 'switching off' the coordinates u, ω^0, π^A, and so on.

Theorem 3.10 ([20])**.** *A self-dual null structure on some open subset $\mathcal{U}$ of $\mathcal{Q}^{2m}$ gives rise to an m-dimensional complex submanifold of $\widehat{\mathcal{U}} \subset \mathbb{PT}$ intersecting $\hat{x}$ in $\widehat{\mathcal{U}}$ transversely for each $x \in \mathcal{U}$. Conversely, any self-dual null structure locally arises in this way.*

4 Examples

We now give two examples of co-integrable null structures that will illustrate the mechanism of Theorems 3.6 and 3.10. These arise in connections with conformal Killing spinors and conformal Killing–Yano 2-forms, and are more transparently constructed in the language of tractor bundles reviewed in Section 2.1.1. As before, we work in the holomorphic category.

4.1 Conformal Killing spinors

For definiteness, let us stick to odd dimensions, i.e., $n = 2m + 1$. The even-dimensional case is similar. A (holomorphic) *conformal Killing spinor* on $\mathcal{Q}^n$ is a section $\xi^{\mathbf{A}}$ of $\mathcal{O}^{\mathbf{A}}$ that satisfies

$$\nabla_a \xi^{\mathbf{A}} + \tfrac{1}{\sqrt{2}} \boldsymbol{\gamma}_{a\mathbf{B}}{}^{\mathbf{A}} \zeta^{\mathbf{B}} = 0, \tag{4.1}$$

where $\zeta^{\mathbf{A}} = \frac{\sqrt{2}}{n} \boldsymbol{\gamma}^a{}_{\mathbf{B}}{}^{\mathbf{A}} \nabla_a \xi^{\mathbf{B}}$ is a section of $\mathcal{O}^{\mathbf{A}}[-1]$.

The prolongation of equation (4.1) is given by (see for instance [6] and references therein)

$$\nabla_a \xi^{\mathbf{A}} + \tfrac{1}{\sqrt{2}} \boldsymbol{\gamma}_{a\mathbf{B}}{}^{\mathbf{A}} \zeta^{\mathbf{B}} = 0, \qquad \nabla_a \zeta^{\mathbf{A}} + \tfrac{1}{\sqrt{2}} \mathrm{P}_{ab} \boldsymbol{\gamma}^b{}_{\mathbf{B}}{}^{\mathbf{A}} \xi^{\mathbf{B}} = 0. \tag{4.2}$$

These equations are equivalent to the tractor spinor $\Xi^{\boldsymbol{\alpha}} = (\xi^{\mathbf{A}}, \zeta^{\mathbf{A}})$ being parallel with respect to the tractor spinor connection, i.e., $\nabla_a \Xi^{\boldsymbol{\alpha}} = 0$. In a conformal scale for which the metric is flat, integration of (4.2) yields

$$\xi^{\mathbf{A}} = \mathring{\xi}^{\mathbf{A}} - \tfrac{1}{\sqrt{2}} x^a \gamma_{a\mathbf{B}}{}^{\mathbf{A}} \mathring{\zeta}^{\mathbf{B}}, \qquad \zeta^{\mathbf{A}} = \mathring{\zeta}^{\mathbf{A}}, \tag{4.3}$$

where $\mathring{\xi}^{\mathbf{A}}$ and $\mathring{\zeta}^{\mathbf{A}}$ denote the constants of integrations at the origin.

A *pure* conformal Killing spinor $\xi^{\mathbf{A}}$ defines an almost null structure. The following proposition combines results from [34, 35] recast in the language of tractors using Lemmata 2.19 and 2.22. It is valid on any conformal manifold of any dimension.

Proposition 4.1 ([34, 35])**.** *The almost null structure of a pure conformal Killing spinor is locally integrable and co-integrable if and only if its associated tractor spinor is pure.*

By Theorems 3.6 and 3.10 one can associate to any such conformal Killing spinor on $\mathcal{Q}^n$ a complex submanifold in $\mathbb{PT}$. These are described in the next two propositions.

4.1.1 Odd dimensions

Proposition 4.2. *Let $\Xi^{\boldsymbol{\alpha}} = (\xi^{\mathbf{A}}, \zeta^{\mathbf{A}})$ be a constant pure tractor spinor on $\mathcal{Q}^{2m+1}$, Ξ its associated twistor in $\mathbb{PT}$, $\check{\Xi}$ its corresponding γ-plane in $\mathcal{Q}^{2m+1}$, and $\mathcal{U} := \mathcal{Q}^{2m+1} \setminus \check{\Xi}$. Then $\xi^{\mathbf{A}}$ is a pure conformal Killing spinor on $\mathcal{Q}^{2m+1}$ with zero set $\check{\Xi}$, and its associated integrable and co-integrable almost null structure N_ξ on $\mathcal{U}$ arises from the submanifold $\mathbf{D}_\Xi \setminus \{\Xi\}$ in $\widehat{\mathcal{U}} \subset \mathbb{PT}$, where $\mathbf{D}_\Xi$ is given by (2.17). In particular, each leaf of N_ξ consists of a γ-plane intersecting $\check{\Xi}$ in an $(m-1)$-plane. Each leaf of $N_\xi^\perp$ consists of a 1-parameter family of γ-planes intersecting in an $(m-1)$-plane. Any two γ-planes contained in two distinct leaves of $N_\xi^\perp$ intersect in an $(m-2)$-plane.*

Proof. The line spanned by $\Xi^{\boldsymbol{\alpha}}$ descends to a point Ξ (i.e., $[\Xi^{\boldsymbol{\alpha}}]$) in $\mathbb{PT}$, and thus singles out a γ-plane $\check{\Xi}$ in $\mathcal{Q}^n$, which by (4.3) can be immediately identified with the zero set of $\xi^{\mathbf{A}}$. Off that set, Proposition 4.1 tells us that N_ξ is integrable and co-integrable. Correspondingly, the conformal Killing spinor $\xi^{\mathbf{A}}$ gives rise to a section $[\xi^{\mathbf{A}}]$ of $\mathbb{F}$, which we can re-express as

$$\Gamma_\xi = \left\{([X^{\mathcal{A}}],[Z^{\boldsymbol{\alpha}}]) \in \mathcal{U} \times \mathbb{PT} \colon Z^{\boldsymbol{\alpha}} = X^{\mathcal{A}} \Xi_{\mathcal{A}}^{\boldsymbol{\alpha}}\right\} \subset \mathbb{F}.$$

Clearly, a point on Γ_ξ descends to a twistor Z on $\mathbf{D}_\xi \setminus \{\Xi\}$ with γ-plane $\check{Z}$ tangent to N_ξ. Thus, for each Z on $\mathbf{D}_\Xi \setminus \{\Xi\}$ in $\widehat{\mathcal{U}} \subset \mathbb{PT}$, $\check{Z}$ is precisely a leaf of N_ξ. The point Ξ itself must be excluded from $\mathbf{D}_\Xi$ since the foliation becomes singular there in the sense the leaves intersect in $\check{\Xi}$. The geometric interpretation of the leaves of N_ξ and $N_\xi^\perp$ follows directly from Theorem 2.11 and Corollary 2.14. In particular, each distinguished curve on $\mathbf{D}_\Xi$ can be identified with a leaf of $N_\xi^\perp$. ■

Local form. Let us re-express the $(m+1)$-plane $\mathbf{D}_\Xi$ as (2.18). We work in a conformal scale for which g_{ab} is the flat metric. Since $\Xi^{\boldsymbol{\alpha}}$ is constant, we can substitute the fields for their constants of integration at the origin. Using (2.30) and $\Xi^{\boldsymbol{\alpha}} = I_{\mathbf{A}}^{\boldsymbol{\alpha}} \mathring{\xi}^{\mathbf{A}} + O_{\mathbf{A}}^{\boldsymbol{\alpha}} \mathring{\zeta}^{\mathbf{A}}$, we obtain, in the obvious notation,

$$\begin{aligned}
&\omega^{a\mathbf{A}} \mathring{\xi}_a^{\mathbf{B}} + 2\mathring{\xi}^{\mathbf{A}} \omega^{\mathbf{B}} - \omega^{\mathbf{A}} \mathring{\xi}^{\mathbf{B}} = 0, \\
&\pi^{a\mathbf{A}} \mathring{\zeta}_a{}^{\mathbf{B}} + 2\mathring{\zeta}^{\mathbf{A}} \pi^{\mathbf{B}} - \pi^{A} \mathring{\zeta}^{\mathbf{B}} = 0, \\
&\omega^{a\mathbf{A}} \mathring{\zeta}_a^{\mathbf{B}} + \omega^{\mathbf{A}} \mathring{\zeta}^{\mathbf{B}} + 4\pi^{[\mathbf{A}} \mathring{\xi}^{\mathbf{B}]} = 0, \\
&\pi^{a\mathbf{A}} \mathring{\xi}_a^{\mathbf{B}} + \pi^{\mathbf{A}} \mathring{\xi}^{\mathbf{B}} + 4\omega^{[\mathbf{A}} \mathring{\zeta}^{\mathbf{B}]} = 0.
\end{aligned} \tag{4.4}$$

Evaluating at $\omega^{\mathbf{A}} = \frac{1}{\sqrt{2}} x^a \gamma_{a\mathbf{B}}{}^{\mathbf{A}} \pi^{\mathbf{B}}$, using the second and third of (4.4) together with the purity of $\Xi^{\boldsymbol{\alpha}}$, we find that $\pi^{\mathbf{A}}$ must be proportional to $\xi^{\mathbf{A}} = \mathring{\xi}^{\mathbf{A}} - \frac{1}{\sqrt{2}} x^a \gamma_{a\mathbf{B}}{}^{\mathbf{A}} \mathring{\zeta}^{\mathbf{B}}$ as expected. This solution then satisfies the first and fourth equations.

