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The notion of exhaustiveness and Ascoli-type theorems

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Abstract

In this paper we introduce the notion of exhaustiveness which applies for both families and nets of functions. This new notion is close to equicontinuity and describes the relation between pointwise convergence for functions and α-convergence (continuous convergence). Using these results we obtain some Ascoli-type theorems dealing with exhaustiveness instead of equicontinuity. Also we deal with the corresponding notions of separate exhaustiveness and separate α-convergence. Finally we give conditions under which the pointwise limit of a sequence of arbitrary functions is a continuous function.

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Keywords: Exhaustiveness; Weak exhaustiveness; Equicontinuity; Joint continuity

0. Introduction

The notion of α-convergence (otherwise continuous convergence or "stetige Konvergenz") has been known by the beginning of the 20th century (see [9,10]). Around 1950s Stoilov [7] and Arens [1] came up with some results which characterize α-convergence and are very helpful for this paper. Also this type of convergence was considered in connection with some other types of convergence in [3]. In Section 1 we state the basic facts about α-convergence.

In Section 2 we introduce the notion of *exhaustiveness* which goes through the rest of this parer. This is closely connected to the notion of equicontinuity. We first apply it for families and sequences of functions. This new notion enables us to view the convergence of a sequence of functions in terms of *properties of the sequence* and not of properties of functions as single members. An example of this is Theorem 2.6 which measures the step from pointwise convergence to α-convergence using the notion of exhaustiveness.

Section 3 is divided into two parts. In the first part we use Theorem 2.6 in order to give a generalization of the classical Ascoli theorem (Theorem 3.1.1). In the second part we extend the notions of α-convergence and exhaustiveness to nets of functions. In connection with [1] we consider conditions under which α-convergence follows from a topology (see Theorem 3.2.5 and Corollary 3.2.7). Afterwards we introduce the notion of an exhaustive net of functions and give the analogue of Theorem 2.6 (Theorem 3.2.12). From this we derive that the α-limit of a net of functions is

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a continuous function (Corollary 3.2.13). Using the previous results and Theorem 3.2.12 we obtain some Ascoli-type theorems (3.2.19, 3.2.20). Also we derive a characterization of locally compact regular spaces (Corollary 3.2.15).

In the first part of Section 4 we consider functions defined on products $X \times Y$ and the corresponding notions of *separate exhaustiveness* and *separate α-convergence*. We give a Namioka-type theorem for exhaustiveness from which (using again Theorem 2.6) we derive the corresponding result for α-convergence (Theorem 4.1.3 and Corollary 4.1.5). In the second part we consider the notion of *weak exhaustiveness* for sequences of functions. It is well known that the pointwise limit of a sequence of continuous functions is not necessarily a continuous function. Following the method of considering the properties of the sequence instead of the properties of each function, we give Theorem 4.2.3. This answers to the problem of finding conditions under which the pointwise limit of a sequence of functions is a continuous function.

1. Basic facts about α-convergence

Let us begin with some comments on notation. With X and Y we mean metric spaces, unless stated otherwise. If it is not mentioned explicitly the symbol d stands for the metric on X and the symbol p for the metric on Y.

If x is a member of X and δ is a positive number, with $S(x,\delta)$ we mean the (open) ball of radious δ, i.e. $S(x,\delta) = \{y \in X \,/\, d(y,x) < \delta\}$. Also if X and Y are metric spaces we denote with $C(X,Y)$ the set of all continuous functions from X to Y.

We now give the definition of α-convergence (continuous convergence) [3].

Definition 1.1. Let $f, f_n, n \in \mathbb{N}$ be functions from X to Y. The sequence $(f_n)_{n\in\mathbb{N}}$ α-converges to f iff for every $x \in X$ and for every sequence $(x_n)_{n\in\mathbb{N}}$ of points of X converging to x, the sequence $(f_n(x_n))_{n\in\mathbb{N}}$ converges to $f(x)$.

We shall write $f_n \xrightarrow{\alpha} f$ to denote that $(f_n)_{n\in\mathbb{N}}$ α-converges to f. Also we will keep the analogous notation about pointwise and uniform convergence, i.e., we will denote them with $f_n \xrightarrow{pw} f$ and $f_n \xrightarrow{u} f$ respectively.

Remarks 1.2.

(1) It is obvious that α-convergence is stronger than pointwise convergence.
(2) The usual convergences such as pointwise and uniform do not require a topology for the domain space. However a topology is needed for α-convergence.
(3) Take $f : \mathbb{R} \to \mathbb{R}$ any non-continuous function and $x_n \to x$ such that the sequence $(f(x_n))_{n\in\mathbb{N}}$ does not converge to $f(x)$. If we put $f_n \equiv f$ for all $n \in \mathbb{N}$, we see that $(f_n)_{n\in\mathbb{N}}$ does not α-converge to f although the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f.
(4) For all $n \in \mathbb{N}$ define $f_n : (0,1] \to \mathbb{R}$ such that $f_n(x) = 1 - nx$, for $x \leqslant \frac{1}{n}$ and $f_n(x) = 0$, for $x > \frac{1}{n}$. Then we can see that the sequence (f_n) α-converges to zero function but does not converge uniformly.

The next proposition is due to Stoilov [7] except the last assertion and describes some interesting results about α-convergence.

Proposition 1.3. *Let (X,d), (Y,p) be metric spaces and functions f, f_n, $n \in \mathbb{N}$, from X to Y.*

(1) *If the sequence $(f_n)_{n\in\mathbb{N}}$ α-converges to f, then f is continuous.*
(2) *The sequence $(f_n)_{n\in\mathbb{N}}$ α-converges to f if and only if f is continuous and $(f_n)_{n\in\mathbb{N}}$ converges to f uniformly on every compact subset of X.*
In particular:
(3) *If $(f_n)_{n\in\mathbb{N}}$ converges to f uniformly and f is continuous, then $(f_n)_{n\in\mathbb{N}}$ α-converges to f.*
And also:
(4) *If X is compact and $(f_n)_{n\in\mathbb{N}}$ α-converges to f, then $(f_n)_{n\in\mathbb{N}}$ converges to f uniformly.*
The following result is due to Holá–Šalát [5].
(5) *A metric space X is compact if and only if for all functions f, f_n, $n \in \mathbb{N}$, from X to Y, if $(f_n)_{n\in\mathbb{N}}$ α-converges to f, then $(f_n)_{n\in\mathbb{N}}$ converges to f uniformly.*

The following is very useful for latter on.

Proposition 1.4. *For all functions f, f_n, $n \in \mathbb{N}$, from X to Y, if $(f_n)_{n\in\mathbb{N}}$ α-converges to f then each subsequence $(f_{k_n})_{n\in\mathbb{N}}$ also α-converges to f.*

Proof. Let $x \in X$ and $x_n \to x$. Define $y_n = x_i$ if $n = k_i$ for some $i \in \mathbb{N}$ and $y_n = x$ otherwise. We have that $y_{k_n} = x_n$ for each $n \in \mathbb{N}$ and also $y_n \to x$. Since $f_n \xrightarrow{\alpha} f$ we obtain that $f_n(y_n) \to f(x)$. Therefore $f_{k_n}(y_{k_n}) \to f(x)$. Since $y_{k_n} = x_n$ we have that $f_{k_n}(x_n) \to f(x)$.

Notice that in fact we have proved the following: if for all $(x_n)_{n\in\mathbb{N}}$ in X with $x_n \to x$ we have that $f_n(x_n) \to f(x)$, then for all $(x_n)_{n\in\mathbb{N}}$ in X with $x_n \to x$ we have that $f_{k_n}(x_n) \to f(x)$. (We will use this in Remark 2.7.) □

2. Exhaustiveness

We now introduce a new notion which is close to the notion of equicontinuity.

Definition 2.1. Let (X, d), (Y, p) be metric spaces, $x \in X$, $\mathcal{F}$ be a family of functions from X to Y and $f_n : X \to Y$, $n \in \mathbb{N}$.

(1) If $\mathcal{F}$ is infinite, we call the family $\mathcal{F}$ *exhaustive at* x iff for every $\varepsilon > 0$ there exists $\delta > 0$ and A a finite subset of $\mathcal{F}$ such that: for every $y \in S(x, \delta)$ and for every $f \in \mathcal{F} \setminus A$ we have that $p(f(y), f(x)) < \varepsilon$.
(2) In case where $\mathcal{F}$ is finite we define $\mathcal{F}$ to be exhaustive at x iff each member of $\mathcal{F}$ is continuous function at X.
(3) $\mathcal{F}$ is *exhaustive* iff $\mathcal{F}$ is exhaustive at every x.
(4) The *sequence* $(f_n)_{n\in\mathbb{N}}$ is called *exhaustive at* x iff for all $\varepsilon > 0$ there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for all $y \in S(x, \delta)$ and all $n \geqslant n_0$ we have that $p(f_n(y), f_n(x)) < \varepsilon$.
(5) The sequence $(f_n)_{n\in\mathbb{N}}$ is called *exhaustive* iff it is exhaustive at every $x \in X$.

Notice that in the most interesting case where $(f_n)_{n\in\mathbb{N}}$ is a sequence of functions for which $f_n \neq f_m$ for $n \neq m$, then the family $\mathcal{F} = \{f_n \,/\, n \in \mathbb{N}\}$ is exhaustive at some $x_0 \in X$ if and only if the sequence $(f_n)_{n\in\mathbb{N}}$ is exhaustive at x_0.

Remarks 2.2.

(1) An equicontinuous family is an exhaustive family such that for every $\varepsilon > 0$ the finite set A in Definition 2.1(1) can be taken to be the empty set. So equicontinuity implies exhaustiveness.
(2) Saying that $\mathcal{F}$ is exhaustive does not imply that there exists a finite subset of $\mathcal{F}$ (call it A) such that $\mathcal{F} \setminus A$ is equicontinuous. (That is because the set A in the definition depends on $\varepsilon > 0$.) See also Example 2.4.

Proposition 2.3. *Let (X, d), (Y, p) be metric spaces, $x \in X$, $\mathcal{F}$ a family of functions from X to Y and $f_n : X \to Y$, $n \in \mathbb{N}$.*

(1) *$\mathcal{F}$ is equicontinuous at x if and only if $\mathcal{F}$ is exhaustive at x and for each $f \in \mathcal{F}$, f is continuous at x.*
(2) *The family $\{f_n \,/\, n \in \mathbb{N}\}$ is equicontinuous at x if and only if the sequence $(f_n)_{n\in\mathbb{N}}$ is exhaustive at x and each f_n is continuous at x.*

Proof. We will prove only (1) since the argument for (2) is the same. Also we may assume that $\mathcal{F}$ is infinite.

The ($\Rightarrow$) direction is obvious. For the inverse direction: Let $\varepsilon > 0$, then there exist $\delta_1 > 0$ and A finite subset of $\mathcal{F}$ such that for every $y \in S(x, \delta_1)$ and for every $f \in \mathcal{F} \setminus A$ we have $p(f(y), f(x)) < \varepsilon$. Since each f is continuous at x there exists $\delta_f > 0$ such that for every $y \in S(x, \delta_f)$ we have $p(f(y), f(x)) < \varepsilon$. Put $\delta = \min\{\delta_1, \delta_f \,/\, f \in A\} > 0$. One can check that for every $y \in S(x, \delta)$ and for every $f \in \mathcal{F}$ we have that $p(f(y), f(x)) < \varepsilon$. □

The preceding proposition suggests that there exists an exhaustive sequence (similarly family) which contains no continuous functions. Indeed this happens as we can see in the following example.

Example 2.4. For $n \in \mathbb{N}$ define $f_n : \mathbb{R} \to \mathbb{R}$ such that $f_n(x) = \frac{1}{n}$, for $x \leqslant 0$ and $f_n(x) = \frac{1}{2n}$, for $x > 0$. Of course no f_n is continuous at 0. We claim that the sequence $(f_n)_{n\in\mathbb{N}}$ is exhaustive at 0. Let $\varepsilon > 0$, then there exists an integer $n_0 > \frac{1}{2\varepsilon}$ such that for $\delta = 1$, for all $y \in (-1, 1)$ and for all $n \geqslant n_0$ we have that $|f_n(y) - f_n(0)| \leqslant \frac{1}{2n} < \varepsilon$.

So we obtain the following picture for an exhaustive family of functions $\mathcal{F}$: the family $\mathcal{F}$ is equicontinuous "as a whole" with the continuity of each member of $\mathcal{F}$ erased.

