Rend. Sem. Mat. Univ. Pol. Torino
Vol. 73/2, 3–4 (2015), 269 – 316

M. Rossi and L. Terracini*

MAPLE SUBROUTINES FOR COMPUTING MILNOR AND TYURINA NUMBERS OF HYPERSURFACE SINGULARITIES WITH APPLICATION TO ARNOL'D ADJACENCIES.

Abstract. In the present paper MAPLE subroutines computing Milnor and Tyurina numbers of an isolated algebraic hypersurface singularity are presented and described. They represent examples, and perhaps the first ones, of a MAPLE implementation of *local monomial ordering*.
As an application, the last section is devoted to writing down equations of algebraic stratifications of Kuranishi spaces of simple Arnol'd singularities: geometrically, they represent, by means of inclusions of algebraic subsets, the partial ordering on classes of simple singularities induced by the *adjacency* relation.

Introduction

Two basic invariants of a complex analytic complete intersection singularity $p \in \overline{U} = \mathbf{f}^{-1}(0)$ are its *Milnor number* $\mu(p)$ and its *Tyurina number* $\tau(p)$. The former essentially "counts the number" of vanishing cycles in the intermediate cohomology of a nearby smoothing $U_t = \mathbf{f}^{-1}(t)$ of $\overline{U}$, which actually turns out to be the *multiplicity of p as a critical point* of the map $\mathbf{f}$. The latter "counts the dimension" of the base space of a versal deformation of $p \in \overline{U}$, which actually turn out to be the *multiplicity of p as a singularity* of the complex space $\overline{U}$. Since the Looijenga–Steenbrink Theorem [17] it is a well known fact that $\tau(p) \leq \mu(p)$.

The purpose of the present paper is to present subroutines ([1] and [24] for a detailed description of MAPLE procedures) allowing to compute these invariants in the case of an isolated algebraic hypersurface singularity (i.h.s.). In fact, from a computational point of view, their calculation could be very intricate and the use of a computer may be needed in most situations. Let us underline that the actual originality of our procedure is not so much based on the effective computation of these invariants as on its implementation in a mathematical software like MAPLE, which is universally known and used in the scientific community. In fact computer algebra packages computing these invariants of singularities already exist (an example is SINGULAR [2]). But our hope is that the routines presented here could be useful to all those who are interested, for any reason, in a concrete evaluation of these invariants without being motivated to learn how to use an entire computer algebra package. Once implemented, the present routine is so easy that even an undergraduate student may use it!

In other words, we believe that our routine could be an interesting and, as far as we know, the first example of a MAPLE implementation of *local monomial ordering*

*The authors were partially supported by the MIUR-PRIN "Geometria delle varietà algebriche"2010-2011 Research Funds. The first author is also supported by the INDAM as a member of GNSAGA.

(l.m.o.). In fact, term orders, i.e. usual monomial ordering (recently called *global* in contrast with the word *local*), are already implemented in MAPLE with the command `MonomialOrder` in the `Groebner` package. Actually, user-defined term orders are also allowed and, in particular, l.m.o.'s can be easily defined as the *opposite* of a standard g.m.o. (pure lexicographic, graded lexicographic, reverse, etc.). The problem is that the Buchberger S-procedure may not end up determining a *normal form* since a l.m.o. is not a well-order in contrast to g.m.o.'s. So the present MAPLE routines start with an implementation of the Mora algorithm for determining a *weak normal form* and then monomial bases of Milnor and Tyurina ideals in the complex ring of convergent power series $\mathbb{C}\{x_1,\dots,x_{n+1}\}$ (see [20], [13] Algorithm 1.7.6, [8] Algorithm 9.22). As a consequence, a monomial basis of the Kuranishi space, parameterizing small versal deformations of the given i.h.s., is obtained, allowing to concretely write down these deformations even for more intricate cases. Actually we get procedures which are able to perform calculations in $(\mathbb{C}[\lambda])[\mathbf{x}]$, where $\lambda = (\lambda_1,\dots,\lambda_r)$ is an r-tuple of parameters e.g. coordinates of the Kuranishi space.

An interesting application of this last feature is that of *writing down equations* of algebraic stratifications in Kuranishi spaces of Arnol'd simple singularities, giving an *explicit* geometric interpretation, by means of inclusions of algebraic subsets, of *Arnol'd's adjacency partial order relation* over classes of simple singularities (see Section 6). This section ends up with an explicit list of the most specialized 1-parameter deformations of simple singularities realizing adjacencies between distinct classes of simple singularities (see 6.7). The interested reader is referred to [24] for detailed calculations.

1. Milnor number of an isolated hypersurface singularity

From the topological point of view a *good representative* $\overline{U}$ of an *isolated hypersurface singularity (i.h.s.)* is the zero locus of a holomorphic map

$$f:\mathbb{C}^{n+1}\longrightarrow\mathbb{C}\quad,\quad n\geq 0 \tag{1}$$

admitting an isolated critical point in $0\in\mathbb{C}^4$.
Set $U_T := f^{-1}(T)$ where T is a small enough neighborhood of $0\in\mathbb{C}$. Then we can assume that *f is a submersion over $U_T\setminus\{0\}$*: therefore $\overline{U}=f^{-1}(0)=U_0$ and U_t is a *local smoothing* of $\overline{U}$, for $0\neq t\in T$.

DEFINITION 1. *Let $X\subset\mathbb{C}^m$ be a subset. The following subset of $\mathbb{C}^m$*

$$\mathrm{Cn}_0(X) := \{tx \mid \forall t\in[0,1]\subset\mathbb{R}\ ,\ \forall x\in X\}$$

will be called the cone projecting X.

THEOREM 1 (Local topology of a isolated hypersurface singularity, [18] Theorem 2.10, Theorem 5.2). *Let D_ε denote the closed ball of radius $\varepsilon>0$, centered in $0\in\mathbb{C}^{n+1}$, whose boundary is the $2n+1$–dimensional sphere S^{2n+1}_ε. Then, for ε small enough,* *the intersection* $\overline{B}_\varepsilon := \overline{U} \cap D_\varepsilon$ *is homeomorphic to the cone* $\mathrm{Cn}_0(K)$ *projecting* $K := \overline{U} \cap S^{2n+1}_\varepsilon$*, which is called the* knot *or* link of the singularity $0 \in \overline{U}$.

THEOREM–DEFINITION 2 (Local homology type of the smoothing [18], Theorems 5.11, 6.5, 7.2). Set $\widetilde{U} := U_t$ for some $0 \neq t \in T$. Then, for ε small enough, the intersection

$$\widetilde{B}_\varepsilon := \widetilde{U} \cap D_\varepsilon$$

(called the *Milnor fibre* of f) has the homology type of a bouquet of n–dimensional spheres. In particular the n–th Betti number $b_n(\widetilde{B}_\varepsilon)$ (called the *Milnor number* m_p of p) coincides with the *multiplicity* of the critical point $0 \in \mathbb{C}^{n+1}$ of f as a solution to the following collection of equations

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \cdots = \frac{\partial f}{\partial x_{n+1}} = 0\ .$$

1.1. Milnor number from the algebraic point of view

Theorem 2 allows the following algebraic interpretation of the Milnor number.
Let $\mathcal{O}_0$ be the local ring of germs of holomorphic function of $\mathbb{C}^{n+1}$ at the origin. By definition of holomorphic function and the identity principle we have that $\mathcal{O}_0$ *is isomorphic to the ring of convergent power series* $\mathbb{C}\{x_1, \ldots, x_{n+1}\}$. A *germ of hypersurface singularity* is defined as the Stein complex space

$$U_0 := \mathrm{Spec}(\mathcal{O}_{f,0}) \tag{2}$$

where $\mathcal{O}_{f,0} := \mathcal{O}_0/(f)$ and f is the germ represented by the series expansion of the holomorphic function (1).

DEFINITION 2 (Milnor number of an i.h.s, see e.g. [16]). *The* Milnor number *of the hypersurface singularity* $0 \in U_0$ *is defined as* the multiplicity of the critical point $0 \in \mathbb{C}^{n+1}$ of f as a solution of the system of partials of f *([18] §7) which is*

$$\mu_f(0) = \dim_{\mathbb{C}}(\mathcal{O}_0/J_f) = \dim_{\mathbb{C}}(\mathbb{C}\{x_1, \ldots, x_{n+1}\}/J_f) \tag{3}$$

where $\dim_{\mathbb{C}}$ *means "dimension as a* $\mathbb{C}$*–vector space" and* J_f *is the jacobian ideal* $J_f := \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_{n+1}}\right)$*. For shortness we will denote the Milnor number (3) by* $\mu(0)$ *whenever* f *is clear.*

2. Tyurina number of an isolated hypersurface singularity

2.1. Deformations of complex spaces

Let $\mathcal{X} \xrightarrow{x} B$ be a *flat*, surjective and proper map of complex spaces such that B is connected and there exists a special point $0 \in B$ whose fibre $X = x^{-1}(0)$ may be singular. Then $\mathcal{X}$ is called *a deformation family of* X. If the fibre $X_b = x^{-1}(b)$ is smooth, for some $b \in B$, then X_b is called *a smoothing of* X.

Let Ω_X be the sheaf of holomorphic differential forms on X and consider the *Lichtenbaum–Schlessinger cotangent sheaves* [15] of X, $\Theta^i_X = \mathcal{E}xt^i(\Omega_X, \mathcal{O}_X)$. Then

$$\Theta^0_X = \mathcal{H}om(\Omega_X, \mathcal{O}_X) =: \Theta_X$$

is the "tangent" sheaf of X and Θ^i_X is supported over $\mathrm{Sing}(X)$, for any $i > 0$. Consider the associated *local* and *global deformation objects*

$$T^i_X := H^0(X, \Theta^i_X) \quad , \quad \mathbb{T}^i_X := \mathrm{Ext}^i\left(\Omega^1_X, \mathcal{O}_X\right) \ , \ i = 0, 1, 2.$$

Then by the *local to global spectral sequence* relating the global Ext and sheaf $\mathcal{E}xt$ (see [14] and [10] II, 7.3.3) we get

$$E_2^{p,q} = H^p\left(X, \Theta^q_X\right) \Longrightarrow \mathbb{T}^{p+q}_X$$

giving that

(4) $\mathbb{T}^0_X \cong T^0_X \cong H^0(X, \Theta_X)$,

(5) if X is smooth then $\mathbb{T}^i_X \cong H^i(X, \Theta_X)$,

(6) if X is Stein then $T^i_X \cong \mathbb{T}^i_X$.

Recall that $\mathcal{X} \xrightarrow{x} B$ is called a *versal* deformation family of X if for any deformation family $(\mathcal{Y}, X) \xrightarrow{y} (C, o)$ of X there exists a map of pointed complex spaces $h : (U, o) \to (B, 0)$, defined on a neighborhood $o \in U \subset C$, such that $\mathcal{Y}|_U$ is the *pull–back* of $\mathcal{X}$ by h i.e.

$$\begin{array}{ccccc} & & \mathcal{Y}|_U = U \times_B \mathcal{X} & \longrightarrow & \mathcal{X} \\ & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle x} \\ C & \hookleftarrow & U & \xrightarrow{h} & B \end{array}$$

THEOREM 3 (Douady–Grauert–Palamodov [9], [11], [21] and [22] Theorems 5.4 and 5.6). *Every compact complex space X has an effective versal deformation $\mathcal{X} \xrightarrow{x} B$ which is a proper map and a versal deformation of each of its fibers. Moreover the germ of analytic space $(B, 0)$ (the* Kuranishi space *of X) is isomorphic to the germ of analytic space $(q^{-1}(0), 0)$, where $q : \mathbb{T}^1_X \to \mathbb{T}^2_X$ is a suitable holomorphic map (the* obstruction map*) such that $q(0) = 0$.*

In particular if $q \equiv 0$ (e.g. when $\mathbb{T}^2_X = 0$) then $(B, 0)$ turns out to be isomorphic to the germ of a neighborhood of the origin in $\mathbb{T}^1_X$.

2.2. Deformations of an i.h.s.

Let us consider the germ of i.h.s. $U_0 := \mathrm{Spec}(\mathcal{O}_{f,0})$ as defined in (2).

DEFINITION 3 (Tyurina number of an i.h.s.). *The* Tyurina number *of the i.h.s.* $0 \in U_0$ *is*

$$\tau_f(0) := \dim_{\mathbb{C}} \mathbb{T}^1_{U_0} \overset{(6)}{=} \dim_{\mathbb{C}} T^1_{U_0} = h^0(U_0, \Theta^1_{U_0})$$

often denoted simply by $\tau(0)$ *whenever* f *is clear. Since* U_0 *is Stein, the obstruction map* q *in Theorem 3 is trivial and the Tyurina number* $\tau(0)$ *turns out to give* the dimension of the Kuranishi space of U_0.

PROPOSITION 1 (see e.g. [26]). *If* $0 \in U_0 = \operatorname{Spec}(O_{f,0})$ *is the germ of an i.h.s. then* $\mathbb{T}^1_{U_0} \cong O_{f,0}/J_f$ *and*

$$\tau_f(0) = \dim_{\mathbb{C}}\left(\mathbb{C}\{x_1,\ldots,x_{n+1}\}/I_f\right) . \tag{7}$$

where $I_f := (f) + J_f$. *Then, recalling (3),* $\tau(0) \leq \mu(0)$. *In particular* $\tau(0)$ *gives the multiplicity of* 0 *as a singular point of the complex space germ* U_0.

3. Milnor and Tyurina numbers of a polynomial

Let us consider the polynomial algebra $\mathbb{C}[\mathbf{x}] := \mathbb{C}[x_1,...,x_{n+1}]$ and let $I \subset \mathbb{C}[\mathbf{x}]$ be an ideal. If we consider the natural inclusion $\mathbb{C}[\mathbf{x}] \subset \mathbb{C}\{\mathbf{x}\} := \mathbb{C}\{x_1,\ldots,x_{n+1}\}$ then we get

$$\dim_{\mathbb{C}}\left(\mathbb{C}\{\mathbf{x}\}/I\cdot\mathbb{C}\{\mathbf{x}\}\right) \leq \dim_{\mathbb{C}}\left(\mathbb{C}[\mathbf{x}]/I\right) \tag{8}$$

since the algebra $\mathbb{C}\{\mathbf{x}\}$ contains also non constant units.

EXAMPLE 1. Let us consider the ideal $I = (x+x^2) \subset \mathbb{C}[x]$. Then $\mathbb{C}[x]/I = \langle 1,x\rangle_{\mathbb{C}}$ while $\mathbb{C}\{x\}/I\cdot\mathbb{C}\{x\} = \langle 1\rangle_{\mathbb{C}}$, since $x \in I\cdot\mathbb{C}\{x\}$ being associated with the generator $x+x^2$ by the unit $1+x \in \mathbb{C}\{x\}$. This means that

$$1 = \dim_{\mathbb{C}}\left(\mathbb{C}\{x\}/I\cdot\mathbb{C}\{x\}\right) < \dim_{\mathbb{C}}\left(\mathbb{C}[x]/I\right) = 2 . \tag{9}$$

In particular the first equality means that, recalling Definition 2 and formula (3), the Milnor number of $0 \in \mathbb{C}$, as a critical point of the polynomial map $f(x) = (\frac{1}{2} + \frac{1}{3}x)x^2$, turns out to be $\mu(0) = 1$.

Let us then set the following

DEFINITION 4 (Milnor and Tyurina numbers of a polynomial). *Given a polynomial* $f \in \mathbb{C}[\mathbf{x}]$ *the following dimension*

$$\mu(f) := \dim_{\mathbb{C}}\left(\mathbb{C}[\mathbf{x}]/J_f\right) \tag{10}$$

is called the Milnor number of the polynomial f. *Analogously the dimension*

$$\tau(f) := \dim_{\mathbb{C}}\left(\mathbb{C}[\mathbf{x}]/I_f\right) \tag{11}$$

where $I_f := (f) + J_f$*, is called* the Tyurina number of the polynomial f. *Then clearly* $\tau(f) \leq \mu(f)$.

The inequality (8) then gives that $\mu_f(p) \leq \mu(f)$ and $\tau_f(p) \leq \tau(f)$, for any point $p \in \mathbb{C}^{n+1}$ and any polynomial $f \in \mathbb{C}[\mathbf{x}]$. Moreover Milnor and Tyurina numbers of points and polynomials are related by the following

PROPOSITION 2 (see e.g. [13] §A.9). *For any* $f \in \mathbb{C}[x_1, \dots, x_{n+1}]$

$$\mu(f) = \sum_{p \in \mathbb{C}^{n+1}} \mu_f(p) \quad \textit{and} \quad \tau(f) = \sum_{p \in \mathbb{C}^{n+1}} \tau_f(p) \tag{12}$$

REMARK 1. Observe that sums on the right terms of (12) are actually finite and well–defined since $\mu(p) \neq 0$ if and only if p is a critical point of the polynomial map f and $\tau(p) \neq 0$ if and only if p is a singular point of the n–dimensional algebraic hypersurface $f^{-1}(0) \subset \mathbb{C}^{n+1}$.

EXAMPLE 2 (Example 1 continued). Critical points of the polynomial map $f(x) = (\frac{1}{2} + \frac{1}{3}x)x^2$ are given by 0 and -1. The translation $x \mapsto x-1$ transforms f into the polynomial $g(x) = \frac{1}{3}(\frac{1}{2} + x)(x-1)^2$ and

$$\mu_f(-1) = \mu_g(0) = \dim_{\mathbb{C}} \left(\mathbb{C}\{x\}/(x^2 - x)\right) = 1 \ .$$

The inequality (9) then gives $2 = \mu(f) = \mu_f(0) + \mu_f(-1)$, according with the first equality in (12). Moreover 0 is the unique singular point of the 0–dimensional hypersurface $f^{-1}(0) = \{-3/2, 0\} \subset \mathbb{C}$ and

$$\begin{aligned} \tau(f) &= \dim_{\mathbb{C}} \left(\mathbb{C}[x] \Big/ \left(\left(\frac{1}{2} + \frac{1}{3}x \right) x^2, x + x^2 \right) \right) = 1 \\ \tau_f(0) &= \dim_{\mathbb{C}} \left(\mathbb{C}\{x\} \Big/ \left(\left(\frac{1}{2} + \frac{1}{3}x \right) x^2, x + x^2 \right) \right) = 1 \end{aligned}$$

according with the second equality in (12).