Let us now work in the coordinate chart $(\mathcal{V}_0, (\omega^0, \omega^A, \pi^A, \pi^{AB}))$ as defined in Section A.1, and write

$$\begin{aligned}
\mathring{\xi}^{\mathbf{A}} &= \mathring{\xi}^0 o^{\mathbf{A}} + \mathrm{i}\frac{1}{2} \mathring{\xi}^A \delta_A^{\mathbf{A}} - \frac{1}{4} \mathring{\xi}^{AB} \delta_{AB}^{\mathbf{A}} + \cdots, \\
\mathring{\zeta}^{\mathbf{A}} &= \frac{1}{\sqrt{2}} \left(\mathrm{i}\mathring{\zeta}^0 o^{\mathbf{A}} + \mathring{\zeta}^A \delta_A^{\mathbf{A}} - \frac{\mathrm{i}}{4\mathring{\xi}^0} \left(\mathring{\xi}^{AB} \mathring{\zeta}^0 - 2\mathring{\xi}^A \mathring{\zeta}^B \right) \delta_{AB}^{\mathbf{A}} + \cdots \right),
\end{aligned} \tag{4.5}$$

where the remaining components of $\mathring{\zeta}^{\mathbf{A}}$ and $\mathring{\xi}^{\mathbf{A}}$ depend only on $\mathring{\zeta}^0$, $\mathring{\zeta}^A$, $\mathring{\xi}^A$ and $\mathring{\xi}^{AB}$ by the purity of $\Xi^{\boldsymbol{\alpha}}$, and where we have assumed $\mathring{\xi}^0 \neq 0$. Substituting (A.7) and (4.5) into the last of equations (4.4) yields

$$\mathring{\xi}^0 \pi^A - \mathring{\xi}^A + \mathring{\zeta}^0 \omega^A - \omega^0 \mathring{\zeta}^A = 0, \qquad \mathring{\xi}^0 \pi^{AB} - \mathring{\xi}^{AB} + 2\omega^{[A} \mathring{\zeta}^{B]} = 0,$$

while the remaining equations do not yield any new information. Now, at every point Z of $\mathbf{D}_\Xi$, the 1-forms

$$\boldsymbol{\beta}^A := \mathring{\xi}^0 \mathrm{d}\pi^A + \mathring{\zeta}^0 \mathrm{d}\omega^A - \mathring{\zeta}^A \mathrm{d}\omega^0, \qquad \boldsymbol{\beta}^{AB} := \mathring{\xi}^0 \mathrm{d}\pi^{AB} + 2\mathrm{d}\omega^{[A}\mathring{\zeta}^{B]},$$

annihilate the vectors tangent to $\mathbf{D}_\Xi$ at Z and the line in D_Z spanned by

$$\boldsymbol{V} := V^0 \boldsymbol{Y} + V^A \boldsymbol{Y}_A, \tag{4.6}$$

where $V^0 := \mathring{\xi}^0 + \frac{1}{2}\mathring{\zeta}^0\omega^0$ and $V^A := \mathring{\zeta}^A + \frac{1}{2}\mathring{\zeta}^0\pi^A$. This corroborates the claims of Theorem 3.6 and Proposition 4.2. Note that the vector field $\boldsymbol{V}$ vanishes at the point $[\Xi^\alpha]$ of $\mathbf{D}_\Xi$. With no loss, we can set $\mathring{\zeta}^0 = -2$. The integral curve, with complex parameter t, of (4.6) passing through the point

$$\begin{aligned}&\left(\omega^0, \omega^A, \pi^A, \pi^{AB}\right)\\ &\quad = \left(\mathring{\xi}^0 + \alpha, -\frac{1}{2}\left(\mathring{\xi}^A + \alpha\mathring{\zeta}^A - \mathring{\xi}^0\alpha^A\right), \mathring{\zeta}^A + \alpha^A, \frac{1}{\mathring{\xi}^0}\left(\mathring{\xi}^{AB} + \mathring{\xi}^{[A}\mathring{\zeta}^{B]}\right) - \alpha^{[A}\mathring{\zeta}^{B]}\right),\end{aligned}$$

for some α, α^A, is given by

$$\begin{aligned}\left(\omega^0(t), \omega^A(t), \pi^A(t), \pi^{AB}(t)\right) &= \left(\mathring{\xi}^0, -\frac{1}{2}\mathring{\xi}^A, \mathring{\zeta}^A, \frac{1}{\mathring{\xi}^0}\left(\mathring{\xi}^{AB} + \mathring{\xi}^{[A}\mathring{\zeta}^{B]}\right)\right)\\ &\quad + \left(\alpha, -\frac{1}{2}\left(\alpha\mathring{\zeta}^A - \mathring{\xi}^0\alpha^A\right), \alpha^A, -\alpha^{[A}\mathring{\zeta}^{B]}\right)\mathrm{e}^{-t},\end{aligned}$$

Writing $A^{\mathcal{A}} = aY^{\mathcal{A}} + A^a Z_a^{\mathcal{A}} + bX^{\mathcal{A}}$ and $A^a = A^A\delta_A^a + A_A\delta^{aA} + A^0u^a$ with

$$\begin{aligned}\alpha &= -a - \frac{1}{2}A^0\mathring{\xi}^0 + \frac{1}{2}A_C\mathring{\xi}^C,\\ \alpha^A &= \frac{1}{2\mathring{\xi}^0}\left(A_C\mathring{\xi}^C\mathring{\zeta}^A - A^0\mathring{\xi}^0\mathring{\zeta}^A - 2\mathring{\xi}^{AB}A_B - A^0\mathring{\xi}^A - 2\mathring{\xi}^0A^A\right),\\ b &= \frac{1}{\mathring{\xi}^0}\left(A^0 - \mathring{\zeta}^C A_C\right),\end{aligned}$$

one can recast this integral curve tractorially as $Z^{\boldsymbol{\alpha}}(t) = \frac{\mathrm{i}}{\sqrt{2}}\left(\mathring{\Xi}^{\boldsymbol{\alpha}} + \frac{\mathrm{i}}{2}\mathrm{e}^{-t}\mathring{A}^{\mathcal{A}}\mathring{\Xi}_{\mathcal{A}}^{\boldsymbol{\alpha}}\right)$, which is one of the distinguished curves of Lemma 2.10 as expected.

4.1.2 Even dimensions

In even dimensions, the story is entirely analogous except for the choice of chirality of the tractor spinor. We leave the details to the reader.

Proposition 4.3. *Let $\Xi^{\boldsymbol{\alpha}'} = (\xi^{\mathbf{A}'}, \zeta^{\mathbf{A}})$ be a constant pure tractor spinor on $\mathcal{Q}^{2m}$, and let $\mathcal{U} := \mathcal{Q}^{2m} \setminus \check{\Xi}$ where $\check{\Xi}$ is the β-plane defined by $\Xi^{\boldsymbol{\alpha}'}$. Then $\xi^{\mathbf{A}'}$ is a pure conformal Killing spinor on $\mathcal{Q}^{2m}$, and its associated null structure N_ξ on $\mathcal{U}$ arises from the submanifold in $\widehat{\mathcal{U}} \subset \mathbb{PT}$ defined by*

$$\Gamma^{(k)}_{\boldsymbol{\alpha}\boldsymbol{\beta}'} Z^{\boldsymbol{\alpha}}\Xi^{\boldsymbol{\beta}'} = 0, \qquad \text{for} \quad k < m, \quad k \equiv m \pmod 2. \tag{4.7}$$

Each leaf of N_ξ consists of an α-plane intersecting $\check{\Xi}$ in an $(m-1)$-plane.

Remark 4.4. In four dimensions, tractor-spinors are always pure, and so almost null structures associated to conformal Killing spinors are always integrable. In this case, the submanifold (4.7) is a complex projective hyperplane in $\mathbb{PT} \cong \mathbb{CP}^3$ given by $\Xi_{\boldsymbol{\alpha}}Z^{\boldsymbol{\alpha}} = 0$ where we have used the canonical isomorphism $\mathbb{PT}^* \cong \mathbb{PT}'$. This example was highly instrumental in the genesis of twistor theory [29]. The null structure arising from the intersection of this submanifold with *real* twistor space generates a shearfree congruence of null geodesics in Minkowski space known as the *Robinson congruence*.