Some of the results of equicontinuity apply for exhaustiveness. For example we know that the pointwise limit of an equicontinuous sequence of functions is a continuous function. The same holds if we replace equicontinuity with exhaustiveness.

Proposition 2.5. *Let (X, d), (Y, p) be metric spaces and f, f_n, $n = 1, 2, \ldots$, Y-valued functions defined on X. If the sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f and $(f_n)_{n\in\mathbb{N}}$ is exhaustive at $x \in X$ then f is continuous at x.*

Proof. Since $(f_n)_{n\in\mathbb{N}}$ is exhaustive at x there exists $\delta > 0$ and there exists $n_0 \in \mathbb{N}$ such that for all $y \in S(x, \delta)$ and all $n \geqslant n_0$ we have that $p(f_n(y), f_n(x)) < \frac{\varepsilon}{3}$.

Let $y \in S(x, \delta)$. Since $f_n \xrightarrow{pw} f$ there exists $n_1 \in \mathbb{N}$ such that for all $n \geqslant n_1$ it holds that $p(f_n(y), f(y)) < \frac{\varepsilon}{3}$ and $p(f_n(x), f(x)) < \frac{\varepsilon}{3}$. Put $n_2 = \max\{n_0, n_1\}$. Then

$$p\big(f(y), f(x)\big) \leqslant p\big(f(y), f_{n_2}(y)\big) + p\big(f_{n_2}(y), f_{n_2}(x)\big) + p\big(f_{n_2}(x), f(x)\big) < \varepsilon. \qquad \square$$

If in the previous proposition we replace the condition "$(f_n)_{n\in\mathbb{N}}$ is exhaustive at x" with "$\{f_n \,/\, n \in \mathbb{N}\}$ is exhaustive at x" the same conclusion will still hold. To see that take the interesting case where the set $\{f_n \,/\, n \in \mathbb{N}\}$ is infinite. Choose naturals $k_1 < k_2 < \cdots < k_n < \cdots$ such that $f_{k_n} \neq f_{k_m}$ for $n \neq m$. The sequence $(f_{k_n})_{n\in\mathbb{N}}$ is exhaustive at x and then follow the proof of Proposition 2.5.

The notion of exhaustiveness is connected with the notion of α-convergence.

Theorem 2.6. *Let (X, d), (Y, p) be metric spaces and functions $f_n, f : X \to Y$, $n \in \mathbb{N}$. The following are equivalent*:

(1) *The sequence $(f_n)_{n\in\mathbb{N}}$ α-converges to f.*
(2) *The sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f and $(f_n)_{n\in\mathbb{N}}$ is exhaustive.*

Proof. (1) ⇒ (2) Let us assume that $(f_n)_{n\in\mathbb{N}}$ is not exhaustive at some point x. This means that there exists $\varepsilon > 0$ such that for every $\delta > 0$ and for every $n \in \mathbb{N}$ there exist $x_{n,\delta} \in S(x, \delta)$ and $k_n \geqslant n$ such that $p(f_{k_n}(x_{n,\delta}), f_{k_n}(x)) \geqslant \varepsilon$. By induction we can define a sequence (x_n) and a set of natural numbers $\{k_1 < k_2 < \cdots < k_n < \cdots\}$ such that $d(x_n, x) < \frac{1}{n}$ and $p(f_{k_n}(x_n), f_{k_n}(x)) \geqslant \varepsilon$ for each $n \in \mathbb{N}$ $(*)$.

Since $f_n \xrightarrow{\alpha} f$ and $(f_{k_n})_{n\in\mathbb{N}}$ is a subsequence of $(f_n)_{n\in\mathbb{N}}$ from Proposition 1.4 we obtain that $f_{k_n} \xrightarrow{\alpha} f$. We have that $x_n \to x$, hence $f_{k_n}(x_n) \to f(x)$. Also $f_{k_n}(x) \to f(x)$ because α-convergence is stronger than pointwise convergence. From the last two statements we obtain that $p(f_{k_n}(x_n), f_{k_n}(x)) \to 0$ contradicting $(*)$.

(2) ⇒ (1) Let $x \in X$ and $x_n \to x$. We need to prove that $f_n(x_n) \to f(x)$. Assume $\varepsilon > 0$, since $f_n \xrightarrow{pw} f$ there exists $n_1 \in \mathbb{N}$ such that for all $n \geqslant n_1$ we have that $p(f_n(x), f(x)) < \frac{\varepsilon}{2}$. Also $(f_n)_{n\in\mathbb{N}}$ is exhaustive and so there exist $\delta > 0$ and $n_2 \in \mathbb{N}$ such that for all $y \in S(x, \delta)$ and for all $n \geqslant n_2$ it follows that $p(f_n(y), f_n(x)) < \frac{\varepsilon}{2}$.

Since $x_n \to x$, for $\delta > 0$ there exist $n_3 \in \mathbb{N}$ such that for all $n \geqslant n_3$ we have that $d(x_n, x) < \delta$. Therefore if $n \geqslant \max\{n_2, n_3\}$ from the previous two statements we have that $p(f_n(x_n), f_n(x)) < \frac{\varepsilon}{2}$. Put $n_0 = \max\{n_1, n_2, n_3\}$ and let $n \geqslant n_0$, then

$$p\big(f_n(x_n), f(x)\big) \leqslant p\big(f_n(x_n), f_n(x)\big) + p\big(f_n(x), f(x)\big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \square$$

Remark 2.7. A careful look on the proof of the preceding theorem and Proposition 1.4 reveals that we have actually proved the following.

Let (X, d), (Y, p) be metric spaces, $x \in X$ and functions $f_n, f : X \to Y$, $n \in \mathbb{N}$. The following are equivalent.

(1) For all sequences $(x_n)_{n\in\mathbb{N}}$ in X with $x_n \to x$ we have that $f_n(x_n) \to f(x)$.
(2) The sequence $(f_n(x))_{n\in\mathbb{N}}$ converges to $f(x)$ and the sequence $(f_n)_{n\in\mathbb{N}}$ is exhaustive at x.

We put this remark separately from Theorem 2.6 since we will refer to it only once in Corollary 4.1.5.

Theorem 2.6 is interesting not only because it measures the step from pointwise converge to α-converge but also because it gives some useful results for the α-limit. For example Theorem 2.6 and Proposition 2.5 imply that the α-limit is a continuous function (which is Proposition 1.3(1) already known). In the next section we will extend this theorem and we will use both results in order to give some Ascoli-type theorems.

3. Connections with general topology

3.1. A generalization of the classical Ascoli theorem

Recall that if X is a metric space we define the space of *bounded functions* on X,

$$Bd(X) = \Big\{ f : X \to \mathbb{R} \,/\, \sup_{x\in X} \big|f(x)\big| < \infty \Big\}.$$

The *supremum norm* on $Bd(X)$ is defined by $\|f\| = \sup_{x\in X} |f(x)|$. We shall denote the corresponding metric space with $(Bd(X), \|\cdot\|)$. Of course the topology induced from this norm is the topology of uniform convergence. In case where X is compact the set $C(X,\mathbb{R})$ is a subset of $Bd(X)$ and therefore we can view it with the norm $\|\cdot\|$.

Also recall the classical Ascoli theorem: if X is a compact metric space and $\mathcal{F} \subseteq C(X,\mathbb{R})$ then $\mathcal{F}$ is compact iff $\mathcal{F}$ is closed, bounded and equicontinuous.

Theorem 3.1.1 *(Generalized Ascoli theorem). Let X be a compact metric space and let $\mathcal{F}$ be an infinite subset of $(Bd(X), \|\cdot\|)$. The following are equivalent*:

(1) *If $\mathcal{F}$ is closed, bounded and exhaustive then $\mathcal{F}$ is compact.*
(2) *If moreover every cluster point of $\mathcal{F}$ is a continuous function then the converse of* (1) *is also true.*

Note that $\mathcal{F}$ is not necessarily a subset of $C(X,\mathbb{R})$. Using the fact that an exhaustive family $\mathcal{F}$ which consists of continuous functions is equicontinuous (Proposition 2.3(1)) it is clear that this theorem is indeed a generalization of the classical Ascoli theorem.

Proof. (1) The main frame is the same with the classical proof. It is enough to prove that $\mathcal{F}$ is sequentially compact. Let $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}$ and $(x_n)_{n\in\mathbb{N}}$ a dense subset of X. The sequence $(f_n(x_1))_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded since $\mathcal{F}$ is bounded. Therefore there exists a convergent subsequence $(f_{k_n^1}(x_1))_{n\in\mathbb{N}}$; the sequence $(f_{k_n^1}(x_2))_{n\in\mathbb{N}}$ is bounded and so there exists a convergent subsequence $((f_{k_n^2}(x_2))_{n\in\mathbb{N}}$.

Inductively we obtain sequences of naturals $\cdots \subseteq (k_n^{j+1}) \subseteq (k_n^j) \subseteq \cdots \subseteq (k_n^1)$ such that for each $j \in \mathbb{N}$ the sequence $(f_{k_n^j}(x_j))_{n\in\mathbb{N}}$ is convergent. One can check that for each $j \in \mathbb{N}$ the diagonal sequence $(f_{k_n^n}(x_j))_{n\in\mathbb{N}}$ is also convergent. Using the fact that $\mathcal{F}$ is exhaustive we obtain that for each $x \in X$ the sequence $(f_{k_n^n}(x))_{n\in\mathbb{N}}$ is a Cauchy sequence (the method for this, is very much the same with the classical one).

Put $f(x) = \lim_{n\in\mathbb{N}} f_{k_n^n}(x)$; using the exhaustiveness and the compactness of X from Theorem 2.6 and Proposition 1.3(4) it follows that $f_{k_n^n} \xrightarrow{u} f$. Since $\mathcal{F}$ is closed we have that $f \in \mathcal{F}$ and so $\mathcal{F}$ is compact.

(2) Assume that $\mathcal{F}$ is compact but not exhaustive at some point x. Then by definition there exists $\varepsilon > 0$ such that for every $\delta > 0$ and for every finite A subset of $\mathcal{F}$, there exists $x_{\delta,A} \in S(x,\delta)$ and $f_{\delta,A} \in \mathcal{F} \setminus A$ such that $|f_{\delta,A}(x_{\delta,A}) - f_{\delta,A}(x)| \geqslant \varepsilon$.

By induction we define two sequences $(x_n)_{n\in\mathbb{N}} \subseteq X$ and $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{F}$ such that: $f_n \neq f_m$ for $n \neq m$, $x_n \to x$ and $|f_n(x_n) - f_n(x)| \geqslant \varepsilon$ for each $n \in \mathbb{N}$ $(*)$. Since $\mathcal{F}$ is compact there exists a subsequence $(f_{k_n})_{n\in\mathbb{N}}$ and $f \in \mathcal{F}$ such that $f_{k_n} \xrightarrow{u} f$. From hypothesis the function f is continuous and so $f(x_{k_n}) \xrightarrow{n\in\mathbb{N}} f(x)$.

For the $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ we have that $|f(x_{k_n}) - f(x)| < \frac{\varepsilon}{3}$. Also there exists $n_1 \in \mathbb{N}$ such that for all $n \geqslant n_1$ it holds that $\|f_{k_n} - f\| < \frac{\varepsilon}{3}$. Let $n = \max\{n_0, n_1\}$, then $|f_{k_n}(x_{k_n}) - f_{k_n}(x)| \leqslant |f_{k_n}(x_{k_n}) - f(x_{k_n})| + |f(x_{k_n}) - f(x)| + |f(x) - f_{k_n}(x)| \leqslant 2 \cdot \|f_{k_n} - f\| + |f(x_{k_n}) - f(x)| < \varepsilon$, contradicting $(*)$. □

3.2. *Exhaustive nets of functions and other Ascoli-type results*

Here we deal with functions defined on a topological space. First we define α-convergence for a net of functions using a condition which also appears in [1] and show that this is indeed an extension of α-convergence as defined in Definition 1.1. Then we introduce the notion of an exhaustive net of functions and prove the analogue of Theorem 2.6. From this we derive Corollary 3.2.13 which ensures that the α-limit of a net of functions is a continuous function.

A natural question to ask is whether α-convergence follows from a topology, i.e., if there is a topology $\mathcal{T}$ for which $f_i \xrightarrow{\mathcal{T}} f$ iff $f_i \xrightarrow{\alpha} f$. This question has been settled by Arens in [1]. Using results of this type and the notion of exhaustiveness we derive two Ascoli-type results (Theorems 3.2.19, 3.2.20).

