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A dichotomy result for a pointwise summable sequence of operators[☆]

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ABSTRACT

Let X be a separable Banach space and Q be a coanalytic subset of $X^{\mathbb{N}} \times X$. We prove that *the set of sequences* $(e_i)_{i\in\mathbb{N}}$ *in* X *which are weakly convergent to some* $e \in X$ *and* $Q((e_i)_{i\in\mathbb{N}}, e)$ *is a coanalytic subset of* $X^{\mathbb{N}}$. The proof applies methods of effective descriptive set theory to Banach space theory. Using Silver's Theorem [J. Silver, Every analytic set is Ramsey, J. Symbolic Logic 35 (1970) 60–64], this result leads to the following dichotomy theorem: *if* X *is a Banach space,* $(a_{ij})_{i,j\in\mathbb{N}}$ *is a regular method of summability and* $(e_i)_{i\in\mathbb{N}}$ *is a bounded sequence in* X, *then there exists a subsequence* $(e_i)_{i\in L}$ *such that either (I) there exists* $e \in X$ *such that every subsequence* $(e_i)_{i\in H}$ *of* $(e_i)_{i\in L}$ *is weakly summable w.r.t.* $(a_{ij})_{i,j\in\mathbb{N}}$ to e and $Q((e_i)_{i\in H}, e)$; *or (II) for every subsequence* $(e_i)_{i\in H}$ *of* $(e_i)_{i\in L}$ *and every* $e \in X$ *with* $Q((e_i)_{i\in H}, e)$ *the sequence* $(e_i)_{i\in H}$ is not weakly summable to e w.r.t. $(a_{ij})_{i,j\in\mathbb{N}}$. This is a version for weak convergence of an Erdös–Magidor result, see [P. Erdös, M. Magidor, A note on Regular Methods of Summability, Proc. Amer. Math. Soc. 59 (2) (1976) 232–234]. Both theorems obtain some considerable generalizations.

1. Introduction and results

The aim of this paper is to exhibit some applications of the so-called *effective* descriptive set theory into Banach space theory. The methods of effective descriptive set theory have already been used in analysis by Debs (see [1]) for instance; however this kind of approach is not used very often.

Let us begin with some comments on notation. If X and Y are Banach spaces we denote by $L(X, Y)$ the set of all linear and bounded operators from X to Y. It is well known that $L(X, Y)$ with the operator norm is a Banach space. If X is a Banach space and A is a subset of X we let $\mathrm{span}A$ be the subspace which is generated from A. By $\overline{\mathrm{span}}A$ we mean the closure of $\mathrm{span}A$ in X. We assume that the set of natural numbers $\mathbb{N}$ starts with 0. Recall that a topological space $\mathcal{X}$ is called *Polish* iff it is separable and metrizable by some metric d such that $(\mathcal{X}, d)$ is complete. If X is a metric space we define $X^{\mathbb{N}}$ to be the product $X \times X \times X \times \cdots$ with the usual product topology. It is well known that if $\mathcal{X}$ is Polish then so is $\mathcal{X}^{\mathbb{N}}$. If K is a compact metric space we denote by $C(K)$ the set of continuous real functions defined on K. We think of $C(K)$ with the supremum norm $\|f\|_\infty = \sup\{|f(x)| \,/\, x \in K\}$. It is well known that $(C(K), \|.\|_\infty)$ is a Polish space. We will always think of $C(K)$ with this norm without mentioning it explicitly. The first basic result of this paper is the following.

Theorem 1.1. *Let X be a separable Banach space and let Q be a coanalytic subset of $X^{\mathbb{N}} \times X$. Then the set*

$$P_Q = \{(y_i)_{i\in\mathbb{N}} \in X^{\mathbb{N}} \,/\, (y_i)_{i\in\mathbb{N}} \text{ is weakly convergent to some } y \text{ and } Q((y_i)_{i\in\mathbb{N}}, y)\}$$

is a coanalytic subset of $X^{\mathbb{N}}$.

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The original statement of Theorem 1.1 was about the set of weakly convergent sequences without any additional properties; i.e. $Q = X^{\mathbb{N}} \times X$. The author would like to express his gratitude to the referee for finding a simpler proof for this case which does not use any effective arguments. Some special cases of the results of this paper can be proved classically, i.e. with no effective arguments. However the effective method provides these results to their full extend and in a more direct way.

Notice that if A is a coanalytic subset of X the corresponding set $\{(y_i)_{i\in\mathbb{N}} \in X^{\mathbb{N}} \;/\; (y_i)_{i\in\mathbb{N}}$ is weakly convergent to some $y \in A\}$ is also coanalytic, as we can view A as a coanalytic subset of $X^{\mathbb{N}} \times X$ via $((x_i)_{i\in\mathbb{N}}, x) \mapsto x$. We shall denote this set again by P_A.

Theorem 1.1 can be extended into two different directions. Recall that if X is Banach space then the closed unit ball of X^* is defined to be the set $B_{X^*} = \{x^* \in X^* \;/\; \|x^*\| \le 1\}$. It is well known that if X is separable then B_{X^*} with the weak* topology is a compact metric space. The first idea is to think of any element y of a separable Banach space X as a function defined on the compact metric space (B_{X^*}, w^*). The arguments for proving Theorem 1.1 can be used in order to prove the following statement $(*)$. Let K be a compact metric space and let Q be a coanalytic subset of $C(K)^{\mathbb{N}} \times C(K)$. Then the set $P_Q = \{(f_i)_{i\in\mathbb{N}} \in C(K)^{\mathbb{N}} \;/$ there exists $f \in C(K)$ s.t. $f_i \overset{pw}{\to} f$ and $Q((f_i)_{i\in\mathbb{N}}, f)\}$ is coanalytic.

Recall that a sequence $(f_i)_{i\in\mathbb{N}}$ in the separable Banach space $C([0, 1])$ is weakly convergent exactly when $(f_i)_{i\in\mathbb{N}}$ is bounded and pointwise convergent to some continuous function. Thus statement $(*)$ is an extension of Theorem 1.1 in the sense that the sequence $(f_i)_{i\in\mathbb{N}}$ need not be bounded. Nevertheless this obtains some interesting examples.

Examples 1.2. The following sets are coanalytic . To see why the corresponding sets Q are coanalytic refer to [7].

(1) Let P be the set of sequences $(f_i)_{i\in\mathbb{N}}$ in $C([0, 1])$ which are differentiable, pointwise convergent to a differentiable f and furthermore $f_i' \overset{pw}{\to} f'$. Then P is coanalytic.

The author is aware of a classic proof of the latter example. However a classic proof for the next example does not seem to be known.

(2) Let $\pi : C([0, 1]) \to C([0, 1])$ be a Borel measurable function. Then the set

$$P = \left\{(f_i)_{i\in\mathbb{N}} \in C([0, 1])^{\mathbb{N}} \;/\; (f_i)_{i\in\mathbb{N}} \text{ is pointwise convergent to some } f \text{ and} (\pi(f_i))_{i\in\mathbb{N}} \text{ is pw-convergent to } \pi(f)\right\}$$

is coanalytic.

The second idea in order to extend Theorem 1.1 is the following. We view an element y of a separable Banach space X as an operator in $L(X^*, \mathbb{R})$; i.e. this time we drop the compactness and focus on the linearity. In this case the extension is not so immediate since the space $L(X^*, \mathbb{R})$ need not be Polish. We shall overcome this fact with some additional assumptions.

Definition 1.3. Let X be a Banach space, Y be a separable Banach space and B be a closed and separable subspace of $L(X, Y)$. We say that *the triple* (X, Y, B) *is a Polish system* iff there exists a sequence $(D_n)_{n\in\mathbb{N}} \subseteq B$ which is norm-dense in B and a Polish topology $\mathcal{T}$ on B_X such that the restriction $D_n \upharpoonright B_X : B_X \to Y$ is Borel measurable for all $n \in \mathbb{N}$.