4.2 Conformal Killing–Yano 2-forms

A (holomorphic) *conformal Killing–Yano* (*CKY*) *2-form* on $\mathcal{Q}^n$ is a section σ_{ab} of $\mathcal{O}_{[ab]}[3]$ that satisfies

$$\nabla_a \sigma_{bc} - \mu_{abc} - 2\mathbf{g}_{a[b}\varphi_{c]} = 0, \tag{4.8}$$

where $\mu_{abc} = \nabla_{[a}\sigma_{bc]}$ and $\varphi_a = \frac{1}{n-2}\nabla^b \sigma_{ba}$. The CKY 2-form equation (4.8) is prolonged to the following system

$$\begin{aligned} &\nabla_a \sigma_{bc} - \mu_{abc} - 2\mathbf{g}_{a[b}\varphi_{c]} = 0, \\ &\nabla_a \mu_{bcd} + 3\mathbf{g}_{a[b}\rho_{cd]} + 3\mathrm{P}_{a[b}\sigma_{cd]} = 0, \\ &\nabla_a \varphi_b - \rho_{ab} + \mathrm{P}_a{}^c \sigma_{cb} = 0, \\ &\nabla_a \rho_{bc} - \mathrm{P}_a{}^d \mu_{dbc} + 2\mathrm{P}_{a[b}\varphi_{c]} = 0. \end{aligned} \tag{4.9}$$

This system can be seen to be equivalent to the existence of a parallel tractor 3-form, i.e.,

$$\nabla_a \Sigma_{\mathcal{ABC}} = 0, \tag{4.10}$$

where $\Sigma_{\mathcal{ABC}} := (\sigma_{ab}, \mu_{abc}, \varphi_a, \rho_{ab}) \in \mathcal{O}_{[\mathcal{ABC}]} \cong \mathcal{O}_{[ab]}[3] + (\mathcal{O}_{[abc]}[3] \oplus \mathcal{O}_a[1]) + \mathcal{O}_{[ab]}[1]$. For an arbitrary conformal manifold, equation (4.10) no longer holds in general, and necessitates the addition of a 'deformation' term as explained in [15].

In flat space, i.e., with $\mathrm{P}_{ab} = 0$, we can integrate equations (4.9) to obtain

$$\begin{aligned} &\sigma_{ab} = \mathring{\sigma}_{ab} + 2x_{[a}\mathring{\varphi}_{b]} + \mathring{\mu}_{abc}x^c - 2\left(x_{[a}\mathring{\rho}_{b]c}x^c + \tfrac{1}{4}\left(x^c x_c\right)\mathring{\rho}_{ab}\right), \\ &\mu_{abc} = \mathring{\mu}_{abc} - 3x_{[a}\mathring{\rho}_{bc]}, \\ &\varphi_a = \mathring{\varphi}_a - \mathring{\rho}_{ab}x^b, \\ &\rho_{ab} = \mathring{\rho}_{ab}, \end{aligned} \tag{4.11}$$

for some constants $\mathring{\sigma}_{ab}$, $\mathring{\mu}_{abc}$, $\mathring{\varphi}_a$ and $\mathring{\rho}_{ab}$.

Remark 4.5. In three dimensions, conformal Killing–Yano 2-forms are Hodge dual to conformal Killing vector fields. These latter are in one-to-one correspondence with parallel sections of tractor 2-forms.

In four dimensions, a 2-form σ_{ab} is a CKY 2-form if and only if its self-dual part σ^+_{ab} and its anti-self-dual part σ^-_{ab} are CKY 2-forms, with, in the obvious notation, $\mu^\pm_{abc} = (*\varphi^\pm)_{abc}$. Self-duality obviously carries over to tractor 3-forms.

4.2.1 Eigenspinors of a 2-form

Let us first assume $n = 2m+1$. We recall that an *eigenspinor* $\xi^{\mathbf{A}}$ of a 2-form σ_{ab} is a spinor satisfying

$$\sigma_{ab}\gamma^{ab}{}_{\mathbf{C}}{}^{[\mathbf{A}}\xi^{\mathbf{B}]}\xi^{\mathbf{C}} = 0, \tag{4.12}$$

i.e., $\sigma_{ab}\gamma^{ab}{}_{\mathbf{C}}{}^{\mathbf{A}}\xi^{\mathbf{C}} = \lambda\xi^{\mathbf{A}}$ for some function λ. Here, $\gamma^{ab}{}_{\mathbf{C}}{}^{\mathbf{A}} := \gamma^{[a}{}_{\mathbf{C}}{}^{\mathbf{B}}\gamma^{b]}{}_{\mathbf{B}}{}^{\mathbf{A}}$. When $\xi^{\mathbf{A}}$ is pure, another convenient way to express the eigenspinor equation (4.12) is given by

$$\sigma^{ab}\gamma^{(m+1)}_{abc_3\ldots c_{m+1}\mathbf{AB}}\xi^{\mathbf{A}}\xi^{\mathbf{B}} = 0.$$

Therefore, to any 2-form σ_{ab}, we can associate a complex submanifold of $\mathbb{F}$ given by the graph

$$\Gamma_\sigma := \left\{\left(x^a, [\pi^{\mathbf{A}}]\right) \in \mathbb{CE}^n \times \mathbb{PT}_{(2m-1)} \colon \sigma^{ab}\gamma^{(m+1)}_{abc_3\ldots c_{m+1}\mathbf{AB}}\pi^{\mathbf{A}}\pi^{\mathbf{B}} = 0\right\}. \tag{4.13}$$

For σ_{ab} generic, this submanifold will have many connected components, each of which corresponding to a local section of $\mathbb{F} \to \mathcal{Q}^{2m+1}$, i.e., a projective pure spinor field that is an eigenspinor of σ_{ab}. To be precise, in $2m+1$ dimensions, a generic 2-form σ_{ab} viewed as an endomorphism $\sigma_a{}^b$ of the tangent bundle, always has m distinct pairs of non-zero eigenvalues opposite to each other, i.e., $(\lambda, -\lambda)$, and a zero eigenvalue. In this case, a generic 2-form viewed as an element of the Clifford algebra has 2^m distinct eigenvalues, and thus 2^m distinct eigenspaces, all of whose elements are pure [26].

When $n = 2m$, the analysis is very similar: the pure eigenspinor equation is now

$$\sigma^{ab}\gamma^{(m)}_{abc_3\dots c_m\mathbf{A}'\mathbf{B}'}\xi^{\mathbf{A}'}\xi^{\mathbf{B}'} = 0,$$

and similarity for spinors of the opposite chirality. Such a 2-form generically has m distinct pairs of non-zero eigenvalues opposite to each other, and as an element of the Clifford algebra, has 2^m eigenspaces that split into two sets of 2^{m-1} eigenspaces according to the chirality of the eigenspinors. The eigenspinor equation lifts to a submanifold $\Gamma_\sigma := \{(x^a, [\pi^{\mathbf{A}'}]) \in \mathbb{CE}^n \times \mathbb{PT}_{(2m-2)} \colon \sigma^{ab}\gamma^{(m)}_{abc_3\dots c_m\mathbf{A}'\mathbf{B}'}\pi^{\mathbf{A}'}\pi^{\mathbf{B}'} = 0\}$ of $\mathbb{F}$, whose connected components correspond to the distinct primed spinor eigenspaces of σ_{ab}.

4.2.2 The null structures of a conformal Killing–Yano 2-forms

The next question to address is when the almost null structure of an eigenspinor of a 2-form is integrable and co-integrable.

Proposition 4.6 ([26])**.** *Let σ_{ab} be a generic conformal Killing 2-form on $\mathcal{Q}^n$ (or any complex Riemannian manifold). Let $\mu_{abc} := \nabla_{[a}\sigma_{bc]}$. Let N be the almost null structure of some eigenspinor of σ_{ab}, and suppose that $\mu_{abc}X^aY^bZ^c = 0$ for any sections X^a, Y^a, Z^a of $N^\perp$. Then N is integrable and, when n is odd, co-integrable too.*

In the light of Theorems 3.6 and 3.10, the foliations arising from the eigenspinors of a CKY 2-form σ_{ab} can be encoded as complex submanifolds of the twistor space $\mathbb{PT}$ of $\mathcal{Q}^n$. As we shall see in a moment, these submanifolds can be constructed from the corresponding tractor $\Sigma_{\mathcal{ABC}}$.