3.1. Milnor and Tyurina numbers of weighted homogeneous polynomials

Let us recall that a polynomial $f \in \mathbb{C}[x_1, \dots, x_{n+1}]$ is called *weighted homogeneous (w.h.p.)* or *quasi–homogeneous* if there exist $n+1$ *positive* rational numbers $\mathbf{w} = (w_1, \dots, w_{n+1}) \in \mathbb{Q}^{n+1}$ such that $\sum_{i=1}^{n+1} w_i \alpha_i = 1$ for any monomial $\mathbf{x}^\alpha := \prod_{i=1}^{n+1} x_i^{\alpha_i}$ appearing in f; $\mathbf{w}$ is then called *the vector of (rational) weights of* f and the *generalized Euler formula*

$$f = \sum_{i=1}^{n+1} w_i x_i \frac{\partial f}{\partial x_i} \tag{13}$$

follows immediately for any w.h.p. $f \in \mathbb{C}[\mathbf{x}]$ admitting the same vector of weights. For any $i = 1, \dots, n+1$, let $(p_i, q_i) \in \mathbb{N}^2$ be the unique ordered couple of positive coprime integers such that p_i/q_i is the reduced fraction representing the positive rational number w_i. Calling d the least common factor of denominators $q_1, \dots, q_{n+1}$, the positive integers $d_i := dw_i$ satisfy the following *weighted homogeneity relation*

$$\forall \lambda \in \mathbb{C} \quad f\left(\lambda^{d_1} x_1, \dots, \lambda^{d_{n+1}} x_{n+1}\right) = \lambda^d f(x_1, \dots, x_{n+1}) \ . \tag{14}$$

For this reason $\mathbf{d} = (d_1, \dots, d_{n+1})$ is called *the vector of integer weights of f* and $d = |\mathbf{d}|$ is called the *degree of f*.

PROPOSITION 3. *Given a polynomial $f \in \mathbb{C}[\mathbf{x}]$ with a finite number of critical points, the following assertions are equivalent:*

(a) *f is a w.h.p.,*

(b) *$\tau(f) = \mu(f)$,*

(c) *$\forall p \in \mathbb{C}^{n+1} \quad \tau(p) = \mu(p)$.*

In particular (c) implies that the set of critical points of f coincides with the set of singular points of $f^{-1}(0)$. Moreover (a) means actually that the origin $0 \in \mathbb{C}^{n+1}$ is the unique possible critical point of f and then the unique possible singular point of $f^{-1}(0)$. Therefore (b) and ((c) give that

$$\mu(0) = \mu(f) = \tau(f) = \tau(0) \; . \tag{15}$$

Proof. $(a) \Rightarrow (c)$. The generalized Euler formula (13) implies that $I_f = J_f$ and (c) follows immediately by (3) and (7). In particular if $\mathbf{x} = (x_1, \ldots, x_{n+1})$ is a critical point of f, then (13) and (14) give that $(\lambda^{d_1} x_1, \ldots, \lambda^{d_{n+1}} x_{n+1})$ is a critical point of f for any complex number λ. Then $\mathbf{x} = 0$ since f admits at most a finite number of critical points, meaning that f admits at most the origin as a critical point.
$(c) \Rightarrow (b)$. This follows immediately by Proposition 2.
$(b) \Rightarrow (a)$. Since $J_f \subseteq I_f$ then we get the natural surjective map of $\mathbb{C}$–algebras

$$\mathbb{C}[\mathbf{x}]/J_f \longrightarrow\!\!\!\!\!\rightarrow \mathbb{C}[\mathbf{x}]/I_f$$

The hypothesis $\tau(f) = \mu(f)$ implies that it is also injective which suffices to show that $J_f = I_f$. Hence $f \in J_f$ and a famous result by K. Saito [25] allows to conclude that f is a w.h.p.. □

DEFINITION 5 (Weighted homogeneous singularity - w.h.s.). *A n–dimensional i.h.s. $0 \in U_0 = \mathrm{Spec}\, \mathcal{O}_{f,0}$ is called* weighted homogeneous *(or* quasi-homogeneous*) if there exists a w.h.p. $F \in \mathbb{C}[x_1, \ldots, x_{n+1}]$ such that $U_0 \cong \mathrm{Spec}\, \mathcal{O}_{F,0}$, as germs of complex spaces.*

REMARK 2. Definition 5 is equivalent to require that there exists an automorphism ϕ^* of $\mathcal{O}_0 = \mathbb{C}\{\mathbf{x}\}$, induced by a biholomorphic local coordinates change $(\mathbb{C}^{n+1}, 0) \xrightarrow{\phi} (\mathbb{C}^{n+1}, 0)$, such that $\phi^*(f) := f \circ \phi = F$ (see [12] Lemma 2.13).

PROPOSITION 4 (Characterization of a w.h.s). $0 \in U_0 = \mathrm{Spec}\, \mathcal{O}_{f,0}$ *is a w.h.s. if and only if* $\tau_f(0) = \mu_f(0)$.

Proof. The statement follows immediately by Proposition 3, keeping in mind Remark 2 and observing that the jacobian ideals J_f and J_F can be obtained each other by multiplying the jacobian matrix of the coordinate change ϕ, which is clearly invertible in a neighborhood of 0. □

3.2. An example: weighted homogeneous cDV singularities

An example of an isolated hypersurface singularity is given by a *compound Du Val* (cDV) singularity which is a 3–fold point p such that, for a hyperplane section H through p, $p \in H$ is a Du Val surface singularity i.e. an A–D–E singular point (see [23], §0 and §2, and [6], chapter III). Then a cDV point p is a germ of hypersurface singularity $0 \in U_0 := \operatorname{Spec}(\mathcal{O}_{f,0})$, where f is the polynomial

$$f(x,y,z,t) := x^2 + q(y,z) + t\, g(x,y,z,t)\ , \tag{16}$$

such that $g(x,y,z,t)$ is a generic element of the maximal ideal $\mathfrak{m}_0 := (x,y,z,t) \subset \mathbb{C}[x,y,z,t]$ and

$$\begin{array}{lcll} A_n & : & q(y,z) := y^2 + z^{n+1} & \text{for } n \geq 1 \\ D_n & : & q(y,z) := y^2 z + z^{n-1} & \text{for } n \geq 4 \\ E_6 & : & q(y,z) := y^3 + z^4 & \\ E_7 & : & q(y,z) := y^3 + yz^3 & \\ E_8 & : & q(y,z) := y^3 + z^5 & \end{array} \tag{17}$$

In particular if

$$g(x,y,z,t) = t \tag{18}$$

then $f = 0$ in (16) *is said to define an Arnol'd simple (threefold) singularity* ([3], [5] §15 and in particular [4] §I.2.3) denoted by $A_n, D_n, E_{6,7,8}$, respectively.

The index $(n,6,7,8)$ turns out to be the Milnor number of the surface Du Val singularity $0 \in U_0 \cap \{t = 0\}$ or equivalently its Tyurina number, since a Du Val singular point always admits a weighted homogeneous local equation. When a cDV point is defined by a weighted homogeneous polynomial f, a classical result of J. Milnor and P. Orlik allows to compare this index with its Milnor (and then Tyurina) number. In particular we get

$$\begin{array}{rcl} w(x) & = & 1/2 \\ w(y) & = & \left\{ \begin{array}{cl} 1/2 & \text{if } p \text{ is } cA_n, \\ (n-2)/(2n-2) & \text{if } p \text{ is } cD_n, \\ 1/3 & \text{if } p \text{ is } cE_{6,7,8}. \end{array} \right. \\ w(z) & = & \left\{ \begin{array}{cl} 1/(n+1) & \text{if } p \text{ is } cA_n, \\ 1/(n-1) & \text{if } p \text{ is } cD_n, \\ 1/4 & \text{if } p \text{ is } cE_6, \\ 2/9 & \text{if } p \text{ is } cE_7, \\ 1/5 & \text{if } p \text{ is } cE_8. \end{array} \right. \end{array} \tag{19}$$

THEOREM 4 (Milnor–Orlik [19], Thm. 1). *Let $f(x_1,\ldots,x_{n+1})$ be a w.h.p., with rational weights $w_1,\ldots,w_{n+1}$, admitting an isolated critical point at the origin. Then the Milnor number of the origin is given by*

$$\mu(0) = \left(w_1^{-1} - 1\right)\left(w_2^{-1} - 1\right)\cdots\left(w_{n+1}^{-1} - 1\right)\ .$$

By putting weights (19) in the previous Milnor–Orlik formula we get the following

COROLLARY 1. *Le $0 \in U_0$ be a w.h. cDV point of index n. Then*

$$\tau(0) = \mu(0) = n\left(w(t)^{-1} - 1\right) .$$

In particular for Arnol'd simple singularities we get $n = \tau(0) = \mu(0)$, as can also be directly checked by the definition.

3.3. The algebraic computation via Gröbner basis

Let us consider an ideal $I \subset \mathbb{C}[\mathbf{x}]$ and let $L(I)$ denote the ideal generated by *leading monomials*, with respect to a fixed *monomial order*, of elements in I. It is a well known fact (see e.g. [7, §5.3]) that the following are isomorphisms of $\mathbb{C}$–vector spaces

$$\mathbb{C}[x_1,\ldots,x_{n+1}]/I \xrightarrow{\cong} \mathbb{C}[x_1,\ldots,x_{n+1}]/L(I) \xrightarrow{\cong} \langle M \setminus L(I)\rangle_{\mathbb{C}} \tag{20}$$

where M is the set (actually a multiplicative monoid) of all monomials $\mathbf{x}^\alpha$. Consider a polynomial $f \in \mathbb{C}[\mathbf{x}]$. By Definition 4 and isomorphisms (20), the computation of $\mu(f)$ and $\tau(f)$ reduces to calculate

$$\dim_{\mathbb{C}}\left(\mathbb{C}[x_1,\ldots,x_{n+1}]/L(I)\right) = |M \setminus L(I)|$$

where $I \subset \mathbb{C}[x_1,\ldots,x_{n+1}]$ is either the jacobian ideal J_f or the ideal $I_f = (f) + J_f$, respectively. The point is then *determining a Gröbner basis of I w.r.t. the fixed monomial order*, which can be realized e.g. by the `Groebner Package` of MAPLE.

REMARK 3. The MAPLE computation of Milnor and Tyurina numbers of polynomials is realized by procedures `PolyMilnor` and `PolyTyurina`, whose concrete description is postponed to section 5. Their usefulness is clarified by Proposition 3 and in particular by equations (15). In fact in the case of a w.h.p. f admitting an isolated critical point in $0 \in \mathbb{C}^{n+1}$ there is no need of working with power series (and then with *local monomial orders*) to determine $\mu(0)$ and $\tau(0)$. Since the Buchberger algorithm implemented with MAPLE turns out to be more efficient when running by usual term orders, the use of *global* procedures `PolyMilnor` and `PolyTyurina` has to be preferred to the use of their *local* counterparts `Milnor` and `Tyurina`, when possible.

4. Monomial Ordering

First of all let us recall what is usually meant by a monomial order. For more details the interested reader is remanded to e.g. [7, §2.2] , [13, §9] and [8, §1].

Let M be the multiplicative monoid of monomials $\mathbf{x}^\alpha = \prod_{i=1}^{n+1} x_i^{\alpha_i}$: clearly $\log : M \xrightarrow{\cong} \mathbb{N}^{n+1}$. A *(global) monomial order on* $\mathbb{C}[\mathbf{x}]$ is a *total order relation* $\leq$ on M which is

(i) *multiplicative* i.e. $\forall \alpha,\beta,\gamma \in \mathbb{N}^{n+1} \quad \mathbf{x}^\alpha \leq \mathbf{x}^\beta \Rightarrow \mathbf{x}^\alpha \cdot \mathbf{x}^\gamma \leq \mathbf{x}^\beta \cdot \mathbf{x}^\gamma$,

(ii) *a well-ordering* i.e. every nonempty subset of M has a *smallest element*.

Since $\mathbb{C}[\mathbf{x}]$ is a noetherian ring a multiplicative total order on M is a well-ordering if and only if

(ii') $\forall i = 1, \ldots, n+1 \quad 1 < x_i$.

DEFINITION 6 (Local and global monomial orders, [13, Definition 1.2.4]). *In the following a m.o. on* $\mathbb{C}[\mathbf{x}]$ *will denote simply a total order relation* $\leq$ *on* M *which is multiplicative i.e. satisfying (i). A m.o. will be called* global (g.m.o.) *if also (ii), or equivalently (ii'), is satisfied. Moreover a m.o. will be called* local (l.m.o.) *if*

(ii") $\forall i = 1, \ldots, n+1 \quad x_i < 1$.

4.1. Localizations in $\mathbb{C}[\mathbf{x}]$ and rings implemented by monomial orders

Given a m.o. $\leq$ on M consider the following subset of $\mathbb{C}[\mathbf{x}]$

$$S := \{f \in \mathbb{C}[\mathbf{x}] \mid L(f) \in \mathbb{C} \setminus \{0\}\}$$

where $L(f)$ is the *leading monomial* of f w.r.t. $\leq$. Since S is a multiplicative subset of $\mathbb{C}[\mathbf{x}]$ we can consider the localization $S^{-1}\mathbb{C}[\mathbf{x}]$. Then (ii') and (ii") give immediately the following

PROPOSITION 5.

$$\begin{array}{rclcl} S^{-1}\mathbb{C}[\mathbf{x}] & = & \mathbb{C}[\mathbf{x}] & \Leftrightarrow & \leq \textit{ is a g.m.o.} \\ S^{-1}\mathbb{C}[\mathbf{x}] & = & \mathbb{C}[\mathbf{x}]_{(\mathbf{x})} & \Leftrightarrow & \leq \textit{ is a l.m.o.} \end{array}$$

where $\mathbb{C}[\mathbf{x}]_{(\mathbf{x})}$ *is the localization of* $\mathbb{C}[\mathbf{x}]$ *at the maximal ideal* $(\mathbf{x}) \subset \mathbb{C}[\mathbf{x}]$.

By Taylor power series expansion of locally holomorphic functions, there is a natural inclusion $\mathbb{C}[\mathbf{x}]_{(\mathbf{x})} \subset \mathbb{C}\{\mathbf{x}\}$ giving the following commutative diagram, for every ideal $I \subset \mathbb{C}[\mathbf{x}]_{(\mathbf{x})}$:

$$\begin{tikzcd} \mathbb{C}[\mathbf{x}]_{(\mathbf{x})} \arrow[r, hook] \arrow[d, two heads] & \mathbb{C}\{\mathbf{x}\} \arrow[d, two heads] \\ \mathbb{C}[\mathbf{x}]_{(\mathbf{x})}/I \arrow[r, hook] & \mathbb{C}\{\mathbf{x}\}/I \cdot \mathbb{C}\{\mathbf{x}\} \end{tikzcd}$$

PROPOSITION 6 (for a proof see e.g. [8, Proposition 9.4]). *If* $\dim_{\mathbb{C}} \left(\mathbb{C}[\mathbf{x}]_{(\mathbf{x})}/I\right)$ *is finite then the inclusion* $\mathbb{C}[\mathbf{x}]_{(\mathbf{x})}/I \subset \mathbb{C}\{\mathbf{x}\}/I \cdot \mathbb{C}\{\mathbf{x}\}$ *is an isomorphism of* $\mathbb{C}$*–algebras. In particular both the underlying vector spaces have the same dimension.*

THEOREM 5 ([8, Theorem 9.29]). *Let* $\leq$ *be a m.o. on* $\mathbb{C}[\mathbf{x}]$ *and* $I \subset S^{-1}\mathbb{C}[\mathbf{x}]$ *be an ideal. Then*

$$\dim_{\mathbb{C}} \left(S^{-1}\mathbb{C}[\mathbf{x}]/I\right) = \dim_{\mathbb{C}} \left(\mathbb{C}[\mathbf{x}]/L(I)\right)$$

where $L(I)$ is the ideal generated by leading monomials*, with respect to $\leq$, of polynomials in I. In particular if it is finite then $M \setminus L(I)$ represents a basis of the vector space $S^{-1}\mathbb{C}[\mathbf{x}]$.*

COROLLARY 2. *Let $f \in \mathbb{C}[\mathbf{x}]$ admit an isolated critical point at $0 \in \mathbb{C}^{n+1}$, $\leq$ be a m.o. on $\mathbb{C}[\mathbf{x}]$ and $I = J_f$ (resp. $I = I_f$). Then*

$$\dim_{\mathbb{C}} (\mathbb{C}[\mathbf{x}]/L(I)) = \begin{cases} \mu(f) \ (\textit{resp. } \tau(f)) & \textit{if } \leq \textit{ is global} \\ \mu_f(0) \ (\textit{resp. } \tau_f(0)) & \textit{if } \leq \textit{ is local} \end{cases}$$

REMARK 4. The point is then determining a *standard basis of I* w.r.t. the fixed m.o. $\leq$. If the latter is a *global* one, a standard basis is a usual Gröbner basis which is obtained by applying the Buchberger algorithm. In MAPLE this is implemented by the `Groebner Package`.

On the other hand, if $\leq$ is a *local* m.o. the Buchberger algorithm does no more work. In fact $\leq$ is no more a well-ordering and the division algorithm employed by the Buchberger algorithm for determining *normal forms* of *S-polynomials* may do not terminate. This problem can be dodged by means of a *weak normal form algorithm* (*weakNF*), firstly due to F. Mora [20], and of a *standard basis algorithm* (*SB*) which replaces the Buchberger algorithm. This is precisely what has been implemented in SINGULAR since 1990 (see [13, §1.7] and references thereof). The aim of the following section 5 is to present a MAPLE implementation of *weakNF* and *SB* algorithms to yield a procedure computing Milnor and Tyurina numbers of points.

5. MAPLE subroutines in detail

The present section is devoted to present and describe MAPLE subroutines computing Milnor and Tyurina numbers of critical points of a polynomial f. They are available as MAPLE 12 file `.mw` at [1] and presented in details in [24, §5.1]. They are composed by several procedures, the most important of which are the following:

- `weakNF` which is the MAPLE implementation of the weak normal form Algorithm 1.7.6 in [13];
- `SB` which is the MAPLE implementation of the standard basis Algorithm 1.7.1 in [13];
- `Milnor` which is the procedure computing the Milnor number of an isolated critical point;
- `PolyMilnor` which is the procedure computing the Milnor number of a polynomial;
- `Tyurina` which is the procedure computing the Tyurina number of an isolated singular point;
- `PolyTyurina` which is the procedure computing the Tyurina number of a polynomial.