It is clear that if X is a separable Banach space then it is a Polish space. Since B is pre-assumed separable it follows that the triple (X, Y, B) is a Polish system. Also notice that in the case where $B = \overline{\text{span}}\{T_i \;/ i \in \mathbb{N}\}$ it is enough to find a Polish topology $\mathcal{T}$ such that every linear combination of T_i's is a Borel measurable function.

We now give one more example of a Polish system. Assume that E is a separable Banach space. Then B_{E^*} with the weak* topology is a compact Polish space. Regard E as a subspace of E^{**}. Since every function $x : (B_{E^*}, w^*) \to \mathbb{R} : x(x^*) = x^*(x)$ is continuous it follows that the triple $(E^*, \mathbb{R}, E)$ is a Polish system. Similarly if Y is an arbitrary separable Banach space and B is a closed separable subspace of $L(X, Y)$ which consists of (weak*, $\|\;\|_Y$) continuous functions from E^* to Y then (E^*, Y, B) is a Polish system.

Theorem 1.4. *Let X and Y be Banach spaces with Y being separable. Also let B be a norm closed and separable subspace of $L(X, Y)$ and Q be a coanalytic subset of $B^{\mathbb{N}} \times B$. Assume that the triple (X, Y, B) is a Polish system. Then the set*

$$P_Q = \{(T_i)_{i\in\mathbb{N}} \in B^{\mathbb{N}} \;/\; \textit{there exists } T \in B \textit{ s.t. } T_i \overset{pw}{\to} T \textit{ and } Q((T_i)_{i\in\mathbb{N}}, T)\}$$

is a coanalytic subset of $B^{\mathbb{N}}$.

It is clear now that Theorem 1.1 is a special case of Theorem 1.4. Theorem 1.1 was inspired by an attempt to get a similar version of a result of Erdös and Magidor, (Theorem 1.6). Let us first recall some basic notions.

A double sequence $(a_{ij})_{i,j\in\mathbb{N}}$ of real numbers is called *a regular method of summability* iff whenever X is a Banach space and $(e_i)_{i\in\mathbb{N}}$ is a sequence in X converging in norm to $e \in X$, then for all $i \in \mathbb{N}$ the series $\sum_{j=0}^{\infty} a_{ij}e_j = y_i$ is convergent and the sequence $(y_i)_{i\in\mathbb{N}}$ is also norm convergent to e. An example of a regular method of summability is the *Cesàro* method: $(a_{0,j})_{j\in\mathbb{N}} = (1, 0, 0, 0, \dots,)$, $(a_{1,j})_{j\in\mathbb{N}} = (\frac{1}{2}, \frac{1}{2}, 0, 0, \dots,)$, $(a_{2,j})_{j\in\mathbb{N}} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, \dots,)$, etc.

Let $(a_{ij})_{i,j\in\mathbb{N}}$ be a regular method of summability and X be a Banach space. From Remark 3.1 one can see that if $(e_i)_{i\in\mathbb{N}}$ is a bounded sequence in X then for all $i \in \mathbb{N}$ the series $\sum_{j=0}^{\infty} a_{ij}e_j$ is convergent in X. Call a bounded sequence $(e_i)_{i\in\mathbb{N}}$ *summable to $e \in X$ with respect to $(a_{ij})_{i,j\in\mathbb{N}}$* iff the sequence $(y_i)_{i\in\mathbb{N}}$ defined by $y_i = \sum_{j=0}^{\infty} a_{ij}e_j$, for all i, is norm convergent to e. Also say that a sequence $(e_i)_{i\in\mathbb{N}}$ in X is *summable w.r.t. $(a_{ij})_{i,j\in\mathbb{N}}$* iff it is summable to some $e \in X$, w.r.t. $(a_{ij})_{i,j\in\mathbb{N}}$. It comes straight from the definition that norm convergent sequences are summable to the same limit. Also a sequence is summable w.r.t. Cesàro method iff the sequence of the mean values is convergent. It is well known that the notion of summability is not preserved under subsequences. Erdös and Magidor [4] established the following result. Let $(a_{ij})_{i,j\in\mathbb{N}}$ be a regular method of summability and $(e_i)_{i\in\mathbb{N}}$ be a bounded sequence in a Banach space X. Then there exists a subsequence $(e_{k_i})_{i\in\mathbb{N}}$ of $(e_i)_{i\in\mathbb{N}}$ such that: either (I) every subsequence of $(e_{k_i})_{i\in\mathbb{N}}$ is summable w.r.t. $(a_{ij})_{i,j\in\mathbb{N}}$ and each being summed to the same limit; or (II) no subsequence of $(e_{k_i})_{i\in\mathbb{N}}$ is summable w.r.t. $(a_{ij})_{i,j\in\mathbb{N}}$. We now give a version of summability for linear operators.

Definition 1.5. Let $(a_{ij})_{i,j\in\mathbb{N}}$ be a regular method of summability and let X, Y be Banach spaces. Also let B be a closed subspace of $L(X, Y)$. A bounded sequence $(T_i)_{i\in\mathbb{N}}$ in B is *pointwise summable to* $T \in B$ *w.r.t.* $(a_{ij})_{i,j\in\mathbb{N}}$ iff the sequence $(F_i)_{i\in\mathbb{N}}$ defined by $F_i = \sum_{j=0}^{\infty} a_{ij}T_j$ for all $i \in \mathbb{N}$, is pointwise convergent to T.

In the case where $(X, Y, B) = (E^*, \mathbb{R}, E)$ for some separable Banach space E, $T_i = e_i \in E$ for all $i \in \mathbb{N}$ and $T = e \in E$ we will say that the sequence $(e_i)_{i\in\mathbb{N}}$ is *weakly summable to e* instead of pointwise summable to e.

Theorem 1.6. *Let X be a Banach space, $(a_{ij})_{i,j\in\mathbb{N}}$ be a regular method of summability, $(e_i)_{i\in\mathbb{N}}$ be a bounded sequence in X and let $Q \subseteq X^{\mathbb{N}} \times X$ be a coanalytic set. Then there exists a subsequence $(e_i)_{i\in L}$ of $(e_i)_{i\in\mathbb{N}}$ such that:*

(I) *either there exists $e \in X$ such that every subsequence $(e_i)_{i\in H}$ of $(e_i)_{i\in L}$ is weakly summable w.r.t. $(a_{ij})_{i,j\in\mathbb{N}}$ to e and $Q((e_i)_{i\in H}, e)$; or*

(II) *for every subsequence $(e_i)_{i\in H}$ of $(e_i)_{i\in L}$ and every $e \in X$ with $Q((e_i)_{i\in H}, e)$ the sequence $(e_i)_{i\in H}$ is not weakly summable to e w.r.t. $(a_{ij})_{i,j\in\mathbb{N}}$.*

In the case where $Q = X^{\mathbb{N}} \times X$ Theorem 1.6 can be proved differently using the well-known Rosenthal's l_1 dichotomy theorem: for every bounded sequence $(e_i)_{i\in\mathbb{N}}$ in X there exists a subsequence which is either weakly Cauchy or equivalent to the unit basis of l_1. To see this notice that (i) a weakly Cauchy sequence is weak*-convergent in some $x^{**} \in X^{**}$ and that (ii) every sequence which is equivalent to the unit basis of l_1 is not weakly summable w.r.t. $(a_{ij})_{i,j\in\mathbb{N}}$. Also notice that in the previous theorem we may assume that X is separable; for otherwise we may take the following subspace $X_0 = \overline{\text{span}}\{e_i \,/ i \in \mathbb{N}\}$ instead of X.

Example 1.7. Consider the separable Banach space $C([0, 1])$. Then weak convergence is equivalent to pointwise convergence of a bounded sequence. Put $Q((f_n)_{n\in\mathbb{N}}, f)$ iff the functions f and f_n for all $n \in \mathbb{N}$ are continuously differentiable and the sequence $(f'_n)_{n\in\mathbb{N}}$ is pointwise-Cesàro summable to f'. We now apply Theorem 1.6.

Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence of differentiable functions in $C([0, 1])$. Then there exists a subsequence $(f_n)_{n\in H}$ such that: either (I) there exists a differentiable f such that for every $L \subseteq H$ the sequences $(f_n)_{n\in L}$ and $(f'_n)_{n\in L}$ are pointwise-Cesàro summable to f and f' respectively; or (II) for every differentiable f and for every $L \subseteq H$ if the sequence $(f_n)_{n\in L}$ is pointwise-Cesàro summable to f then the sequence $(f'_n)_{n\in L}$ is not pointwise-Cesàro summable to f'. Notice now that we cannot apply Rosenthal's l_1 dichotomy theorem to the sequence $(f'_n)_{n\in\mathbb{N}}$ simply because it may fail to be bounded. Notice also that one can pursue the dichotomy further by applying it to case (II) for the sequence $(f_n)_{n\in L}$ and $Q = X^{\mathbb{N}} \times X$. However we will not follow this direction.

Of course there is no point in stating an analogue version of Theorem 1.6 for sequences $(f_i)_{i\in\mathbb{N}}$ in $C([0, 1])$ since $(f_i)_{i\in\mathbb{N}}$ is a priori bounded. However we may give an extension of Theorem 1.6 using linear operators.

Theorem 1.8. *Let X and Y be Banach spaces with Y being separable, let $(a_{ij})_{i,j\in\mathbb{N}}$ be a regular method of summability and let $(T_i)_{i\in\mathbb{N}}$ be a bounded sequence of linear and bounded operators from X to Y. Put $B = \overline{span}\{T_i \,/ i \in \mathbb{N}\}$ and assume that the triple (X, Y, B) is a Polish system. Also let $Q \subseteq B^{\mathbb{N}} \times B$. Then there exists a subsequence $(T_i)_{i\in L}$ of $(T_i)_{i\in\mathbb{N}}$ such that:*

(I) *either there exists some T in B such that every subsequence $(T_i)_{i\in H}$ of $(T_i)_{i\in L}$ is pointwise summable to T w.r.t. $(a_{ij})_{i,j\in\mathbb{N}}$ and $Q((T_i)_{i\in H}, T)$; or*

(II) *for every subsequence $(T_i)_{i\in H}$ of $(T_i)_{i\in L}$ and every $T \in B$ with $Q((T_i)_{i\in H}, T)$ the sequence $(T_i)_{i\in H}$ is not pointwise summable to T w.r.t. $(a_{ij})_{i,j\in\mathbb{N}}$.*

Below we refer to notions of effective descriptive set theory, in particular we refer to Δ^1_1 *computability*. In the next section we prove – under the appropriate hypothesis – that the pointwise limit of a sequence of operators $(T_i)_{i\in\mathbb{N}}$ is Δ^1_1-computable from the sequence $(T_i)_{i\in\mathbb{N}}$, (Theorem 2.3). From this using the Theorem on Restricted Quantification 2.1 we prove Theorem 1.4. Theorem 1.1 becomes then an easy consequence. We will not bother to give the proof for the case where we have a sequence of functions in $C(K)$. It shall become evident to the reader that this case can be proved similarly — in fact even easier. We also give some examples of sets of the form P_Q as in Theorem 1.1 which are not Borel; thus the fact that a set of the form P_Q is coanalytic is the best we can say about it. However in some cases P_Q becomes Borel, see Proposition 2.5.

In Section 3 we give some applications. First we prove the dichotomy Theorem 1.8 from which Theorem 1.6 follows. Then we consider the special case of Theorem 1.6 where $Q = X^{\mathbb{N}} \times X$ and examine some of its consequences. The outcome is the following corollary which in part characterizes the reflexive Banach spaces. (Recall that a Banach space is reflexive iff the closed unit ball B_X is weakly compact).

Corollary 1.9. *Let X be a Banach space and let K be a bounded and weakly closed subset of X. Also let $(a_{ij})_{i,j\in\mathbb{N}}$ be a regular method of summability. Then K is weakly compact if and only if for all sequences $(x_i)_{i\in\mathbb{N}}$ in K there exists a subsequence $(x_{k_i})_{i\in\mathbb{N}}$ which is weakly summable w.r.t. $(a_{ij})_{i,j\in\mathbb{N}}$.*

Notice that in the special case where K is convex and $(a_{ij})_{i,j\in\mathbb{N}}$ is positive (i.e. $a_{ij} \geq 0$ for all $i, j \in \mathbb{N}$) this corollary is a weaker version of a theorem in [2] (p.82). The latter is substantially based on James' characterization of weakly compact sets given in [6] (Theorem 5). On the other hand Corollary 1.9 uses elementary methods from the point of view of functional analysis. Finally we use a bit more of effective theory. In particular we prove the following corollary which makes a substantial use of Debs' Theorem, (see [1]).

Corollary 1.10. *Let X be a Banach space and let $(e_i)_{i\in\mathbb{N}}$ be a bounded sequence in X for which case (II) of Theorem 1.6 fails for $Q = X^{\mathbb{N}} \times X$. Then there exists a Borel measurable function $f : [\mathbb{N}]^{\omega} \to [\mathbb{N}]^{\omega}$ such that the sequence $(e_i)_{i\in f(L)}$ is a weakly convergent subsequence of $(e_i)_{i\in L}$ for all $L \in [\mathbb{N}]^{\omega}$.*

For the definition of $[\mathbb{N}]^{\omega}$ refer to the first lines of Section 3.

2. The set of pointwise convergent operators

In this section we prove Theorem 1.4 and consequently Theorem 1.1. We describe the general setting. As usual if X is a Banach space, then by B_X we denote the closed unit ball of X. We will deal with sets of operators from some Banach space X to a separable Banach space Y. In order to proceed we need the operators to be members of a Polish space. Furthermore since we deal with pointwise convergence we need to quantify over B_X. So it is necessary to have a Polish topology for B_X. The notion of a Polish system given in the Introduction (see Definition 1.3) is exactly to meet those requirements.

Now we have to get into effective descriptive set theory. For a complete introduction to the effective theory in Polish spaces, see [9] ch. 3. We denote by Σ^1_1, Π^1_1 and Δ^1_1 the corresponding pointclasses of *lightface* analytic, coanalytic and Borel sets. The analogous symbols are used for the relativized case. We shall also write $y \in \Delta^1_1(x)$ if y is computed from a set in $\Sigma^1_1(x)$ and a set in $\Pi^1_1(x)$. We will use the following deep result of the effective theory, which is originally proved in [8] but can also be found in [9] 4D.3 in a form which is closer to the present.

Theorem 2.1 (*The Theorem on Restricted Quantification*). *Let $\mathcal{X}$ and $\mathcal{Y}$ be recursively representable Polish spaces and let $Q \subseteq \mathcal{X} \times \mathcal{Y}$ be in $\Pi^1_1(\varepsilon)$ for some $\varepsilon \in \mathcal{N}$. Put $P(x) \Leftrightarrow (\exists y \in \Delta^1_1(\varepsilon, x))Q(x, y)$. Then P is also in $\Pi^1_1(\varepsilon)$.*

Intuitively the preceding theorem gives a condition under which the projection of a coanalytic set is also coanalytic. Let us now describe how one can give a recursive presentation of some Polish space $\mathcal{X}$. We fix once and for all a recursive enumeration of the rationals $\{r_s \ / \ s \in \mathbb{N}\}$ for example $r_s = (-1)^{(s)_0} \cdot \dfrac{(s)_1}{(s)_2 + 1}$. We think of the set of the reals $\mathbb{R}$ with this recursive presentation. Let d be a suitable metric for $\mathcal{X}$ and $D = \{x_n \ / \ n \in \mathbb{N}\}$ be a dense subset of $\mathcal{X}$. We define the irrationals $\varepsilon_1, \varepsilon_2 \in \mathcal{N}$ as follows: $\varepsilon_1(\langle s, n, m\rangle) = 1$ if $d(x_n, x_m) < r_s$ and 0 otherwise; $\varepsilon_2(\langle s, n, m\rangle) = 1$ if $d(x_n, x_m) \le r_s$ and 0 otherwise. Then $\mathcal{X}$ admits a presentation which is recursive in $\varepsilon_{\mathcal{X}} = \langle \varepsilon_1, \varepsilon_2\rangle$. The basic neighborhoods are the sets $N^{\mathcal{X}}_{\langle n,s\rangle} = N^{\mathcal{X}}(x_n, r_s) = \{x \in X \ / \ d(x, x_n) < r_s\}$. It follows that the relation $P(x, n, s) \Leftrightarrow d(x, x_n) < r_s$ is semirecursive in $\varepsilon_{\mathcal{X}}$, (see [9], 3C.1). In most cases $\mathcal{X}$ will be a separable Banach space, hence d can be chosen to be the function $d(x, y) = \|x - y\|$. Also we may choose the countable dense set D so that $D + D \subseteq D$ and $r_s \cdot D \subseteq D$ for all $s \in \mathbb{N}$. We then define the irrationals $\varepsilon_3, \varepsilon_4$ as follows: $\varepsilon_3(\langle n, m, k\rangle) = 1$ if $x_n + x_m = x_k$ and 0 otherwise; $\varepsilon_4(\langle s, n, k\rangle) = 1$ if $r_s \cdot x_n = x_k$ and 0 otherwise. In this case we define $\varepsilon_{\mathcal{X}} = \langle \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\rangle$. Using this suggested recursive presentation for Banach spaces one can verify that the addition and scalar multiplication are recursive functions.