The additional condition on μ_{abc} in Proposition 4.6 can also be understood in terms of the graph of a connected component of Γ_σ defined by (4.13). For such a graph to descend to a complex submanifold of $\mathbb{PT}$, its defining equations should be annihilated by the vectors tangent to $\mathbb{F} \to \mathbb{PT}$. Such a condition, in odd dimensions, can be expressed as $0 = \pi^{[\mathbf{C}}\pi^{c\mathbf{D}]}\nabla_c(\sigma_{ab}\pi^{a\mathbf{A}}\pi^{b\mathbf{B}})$, and using (4.8) gives $\mu_{abc}\pi^{a\mathbf{A}}\pi^{b\mathbf{B}}\pi^{b\mathbf{C}} = 0$. Thus, we shall be interested in the local sections of $\mathbb{F} \to \mathcal{Q}^n$ defined by

$$\begin{aligned}\Gamma_{\sigma,\mu} := \big\{ \big(x^a, \big[\pi^{\mathbf{A}}\big]\big) \in \mathbb{CE}^n \times \mathbb{PT}_{(2m-1)} \colon \\ \sigma^{ab}\gamma^{(m+1)}_{abc_3\dots c_{m+1}\mathbf{AB}}\pi^{\mathbf{A}}\pi^{\mathbf{B}} = 0,\ \mu^{abc}\gamma^{(m+1)}_{abcd_4\dots d_{m+1}\mathbf{AB}}\pi^{\mathbf{A}}\pi^{\mathbf{B}} = 0\big\}.\end{aligned} \tag{4.14}$$

In even dimensions, this is entirely analogous except that (4.14) is now

$$\begin{aligned}\Gamma_{\sigma,\mu} := \big\{ \big(x^a, \big[\pi^{\mathbf{A}'}\big]\big) \in \mathbb{CE}^n \times \mathbb{PT}_{(2m-2)} \colon \\ \sigma^{ab}\gamma^{(m)}_{abc_3\dots c_m\mathbf{A}'\mathbf{B}'}\pi^{\mathbf{A}'}\pi^{\mathbf{B}'} = 0,\ \mu^{abc}\gamma^{(m)}_{abcd_4\dots d_m\mathbf{A}'\mathbf{B}'}\pi^{\mathbf{A}'}\pi^{\mathbf{B}'} = 0\big\}.\end{aligned}$$

Proposition 4.7. *Set $n = 2m + \epsilon$, where $\epsilon \in \{0, 1\}$. Let σ_{ab} be a generic conformal Killing–Yano 2-form on some open subset $\mathcal{U}$ of $\mathcal{Q}^n$, with associated tractor 3-form $\Sigma_{\mathcal{ABC}}$. Then if the almost null structure associated to some eigenspinor of σ_{ab} is integrable and co-integrable, it must arise from the submanifold in $\widehat{\mathcal{U}} \subset \mathbb{PT}$ defined by*

$$\Sigma^{\mathcal{ABC}}\Gamma^{(m+1+\epsilon)}_{\mathcal{ABCD}_4\dots\mathcal{D}_{m+1+\epsilon}\boldsymbol{\alpha\beta}}Z^{\boldsymbol{\alpha}}Z^{\boldsymbol{\beta}} = 0. \tag{4.15}$$

Proof. We focus on the odd-dimensional case only, and leave the even-dimensional case to the reader. Let us write

$$\Sigma_{\mathcal{ABC}} = 3Y_{[\mathcal{A}}Z^b_{\mathcal{B}}Z^c_{\mathcal{C}]}\sigma_{bc} + \left(Z^a_{\mathcal{A}}Z^b_{\mathcal{B}}Z^c_{\mathcal{C}}\mu_{abc} + 6X_{[\mathcal{A}}Y_{\mathcal{B}}Z^c_{\mathcal{C}]}\varphi_c\right) + 3X_{[\mathcal{A}}Z^b_{\mathcal{B}}Z^c_{\mathcal{C}]}\rho_{bc}.$$

Since $\Sigma_{\mathcal{ABC}}$ is constant, we can substitute the fields for their constants of integration at the origin, $\mathring{\sigma}_{ab}$, $\mathring{\mu}_{abc}$, $\mathring{\varphi}_a$ and $\mathring{\rho}_{ab}$, so that using (2.30) we can re-express (4.15) as

$$\begin{aligned}
0 &= -3\sqrt{2}\mathring{\sigma}^{ab}\gamma^{(m+1)}_{abd_4\dots d_{m+2}\mathbf{AB}}\pi^{\mathbf{A}}\pi^{\mathbf{B}} + 2\mathring{\mu}^{abc}\gamma^{(m+2)}_{abcd_4\dots d_{m+2}\mathbf{AB}}\omega^{\mathbf{A}}\pi^{\mathbf{B}} - 12\mathring{\varphi}^a\gamma^{(m)}_{ad_4\dots d_{m+2}\mathbf{AB}}\omega^{\mathbf{A}}\pi^{\mathbf{B}} \\
&\quad + 3\sqrt{2}\mathring{\rho}^{ab}\gamma^{(m+1)}_{abd_4\dots d_{m+2}\mathbf{AB}}\omega^{\mathbf{A}}\omega^{\mathbf{B}}, \\
0 &= \sqrt{2}\mathring{\mu}^{abc}\gamma^{(m+1)}_{abcd_4\dots d_{m+1}\mathbf{AB}}\pi^{\mathbf{A}}\pi^{\mathbf{B}} - 6\mathring{\rho}^{ab}\gamma^{(m)}_{abd_4\dots d_{m+1}\mathbf{AB}}\omega^{\mathbf{A}}\pi^{\mathbf{B}}, \\
0 &= -\sqrt{2}\mathring{\mu}^{abc}\gamma^{(m+1)}_{abcd_4\dots d_{m+1}\mathbf{AB}}\omega^{\mathbf{A}}\omega^{\mathbf{B}} + 6\mathring{\sigma}^{ab}\gamma^{(m)}_{abd_4\dots d_{m+1}\mathbf{AB}}\omega^{\mathbf{A}}\pi^{\mathbf{B}}, \\
0 &= 2\mathring{\mu}^{abc}\gamma^{(m)}_{abcd_4\dots d_m\mathbf{AB}}\omega^{\mathbf{A}}\pi^{\mathbf{B}}.
\end{aligned}$$

Evaluating this system of equations on the intersection of (4.15) and $\widehat{\mathcal{U}}$ amounts to setting $\omega^{\mathbf{A}} = \frac{1}{\sqrt{2}}x^a\gamma_{a\mathbf{B}}{}^{\mathbf{A}}\pi^{\mathbf{B}}$, and we find, after some algebraic manipulations,

$$\begin{aligned}
0 &= -3\sqrt{2}\left(\sigma^{ab}\gamma^{(m+1)}_{abd_4\dots d_{m+2}\mathbf{AB}}\pi^{\mathbf{A}}\pi^{\mathbf{B}}\right) + \sqrt{2}(m-1)\left(x_{[d_4|}\mu^{abc}\gamma^{(m+1)}_{abc|d_5\dots d_{m+2}]\mathbf{AB}}\pi^{\mathbf{A}}\pi^{\mathbf{B}}\right), \\
0 &= \sqrt{2}\mu^{abc}\gamma^{(m+1)}_{abcd_4\dots d_{m+1}\mathbf{AB}}\pi^{\mathbf{A}}\pi^{\mathbf{B}}, \\
0 &= -\frac{(x^e x_e)}{\sqrt{2}}\mu^{abc}\gamma^{(m+1)}_{abcd_4\dots d_{m+1}\mathbf{AB}}\pi^{\mathbf{A}}\pi^{\mathbf{B}} + 3\sqrt{2}\sigma^{ab}x^c\gamma^{(m+1)}_{abcd_4\dots d_{m+1}\mathbf{AB}}\pi^{\mathbf{B}}\pi^{\mathbf{B}} \\
&\quad + \sqrt{2}(m-2)x_{[d_4|}\mu^{abc}x^f\gamma^{(m+1)}_{abcf|d_5\dots d_{m+1}]\mathbf{AB}}\pi^{\mathbf{B}}\pi^{\mathbf{B}}, \\
0 &= \sqrt{2}\mu^{abc}x^d\gamma^{(m+1)}_{abcde_5\dots e_{m+1}\mathbf{AB}}\pi^{\mathbf{A}}\pi^{\mathbf{B}},
\end{aligned}$$

where we have made use of (4.11) and the identity

$$\tfrac{1}{4}\left(x^c\gamma_{c\mathbf{C}}{}^{\mathbf{A}}\right)\left(\mathring{\rho}_{ab}\gamma^{ab}{}_{\mathbf{A}}{}^{\mathbf{B}}\right)\left(x^d\gamma_{d\mathbf{B}}{}^{\mathbf{D}}\right) = \left(x_a\mathring{\rho}_{bc}x^c + \tfrac{1}{4}(x^c x_c)\mathring{\rho}_{ab}\right)\gamma^{ab}{}_{\mathbf{C}}{}^{\mathbf{D}}.$$

In particular, we immediately recover, that on the intersection of the twistor submanifold (4.15) with $\widehat{\mathcal{U}}$,

$$\sigma^{ab}\gamma^{(m+1)}_{abc_3\dots c_{m+1}\mathbf{AB}}\pi^{\mathbf{A}}\pi^{\mathbf{B}} = 0, \qquad \mu^{abc}\gamma^{(m+1)}_{abcd_4\dots d_{m+1}\mathbf{AB}}\pi^{\mathbf{A}}\pi^{\mathbf{B}} = 0.$$

But these are precisely the zero set (4.14) corresponding to the eigenspinors of σ_{ab}. ■

Remark 4.8. In three dimensions, the twistor submanifold is simply a smooth quadric in $\mathbb{PT} \cong \mathbb{CP}^3$.