5.1. The subroutines

Preambles to introduce the useful MAPLE packages:

```
> with(Groebner):

> with(PolynomialIdeals):

> with(Ore_algebra):
```

A first control procedure :

```
> localorglobal := proc (STO, variables)
> A:= poly_algebra(op(variables)); TP := MonomialOrder(A, STO);
> nuu:= 1; muu := 1;
> for i to nops(variables) do if TestOrder(1,variables[i], TP)
> then nuu := 0 else muu := 0 end if end do;
> if nuu = 1 then Lo else if muu = 1 then Gl else Mi
> end if end if end proc:
```

Actually `localorglobal` is not an essential procedure in the present routine: its meaning is simply that of giving a feedback about what kind of m.o. the user is employing and stopping the procedure running with a wrong term order: in fact `Milnor` and `Tyurina` may give wrong output when running with a g.m.o.; on the other hand `PolyMilnor` and `PolyTyurina` may not terminate when running with a l.m.o..

Implementing local monomial orders

The following three procedures give the core of the *MAPLE implementation of l.m.o.'s* for determining standard basis of ideals in $\mathbb{C}[\mathbf{x}]_{(\mathbf{x})}$. The first procedure introduces the *ecart* concept which is the main ingredient in the Mora algorithm `weakNF`. It is defined following [13] Definition 1.7.5:

```
> ecart := proc (f, variables, STO)
> degree(f, variables)-degree(LeadingMonomial(f, STO), variables)
> end proc:
```

Then we give the Mora algorithm for determining a weak normal form of a polynomial $f \in \mathbb{C}[\mathbf{x}]$ w.r.t. a finite subset of polynomials $G \subset \mathbb{C}[\mathbf{x}]$ (see [13, Algorithm 1.7.6] and [8, Algorithm 9.22]):

```
> weakNF := proc (f, G, variables, STO)
> h := f;
> TT := G;
> TTh := {}; for i to nops(TT) do
> if divide(LeadingMonomial(h, STO),LeadingMonomial(TT[i], STO))
> then TTh := {TT[i], op(TTh)}
> end if end do;
> while (h <> 0 and TTh <> {}) do
> L := [op(TTh)];
> L1 := sort(L, proc (t1, t2) options operator, arrow;
> ecart(t1,variables,STO) <= ecart(t2,variables,STO) end proc);
> g := L1[1];
> if ecart(h, variables, STO) < ecart(g, variables, STO)
> then TT := {h, op(TT)} end if;
> h := SPolynomial(h, g, STO);
> TTh := {};
> for i to nops(TT) do
> if divide(LeadingMonomial(h, STO), LeadingMonomial(TT[i],STO))
> then TTh := {TT[i], op(TTh)} end if end do end do;
> h end proc
```

At last the standard basis procedure giving the analogue of Buchberger algorithm with l.m.o.'s:

```
> SB := proc (G, variables, STO)
> S := G;
> P:= {seq(seq({G[i], G[j]},j=i+1..nops(G)),i=1..nops(G))};
> while P <> {} do
> P1 := P[1];
> P := `minus`(P,{P1});
> h := weakNF(SPolynomial(P1[1],P1[2],STO), S, variables, STO);
> if h <> 0 then P := {seq({h, S[i]}, i = 1 ..  nops(S)),op(P)};
> S := {h, op(S)} end if end do;
> S end proc:
```

Computing Milnor and Tyurina numbers

We are now in a position to introduce the procedures computing Milnor and Tyurina numbers of critical points of a polynomial $F \in \mathbb{C}[\mathbf{x}]$. $F \in \mathbb{C}[\mathbf{x}]$. They actually give some more output. Precisely, saying $I = J_F$ (resp. $I = I_F$) `Milnor` (resp. `Tyurina`) returns:

- a standard basis G of the ideal $I \cdot \mathbb{C}[\mathbf{x}]_{(\mathbf{x})} \subset \mathbb{C}[\mathbf{x}]_{(\mathbf{x})}$,
- the leading monomial basis $L(G)$ w.r.t. a fixed l.m.o.,
- a basis of the quotient vector space $\mathbb{C}[\mathbf{x}]_{(\mathbf{x})} / I \cdot \mathbb{C}[\mathbf{x}]_{(\mathbf{x})}$

- and its dimension over $\mathbb{C}$, which is $\mu_F(0)$ (resp. $\tau_F(0)$).

The procedures described below require three input: the polynomial F and two further optional input, precisely

- a set of variables, by default set as the variables appearing in F,
- a monomial order, by default set either as `tdeg(variables)`, that is the graduated reverse lexicographic g.m.o., or as `tdeg_ min(variables)`, which is the l.m.o. defined as the `tdeg` opposite: the former is clearly introduced in `PolyMilnor` and `PolyTyurina` and the latter in `Milnor` and `Tyurina`.

The Milnor procedure:

INPUT:
a polynomial F,
(optional) a set of variables $\texttt{variables} = \{x_1, ..., x_r\}$
(by default `variables` is the set of indeterminates appearing in F)
(optional) a l.m.o. U (by default U is `tdeg_ min`)

1. Set $J := [\partial F/\partial x_1, ..., \partial F/\partial x_r]$
2. Set
   G := $SB(J, \texttt{variables}, U)$
   Ini := [leading monomials w.r.t. U of elements in G]
3. Check that the ideal $Ini \subseteq \mathbb{C}[x_1, ...x_r]$ is zero-dimensional; else **error**;
4. Set
   m := $r \cdot \max\{$degree of monomials in $Ini\}$
   L := list of all monomials of degree $\leq m$
   M := $L \setminus \{$monomials divisible for some monomial in $Ini\}$
5. Return $[G, Ini, M, |M|]$

The Tyurina procedure:

INPUT:
a polynomial F,
(optional)a set of variables $\texttt{variables} = \{x_1, ..., x_r\}$
(by default `variables` is the set of indeterminates appearing in F)
(optional) a l.m.o. U (by default U is `tdeg_ min`)

1. Set $I := [F, \partial F/\partial x_1, ..., \partial F/\partial x_r]$
2. Proceed as in `Milnor` procedure from Step 2, replacing J by I.

Ultimately the following procedures allows to compute Milnor and Tyurina numbers $\mu(F)$ and $\tau(F)$ of a polynomial $F \in \mathbb{C}[\mathbf{x}]$. They give the same output of `Milnor` and `Tyurina` but for an ideal $I \subset \mathbb{C}[\mathbf{x}]$, since they works with a g.m.o..

The PolyMilnor procedure

INPUT:
a polynomial F,
(optional) a set of variables `variables` $= \{x_1,...,x_r\}$
(by default `variables` is the set of indeterminates appearing in F)
(optional) a g.m.o. U (by default U is `tdeg`)

1. Set $J := [\partial F/\partial x_1,...,\partial F/\partial x_r]$;
2. Set
   G := Gröbner basis of J w.r.t. variables in `variables` and monomial order U
   Ini := [leading monomials w.r.t. U of elements in G]
3. Proceed as in `Milnor` procedure from Step 3.

The PolyTyurina procedure

INPUT:
a polynomial F,
(optional) a set of variables `variables` $= \{x_1,...,x_r\}$
(by default `variables` is the set of indeterminates appearing in F)
(optional) a g.m.o. U (by default U is `tdeg`)

1. Set $I := [F,\partial F/\partial x_1,...,\partial F/\partial x_r]$;
2. Set
   G := Gröbner basis of I w.r.t. variables in `variables` and monomial order U
   Ini := [leading monomials w.r.t. U of elements in G]
3. Proceed as in `Milnor` procedure from Step 3, replacing J by I.

The interested reader can find the MAPLE implementations of the above algorithms in [1]

5.2. Some user friendly examples

Once implemented the routines 5.1 needs quite simple and minimal commands to work. As a first example let us start, for comparison, by a problem already studied by using SINGULAR in [12] Example 2.7.2(2).

EXAMPLE 3. Let us study critical points of $F(x,y) := x^5+y^5+x^2y^2$ and singularities of $F^{-1}(0)$. Hence we have to type: Let us find, at first, the critical points of F, by solving the algebraic system of partial derivatives:

```
>  solve({diff(F, x), diff(F, y)}, [x,y]);
```

$$
\begin{aligned}
&[[x=0,y=0],[x=0,y=0],[x=0,y=0],[x=0,y=0],[x=0,y=0],\\
&[x=0,y=0],[x=0,y=0],[x=0,y=0],[x=-2/5,y=-2/5],\\
&[x=2/5-2/5\left(RootOf\left(\_Z^4-\_Z^3+\_Z^2-\_Z+1,label=\_L2\right)\right)^3\\
&\quad+2/5\left(RootOf\left(\_Z^4-\_Z^3+\_Z^2-\_Z+1,label=\_L2\right)\right)^2\\
&\quad-2/5\,RootOf\left(\_Z^4-\_Z^3+\_Z^2-\_Z+1,label=\_L2\right),\\
&\ y=2/5\,RootOf\left(\_Z^4-\_Z^3+\_Z^2-\_Z+1,label=\_L2\right)]]
\end{aligned}
$$

Then F admits 6 critical points: the repetition of the solution in the origin means that this point is a multiple solution. Then we have to expect $\mu(0)>0$.
Singular points of $F^{-1}(0)$ are given by:

```
>  solve({F, diff(F, x), diff(F, y)}, [x,y])
```

We find that $F^{-1}(0)$ has a unique singular point in the origin. Therefore, by the second formula in (12), $\tau(F)=\tau(0)$. In fact we obtain

```
>  Tyurina(F)[4]
```

$$10$$

and

```
>  PolyTyurina(F)[4]
```

$$10$$

- *In general, if $F^{-1}(0)$ admits a unique singular point the procedure* `PolyTyurina` *has to be preferred, since it turns out to be more efficient.*

To compute Milnor numbers let us start by $\mu(F)$, by typing

```
>  PolyMilnor(F)[4]
```

$$16$$

Since $\tau(0)\leq\mu(0)$ and F admits 6 critical points, by (12) we have to expect $10\leq\mu(0)\leq 11$. Furthermore the 5 critical points different from the origin can be exchanged each other under the action of the order 5 cyclic group

$$\left\langle\begin{pmatrix}-\varepsilon^3 & 0\\ 0 & \varepsilon^2\end{pmatrix}^i \mid \varepsilon^5+1=0\ ,\ 1\leq i\leq 5\right\rangle\subset \mathrm{GL}(2,\mathbb{C})$$

which is also a subgroup of $\mathrm{Aut}(F)$. Then they cannot assume Milnor number greater than 1, giving $\mu(0)=11$. In fact:

```
>  Milnor(F)[4]
```

$$11$$

Then in this case, the use of `solve`, `PolyTyurina` and `PolyMilnor`, may avoid to employ `Tyurina` and `Milnor` *which turn out to be in general less efficient procedures.*

EXAMPLE 4 (w.h. polynomials). What observed at the end of the previous Example 3 *is obviously true for a w.h.p.*, after Proposition 3. Let us in fact consider the E_6 3–dimensional singularity $0 \in F^{-1}(0)$ where

$$F(x,y,z,t) = x^2 + y^3 + z^4 + t^2 \ .$$

By Corollary 1, since $w(t) = 1/2$, one has to expect $\tau(0) = \mu(0) = 6$. This fact can be checked by the quicker procedure `PolyMilnor`:

```
> F:=x^2+y^3+z^4+t^2:
> PolyMilnor(F)[4];
```

$$6$$

EXAMPLE 5 (Non–isolated singularities). All the procedures presented in 5.1 stop, giving an error message, if the considered polynomial admits *non–isolated singularities*. Consider, in fact, $F = x^2z^2 + y^2z^2 + x^2y^2$ admitting the union of the three coordinate axes as the locus of critical (and singular, since F is homogeneous) points as can easily checked by typing:

```
> F:=x^2*z^2+y^2*z^2+x^2*y^2:
> solve({diff(F, x), diff(F, y), diff(F,z)}, [x, y, z]);
```

$$[[x = 0, y = y, z = 0], [x = 0, y = y, z = 0], [x = x, y = 0, z = 0], [x = 0, y = 0, z = z]]$$

Then:

```
> PolyMilnorNumber(F);
```

Error, (in PolyMilnor) there are non isolated critical points

The user will obtain the similar error messages by running any of the other procedures.

REMARK 5 (Be careful with variables!). The second input of any procedure in 5.1 is the set `variables` of variables one wants to work with. It is an *optional input* meaning that by default `variables` is assumed to be the set `indets(F)` of variables appearing in the polynomial F. This means that if the user is interested in consider the cylinder $F^{-1}(0)$ where $F : \mathbb{C}^3 \to \mathbb{C}$ is the polynomial map $F(x,y,z) = y^2 - x(x-1)(x-2)$ then he has to type:

```
> F := y^2-x*(x-1)*(x-2):
> PolyMilnor(F, {x, y, z});
```

Error, (in PolyMilnor) there are non isolated critical points
which is right since F do not admit isolated critical points as can be checked by:

```
> solve({diff(F, x), diff(F, y), diff(F,z)}, [x,y,z]);
```

$$[[x = RootOf\left(3\_Z^2 - 6\_Z + 2\right), y = 0, z = z]]$$

If the set of variables is not specified then by default it is assumed to be $\{x, y\}$, meaning that F is considered as a polynomial map from $\mathbb{C}^2$ to $\mathbb{C}$. In this case F admits only the two isolated critical points

```
>  solve(diff(F, x), diff(F, y), [x,y])
```

$$[[x = RootOf\left(3\_Z^2 - 6\_Z + 2, label = \_L1\right), y = 0]]$$

In fact

```
>  PolyMilnor(F);
```

$$[[y, 3x^2 - 6x + 2], [y, x^2], [1, x], 2]$$

Read the output as follows: the first output is the Gröbner basis G of J_F w.r.t the g.m.o. tdeg, the second output is the list of leading monomials of elements in G, the third output is $M \setminus L(J_F)$ whose cardinality is precisely the fourth output. Since the problem is symmetric w.r.t. the y–axis, $\mu(F) = 2$ implies that each critical point has Milnor number 1.

Observe that the origin is not a critical point of F both as a polynomial map from $\mathbb{C}^3$ and from $\mathbb{C}^2$. In fact

```
>  Milnor(F);
```

$$[[-(x-1)(x-2) - x(x-2) - x(x-1), 2y], [1, y], [], 0]$$

```
>  Milnor(F, {x, y, z})
```

$$[[0, -(x-1)(x-2) - x(x-2) - x(x-1), 2y], [1, 1, y], [], 0]$$

In this case the first output is a standard bases of J_F w.r.t. the l.m.o. tdeg_ min whose leading monomials give the second output.

At last let us observe that the zero locus $F^{-1}(0)$ is smooth both as a subset of $\mathbb{C}^2$ and of $\mathbb{C}^3$, in fact

```
>  PolyTyurina(F);
```

$$[[1], [1], [], 0]$$

```
>  PolyTyurina(F, {x, y, z});
```

$$[[1], [1], [], 0]$$

5.3. Optional input: some more subtle utilities

Introducing different choices for the optional input may show interesting possibilities of our subroutine.

Monomial ordering

It is a well known fact that the graduated reverse lexicographic g.m.o. is in general the more efficient monomial ordering for Buchberger algorithm: this is the reason for the default choices in PolyMilnor and PolyTyurina. Anyway, if needed, these procedures may run with many further g.m.o.: e.g. if, for any reason, the user will prefer to run PolyTyurina w.r.t. the pure lexicographic g.m.o. he will have to type, in the following case of a deformation of a threefold E_8 singularity:

```
> F:=x^2+y^3+z^5+t^2+y*z+y^2+z^2+y*z^2+z^3+y*z^3+z^4;
```

$$F := x^2+y^3+z^5+t^2+yz^2+z^3+yz^3+z^4$$

```
> PolyMilnor(F);
```

$$[[x,t,3yz^2-15y^2z+4z^2+2yz+3y^2,3y^2+z^2+z^3,\\ 3375y^4-1677y^2z-639y^3+224z^2+124yz+114y^2,\\ 225y^3z-324y^2z-18y^3+88z^2+38yz+93y^2],\\ [x,t,yz^2,z^3,y^4,y^3z],[1,y,z,y^2,yz,z^2,y^3,y^2z],8]$$

```
> PolyMilnor(F,plex(x,y,z,t));
```

$$[[t,31z^3+88z^4+159z^5+129z^6+75z^7,\\ 128z^3+319z^4+237z^5+225z^6+69z^2+46yz,3y^2+z^2+z^3,x],\\ [t,z^7,yz,y^2,x],[1,z,z^2,z^3,z^4,z^5,z^6,y],8]$$

Observe how different are the two Gröbner bases and consequently the leading monomial bases and associated bases of quotient vector spaces. The user may also verify how much slower is plex w.r.t. the default tdeg by running by himself the routines.
In particular, running Tyurina w.r.t. different l.m.o.'s gives *different monomial basis of the Kuranishi space*:

```
> Tyurina(F)[3];
```

$$[z^2,z,y,1]$$

```
> Tyurina(F,plex_min(x,y,z,t))[3];
```

$$[y^2,y,z,1]$$

REMARK 6. For what concerns efficiency of l.m.o.'s in Milnor and Tyurina it turns out that sometimes plex_min is more efficient than tdeg_min, as observed in the last Section 6 when proving Theorems 10 and 11. But we do not know if this is a general fact, then we keep tdeg_min as the default l.m.o. both in Milnor and Tyurina, for coherence with the default choice of tdeg for their global counterparts.

Variables

The default choice for the optional input variables as the set of those variables appearing in the given polynomial F has been thought to make our routine more user friendly. Anyway this choice may hide some important subtleties, as already pointed out in Remark 5 in the case variables has been chosen as a greater set of variables w.r.t. the set of indeterminates in F. Here we want to underline a significant potentiality of our routine when variables is chosen to be a *strictly smaller subset of* the indeterminates in F.