Since the results about the pointclass Δ^1_1 are carried out in a straightforward way to the relativized pointclass $\Delta^1_1(\varepsilon_{\mathcal{X}})$, we may assume that each given Polish space is already recursively presented. It is also necessary to code sequences in Polish spaces. If $\bar{y} = (y_i)_{i\in\mathbb{N}}$ is a sequence in a Polish space $\mathcal{X}$ we define the irrational $\delta(\bar{y})$ as follows: $\delta(\bar{y})(\langle i, n, s\rangle) = 1$ if $d(y_i, x_n) < r_s$ and 0 otherwise; where $\{x_n \ / \ n \in \mathbb{N}\}$ is the countable dense subset of $\mathcal{X}$ which comes with the recursive presentation of $\mathcal{X}$. Thinking of a sequence as a member of the Baire space makes it easier for the computations to come. However one should have in mind that a sequence $(y_i)_{i\in\mathbb{N}}$ is also a member of $\mathcal{X}^{\mathbb{N}}$. We will abuse the notation by writing $\Delta^1_1((y_i))$ instead of $\Delta^1_1(\delta(\bar{y}))$. In fact one can give a precise meaning for $\Delta^1_1((y_i))$: it is the class of points which are Δ^1_1-computable from $\bar{y} = (y_i)_{i\in\mathbb{N}} \in \mathcal{X}^{\mathbb{N}}$. It is not hard to see that the irrational $\delta(\bar{y})$ is in $\Delta^1_1((y_i))$ and that the sequence $(y_i)_{i\in\mathbb{N}}$ is in $\Delta^1_1(\delta(\bar{y}))$. Hence a point is in $\Delta^1_1(\delta(\bar{y}))$ exactly when it is in $\Delta^1_1((y_i))$. This justifies the interchange of notations. In the case we are interested in, the sequence $(y_i)_{i\in\mathbb{N}}$ is going to be a sequence of operators.

Now let us get into the proof of Theorem 1.4. We fix the spaces X, Y and B such that (X, Y, B) is a Polish system. Also we fix a sequence $(D_n)_{n\in\mathbb{N}}$ which is dense in B and a Polish topology $\mathcal{T}$ on B_X such that the restriction of every D_n on $(B_X, \mathcal{T})$ is Borel measurable. As pointed out before we may assume that each Polish space we deal with is already recursively presented and in particular the set $\{D_n \ / \ n \in \mathbb{N}\}$ is the one which comes with the recursive presentation of B. When we refer to B_X we always mean the topological space $(B_X, \mathcal{T})$. The function $f : \mathbb{N} \times B_X \to Y : f(n, x) = D_n(x)$ is Borel measurable. Using a well-known theorem of Kleene (see [9] 7A.1) there is an irrational ε^* such that f is $\Delta^1_1(\varepsilon^*)$-recursive. Again we may assume that f is Δ^1_1-recursive. Hence the relation $P \subseteq B_X \times \mathbb{N}^3$ defined by $P(x, n, m, s) \Leftrightarrow \|D_n(x) - D_m(x)\| < r_s$ is in Δ^1_1. Using the recursive presentation of B it is not hard to see that the relation $Q \subseteq B \times B \times \mathbb{N}$ defined by $Q(T, L, s) \Leftrightarrow \|T - L\| < r_s$ is semirecursive. The next proposition is the corresponding for the pointwise evaluation of operators.

Proposition 2.2. *Define $P, P_{\le} \subseteq B_X \times B \times B \times \mathbb{N}$ by $P(x, T, L, s) \Leftrightarrow \|T(x) - L(x)\| < r_s$ and $P_{\le}(x, T, L, s) \Leftrightarrow \|T(x) - L(x)\| \le r_s$. Then P and $P_{\le}$ are both in Δ^1_1.*

Proof. We claim that

$$\|T(x) - L(x)\| < r_s \Leftrightarrow (\exists k)(\exists m)(\exists n)\Big[\ \|T - D_m\| < \frac{1}{k+1} \ \& \ \|L - D_n\| < \frac{1}{k+1} \\ \& \ \|D_m(x) - D_n(x)\| < r_s - \frac{2}{k+1}\ \Big]$$

Both directions are straightforward; for the left-to-right-hand direction notice that one has to choose $k \in \mathbb{N}$ s.t. $\|T(x) - L(x)\| < r_s - \frac{4}{k+1}$ and $m, n \in \mathbb{N}$ s.t. $\|D_m - T\| < \frac{1}{k+1}$, $\|L - D_n\| < \frac{1}{k+1}$. Using this equivalence one can see that both P and $P_{\leq}$ are in Δ^1_1. ⊣

Now we are going to compute the complexity of the pointwise limit of a sequence of operators. The following is the essential step for using the Theorem on Restricted Quantification 2.1.

Theorem 2.3. *Let a sequence $(T_i)_{i\in\mathbb{N}}$ in B, $T \in B$ and assume that the sequence $(T_i)_{i\in\mathbb{N}}$ converges pointwise to T, i.e. $T_i(x) \overset{i\in\mathbb{N}}{\to} T(x)$ for all $x \in X$. Then T is in $\Delta^1_1((T_i))$. In particular if E is a Banach space, $e \in E$ and $(e_i)_{i\in\mathbb{N}}$ is a sequence in E which is weakly convergent to e, then e is in $\Delta^1_1((e_i))$. Similarly if K is a compact metric space and $(f_i)_{i\in\mathbb{N}}$ is a sequence in $C(K)$ which is pointwise convergent to $f \in C(K)$ then f is in $\Delta^1_1((f_i))$.*

Lemma 2.4. *Let $T \in B$. Put $R^T(x, n, s) \Leftrightarrow \|T(x) - D_n(x)\| \leq r_s$. If $(T_i)_{i\in\mathbb{N}}$ is a sequence in B which converges pointwise to T then the relation R^T is in $\Delta^1_1((T_i))$.*

Proof. First of all notice that

$$\begin{aligned} F = T_i &\Leftrightarrow (\forall s)(\forall n)[\ \|F - D_n\| < r_s \leftrightarrow \|T_i - D_n\| < r_s\ ] \\ &\Leftrightarrow (\forall s)(\forall n)[\ \|F - D_n\| < r_s \leftrightarrow \delta(\bar{T})(\langle i, n, s\rangle) = 1\ ] \end{aligned}$$

where $\delta(\bar{T})$ is the irrational assigned to the sequence $\bar{T} = (T_i)_{i\in\mathbb{N}}$. Thus the relation $Eq(i, F) \Leftrightarrow F = T_i$ is in $\Delta^1_1((T_i))$. Now

$$\|T(x) - D_n(x)\| \leq r_s \ \Leftrightarrow \lim_{i\in\mathbb{N}} \|T_i(x) - D_n(x)\| \leq r_s.$$

Using the relation $P_{\leq}$ defined in Proposition 2.2 and the relation Eq one can see that the relation R^T is in $\Delta^1_1((T_i))$. ⊣

Proof of Theorem 2.3. We need to show that the relation $P(n, t) \Leftrightarrow \|T - D_n\| < r_t$ is both in $\Sigma^1_1((T_i))$ and $\Pi^1_1((T_i))$.