The most obvious way to define α-convergence for a net of functions $(f_i)_{i\in I}$ is to give the following condition: for all $x \in X$ and all nets $(x_i)_{i\in I}$ with $x_i \to x$ it follows that $f_i(x_i) \to f(x)$. In fact this condition is the definition of *continuous convergence* for a net of functions (see [8, p. 241]). In our case though it will be more suitable to give a stronger condition.

Recall that if $(I, \leqslant_I)$ and $(K, \leqslant_K)$ are two directed sets we define the *product pre-ordering* $\preccurlyeq$ on $I \times K$ as follows:

$$(i_1, \kappa_1) \preccurlyeq (i_2, \kappa_2) \quad \Leftrightarrow \quad i_1 \leqslant_I i_2 \quad \text{and} \quad \kappa_1 \leqslant_K \kappa_2.$$

It is clear that the space $(I \times K, \preccurlyeq)$ is directed. From now on we will write just $I \times K$ without mentioning explicitly the pre-ordering described above. Also we will refer to a pre-ordering with the symbol $\leqslant$; it should be clear from the context were $\leqslant$ refers to.

Definition 3.2.1. (See also [1].) Let X be a topological space, (Y, p) be a metric space, a function $f : X \to Y$ and a net $(f_i)_{i\in I}$ of functions from X to Y. We say that the net $(f_i)_{i\in I}$ α-converges to f iff for all $x \in X$ and all nets $(x_\kappa)_{\kappa\in K}$ in X such that $x_\kappa \to x$ the net $(y_{(i,\kappa)})_{(i,\kappa)\in I\times K}$ defined by $y_{(i,\kappa)} = f_i(x_\kappa)$ converges to $f(x)$ (in Y), i.e., for all $\varepsilon > 0$ there exist $i_0 \in I$ and $\kappa_0 \in K$ such that for all $i \in I$ and $\kappa \in K$ with $i_0 \leqslant i$ and $\kappa_0 \leqslant \kappa$, we have that $p(f_i(x_\kappa), f(x)) < \varepsilon$.

As before we shall write $f_i \xrightarrow{\alpha} f$ in case where $(f_i)_{i\in I}$ α-converges to f.

It turns out that continuous convergence mentioned above is indeed weaker than α-convergence. In fact—despite its name—the continuous limit of a net of functions is not necessarily a continuous function (see Example 3.2.4), in contradiction with the α-limit (see Corollary 3.2.13).

It is clear that a subnet of an α-convergent net is also α-convergent. Now we have to make sure that the new definition coincides with Definition 2.1 in case where we have a sequence of functions and X is a metric space.

Proposition 3.2.2. *Let X and Y be metric spaces and also let functions $f_n, f : X \to Y$. The net $(f_n)_{n\in\mathbb{N}}$ α-converges to f (in the sense of the previous definition) if and only if $(f_n)_{n\in\mathbb{N}}$ α-converges to f as a sequence (i.e., in the sense of Definition 2.1).*

Proof. While it is easy to verify that if $(f_n)_{n\in\mathbb{N}}$ α-converges to f as a net then $(f_n)_{n\in\mathbb{N}}$ α-converges to f as a sequence, the inverse direction needs a little attention.

Assume that $(f_n)_{n\in\mathbb{N}}$ α-converges to f as a sequence. Let $x \in X$ and a net $(x_\kappa)_{\kappa\in K}$ in X such that $x_\kappa \to x$. We will prove that $f_n(x_\kappa) \to f(x)$.

Suppose that $f_n(x_\kappa) \nrightarrow f(x)$. Then there exists an open subset of Y call it U, which contains $f(x)$ and: for all $n \in \mathbb{N}$ and all $\kappa \in K$ there exist n', κ' such that $n \leqslant n'$, $\kappa \leqslant \kappa'$ and $f_{n'}(x_{\kappa'}) \notin U$ $(*)$.

So for some $m_1 \geqslant 1$ and some $\kappa_1 \in K$ $f_{m_1}(x_{\kappa_1}) \notin U$. Since $x_\kappa \to x$ there exists some $\lambda_1 \in K$ with $\kappa_1 \leqslant \lambda_1$ and for all $\lambda \in K$ with $\lambda_1 \leqslant \lambda$ we have that $d(x_\lambda, x) < \frac{1}{2}$ (where d is the metric on X).

Taking $n = m_1 + 1$ and $\kappa = \lambda_1$ in $(*)$ there exist $m_2 > m_1$ and $\kappa_2 \in K$ with $\lambda_1 \leqslant \kappa_2$ such that $f_{m_2}(x_{\kappa_2}) \notin U$.

Since $\lambda_1 \leqslant \kappa_2$ we also have that $d(x_{\kappa_2}, x) < \frac{1}{2}$. Now take some $\lambda_2 \in K$ with $\kappa_2 \leqslant \lambda_2$ such that for all $\lambda \in K$ with $\lambda_2 \leqslant \lambda$ it follows that $d(x_\lambda, x) < \frac{1}{3}$.

Applying $(*)$ again there exist $m_3 > m_2$ and $\kappa_3 \in K$ with $\lambda_2 \leqslant \kappa_3$ such that $f_{m_3}(x_{\kappa_3}) \notin U$. Also we have that $d(x_{\kappa_3}, x) < \frac{1}{3}$.

Proceeding inductively we find naturals $m_1 < m_2 < \cdots < m_n < \cdots$ and elements of K, $\kappa_1 \leqslant \kappa_2 \leqslant \cdots \leqslant \kappa_n \leqslant \cdots$ such that for all $n \in \mathbb{N}$ $f_{m_n}(x_{\kappa_n}) \notin U$ and $d(x_{\kappa_n}, x) < \frac{1}{n}$.

Define $z_n = x_{\kappa_n}$ for all $n \in \mathbb{N}$. Then $(z_n)_{n\in\mathbb{N}}$ is a sequence in X which converges to x. Since $(f_n)_{n\in\mathbb{N}}$ α-converges to f as a sequence it follows that the subsequence $(f_{m_n})_{n\in\mathbb{N}}$ α-converges to f as well (see Proposition 1.4). It follows that $f_{m_n}(z_n) = f_{m_n}(x_{\kappa_n}) \xrightarrow{n\in\mathbb{N}} f(x)$, contradicting to the fact that $f_{m_n}(x_{\kappa_n}) \notin U$, for all $n \in \mathbb{N}$. □

Notice that in the previous proposition we may assume that X is just a first countable topological space, i.e., each $x \in X$ has a countable neighborhood basis. The previous proof is very tempting for someone to make the following conjecture whenever X is a first countable topological space: in order to achieve α-convergence using Definition 3.2.1 it is enough to use sequences $(x_n)_{n\in\mathbb{N}}$ instead of arbitrary nets $(x_\kappa)_{\kappa\in K}$ and then check that $f_i(x_n) \to f(x)$.

However this is not true. The problem is that although a sequence of naturals $m_1 < m_2 < \cdots < m_n < \cdots$ defines a subsequence of some $(f_n)_{n\in\mathbb{N}}$, a corresponding sequence $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n \leqslant \cdots$ of elements of some directed I may not define a subnet of a net $(f_i)_{i\in I}$.

Example 3.2.3. Here we give an example of functions $f_i, f : [0,1] \to [0,1]$, $i \in I$, such that the net $(f_i)_{i\in I}$ does not α-converge to f, however it has the following property: for all $x \in [0,1]$ and all sequences $(x_n)_{n\in\mathbb{N}}$ in $[0,1]$ with $x_n \to x$ it holds that $f_i(x_n) \to f(x)$.

Therefore it is essential to use arbitrary nets $(x_\kappa)_{\kappa\in K}$ in Definition 3.2.1 even if we regard X as a metric space.

Let ω_1 be the first uncountable ordinal with the usual well ordering. For each $n \in \mathbb{N}$ define $\Delta_n = (\frac{1}{n+1}, \frac{1}{n})$ and let $\pi_n : \omega_1 \to \Delta_n$ which is one-to-one. Notice that for each $x \in [0,1]$ there exists at most one pair (n,ξ) such that $\pi_n(\xi) = x$.

Put $I = \omega_1$. Of course I is directed. For each $\xi \in I$ define $f_\xi : [0,1] \to \{0,1\}$ such that

$$f_\xi(x) = \begin{cases} 1, & \text{if for some } n \in \mathbb{N} \text{ we have that } x = \pi_n(\xi), \\ 0, & \text{otherwise.} \end{cases}$$

Also put $f \equiv 0$.

Now let a sequence $(x_n)_{n\in\mathbb{N}}$ in $[0,1]$ and $x \in [0,1]$ such that $x_n \to x$. The set $A = \{\xi \in \omega_1 \,/\, \text{for some } k, n \in \mathbb{N} \text{ we have that } x_n = \pi_k(\xi)\}$ is countable. Therefore if we set $\xi_0 = \sup A + 1$ then $\xi_0 < \omega_1$.

Now for each $\xi \in \omega_1$ with $\xi_0 \leqslant \xi$ we have that $f_\xi(x_n) = 0$ for all $n \in \mathbb{N}$. Therefore $f_\xi(x_n) \to f(x)$.

We will prove that the net $(f_\xi)_{\xi\in\omega_1}$ does not α-converge to f.

Think of $\mathbb{N} \times \omega_1$ with the product ordering. For each $(n,\xi) \in \mathbb{N} \times \omega_1$ define $x_{(n,\xi)} = \pi_n(\xi)$. Since $\pi_n[\omega_1] \subseteq \Delta_n$ it is easy to see that $x_{(n,\xi)} \to 0$. We will show that $f_i(x_{(n,\xi)}) \nrightarrow 0$. Let $i \in \omega_1$ and $(n,\xi) \in \mathbb{N} \times \omega_1$. Take $\xi' = \max\{i,\xi\} \in \omega_1$ and $i' = \xi'$. Then $i \leqslant i'$, $(n,\xi) \leqslant (n,\xi')$ and $f_{i'}(x_{(n,\xi')}) = f_{\xi'}(x_{(n,\xi')}) = f_{\xi'}(\pi_n(\xi')) = 1$, i.e., $f_i(x_{(n,\xi)}) \nrightarrow 0$.

We also give an example which distinguishes α-convergence from continuous convergence.

Example 3.2.4. Let ω be the first infinite ordinal and set $I = \omega + 1 = \{n \,/\, n \in \omega\} \cup \{\omega\}$ with the usual well ordering. Notice that if $(x_i)_{i\in I}$ is a net of elements of $[0,1]$ such that $x_i \xrightarrow{i\in I} x$ for some $x \in [0,1]$, then $x_\omega = x$.

Now take any function $f : [0,1] \to [0,1]$ which is not continuous and define $f_i = f$ for all $i \in I$. Using the remark above it is easy to check that the net $(f_i)_{i\in I}$ converges to f continuously. Since the function f is not continuous it follows that f cannot be an α-limit (see Corollary 3.2.13).

Let us give another example in which I does not have a maximum element. Put $I = \omega_1$, where ω_1 is the first uncountable ordinal. Notice that if $(x_i)_{i\in\omega_1}$ is a net in $[0,1]$ which converges to some $x \in [0,1]$, then there exists some $\xi \in \omega_1$ such that $x_\lambda = x$ for all $\xi \leqslant \lambda < \omega_1$. One can now apply to the previous example taking this I.

As we mentioned in the beginning it is interesting to ask whether we can find a topology $\mathcal{T}$ for a family of functions $\mathcal{F}$ from which α-convergence follows. First observe that this family $\mathcal{F}$ must consist of continuous functions. To see this take $f_n = f$, for all $n \in \mathbb{N}$. Then $f_n \xrightarrow{\mathcal{T}} f$ and so $f_n \xrightarrow{\alpha} f$. From Proposition 1.3(1) we have that f is continuous.

Let us recall a topological notion. Let $\mathcal{F}$ be a family of continuous functions from X to Y. Each topology for $\mathcal{F}$ is giving rise to a product topology for $X \times \mathcal{F}$. A topology $\mathcal{T}$ for $\mathcal{F}$ is called *jointly continuous* iff the evaluation function $E : X \times \mathcal{F} \to Y$: $E(x,f) = f(x)$ is continuous (see [8, p. 223]). For example the discrete topology on $\mathcal{F}$ is jointly continuous, since $\mathcal{F}$ consists of continuous functions. (Also notice that if there exists a jointly continuous topology on an arbitrary $\mathcal{F}'$ then $\mathcal{F}'$ consists of continuous functions.)

