A model for an aquatic ecosystem

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Abstract

An ecosystem made of nutrients, plants, detritus and dissolved oxygen is presented. Its equilibria are established. Sufficient conditions for the existence of the coexistence equilibrium are derived and its feasibility is discussed in every detail.

Keywords: aquatic plants, dissolved oxygen, nutrient recycling

1. Introduction

Eutrophication is the large increase of aquatic plants due to the discharge in waterbodies of chemical elements such as nitrogen and phosphorus. Dead aquatic plants generate detritus, exploiting large amount of dissolved oxygen for its decomposition, and therefore endangering aquatic life and fish survival. We introduce here a mathematical model for such an ecosystem and investigate its behavior.

2. Model formulation

Let N, D and O respectively denote the nutrients, detritus and dissolved oxygen amount in the water. Further, let P denote the aquatic plants population. The nutrients N are assumed to be discharged into the lake at rate q; they also are generated by the reconversion of decomposed detritus, at rate u, a fraction 0 < f < 1 of which becomes new nutrient. This process involves also the usage of dissolved oxygen, and therefore the term representing it must contain the product of both the required quantities, O and D. In addition, nutrients can be washed away from the body water by emissaries, at rate a and there are further losses due to nutrients uptake by the plants, that need them for surviving. The uptake rate is denoted by b. The plants reproduce because they assume nutrients, this grazing being expressed by the product NP, but only a fraction 0 < e < 1 of the assumed nutrients contribute to increase the plant biomass. The plants have a natural mortality rate m, and experience intraspecific competition at rate c. The detritus originates by dead plants. A proportion 0 < g < 1 of the plants that die sink to the bottom of the water body and become detritus. In part the latter may be washed away by outgoing streams, at rate w; in addition, using dissolved oxygen, detritus at rate u may be decomposed by microorganisms into its basic elements, which contribute to the formation of nutrients. Finally, dissolved oxygen may be contained into the streams flowing into the lake, we assume this to occur at rate h, but it is also produced by the aquatic plants through photosynthesis, at rate s. Because most of aquatic plants float on the surface of a lake, large amount of the produced oxygen may go into the air, justifying the fact that the rate s may be small. In addition, there could be a shading effect, for which only the plants near the surface get enough light, while the ones in the lower layers of water are in part shaded from direct sunlight by the topmost ones. In order for photosynthesis to be viable, plants need clorophyll and consume nutrients for its production. Thus, in the model we must model this process via the bilinear term PN. Further, there is a natural upper limit on their oxygen production capacity, that produces a saturation limit, a fact that is expressed by using a Holling type II function to model it, k being the half saturation constant. In addition, dissolved oxygen is also washed away at rate v. Other losses are due to the decomposition of detritus into its basic components, a process that we said requires microorganisms that consume oxygen, at rate z. With these assumptions, the model takes the following form:

$$\frac{dN}{dt} = q + fuOD - aN - bNP, \quad \frac{dP}{dt} = ebNP - mP - cP^{2}, \tag{1}$$
$$\frac{dD}{dt} = g(mP + cP^{2}) - wD - uOD, \quad \frac{dO}{dt} = h - vO + \frac{sPN}{k + PN} - zOD$$

3. Boundedness of the system's trajectories

We now evaluate the region of attraction for all solutions initiating in the positive octant. Let T = N + P + D + O. Adding all the equilibrium equations, we obtain

$$\frac{d}{dt}T = q + h - aN - m(1 - g)P - c(1 - g)P^2 - b(1 - e)NP - u(1 - f)OD - zOD - wD - vO + s\frac{PN}{k + PN}.$$

Dropping some of the negative terms, taking into account that $0 \le e, f, g \le 1$ and that the last fraction is smaller than one and defining $\beta_m = min\{a, m(1-g), w, v\}$ and H = q + h + s, the above differential equations becomes the following differential inequality

$$\frac{d}{dt}T \le q+h+s-aN-m(1-g)P-wD-vO \le H-\beta_mT.$$

The solution of the associated differential equation is $T = H\beta_m^{-1}(1 - \exp(-\beta_m t) + T(0)\exp(-\beta_m t))$. Thus we obtain the final estimate that defines the region of attraction for all solutions initiating in the interior of the positive orthant:

$$T(t) \le \max\{H\beta_m^{-1}, T(0)\}.$$

4. System's equilibria

The model (1) has only two equilibria: the environment containing only oxygen and nutrients, $E_3 = (qa^{-1}, 0, 0, hv^{-1})$, and the system in which all the components are at nonzero level $E^* = (N^*, P^*, D^*, O^*)$. In addition to the general case, there are several particular cases, that arise when one or both of the exogenous inputs are blocked: $E_0 = (0, 0, 0, 0)$ arising with q = 0, h = 0, $E_1 = (qa^{-1}, 0, 0, 0)$ whenever h = 0, $E_2 = (0, 0, 0, hv^{-1})$ obtained for q = 0.