In four dimensions, the submanifold (4.15) restricts to an anti-self-dual tractor 3-form $\Sigma^-_{\mathcal{ABC}}$ corresponding to a self-dual CKY 2-form σ_{ab}. Setting $\Sigma^-_{\boldsymbol{\alpha\beta}} := \Sigma^-_{\mathcal{ABC}}\Gamma^{\mathcal{ABC}}{}_{\boldsymbol{\alpha\beta}}$, we recover the quadratic polynomial $\Sigma^-_{\boldsymbol{\alpha\beta}}Z^{\boldsymbol{\alpha}}Z^{\boldsymbol{\beta}} = 0$ given in [31]. Under appropriate reality conditions, this submanifold produces a shearfree congruence of null geodesics in Minkowski space known as the *Kerr congruence*. A suitable perturbation of Minkowski space by the generator of such a congruence leads to the solution of Einstein's equations known as the *Kerr metric* [22, 23]. A Euclidean analogue is also given in [32].

In six dimensions, we have a splitting of $\mu_{abc} = \mu^+_{abc} + \mu^-_{abc}$ into a self-dual part and an anti-self-dual part. Since $\xi^{a\mathbf{A}}\xi^{b\mathbf{B}}\xi^{c\mathbf{C}}\mathring{\mu}^+_{abc} = 0$ for any $\xi^{\mathbf{A}'}$, the obstruction to the integrability of a positive eigenspinor of a generic CKY 2-form σ_{ab} is the anti-self-dual part μ^-_{abc} of μ_{abc}.

5 Curved spaces

Let $\mathcal{M}$ be a complex manifold equipped with a holomorphic non-degenerate symmetric bilinear form g_{ab}. The pair $(\mathcal{M}, g_{ab})$ will be referred to as a *complex Riemannian manifold*. We assume that $\mathcal{M}$ is equipped with a holomorphic complex orientation and a holomorphic spin structure. We may also assume that one merely has a holomorphic conformal structure rather than a metric one. For definiteness, we set $n = 2m + 1$ as the dimension of $\mathcal{M}$. The analogue of the correspondence space $\mathbb{F}$ is the projective pure spinor bundle $\nu\colon \mathcal{F} \to \mathcal{M}$: for any $x \in \mathcal{M}$, a point p in a fiber $\nu^{-1}(x)$ is a totally null m-plane in $\mathrm{T}_x\mathcal{M}$, and sections of $\mathcal{F}$ are almost null structures on $\mathcal{M}$. To define the twistor space of $(\mathcal{M}, g_{ab})$, one must replace the notion of γ-plane by that of γ-*surface*, i.e., an m-dimensional complex submanifold of $\mathcal{M}$ such that at any point of such a surface, its tangent space is totally null with respect to the metric and totally geodetic with respect to the metric connection. The integrability condition for the existence of a γ-surface $\mathcal{N}$ through a point x is [35]

$$C_{abcd}X^aY^bZ^cW^d = 0, \qquad \text{for all} \quad X^a, Y^a, Z^c \in \mathrm{T}_x\mathcal{N}, \quad W^a \in \mathrm{T}_x\mathcal{N}. \tag{5.1}$$

If we define the twistor space of $(\mathcal{M}, g_{ab})$ to be the $\frac{1}{2}(m+1)(m+2)$-dimensional complex manifold parametrising the γ-surfaces of $(\mathcal{M}, g_{ab})$, we must have a $\frac{1}{2}m(m+1)$-parameter family of γ-surfaces through each point of $\mathcal{M}$. From the integrability condition (5.1), we must conclude that for the twistor space of $(\mathcal{M}, g_{ab})$ to exist, $(\mathcal{M}, g_{ab})$ must be conformally flat in odd dimensions greater than three. In even dimensions the story is similar: one replaces the notion of α-plane by that of an α-surface in the obvious way. We then find that for $(\mathcal{M}, g_{ab})$ to admit a twistor space, it must be conformally flat in even dimensions greater than four, and anti-self-dual in dimension four.

Curved twistor theory in dimensions three and four is pretty well-known. In dimension four, we have the *Penrose correspondence*, whereby twistor space is a three-dimensional complex manifold containing a complete analytic family of rational curves with normal bundle $\mathcal{O}(1)\oplus\mathcal{O}(1)$ parameterised by the points of an anti-self-dual complex Riemannian manifold [30]. In dimension three, the *LeBrun correspondence* can be seen as a special case of the Penrose correspondence: if we endow twistor space with a holomorphic 'twisted' contact structure, then a three-dimensional conformal manifold arises as the umbilic conformal infinity of an Einstein anti-self-dual four-dimensional complex Riemannian manifold [25]. Finally, the mini-twistor space in the *Hitchin correspondence* is a two-dimensional complex manifold containing a complete analytic family of rational curves with normal bundle $\mathcal{O}(2)$ parameterised by the points of an Einstein–Weyl space [18, 21].

Theorems 3.5 (or 3.6), 3.8 and 3.10 can be adapted to the curved setting by interpreting the leaf space of a totally geodetic null foliation as a complex submanifold of twistor space. See [8] for an application of a 'curved' Theorem 3.8 in the investigation of three-dimensional Einstein–Weyl spaces.

A Coordinate charts on twistor space and correspondence space

In this appendix, we construct atlases of coordinates charts covering $\mathbb{PT}$ and $\mathbb{F}$. We refer to the setup of Section 2 throughout. In particular, we work with the splittings (2.3), (2.8) and (2.12).

A.1 Odd dimensions

Let us introduce a splitting of $\mathbb{V}_0$ as

$$\mathbb{V}_0 \cong \mathbb{W} \oplus \mathbb{W}^* \oplus \mathbb{U}, \tag{A.1}$$

where $\mathbb{W} \cong \mathbb{C}^m$ is a totally null m-plane of $(\mathbb{V}_0, g_{ab})$, and $\mathbb{U} \cong \mathbb{C}$ is the one-dimensional complement of $\mathbb{W} \oplus \mathbb{W}^*$ in $\mathbb{V}_0$. Elements of $\mathbb{W}$ and $\mathbb{W}^*$ will carry upstairs and downstairs upper-case Roman indices respectively, i.e., $V^A \in \mathbb{W}$, and $W_A \in \mathbb{W}^*$. The vector subspace $\mathbb{U}$ will be spanned by a unit vector u^a. Denote by δ^{aA} the injector from $\mathbb{W}^*$ to $\mathbb{V}_0$, and δ^a_A the injector from $\mathbb{W}$ to $\mathbb{V}_0$ satisfying $\delta^A_a \delta^a_B = \delta^A_B$, where δ^A_B is the identity on $\mathbb{W}$ and $\mathbb{W}^*$. We shall think of $\{\delta^{aA}\}$ as a basis for $\mathbb{W}$ with dual basis $\{\delta^a_A\}$ for $\mathbb{W}^*$. The splitting (A.1) allows us to identify the two copies $\mathbb{S}_{\pm\frac{1}{2}}$ of the spinor space of $(\mathbb{V}_0, g_{ab})$ with its Fock representation, i.e.,

$$\mathbb{S}_{\pm\frac{1}{2}} \cong \wedge^m \mathbb{W} \oplus \wedge^{m-1}\mathbb{W} \oplus \cdots \oplus \mathbb{W} \oplus \mathbb{C}.$$

This is essentially the strategy adopted in Section 2.2 for the spinors of $\mathrm{Spin}(2m+3, \mathbb{C})$. To realise it explicitly, we proceed as follows: let $o^{\mathbf{A}}$ be a (pure) spinor annihilating $\mathbb{W}$ so that $o^{\mathbf{A}}$ is a spanning element of $\wedge^m \mathbb{W}$. A (Fock) basis for $\mathbb{S}_{\pm\frac{1}{2}}$ can then be produced by acting on $o^{\mathbf{A}}$ by basis elements of $\wedge^\bullet \mathbb{W}^*$, i.e.,

$$\mathbb{S}_{\pm\frac{1}{2}} = \left\langle o^{\mathbf{A}}, \delta^{\mathbf{A}}_{A_1}, \delta^{\mathbf{A}}_{A_1 A_2}, \dots \right\rangle, \tag{A.2}$$

where

$$\delta^{\mathbf{A}}_{A_1\dots A_k} := \delta^{a_1}_{[A_1} \cdots \delta^{a_k}_{A_k]} o^{\mathbf{A_0}} \gamma_{a_1 \mathbf{A_0}}{}^{\mathbf{A_1}} \cdots \gamma_{a_k \mathbf{A_{k-1}}}{}^{\mathbf{A}},$$

for each $k = 1, \dots, m$. With this notation, the Clifford multiplication of $\mathbb{V}_0 \subset \mathcal{C}\ell(\mathbb{V}_0, g_{ab})$ on $\mathbb{S}_{-\frac{1}{2}}$ is given explicitly by

$$\begin{aligned}
&\delta^{aA}\gamma_{a\mathbf{B}}{}^{\mathbf{C}}\delta^{\mathbf{B}}_{B_1\dots B_p} = -2p\delta^{\mathbf{C}}_{[B_1\dots B_{p-1}}\delta^A_{B_p]}, && \delta^a_A\gamma_{a\mathbf{B}}{}^{\mathbf{C}}\delta^{\mathbf{B}}_{B_1\dots B_p} = \delta^{\mathbf{C}}_{B_1\dots B_p A},\\
&u^a\gamma_{a\mathbf{B}}{}^{\mathbf{C}}o^{\mathbf{B}} = \mathrm{i}o^{\mathbf{C}}, && u^a\gamma_{a\mathbf{B}}{}^{\mathbf{C}}\delta^{\mathbf{B}}_{B_1\dots B_p} = (-1)^p\mathrm{i}\delta^{\mathbf{C}}_{B_1\dots B_p}.
\end{aligned} \tag{A.3}$$