Similarly a topology $\mathcal{T}$ for $\mathcal{F}$ is called *jointly continuous on compacta* iff for each compact $K \subseteq X$ the restriction of the evaluation function $E_K : K \times \mathcal{F} \to Y$: $E_K(x, f) = f(x)$ is continuous. Of course if $\mathcal{T}$ is jointly continuous then it is jointly continuous on compacta. It is not hard to see that the inverse is also true in case where X is locally compact.

It is also easy to see that if $\mathcal{T}_0$ is jointly continuous and $\mathcal{T}_1$ is a larger topology, then $\mathcal{T}_1$ is also jointly continuous. Therefore a natural question to ask is whether there exists a least jointly continuous topology for $\mathcal{F}$.

Recall the *compact open* topology for $\mathcal{F}$, i.e., the topology which is generated from the sets

$$W(K,U) = \{ f \in \mathcal{F} \,/\, f[K] \subseteq U \},$$

where $K \subseteq X$ is compact and $U \subseteq Y$ is open. It is well known that the compact open topology for $\mathcal{F}$ is jointly continuous on compacta and in fact it is the least one with this property [8, Theorem 5, p. 223]. Hence whenever X is locally compact the compact open topology is the least jointly continuous topology for $\mathcal{F}$.

The following theorem is essentially contained in [1, Theorem 4]; here we are just making a refinement which is more suitable for later on.

Theorem 3.2.5. *Let X be a topological space, Y be a metric space, $\mathcal{F} \subseteq C(X,Y)$ and $\mathcal{T}_0$ a topology for $\mathcal{F}$. The following are equivalent*:

(1) *The topology $\mathcal{T}_0$ is the least jointly continuous topology on $\mathcal{F}$.*
(2) *For all nets $(f_i)_{i\in I}$ of functions in $\mathcal{F}$ and all functions $f \in \mathcal{F}$, $f_i \xrightarrow{\alpha} f$ if and only if $f_i \xrightarrow{\mathcal{T}_0} f$.*

In particular α-convergence in $\mathcal{F}$ follows from a topology if and only if there exists the least jointly continuous topology for $\mathcal{F}$.

Before proving this theorem let us state the following proposition. Notice that it is also given in [8] (p. 241, M(a)) with one difference: the author refers to continuous convergence instead of α-convergence that we refer to.

Proposition 3.2.6. *Let $\mathcal{F}$ be a family of continuous functions from X to Y and $\mathcal{T}$ a topology on $\mathcal{F}$. The following are equivalent*:

(1) *The topology $\mathcal{T}$ is jointly continuous.*
(2) *For all nets $(f_i)_{i\in I}$ of functions in $\mathcal{F}$ and all functions $f \in \mathcal{F}$, if $f_i \xrightarrow{\mathcal{T}} f$ then $f_i \xrightarrow{\alpha} f$.*

Proof of Theorem 3.2.5. (1) $\Rightarrow$ (2) Let a net of functions $(f_i)_{i\in I}$ in $\mathcal{F}$ and a function $f \in \mathcal{F}$. If $f_i \xrightarrow{\mathcal{T}_0} f$ then $f_i \xrightarrow{\alpha} f$ from Proposition 3.2.6 since $\mathcal{T}_0$ is jointly continuous.

For the inverse direction let $f_i \xrightarrow{\alpha} f$. We will prove that $f_i \xrightarrow{\mathcal{T}_0} f$. For each $i \in I$ define $W_i = \{f_j \,/\, i \leqslant j\} \cup \{f\}$. Let $\mathcal{T}$ be the topology which is generated from the sets W_i and $\{g\}$, for $i \in I$ and $g \in \mathcal{F}$ with $g \neq f$.

We will prove that the topology $\mathcal{T}$ is jointly continuous and that $f_i \xrightarrow{\mathcal{T}} f$. From this it follows that $\mathcal{T}_0 \subseteq \mathcal{T}$ and so $f_i \xrightarrow{\mathcal{T}_0} f$.

Since the $\mathcal{T}$-subbasic sets which contain f are exactly the W_i's, it is obvious that $f_i \xrightarrow{\mathcal{T}} f$. Now assume that $x_\kappa \xrightarrow{\kappa\in K} x$ and $g_\kappa \xrightarrow{\mathcal{T}} g$. We will prove that $g_\kappa(x_\kappa) \xrightarrow{\kappa\in K} g(x)$.

If $g \neq f$ then since $\{g\} \in \mathcal{T}$ and g is continuous (as a member of $\mathcal{F}$) the result follows. So assume that $g = f$.

Let $\varepsilon > 0$. Since $x_\kappa \to x$ and $f_i \xrightarrow{\alpha} f$ we have that $f_i(x_k) \to f(x) = g(x)$. Therefore there exist $i_0 \in I$ and $\kappa_0 \in K$ such that $p(f_i(x_\kappa), g(x)) < \varepsilon$ for all i, κ with $i_0 \leqslant i$ and $\kappa_0 \leqslant \kappa$ $(*)$.

Also $g_\kappa \xrightarrow{\mathcal{T}} f$, hence for the set W_{i_0} there exists $\kappa_1 \in K$ such that for all $\kappa \in K$ with $\kappa_1 \leqslant \kappa$ we have that $g_\kappa \in W_{i_0}$.

The function f is continuous as a member of $\mathcal{F}$. Therefore there exists some $\kappa_2 \in K$ such that for all $\kappa \in K$ with $\kappa_2 \leqslant \kappa$ we have that $p(f(x_\kappa), f(x)) < \varepsilon$.

Pick $\kappa_3 \in K$ with $\kappa_0, \kappa_1, \kappa_2 \leqslant \kappa_3$ and let $\kappa \in K$ such that $\kappa_3 \leqslant \kappa$. We will show that $p(g_\kappa(x_\kappa), g(x)) < \varepsilon$.

Since $g_\kappa \in W_{i_0}$ either $g_\kappa = f_i$ some $i \in I$ with $i_0 \leqslant i$, or $g_\kappa = f$. Take the first case. From $(*)$ since $\kappa_0 \leqslant \kappa$ and $i_0 \leqslant i$ we have that $p(f_i(x_\kappa), g(x)) < \varepsilon$, i.e. $p(g_\kappa(x_\kappa), g(x)) < \varepsilon$. For the case $g_\kappa = f$ since $\kappa_2 \leqslant \kappa$ we have that $p(g_\kappa(x_\kappa), g(x)) = p(f(x_\kappa), f(x)) < \varepsilon$.

(2) ⇒ (1) Assume a topology $\mathcal{T}$ for $\mathcal{F}$ which is jointly continuous. We will prove that $\mathcal{T}_0 \subseteq \mathcal{T}$. It is enough to show that each $\mathcal{T}$-convergent net is also $\mathcal{T}_0$-convergent.

Let a net $(f_i)_{i\in I}$ in $\mathcal{F}$ such that $f_i \xrightarrow{\mathcal{T}} f$, for some $f \in \mathcal{F}$. From Proposition 3.2.6 we have that $f_i \xrightarrow{\alpha} f$ and hence from (2) it follows that $f_i \xrightarrow{\mathcal{T}_0} f$. □

It follows that if X is locally compact α-convergence in $\mathcal{F}$ follows from the compact open topology [1, Theorem 4(b)]. Recall the topology of *uniform convergence on compact sets* for an arbitrary $\mathcal{F}$, whose basis consists of the sets

$$W(f, K, \varepsilon) = \{g : X \to Y \,/\, p\big(f(x), g(x)\big) < \varepsilon, \text{ for all } x \in K\},$$

where $f \in \mathcal{F}$, K is a compact subset of X and $\varepsilon > 0$.

It is well known that in case where $\mathcal{F}$ consists of continuous functions the compact open topology for $\mathcal{F}$ coincides with the topology of uniform convergence on compact sets (see [8, Theorem 11, p. 230] and [1, Theorem 6]).

Putting Theorem 3.2.5 and all these remarks together the next corollary is self-evident.

Corollary 3.2.7. *(See [1].) Let X be a locally compact topological space and Y be a metric space. Then for each net of functions $(f_i)_{i\in I}$ in $C(X, Y)$ and each $f \in C(X, Y)$ we have that $f_i \xrightarrow{\alpha} f$ iff $f_i \to f$ uniformly on each compact subset of X.*

Notice the resemblance with Stoilov's result Proposition 1.3(2). Later on using the notion of exhaustiveness we will extend this corollary to arbitrary functions f_i, i.e., not necessarily continuous (Theorem 3.2.14).

As mentioned the local compactness of X is sufficient to ensure the existence of the least jointly continuous topology for some $\mathcal{F}$. Arens has also proved in [1] that in case where $\mathcal{F} = C(X, [0, 1])$ (with X being completely regular) this condition is also necessary [1, Theorem 3].

Corollary 3.2.8. *(See [1].) Let X be a completely regular space. Then X is locally compact iff for all nets $(f_i)_{i\in I}$ in $C(X, [0, 1])$ and all $f \in C(X, [0, 1])$ we have that $f_i \xrightarrow{\alpha} f$ iff $f_i \to f$ uniformly on each compact subset of X.*

We now proceed to the notion of an exhaustive net of functions. For the rest of this section with X we will mean a topological space and Y a metric space.

Definition 3.2.9. A net $(f_i)_{i\in I}$ of functions from X to (Y, p) is called exhaustive at some $x_0 \in X$ iff for all $\varepsilon > 0$ there exists some open set V containing x_0 and $i_0 \in I$ such that for all $x \in V$ and all $i \in I$ with $i_0 \leqslant i$ we have that $p(f_i(x), f_i(x_0)) < \varepsilon$.

A net $(f_i)_{i\in I}$ is called exhaustive iff it is exhaustive at all $x \in X$.

As in Section 2, if I is directed and the family $\mathcal{F} = \{f_i \,/\, i \in I\}$ is equicontinuous at x_0 then the net $(f_i)_{i\in I}$ is exhaustive at x_0. The inverse fails even if each f_i is a continuous function (see the next example). Hence we cannot have the analogue of Proposition 2.3(2).

Example 3.2.10. Take $I = \mathbb{N} \times \mathbb{N}$ and define $(m_1, n_1) \leqslant (m_2, n_2)$ iff $m_1 \cdot n_2 \leqslant n_1 \cdot m_2$. Then I is directed.

Define $f_{(m,n)} : [1, 4] \to \mathbb{R} : x \mapsto \max\{x^{\frac{n}{m}}, x\}$. Notice that $f_{(1,n)}(x) = x^n$ for all $n \in \mathbb{N}$ and all $x \in [1, 4]$. It is easy to check that the sequence $(f_{(1,n)})_{n\in\mathbb{N}}$ is not equicontinuous at 2 and so the family $\mathcal{F} = \{f_{(m,n)} \,/\, (m, n) \in I\}$ is also not equicontinuous at 2. (In fact $\mathcal{F}$ is not even exhaustive since it consists of continuous functions.)

However if $(2, 1) \leqslant (m, n)$ then $2 \cdot n \leqslant m$ and so $f_{(m,n)}(x) = x$ for all $x \in [1, 4]$. Therefore the net $(f_i)_{i\in I}$ converges uniformly to the identity function. It is now easy to see that the net $(f_{(m,n)})_{(m,n)\in I}$ is exhaustive at 2.

As expected the analogues of Proposition 2.5 and Theorem 2.6 hold for exhaustive nets.

Theorem 3.2.11. *Let a function $f : X \to Y$ and a net $(f_i)_{i\in I}$ of functions from X to Y. If the net $(f_i)_{i\in I}$ is exhaustive and converges pointwise to f then f is a continuous function.*

Theorem 3.2.12. *Let a function $f : X \to Y$ and a net $(f_i)_{i\in I}$ of functions from X to Y. The following are equivalent*:

(1) $f_i \xrightarrow{\alpha} f$.
(2) $f_i \xrightarrow{pw} f$ *and the net* $(f_i)_{i\in I}$ *is exhaustive.*

Corollary 3.2.13. *If f is the α-limit of a net, then f is a continuous function.*

This corollary makes an essential use of the notion of exhaustiveness. No diagonal arguments which work for nets of the form $(f_i(x_i))_{i\in I}$ would get us this result. This is because the continuous limit of a net of functions is not necessarily continuous (see Example 3.2.4).

Proof. The only point which is not entirely the same with the proofs of Section 2 is direction $1 \Rightarrow 2$ of Theorem 3.2.12.