Feasibility of each E_i , i = 0, ..., 3 is obvious. To analyze coexistence, from the second equilibrium equation of (1)

$$N^* = (m + cP)(eb)^{-1} \ge 0, \quad NP = P(m + cP)(eb)^{-1}$$
(2)

are obtained. Now substituting into the first equilibrium equation, we find

$$OD = \Psi(P)(befu)^{-1}, \quad \Psi(P) \equiv bcP^2 + (ac + bm)P + am - beq.$$
(3)

The third equilibrium equation now gives $D^* = [g(mP + cP^2) - uOD]w^{-1}$. Using (3), we have explicitly:

$$D^* = \frac{1}{befw} \Phi(P), \quad \Phi(P) \equiv AP^2 + BP + C, \quad A = bc (efg - 1) < 0, \quad B = befgm - (ac + bm) = bm(efg - 1) - ac < 0.$$
(4)

$$C = beq - am.$$

Now, the sign of *OD* only depends on $\Psi(P)$. From (3), when C > 0, (4), to guarantee $\Psi(P) \ge 0$, if P_2 is its positive root, we need $P \ge P_2$. When C < 0 instead, $\Psi(P) \ge 0$ for every $P \ge 0$. To impose $D \ge 0$, we study the concave parabola $\Phi(P)$, with roots $P_3 \le P_4$. For C > 0, we have that $\Phi(P) \ge 0$ for $0 \le P \le P_4$, going down from (0, *C*) to ($P_4, 0$). From (3), (4):

$$O_1(P) = \frac{OD}{D} = \frac{w}{u} \frac{\Psi(P)}{\Phi(P)}.$$
(5)

To guarantee its nonnegativity, we combine the above results for $\Psi(P)$ and $\Phi(P)$, to get the following condition and the corresponding nonnegativity interval:

(a)
$$C > 0$$
, $B < 0$, $P_2 \le P \le P_4$. (6)

Let $\chi(P) = behk + m(h+s)P + c(h+s)P^2 > 0$, $\Pi(P) = bek + mP + cP^2 > 0$, $\Gamma(P) = Q + MP + LP^2$, Q = befvw + zC, M = zB < 0, L = zA < 0 and the discriminant of Γ be $\Delta_{\Gamma} = M^2 - 4LQ > M^2$. Eliminating *NP* from (2), the fourth equilibrium equation instead gives:

$$O_2(P) = \left[h + sPN^*(k + PN^*)^{-1}\right](v + zD^*)^{-1} = befw\chi(P)[\Pi(P)\Gamma(P)]^{-1},$$
(7)

To find sufficient conditions for the intersection of O_1 and O_2 for $P \ge 0$, we study each function separately.

 O_1 is positive if conditions (6) hold. In the case (a) O_1 increases from a zero at $P = P_2$ toward the vertical asymptote at $P = P_4$.

The sign of O_2 is instead determined just by the one of Γ . In principle, there is the following possible case guaranteeing that the concave parabola Γ is positive. (1) Q > 0, M < 0 which implies a decreasing branch of this parabola lies in the first quadrant between the vertical axis and its positive root γ_+ , joining (0, Q) with $(\gamma_+, 0)$.

Correspondingly, there is a possible case for O_2 being nonnegative when $P \ge 0$: it raises up from a positive height at the origin to the vertical asymptote at $P = \gamma_+$.