- **The $\Pi^1_1((T_i))$ case.** Recall that for each $L \in B$ we have that $\|L\| = \sup\{\|L(x)\| \ / \ x \in B_X\}$. So

$$\begin{aligned} \|T - D_n\| \leq r_s &\Leftrightarrow (\forall x)[\ \|T(x) - D_n(x)\| \leq r_s\ ] \\ &\Leftrightarrow (\forall x)[\ R^T(x, n, s)\ ] \end{aligned}$$

where R^T is as in Lemma 2.4. Therefore the relation P is in $\Pi^1_1((T_i))$.

$$\begin{aligned} P(n, t) &\Leftrightarrow (\exists s)[\ r_s < r_t \ \&\ \|T - D_n\| \leq r_s\ ] \\ &\Leftrightarrow (\exists s)[\ r_s < r_t \ \&\ (\forall x) R^T(x, n, s)\ ] \end{aligned}$$

- **The $\Sigma^1_1((T_i))$ case.** We first claim that $(*)$

$$\begin{aligned} \|T - D_n\| < r_t \ \Leftrightarrow (\exists k)(\forall m)(\exists x) \Bigg\{ &\|D_m(x) - D_n(x)\| > \|D_m - D_n\| - \frac{1}{k+1} \\ &\& \left[\|D_m(x) - T(x)\| < \frac{1}{k+1} \to \|T(x) - D_n(x)\| < r_t - \frac{3}{k+1} \right] \Bigg\} \end{aligned}$$

For the $(\Rightarrow)$ direction: choose $k \in \mathbb{N}$ s.t. $\|T - D_n\| < r_t - \frac{3}{k+1}$. Let $m \in \mathbb{N}$ be given. Since $\|D_m - D_n\| = \sup\{\|D_m(x) - D_n(x)\| \ / \ x \in B_X\}$ there exists some $x \in B_X$ such that $\|D_m(x) - D_n(x)\| > \|D_m - D_n\| - \frac{1}{k+1}$. Now $\|T(x) - D_n(x)\| \leq \|T - D_n\| < r_t - \frac{3}{k+1}$, (regardless of the value of $\|D_m(x) - T(x)\|$). For the $(\Leftarrow)$ direction: let k be given as above and choose m s.t. $\|D_m - T\| < \frac{1}{k+1}$, (such m exists since the set $\{D_m \ / \ m \in \mathbb{N}\}$ is dense in X). For this m choose some $x \in B_X$ which comes from hypothesis. Since $\|D_m(x) - T(x)\| \leq \|D_m - T\| < \frac{1}{k+1}$ we have that $\|T(x) - D_n(x)\| < r_t - \frac{3}{k+1}$. Also we have that $\|D_m(x) - D_n(x)\| > \|D_m - D_n\| - \frac{1}{k+1}$. It follows that

$$\begin{aligned} \|D_m - D_n\| &< \|D_m(x) - D_n(x)\| + \frac{1}{k+1} \\ &\leq \|D_m(x) - T(x)\| + \|T(x) - D_n(x)\| + \frac{1}{k+1} \\ &< \frac{1}{k+1} + r_t - \frac{3}{k+1} + \frac{1}{k+1} \\ &= r_t - \frac{1}{k+1} \ \text{; therefore} \\ \|T - D_n\| &\leq \|T - D_m\| + \|D_m - D_n\| \\ &< \frac{1}{k+1} + r_t - \frac{1}{k+1} = r_t. \end{aligned}$$

Hence we have proved the equivalence. From this it follows that P is in $\Sigma^1_1((T_i))$. ⊣

In the case of $(f_i)_{i\in\mathbb{N}}$ and f in $C(K)$ the previous proof becomes much simpler. Just take a sequence $(x_m)_{m\in\mathbb{N}}$ which is dense in K and notice that $\|f - d_n\| \le r_s \Leftrightarrow (\forall m)[\ |f(x_m) - d_n(x_m)| \le r_s\ ]$. Then apply the analogous of Lemma 2.4. We are now ready for the proof of the first main theorem.

Proof of Theorem 1.4. Recall that the space $B^{\mathbb{N}}$ is a Polish space and that we have assumed that it is recursively presented. The set Q which comes from the statement of the theorem is a coanalytic set. Thus it is in some $\Pi^1_1(\varepsilon)$. We may assume that it is in fact in Π^1_1. Put $Q_0(x, T, (T_i), j, s) \Leftrightarrow \|T(x) - T_j(x)\| \le r_s$. One can verify that Q is in Δ^1_1. Now define $S(x, T, (T_i)) \Leftrightarrow T(x) = \lim_{i\in\mathbb{N}} T_i(x)$; then S is also in Δ^1_1 since $S(x, T, (T_i)) \Leftrightarrow (\forall s)[\ r_s > 0 \to (\exists j_0)(\forall j \ge j_0)Q_0(x, T, (T_i), j, s)\ ]$. Using Theorem 2.3 we have that

$$\begin{aligned}
P_Q((T_i)) &\Leftrightarrow (T_i) \text{ is pointwise convergent to some } T \in B \,\&\, Q((T_i)_{i\in\mathbb{N}}, T)\\
&\Leftrightarrow (\exists T)(\forall x)[T(x) = \lim_{i\in\mathbb{N}} T_i(x) \,\&\, Q((T_i)_{i\in\mathbb{N}}, T)]\\
&\Leftrightarrow (\exists T \in \Delta^1_1((T_i)))(\forall x)[T(x) = \lim_{i\in\mathbb{N}} T_i(x) \,\&\, Q((T_i)_{i\in\mathbb{N}}, T)]\\
&\Leftrightarrow (\exists T \in \Delta^1_1((T_i)))(\forall x)[S(x, T, (T_i)) \,\&\, Q((T_i)_{i\in\mathbb{N}}, T)]
\end{aligned}$$

From the Theorem on Restricted Quantification 2.1 it follows that P_Q is in Π^1_1 and thus it is a coanalytic set. $\dashv$

The sets of the form P_Q are not Borel in general even if Q is the whole space. We outline the proof for some examples. Let Tr denote the Polish space of *trees on* $\mathbb{N}$. Also denote by WF the set of well-founded trees. The set WF is a coanalytic subset of Tr. A classic method for proving that a set $P \subseteq \mathcal{X}$ is not a Borel set is finding a continuous function $\pi : Tr \to \mathcal{X}$ such that $T \in WF \Leftrightarrow \pi(T) \in P$; in fact this proves that P is $\mathbf{\Pi}^1_1$-complete (see [7], 22.9).

In [7] (see 33.11) one can find the construction of a continuous function $\pi : Tr \to C([0, 1])^{\mathbb{N}}$, $\pi(T) \equiv (f_i^T)_{i\in\mathbb{N}}$ such that for all T, (1) the sequence $(f_i^T)_{i\in\mathbb{N}}$ is bounded and (2) $T \in WF$ if and only if the sequence $(f_i^T)_{i\in\mathbb{N}}$ is pointwise convergent – and in fact – if and only if the sequence $(f_i^T)_{i\in\mathbb{N}}$ is pointwise convergent to 0. This proves the $\mathbf{\Pi}^1_1$-completeness of the set $P_1 = \{(f_i)_{i\in\mathbb{N}} \in C([0, 1])^{\mathbb{N}} \ /\ \exists f \in C([0, 1]) \text{ such that } f_i \overset{pw}{\to} f\}$. Using condition (1) above it follows that the set $P_2 = \{(f_i)_{i\in\mathbb{N}} \in C([0, 1])^{\mathbb{N}} \ /\ (f_i)_{i\in\mathbb{N}} \text{ is weakly convergent in } C([0, 1])\}$ is also $\mathbf{\Pi}^1_1$-complete. By letting $F_n^T(x) = \int_0^x f_n^T(t)$ and $F^T(x) = \int_0^x f^T(t)$ one can verify the $\mathbf{\Pi}^1_1$-completeness of the set $P_3 = \{(F_i)_{i\in\mathbb{N}} \in C([0, 1])^{\mathbb{N}} \ /\ \exists F \in C([0, 1]) \text{ such that } F_i \overset{pw}{\to} F,$ the F_i's and F are differentiable and $F_i' \overset{i\in\mathbb{N}}{\to} F'\}$. In some cases though the sets of the form P_Q become Borel.

