ON CERTAIN BOUNDS FOR FIRST-CROSSING-TIME PROBABILITIES OF A JUMP-DIFFUSION PROCESS

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Received April 5, 2006

Abstract. We consider the first-crossing-time problem through a constant boundary for a Wiener process perturbed by random jumps driven by a counting process. On the base of a sample-path analysis of the jump-diffusion process we obtain explicit lower bounds for the first-crossing-time density and for the first-crossing-time distribution function. In the case of the distribution function, the bound is improved by use of processes comparison based on the usual stochastic order. The special case of constant jumps driven by a Poisson process is thoroughly discussed.

1 Introduction In a variety of applied contexts a relevant role is played by jump-diffusion processes, i.e. by diffusion processes to which jumps occurring at random times are superimposed. Indeed, such processes are for example invoked for the description of stochastic neuronal activity (see Giraudo and Sacerdote [13] and Giraudo *et al.* [15]), of complex queueing systems (see Perry and Stadje [17]), of random assets in mathematical finance (see Ball and Roma [4]), of surplus of insurance companies in ruin theory (see Gerber and Landry [11]), of acto-myosin interaction in biomathematics (see Buonocore *et al.* [6]). Despite the relevance of the first-crossing-time (FCT) problem for jump-diffusion processes in such contexts, only few analytical results are available on their probability density function (pdf), even in the case of very simple boundaries. The available results are mainly focused on equations involving the FCT moments (see, for instance, Abundo [2], Giraudo and Sacerdote [12], and Tuckwell [21]), which however seem to be hardly manageable for practical purposes. Other results concerning bounds, obtained by use of Laplace transform, are limited to upper and lower bounds for the ruin probability in a jump-diffusion process involved in a risk model perturbed by Brownian motion (see Yin and Chiu [22]), and for the mean and the variance of the hitting time in certain jump-diffusion processes (see Schäl [19]). Analytical results on FCT pdf's are also very rare (see Kou and Wang [16], where explicit solutions of the Laplace transform of the distributions of the first-crossing times are disclosed for a Brownian motion perturbed by double exponentially distributed jumps). Hence, as a viable alternative, efficient algorithms have been devised in order to evaluate FCT densities (cf. the recent contributions by Atiya and Metwally [3], and by Di Crescenzo *et al.* [7]).

In order to evaluate the performance of simulation algorithms, one needs to come up with some sample cases in which FCT densities and distribution functions obtained by simulation can be compared with the corresponding bounds analytically determined. Hence, in the present paper lower bounds for FCT densities and distribution functions will be determined for jump-diffusion processes based on the Wiener process in the presence of a constant boundary. Our results are based on a sample-path analysis of the jump-diffusion process and on some specific features of the underlying Wiener process, such as the space and time

2000 *Mathematics Subject Classification.* 60G40, 60J65, 60E15.

Key words and phrases. Jump-diffusion process, Wiener process, first-crossing time, usual stochastic order.

homogeneity and the availability of a closed form of the FCT density through a constant boundary. It must be pointed out that our approach is at all different from that of Bischoff and Hashorva [5], where the Cameron-Martin-Girsanov formula is exploited to obtain a lower bound for the boundary crossing probability of Brownian bridge with trend.

In Section 2 we formally describe the jump-Wiener model, that consists of the superposition of a Wiener process and of a jump process with generally distributed jumps occurring at the occurrences of a counting process. The FCT problem for such process through a constant boundary is then addressed and a lower bound for the FCT pdf is then determined in Section 3. Use of such a bound is then made for the special case when upward and downward jumps have constant amplitudes and occur according to a Poisson process. Section 4 presents a lower bound for the FCT cumulative distribution function (cdf). The special case when only upward constant jumps are allowed, and are separed by random times having exponential distribution is thoroughly investigated. In this case the bound is improved by making use of a technique based on the comparison of FCT's by the "usual stochastic order". Finally, in Section 5 some remarks on the computational aspects are given.