Let us set $F = x^3 + x^4 + xy^2$. Then we get:

```
> F :=x^3+y^4+x*y^2:
> Milnor(F);
```

$$[\{6x^3 - 4y^4, 3x^2 + y^2, 4y^3 + 2xy\}, [y^2, xy, x^3], [x^2, x, y, 1], 4]$$

```
> Tyurina(F);
```

$$[\{x^3 + y^4 + xy^2, 3x^2 + y^2, 4y^3 + 2xy, 2x^3 - y^4\}, [y^2, xy, xy^2, x^3], [x^2, x, y, 1], 4]$$

Then $T^1 \cong \langle x^2, x, y, 1\rangle_{\mathbb{C}}$ and

$$F_t := F + tx^2 = x^3 + y^4 + xy^2 + tx^2 \tag{21}$$

is a non–trivial 1–parameter small deformation of F such that, for any fixed $t \in \mathbb{C}$, F_t has a critical point in $0 \in \mathbb{C}^2$ which is also a singular point of the plane curve $F_t^{-1}(0)$. We are interested in studying Milnor and Tyurina numbers of this singularity *for any* $t \in \mathbb{C}$. This can be performed by a careful use of the variables input. Let us first of all observe that if no optional input are added then we get

```
> Ft := F+t*x^2;
```

$$Ft := x^3 + y^4 + xy^2 + tx^2$$

```
> Milnor(Ft);
```

Error, (in Milnor) the given critical point is not isolated

In fact, by default `Milnor` considers F_t as a polynomial map defined over $\mathbb{C}^3(x, y, t)$. By forcing `Milnor` to work with variables $\{x, y\}$ only, then F_t is considered as a polynomial map defined over $\mathbb{C}^2$ with coefficient ring $\mathbb{C}[t]$, i.e. $F_t \in (\mathbb{C}[t])[x, y]$, giving:

```
> Milnor(Ft, {x, y})
```

$$[\{3x^2 + y^2 + 2tx, (1 - 4t)y^3 + 3yx^2, 4y^3 + 2xy\}, [x, y^3, xy], [y^2, y, 1], 3]$$

```
> Tyurina(Ft, {x, y})
```

$$[\{3x^2 + y^2 + 2tx, x^3 + y^4 + xy^2 + tx^2, (1 - 4t)y^3 + 3yx^2, 4y^3 + 2xy\},$$
$$[x, y^3, x^2, xy], [y^2, y, 1], 3]$$

This means that, *for generic* * t, $\tau_{F_t}(0) = 3 = \mu_{F_t}(0)$. Moreover *by looking at the leading coefficients of the given standard basis of* J_F *we get all the relations defining non–generic values for* t, precisely:

```
> MB := Milnor(Ft, {x, y})[1]
```

$$MB := \{3x^2 + y^2 + 2tx, (1 - 4t)y^3 + 3yx^2, 4y^3 + 2xy\}$$

```
> for i from 1 to nops(MB) do
> LeadingTerm(MB[i],tdeg_min(x,y)) end do;
```

$$2t, x$$

* t is treated as a variable without any evaluation.

$$1-4t,\, y^3$$
$$2,\, xy$$

For $t=0$ we do not have any deformation of F, giving $\tau_{F_0}(0)=4=\mu_{F_0}$. But the further relation $1-4t=0$ gives:

```
> t := 1/4: Milnor(Ft, {x, y})
```

$$[\left\{3x^2+y^2+1/2x, 36y^3x^2+12y^5, 4y^3+2xy\right\}, [x,y^5,xy], [y^4,y^3,y^2,y,1], 5]$$

```
> Tyurina(Ft, {x, y})
```

$$\left[\left\{x^3+y^4+xy^2+1/4x^2, \frac{3}{64}y^4x^2+\frac{1}{64}y^6-\frac{27}{256}x^6-\frac{9}{256}x^4y^2, 4y^3+2xy,\right.\right.$$
$$\left.\left. 36y^3x^2+12y^5, 3x^2+y^2+1/2x\right\}, [x,y^6,x^2,y^5,xy], [y^4,y^3,y^2,y,1], 5\right]$$

Therefore

$$\tau_{F_t}(0)=\mu_{F_t}(0)=\begin{cases} 5 & \text{for } t=1/4\ , \\ 4 & \text{for } t=0\ , \\ 3 & \text{otherwise}\ . \end{cases}$$

In particular Proposition 4 implies that, *for any t,* $0\in \operatorname{Spec} O_{F_t,0}$ *is a w.h. singularity*, in spite of the fact that F_t is never a w.h. polynomial.

5.4. An efficiency remark: global to local subroutines

After numerous applications of the previous routines the reader will convince himself that the `SB` procedure turns out to be less efficient than the Buchberger algorithm as implemented in MAPLE. As a consequence our routines can be arranged in the following decreasing sequence of efficiency:

$$\texttt{PolyMilnor} > \texttt{PolyTyurina} > \texttt{Milnor} > \texttt{Tyurina}\ .$$

A slight improvement of `Milnor` and `Tyurina` efficiency can be obtained by applying the `SB` algorithm to a Gröbner basis of J_f and I_f rather than to their original generators. What is obtained is a sort of “pasting" of global and local routines, giving rise to `MILNOR` and `TYURINA` procedures described below.

The `MILNOR` procedure: `MILNOR` is a procedure computing Milnor numbers of both a polynomial f and of a critical point of f;

INPUT:
a polynomial F,
(optional) a set of variables `variables` $=\{x_1,...,x_r\}$
(by default `variables` is the set of indeterminates appearing in F)
(optional) a l.m.o. U (by default U is `tdeg_ min`)
(optional) a g.m.o. V (by default V is `tdeg`)

1. Set $J := [\partial F/\partial x_1, ..., \partial F/\partial x_r]$;
2. Set

$$\begin{array}{lll} J & := & \text{Gröbner basis of } J \text{ w.r.t. V} \\ IniJ & := & [\text{leading monomials w.r.t. } V \text{ of elements in } J] \end{array}$$

3. Check that the ideal $IniJ \subseteq \mathbb{C}[x_1, ...x_r]$ is zero-dimensional; else **error**;
4. Set

$$\begin{array}{lll} m & := & r \cdot \max\{\text{degree of monomials in } IniJ\} \\ L & := & \text{list of all monomials of degree } \leq m \\ N & := & L \setminus \{\text{monomials divisible for some monomial in } IniJ\} \\ G & := & SB(J, \texttt{variables}, U) \\ Ini & := & [\text{leading monomials w.r.t. } U \text{ of elements in } G] \\ m & := & r \cdot \max\{\text{degree of monomials in } Ini\} \\ L & := & \text{list of all monomials of degree } \leq m \\ M & := & L \setminus \{\text{monomials divisible for some monomial in } Ini\} \end{array}$$

5. Return $[J, IniJ, N, |N|, G, Ini, M, |M|]$

The `TYURINA` **procedure:** `TYURINA` is a the procedure computing the Tyurina numbers of both a polynomial f and of a critical point of f:

INPUT:
a polynomial F,
(optional) a set of variables $\texttt{variables} = \{x_1, ..., x_r\}$
(by default `variables` is the set of indeterminates appearing in F)
(optional) a l.m.o. U (by default U is `tdeg_ min`)
(optional) a g.m.o. V (by default V is `tdeg`)

1. $J = [F, \partial F/\partial x_1, ..., \partial F/\partial x_r]$
2. Proceed as in `MILNOR` procedure from Step 2.

The interested reader can find the details of these procedure in [24, §5.4]

An example of application.

Let us consider the same polynomial $F(x,y) = x^3 + y^4 + xy^2$ given in 5.3. Procedures `MILNOR` and `TYURINA` give *all* the information we could get by applying all the introduced routines, precisely:

> `MILNOR(F)`

$$[[3x^2 + y^2, 2y^3 + xy], [x^2, y^3], [1, y, x, y^2, xy, xy^2], 6,$$
$$\left\{-2y^4 + 3x^3, 2y^3 + xy, 3x^2 + y^2\right\}, [y^2, xy, x^3], [x^2, x, y, 1], 4]$$

> `TYURINA(F)`

$$[[xy, 3x^2+y^2, y^3], [xy, x^2, y^3], [1, y, x, y^2], 4,$$
$$\left\{xy, 3x^3, 3x^2+y^2, y^3\right\}, [y^2, x^3y^3, xy], [x^2, x, y, 1], 4]$$

In particular it turns out that F admits some further critical point which is not a singular point of $F^{-1}(0)$. By

```
> solve({diff(F,x),diff(F,y)},[x,y]);
```

$$[[x=0, y=0], [x=0, y=0], [x=-1/2\left(RootOf\left(3\_Z^2+1, label=\_L1\right)\right)^2,$$
$$y=1/2\,RootOf\left(3\_Z^2+1, label=\_L1\right)], [x=0, y=0]]$$

it follows that there are precisely two further critical points of F having Milnor number 1 and Tyurina number 0. Let us now type:

```
> TyB := Tyurina(F)[3]
```

$$TyB := [x^2, x, y, 1]$$

```
> T := nops(TyB):
> F[Lambda] := F+sum(lambda[i]*TyB[T-i],i= 0 .. T-1);
```

$$F_\Lambda := x^3+y^4+xy^2+\lambda_0+\lambda_1 y+\lambda_2 x+\lambda_3 x^2$$

```
> TYURINA(F[Lambda], {x, y});
```

$$[[1], [1], [], 0, [1], [1], [], 0]$$

```
> Tyurina(F[Lambda], {x, y});
Warning, computation interrupted
```

We had to interrupt the calculation of `Tyurina` since it wasn't able to produce any output after considerable time, on the contrary of `TYURINA` which quickly produced the given (trivial) output.

REMARK 7. What observed in 5.2 and 5.3, about optional inputs for `Milnor` and `Tyurina`, applies analogously for `MILNOR` and `TYURINA`.

6. Application: adjacencies of Arnol'd simple singularities

Let us consider the classes of Arnol'd simple singularities A_n, D_n, E_6, E_7, E_8. Recall that a class of singularities B is said to be *adjacent* to a class of singularities A (notation $A \leftarrow B$) if any singularity in B can be deformed to a singularity in A by an arbitrarily small deformation (see [4] §I.2.7, [5] §15.0 and [16] §7.C). Adjacency turns out to be a *partial order relation* on the set of singularities' equivalence classes.

In the following we will employ the optional input on variables, as observed in 5.3, to show explicit equations of algebraic *stratifications* of Kuranishi spaces verifying the

following Arnol'd adjacency diagram
(22)

$$
\begin{tikzcd}
A_1 & A_2 \arrow[l] & A_3 \arrow[l] & A_4 \arrow[l] & A_5 \arrow[l] & A_6 \arrow[l] & A_7 \arrow[l] & \cdots A_{n-1} \cdots \arrow[l] \\
 & & D_4 \arrow[u] & D_5 \arrow[l] \arrow[u] & D_6 \arrow[l] \arrow[u] & D_7 \arrow[l] \arrow[u] & D_8 \arrow[l] \arrow[u] & \cdots D_n \cdots \arrow[l] \arrow[u] \\
 & & & E_6 \arrow[u] \arrow[uur] & E_7 \arrow[l] \arrow[u] \arrow[uur] & E_8 \arrow[l] \arrow[u] \arrow[uur] & &
\end{tikzcd}
$$

(see [3], [5] and in particular [4] §I.2.7). Such a stratification gives a geometric interpretation, by means of inclusions of algebraic subsets, of the partial order relation induced by adjacency.

Let us first of all observe that the Kuranishi space T^1 of an Arnol'd simple m–fold singularity of type A_n, D_n, E_6, E_7, E_8 do not depend on its dimension m, since partials of quadratic terms are linear generators of I_f eliminating the associated variable from $M \setminus L(I_f)$. Therefore the study of diagram (22) can be reduced to the case of Arnol'd simple *curve* singularities whose local equations are given in (17): this is actually guaranteed by the following Morse Splitting Lemma 1.

In the sequel we will need the following notation and results, essentially due to V.I. Arnol'd [5]. We refer the interested reader to books [4], [5], [16] and [12] for details and proofs.

DEFINITION 7 (Co-rank of a critical point). *The* co-rank of a critical point *p of a holomorphic function f defined over an open subset of $\mathbb{C}^n$ is the number*

$$\mathrm{crk}_f(p) := n - [,(]\mathrm{Hess}_f(p))$$

where $\mathrm{Hess}_f(p)$ *is the Hessian matrix of f in p (for shortness the function f will be omitted when clear from the context). In particular if* $\mathrm{crk}(p) = 0$ *then p is called a* non–degenerate*, or* Morse*,* critical point*.*

LEMMA 1 (Morse Splitting Lemma, [5] §11.1, [16] (7.16) and [12] Theorem I.2.47). *Assume that $f \in \mathfrak{m}^2 \subset \mathbb{C}\{\mathbf{x}\}$ and* $\mathrm{crk}_f(0) = n-k$. *Then f is equivalent* † *to the following germ of singularity*

$$\sum_{i=1}^{k} x_i^2 + g(x_{k+1}, \ldots, x_n)$$

where $g \in \mathfrak{m}^3$ is uniquely determined (up to equivalence).

THEOREM 6 ([12], Theorems I.2.46, I.2.48, I.2.51, I.2.53).

1. *For $f \in \mathfrak{m}^2 \subset \mathbb{C}\{\mathbf{x}\}$ the following facts are equivalent:*
    - $\mathrm{crk}(0) = 0$ *i.e.* 0 *is a non–degenerate critical point of f,*

† In the sense of Definition 5 and Remark 2.

- $\mu(0) = 1 = \tau(1)$,
- $0 \in f^{-1}(0)$ *is, up to equivalence, a node i.e. a simple* A_1 *singularity.*

2. *For* $f \in \mathfrak{m}^2 \subset \mathbb{C}\{\mathbf{x}\}$ *the following facts are equivalent:*

   - $\mathrm{crk}(0) \leq 1$ *and* $\mu(0) = m$,
   - $0 \in f^{-1}(0)$ *is equivalent to a simple* A_m *singularity.*

3. *For* $f \in \mathfrak{m}^3 \subset \mathbb{C}\{y,z\}$ *the following facts are equivalent:*

   - *the 3–jet* $f^{(3)}$ *of* f *factors into at least two distinct linear factors and* $\mu(0) = m \geq 4$,
   - $0 \in f^{-1}(0)$ *is equivalent to a simple* D_m *singularity.*

4. *For* $f \in \mathfrak{m}^3 \subset \mathbb{C}\{y,z\}$ *the following facts are equivalent:*

   - *the 3–jet* $f^{(3)}$ *of* f *has a unique liner factor (of multiplicity 3) and* $\mu(f) \leq 8$,
   - $0 \in f^{-1}(0)$ *is equivalent to a simple singularity of type* E_6, E_7 *or* E_8 *and* $\mu(0) = 6$, 7 *or* 8 *respectively.*

Let us now introduce a non-standard notation, useful to describe a nice geometric property of stratifications via algebraic subsets of a simple Arnol'd singularity's Kuranishi space, as explained in the following statements. Consider the following square of subset inclusions

(23)
$$\begin{tikzcd} B \arrow[r, hook] & D \\ A \arrow[u, hook] \arrow[r, hook] & C \arrow[u, hook] \end{tikzcd}$$

Then $A \subseteq B \cap C$, necessarily.

DEFINITION 8 (Complete Intersection Property - (c.i.p.)). *A square of subset inclusions (23) is said to admit the* complete intersection property *if*

$$A = B \cap C \, .$$

For shortness we will say that (23) is a c.i.p. square. *The geometric meaning of c.i.p. in (23) is explained by Figure 1, while Figure 2 describes geometrically the following* sequence of two c.i.p. squares

(24)
$$\begin{tikzcd} B \arrow[r, hook] & D \arrow[r, hook] & F \\ A \arrow[u, hook] \arrow[r, hook] & C \arrow[u, hook] \arrow[r, hook] & E \arrow[u, hook] \end{tikzcd}$$

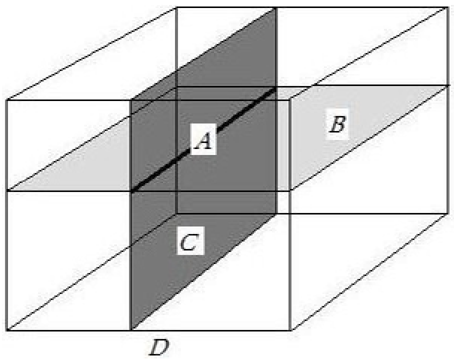

Figure 1: The c.i.p. for the inclusions' square (23)

meaning that $A = B \cap C = B \cap D \cap E$*. Moreover the following inclusions' diagram*

(25)
$$\begin{tikzcd}
B \cup F \arrow[dr, hook] & E \arrow[l, hook'] \arrow[dr, hook] & \\
A \arrow[u, hook] \arrow[dr, hook] & D & G \arrow[l, hook'] \\
& C \arrow[u, hook] &
\end{tikzcd}$$

is called a union of c.i.p. squares *if* $A = B \cap C$ *and* $E = F \cap G$*. A particular case, occurring in the following, is when* $C = G$ *: then diagram (25) becomes the following one*

(26)
$$\begin{tikzcd}
& B \cup F \arrow[r, hook] & D \\
& E \arrow[u, hook] \arrow[r, hook] & C = G \arrow[u, hook] \\
A \arrow[ur, hook, bend left] \arrow[urr, hook, bend right] & &
\end{tikzcd}$$

and we will say this diagram to represent a hinged union of c.i.p. squares*, whose hinge is the inclusion* $C \hookrightarrow D$*.*

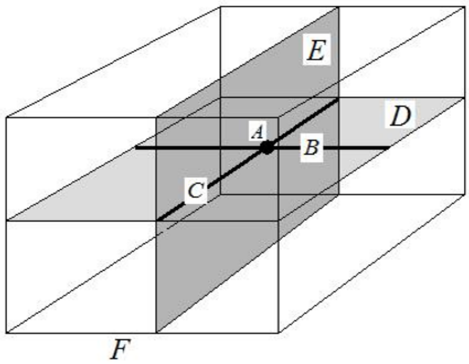

Figure 2: The sequence (24) of two c.i.p. squares

At last the following inclusions' diagram

(27)
$$\begin{tikzcd} & B \arrow[r, hook] & D \\ & E \arrow[u, hook] \arrow[r, hook] & C \arrow[u, hook] \\ A \arrow[uur, hook, bend left] \arrow[urr, hook, bend right] & & \end{tikzcd}$$

is called a reducible c.i.p. square *if* $A \cup E = B \cap C$ *i.e. if*

$$\begin{tikzcd} B \arrow[r, hook] & D \\ A \cup E \arrow[u, hook] \arrow[r, hook] & C \arrow[u, hook] \end{tikzcd}$$

is a c.i.p. square.

6.1. Outline of the following results

Statements and proofs of the following Theorems 7 , 8 , 9 , 10 and 11 have the same structure we are going to outline here. Precisely their statements describe set theoretical stratifications by algebraic subsets of the Kuranishi space T^1 of simple hypersurface singularities A_n, D_n and E_n (the latter with $6 \leq n \leq 8$). Their proofs go on by the following steps:

1. look for the critical points of a generic small deformation of our initial simple singularity, by solving the polynomial system assigned by partial derivatives (Jacobian ideal generators): they turn out to be precisely n points (with $6 \leq n \leq 8$ in the E_n case);

2. imposing one of the previous critical points to be actually a singular point means defining a hypersurface $\mathcal{L} \subset T^1$: we have then n of such hypersurfaces, one for each critical point;

3. any of those hypersurfaces is then stratified by nested algebraic subsets defined by a progressive vanishing of leading coefficients in Jacobian ideals' standard bases of more and more specialized deformations. More precisely, the general strategy is that of looking at the leading monomials ordered by the choice of a suitable l.m.o.: then imposing the vanishing of only the leading coefficient associated with the smallest leading monomial realizes "horizontal adjacencies", while imposing the vanishing of all the leading coefficients gives "vertical adjacencies", in diagrams (29), (35), (42), (51) and (62).