Proposition 2.5. *Let X be a separable Banach space and Y, B be such that the triple (X, Y, B) is a Polish system. Also let Q be a Borel subset of $Q \subseteq B^{\mathbb{N}} \times B$. Then the set P_Q defined in Theorem 1.4 is a Borel subset of $B^{\mathbb{N}}$.*

In particular if E is Banach space such that E^* is separable and Q is a Borel subset of $E^{\mathbb{N}} \times E$ then the set P_Q is also a Borel subset of $E^{\mathbb{N}}$.

Proof. Define $R((T_i), T) \Leftrightarrow$ *the sequence $(T_i)_{i\in\mathbb{N}}$ is pointwise convergent to T* $\Leftrightarrow$ (T_i) is bounded $\&(\forall n)[T_i(x_n) \overset{i\in\mathbb{N}}{\to} T(x_n)]$; where $(x_n)_{n\in\mathbb{N}}$ is norm dense in B_X. Hence R and $R \cap Q$ are Borel subsets of B. Notice that $P_Q = pr[R \cap Q]$ where $pr : B^{\mathbb{N}} \times B \to B^{\mathbb{N}}$ is the projection function and that the function pr is one-to-one on R. Using a classic theorem of Souslin (see [9]) we have that P_Q is also a Borel set as an one-to-one image of a Borel set. $\dashv$

Notice that no effective set theoretic arguments were used in the previous proof. Thus whenever X is separable and Q is Borel no effective theory is needed for the proof of Theorem 1.4.

3. Applications

In this section we present some applications of the theorems proved in the previous section. Let us begin with some basic notions. Denote with $[\mathbb{N}]^\omega$ the set of all infinite subsets of $\mathbb{N}$. Consider the topology on $[\mathbb{N}]^\omega$ which is generated from the sets $A_n = \{L \in [\mathbb{N}]^\omega \ /\ n \in L\}$, $B_n = \{L \in [\mathbb{N}]^\omega \ /\ n \notin L\}$, $n \in \mathbb{N}$. It is well known that $[\mathbb{N}]^\omega$ with that topology is a Polish space. If L is an infinite subset of $\mathbb{N}$ we denote by $[L]$ the set of all infinite subsets of L. The dichotomy we achieve is essentially based on the following theorem given by Silver. Let $\mathcal{A}$ be an analytic (equivalently coanalytic) subset of $[\mathbb{N}]^\omega$. Then for all $M \in [\mathbb{N}]^\omega$ there exists an infinite $L \subseteq M$ such that: either (I) $[L] \subseteq \mathcal{A}$; or (II) $[L] \subseteq [\mathbb{N}]^\omega \setminus \mathcal{A}$, see [11]. For the version of this theorem which refers to Borel sets instead of analytic sets see [5]. (The original proof uses the method of forcing; for a classical proof refer to [12]). Let us proceed with a useful characterization of a regular method of summability.

Remark 3.1. It is well known [3] that a double sequence of reals $(a_{ij})_{i,j\in\mathbb{N}}$ is a regular method of summability if and only if the following hold: (1) $\sup_{i\in\mathbb{N}} \sum_{j=0}^{\infty} |a_{ij}| < M < \infty$; (2) $\lim_{i\in\mathbb{N}} a_{ij} = 0$, for all $j \in \mathbb{N}$ and (3) $\lim_{i\in\mathbb{N}} \sum_{j=0}^{\infty} a_{ij} = 1$.

Now we are ready for the proof of Theorem 1.8.

Proof of Theorem 1.8. Define the set

$$\begin{aligned}
\mathcal{A} = \{L = \{k_0 < k_1 < \cdots < k_i < \cdots\} \in [\mathbb{N}]^\omega \ /\ &\text{the sequence } (T_{k_i})_{i\in\mathbb{N}} \text{ is}\\
&\text{pointwise summable to some } T \in B \text{ w.r.t. } (a_{ij})_{i,j\in\mathbb{N}} \text{ and } Q((T_{k_i})_{i\in\mathbb{N}}, T)\}.
\end{aligned}$$

For every $i \in \mathbb{N}$ and every $L = \{k_j \,/\, j \in \mathbb{N}\} \in [\mathbb{N}]^\omega$ put $f(i, L) = \sum_{j=0}^{\infty} a_{ij} T_{k_j}$. Using Remark 3.1 one can see that this f is well defined and continuous, (see [4], [10]). Put $\bar{f}(L) = (f(i, L))_{i\in\mathbb{N}}$ and $\bar{h}(L) = (T_i)_{i\in L}$; it is clear that both functions $\bar{f}$ and $\bar{h}$ are continuous.

Let R_Q be the subset of $B^{\mathbb{N}} \times B^{\mathbb{N}}$ which is defined by

$$R_Q((F_i), (S_i)) \Leftrightarrow \text{there exists some } T \in B \text{ such that the sequence } (F_i)_{i\in\mathbb{N}} \text{ is pointwise convergent to } T \text{ and } Q((S_i), T)$$

With arguments similar to these of the proof of Theorem 1.4 one can see that the set R_Q is coanalytic. Now compute

$$\begin{aligned} L \in \mathcal{A} &\Leftrightarrow \bar{f}(L) \text{ is pointwise convergent to some } T \in B \text{ and } Q((T_i)_{i\in L}, T) \\ &\Leftrightarrow \bar{f}(L) \text{ is pointwise convergent to some } T \in B \text{ and } Q(\bar{h}(L), T) \\ &\Leftrightarrow R_Q(\bar{f}(L), \bar{h}(L)) \end{aligned}$$

Using the continuity of $\bar{f}$, $\bar{h}$ we have that the set $\mathcal{A}$ is coanalytic. The dichotomy follows from Silver's Theorem.

Now we will prove the uniqueness of the limit for the case (I). Let M be such that $[M] \subseteq \mathcal{A}$; i.e. if $H \in [M]$ there exists $T \in B$ such that the sequence $(T_i)_{i\in H}$ is pointwise summable to T w.r.t. $(a_{ij})_{i,j\in\mathbb{N}}$ and $Q((T_i)_{i\in H}, T)$. We will show how one can choose a subsequence of $(T_i)_{i\in M}$ such that every one of its subsequences is pointwise summable to the same limit. We will not worry about the additional property Q since for every $H \in [M]$ and every $T \in B$ if $(T_i)_{i\in H}$ is pointwise summable to T then $Q((T_i)_{i\in H}, T)$. The latter is because we are in case (I) and also because the pointwise limit is unique. The method for proving the uniqueness is essentially the same with this in [4] and [10]. The following remark is going to be very useful.

Remark 3.2. Using (2) of Remark 3.1 it is easy to see that if a sequence $(F_n)_{n\in\mathbb{N}}$ in B is pointwise summable to some $F \in B$ and $S_0, S_1, \dots, S_N \in B$, then the sequence $(S_0, S_1, \dots, S_N, F_{N+1}, F_{N+2}, \dots, F_n, \dots)$ is also pointwise summable to F.