An arbitrary spinor $\pi^{\mathbf{A}}$ in $\mathbb{S}_{\frac{1}{2}}$ can then be expressed in the Fock basis (A.2) as

$$\begin{aligned}
\pi^{\mathbf{A}} &= \pi^0 o^{\mathbf{A}} + \sum_{k=1}^{[m/2]} \left(-\frac{1}{4}\right)^k \frac{1}{k!}\pi^{A_1\dots A_{2k}}\delta^{\mathbf{A}}_{A_1\dots A_{2k}}\\
&\quad + \frac{\mathrm{i}}{2}\sum_{k=0}^{[m/2]} \left(-\frac{1}{4}\right)^k \frac{1}{k!}\pi^{A_1\dots A_{2k+1}}\delta^{\mathbf{A}}_{A_1\dots A_{2k+1}}, \qquad m > 1,\\
\pi^{\mathbf{A}} &= \pi^0 o^{\mathbf{A}} + \tfrac{\mathrm{i}}{2}\pi^A\delta^{\mathbf{A}}_A, \qquad m = 1,
\end{aligned} \tag{A.4}$$

where $\left[\frac{m}{2}\right]$ is $\frac{m}{2}$ when m is even, $\frac{m-1}{2}$ when m is odd, and π^0 and $\pi^{A_1A_2\dots A_k} = \pi^{[A_1A_2\dots A_k]}$ are the components of $\pi^{\mathbf{A}}$. Let us now assume that $\pi^{\mathbf{A}}$ is pure, i.e., satisfies (2.31a). When $m = 1$ and 2, there are no algebraic constraints, and the space of projective pure spinors is isomorphic to $\mathbb{CP}^1$ and $\mathbb{CP}^3$ respectively. When $m > 2$, the pure spinor variety is then given by the complete intersections of the quadric hypersurfaces

$$\begin{aligned}
\pi^0\pi^{A_1A_2\dots A_{2k+1}} &= \pi^{[A_1}\pi^{A_2\dots A_{2k+1}]}, && k = 1, \dots, [m/2],\\
\pi^0\pi^{A_1A_2A_3\dots A_{2k}} &= \pi^{[A_1A_2}\pi^{A_3\dots A_{2k}]}, && k = 1, \dots, [m/2],
\end{aligned} \tag{A.5}$$

in $\mathbb{CP}^{2^m-1}$. We can therefore cover a fibre of $\mathbb{F}$ with 2^m open subsets $\mathcal{U}_0$, $\mathcal{U}_{A_1\dots A_k}$, where $\pi^0 \neq 0$ on $\mathcal{U}_0$ and $\pi^{A_1\dots A_k} \neq 0$ on $\mathcal{U}_{A_1\dots A_k}$, and thus obtain 2^m coordinate charts in the obvious way. This induces an atlas of charts on $\mathbb{F}_{\mathbb{CE}^n}$ given by the open subsets $\mathbb{CE}^n \times \mathcal{U}_0$, $\mathbb{CE}^n \times \mathcal{U}_{A_1\dots A_k}$.

Let us now write the spinor $\omega^{\mathbf{A}}$ in $\mathbb{S}_{-\frac{1}{2}}$ in the Fock basis as

$$\omega^{\mathbf{A}} = \frac{\mathrm{i}}{\sqrt{2}}\omega^0 o^{\mathbf{A}} + \frac{1}{\sqrt{2}}\omega^A \delta_A^{\mathbf{A}}, \qquad m=1,$$

$$\omega^{\mathbf{A}} = \frac{\mathrm{i}}{\sqrt{2}}\omega^0 o^{\mathbf{A}} + \frac{\mathrm{i}}{2\sqrt{2}}\sum_{k=1}^{[m/2]}\left(-\frac{1}{4}\right)^{k-1}\frac{1}{(k-1)!}\omega^{A_1\dots A_{2k}}\delta_{A_1A_2\dots A_{2k}}^{\mathbf{A}}$$
$$+\frac{1}{\sqrt{2}}\sum_{k=0}^{[m/2]}\left(-\frac{1}{4}\right)^{k}\frac{1}{k!}\omega^{A_1\dots A_{2k+1}}\delta_{A_1\dots A_{2k+1}}^{\mathbf{A}}, \qquad m>1, \tag{A.6}$$

where ω^0 and $\omega^{A_1A_2\dots A_k} = \omega^{[A_1A_2\dots A_k]}$ are the components of $\omega^{\mathbf{A}}$. The condition for $Z^{\boldsymbol{\alpha}} = (\omega^{\mathbf{A}}, \pi^{\mathbf{A}})$ to be pure, so that (2.31) hold, is that the relations

$$\pi^0\omega^{A_1\dots A_{2k-1}A_{2k}} = \pi^{[A_1\dots A_{2k-1}}\omega^{A_{2k}]} - \tfrac{1}{2k}\pi^{A_1\dots A_{2k}}\omega^0,$$
$$\pi^0\omega^{A_1\dots A_{2k}A_{2k+1}} = \pi^{[A_1\dots A_{2k}}\omega^{A_{2k+1}]},$$

hold for $k \geq 1$ when $m>1$, and that (A.5) hold too when $m>2$. Hence, we can cover $\mathbb{PT}_{\backslash\widehat{\infty}}$ with 2^m open subsets $\mathcal{V}_0$, where $\pi^0 \neq 0$, and $\mathcal{V}_{A_1\dots A_k}$ where $\pi^{A_1\dots A_k}\neq 0$ in the obvious way. Coordinates on the complement $\widehat{\infty}$ parametrised by $[\omega^{\mathbf{A}}, 0]$ satisfy the conditions

$$\omega^0\omega^{A_1\dots A_{2k}A_{2k+1}} = -2k\omega^{[A_1\dots A_{2k}}\omega^{A_{2k+1}]}, \qquad \omega^{[A_1\dots A_{2k-1}}\omega^{A_{2k}]} = 0.$$

Let (z^A, z_A, u) be null coordinates on $\mathbb{CE}^n$ in the sense that $x^a = z^A\delta_A^a + z_A\delta^{aA} + uu^a$ so that the flat metric on $\mathbb{CE}^n$ takes the form $\boldsymbol{g} = 2\mathrm{d}z^A \odot \mathrm{d}z_A + \mathrm{d}u\otimes\mathrm{d}u$. Then the incidence relation (2.33) reads

$$\omega^0 = \pi^0 u - \pi^B z_B,$$
$$\omega^A = \pi^0 z^A + \pi^{AB}z_B + \tfrac{1}{2}\pi^A u,$$
$$\omega^{A_1\dots A_{2k-1}A_{2k}} = \pi^{[A_1\dots A_{2k-1}}z^{A_{2k}]} + \tfrac{4k+2}{4k}\pi^{A_1\dots A_{2k-1}A_{2k}A_{2k+1}}z_{A_{2k+1}} - \tfrac{1}{2k}\pi^{A_1\dots A_{2k}}u,$$
$$\omega^{A_1\dots A_{2k}A_{2k+1}} = \pi^{[A_1\dots A_{2k}}z^{A_{2k+1}]} + \pi^{A_1\dots A_{2k}A_{2k+1}A_{2k+2}}z_{A_{2k+2}} + \tfrac{1}{2}\pi^{A_1\dots A_{2k+1}}u.$$