The last step (3) is obtained by a systematic use of routines `Milnor`, `Tyurina`, `MILNOR` and `TYURINA` previously described, running with suitable l.m.o.'s defined on a strict subset of variables appearing in the polynomial equation of a generic small deformation, as explained in 5.3. Such a procedure allows to explicitly write down relations on deformation parameters and then equations of the algebraic stratifications.

6.2. Simple singularities of A_n type.

THEOREM 7. *Let T^1 be the Kuranishi space of a simple N–dimensional singular point $0 \in f^{-1}(0)$ with*

$$f(x_1,\dots,x_{N+1}) = \sum_{i=1}^{N} x_i^2 + x_{N+1}^{n+1} \quad (\textit{for } n \geq 1)\ .$$

The subset of T^1 parameterizing small deformations of $0 \in f^{-1}(0)$ to a simple node (i.e. an A_1 singularity) is the union of n hypersurfaces. Moreover, calling $\mathcal{L}$ any of those hypersurfaces, there exists a stratification of nested algebraic subsets

$$\mathcal{L} \hookleftarrow \mathcal{V}_2 \hookleftarrow \cdots \hookleftarrow \mathcal{V}_2^m \hookleftarrow \cdots \hookleftarrow \mathcal{V}_2^n \tag{28}$$

verifying the Arnol'd's adjacency diagram

$$A_1 \longleftarrow A_2 \longleftarrow \cdots \longleftarrow A_m \longleftarrow \cdots \longleftarrow A_n \tag{29}$$

where

- *$\mathcal{L}$ is the hypersurface of T^1 defined by equation (31), keeping in mind (30),*
- *$\mathcal{V}_2^m := \bigcap_{k=2}^m \mathcal{V}_k$ where $\mathcal{V}_k$ are hypersurfaces of $\mathcal{L}$ defined by the vanishing of variables v_k introduced by (33).*

Proof. Let us follow the outline previously exposed in 6.1.

(1) By the Morse Splitting Lemma 1 we can reduce to the case $N = 1$ with $f(y,z) = y^2 + z^{n+1}$ for $n \geq 1$. Then Proposition 1 gives

$$T^1 \cong \mathbb{C}[y,z]/(y,z^n) \cong \langle 1, z, \dots, z^{n-1} \rangle_{\mathbb{C}}$$

and, given $\Lambda=(\lambda_0,\dots,\lambda_{n-1})\in T^1$, the associated deformation of $U_0=\mathrm{Spec}(O_{f,0})$ is

$$U_\Lambda=\{f_\Lambda(y,z):=f(y,z)+\sum_{i=0}^{n-1}\lambda_i z^i=0\}\ .$$

A solution of the jacobian system of partial derivatives is then given by $p_\Lambda=(0,z_\Lambda)$ where z_Λ is a zero of the following polynomial

$$(n+1)z^n+\sum_{i=1}^{n-1}i\lambda_i z^{i-1}\in\mathbb{C}[\lambda][z]\ . \tag{30}$$

This means that f_Λ admits precisely n critical points.

(2) Imposing $p_\Lambda\in U_\Lambda$, which is asking for one of the previous critical points to be actually a singular point of U_Λ, defines the following hypersurface in T^1

$$p_\Lambda\in U_\Lambda\iff\Lambda\in\mathcal{L}:=\{z_\Lambda^{n+1}+\sum_{i=0}^{n-1}\lambda_i z_\Lambda^i=0\}\subset T^1\ . \tag{31}$$

Notice that we get n such hypersurfaces, not necessarily distinct, one for each critical point of f_Λ.

- *$\mathcal{L}$ is a hypersurface of the Kuranishi space parameterizing small deformations of U_0 admitting a singular point of type at least A_1 in the origin.*

After translating $z\mapsto z+z_\Lambda$, we get

$$\begin{aligned}
f_\Lambda(y,z+z_\Lambda)=f(y,z)\quad &+\quad \left(z_\Lambda^{n+1}+\sum_{i=0}^{n-1}\lambda_i z_\Lambda^i\right)\\
&+\quad \left((n+1)z_\Lambda^n+\sum_{i=1}^{n-1}i\lambda_i z_\Lambda^{i-1}\right)z\\
&+\quad \sum_{k=2}^{n-1}\left(\binom{n+1}{k}z_\Lambda^{n+1-k}+\sum_{i=k}^{n-1}\binom{i}{k}\lambda_i z_\Lambda^{i-k}\right)z^k\\
&+\quad (n+1)z_\Lambda\, z^n\\
\overset{(30),(31)}{=}f(y,z)\quad &+\quad \sum_{k=2}^{n}v_k z^k
\end{aligned} \tag{32}$$

where

$$\begin{aligned}
v_k &:= \binom{n+1}{k}z_\Lambda^{n+1-k}+\sum_{i=k}^{n-1}\binom{i}{k}\lambda_i z_\Lambda^{i-k}\quad,\quad k=2,\dots,n-1\\
v_n &:= z_\Lambda\ .
\end{aligned} \tag{33}$$

(3) Define codimension 1 subvarieties $\mathcal{V}_k:=\{v_k=0\}$ of $\mathcal{L}$. Then

- *$p_\Lambda \in U_\Lambda$ turns out to be a A_m $(2 \leq m \leq n)$ simple hypersurface singularity if and only if Λ is the generic element of the codimension m subvariety $\mathcal{V}_2^m := \bigcap_{k=2}^m \mathcal{V}_k \subset T^1$.*

In fact one can check that the origin (i.e. $p_\Lambda \in U_\Lambda$) admits Milnor number $\mu(0) = m$ if and only if Λ is the generic element of $\mathcal{V}_2^m$ (see [24] §6.2, for details when $n \leq 7$). Moreover $\mathrm{crk}(0) \leq 1$, since the rank of the Hessian matrix is always at least 1 for the contribution of y^2. Then Theorem 6(2) applies. This gives precisely the nested stratification (28) verifying the top row (29) in diagram (22) of Arnol'd's adjacencies.
Let us observe that, if Λ is the generic point of $\mathcal{L}$ then the standard basis of J_{f_Λ} is given by $\{2y, (n+1)z^n + 2v_2 z + 3v_3 z^2 + \cdots + nv_n z^{n-1}\}$ whose leading coefficient w.r.t. the default l.m.o. is $2v_2$. Moreover, if Λ is the generic point of $\mathcal{V}_2^m$ then $(m+1)v_{m+1}$ turns out to be the leading coefficient of the standard basis of J_{f_Λ} which is now given by $\{2y, (n+1)z^n + (m+1)v_{m+1}z^m + (m+2)v_{m+2}z^{m+1} + \cdots + nv_n z^{n-1}\}$. Then the nested stratification (28) is obtained by the progressive vanishing of such leading coefficients. □

6.3. Simple singularities of D_n type.

THEOREM 8. *Let T^1 be the Kuranishi space of a simple N–dimensional singular point $0 \in f^{-1}(0)$ with*

$$f(x_1, \ldots, x_{N+1}) = \sum_{i=1}^{N-1} x_i^2 \; + \; x_N^2 \, x_{N+1} + x_{N+1}^{n-1} \quad \textit{(for } n \geq 4\textit{)}$$

The subset of T^1 parameterizing small deformations of $0 \in f^{-1}(0)$ to a simple node is the union of n hypersurfaces. Moreover, calling $\mathcal{L}$ any of those hypersurfaces, there exists a stratification of nested algebraic subsets giving rise to the following sequence of inclusions and c.i.p. squares

(34)
$$\begin{tikzcd}
\mathcal{L} & \mathcal{W}_2 \arrow[l, hook'] & \mathcal{V}_0^1 \cup \mathcal{W}_2^3 \arrow[l, hook'] & \cdots \mathcal{W}_2^m \cdots \arrow[l, hook'] & \mathcal{W}_2^{n-1} \arrow[l, hook'] \\
 & & \mathcal{V}_0^2 \arrow[u, hook] & \cdots \mathcal{V}_0^{m-1} \cdots \arrow[l, hook'] \arrow[u, hook] & \mathcal{V}_0^{n-2} = \{0\} \arrow[l, hook'] \arrow[u, hook]
\end{tikzcd}$$

verifying the Arnol'd's adjacency diagram

(35)
$$\begin{tikzcd}
A_1 & A_2 \arrow[l] & A_3 \arrow[l] & \cdots A_m \cdots \arrow[l] & A_{n-1} \arrow[l] \\
 & & D_4 \arrow[u] & \cdots D_m \cdots \arrow[l] \arrow[u] & D_n \arrow[l] \arrow[u]
\end{tikzcd}$$

where

- *$\mathcal{L}$ is the hypersurface of T^1 defined by equation (37), keeping in mind (36),*

- $\mathcal{V}_0^m := \bigcap_{k=0}^m \mathcal{V}_k$ *and* $\mathcal{W}_2^m := \bigcap_{k=2}^m \mathcal{W}_k$ *where* $\mathcal{V}_k, \mathcal{W}_k$ *are hypersurfaces of* $\mathcal{L}$ *defined by equations (40), keeping in mind definitions (39).*

Proof. Following the outline 6.1.

(1) By the Morse Splitting Lemma 1, our problem can be reduced to the case $N = 1$ with $f(y,z) = y^2z + z^{n-1}$ for $n \geq 4$. Then the Kuranishi space T^1 is given by

$$T^1 = \langle 1, y, z, \ldots, z^{n-2} \rangle_{\mathbb{C}} .$$

Given $\Lambda = (\lambda_0, \lambda, \lambda_1, \ldots, \lambda_{n-2}) \in T^1$, the associated small deformation of U_0 is

$$U_\Lambda = \{ f_\Lambda(y,z) := f(y,z) + \lambda y + \sum_{i=0}^{n-2} \lambda_i z^i = 0 \} .$$

A solution of the jacobian system of partials is then given by a solution $p_\Lambda = (y_\Lambda, z_\Lambda)$ of the following polynomial system in $\mathbb{C}[\lambda][y,z]$

$$\left\{ \begin{array}{c} 2yz + \lambda = 0 \\ (n-1)z^{n-2} + y^2 + \sum_{i=1}^{n-2} i\lambda_i z^{i-1} = 0 \end{array} \right. \tag{36}$$

giving precisely n critical points for f_Λ.

(2) Imposing that one of those critical points, say p_Λ, is actually a singular point of U_Λ means to require that

$$p_\Lambda \in U_\Lambda \iff \Lambda \in \mathcal{L} := \{ y_\Lambda^2 z_\Lambda + z_\Lambda^{n-1} + \lambda y_\Lambda + \sum_{i=0}^{n-2} \lambda_i z_\Lambda^i = 0 \} \subset T^1 \tag{37}$$

where, as above, $\mathcal{L}$ is one of the n hypersurfaces of T^1 parameterizing small deformations of $0 \in U_0$ to nodes. After translating $y \mapsto y + y_\Lambda, z \mapsto z + z_\Lambda$, we get

$$\begin{aligned} f_\Lambda(y + y_\Lambda, z + z_\Lambda) = f \quad & + \quad \left(y_\Lambda^2 z_\Lambda + z_\Lambda^{n-1} + \lambda y_\Lambda + \sum_{i=0}^{n-2} \lambda_i z_\Lambda^i \right) \\ & + \quad (2y_\Lambda z_\Lambda + \lambda)\, y \\ & + \quad \left((n-1) z_\Lambda^{n-2} + y_\Lambda^2 + \sum_{i=1}^{n-2} i\lambda_i z_\Lambda^{i-1} \right) z \\ & + \quad 2y_\Lambda\, yz \ + \ z_\Lambda\, y^2 \\ & + \quad \sum_{k=2}^{n-2} \left(\binom{n-1}{k} z_\Lambda^{n-1-k} + \sum_{i=k}^{n-2} \binom{i}{k} \lambda_i z_\Lambda^{i-k} \right) z^k \\ \stackrel{(36),(37)}{=} f(y,z) \quad & + \quad v_0 yz + v_1 y^2 + \sum_{k=2}^{n} v_k z^k \end{aligned} \tag{38}$$

where

$$
(39)\qquad \begin{aligned}
v_0 &:= 2y_\Lambda \\
v_1 &:= z_\Lambda \\
v_k &:= \binom{n-1}{k} z_\Lambda^{n-1-k} + \sum_{i=k}^{n-2} \binom{i}{k} \lambda_i z_\Lambda^{i-k}\ ,\ k = 2,\dots,n-2
\end{aligned}
$$

(3) Define the following codimension 1 subvarieties of $\mathcal{L}$

$$
(40)\qquad \begin{aligned}
\mathcal{V}_k &:= \{v_k = 0\}\ ,\quad 0 \le k \le n-2 \\
\mathcal{W}_k &:= \begin{cases} \{4v_1v_2 - v_0^2 = 0\} & \text{for } k = 2 \text{ (vanishing of } \det(\mathrm{Hess})) \\ \{v_1 v_k + v_{k-1} = 0\} & \text{for } 3 \le k \le n-2 \\ \{v_1 + v_{n-2} = 0\} & \text{for } k = n-1 \end{cases}
\end{aligned}
$$

Notice that:

- $4v_1v_2 - v_0^2$ is the leading coefficient associated with the leading monomial y of one of the three generators in the standard Gröbner basis of the jacobian ideal J_{f_Λ} w.r.t. the l.m.o. `tdeg_min`(y,z), for a generic choice of $\Lambda \in \mathcal{L}$; since $4v_1v_2 - v_0^2 = \det(\mathrm{Hess}_{f_\Lambda}(0))$ then Theorem 6(2) gives that $0 \in f_\Lambda^{-1}(0)$ *is a simple* A_2 *singularity for generic* $\Lambda \in \mathcal{W}_2$;
- v_0 and v_1 are the leading coefficients with respect to the leading monomial z in the remaining two generators in the standard basis of J_{f_Λ}, for a generic choice of $\Lambda \in \mathcal{L}$; then $0 \in f_\Lambda^{-1}(0)$ *is a simple* A_3 *singularity for generic* $\Lambda \in \mathcal{V}_0^1 \subset \mathcal{W}_2$;
- $\forall k : 3 \le k \le n-2\ ,\ v_1(v_1v_k + v_{k-1})$ is proportional to the leading coefficient associated with the leading monomial y^{k-1} of one of the three generators in the standard Gröbner basis of J_{f_Λ} for a generic choice of $\Lambda \in \mathcal{W}_2^{k-1}$; then $0 \in f_\Lambda^{-1}(0)$ *is a simple* A_k *singularity for generic* $\Lambda \in \mathcal{W}_2^k$;
- $v_1(v_1 + v_{n-2})$ is proportional to the leading coefficient associated with the leading monomial y^{n-2} of one of the three generators in the standard Gröbner basis of J_{f_Λ} for a generic choice of $\Lambda \in \mathcal{W}_2^{n-2}$; then $0 \in f_\Lambda^{-1}(0)$ *is a simple* A_{n-1} *singularity for generic* $\Lambda \in \mathcal{W}_2^{n-1}$;
- $\forall k : 2 \le k \le n-2\ ,\ v_1^{k-2}v_k$ is proportional to the leading coefficient associated with the leading monomial z in one of the remaining generators in the standard basis of J_{f_Λ}, for a generic choice of $\Lambda \in \mathcal{V}_0^{k-1}$; then Theorem 6(3) gives that $0 \in f_\Lambda^{-1}(0)$ *is a simple* D_{k+2} *singularity for generic* $\Lambda \in \mathcal{V}_0^k$.

For details the reader is referred to [24] §6.3 where Gröbner basis, leading coefficients and monomials and Milnor numbers are explicitly computed via Maple procedures until the case $n = 6$: this in enough to establish the recursion.

Putting all together we get the relation between diagrams (34) and (35). To end up the proof observe that $\mathcal{V}_0^k = \mathcal{V}_0^{k-1} \cap \mathcal{W}_2^{k+1}$, for any $3 \le k \le n-2$, showing the c.i.p. property for any square in diagram (34). $\square$

6.4. Simple singularities of E_6 type.

THEOREM 9. *Let T^1 be the Kuranishi space of a simple N–dimensional singular point $0 \in f^{-1}(0)$ with*

$$f(x_1,\ldots,x_{N+1}) = \sum_{i=1}^{N-1} x_i^2 \ + \ x_N^3 + x_{N+1}^4$$

The subset of T^1 parameterizing small deformations of $0 \in f^{-1}(0)$ to a simple node is the union of 6 hypersurfaces. Moreover, calling $\mathcal{L}$ any of those hypersurfaces, there exists a stratification of nested algebraic subsets giving rise to the following sequence of inclusions and c.i.p. squares

(41)
$$\begin{tikzcd}
\mathcal{L} & \mathcal{W}_2 \arrow[l, hook'] & \mathcal{W}_2^3 \arrow[l, hook'] & \widetilde{\mathcal{W}_2^4} \arrow[l, hook'] & \mathcal{W} \cap \left(\bigcap_{k=0}^2 \mathcal{V}_{2k}\right) \arrow[l, hook'] \\
 & & \mathcal{V}_0^2 \arrow[u, hook] & \mathcal{V} \cap \mathcal{V}_0^2 \arrow[l, hook'] \arrow[u, hook] & \\
 & & & \{0\} \arrow[u, hook] \arrow[ruu, hook] &
\end{tikzcd}$$

verifying the Arnol'd's adjacency diagram

(42)
$$\begin{tikzcd}
A_1 & A_2 \arrow[l] & A_3 \arrow[l] & A_4 \arrow[l] & A_5 \arrow[l] \\
 & & D_4 \arrow[u] & D_5 \arrow[l] \arrow[u] & \\
 & & & E_6 \arrow[u] \arrow[ruu] &
\end{tikzcd}$$

where

- *$\mathcal{L}$ is the hypersurface of T^1 defined by equation (44), keeping in mind (43),*
- *$\mathcal{V}_0^m := \bigcap_{k=0}^m \mathcal{V}_k$ and $\mathcal{W}_2^m := \bigcap_{k=2}^m \mathcal{W}_k$ where $\mathcal{V}_k, \mathcal{W}_k$ are hypersurfaces of $\mathcal{L}$ defined by equations (46),*
- *$\widetilde{\mathcal{W}_2^4}$ is a codimension 3 Zariski closed subset of $\mathcal{L}$ defined in (47),*
- *$\mathcal{V}$ is a hypersurfaces of $\mathcal{L}$ defined by equation (48),*
- *$\mathcal{W}$ is a hypersurfaces of $\mathcal{L}$ defined by equation (49).*

Proof. Following the outline 6.1.