Since B is separable for each $k \geq 1$ there exists a sequence $(B_n^k)_{n\in\mathbb{N}}$ of open balls of radius $\frac{1}{k}$ s.t. $B = \bigcup_{n\in\mathbb{N}} B_n^k$. Notice that if $(n_k)_{k\in\mathbb{N}}$ is a sequence of natural numbers then the intersection $\bigcap_{k=1}^{\infty} B_{n_k}^k$ is at most a singleton. Define

$$\mathcal{A}(B_n^k) = \{L = \{k_0 < k_1 < \dots < k_i < \dots\} \subseteq M \,/\, \text{the sequence } (T_{k_i})_{i\in\mathbb{N}} \text{ is } \textit{pointwise} \text{ summable w.r.t. } (a_{ij})_{i,j\in\mathbb{N}} \text{ to } \textit{some } T \in B_n^k\}$$

Since B_n^k is an open subset of X, from Theorem 1.4 we have that the set $\mathcal{A}(B_n^k)$ is coanalytic for every k, n. From Silver's Theorem there exists some $H_0^1 \subseteq M$ such that: either (a) every subsequence of $(T_i)_{i\in H_0^1}$ is pointwise summable w.r.t. $(a_{ij})_{i,j\in\mathbb{N}}$ to some $T \in B_0^1$; or (b) every subsequence of $(T_i)_{i\in H_0^1}$ is pointwise summable w.r.t. $(a_{ij})_{i,j\in\mathbb{N}}$ to some $T \notin B_0^1$.

Repeating the same arguments we find $M \supseteq H_0^1 \supseteq H_1^1 \supseteq \dots \supseteq H_n^1 \supseteq \cdots$ such that for every $n \in \mathbb{N}$ either (A) every subsequence of $(T_i)_{i\in H_n^1}$ is pointwise summable w.r.t. $(a_{ij})_{i,j\in\mathbb{N}}$ to some $T \in B_n^1$; or (B) every subsequence of $(T_i)_{i\in H_n^1}$ is pointwise summable w.r.t. $(a_{ij})_{i,j\in\mathbb{N}}$ to some $T \notin B_n^1$.

Let $L_1 = \{l_0^1 < l_1^1 < \dots < l_n^1 < \cdots\}$ be the diagonal sequence which comes from $(H_n^1)_{n\in\mathbb{N}}$, i.e. l_n^1 is the nth term H_n^1. The sequence $(T_i)_{i\in L_1}$ is a subsequence of $(T_i)_{i\in M}$ and thus it is pointwise summable to some $T \in B = \bigcup_{n\in\mathbb{N}} B_n^1$. Take $n_1 \in \mathbb{N}$ such that $T \in B_{n_1}^1$. We claim that for this n_1 the case (A) above occurs. We need to show that (B) does not hold and in order to do so it is enough to find a subsequence of $(T_i)_{i\in H_{n_1}^1}$ which is summable to $T \in B_{n_1}^1$. Take the first $n_1 - 1$ terms of $(T_i)_{i\in L_1}$ and replace them with the first $n_1 - 1$ terms of $(T_i)_{i\in H_{n_1}^1}$; (if $n_1 = 0$ do nothing, if $n_1 = 1$ replace just the first term and so on). This gives rise to a subsequence $(T_i)_{i\in N}$ of $(T_i)_{i\in H_{n_1}^1}$ which differs from $(T_i)_{i\in L_1}$ by $n_1 - 1$ terms at most. Since the sequence $(T_i)_{i\in L_1}$ is pointwise summable to T using Remark 3.2 we obtain that $(T_i)_{i\in N}$ is pointwise summable to T as well.

We continue with $k = 2$ and find sets $L_1 \supseteq H_0^2 \supseteq H_1^2 \supseteq \dots \supseteq H_n^2 \supseteq \cdots$ such that for every $n \in \mathbb{N}$ either (A) every subsequence of $(T_i)_{i\in H_n^2}$ is pointwise summable w.r.t. $(a_{ij})_{i,j\in\mathbb{N}}$ to some $T \in B_n^2$; or (B) every subsequence of $(T_i)_{i\in H_n^2}$ is pointwise summable w.r.t. $(a_{ij})_{i,j\in\mathbb{N}}$ to some $T \notin B_n^2$. As before we define L_2 and find n_2 for which (A) holds. Proceeding inductively we find sequences of sets $(L_k)_{k\in\mathbb{N}}$, $(H_n^k)_{k,n}$ such that for all $k \geq 1$ we have that

- L_k is the diagonal sequence which comes from $(H_n^k)_{n\in\mathbb{N}}$,
- $L_{k-1} \supseteq H_0^k \supseteq H_1^k \supseteq \dots \supseteq H_n^k \supseteq \dots$ and
- for all $n \in \mathbb{N}$ we have that either $(A : k)$ every subsequence of $(T_i)_{i\in H_n^k}$ is pointwise summable w.r.t. $(a_{ij})_{i,j\in\mathbb{N}}$ to some $T \in B_n^k$; or $(B : k)$ every subsequence of $(T_i)_{i\in H_n^k}$ is pointwise summable w.r.t. $(a_{ij})_{i,j\in\mathbb{N}}$ to some $T \notin B_n^k$.

Also we define a sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$ for which $(A : k)$ holds for n_k. Define L_∞ to be the diagonal sequence which comes from $(L_k)_{k\in\mathbb{N}}$. We claim that each subsequence $(T_i)_{i\in L_\infty}$ is pointwise summable in $\bigcap_{k=1}^{\infty} B_{n_k}^k$. Since the last intersection is at most a singleton if we prove this we are done.

Let N be an infinite subset of L_∞ and $k \geq 1$. Since L_∞ is the diagonal sequence which comes from $(L_k)_{k\in\mathbb{N}}$ we have that if $i \in N$ then $i \in L_k$ for all large i. By repeating the same argument for L_k instead for L_∞ we have that if $i \in N$ then $i \in H_{n_k}^k$ for all large i. Let m be the number of naturals which are in N and not in $H^k_{n_k}$. We replace the first m terms of N by the first m terms of $H^k_{n_k}$ — if $m = 0$ we do nothing. Thus we have a subsequence $(T_i)_{i\in N'}$ of $(T_i)_{i\in H^k_{n_k}}$. Since $(A : k)$ holds for n_k we have that the sequence $(T_i)_{i\in N'}$ is pointwise summable in $B^k_{n_k}$. From Remark 3.2 we have that the sequence $(T_i)_{i\in N}$ is also pointwise summable in $B^k_{n_k}$. ⊣

In the rest of this section we give some applications of Theorem 1.6 taking $Q = X^{\mathbb{N}} \times X$. From now on we always take this Q when referring to Theorem 1.6. The dichotomy result of this theorem allows us to give a weak-convergence version of a theorem of James which characterizes a reflexive Banach space. First notice the following.

Proposition 3.3. *Let E be a finite dimensional normed space, $(b_i)_{i\in\mathbb{N}}$ be a bounded sequence in E, $b \in E$ and $(a_{ij})_{i,j\in\mathbb{N}}$ be a regular method of summability. Assume that every subsequence of $(b_i)_{i\in\mathbb{N}}$ is summable to b w.r.t. $(a_{ij})_{i,j\in\mathbb{N}}$. Then the sequence $(b_i)_{i\in\mathbb{N}}$ converges to b in norm.*

Proof. Assume not, then there exist some $\varepsilon > 0$ and a subsequence $(c_i)_{i\in\mathbb{N}}$ such that $\|c_i - b\| \geq \varepsilon$ for all $i \in \mathbb{N}$. Since $(c_i)_{i\in\mathbb{N}}$ is bounded and E is finite dimensional there exists a subsequence $(d_i)_{i\in\mathbb{N}}$ which converges in norm to some $d \in E$. It follows that the sequence $(e_i)_{i\in\mathbb{N}}$ defined by $e_i = \sum_{j=0}^{\infty} a_{ij} d_j$ $(i \in \mathbb{N})$ is also convergent to d.

Also since $(d_i)_{i\in\mathbb{N}}$ is a subsequence of $(b_i)_{i\in\mathbb{N}}$ from hypothesis we have that $(d_i)_{i\in\mathbb{N}}$ is summable to b w.r.t. $(a_{ij})_{i,j\in\mathbb{N}}$, i.e. the sequence $(e_i)_{i\in\mathbb{N}}$ defined above converges to b. Hence $d = b$ and $d_i \overset{\|\cdot\|}{\to} b$ which is a contradiction since $(d_i)_{i\in\mathbb{N}}$ is a subsequence of $(c_i)_{i\in\mathbb{N}}$ and $\|c_i - b\| \geq \varepsilon$ for all $i \in \mathbb{N}$. ⊣

Corollary 3.4. *Let X be a Banach space and $(a_{ij})_{i,j\in\mathbb{N}}$ be a regular method of summability. Then X is reflexive if and only if every bounded sequence in X has a subsequence which is weakly summable w.r.t. $(a_{ij})_{i,j\in\mathbb{N}}$.*

Proof. First recall that X is reflexive if and only if every bounded sequence in X has a weakly convergent subsequence. So the left-to-right-hand direction is immediate.