Assume that $f_i \xrightarrow{\alpha} f$ and that f is not exhaustive at some $x_0 \in X$. Then for some $\varepsilon > 0$ we have that for all open neighborhoods V of x_0 and all $i \in I$ there exist some $x \in V$ and some $\kappa \in I$ with $i \leqslant k$ such that $p(f_\kappa(x), f_\kappa(x_0)) \geqslant \varepsilon$ $(*)$.

Denote with $\mathcal{V}_{x_0}$ the family of open neighborhoods of x_0. For $U_1, U_2 \in \mathcal{V}_{x_0}$ define $U_1 \leqslant U_2$ iff $U_2 \subseteq U_1$. Of course the set $\mathcal{V}_{x_0}$ with $\leqslant$ is directed.

Define $M = \{(\kappa, V) \in I \times \mathcal{V}_{x_0} /$ there exists $x \in V$ such that $p(f_\kappa(x), f_\kappa(x_0)) \geqslant \varepsilon\}$.

Consider the product pre-ordering (described before Definition 3.2.1) for $I \times \mathcal{V}_{x_0}$ and then the restriction on M. Using $(*)$ one can verify that M with this relation is directed; in fact for all $i \in I$ and for all $V \in \mathcal{V}_{x_0}$ there exists some $k \in I$ with $i \leqslant k$ and $(k, V) \in M$ $(**)$.

From the Axiom of Choice we get a net $(x_{(\kappa,V)})_{(\kappa,V)\in M}$ in X such that $p(f_\kappa(x_{(\kappa,V)}), f_\kappa(x_0)) \geqslant \varepsilon$ and $x_{(\kappa,V)} \in V$, for all $(\kappa, V) \in M$.

It is clear that $x_{(\kappa,V)} \xrightarrow{(\kappa,V)\in M} x_0$ and from our hypothesis for α-convergence we get that

$$f_i(x_{(\kappa,V)}) \xrightarrow{(i,\kappa,V)\in I\times M} f(x_0).$$

Since $f_i(x_0) \to f(x_0)$ it is easy to check that there exists some $i_0 \in I$ and $(\kappa_0, V_0) \in M$ such that for all $i \in I$ and $(\kappa, V) \in M$ with $i_0 \leqslant i$, $\kappa_0 \leqslant \kappa$, $V \subseteq V_0$ we have that $p(f_i(x_{(\kappa,V)}), f_i(x_0)) < \varepsilon$.

Now use $(**)$ to get some $\kappa \in I$ with $i_0, \kappa_0 \leqslant \kappa$ and $(\kappa, V_0) \in M$. Therefore $p(f_\kappa(x_{(\kappa,V_0)}), f_\kappa(x_0)) < \varepsilon$, which is a contradiction from the choice of $x_{(\kappa,V_0)}$. □

In Corollary 3.2.7 the only reason one has to assume that the functions f_i are continuous is because it is necessary to have a topology from which α-convergence follows. The notion of exhaustiveness allows us to overcome this.

Theorem 3.2.14. *Let X be a locally compact topological space and (Y, p) be a metric space. Also let functions $f_i, f : X \to Y$, $i \in I$, with I directed. The following are equivalent*:

(1) *The net $(f_i)_{i\in I}$ α-converges to f.*
(2) *The function f is continuous and for each compact $K \subseteq X$ the net $(f_i)_{i\in I}$ converges to f uniformly on K.*

It follows that if X is locally compact then α-convergence in $C(X, Y)$ follows from the topology of uniform convergence on compact sets.

Before proving this theorem we give a remark that we will use regularly. If I is directed and $J \subseteq I$ is cofinal in I then J is also directed. Hence if $(x_i)_{i\in I}$ is a net then each J as above gives rise to a subnet $(x_j)_{j\in J}$.

Proof. (1) $\Rightarrow$ (2) The function f is continuous from Corollary 3.2.13. Let K be a compact subset of X. If the net $(f_i)_{i\in I}$ does not converge to f uniformly on K, then for some $\varepsilon > 0$ and for all $i \in I$ there exist $j \in I$ with $i \leqslant j$ and there exists $x \in K$ such that $p(f_j(x), f(x)) \geqslant \varepsilon$.

Define $J = \{j \in I /$ there exists some $x \in K$ such that $p(f_j(x), f(x)) \geqslant \varepsilon\}$, then J is cofinal in I. From the Axiom of Choice there exists a net $(x_j)_{j\in I}$ in K such that $p(f_j(x_j), f(x_j)) \geqslant \varepsilon$ for all $j \in J$ $(*)$.

Since K is compact there exists a subnet $(x_{j_\mu})_{\mu\in M}$ and $x_0 \in K$ such that $x_{j_\mu} \xrightarrow{\mu\in M} x_0$. As mentioned before α-convergence is preserved under subnets; hence we have that $f_{j_\mu} \xrightarrow{\alpha} f$ and so $f_{j_\mu}(x_{j_\mu}) \xrightarrow{\mu\in M} f(x_0)$. Also from the continuity of f it follows that $f(x_{j_\mu}) \xrightarrow{\mu\in M} f(x_0)$. Therefore $p(f_{x_{j_\mu}}(x_{j_\mu}), f(x_{j_\mu})) \xrightarrow{\mu\in M} 0$ contradicting $(*)$.

(2) $\Rightarrow$ (1) Let a net $(x_\mu)_{\mu\in M}$ in X which converges to some x_0. Since X is locally compact there exists some open U such that $x_0 \in U$ and $\overline{U}$ is compact. Pick some $\mu_0 \in M$ such that for all $\mu \in M$ with $\mu_0 \leqslant \mu$ we have that $x_\mu \in U$.

Define $L = \{x_\mu \,/\, \mu_0 \leqslant \mu\}$ and $K = \overline{L}$. Then the set K is a closed subset of $\overline{U}$ and thus compact. From the hypothesis it follows that the net $(f_i)_{i\in I}$ converges to f uniformly on K. Using the continuity of f it is easy to see that $p(f_i(x_\mu), f(x_0)) \to 0$. □

We can also give the analogue of 3.2.8.

Corollary 3.2.15. *Let X be a regular space and Y be a metric space with at least two elements. The following are equivalent*:

(1) *X is locally compact.*
(2) *For all nets $(f_i)_{i\in I}$ of functions from X to Y and for all $f : X \to Y$, the net $(f_i)_{i\in I}$ α-converges to f if and only if f is continuous and $(f_i)_{i\in I}$ converges to f uniformly on each compact subset of X.*

Proof. The direction (1) $\Rightarrow$ (2) is Theorem 3.2.14.

(2) $\Rightarrow$ (1) Assume towards a contradiction that X is not locally compact. We will define a net of functions which converges to some continuous f uniformly on every compact subset of X and does not α-converge to f.

Take $\mathcal{K}$ to be the family of all compact subsets of X. For $K_1, K_2 \in \mathcal{K}$ define $K_1 \leqslant K_2$ iff $K_1 \subseteq K_2$. Then $(\mathcal{K}, \leqslant)$ is directed.

Fix some $y_0, y_1 \in Y$ with $y_0 \neq y_1$. For $K \in \mathcal{K}$ define $f_K : X \to Y$ such that $f_K(x) = y_0$ if $x \in K$ and $f_K(x) = y_1$ if $x \notin K$. Also put $f(x) = y_0$ for all $x \in X$.

We will prove that the net $(f_K)_{K\in\mathcal{K}}$ does not α-converge to f.

Since X is not locally compact there exists some $x_0 \in X$ such that for each closed F with $x_0 \in F^\circ$ there exists a net $(x_i^F)_{i\in I^F}$ in F with no convergent subnet. Now take I to be the disjoint union of all those I^F's; i.e., for all $i \in I$ there exists a unique F such that $i \in I^F$.

For $i, j \in I$ define

$$i \leqslant j \quad \Leftrightarrow \quad i \in I^F \quad \text{and} \quad j \in I^C \quad \text{and} \quad [\text{either } F = C \text{ and } i \leqslant_F j \text{ or } C \subsetneqq F].$$

It is not hard to see that $\leqslant$ is a pre-ordering on I.

Also I is directed. Let $i \in I^F$, $j \in I^C$ and define $L = F \cap C$. Then $x_0 \in L^\circ$.

Assume that $F \cap C = F$. Then $F \subseteq C$. If $F \subsetneqq C$ then $j \leqslant i$. If $F = C$ then since $(I^F, \leqslant_F)$ is directed there exists some $\kappa \in I^F$ such that $i, j \leqslant_F \kappa$. Since $\kappa \in I^F$ it follows that $i, j \leqslant \kappa$.

Assume now that $F \cap C \subsetneqq F$. If $F \cap C = C$ then $C \subsetneqq F$ and so $i \leqslant j$. If $F \cap C \subsetneqq C$ then any $\kappa \in I^{F\cap C} = I^L$ suffices for the relation $i, j \leqslant \kappa$.

Now for $i \in I$ define $x_i = x_i^F$, where F is the unique closed F with $x_0 \in F^\circ$ and $i \in I^F$. We claim that $x_i \xrightarrow{i\in I} x_0$. Indeed if U is open with $x_0 \in U$, using the regularity of X there exists some closed F with $x_0 \in F^\circ \subseteq F \subseteq U$. Let any $i_0 \in I^F$. If $i \in I$ with $i_0 \leqslant i$ and $i \in I^C$ then $C \subseteq F$ and from the choice of x_i^C we have that $x_i = x_i^C \in C \subseteq F \subseteq U$.

Finally we prove that $f_K(x_i) \nrightarrow f(x_0) = y_0$. Let $K \in \mathcal{K}$ and $i \in I$ with $i \in I^F$. It is enough to find some $j \in I$ such that $i \leqslant j$ and $f_K(x_j) = y_1$, i.e., $x_j \notin K$.

Define $J^F = \{j \in I^F \,/\, i \leqslant_F j\}$ and notice that J^F is directed. It is easy to see that each $J \subseteq J^F$ which is cofinal in J^F is also cofinal in I^F. Hence each subnet of $(x_j^F)_{j\in J^F}$ is also a subnet of $(x_i^F)_{i\in I^F}$. Now K cannot contain the net $(x_j^F)_{j\in J^F}$ for otherwise from the compactness of K we would get a convergent subnet of $(x_i^F)_{i\in I^F}$. However this contradicts to the choice of the net $(x_i^F)_{i\in I^F}$. Therefore there exists some $j \in J^F$ with $x_j^F \notin K$. Hence $i \leqslant j$ and $x_j = x_j^F \notin K$. □

These results together with Theorem 3.2.5 will provide us with the tools to give some Ascoli-type theorems. In Section 3.1 we gave an analogous result for not necessarily continuous functions. However it was essential to have a metric on the set $\mathcal{F}$. Now we will not have to assume any metrizability at all; the topology of α-convergence will suffice. The payoff though is that we must restrict ourselves to continuous functions.

First let us state some notations. With $\mathcal{P}$ we denote the *topology of pointwise convergence* on the set of functions from X to Y. Also if $\mathcal{T}$ is a topology on some $\mathcal{F}_0$ and $\mathcal{F} \subseteq \mathcal{F}_0$ then $\mathcal{T}^{\mathcal{F}}$ stands for the *restriction* of the topology $\mathcal{T}$ on $\mathcal{F}$. Also we denote with $\mathrm{cl}_{\mathcal{T}}\,\mathcal{F}$ the *closure* of $\mathcal{F}$ with respect to $\mathcal{T}$.

The following lemma is useful for the proof of Theorem 3.2.19. It is also interesting on its own right.

Lemma 3.2.16. *Let X be a topological space, Y be a metric space and let some $\mathcal{F}_0$ which is a subset of $C(X,Y)$. Assume that there exists a topology $\mathcal{T}_0$ for $\mathcal{F}_0$ which is the least jointly continuous topology. Let $\mathcal{F} \subseteq \mathcal{F}_0$ with the following property: for each net of functions in $\mathcal{F}$ which is pointwise convergent to a function f from X to Y there exists an exhaustive subnet. Then the following hold:*

(1) *The restriction of the topology $\mathcal{T}_0$ on $\mathcal{F}$ coincides with the topology of pointwise convergence, i.e. $\mathcal{T}_0^{\mathcal{F}} = \mathcal{P}^{\mathcal{F}}$. Hence α-convergence in $\mathcal{F}$ coincides with pointwise convergence.*
(2) *The $\mathcal{T}_0$-closure of $\mathcal{F}$ is equal to the pointwise closure of $\mathcal{F}$ in $\mathcal{F}_0$, i.e. $\mathrm{cl}_{\mathcal{T}_0}\,\mathcal{F} = \mathcal{F}_0 \cap \mathrm{cl}_{\mathcal{P}}\,\mathcal{F}$.*
In fact if $\mathcal{F}_0$ is $\mathcal{P}$-closed in $C(X,Y)$ (i.e., if the function f is in $C(X,Y)$ and $f \in \mathrm{cl}_{\mathcal{P}}\,\mathcal{F}_0$ then we have that $f \in \mathcal{F}_0$), then $\mathrm{cl}_{\mathcal{T}_0}\,\mathcal{F} = \mathrm{cl}_{\mathcal{P}}\,\mathcal{F}$.