(1) By the Morse Splitting Lemma 1, our problem can be reduced to the case $N = 1$ with $f(y,z) = y^3 + z^4$. Therefore

$$T^1 \cong \langle 1, y, z, yz, z^2, yz^2 \rangle_{\mathbb{C}} \ .$$

Given $\Lambda = (\lambda_0, \lambda_1, \dots, \lambda_5) \in T^1$, the associated deformation of U_0 is

$$U_\Lambda = \{ f_\Lambda(y,z) := f(y,z) + \lambda_0 + \lambda_1 y + \lambda_2 z + \lambda_3 yz + \lambda_4 z^2 + \lambda_5 yz^2 = 0 \}$$

A solution of the jacobian system of partials is then given by a solution $p_\Lambda = (y_\Lambda, z_\Lambda)$ of the following polynomial system in $\mathbb{C}[\lambda][y,z]$

$$\left\{ \begin{array}{c} 3y^2 + \lambda_1 + \lambda_3 z + \lambda_5 z^2 = 0 \\ 4z^3 + 2\lambda_4 z + \lambda_2 + y(\lambda_3 + 2\lambda_5 z) = 0 \end{array} \right. \tag{43}$$

giving precisely 6 critical points for f_Λ.

(2) Imposing that one of those critical points, say p_Λ, is actually a singular point of U_Λ means to require that

$$\begin{gathered} p_\Lambda \in U_\Lambda \iff \Lambda \in \mathcal{L} \quad \text{where} \\ \mathcal{L} := \{ y_\Lambda^3 + z_\Lambda^4 + \lambda_0 + \lambda_1 y_\Lambda + \lambda_2 z_\Lambda + \lambda_3 y_\Lambda z_\Lambda + \lambda_4 z_\Lambda^2 + \lambda_5 y_\Lambda z_\Lambda^2 = 0 \} \quad \subset T^1 \end{gathered} \tag{44}$$

which is *one of the 6 hypersurfaces of* T^1 *parameterizing small deformations of* $0 \in U_0$ *to nodes.* After translating $y \mapsto y + y_\Lambda, z \mapsto z + z_\Lambda$, we get

$$\begin{array}{rcl} f_\Lambda(y + y_\Lambda, z + z_\Lambda) & = & \\ f(y,z) & + & \left(y_\Lambda^3 + z_\Lambda^4 + \lambda_0 + \lambda_1 y_\Lambda + \lambda_2 z_\Lambda + \lambda_3 y_\Lambda z_\Lambda + \lambda_4 z_\Lambda^2 + \lambda_5 y_\Lambda z_\Lambda^2 \right) \\ & + & \left(3y_\Lambda^2 + \lambda_1 + \lambda_3 z_\Lambda + \lambda_5 z_\Lambda^2 \right) y \\ & + & \left(4z_\Lambda^3 + 2\lambda_4 z_\Lambda + \lambda_2 + y_\Lambda(\lambda_3 + 2\lambda_5 z_\Lambda) \right) z \\ & + & (\lambda_3 + 2\lambda_5 z_\Lambda) yz + 3y_\Lambda\, y^2 + \left(6z_\Lambda^2 + \lambda_4 + \lambda_5 y_\Lambda \right) z^2 \\ & + & 4z_\Lambda\, z^3 + \lambda_5\, yz^2 \\ \overset{(43),(44)}{=} f(y,z) & + & v_0 yz + v_1 y^2 + v_2 z^2 + v_3 z^3 + v_4 z^4 \end{array}$$

where

$$v_0 = \lambda_3 + 2\lambda_5 z_\Lambda \ , \ v_1 = 3y_\Lambda \ , \ v_2 = 6z_\Lambda^2 + \lambda_4 + \lambda_5 y_\Lambda \ , \ v_3 = \lambda_5 \ , \ v_4 = 4z_\Lambda \tag{45}$$

(3) Define the following codimension 1 subvarieties of $\mathcal{L}$

$$\begin{array}{rcl} \mathcal{V}_k & := & \{ v_k = 0 \} \ , \quad 0 \leq k \leq 4 \\ \mathcal{W}_2 & := & \left\{ 4v_1 v_2 - v_0^2 = 0 \right\} \ , \quad \text{(vanishing of det(Hess))} \\ \mathcal{W}_3 & := & \left\{ v_1^3 v_4^2 - v_2 (v_1 v_3 + v_2)^2 = 0 \right\} \\ \mathcal{W}_4 & := & \left\{ 4v_1^3 - (v_1 v_3 + 3v_2)^2 = 0 \right\} \\ \mathcal{W}_5 & = & \{ v_1 v_3 + 3v_2 = 0 \} \ . \end{array} \tag{46}$$

Notice that:

- $4v_1v_2 - v_0^2$ is the leading coefficient associated with the leading monomial y of one of the three generators in the standard Gröbner basis of the jacobian ideal J_{f_Λ} w.r.t. the l.m.o. `tdeg_min`(y,z), for a generic choice of $\Lambda \in \mathcal{L}$; since $4v_1v_2 - v_0^2 = \det(\mathrm{Hess}_{f_\Lambda}(0))$ then Theorem 6(2) gives that $0 \in f_\Lambda^{-1}(0)$ *is a simple A_2 singularity for generic $\Lambda \in \mathcal{W}_2$*;
- imposing $v_1^3v_4^2 - v_2(v_1v_3 + v_2)^2 = 0$ means to annihilate the leading coefficient associated with the leading monomial y^2 of one of the three generators in the standard Gröbner basis of the jacobian ideal J_{f_Λ} for a generic choice of $\Lambda \in \mathcal{W}_2$; in this case a direct calculation gives $\mu(0) = 0$ and Theorem 6(2) implies that $0 \in f_\Lambda^{-1}(0)$ *is a simple A_3 singularity for generic $\Lambda \in \mathcal{W}_2^3$*;
- the leading monomial of the remaining two generators in the standard Gröbner basis of J_{f_Λ} for a generic choice of Λ in $\mathcal{L}, \mathcal{W}_2$ or $\mathcal{W}_2^3$ is always given by z: annihilating the associated leading coefficients increases the Milnor number only for $\Lambda \in \mathcal{W}_2^3$; in this case the leading coefficients are proportional to v_1v_2 and v_2, respectively; observe that Milnor number increases only if either all coefficients v_0, v_1 and v_2 annihilates or if $4v_1^3 - (v_1v_3 + 3v_2)^2 = 0$, which actually imposes the vanishing of the leading coefficient associated with the leading monomial y^3 of the remaining third generator in the standard basis of J_{f_Λ} for generic $\Lambda \in \mathcal{W}_2^3$; observe that $\mathcal{V}_0^2 \subseteq \mathcal{W}_2^4$ and that $\mathrm{codim}_{\mathcal{L}}\, \mathcal{V}_0^2 = \mathrm{codim}_{\mathcal{L}}\, \mathcal{W}_2^4 = 3$ meaning that $\mathcal{V}_0^2$ is an algebraic component of $\mathcal{W}_2^4$; let $\widetilde{\mathcal{W}_2^4}$ be the complementary algebraic component which is the algebraic closure

  (47) $$\widetilde{\mathcal{W}_2^4} := \overline{\mathcal{W}_2^4 \setminus \mathcal{V}_0^2}\ ;$$

  then points (2) and (3) in Theorem 6 gives that $0 \in f_\Lambda^{-1}(0)$ *is a simple A_4 singularity for generic $\Lambda \in \widetilde{\mathcal{W}_2^4}$ and a simple D_4 singularity for generic $\Lambda \in \mathcal{V}_0^2$*;
- the leading coefficient associated with the leading monomial y^3 of one of the four generators of the standard Gröbner basis of J_{f_Λ}, for generic $\Lambda \in \mathcal{V}_0^2$, is proportional to $4v_3^3 + 27v_4^2$; setting

  (48) $$\mathcal{V} := \left\{4v_3^3 + 27v_4^2 = 0\right\}$$

  Theorem 6(3) gives that $0 \in f_\Lambda^{-1}(0)$ *is a simple D_5 singularity for generic $\Lambda \in \mathcal{V} \cap \mathcal{V}_0^2$*; in particular $\mathcal{V} \cap \mathcal{V}_0^2$ turns out to be the intersection of the two algebraic components $\mathcal{V}_0^2$ and $\widetilde{\mathcal{W}_2^4}$ in $\mathcal{W}_2^4$;
- for generic $\Lambda \in \widetilde{\mathcal{W}_2^4}$, the standard Gröbner basis of J_{f_Λ} has three generators whose leading coefficients vanish when cutting with $\mathcal{W}_5$; observing that

  $$\widetilde{\mathcal{W}_2^4} \cap \mathcal{W}_5 = \mathcal{V} \cap \mathcal{V}_0^2$$

  we are then reduced to D_5 singularities already considered above; on the other hand imposing the vanishing of only the leading coefficients of the two generators admitting z as a leading monomial gives that $0 \in f_\Lambda^{-1}(0)$ *is a simple A_5*

singularity for generic $\Lambda \in \mathcal{W} \cap \left(\bigcap_{k=0}^{2} \mathcal{V}_{2k}\right)$ *where*

$$\mathcal{W} := \{v_3^2 - 4v_1 = 0\}\ ; \tag{49}$$

- the last step is now obtained by observing that $\left(\mathcal{V} \cap \mathcal{V}_0^2\right) \cap \left(\mathcal{W} \cap \left(\bigcap_{k=0}^{2} \mathcal{V}_{2k}\right)\right) = \{0\}$.

For details the reader is referred to [24] §6.4 where Gröbner basis, leading coefficients and monomials and Milnor numbers are explicitly computed via Maple procedures. □

6.5. Simple singularities of E_7 type.

THEOREM 10. *Let T^1 be the Kuranishi space of a simple N–dimensional singular point $0 \in f^{-1}(0)$ with*

$$f(x_1, \ldots, x_{N+1}) = \sum_{i=1}^{N-1} x_i^2 \ + \ x_N^3 + x_N x_{N+1}^3$$

The subset of T^1 parameterizing small deformations of $0 \in f^{-1}(0)$ to a simple node is the union of n hypersurfaces. Moreover, calling $\mathcal{L}$ any of those hypersurfaces, there exists a stratification of nested algebraic subsets giving rise to the following sequence of inclusions, c.i.p. squares and a hinged union of c.i.p. squares

$$\begin{tikzcd}
\mathcal{L} & \mathcal{W}_2 \arrow[l, hook'] & \mathcal{W}_2^3 \arrow[l, hook'] & \widetilde{\mathcal{W}}_2^4 \arrow[l, hook'] & \widetilde{\mathcal{W}}_2^5 \cup \widetilde{\mathcal{W}'}_2^5 \arrow[l, hook'] & \widetilde{\mathcal{W}'}_2^6 \arrow[l, hook'] \\
 & & \mathcal{V}_0^2 \arrow[u, hook] & \mathcal{V} \cap \mathcal{V}_0^2 \arrow[l, hook'] \arrow[u, hook] & \mathcal{V}' \cap \mathcal{V}_0^2 \arrow[l, hook'] \arrow[u, hook] & \\
 & & & \mathcal{V} \cap \mathcal{V}_0^3 \arrow[u, hook] \arrow[uur, hook] & \{0\} \arrow[l, hook'] \arrow[u, hook] \arrow[uur, hook] &
\end{tikzcd} \tag{50}$$

verifying the Arnol'd's adjacency diagram

$$\begin{tikzcd}
A_1 & A_2 \arrow[l] & A_3 \arrow[l] & A_4 \arrow[l] & A_5 \arrow[l] & A_6 \arrow[l] \\
 & & D_4 \arrow[u] & D_5 \arrow[l] \arrow[u] & D_6 \arrow[l] \arrow[u] & \\
 & & & E_6 \arrow[u] \arrow[uur] & E_7 \arrow[l] \arrow[u] \arrow[uur] &
\end{tikzcd} \tag{51}$$

where

- *$\mathcal{L}$ is the hypersurface of T^1 defined by equation (54), keeping in mind (53),*
- *$\mathcal{V}_0^m := \bigcap_{k=0}^m \mathcal{V}_k$ and $\mathcal{W}_2^m := \bigcap_{k=2}^m \mathcal{W}_k$ where $\mathcal{V}_k$, $\mathcal{W}_k$ and $\mathcal{V}$ are hypersurfaces of $\mathcal{L}$ defined by equations (56),*
- *$\mathcal{V}'$ is a codimension 2 complete intersection in $\mathcal{L}$ defined by equation (58),*
- *$\widetilde{\mathcal{W}_2^k}$ and $\widetilde{\mathcal{W}'}_2^k$ are Zariski closed subsets of $\mathcal{L}$ defined by (57), (59) and (60).*

In particular complete intersection properties in diagram (50) are summarized by the following relations:

$$\widetilde{\mathcal{W}_2^4} \cap \mathcal{V}_0^2 = \mathcal{V} \cap \mathcal{V}_0^2 \ , \ \widetilde{\mathcal{W}_2^5} \cap \left(\mathcal{V} \cap \mathcal{V}_0^2\right) = \mathcal{V}' \cap \mathcal{V}_0^2 \ , \ \widetilde{\mathcal{W}'}_2^5 \cap \left(\mathcal{V} \cap \mathcal{V}_0^2\right) = \mathcal{V} \cap \mathcal{V}_0^3$$
$$\left(\mathcal{V}' \cap \mathcal{V}_0^2\right) \cap \left(\mathcal{V} \cap \mathcal{V}_0^3\right) = \{0\} = \widetilde{\mathcal{W}'}_2^6 \cap \left(\mathcal{V}' \cap \mathcal{V}_0^2\right) \ . \tag{52}$$

Proof. Following the outline 6.1.

(1) By the Morse Splitting Lemma 1, our problem can be reduced to the case $N = 1$ with $f(y,z) = y^3 + yz^3$. Therefore

$$T^1 \cong \langle 1, y, z, yz, z^2, z^3, z^4 \rangle_{\mathbb{C}} \ .$$

Given $\Lambda = (\lambda_0, \lambda_1, \dots, \lambda_6) \in T^1$ the associated deformation of U_0 is

$$\begin{array}{rcl} U_\Lambda & = & \{f_\Lambda(y,z) = 0\} \quad \text{where} \\ f_\Lambda(y,z) & := & f(y,z) + \lambda_0 + \lambda_1 y + \lambda_2 z + \lambda_3 yz + \lambda_4 z^2 + \lambda_5 z^3 + \lambda_6 z^4 \ . \end{array}$$

A solution of the jacobian system of partials is then given by a solution $p_\Lambda = (y_\Lambda, z_\Lambda)$ of the following polynomial system in $\mathbb{C}[\lambda][y,z]$

$$\left\{ \begin{array}{c} 3y^2 + z^3 + \lambda_1 + \lambda_3 z \equiv 0 \\ 3yz^2 + \lambda_2 + \lambda_3 y + 2\lambda_4 z + 3\lambda_5 z^2 + 4\lambda_6 z^3 = 0 \end{array} \right. \tag{53}$$

giving precisely 7 critical points for f_Λ.

(2) Imposing that one of those critical points, say p_Λ, is actually a singular point of U_Λ means to require that

$$p_\Lambda \in U_\Lambda \iff \Lambda \in \mathcal{L} \subset T^1 \quad \text{where} \tag{54}$$
$$\mathcal{L} := \{y_\Lambda^3 + y_\Lambda z_\Lambda^3 + \lambda_0 + \lambda_1 y_\Lambda + \lambda_2 z_\Lambda + \lambda_3 y_\Lambda z_\Lambda + \lambda_4 z_\Lambda^2 + \lambda_5 z_\Lambda^3 + \lambda_6 z_\Lambda^4 = 0\}$$

which is *one of the 7 hypersurfaces of T^1 parameterizing small deformations of* $0 \in U_0$ *to nodes.* After translating $y \mapsto y + y_\Lambda, z \mapsto z + z_\Lambda$, we get

$$\begin{aligned}
&f_\Lambda(y+y_\Lambda, z+z_\Lambda) = \\
&\qquad f(y,z) &+& \left(y_\Lambda^3 + y_\Lambda z_\Lambda^3 + \lambda_0 + \lambda_1 y_\Lambda + \lambda_2 z_\Lambda + \lambda_3 y_\Lambda z_\Lambda + \lambda_4 z_\Lambda^2 + \lambda_5 z_\Lambda^3 + \lambda_6 z_\Lambda^4\right) \\
&&+& \left(3y_\Lambda^2 + z_\Lambda^3 + \lambda_1 + \lambda_3 z_\Lambda\right) y \\
&&+& \left(3y_\Lambda z_\Lambda^2 + \lambda_2 + \lambda_3 y_\Lambda + 2\lambda_4 z_\Lambda + 3\lambda_5 z_\Lambda^2 + 4\lambda_6 z_\Lambda^3\right) z \\
&&+& \left(3z_\Lambda^2 + \lambda_3\right) yz + 3y_\Lambda\, y^2 + \left(3y_\Lambda z_\Lambda + \lambda_4 + 3\lambda_5 z_\Lambda + 6\lambda_6 z_\Lambda^2\right) z^2 \\
&&+& 3z_\Lambda\, yz^2 + (\lambda_5 + 4\lambda_6 z_\Lambda + y_\Lambda) z^3 + \lambda_6\, z^4 \\
&\overset{(53),(54)}{=} f(y,z) &+& v_0 yz + v_1 y^2 + v_2 z^2 + v_3 yz^2 + v_4 z^3 + v_5 z^4
\end{aligned}$$

where

$$\begin{gathered}
v_0 = 3z_\Lambda^2 + \lambda_3 \quad , \quad v_1 = 3y_\Lambda \quad , \quad \nu_2 = 3y_\Lambda z_\Lambda + \lambda_4 + 3\lambda_5 z_\Lambda + 6\lambda_6 z_\Lambda^2 \, , \\
v_3 = 3z_\Lambda \quad , \quad v_4 = \lambda_5 + 4\lambda_6 z_\Lambda + y_\Lambda \quad , \quad v_5 = \lambda_6 \, .
\end{gathered} \tag{55}$$

(3) Define the following codimension 1 subvarieties of $\mathcal{L}$

$$\begin{aligned}
\mathcal{V}_k &:= \{v_k = 0\} \ , \quad 0 \le k \le 5 \\
\mathcal{W}_2 &:= \left\{4v_1v_2 - v_0^2 = 0\right\} \ , \quad \text{(vanishing of det(Hess))} \\
\mathcal{W}_3 &:= \left\{v_1^3 v_4^2 - v_2(v_1v_3 + v_2)^2 = 0\right\} \ , \\
\mathcal{W}_4 &:= \left\{16v_1^5 v_2 - [(v_1v_2 + 3v_2)^2 - 4v_1^3 v_5]^2 = 0\right\} \ , \\
\mathcal{W}_5 &:= \left\{v_1 v_5^2 - v_2 = 0\right\} \quad , \quad \mathcal{W}_5' := \left\{v_1(v_1^2 - 9v_2v_5)^2 - 81v_2^3 = 0\right\} \ , \\
\mathcal{W}_6 &:= \left\{16v_1^5 - 729v_2^3 = 0\right\} \ , \\
\mathcal{V} &:= \left\{4v_3^3 + 27v_4^2 = 0\right\} .
\end{aligned} \tag{56}$$

Then the proof goes on exactly as in the E_6 case, with the only difference that, for efficiency reason, computation are performed with the l.m.o. `plex_min(z,y)`, defined as the opposite of the pure lexicographic g.m.o. with $y < z$: hence in the following $z < y$. Namely we get:

- $0 \in f_\Lambda^{-1}(0)$ *is a simple* A_2 *singularity for generic* $\Lambda \in \mathcal{W}_2$;
- $0 \in f_\Lambda^{-1}(0)$ *is a simple* A_3 *singularity for generic* $\Lambda \in \mathcal{W}_2^3$;
- $0 \in f_\Lambda^{-1}(0)$ *is a simple* D_4 *singularity for generic* $\Lambda \in \mathcal{V}_0^2$;
- let $\widetilde{\mathcal{W}_2^4}$ be the complementary algebraic component of $\mathcal{V}_0^2 \subseteq \mathcal{W}_2^4$, which is the algebraic closure

  $$\widetilde{\mathcal{W}_2^4} := \overline{\mathcal{W}_2^4 \setminus \mathcal{V}_0^2} \ , \tag{57}$$

  then $0 \in f_\Lambda^{-1}(0)$ *is a simple* A_4 *singularity for generic* $\Lambda \in \widetilde{\mathcal{W}_2^4}$;

- $0 \in f_\Lambda^{-1}(0)$ *is a simple* D_5 *singularity for generic* $\Lambda \in \mathcal{V} \cap \mathcal{V}_0^2$.