We now work in the chart $\mathcal{U}_0$, and since $\pi^0\neq 0$ there, we can set with no loss of generality $\pi^0 = 1$. Let (x,π) be a point in $\mathbb{F}_{\mathbb{CE}^n}$ and let $(\mathcal{U}_0, (\pi^A, \pi^{AB}))$ be a coordinate chart containing $\pi\in\mathbb{F}_x$. Let (ω,π) be the image of (x,π) under the projection $\mu\colon \mathbb{F}\to\mathbb{PT}$ so that $(\mathcal{V}_0, (\omega^0, \omega^A, \pi^A, \pi^{AB}))$ is a coordinate chart containing (ω,π). Then, in these charts, (A.6) and (A.4) reduce to

$$\omega^{\mathbf{A}} = \tfrac{1}{\sqrt{2}}\big(\mathrm{i}\omega^0 o^{\mathbf{A}} + \omega^A\delta_A^{\mathbf{A}} - \tfrac{\mathrm{i}}{4}\big(\pi^{AB}\omega^0 - 2\pi^A\omega^B\big)\delta_{AB}^{\mathbf{A}} + \cdots\big), \tag{A.7a}$$
$$\pi^{\mathbf{A}} = o^{\mathbf{A}} + \tfrac{\mathrm{i}}{2}\pi^A\delta_A^{\mathbf{A}} - \tfrac{1}{4}\pi^{AB}\delta_{AB}^{\mathbf{A}} + \cdots. \tag{A.7b}$$

More succinctly, $\pi^{\mathbf{A}} = \exp(-\frac{1}{4}\pi^{ab}\gamma_{ab\mathbf{B}}{}^{\mathbf{A}})o^{\mathbf{B}}$, where $\pi^{ab} = \pi^{AB}\delta_A^a\delta_B^b + 2\pi^A\delta_A^{[a}u^{b]}$ belongs to the complement of the stabiliser of $o^{\mathbf{A}}$ in $\mathfrak{so}(\mathbb{V}_0, g_{ab})$, i.e., (π^A, π^{AB}) are coordinates on a dense open subset of the homogeneous space P/Q. We can also rewrite $\omega^{\mathbf{A}}$ more compactly in the two alternative forms

$$\omega^{\mathbf{A}} = \tfrac{1}{\sqrt{2}}\big(\omega^A\delta_A^a + \tfrac{1}{2}\omega^0u^a\big)\pi_a^{\mathbf{A}} + \tfrac{\mathrm{i}}{2\sqrt{2}}\omega^0\pi^{\mathbf{A}},$$
$$\omega^{\mathbf{A}} = \tfrac{1}{\sqrt{2}}\omega^a\pi_a^{\mathbf{A}}, \qquad \text{where} \quad \omega^a := \big(\omega^A - \tfrac{1}{2}\omega^0\pi^A\big)\delta_A^a + \omega^0u^a,$$

from which it is easy to check that $\pi^{\mathbf{A}}$ and $\omega^{\mathbf{A}}$ indeed satisfy the conditions given in Lemma 2.19.

Finally, in the coordinate chart $(\mathbb{CE}^n \times \mathcal{U}_0, (z^A, z_A, u; \pi^A, \pi^{AB}))$, we have

$$x^a \pi_a^{\mathbf{A}} = \mathrm{i}\big(u - \pi^B z_B\big) o^{\mathbf{A}} + \big(z^B + \pi^{BC} z_C + \tfrac{1}{2} u \pi^B\big) \delta_B^{\mathbf{A}} + \cdots,$$

so that the incidence relation (2.33) reduces to

$$\omega^A = z^A + \pi^{AB} z_B + \tfrac{1}{2}\pi^A u, \qquad \omega^0 = u - \pi^B z_B. \tag{A.8}$$

Tangent and cotangent spaces. Let us introduce the short-hand notation

$$\partial_A := \frac{\partial}{\partial z^A} = \delta_A^a \nabla_a, \qquad \partial^A := \frac{\partial}{\partial z_A} = \delta^{aA} \nabla_a, \qquad \partial := \frac{\partial}{\partial u} = u^a \nabla_a,$$

so that $\mathrm{T}_{(x,\pi)} \mathcal{Q}^n \cong \mathfrak{p}_{-1} = \langle \partial_A, \partial^A, \partial \rangle$, and define 1-forms

$$\boldsymbol{\alpha}^A := \mathrm{d}\omega^A + \tfrac{1}{2}\pi^A \mathrm{d}\omega^0 - \tfrac{1}{2}\omega^0 \mathrm{d}\pi^A, \qquad \boldsymbol{\alpha}^{AB} := \mathrm{d}\pi^{AB} - \pi^{[A} \mathrm{d}\pi^{B]}, \tag{A.9}$$

and vectors

$$\begin{aligned} \boldsymbol{X}_A &:= \frac{\partial}{\partial \omega^A}, & \boldsymbol{X}_{AB} &:= \frac{\partial}{\partial \pi^{AB}}, \\ \boldsymbol{Y} &:= \frac{\partial}{\partial \omega^0} - \frac{1}{2}\pi^C \frac{\partial}{\partial \omega^C}, & \boldsymbol{Y}_A &:= \frac{\partial}{\partial \pi^A} - \pi^B \frac{\partial}{\partial \pi^{AB}} + \frac{1}{2}\omega^0 \frac{\partial}{\partial \omega^A}. \end{aligned} \tag{A.10}$$

Then bases for the cotangent and tangent spaces of $\mathbb{PT}$ at (ω, π) are given by

$$\begin{aligned} &\mathrm{T}^*_{(\omega,\pi)} \mathbb{PT} \cong \mathfrak{r}_1^* \oplus \mathfrak{r}_2^* = \big\langle \mathrm{d}\omega^0, \mathrm{d}\pi^A \big\rangle \oplus \big\langle \boldsymbol{\alpha}^A, \boldsymbol{\alpha}^{AB} \big\rangle, \\ &\mathrm{T}_{(\omega,\pi)} \mathbb{PT} \cong \mathfrak{r}_{-2} \oplus \mathfrak{r}_{-1} = \big\langle \boldsymbol{X}_A \boldsymbol{X}_{AB} \big\rangle \oplus \big\langle \boldsymbol{Y}, \boldsymbol{Y}_A \big\rangle, \end{aligned}$$

respectively.

Remark A.1. Using (A.7), one can check that the expressions for the set (A.9) of $\frac{1}{2}m(m+1)$ 1-forms are none other than the 1-forms (2.32), and thus (2.25). These forms annihilate the rank-$(m+1)$ canonical distribution D on $\mathbb{PT}$ spanned by $\boldsymbol{Y}$ and $\boldsymbol{Y}_A$. Further, the vector $\boldsymbol{Y}$ clearly coincides with (2.37) to describe mini-twistor space – this can be checked by using transformations (A.7).

Now, define the 1-forms and vectors

$$\begin{aligned} &\boldsymbol{\theta}^A := \mathrm{d}z^A + \big(\pi^{AD} - \tfrac{1}{2}\pi^A \pi^D\big)\mathrm{d}z_D + \pi^A \mathrm{d}u, && \boldsymbol{\theta}^0 := \mathrm{d}u - \pi^C \mathrm{d}z_C, \\ &\boldsymbol{Z}^A := \partial^A + \big(\pi^{AD} - \tfrac{1}{2}\pi^A \pi^D\big)\partial_D + \pi^A \partial, && \boldsymbol{U} := \partial - \pi^D \partial_D, \\ &\boldsymbol{W}_A := \frac{\partial}{\partial \pi^A} - \pi^B \frac{\partial}{\partial \pi^{AB}}. \end{aligned}$$

Then bases for the cotangent and tangent spaces of $\mathbb{F}$ at (x, π) are given by

$$\begin{aligned} &\mathrm{T}^*_{(x,\pi)} \mathbb{F} \cong \mathfrak{q}_1^{*E} \oplus \mathfrak{q}_1^{*F} \oplus \mathfrak{q}_2^{*E} \oplus \mathfrak{q}_2^{*F} \oplus \mathfrak{q}_3^* = \langle \mathrm{d}z_A \rangle \oplus \big\langle \mathrm{d}\pi^A \big\rangle \oplus \langle \boldsymbol{\theta}^0 \rangle \oplus \big\langle \boldsymbol{\alpha}^{AB} \big\rangle \oplus \big\langle \boldsymbol{\theta}^A \big\rangle, \\ &\mathrm{T}_{(x,\pi)} \mathbb{F} \cong \mathfrak{q}_{-3} \oplus \mathfrak{q}_{-2}^F \oplus \mathfrak{q}_{-2}^E \oplus \mathfrak{q}_{-1}^F \oplus \mathfrak{q}_{-1}^E = \langle \partial_A \rangle \oplus \langle \boldsymbol{X}_{AB} \rangle \oplus \langle \boldsymbol{U} \rangle \oplus \langle \boldsymbol{W}_A \rangle \oplus \big\langle \boldsymbol{Z}^A \big\rangle, \end{aligned}$$

respectively.

We note that the coordinates $(\omega^0, \omega^A, \pi^A, \pi^{AB})$ on $\mathcal{V}_0$ are indeed annihilated by the vectors $\boldsymbol{Z}^A$ tangent to the fibres of $\mathbb{F} \to \mathbb{PT}$. Further, the pullback of $\boldsymbol{\alpha}^A$ to $\mathbb{F}$ is given by $\mu^*(\boldsymbol{\alpha}^A) = \boldsymbol{\alpha}^{AB} z_B + \boldsymbol{\theta}^A$, i.e., the annihilator of $\mathrm{D} = \mathrm{T}^{-1}\mathbb{PT}$ pulls back to the annihilator of $\mathrm{T}_E^{-2}\mathbb{F}$ corresponding to $\mathfrak{q}_{-2}^E \oplus \mathfrak{q}_{-1}^F \oplus \mathfrak{q}_{-1}^E$.