Proof. (1) First notice that the topology $\mathcal{T}_0^{\mathcal{F}}$ is the least jointly continuous topology for $\mathcal{F}$. To see this, for any functions $f_i, f \in \mathcal{F}$ $(i \in I)$, since $\mathcal{F} \subseteq \mathcal{F}_0$ and $\mathcal{T}_0$ is the least jointly continuous topology for $\mathcal{F}_0$, from Theorem 3.2.5 we obtain that $f_i \xrightarrow{\alpha} f$ iff $f_i \xrightarrow{\mathcal{T}_0} f$; i.e., α-convergence in $\mathcal{F}$ follows from the topology $\mathcal{T}_0^{\mathcal{F}}$. Again from Theorem 3.2.5 we have that $\mathcal{T}_0^{\mathcal{F}}$ is the least jointly continuous topology for $\mathcal{F}$.

It is easy to check that $\mathcal{P}^{\mathcal{F}}$ is contained in any topology for $\mathcal{F}$ which is jointly continuous. Using the remark above it is enough to show that the topology $\mathcal{P}^{\mathcal{F}}$ is jointly continuous.

Let $x_i \xrightarrow{i \in I} x$ and $f_i \xrightarrow{pw} f$, with $f_i, f \in \mathcal{F}, i \in I$. Suppose that $f_i(x_i) \nrightarrow f(x)$. Then for some open $U \subseteq Y$ which contains $f(x)$ and some $J \subseteq I$ which cofinal in I we have that $f_j(x_j) \notin U$, for all $j \in J$, $(*)$.

From the hypothesis for $\mathcal{F}$ the net $(f_j)_{j \in J}$ has an exhaustive subnet $(f_{j_\mu})_{\mu \in M}$. Since $f_{j_\mu} \xrightarrow{pw} f$, from Theorem 3.2.12 we have that $f_{j_\mu} \xrightarrow{\alpha} f$. Hence $f_{j_\mu}(x_{j_\mu}) \xrightarrow{\mu \in M} f(x)$ contradicting $(*)$.

(2) In the proof of (1) we mentioned that the topology of pointwise convergence is contained in each jointly continuous topology, hence $\mathcal{P}^{\mathcal{F}_0} \subseteq \mathcal{T}_0$. Therefore $\mathrm{cl}_{\mathcal{T}_0}\,\mathcal{F} \subseteq \mathcal{F}_0 \cap \mathrm{cl}_{\mathcal{P}}\,\mathcal{F}$.

Now let $f \in \mathcal{F}_0$ which is in $\mathrm{cl}_{\mathcal{P}}\,\mathcal{F}$. Then there exists a net $(f_i)_{i \in I}$ in $\mathcal{F}$ such that $f_i \xrightarrow{pw} f$. From the property of $\mathcal{F}$ there exists a subnet $(f_{i_\kappa})_{\kappa \in K}$ which is exhaustive. Applying again Theorem 3.2.12 we have that $f_{i_\kappa} \xrightarrow{\alpha} f$. From Theorem 3.2.5 (since all our functions are members of $\mathcal{F}_0$) we obtain that $f_{i_\kappa} \xrightarrow{\mathcal{T}_0} f$ and so $f \in \mathrm{cl}_{\mathcal{T}_0}\,\mathcal{F}$. Therefore $\mathrm{cl}_{\mathcal{T}_0}\,\mathcal{F} = \mathcal{F}_0 \cap \mathrm{cl}_{\mathcal{P}}\,\mathcal{F}$.

In fact we have shown that if $f \in \mathrm{cl}_{\mathcal{P}}\,\mathcal{F}$ then the function f is the α-limit of a net of functions in $\mathcal{F}$. From Corollary 3.2.13 we have that f is continuous and hence in case where $\mathcal{F}_0$ is $\mathcal{P}$-closed in $C(X,Y)$ it follows that $f \in \mathcal{F}_0$. So in this case $\mathrm{cl}_{\mathcal{P}}\,\mathcal{F} \subseteq \mathcal{F}_0$ and therefore $\mathcal{F}_0 \cap \mathrm{cl}_{\mathcal{P}}\,\mathcal{F} = \mathrm{cl}_{\mathcal{P}}\,\mathcal{F}$. Hence $\mathrm{cl}_{\mathcal{T}_0}\,\mathcal{F} = \mathrm{cl}_{\mathcal{P}}\,\mathcal{F}$. □

Remark 3.2.17. Let $\mathcal{F} \subseteq C(X,Y)$ and P_1, P_2 be the following properties for $\mathcal{F}$:

P_1: for all nets in $\mathcal{F}$ there exists an exhaustive subnet.

P_2: for all nets in $\mathcal{F}$ which are pointwise convergent (not necessarily to a member of $\mathcal{F}$) there exists an exhaustive subnet.

It is obvious that if $\mathcal{F}$ is equicontinuous then P_1 and P_2 hold. In fact if P_1 holds for $\mathcal{F}$ then $\mathcal{F}$ is equicontinuous. Suppose not, then there exists $\varepsilon > 0$, $x_0 \in X$, a net $(x_V)_{V \in \mathcal{V}}$ which converges to x_0 and a net $(f_V)_{V \in \mathcal{V}}$ in $\mathcal{F}$ such that $p(f_V(x_V), f_V(x_0)) \geqslant \varepsilon$, for all $V \in \mathcal{V}$. Using this it is easy to see that no subnet of $(f_V)_{V \in \mathcal{V}}$ is exhaustive at x_0.

However condition P_2 does not imply equicontinuity. Therefore the preceding lemma is stronger than a corresponding lemma which asserts that $\mathcal{F}$ is equicontinuous.

Counterexample: For each $k \in \mathbb{Z}$ with $k \neq 0$ define $f_k : [2,4] \to \mathbb{R}$: $f_k(x) = \max\{x^{k+1}, x\}$ and put $\mathcal{F} = \{f_k \,/\, k \neq 0\}$. Of course $\mathcal{F} \subseteq C([2,4], \mathbb{R})$. The family $\{f_n \,/\, n \geqslant 1\}$ is not equicontinuous and so neither is $\mathcal{F}$. We will show that $\mathcal{F}$ has the property P_2.

Notice that if $f_k(x) = x^{k+1}$ we have that $k \geqslant 1$ and if $f_k(x) = x$ then $k \leqslant -1$.

Let $(f_i)_{i \in I}$ be a net in $\mathcal{F}$ which is pointwise convergent. Choose a family of naturals $(k_i)_{i \in I}$ such that $f_i = f_{k_i}$ for all $i \in I$. We will prove that the whole net $(f_i)_{i \in I}$ is exhaustive. Let $x_0 \in [2,4]$ and $\varepsilon > 0$. Notice that since $x_0 > 1$ the number $r \equiv r(x_0)$ defined by $r = \inf\{|x_0^n - x_0^m| \,/\, n, m \geqslant 1 \text{ and } n \neq m\}$ is positive.

The net $(f_i(x_0))_{i \in I}$ is Cauchy and so there exists $i_0 \in I$ such that for all $i, j \in I$ with $i_0 \leqslant i, j$ we have that $|f_i(x_0) - f_j(x_0)| < r$. From the choice of r we cannot have that $f_i(x_0) = x_0^{k_i+1}$ and $f_j(x_0) = x_0$ for any $i, j \in I$ with $i_0 \leqslant i, j$.

Hence either for all $i, j \in I$ with $i_0 \leqslant i, j$ we have that $f_i(x_0) = x_0^{k_i+1}$ and $f_j(x_0) = x_0^{k_j+1}$ or for all $i, j \in I$ with $i_0 \leqslant i, j$ we have that $f_i(x_0) = f_j(x_0) = x_0$.

Take the first case. We have that $k_i \geqslant 1$ for all $i \in I$ with $i_0 \leqslant i$. Again from the choice of r it follows that $k_i = k_j = k \geqslant 1$ for all $i, j \in I$ with $i_0 \leqslant i, j$. Since $k \geqslant 1$ and $f_i = f_{k_i} = f_k$ we have that $f_i(x) = f_k(x) = \max\{x^{k+1}, x\} = x^{k+1}$ for all $x \in [2,4]$ and all $i \in I$ with $i_0 \leqslant i$.

From the continuity of the function $(x \mapsto x^{k+1})$ there exists some $\delta > 0$ such that for all $x \in (x_0 - \delta, x_0 + \delta) \cap [2,4]$ we have that $|x^{k+1} - x_0^{k+1}| < \varepsilon$. Now let $i \in I$ with $i_0 \leqslant i$ and $x \in (x_0 - \delta, x_0 + \delta) \cap [2,4]$. Then $|f_i(x) - f_i(x_0)| = |x^{k+1} - x_0^{k+1}| < \varepsilon$.

For the second case the result is proved similarly.

From now on we will state the compact open topology with $\mathcal{C}$ and the topology of uniform convergence on compact sets with $\mathcal{U}_c$. As mentioned before Corollary 3.2.7 these topologies coincide on a family $\mathcal{F}$ which consists of continuous functions. Also if X is locally compact and $\mathcal{F} \subseteq C(X,Y)$ then $\mathcal{C}^{\mathcal{F}} = \mathcal{U}_c^{\mathcal{F}}$ is the least jointly continuous topology for $\mathcal{F}$ (see comments before Theorem 3.2.5). Taking $\mathcal{F}_0 = C(X,Y)$ in Lemma 3.2.16 we obtain the following.

Corollary 3.2.18. *Let X be a locally compact topological space, Y be metric space and let some $\mathcal{F}$ which is a subset of $C(X,Y)$. Assume that for each net of functions in $\mathcal{F}$ which is pointwise convergent to a function f from X to Y there exists an exhaustive subnet. Then the following hold*:

(1) *The restriction of the topology $\mathcal{C}$ on $\mathcal{F}$ coincides with the topology of pointwise convergence, i.e., $\mathcal{C}^{\mathcal{F}} = \mathcal{U}_c^{\mathcal{F}} = \mathcal{P}^{\mathcal{F}}$.*
(2) *The $\mathcal{C}$-closure of $\mathcal{F}$ is equal to the pointwise closure of $\mathcal{F}$, i.e., $\mathrm{cl}_{\mathcal{C}}\, \mathcal{F} = \mathrm{cl}_{\mathcal{U}_c}\, \mathcal{F} = \mathrm{cl}_{\mathcal{P}}\, \mathcal{F}$.*

Theorem 3.2.19 *(Ascoli-type (I)). Let X be a topological space, Y be metric space and let some $\mathcal{F}_0$ which is a subset of $C(X,Y)$ and $\mathcal{P}$-closed in $C(X,Y)$. Assume that there exists the least jointly continuous topology $\mathcal{T}_0$ for $\mathcal{F}_0$. Let some $\mathcal{F}$ which is a subset of $\mathcal{F}_0$. The following are equivalent*:

(I) *The set $\mathcal{F}$ is $\mathcal{T}_0$-compact.*
(II) *Each net in $\mathcal{F}$ has a subnet which is α-convergent in $\mathcal{F}$.*
(III) *The following conditions hold*:
 (i) *$\mathcal{F}$ is $\mathcal{T}_0$-closed.*
 (ii) *For each $x \in X$ the set $\mathcal{F}[x] = \{f(x) \,/\, f \in \mathcal{F}\}$ has a compact closure in Y.*
 (iii) *$\mathcal{F}$ is equicontinuous.*
(IV) *The following conditions hold*:
 (1) *$\mathcal{F}$ is $\mathcal{T}_0$-closed.*
 (2) *For each $x \in X$ the set $\mathcal{F}[x] = \{f(x) \,/\, f \in \mathcal{F}\}$ has a compact closure in Y.*
 (3) *For each net of functions in $\mathcal{F}$ which is pointwise convergent to a function f from X to Y there exists an exhaustive subnet.*

Notice that conditions (iii) and (3) above are not equivalent because of the counterexample given in Remark 3.2.17.