To go further, notice that:

- for a generic $\Lambda \in \mathcal{V} \cap \mathcal{V}_0^2$ the standard Gröbner basis of J_{f_Λ} has three generators whose leading monomials are given by z^4, yz and y^2 respectively; the latter admits an associated constant leading coefficient, while cutting with $v_3 = 0$ (or equivalently $v_4 = 0$) annihilates both the leading coefficients of the first two generators; then Theorem 6(4) gives that $0 \in f_\Lambda^{-1}(0)$ *is a singularity of type* E_6 *for generic* Λ *in* $\mathcal{V} \cap \mathcal{V}_0^3 = \mathcal{V}_0^4$*;*

- on the other hand imposing $\Lambda \in \mathcal{V}' \cap \mathcal{V}_0^2$, where $\mathcal{V}'$ is the codimension 2 algebraic subset of $\mathcal{L}$, contained in the hypersurface $\mathcal{V}$ and given by

$$\mathcal{V}' = \left\{ v_3 + 3v_5^2 = v_4 + 2v_5^3 = 0 \right\} \ , \tag{58}$$

annihilates the leading coefficient associated with the leading monomial z^4 of the first generator; then Theorem 6(3) gives that $0 \in f_\Lambda^{-1}(0)$ *is a singularity of type* D_6 *for generic* Λ *in* $\mathcal{V}' \cap \mathcal{V}_0^2$;

- a further specialization here gives the trivial deformation since

$$(\mathcal{V} \cap \mathcal{V}_0^3) \cap (\mathcal{V}' \cap \mathcal{V}_0^2) = \mathcal{V}' \cap \mathcal{V}_0^3 = \{0\} \ ;$$

- coming back to consider A_4 singularities, for a generic $\Lambda \in \widetilde{\mathcal{W}_2^4}$ the standard Gröbner basis of J_{f_Λ} has three generators whose leading monomials are given by y, y and z^4 respectively; setting $v_1 = 0$ annihilates both the leading coefficients associated with y; but

$$\widetilde{\mathcal{W}_2^4} \cap \mathcal{V}_1 = \mathcal{V} \cap \mathcal{V}_0^2$$

and *we are reduced to the already considered case of* D_5 *singularities*; on the other hand the leading coefficient associated with z^4 can be annihilated by imposing three independent conditions, only two of which increase the Milnor number; these last are realized by cutting either with $\mathcal{W}_5$ or with $\mathcal{W}_5'$; observing that $\mathcal{V}_0^2 \subseteq \mathcal{W}_5 \cap \mathcal{W}_5'$, let us define the complementary algebraic components

$$\begin{aligned} \widetilde{\mathcal{W}_2^5} &:= \overline{\mathcal{W}_2^5 \setminus \mathcal{V}_0^2} = \widetilde{\mathcal{W}_2^4} \cap \mathcal{W}_5 \\ \widetilde{\mathcal{W}'}_2^5 &:= \overline{(\mathcal{W}_2^4 \cap \mathcal{W}_5') \setminus \mathcal{V}_0^2} = \widetilde{\mathcal{W}_2^4} \cap \mathcal{W}_5' \ ; \end{aligned} \tag{59}$$

then Theorem 6(2) gives that $0 \in f_\Lambda^{-1}(0)$ *is a singularity of type* A_5 *for generic* $\Lambda \in \widetilde{\mathcal{W}_2^5} \cup \widetilde{\mathcal{W}'}_2^5$;

- for a generic $\Lambda \in \widetilde{\mathcal{W}_2^5}$ the standard Gröbner basis of J_{f_Λ} has three generators whose leading coefficients can be simultaneously annihilated by cutting with $\mathcal{V}_1$; since

$$\widetilde{\mathcal{W}_2^5} \cap \mathcal{V}_1 = \mathcal{V}' \cap \mathcal{V}_0^2$$

we are reduced to the already considered case of D_6 singularities; on the other hand for a generic $\Lambda \in \widetilde{\mathcal{W}'}^{5}_{2}$ the standard Gröbner basis of J_{f_Λ} has three generators whose leading coefficients can be simultaneously annihilated by cutting with $\mathcal{V}_2$; since

$$\widetilde{\mathcal{W}'}^{5}_{2} \cap \mathcal{V}_2 = \mathcal{V}_0^4$$

we are reduced to the already considered case of E_6 singularities; moreover the leading coefficient associated with the leading monomial z^5 of the first generator in the standard basis of J_{f_Λ} can be annihilated by cutting with $\mathcal{W}_6$; define

$$\widetilde{\mathcal{W}'}^{6}_{2} := \widetilde{\mathcal{W}'}^{5}_{2} \cap \mathcal{W}_6 \,, \tag{60}$$

then Theorem 6(2) gives that $0 \in f_\Lambda^{-1}(0)$ *is a simple A_6 singularity for generic* $\Lambda \in \widetilde{\mathcal{W}'}^{6}_{2}$;

- the last step is now obtained by observing that further specializations lead to the trivial deformation, which is $\widetilde{\mathcal{W}'}^{6}_{2} \cap \mathcal{V}_0^4 = \{0\}$.

For details the reader is referred to [24] §6.5 where Gröbner basis, leading coefficients and monomials and Milnor numbers are explicitly computed via Maple procedures. □

6.6. Simple singularities of E_8 type.

THEOREM 11. *Let T^1 be the Kuranishi space of a simple N–dimensional singular point $0 \in f^{-1}(0)$ with*

$$f(x_1, \ldots, x_{N+1}) = \sum_{i=1}^{N-1} x_i^2 \; + \; x_N^3 + x_{N+1}^5$$

The subset of T^1 parameterizing small deformations of $0 \in f^{-1}(0)$ to a simple node is the union of n hypersurfaces. Moreover, calling $\mathcal{L}$ any of those hypersurfaces, there exists a stratification of nested algebraic subsets giving rise to the following sequence of inclusions, c.i.p. squares and reducible c.i.p squares

(61)

$$\begin{tikzcd}
\mathcal{L} & \mathcal{W}_2 \arrow[l, hook'] & \mathcal{W}_2^3 \arrow[l, hook'] & \widetilde{\mathcal{W}}_2^4 \arrow[l, hook'] & \widetilde{\mathcal{W}}_2^5 \arrow[l, hook'] & \widetilde{\mathcal{W}}_2^6 \arrow[l, hook'] & \widetilde{\mathcal{W}}_2^7 \arrow[l, hook'] \\
 & & \mathcal{V}_0^2 \arrow[u, hook] & \mathcal{V} \cap \mathcal{V}_0^2 \arrow[l, hook'] \arrow[u, hook] & \mathcal{V}' \cap \mathcal{V}_0^2 \arrow[l, hook'] \arrow[u, hook] & \mathcal{V}'' \cap \mathcal{V}_0^2 \arrow[l, hook'] \arrow[u, hook] & \\
 & & & \mathcal{V}_0^4 \arrow[u, hook] \arrow[uur, hook] & \mathcal{V}_0^4 \cap \mathcal{V}_6 \arrow[l, hook'] \arrow[u, hook] \arrow[uur, hook] & \{0\} \arrow[l, hook'] \arrow[u, hook] \arrow[uur, hook] &
\end{tikzcd}$$

verifying the Arnol'd's adjacency diagram

$$
\begin{tikzcd}
A_1 & A_2 \arrow[l] & A_3 \arrow[l] & A_4 \arrow[l] & A_5 \arrow[l] & A_6 \arrow[l] & A_7 \arrow[l] \\
 & & D_4 \arrow[u] & D_5 \arrow[l] \arrow[u] & D_6 \arrow[l] \arrow[u] & D_7 \arrow[l] \arrow[u] & \\
 & & & E_6 \arrow[u] \arrow[uur] & E_7 \arrow[l] \arrow[u] \arrow[uur] & E_8 \arrow[l] \arrow[u] \arrow[uur] &
\end{tikzcd}
\tag{62}
$$

where

- *$\mathcal{L}$ is the hypersurface of T^1 defined by equation (65), keeping in mind (64),*
- *$\mathcal{V}_0^m := \bigcap_{k=0}^m \mathcal{V}_k$ and $\mathcal{W}_2^m := \bigcap_{k=2}^m \mathcal{W}_k$ where $\mathcal{V}_k, \mathcal{W}_k$ are hypersurfaces of $\mathcal{L}$ defined by equations (66) and (67),*
- *$\mathcal{V}, \mathcal{V}'$ and $\mathcal{V}''$ are a hypersurface, a codimension 2 and a codimension 3 complete intersections in $\mathcal{L}$, respectively, defined by the latter equation in (67), by (68) and by (69), respectively,*
- *$\widetilde{\mathcal{W}_2^k}$ are Zariski closed subsets of $\mathcal{L}$ defined by (70), (71), (72) and (73).*

In particular, complete intersection properties in diagram (61) are summarized by the following relations:

$$
\begin{aligned}
\widetilde{\mathcal{W}_2^4} \cap \mathcal{V}_0^2 &= \mathcal{V} \cap \mathcal{V}_0^2 , \\
\widetilde{\mathcal{W}_2^5} \cap \left(\mathcal{V} \cap \mathcal{V}_0^2\right) &= (\mathcal{V}' \cap \mathcal{V}_0^2) \cup \mathcal{V}_0^4 , \\
\widetilde{\mathcal{W}_2^6} \cap \left(\mathcal{V}' \cap \mathcal{V}_0^2\right) &= (\mathcal{V}'' \cap \mathcal{V}_0^2) \cup (\mathcal{V}_0^4 \cap \mathcal{V}_6) , \\
\left(\mathcal{V}'' \cap \mathcal{V}_0^2\right) \cap \left(\mathcal{V}_0^4 \cap \mathcal{V}_6\right) &= \widetilde{\mathcal{W}_2^7} \cap \left(\mathcal{V}'' \cap \mathcal{V}_0^2\right) \;=\; \{0\} .
\end{aligned}
\tag{63}
$$

Proof. Following the outline 6.1.

(1) By the Morse Splitting Lemma 1, our problem can be reduced to the case $N = 1$ with $f(y,z) = y^3 + z^5$. Therefore:

$$T^1 \cong \langle 1, y, z, yz, z^2, yz^2, z^3, yz^3 \rangle_{\mathbb{C}} .$$

Given $\Lambda = (\lambda_0, \lambda_1, \dots, \lambda_7) \in T^1$, the associated deformation of U_0 is

$$
\begin{aligned}
U_\Lambda &= \{f_\Lambda(y,z) = 0\} \quad \text{where} \quad f_\Lambda(y,z) := \\
&= f(y,z) + \lambda_0 + \lambda_1 y + \lambda_2 z + \lambda_3 yz + \lambda_4 z^2 + \lambda_5 yz^2 + \lambda_6 z^3 + \lambda_7 yz^3
\end{aligned}
$$

A solution of the jacobian system of partials is then given by a solution $p_\Lambda = (y_\Lambda, z_\Lambda)$ of the following polynomial system in $\mathbb{C}[\lambda][y,z]$

$$
\left\{
\begin{array}{c}
3y^2 + \lambda_1 + \lambda_3 z + \lambda_5 z^2 + \lambda_7 z^3 = 0 \\
5z^4 + \lambda_2 + 2\lambda_4 z + 3\lambda_6 z^2 + y(\lambda_3 + 2\lambda_5 z + 3\lambda_7 z^2) = 0
\end{array}
\right.
\tag{64}
$$

giving 8 critical points for f_Λ.

(2) Imposing that one of those critical points, say p_Λ, is actually a singular point of U_Λ means to require that

$$\begin{aligned} p_\Lambda \quad &\in \quad U_\Lambda \iff \Lambda \in \mathcal{L} \subset T^1 \quad \text{where} \\ \mathcal{L} \quad &:= \quad \{y_\Lambda^3 + z_\Lambda^5 + \lambda_0 + \lambda_1 y_\Lambda + \lambda_2 z_\Lambda + \lambda_3 y_\Lambda z_\Lambda \\ &\qquad + \lambda_4 z_\Lambda^2 + \lambda_5 y_\Lambda z_\Lambda^2 + \lambda_6 z_\Lambda^3 + \lambda_7 y_\Lambda z_\Lambda^3 = 0\} \end{aligned} \tag{65}$$

which is *one of the 8 hypersurfaces of T^1 parameterizing small deformations of* $0 \in U_0$ *to nodes.* After translating $y \mapsto y + y_\Lambda, z \mapsto z + z_\Lambda$, we get

$$\begin{aligned} & f_\Lambda(y+y_\Lambda, z+z_\Lambda) = f(y,z) + \\ + \quad & \left(y_\Lambda^3 + z_\Lambda^5 + \lambda_0 + \lambda_1 y_\Lambda + \lambda_2 z_\Lambda + \lambda_3 y_\Lambda z_\Lambda + \lambda_4 z_\Lambda^2 + \lambda_5 y_\Lambda z_\Lambda^2 + \lambda_6 z_\Lambda^3 + \lambda_7 y_\Lambda z_\Lambda^3\right) \\ + \quad & \left(3y_\Lambda^2 + \lambda_1 + \lambda_3 z_\Lambda + \lambda_5 z_\Lambda^2 + \lambda_7 z_\Lambda^3\right) y \\ + \quad & \left(5z_\Lambda^4 + \lambda_2 + 2\lambda_4 z_\Lambda + 3\lambda_6 z_\Lambda^2 + y_\Lambda(\lambda_3 + 2\lambda_5 z_\Lambda + 3\lambda_7 z_\Lambda^2)\right) z \\ + \quad & \left(\lambda_3 + 2\lambda_5 z_\Lambda + 3\lambda_7 z_\Lambda^2\right) yz + 3y_\Lambda\, y^2 \\ + \quad & \left(10z_\Lambda^3 + \lambda_4 + \lambda_5 y_\Lambda + 3\lambda_6 z_\Lambda + 3\lambda_7 y_\Lambda z_\Lambda\right) z^2 \\ + \quad & (\lambda_5 + 3\lambda_7 z_\Lambda) yz^2 + \left(10 z_\Lambda^2 + \lambda_6 + \lambda_7 y_\Lambda\right) z^3 + \lambda_7\, yz^3 + 5z_\Lambda\, z^4 \\ \overset{(64),(65)}{=} \quad & f(y,z) + v_0 yz + v_1 y^2 + v_2 z^2 + v_3 yz^2 + v_4 z^3 + v_5 yz^3 + v_6 z^4 \end{aligned}$$

where

$$\begin{gathered} v_0 = \lambda_3 + 2\lambda_5 z_\Lambda + 3\lambda_7 z_\Lambda^2 \quad , \quad v_1 = 3y_\Lambda \\ v_2 = 10 z_\Lambda^3 + \lambda_4 + \lambda_5 y_\Lambda + 3\lambda_6 z_\Lambda + 3\lambda_7 y_\Lambda z_\Lambda \quad , \quad v_3 = \lambda_5 + 3\lambda_7 z_\Lambda \\ v_4 = 10 z_\Lambda^2 + \lambda_6 + \lambda_7 y_\Lambda \quad , \quad v_5 = \lambda_7 \quad , \quad v_6 = 5 z_\Lambda \end{gathered} \tag{66}$$

(3) Define the following codimension 1 subvarieties of $\mathcal{L}$

$$\begin{aligned} \mathcal{V}_k &:= \{v_k = 0\} \ , \quad 0 \le k \le 6 \\ \mathcal{W}_2 &:= \left\{4v_1v_2 - v_0^2 = 0\right\} \ , \quad \text{(vanishing of det(Hess))} \\ \mathcal{W}_3 &:= \left\{v_1^3 v_4^2 - v_2(v_1v_3 + v_2)^2 = 0\right\} \ , \\ \mathcal{W}_4 &:= \left\{16 v_1^5 v_2 v_5^2 - [(v_1v_3 + 3v_2)^2 - 4v_1^3 v_6]^2 = 0\right\} \ , \\ \mathcal{W}_5 &:= \Big\{[4v_1^5 v_5^2 (v_1v_3 + 3v_2)^2 + 16 v_1^7 (v_1^2 + v_1v_3v_5 + 3v_2v_5) - 9v_2(v_1v_3 + 3v_2)^4] \cdot \\ &\qquad [4v_1^5 v_5^2 (v_1v_3 + 3v_2)^2 + 16 v_1^7 (v_1^2 - v_1v_3v_5 - 3v_2v_5) - 9v_2(v_1v_3 + 3v_2)^4] = 0\Big\} \ , \\ \mathcal{W}_6 &:= \Big\{[32 v_1^9 - 2v_1^5 v_5 (v_1v_3 + 3v_2)(8v_1^2 - 3v_2v_5 - v_1v_3v_5) + (v_1v_3 + 3v_2)^5] \cdot \\ &\qquad [32 v_1^9 + 2v_1^5 v_5 (v_1v_3 + 3v_2)(8v_1^2 + 3v_2v_5 + v_1v_3v_5) - (v_1v_3 + 3v_2)^5] = 0\Big\} \ , \\ \mathcal{W}_7 &:= \left\{256 v_2 - v_1 v_5^4 = 0\right\} \ , \\ \mathcal{V} &:= \left\{4v_3^3 + 27 v_4^2 = 0\right\} . \end{aligned} \tag{67}$$

Then the proof goes on exactly as in the previous E_7 case until E_6 singularities. Computations are still performed by the l.m.o. `plex_min(z,y)`, hence in the following $z < y$. We have then the inclusions' chain

$$T^1 \supset \mathcal{L} \supset \mathcal{W}_2 \supset \mathcal{W}_2^3 \supset \mathcal{V}_1 \cap \mathcal{W}_2^3 = \mathcal{V}_0^2 \supset \mathcal{V} \cap \mathcal{V}_0^2 \supset \mathcal{V}_0^4$$

of subsets parameterizing small deformations whose generic fibre is either smooth or admits a singularity of type A_1, A_2, A_3 , D_4, D_5 and E_6, respectively.