Mini-twistor space. By Lemma 2.24, the mini-twistor space $\mathbb{MT}$ of $\mathbb{CE}^n$ is the leaf space of the vector field $\boldsymbol{Y}$ defined by (2.37), given in (A.10) in the coordinate chart $(\mathcal{V}_0, (\omega^0, \omega^A, \pi^A, \pi^{AB}))$. Accordingly, we have a local coordinate chart $(\underline{\mathcal{V}}_0, (\underline{\omega}^A, \pi^{AB}, \pi^A))$ on $\mathbb{MT}$ where

$$\underline{\omega}^A = \omega^A + \tfrac{1}{2}\pi^A\omega^0,$$

which can be seen to be annihilated by $\boldsymbol{Y}$. The incidence relation (2.35) or (2.40) can then be expressed as

$$\underline{\omega}^A = z^A + \left(\pi^{AB} - \tfrac{1}{2}\pi^A\pi^B\right)z_B + \pi^A u,$$

which are indeed annihilated by $\boldsymbol{Z}^A$ and $\boldsymbol{U}$. The tangent space of $\mathbb{MT}$ at a point $(\underline{\omega}, \pi)$ in $\underline{\mathcal{V}}_0$ is clearly

$$\mathrm{T}_{(\underline{\omega},\pi)}\mathbb{MT} = \langle \underline{\boldsymbol{X}}_A, \boldsymbol{X}_{AB}, \boldsymbol{W}_A\rangle, \qquad \text{where} \quad \underline{\boldsymbol{X}}_A := \frac{\partial}{\partial \underline{\omega}^A}.$$

Normal bundle of $\hat{x}$ in $\mathbb{PT}_{\backslash\widehat{\infty}}$. Let x be a point in $\mathbb{CE}^n$. In the chart $(\mathcal{V}_0, (\omega^0, \omega^A, \pi^A, \pi^{AB}))$, the corresponding $\hat{x}$ is given by (A.8). In particular, the 1-forms

$$\boldsymbol{\beta}^A(x) := \mathrm{d}\omega^A - \mathrm{d}\pi^{AB}z_B - \tfrac{1}{2}\mathrm{d}\pi^A u, \qquad \boldsymbol{\beta}^0(x) := \mathrm{d}\omega^0 + \mathrm{d}\pi^B z_B,$$

vanish on $\hat{x}$, and the tangent space of $\hat{x}$ at (ω, π) is spanned by the vectors $\boldsymbol{Y}_A - z_A\boldsymbol{Y}$ and $\boldsymbol{X}_{AB} - z_{[A}\boldsymbol{X}_{B]}$. This distinguishes the m-dimensional subspace $\langle \boldsymbol{Y}_A - z_A\boldsymbol{Y}\rangle$ tangent to both $\hat{x}$ and the canonical distribution D at (ω, π).

A.2 Even dimensions

The local description of $\mathbb{F}$ and $\mathbb{PT}$ in even dimensions can be easily derived from the one above. We split $\mathbb{V}_0$ as $\mathbb{V}_0 \cong \mathbb{W} \oplus \mathbb{W}^*$ where $\mathbb{W} \cong \mathbb{C}^m$ is a totally null m-plane of $(\mathbb{V}_0, g_{ab})$, with adapted basis $\{\delta^{aA}, \delta^a_A\}$. The Fock representations of the irreducible spinor spaces $\mathbb{S}_{-\frac{1}{2}}$ and $\mathbb{S}'_{-\frac{1}{2}}$ on $\mathbb{V}_0$ are given by

$$\mathbb{S}_{\frac{1}{2}} \cong \mathbb{S}'_{-\frac{1}{2}} \cong \wedge^m\mathbb{W} \oplus \wedge^{m-2}\mathbb{W} \oplus \cdots, \qquad \mathbb{S}'_{\frac{1}{2}} \cong \mathbb{S}_{-\frac{1}{2}} \cong \wedge^{m-1}\mathbb{W} \oplus \wedge^{m-3}\mathbb{W} \oplus \cdots.$$

Let $o^{\mathbf{A}'}$ be a (pure) spinor annihilating $\mathbb{W}$. Then bases for $\mathbb{S}_{\frac{1}{2}}$ and $\mathbb{S}_{-\frac{1}{2}}$ can then be produced by acting on $o^{\mathbf{A}'}$ by basis elements of $\wedge^{2k}\mathbb{W}^*$ and of $\wedge^{2k-1}\mathbb{W}^*$. Explicitly,

$$\mathbb{S}_{\frac{1}{2}} = \left\langle o^{\mathbf{A}'}, \delta^{\mathbf{A}'}_{A_1A_2}, \dots\right\rangle, \qquad \mathbb{S}_{-\frac{1}{2}} = \left\langle \delta^{\mathbf{A}}_{A_1}, \delta^{\mathbf{A}}_{A_1A_2A_3}, \dots\right\rangle,$$

where

$$\delta^{\mathbf{A}'}_{A_1\dots A_{2k}} := \delta^{a_1}_{[A_1}\cdots\delta^{a_{2k}}_{A_{2k}]}o^{\mathbf{A}'_0}\gamma_{a_1\mathbf{A}'_0}{}^{\mathbf{A_1}}\cdots\gamma_{a_{2k}\mathbf{A_{2k-1}}}{}^{\mathbf{A}'},$$
$$\delta^{\mathbf{A}}_{A_1\dots A_{2k-1}} := \delta^{a_1}_{[A_1}\cdots\delta^{a_{2k-1}}_{A_{2k-1}]}o^{\mathbf{A}'_0}\gamma_{a_1\mathbf{A}'_0}{}^{\mathbf{A_1}}\cdots\gamma_{a_{2k-1}\mathbf{A}'_{\mathbf{2k-2}}}{}^{\mathbf{A}}.$$

The Clifford action of $\mathbb{V}_0 \subset \mathcal{C}\ell(\mathbb{V}_0, g_{ab})$ on $\mathbb{S}_{\pm\frac{1}{2}}$ follows the same lines as (A.3) with appropriate priming of spinor indices.

Coordinate charts in even dimensions can be obtained from the odd-dimensional case by switching off $\pi^{A_1\dots A_k}$ for all odd k, and $\omega^{A_1\dots A_k}$ for all even k. We therefore have a covering of each fibre of $\mathbb{F}$ by 2^{m-1} open subsets $\mathcal{U}_0, \mathcal{U}_{A_1\dots A_{2k}}$, and a covering of $\mathbb{PT}_{\backslash\widehat{\infty}}$ by 2^{m-1} open subsets $\mathcal{V}_0$, $\mathcal{V}_{A_1\dots A_{2k}}$ in the obvious way. In particular, in $(\mathcal{V}_0, (\omega^A, \pi^{AB}))$, the homogeneous coordinates $[\omega^{\mathbf{A}}, \pi^{\mathbf{A}'}]$ are given by

$$\omega^{\mathbf{A}} = \tfrac{1}{\sqrt{2}}\left(\omega^A\delta^{\mathbf{A}}_A - \tfrac{1}{4}\omega^A\pi^{BC}\delta^{\mathbf{A}}_{ABC} + \cdots\right), \qquad \pi^{\mathbf{A}'} = o^{\mathbf{A}'} - \tfrac{1}{4}\pi^{AB}\delta^{\mathbf{A}'}_{AB} + \cdots.$$

where the former can also be rewritten as $\omega^{\mathbf{A}} = \frac{1}{\sqrt{2}}\omega^a \pi_a^{\mathbf{A}}$ with $\omega^a := \omega^A \delta_A^a$. Finally, the even-dimensional version of the incidence relation (2.33) can be rewritten as $\omega^A = z^A + \pi^{AB} z_B$.

As for the tangent spaces of $\mathcal{Q}^{2m}$, its twistor space and their correspondence space, we find, in the obvious notation, $\mathrm{T}_{(x,\pi)}\mathcal{Q}^n \cong \mathfrak{p}_{-1} = \langle \partial_A, \partial^A, \partial \rangle$, $\mathrm{T}_{(x,\pi)}\mathbb{F} \cong \mathfrak{q}_{-2} \oplus \mathfrak{q}_{-1}^F \oplus \mathfrak{q}_{-1}^E = \langle \partial_A \rangle \oplus \langle \boldsymbol{X}_{AB} \rangle \oplus \langle \boldsymbol{Z}^A \rangle$, and $\mathrm{T}_{(\omega,\pi)}\mathbb{PT} \cong \mathfrak{r}_{-1} = \langle \boldsymbol{X}_A, \boldsymbol{X}_{AB} \rangle$, where $\boldsymbol{Z}^A := \partial^A + \pi^{AB}\partial_B$, $\boldsymbol{X}_{AB} := \frac{\partial}{\partial \pi^{AB}}$, $\boldsymbol{X}_A := \frac{\partial}{\partial \omega^A}$, and so on.

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