Proof. The equivalence between (I) and (II) is straightforward. Assume now (I). We will prove the conditions of (III). For (i) notice that since Y is Hausdorff as a metric space then each jointly continuous topology for $\mathcal{F}_0$ is also Hausdorff. Hence $\mathcal{F}$ is $\mathcal{T}_0$-closed as a $\mathcal{T}_0$-compact subset of the Hausdorff space $(\mathcal{F}_0, \mathcal{T}_0)$.

For (ii) take $x \in X$ and notice that the function $E_x\colon \mathcal{F}_0 \to Y\colon E_x(f) = f(x)$ is $\mathcal{T}_0$-continuous since $\mathcal{T}_0$ is jointly continuous. Now $\mathcal{F}[x] = \{f(x) \,/\, f \in \mathcal{F}\} = \{E_x(f) \,/\, f \in \mathcal{F}\} = E_x[\mathcal{F}]$. Since $\mathcal{F}$ is $\mathcal{T}_0$-compact we obtain that $\mathcal{F}[x]$ is a compact subset of Y.

For (iii) we will prove the equivalent condition P_1 in Remark 3.2.17. Take a net $(f_i)_{i\in I}$ in $\mathcal{F}$. Since $\mathcal{F}$ is $\mathcal{T}_0$-compact there exists a subnet $(f_{i_\kappa})_{\kappa\in K}$ which is $\mathcal{T}_0$-convergent in $\mathcal{F}$. From Theorem 3.2.5 the subnet $(f_{i_\kappa})_{\kappa\in K}$ is α-convergent and hence from Theorem 3.2.12 it is exhaustive.

It is also obvious that (III) implies (IV).

Assume now the conditions of (IV). We will prove that $\mathcal{F}$ is $\mathcal{T}_0$-compact. Notice that $\mathcal{F} \subseteq \prod_{x\in X} \overline{\mathcal{F}[x]}$. From (2) and Tychonoff's theorem the set $\prod_{x\in X} \overline{\mathcal{F}[x]}$ is $\mathcal{P}$-compact. Hence $\mathrm{cl}_{\mathcal{P}} \mathcal{F}$ is also $\mathcal{P}$-compact. From condition (3) and Lemma 3.2.16 we have that $\mathrm{cl}_{\mathcal{P}} \mathcal{F} = \mathrm{cl}_{\mathcal{T}_0} \mathcal{F}$ and since $\mathcal{F}$ is $\mathcal{T}_0$-closed it follows that $\mathrm{cl}_{\mathcal{P}} \mathcal{F} = \mathcal{F}$. Therefore $\mathcal{F}$ is $\mathcal{P}$-compact. Again from Lemma 3.2.16 the topology of pointwise convergence on $\mathcal{F}$ coincides with $\mathcal{T}_0^{\mathcal{F}}$. Hence $\mathcal{F}$ is $\mathcal{T}_0$-compact. □

Taking $\mathcal{F}_0$ in the previous theorem to be $C(X, Y)$ and using the remarks above for the compact open topology we obtain the following result (compare with [8, 7.6, p. 224, 7.17, p. 233–234]).

Theorem 3.2.20 *(Ascoli-type (II)). Let X be a locally compact topological space, Y be metric space and let some $\mathcal{F}$ which is a subset of $C(X, Y)$. The following are equivalent*:

(I) *The set $\mathcal{F}$ is $\mathcal{C}$-compact (equivalently $\mathcal{U}_c$-compact).*
(II) *Each net in $\mathcal{F}$ has a subnet which is α-convergent in $\mathcal{F}$.*
(III) *The following conditions hold*:
  (i) *$\mathcal{F}$ is $\mathcal{C}$-closed (equivalently $\mathcal{U}_c$-closed).*
  (ii) *For each $x \in X$ the set $\mathcal{F}[x] = \{f(x) \,/\, f \in \mathcal{F}\}$ has a compact closure in Y.*
  (iii) *$\mathcal{F}$ is equicontinuous.*
(IV) *The following conditions hold*:
  (1) *$\mathcal{F}$ is $\mathcal{C}$-closed (equivalently $\mathcal{U}_c$-closed).*
  (2) *For each $x \in X$ the set $\mathcal{F}[x] = \{f(x) \,/\, f \in \mathcal{F}\}$ has a compact closure in Y.*
  (3) *For each net of functions in $\mathcal{F}$ which is pointwise convergent to a function f from X to Y there exists an exhaustive subnet.*

4. Further applications

4.1. Separate α-convergence

Here we deal with functions defined on products of the form $X \times Y$ and we give conditions under which separate exhaustiveness (α-convergence) of a sequence gives joint exhaustiveness (respectively α-convergence) on some comeager subset of $X \times Y$.

Let X, Y, Z be metric spaces and a function $f : X \times Y \to Z$. For $y \in Y$ we denote with f^y the function $(x \mapsto f(x, y))$ for $x \in X$. Also we denote with f_x the function $(y \mapsto f(x, y))$. In case where we have a sequence of functions $(f_n)_{n\in\mathbb{N}}$ we use the symbols f_n^y and $f_{x,n}$ for the corresponding functions.

It is also useful to think of functions defined on some $G = A \times B$ which is a subset of $X \times Y$. Of course in this case the function f^y is defined on A for $y \in B$. The analogous holds for f_x.

Definition 4.1.1. Let X, Y, Z be metric spaces and functions $f_n, f : X \times Y \to Z$. We say that the sequence $(f_n)_{n\in\mathbb{N}}$ is *separately exhaustive* iff for each $y \in Y$ the sequence $(f_n^y)_{n\in\mathbb{N}}$ is exhaustive and for each $x \in X$ the sequence $(f_{x,n})_{n\in\mathbb{N}}$ is exhaustive.

Also we say that the sequence $(f_n)_{n\in\mathbb{N}}$ *converges α-separately* iff for all x, y the sequence $(f_n^y)_{n\in\mathbb{N}}$ α-converges to f^y and the sequence $(f_{x,n})_{n\in\mathbb{N}}$ α-converges to f_x.

In case where our functions are defined on some $G \subseteq X \times Y$ we keep the same definitions as above, where the sequences $(f_n^y)_{n\in\mathbb{N}}$ and $(f_{x,n})_{n\in\mathbb{N}}$ are taken for suitable x's and y's.

The analogous well-known notions of a *separate continuous function* and a *separate uniformly convergent* sequence are immediate.

As expected separate exhaustiveness does not imply exhaustiveness and α-separate convergence does not imply α-convergence.

Example 4.1.2. For each $n \in \mathbb{N}$ define $f_n, f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$f(x,y) = \begin{cases} \frac{x\cdot y}{x^2+y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

and $f_n = f + \frac{1}{n}$. Notice that $f_n \xrightarrow{u} f$. Since f is not continuous the sequence $(f_n)_{n\in\mathbb{N}}$ does not α-converge to f. From Theorem 2.6 the sequence $(f_n)_{n\in\mathbb{N}}$ is not exhaustive. However we will show that $(f_n)_{n\in\mathbb{N}}$ is separately exhaustive.

Let $x_n \to x \neq 0$. We may assume that $x_n \neq 0$ for all $n \in \mathbb{N}$. For each $y \in \mathbb{R}$ we have that $f_n^y(x_n) = f_n(x_n, y) = \frac{x_n\cdot y}{x_n^2+y^2} + \frac{1}{n} \xrightarrow{n\in\mathbb{N}} \frac{x\cdot y}{x^2+y^2} = f(x,y) = f^y(x)$.

If $x_n \to 0$ and $y = 0$ then $f_n^0(x_n) = 0 + \frac{1}{n} \to 0 = f^0(0)$ and if $y \neq 0$ then $f_n^y(x_n) = \frac{x_n\cdot y}{x_n^2+y^2} + \frac{1}{n} \to 0 = f^y(0)$.

Since $f_n(x,y) = f_n(y,x)$ and $f(x,y) = f(y,x)$ the same things hold for the sequences $(f_{x,n})_{n\in\mathbb{N}}$ $(x \in \mathbb{R})$.

It follows that the sequence $(f_n)_{n\in\mathbb{N}}$ is separately exhaustive. Also from Theorem 2.6 the sequence converges α-separately to f although it does not α-converge.

Recall that a set $A \subseteq X$ is called *nowhere dense* iff $(\overline{A})^\circ = \emptyset$. The set A is called *meager* iff it is the countable union of a sequence of nowhere dense sets. Also A is called *comeager* iff $X \setminus A$ is meager.

The previous example comes from the classical example of a separately continuous function which is not continuous. A well-known result of Namioka says that separate continuity implies continuity on some comeager subset of the domain space $X \times Y$. Furthermore if X and Y are compact this subset can be taken to be of the form $A \times Y$. A small variation of the proof of the first result gives the analogue for exhaustiveness (see [11] and [12]).

Theorem 4.1.3. *Let (X,d), (Y,p), (Z,l) be metric spaces and functions $f_n : X \times Y \to Z$. Assume that for all $y \in Y$ each function $f_n^y : X \to Z$ is continuous and also that the sequence $(f_n)_{n\in\mathbb{N}}$ is separately exhaustive. Then there exists a comeager set $G \subseteq X \times Y$ such that the sequence $(f_n)_{n\in\mathbb{N}}$ is exhaustive at every $(x,y) \in G$.*

Proof. For each $m, k, n_0 \in \mathbb{N}$ define

$$F_{m,k,n_0} = \left\{(x,y) \in X \times Y \,/\, \forall n \geqslant n_0 \quad \text{and} \quad \forall u, v \in S\left(y, \frac{1}{k}\right) \Rightarrow l\big(f_n(x,u), f_n(x,v)\big) \leqslant \frac{1}{m}\right\}.$$

We claim that each F_{m,k,n_0} is closed.

Let $(x_i, y_i) \xrightarrow{i\in\mathbb{N}} (x,y)$ with $(x_i, y_i) \in F_{m,k,n_0}$ for all $i \in \mathbb{N}$. Also let $n \geqslant n_0$ and $u, v \in S(y, \frac{1}{k})$. We will prove that $l(f_n(x,u), f_n(x,v)) \leqslant \frac{1}{m}$.

Since $y_i \to y$ and $u, v \in S(y, \frac{1}{k})$ there exists some $i_0 \in \mathbb{N}$ such that for all $i \geqslant i_0$ we have that $u, v \in S(y_i, \frac{1}{k})$. Since $(x_i, y_i) \in F_{m,k,n_0}$ we have that $l(f_n(x_i,u), f_n(x_i,v)) \leqslant \frac{1}{m}$, for all $i \geqslant i_0$. From hypothesis the functions f_n^u and f_n^v are continuous. Since $x_i \to x$ it follows that $l(f_n(x,u), f_n(x,v)) \leqslant \frac{1}{m}$.

For each $(x,y) \in X \times Y$ the sequence $(f_{x,n})_{n\in\mathbb{N}}$ is exhaustive at y. Using this it is not hard to see that $X \times Y = \bigcap_{m\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} \bigcup_{n_0\in\mathbb{N}} F_{m,k,n_0}$.

For all $y \in Y$ put $F_{m,k,n_0}^y = \{x \in X \,/\, (x,y) \in F_{m,k,n_0}\}$. Define $D = \bigcup_{m,k,n_0} \{(x,y) \,/\, x \in F_{m,k,n_0}^y \setminus (F_{m,k,n_0}^y)^\circ\}$. Of course with D^y we mean the set of all $x \in X$ such that $(x,y) \in D$.

It is obvious that $D \subseteq \bigcup_{m,k,n_0} F_{m,k,n_0} \setminus (F_{m,k,n_0})^\circ$. Since each set F_{m,k,n_0} is closed it follows that the set $F_{m,k,n_0} \setminus (F_{m,k,n_0})^\circ$ is meager. Hence D is meager as well. Put $G = X \times Y \setminus D$. Let $(x,y) \in G$. We will prove that the sequence $(f_n)_{n\in\mathbb{N}}$ is exhaustive at (x,y).

Let $\varepsilon > 0$ and $m \in \mathbb{N}$ such that $\frac{2}{m} < \varepsilon$. Choose $k, n_0 \in \mathbb{N}$ such that $(x,y) \in F_{m,k,n_0}$. Then $x \in F_{m,k,n_0}^y \setminus D^y \subseteq (F_{m,k,n_0}^y)^\circ$. Hence $x \in (F_{m,k,n_0}^y)^\circ$.