For the inverse direction let $(x_i)_{i\in\mathbb{N}}$ be a bounded sequence in X. Notice that from the hypothesis we have that case (II) of Theorem 1.6 fails for $(x_i)_{i\in\mathbb{N}}$. Thus case (I) holds, i.e. there exist some $x \in X$ and a subsequence $(x_{k_i})_{i\in\mathbb{N}}$ such that every one of its subsequences is weakly summable to x w.r.t. $(a_{ij})_{i,j\in\mathbb{N}}$.

We claim that the sequence $(x_{k_i})_{i\in\mathbb{N}}$ is weakly convergent to x. Let $x^* \in X^*$. Define $b_i = x^*(x_{k_i}) - x^*(x)$, for all $i \in \mathbb{N}$. Then $(b_i)_{i\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}$. We will prove that each subsequence of $(b_i)_{i\in\mathbb{N}}$ is summable to $0 \in \mathbb{R}$ w.r.t. $(a_{ij})_{i,j\in\mathbb{N}}$. If we prove this then from Proposition 3.3 we will have that $b_i \to 0$, i.e. $x^*(x_{k_i}) \to x^*(x)$. A subsequence of $(b_i)_{i\in\mathbb{N}}$ is defined by a subsequence $(y_i)_{i\in\mathbb{N}}$ of $(x_{k_i})_{i\in\mathbb{N}}$. We need to prove that $\sum_{j=0}^{\infty} a_{ij}(x^*(y_j) - x^*(x)) \overset{i\in\mathbb{N}}{\to} 0$. We know that $\sum_{j=0}^{\infty} a_{ij} x^*(y_j) \overset{i\in\mathbb{N}}{\to} x^*(x)$, since $(y_i)_{i\in\mathbb{N}}$ is a subsequence of $(x_{k_i})_{i\in\mathbb{N}}$. Also $\sum_{j=0}^{\infty} a_{ij} x^*(x) = x^*(x) \cdot \sum_{j=0}^{\infty} a_{ij} \overset{i\in\mathbb{N}}{\to} x^*(x) \cdot 1 = x^*(x)$, from the properties of a regular method of summability. Hence $\sum_{j=0}^{\infty} a_{ij}(x^*(y_j) - x^*(x)) = \sum_{j=0}^{\infty} a_{ij} x^*(y_j) - \sum_{j=0}^{\infty} a_{ij} x^*(x) \overset{i\in\mathbb{N}}{\to} x^*(x) - x^*(x) = 0$. ⊣

The previous proof can be extended and give Corollary 1.9 mentioned in the introduction. Let us outline the proof of this. First recall that each such set K is weakly compact iff every sequence in K has a weakly convergent subsequence in K. So the left-to-right-hand direction of the corollary is clear. For the converse direction we proceed as in the previous proof; the only thing that needs some attention is where we find a sequence $(x_{k_i})_{i\in\mathbb{N}}$ in K which is weakly convergent to some x. We need to know that x is a member of K as well. However this is immediate since K is weakly closed.

A careful look at the proof of Corollary 3.4 reveals the following: if case (II) of Theorem 1.6 fails for some bounded sequence $(e_i)_{i\in\mathbb{N}}$, then $(e_i)_{i\in\mathbb{N}}$ has a weakly convergent subsequence. It is clear that if case (II) of Theorem 1.6 fails for some bounded sequence $(e_i)_{i\in\mathbb{N}}$, then it also fails for every one of its subsequences. Thus we have actually proved the following: if case (II) of Theorem 1.6 fails for some bounded sequence $(e_i)_{i\in\mathbb{N}}$, then every subsequence of $(e_i)_{i\in\mathbb{N}}$ has a weakly convergent subsequence. We now ask when one can pick this weakly convergent subsequence in a "Borel manner". The basic tool for this is given by the following theorem which is due to Debs (see [1]). Let $\mathcal{X}$ be a recursively presented Polish space and $(f_i)_{i\in\mathbb{N}}$ be a sequence of continuous functions from $\mathcal{X}$ to $\mathbb{R}$ such that: (1) the sequence $(f_i)_{i\in\mathbb{N}}$ is pointwise bounded, (2) every cluster point of $(f_i)_{i\in\mathbb{N}}$ in $\mathbb{R}^{\mathcal{X}}$ with the topology of pointwise convergence is a Borel measurable function and (3) the sequence $(f_i)_{i\in\mathbb{N}}$ is in $\Delta^1_1(\alpha)$. Then there exists $L \in \Delta^1_1(\alpha)$ such that the subsequence $(f_i)_{i\in L}$ is pointwise convergent.

There is something here that needs some explanation. Since $\mathcal{X}$ need not be compact we may not view the sequence $(f_i)_{i\in\mathbb{N}}$ as a member of a Polish space. So one can ask what the abbreviation "$(f_i)_{i\in\mathbb{N}}$ is in $\Delta^1_1(\alpha)$" means. That is the relation of the pointwise evaluation $P(x, i, n, s) \Leftrightarrow f_i(x) \in N^{\mathbb{R}}_{\langle n,s\rangle}$ (where $N^{\mathbb{R}}_{\langle n,s\rangle} = \{y \in \mathbb{R} \,/\, |y - r_n| < r_s\}$) is a $\Delta^1_1(\alpha)$ subset of $\mathcal{X} \times \mathbb{N} \times \mathbb{N}$.

Now let X be a separable Banach space and $(e_i)_{i\in\mathbb{N}}$ be a bounded sequence in X for which case (II) of Theorem 1.6 fails. We view every e_i as a real function defined on $\mathcal{X} = (B_{X^*}, w^*)$. Fix also an $L \subseteq [\mathbb{N}]^{\omega}$. The relation $P(x^*, i, n, s) \Leftrightarrow x^*(e_i) \in N^{\mathbb{R}}_{\langle n,s\rangle}$ is in fact open; thus it is in $\Delta^1_1(\alpha)$ for some α. Therefore the sequence $(e_i)_{i\in L}$ is in $\Delta^1_1(\alpha, L)$ in the sense of Debs' Theorem. Using our hypothesis about the sequence $(e_i)_{i\in\mathbb{N}}$ it is not difficult to verify conditions (1) and (2) in the latter theorem for the sequence $(e_i)_{i\in L}$. Thus we have derived the following. If $(e_i)_{i\in\mathbb{N}}$ is a bounded sequence in a separable Banach space X for which case (II) of Theorem 1.6 fails, then for all L there exists M in $\Delta^1_1(\alpha, L)$ such that the sequence $(e_i)_{i\in M}$ is weakly convergent in X, (where α is a fixed point depending on the initial sequence). Debs' result has a classic interpretation, see [1]. In our case that will be Corollary 1.10.

Proof of Corollary 1.10. Define $P \subseteq [\mathbb{N}]^\omega \times [\mathbb{N}]^\omega$ by $P(L, M) \Leftrightarrow M \subseteq L$ & the sequence $(e_i)_{i\in M}$ is weakly convergent. From Theorem 1.4 it follows that the previous P is a coanalytic set. We may assume that P is in Π^1_1. From the previous remarks we have that for every $L \in [\mathbb{N}]^\omega$ there exists $M \in [\mathbb{N}]^\omega$ in $\Delta^1_1(L)$ such that $P(L, M)$, (as usual we have eliminated the parameter α). From the Strong Δ-Selection Principle (see 4D.6 in [9]) there exists a Π^1_1-recursive function f defined on all $[\mathbb{N}]^\omega$ with values in $[\mathbb{N}]^\omega$ s.t. $P(L, f(L))$ for every $L \in [\mathbb{N}]^\omega$. Now total Π^1_1-recursive functions are Δ^1_1-recursive (see 4C.3 in [9]) and thus Borel. $\dashv$

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