2 FCT problem for a jump-diffusion process Let $\{X(t)\}_{t\geq 0}$ be a jump-diffusion process defined by

$$X(t) = W(t) + Y(t) \tag{1}$$

where $\{W(t)\}_{t\geq 0}$ and $\{Y(t)\}_{t\geq 0}$ are independent stochastic processes and

(i) $\{W(t)\}_{t\geq 0}$ is a Wiener process with drift $\mu \in \mathbb{R}$, and variance $\sigma^2 \in (0, +\infty)$ per unit time, starting at $W(0) = x_0$;

(ii) $\{Y(t)\}_{t\geq 0}$ is a jump process such that $Y(0) = 0$ and $Y(t) = \sum_{i=1}^{N(t)} J_i$, $t > 0$, with $Y(t) = 0$ when $N(t) = 0$ and $J_1, J_2, \ldots$ real-valued i.i.d. r.v.'s such that $\mathrm{P}\{J_i = 0\} < 1$ for all i. The cdf of J_i will be denoted by $F_J(x)$ and the survival function by $\overline{F}_J(x) = 1 - F_J(x)$. By $\{N(t)\}_{t\geq 0}$ we denote a counting process independent of $\{J_1, J_2, \ldots\}$ and characterized by i.i.d. absolutely continuous positive renewals $R_1, R_2, \ldots$ having pdf $f_R(x)$, cdf $F_R(x)$ and survival function $\overline{F}_R(x) = 1 - F_R(x)$.

Let

$$f(x, t \,|\, x_0) = \frac{\partial}{\partial x} \mathrm{P}\left\{X(t) \leq x \,|\, X(0) = x_0\right\}, \qquad t > 0,$$

be the conditional pdf of $\{X(t)\}$. Since

$$\mathrm{P}\left\{W(t) + \sum_{i=1}^{k} J_i \in \mathrm{d}x \,\middle|\, W(0) = x_0\right\} = \int_{\mathbb{R}} f_W(x-u, t \,|\, x_0)\,\mathrm{d}F^{[k]}(u)\,\mathrm{d}x,$$

where $F^{[k]}(u)$ denotes the cdf of $\sum_{i=1}^{k} J_i$ and

$$f_W(x, t \,|\, x_0) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left\{-\frac{(x - x_0 - \mu t)^2}{2\sigma^2 t}\right\}$$

is the pdf of $W(t)$, from the assumptions on $\{X(t)\}$ we have:

$$f(x, t \,|\, x_0) = \sum_{k=0}^{\infty} P[N(t) = k] \int_{\mathbb{R}} f_W(x-u, t \,|\, x_0)\,\mathrm{d}F^{[k]}(u). \tag{2}$$

Denoting by $\phi_N(u,t)$ the probability generating function of $N(t)$ and by $\psi_J(s)$ the moment generating function (m.g.f.) of J_i, the m.g.f. of $\{X(t)\}$ is given by

$$E\left[e^{sX(t)}\,\Big|\,X(0)=x_0\right] = \phi_N[\psi_J(s),t]\,\exp\left\{(x_0+\mu t)s+\frac{\sigma^2 t}{2}s^2\right\}, \tag{3}$$

and

$$E[X(t)\,|\,X(0)=x_0] \;=\; x_0+\mu t+E(J_1)\,E[N(t)], \tag{4}$$
$$\mathrm{Var}[X(t)\,|\,X(0)=x_0] \;=\; \sigma^2 t+\mathrm{Var}(J_1)\,E[N(t)]+E^2(J_1)\;\mathrm{Var}[N(t)]. \tag{5}$$

Assuming $x_0 < S$, hereafter we shall discuss the FCT problem through a constant boundary S for the jump-diffusion process defined in (1). We denote by

$$T_X = \inf\{t\geq 0 : X(t)\geq S\}, \qquad \mathrm{P}\{X(0)=x_0\}=1, \tag{6}$$

the FCT of $\{X(t)\}$ through S from below, and by

$$g_X(S,t\,|\,x_0) = \frac{\partial}{\partial t}\,\mathrm{P}\left\{T_X\leq t\,|\,X(0)=x_0\right\}, \qquad t>0 \tag{7}$$

its pdf. The corresponding cdf will be denoted by

$$G_X(S,t\,|\,x_0) = \mathrm{P}(T_X\leq t\,|\,X(0)=x_0), \qquad t\geq 0. \tag{8}$$

To determine the ultimate FCT probability and the FCT moments one is usually asked to solve appropriate integro-differential equations (see Abundo [2] and Tuckwell [21]). The determination of closed-form expressions for the FCT pdf and cdf is instead a harder problem, because so far no analytical method appear to be available thus for. Hence, in the following sections we shall confine our investigation to determining useful lower bounds for density (7) and cdf (8).