We now combine O_1 and O_2 to find an intersection in the first quadrant. (1) is compatible with (a). Sufficient conditions for (P^*, O^*) to lie in the first quadrant is

$$(1) - (a): \quad Q > 0, \quad B < 0, \quad C > 0, \quad P_2 < P^* < P_4 < \gamma_+.$$
(8)

5. Stability

Let J be the Jacobian of (1) and let J_i denote the matrix M evaluated at each equilibrium E_i :

$$J = \begin{pmatrix} -a - bP & -bN & fuO & fuD \\ ebP & ebN - m - 2cP & 0 & 0 \\ 0 & gm + 2gcP & -uO - w & -uD \\ \frac{skP}{(k+PN)^2} & \frac{skN}{(k+PN)^2} & -zO & -v - zD \end{pmatrix}$$
(9)

For J_0 the eigenvalues are all negative, namely -a, -m, -w, -v. Therefore, E_0 is stable. J_1 has the following eigenvalues -a, $(beq - am)a^{-1}$, -w, -v. From the condition (8), C = beq - am > 0 we can obtain that if the equilibrium E^* exists, E_1 becomes a saddle point. J_2 has four negative eigenvalues -a, -m, $-huv^{-1} - w$, -v so that E_2 is also always locally asymptotically stable. The eigenvalues at the equilibrium E_3 are -a, $(beq - am)a^{-1}$, $-huv^{-1} - w$, -v and again this equilibrium is a saddle point when E^* exists. The stability of the coexistence point E^* is examined numerically.

6. Numerical simulations

The equilibrium E^* is feasible and stable for the parameter values: q = 0.5, f = 0.5, u = 0.5, a = 0.05, b = 0.41, e = 0.9, m = .095, c = 0.08, g = 0.9, w = 0.013, h = 0.3, v = 0.08, s = 0.02, k = 0.02, z = 0.025, obtaining $N^* = 0.7801$, $P^* = 2.4107$, $D^* = 0.3436$, $O^* = 3.6098$.

Letting the input rate of nutrients vary, q, from Figure ?? we observe that with increasing q, the densities of nutrients, aquatic plants and detritus increase. In fact, for q = 0 their densities become zero within a short period. The opposite results occur instead for dissolved oxygen, the concentration of which increases with decreasing q. When q = 0, dissolved oxygen attains its maximum concentration.

Figure ?? depicts instead the effect of rate of conversion of detritus into nutrients on the coexistence equilibrium E^* . A similar situation as for Figure ?? arises. With increasing f, the densities of nutrients, aquatic plants and detritus increase, while the concentration of dissolved oxygen decreases.

7. Discussion

The ecosystem could disappear, as the equilibrium E_0 is locally asymptotically stable, not surprisingly. In fact a pond lacking sufficient inputs becomes dry, and all its components disappear. The no-life equilibrium E_2 is also always stably achievable. E_1 and E_3 can be rendered stable if C < 0. Rewritten as $bem^{-1} < aq^{-1}$ it means that the reduced plants natality, i.e. the ratio of new plants over their mortality, should be bounded above by the reduced nutrients depletion rate, i.e. the ratio of their washout rate to the net input rate. Hence, more nutrients are supplied into the system than they are used for plant reproduction. In such case the nutrients-only equilibrium and the point with only nutrients and dissolved oxygen are stable.

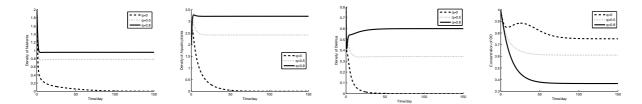


Figure 1: Variations of nutrients, aquatic plants, detritus and dissolved oxygen as functions of q. A reduction of the exogeous input of nutients lowers the equilibrium values of all the ecosystem components but for dissolved oxygen.

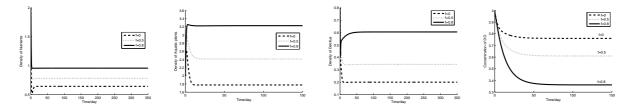


Figure 2: Variations of nutrients, aquatic plants, detritus and dissolved oxygen as functions of f. Reducing the conversion rate of detritus into nutrients enhances dissolved oxygen but hinders nutrients, aquatic plants and unexpectedly also detritus itself.

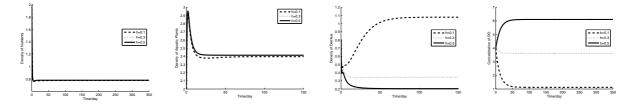


Figure 3: Variations of nutrients, aquatic plants, detritus and dissolved oxygen as functions of h. While nutrients and aquatic plants are scarcely affected by changes in this parameter, for smaller values of h we find a sharp increase in detritus and a corresponding sensible decrease in the dissolved oxygen to very low concentrations. To maintain an inflow of dissolved oxygen appears thus to be necessary to keep the water body healthy.

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