The sequence $(f_n^y)_{n\in\mathbb{N}}$ is exhaustive at x and each f_n^y is a continuous function. From Proposition 2.3 we obtain that the sequence $(f_n^y)_{n\in\mathbb{N}}$ is equicontinuous at x. Therefore there exists $\delta > 0$ such that $S(x,\delta) \subseteq F^y_{m,k,n_0}$ and for all $s \in S(x,\delta)$ and all $n \in \mathbb{N}$ it follows that $l(f_n(s,y), f_n(x,y)) \leqslant \frac{1}{m}$ $(*)$.

Let $(s,t) \in S(x,\delta) \times S(y,\frac{1}{k})$ and $n \geqslant n_0$. We need to prove that $l(f_n(s,t), f_n(x,y)) < \varepsilon$. Since $s \in S(x,\delta) \subseteq F^y_{m,k,n_0}$ we have that $(s,y) \in F_{m,k,n_0}$. From the fact that $t \in S(y,\frac{1}{k})$ and $n \geqslant n_0$ we obtain that $l(f_n(s,t), f_n(s,y)) \leqslant \frac{1}{m}$ $(**)$.

Now from $(*)$ and $(**)$ we have that

$$l\big(f_n(s,t), f_n(x,y)\big) \leqslant l\big(f_n(s,t), f_n(s,y)\big) + l\big(f_n(s,y), f_n(x,y)\big)$$
$$\leqslant \frac{1}{m} + \frac{1}{m} = \frac{2}{m} < \varepsilon. \qquad \square$$

Remark 4.1.4. Notice that if G_0 is a subset of $X \times Y$ of the form $A \times B$ we can repeat the same proof using the restrictions on G_0. However the resulting set $G \subseteq G_0$ is going to be *meager in* G_0, i.e., the set $G_0 \setminus G$ will be meager. Therefore if in the previous theorem our functions are defined only on a set $G_0 = A \times B \subseteq X \times Y$, the conclusion is that there exists a set $G \subseteq G_0$ such that $G_0 \setminus G$ is meager and the sequence $(f_n)_{n\in\mathbb{N}}$ is exhaustive at every $(x,y) \in G$. This remark will help us in the next result.

Corollary 4.1.5. *Let X, Y, Z be metric spaces and functions $f_n, f : X \times Y \to Z$. Assume that each function f_n^y is continuous.*

(1) *If the sequence $(f_n)_{n\in\mathbb{N}}$ converges to f α-separately, then there exists a comeager set $G \subseteq X \times Y$ such that for all $(x,y) \in G$ and all sequences $((x_n,y_n))_{n\in\mathbb{N}}$ in $X \times Y$ with $(x_n,y_n) \to (x,y)$ it follows that $f_n(x_n,y_n) \to f(x,y)$. In particular the sequence $(f_n \restriction G)_{n\in\mathbb{N}}$ α-converges to $f \restriction G$.*
(2) *Assume that X and Y are compact. If each function f_x is continuous and the sequence $(f_n)_{n\in\mathbb{N}}$ converges to f separate uniformly, then there exist a comeager set $G \subseteq X \times Y$ such that for all $(x,y) \in G$ and all sequences $((x_n,y_n))_{n\in\mathbb{N}}$ in G with $(x_n,y_n) \to (x,y)$ it follows that $f_n(x_n,y_n) \to f(x,y)$; i.e., the sequence $(f_n \restriction G)_{n\in\mathbb{N}}$ α-converges to $f \restriction G$.*

Proof. For (1) notice that the sequence $(f_n)_{n\in\mathbb{N}}$ converges to f pointwise. Then use Theorem 4.1.3 and Remark 2.7.

For (2) we need Remark 4.1.4. First notice the function f is separate continuous, since for each $y \in Y$ we have that $f_n^y \xrightarrow{u} f^y$ and each f_n^y is continuous.

From Namioka's result (see [12] and [11]) there exists a comeager set $G_0 \subseteq X \times Y$ such that f is continuous at every $(x,y) \in G_0$. Also since X and Y are compact the set G_0 is of the form $A \times Y$.

It follows that the restriction $f \restriction G_0$ is a continuous function and hence separately continuous. Now using Proposition 1.3(3) the sequence $(f_n \restriction G_0)_{n\in\mathbb{N}}$ converges to $f \restriction G_0$ α-separately. It follows that the sequence $(f_n \restriction G_0)_{n\in\mathbb{N}}$ is separately exhaustive.

Since G_0 is of the form $A \times Y$ from Remark 4.1.4 there exists a set $G \subseteq G_0$ such that $G_0 \setminus G$ is meager and the sequence $(f_n \restriction G_0)_{n\in\mathbb{N}}$ is exhaustive at every $(x,y) \in G$. Applying Remark 2.7 for the restrictions on G_0 we obtain that for all $(x,y) \in G$ and all sequences $((x_n,y_n))_{n\in\mathbb{N}}$ in G_0 with $(x_n,y_n) \to (x,y)$ it follows that $f_n(x_n,y_n) \to f(x,y)$.

Also notice that $X \times Y \setminus G = (X \times Y \setminus G_0) \cup (G \setminus G_0)$. Hence G is comeager in $X \times Y$. $\square$

4.2. *Notions which are derived from exhaustiveness*

The notion of exhaustiveness can lead us to some more definitions with interesting properties. Using these new meanings we will derive a necessary and sufficient condition for the continuity of a function which is the pointwise limit of a sequence of—not necessarily continuous—functions (see Theorem 4.2.3).

Definition 4.2.1. Let (X,d), (Y,p) be metric spaces, $x \in X$ and $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions from X to Y.

(1) The sequence $(f_n)_{n\in\mathbb{N}}$ is *weakly exhaustive at* x iff for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $y \in S(x,\delta)$ there exists $n_y \in \mathbb{N}$ such that for all $n \geqslant n_y$ we have that $p(f_n(y), f_n(x)) < \varepsilon$.
(2) The sequence $(f_n)_{n\in\mathbb{N}}$ is *of vanishing oscillation at* x iff for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ there exists $\delta_n > 0$ such that for all $y \in S(x,\delta_n)$ it follows that $p(f_n(y), f_n(x)) < \varepsilon$.
(3) The sequence $(f_n)_{n\in\mathbb{N}}$ is *weakly exhaustive* iff it is weakly exhaustive at every x. The same for the notion of the vanishing oscillation.

Remark 4.2.2.

(1) The idea of weak exhaustiveness is that the natural n_0 now depends not only on the $\varepsilon > 0$ but also on the y chosen in $S(x,\delta)$. The dual happens for the vanishing oscillation: the $\delta > 0$ does not only depend on $\varepsilon > 0$ but also on the function f_n.
(2) Both new meanings introduced in Definition 4.2.1 are weaker that the exhaustive notion. For counterexamples refer to Example 4.2.5.
(3) It is clear that one might give the analogous definitions for a family of functions $\mathcal{F}$ or a net of functions $(f_i)_{i\in I}$, as we did in Sections 2 and 3. However here we are not interested for a topology for α-convergence so we will not use nets. Also the notion of a weakly exhaustive sequence (instead of a family) reflects better the situation in the next theorem.

A well-known problem is when the pointwise limit of continuous functions is continuous. An answer is obtained for compact spaces in [4]. Here we will present a much more general result.

Theorem 4.2.3. *Let (X,d), (Y,p) be metric spaces, $x \in X$ and functions $f_n, f : X \to Y$, $n \in \mathbb{N}$, such that the sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f. Then f is continuous at x if and only if the sequence $(f_n)_{n\in\mathbb{N}}$ is weakly exhaustive at x.*

Note that we do not assume any continuity for f_n. This leads to a deeper idea: we do not care about each function as a single member but we do care for what is the sequence as a whole.

Proof. ($\Rightarrow$) Let $\varepsilon > 0$, from the continuity of f there exists $\delta > 0$ such that for all $y \in S(x,\delta)$ we have that $p(f(y), f(x)) < \frac{\varepsilon}{2}$. Let $y \in S(x,\delta)$, since $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f there exists $n_y \in \mathbb{N}$ such that for all $n \geqslant n_y$ it follows that $p(f_n(y), f(y)) < \frac{\varepsilon}{4}$ and $p(f_n(x), f(x)) < \frac{\varepsilon}{4}$. So for each $n \geqslant n_y$ we have that $p(f_n(y), f_n(x)) < \varepsilon$.

($\Leftarrow$) Let $\varepsilon > 0$, since $(f_n)_{n\in\mathbb{N}}$ is weakly exhaustive at x there exists $\delta > 0$ such that for all $y \in S(x,\delta)$ there exists $n_y \in \mathbb{N}$ such that for all $n \geqslant n_y$ it follows that $p(f_n(y), f_n(x)) < \frac{\varepsilon}{3}$. Take $y \in S(x,\delta)$, since $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ we have that $p(f_n(y), f(y)) < \frac{\varepsilon}{3}$ and $p(f_n(x), f(x)) < \frac{\varepsilon}{3}$. Therefore $p(f(y), f(x)) < \varepsilon$ and thus f is continuous at x. □

Recall that if we have a function $f : X \to Y$ and a point $x \in X$ the *oscillation of* f *at* x is defined by $osc(f,x) = \inf\{\mathrm{diam}(f[U]) \,/\, U$: open subset of X and $x \in U\}$, where $\mathrm{diam}(A) = \sup\{p(z,w) \,/\, z, w \in A\}$, for $A \subseteq Y$. It is easy to see that f is continuous at x if and only if $osc(f,x) = 0$.

The following proposition is straightforward and we put it down because it helps in Examples 4.2.5.

Proposition 4.2.4. *Let (X,d), (Y,p) be metric spaces, $x \in X$ and functions $f_n : X \to Y$, $n \in \mathbb{N}$. The sequence $(f_n)_{n\in\mathbb{N}}$ is of vanishing oscillation at x if and only if $osc(f_n,x) \xrightarrow{n\in\mathbb{N}} 0$.*

Example 4.2.5. (1) Let $(f_n)_{n\in\mathbb{N}}$ be any sequence of continuous functions, pointwise converging to a function f which is not continuous at a point x. Since each f_n is continuous at x we have that $osc(f_n,x) = 0$ for each $n \in \mathbb{N}$. From Proposition 4.2.4 we obtain that the sequence $(f_n)_{n\in\mathbb{N}}$ is of vanishing oscillation at x. However $(f_n)_{n\in\mathbb{N}}$ is not weakly exhaustive at x because of Theorem 4.2.3 and the fact that f is not continuous at x. It follows also that the sequence $(f_n)_{n\in\mathbb{N}}$ is not exhaustive at x.

(2) For each $n \in \mathbb{N}$ define $f_n : \mathbb{R} \to \mathbb{R}$: $f_n(x) = 0$ for $x \in (-\infty, -\frac{1}{n}) \cup (\frac{1}{n}, +\infty) \cup \{0\}$; the graph of f_n restricted on $[-\frac{1}{n}, 0) \times \mathbb{R}$ is linear connecting the points of $\mathbb{R}^2$: $(-\frac{1}{n}, 0)$ and $(0, 1)$; and the graph of f_n restricted on $(0, \frac{1}{n}] \times \mathbb{R}$ is also linear connecting the points of $\mathbb{R}^2$: $(0, 1)$ and $(\frac{1}{n}, 0)$. The sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise to the zero function which is continuous. Therefore from Theorem 4.2.3 the sequence $(f_n)_{n\in\mathbb{N}}$ is weakly exhaustive at 0. It is not difficult to check that $osc(f_n, 0) \geqslant 1$ for each $n \in \mathbb{N}$. Therefore the sequence $(osc(f_n, 0))_{n\in\mathbb{N}}$ does not converge to 0 and from Proposition 4.2.4 it follows that the sequence $(f_n)_{n\in\mathbb{N}}$ is not of vanishing oscillation at 0. Hence the sequence $(f_n)_{n\in\mathbb{N}}$ is not exhaustive at x.

Open problems.

(1) It would be interesting to study the relation between equal [2] and uniformly equal convergence [6] under the notions of exhaustiveness and weak exhaustiveness. We can ask the same with pointwise and equal or pointwise and uniformly equal convergence.
(2) Let $\mathcal{F}$ be an infinite family of functions from X to Y and let σ stand for any convergence of sequences of functions. Denote with $\mathcal{F}^\sigma$ the family of cluster points of $\mathcal{F}$ under σ-convergence. Under what conditions $\mathcal{F}^\sigma$ is exhaustive?

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