To go further notice that:

- for a generic $\Lambda \in \mathcal{V}_0^4$ the standard Gröbner basis of J_{f_Λ} has three generators whose leading monomials are given by z^4, yz^2 and y^2 respectively; the latter admits an associated constant leading coefficient, while cutting with $v_6 = 0$ annihilates the leading coefficients of the former; then Theorem 6(4) gives that $0 \in f_\Lambda^{-1}(0)$ *is a singularity of type E_7 for generic Λ in $\mathcal{V}_0^4 \cap \mathcal{V}_6$;*
- coming back to consider the standard Gröbner basis of J_{f_Λ} for a generic $\Lambda \in \mathcal{V} \cap \mathcal{V}_0^2$, it is given by three generators whose leading monomial are given by z^4, yz and y^2; imposing $\Lambda \in \mathcal{V}' \cap \mathcal{V}_0^2$, where $\mathcal{V}'$ is the codimension 2 algebraic subset of $\mathcal{L}$, contained in the hypersurface $\mathcal{V}$ and given by

  (68) $$\mathcal{V}' := \left\{ v_3 v_5^2 + 3v_6^2 = v_4 v_5^3 + 2v_6^3 = 0 \right\} \subset \mathcal{V}$$

  annihilates the leading coefficient associated with the leading monomial z^4; then Theorem 6(3) gives that $0 \in f_\Lambda^{-1}(0)$ *is a singularity of type D_6 for generic Λ in $\mathcal{V}' \cap \mathcal{V}_0^2$*;
- moreover the standard Gröbner basis of J_{f_Λ} for a generic $\Lambda \in \mathcal{V}' \cap \mathcal{V}_0^2$, is given by three generators whose leading monomial are given by z^5, yz and y^2; the leading coefficient associated with the former is annihilated by requiring that Λ is the generic element in $\mathcal{V}'' \cap \mathcal{V}_0^2$, where $\mathcal{V}''$ is the section of $\mathcal{V}'$ given by

  (69) $$\mathcal{V}'' := \mathcal{V}' \cap \left\{ 12v_6 + v_5^3 = 0 \right\} \ ;$$

  then, Theorem 6(3) gives that $0 \in f_\Lambda^{-1}(0)$ *is a singularity of type D_7 for generic Λ in $\mathcal{V}'' \cap \mathcal{V}_0^2$*;
- on the other hand $\mathcal{V}' \cap \mathcal{V}_0^3 = \mathcal{V}_0^4 \cap \mathcal{V}_6$ and we get the E_7 singularities already discussed;
- a further specialization here gives the trivial deformation since

  $$(\mathcal{V}_0^4 \cap \mathcal{V}_6) \cap (\mathcal{V}'' \cap \mathcal{V}_0^2) = \{0\} \ ;$$

- let $\widetilde{\mathcal{W}_2^4}$ be the complementary algebraic component of $\mathcal{V}_0^2 \subseteq \mathcal{W}_2^4$, which is the algebraic closure

  (70) $$\widetilde{\mathcal{W}_2^4} := \overline{\mathcal{W}_2^4 \setminus \mathcal{V}_0^2} \ ,$$

  then $0 \in f_\Lambda^{-1}(0)$ *is a simple A_4 singularity for generic $\Lambda \in \widetilde{\mathcal{W}_2^4}$*;

- all the leading coefficients of the three generators in the standard Gröbner basis of J_{f_Λ}, for a generic $\Lambda \in \widetilde{\mathcal{W}_2^4}$ are annihilated by cutting with $\mathcal{V}_1$; by the way $\mathcal{V}_1 \cap \widetilde{\mathcal{W}_2^4} = \mathcal{V} \cap \mathcal{V}_0^2$ obtaining a generic D_5 singularity, as already described above; on the other hand annihilating only the leading coefficient associated with the minimal leading monomial z^4 drives to consider the complementary algebraic component of $\mathcal{V}_0^2 \subseteq \mathcal{W}_2^5$, which is the algebraic closure

$$\widetilde{\mathcal{W}_2^5} := \overline{\mathcal{W}_2^5 \setminus \mathcal{V}_0^2}\ , \tag{71}$$

then $0 \in f_\Lambda^{-1}(0)$ *is a simple* A_5 *singularity for generic* $\Lambda \in \widetilde{\mathcal{W}_2^5}$;

- for generic $\Lambda \in \widetilde{\mathcal{W}_2^5}$, imposing the vanishing of all the leading coefficients leads to impose either $\Lambda \in \mathcal{V}'' \cap \mathcal{V}_0^2$ or $\Lambda \in \mathcal{V}_0^4$ giving the above discussed case of D_6 and E_6 singularities, respectively; on the other hand annihilating only the leading coefficient associated with the minimal leading monomial z^5 drives to consider the complementary algebraic component of $\mathcal{V}_0^2 \subseteq \mathcal{W}_2^6$, which is the algebraic closure

$$\widetilde{\mathcal{W}_2^6} := \overline{\mathcal{W}_2^6 \setminus \mathcal{V}_0^2} = \widetilde{\mathcal{W}_2^5} \cap \mathcal{W}_6\ , \tag{72}$$

then $0 \in f_\Lambda^{-1}(0)$ *is a simple* A_6 *singularity for generic* $\Lambda \in \widetilde{\mathcal{W}_2^6}$;

- for generic $\Lambda \in \widetilde{\mathcal{W}_2^6}$, imposing the vanishing of all the leading coefficients leads to impose either $\Lambda \in \mathcal{V}''' \cap \mathcal{V}_0^2$ or $\Lambda \in \mathcal{V}_0^4 \cap \mathcal{V}_6$ giving the above discussed case of D_7 and E_7 singularities, respectively; on the other hand annihilating only the leading coefficient associated with the minimal leading monomial z^6 drives to consider the complementary algebraic component of $\mathcal{V}_0^2 \subseteq \mathcal{W}_2^7$, which is the algebraic closure

$$\widetilde{\mathcal{W}_2^7} := \overline{\mathcal{W}_2^7 \setminus \mathcal{V}_0^2} = \widetilde{\mathcal{W}_2^6} \cap \mathcal{W}_7\ , \tag{73}$$

then $0 \in f_\Lambda^{-1}(0)$ *is a simple* A_7 *singularity for generic* $\Lambda \in \widetilde{\mathcal{W}_2^7}$;

- a further specialization here gives the trivial deformation since

$$\widetilde{\mathcal{W}_2^7} \cap (\mathcal{V}''' \cap \mathcal{V}_0^2) = \{0\}\ .$$

To check relations (63) is then left to the reader. For details the reader is referred to [24] §6.6 where Gröbner basis, leading coefficients and monomials and Milnor numbers are explicitly computed via Maple procedures. □

6.7. A list of very special adjacencies

As a consequence of the analysis performed in the previous sections, we are now able to concretely write down some very special small 1-parameter deformations of a A_n,

D_n, E_6, E_7 or E_8, realizing adjacencies not directly mentioned in [3] and [5] (except for those in 6.7). The 1-parameter deformations we are going to list in the following are obtained by last steps in proofs of Theorems 7, 8, 9, 10 and 11, giving precisely 1-parameter deformations, after some possible parameter's re-scaling.

$A_{n-1} \longleftarrow D_n$

Assume that $n \geq 4$ is either $n = 2m+4$ when n is even, or $n = 2m+5$ when odd. Then consider the 1-parameter family $\mathcal{X}_t := \{f_t(\mathbf{x}) = 0\}$, $t \in \mathbb{C}$, where

$$f_0(\mathbf{x}) := \sum_{i=1}^{N-1} x_i^2 \; + \; x_N^2\, x_{N+1} + x_{N+1}^{n-1}$$

and either

$$f_t(\mathbf{x}) := f_0(\mathbf{x}) + t(x_N + it^m x_{N+1})^2 + \sum_{k=3}^{2m+2} (-t)^{2m+3-k} x_{N+1}^k \quad (n \text{ even})$$

or

$$f_t(\mathbf{x}) := f_0(\mathbf{x}) + t^2(x_N + t^{2m+1} x_{N+1})^2 + \sum_{k=3}^{2m+3} (-t^2)^{2m+4-k} x_{N+1}^k \quad (n \text{ odd}).$$

Then $\mathcal{X}_0$ is an isolated D_n point and, for generic $t \in \mathbb{C}$, $\mathcal{X}_t$ admits the unique singular point $\mathbf{0} \in f_t^{-1}(0)$ which is of type A_{n-1}.

$A_5 \longleftarrow E_6$

Consider the 1-parameter family $\mathcal{X}_t := \{f_t(\mathbf{x}) = 0\}$, $t \in \mathbb{C}$, with

$$\begin{aligned} f_0(\mathbf{x}) \quad &:= \quad \sum_{i=1}^{N-1} x_i^2 \; + \; x_N^3 + x_{N+1}^4 \\ f_t(\mathbf{x}) \quad &:= \quad f_0(\mathbf{x}) + t^2 x_N^2 + 2t x_N x_{N+1}^2 \; . \end{aligned}$$

Then $\mathcal{X}_0$ is an isolated E_6 singular point and, for generic $t \in \mathbb{C}$, $\mathcal{X}_t$ admits the unique singular point $\mathbf{0} \in f_t^{-1}(0)$ which is of type A_5.

$D_5 \longleftarrow E_6$

Let $f_0(\mathbf{x})$ be as in the previous case and assume

$$f_t(\mathbf{x}) := f_0(\mathbf{x}) - 3t^2 x_N x_{N+1}^2 - 2t^3 x_{N+1}^3 \; .$$

Then, for generic $t \in \mathbb{C}$, $\mathcal{X}_t$ admits the unique singular point $\mathbf{0} \in f_t^{-1}(0)$ which is of type D_5.

$A_6 \longleftarrow E_7$

Consider the 1-parameter family $\mathcal{X}_t := \{f_t(\mathbf{x}) = 0\}$, $t \in \mathbb{C}$, with

$$f_0(\mathbf{x}) := \sum_{i=1}^{N-1} x_i^2 \; + \; x_N^3 + x_N x_{N+1}^3$$

$$f_t(\mathbf{x}) := f_0(\mathbf{x}) + 432t^3(x_N + 4tx_{N+1})^2 - 120t^2 x_N x_{N+1}^2 - 416t^3 x_{N+1}^3 + 7tx_{N+1}^4$$

Then $\mathcal{X}_0$ is an isolated E_7 singular point and, for generic $t \in \mathbb{C}$, $\mathcal{X}_t$ admits the unique singular point $\mathbf{0} \in f_t^{-1}(0)$ which is of type A_6.

$D_6 \longleftarrow E_7$

Let $f_0(\mathbf{x})$ be as in the previous case and assume

$$f_t(\mathbf{x}) := f_0(\mathbf{x}) - 3t^2 x_N x_{N+1}^2 - 2t^3 x_{N+1}^3 + tx_{N+1}^4 \; .$$

Then, for generic $t \in \mathbb{C}$, $\mathcal{X}_t$ admits the unique singular point $\mathbf{0} \in f_t^{-1}(0)$ which is of type D_6.

$A_7 \longleftarrow E_8$

Consider the 1-parameter family $\mathcal{X}_t := \{f_t(\mathbf{x}) = 0\}$, $t \in \mathbb{C}$, with

$$f_0(\mathbf{x}) := \sum_{i=1}^{N-1} x_i^2 \; + \; x_N^3 + x_{N+1}^5$$

$$f_t(\mathbf{x}) := f_0(\mathbf{x}) + t^5(x_N - t^2 x_{N+1})^2 - 5t^4 x_N x_{N+1}^2 + 4t^6 x_{N+1}^3 - 4tx_N x_{N+1}^3 + 5t^3 x_{N+1}^4$$

Then $\mathcal{X}_0$ is an isolated E_8 singular point and, for generic $t \in \mathbb{C}$, $\mathcal{X}_t$ admits the unique singular point $\mathbf{0} \in f_t^{-1}(0)$ which is of type A_7.

$D_7 \longleftarrow E_8$

Let $f_0(\mathbf{x})$ be as in the previous case and assume

$$f_t(\mathbf{x}) := f_0(\mathbf{x}) - 27t^4 x_N x_{N+1}^2 + 54t^6 x_{N+1}^3 - 6tx_N x_{N+1}^3 + 18t^3 x_{N+1}^4 \; .$$

Then, for generic $t \in \mathbb{C}$, $\mathcal{X}_t$ admits the unique singular point $\mathbf{0} \in f_t^{-1}(0)$ which is of type D_7.

Acknowledgment. We are greatly indebt to G. M. Greuel who timely pointed out to us serious mistakes in the first version of this note. His concise and sharp remarks considerably helped us to improve our routine and the final product. The authors would also like to thank A. Albano for enlightening conversations.

References

[1] http://www.maplesoft.com/applications/view.aspx?SID=19341

[2] http://www.singular.uni-kl.de

[3] Arnol'd V. I. "Local normal forms of functions" *Invent. Math.* **35** (1976), 87–109.

[4] Arnol'd V. I., Goryunov V. V., Lyashko O. V. and Vasil'ev V. A. *Singularity theory I*, translated from the 1988 Russian original by A. Iacob, reprint of the original English edition from the series Encyclopaedia of Mathematical Sciences [Dynamical systems. VI, Encyclopaedia Math. Sci., 6, Springer, Berlin, 1993; MR1230637 (94b:58018)] Springer-Verlag, Berlin, 1998.

[5] Arnol'd V. I., Gusein-Zade S. M. and Varchenko A. N. *Singularities of differentiable maps*, Vol. I, *The classification of critcal points caustics and wave fronts*, Monographs in Mathematics **82**, Birkhäuser Boston, Inc., Boston, MA, 1988.

[6] Barth W., Peters C. and Van de Ven A. *Compact complex surfaces* vol. **4** E.M.G, Springer–Verlag (1984)

[7] Cox D., Little J. and O'Shea D. *Ideals, varieties, and algorithms. An introduction to computational algebraic geometry and commutative algebra.* Third edition, Undergraduate Texts in Mathematics, Springer, New York (2007).

[8] Decker W. and Lossen C. *Computing in algebraic geometry. A quick start using SINGULAR* Algorithms and Computation in Mathematics **16**, Springer-Verlag, Berlin; Hindustan Book Agency, New Delhi (2006).

[9] Douady A. "Le problème des modules locaux pour les espaces C–analytiques compacts" *Ann. scient. Éc. Norm. Sup.* 4e série, **7** 569–602 (1974).

[10] Godement R. *Topologie Algébrique et Théorie des Faisceaux*, Hermann, Paris (1958).

[11] Grauert H. "DerSatz von Kuranishi für Kompakte Komplexe Räume" *Invent.Math.* **25**, 107–142 (1974).

[12] Greuel G.-M., Lossen C. and Shustin E. *Introduction to singularities and deformations* Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2007.

[13] Greuel G.-M. and Pfister G. *A Singular introduction to commutative algebra* Springer-Verlag, Berlin, 2002.

[14] Grothendieck A. "Sur quelques points d'algèbre homologique" *Tôhoku Math. J.* **9**, 199–221 (1957).

[15] Lichtenbaum S. and Schlessinger M. "On the cotangent complex of a morphism" *Trans. A.M.S.* **128**, 41–70 (1967).

[16] E. J. N. Looijenga *Isolated singular points on complete intersections* London Mathematical Society Lecture Note Series, **77**, Cambridge University Press, Cambridge, 1984.

[17] E. J. N. Looijenga and J. Steenbrink "Milnor number and Tjurina number of complete intersections" *Math. Ann.* **271** (1) (1985), 121–124.

[18] Milnor J. *Singular points of complex hypersurfaces*, Annals of Math. Studies **61**, Princeton University Press, Princeton (1968).

[19] Milnor J. and Orlik P. "Isolated singularities defined by weighted homogeneous polynomials", *Topology* **9** (1970), 385–393.

[20] Mora F. "An algorithm to compute the equations of tangent cones" in *Computer algebra (Marseille, 1982)*, Lecture Notes in Comput. Sci. **144**, Springer Berlin-New York (1982), 158–165.

[21] Palamodov V. P. "The existence of versal deformations of complex spaces" *Dokl. Akad. Nauk SSSR* **206** (1972), 538–541.

[22] Palamodov V. P. "Deformations of complex spaces" *Russian Math. Surveys* **31**(3) (1976), 129–197; from russian *Uspekhi Mat. Nauk* **31**(3) (1976), 129–194.

[23] Reid M. "Canonical 3–folds" in *Journées de géométrie algébrique d'Angers*, Sijthoff & Norddhoff (1980), 671–689.

[24] Rossi M. and Terracini L. *MAPLE subroutines for computing Milnor and Tyurina numbers of hypersurfaces singulrities with application to Arnol'd adjacencies*, `arXiv:0809.4345`.

[25] Saito K. "Quasihomogene isolierte Singularitäten von Hyperflächen" *Invent. Math.* **14** (1971), 123–142.

[26] J. Stevens *Deformations of singularities* Lecture Notes in Mathematics **1811**, Springer-Verlag, Berlin, 2003.

Lavoro arrivato in redazione il 5-5-16.