Let T_W denote the FCT from below of Wiener process $\{W(t)\}$ from $x_0 < S$ to the constant boundary S. Some well-known results on T_W that will be used later are recalled hereafter:

- FCT cdf of $\{W(t)\}$ through S:

$$\begin{aligned} G_W(S,t\,|\,x_0) \;&=\; \mathrm{P}\{T_W\leq t\,|\,W(0)=x_0\} \\ &=\; \Phi\left(-\frac{S-x_0-\mu t}{\sqrt{\sigma^2 t}}\right)+\exp\left(2\mu\frac{S-x_0}{\sigma^2}\right)\,\Phi\left(-\frac{S-x_0+\mu t}{\sqrt{\sigma^2 t}}\right), \end{aligned} \tag{9}$$

where

$$\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-x^2/2}\mathrm{d}x, \qquad z\in\mathbb{R}; \tag{10}$$

- FCT pdf of $\{W(t)\}$ through S:

$$g_W(S,t\,|\,x_0) = \frac{\partial}{\partial t}\,\mathrm{P}\{T_W\leq t\,|\,W(0)=x_0\} = \frac{S-x_0}{\sqrt{2\pi\sigma^2 t^3}}\exp\left\{-\frac{(S-x_0-\mu t)^2}{2\sigma^2 t}\right\}; \tag{11}$$

- S-avoiding transition pdf of $\{W(t)\}$:

$$\begin{aligned} \alpha_W(x,t\,|\,x_0) \;&=\; \frac{\partial}{\partial x}\,\mathrm{P}\{W(t)\leq x, T_W>t\,|\,W(0)=x_0\} \\ &=\; \frac{1}{\sqrt{2\pi\sigma^2 t}}\,\exp\left\{-\frac{(x-x_0-\mu t)^2}{2\sigma^2 t}\right\}\left[1-\exp\left\{-\frac{2(S-x)(S-x_0)}{\sigma^2 t}\right\}\right]. \end{aligned} \tag{12}$$

3 Lower bound for FCT pdf In order to obtain the preannounced lower bound for pdf (7) let us note that, for all $t > 0$, event $\{T_X \in (t, t + \mathrm{d}t)\}$ can be decomposed into the following three mutually exclusive events:
(i) the first jump occurs after t and the first crossing through S occurs in $(t, t+\mathrm{d}t)$ due to the diffusive component of $\{X(t)\}$;
(ii) in $(0, t)$ the diffusive component of $\{X(t)\}$ does not cross the boundary, the first jump occurs in $(t, t + \mathrm{d}t)$ and it causes the first crossing;
(iii) the first jump occurs at time $\theta \in (0, t)$ and it does not cause the first crossing, the diffusive component of $\{X(t)\}$ having not crossed the boundary in $(0, \theta)$; the first crossing finally occurs in $(t, t + \mathrm{d}t)$.

Hence, for all $t > 0$ the following equation holds:

$$
\begin{aligned}
(13)\ g_X(S,t\,|\,x_0) \;=\; & \overline{F}_R(t)\, g_W(S,t\,|\,x_0) + f_R(t)\int_{-\infty}^{S} \alpha_W(x,t\,|\,x_0)\,\overline{F}_J(S-x)\,\mathrm{d}x \\
& + \int_0^t \mathrm{d}F_R(\theta)\int_{-\infty}^{S}\alpha_W(x,\theta\,|\,x_0)\int_{-\infty}^{S-x} g_X(S,t-\theta\,|\,x+u)\,\mathrm{d}F_J(u)\,\mathrm{d}x.
\end{aligned}
$$

The formal proof of Eq. (13) has been given by Giraudo and Sacerdote [14] in the special case of jumps of constant size at the occurrence of a Poisson process, aiming to find out conditions under which the FCT density becomes multimodal (see also Sacerdote and Sirovich [18] on this topic). As an immediate consequence of (13) a lower bound for the FCT pdf of $\{X(t)\}$ can be obtained. Indeed, for all $t > 0$ the following inequality holds:

$$
\begin{aligned}
(14)\quad g_X(S,t\,|\,x_0) \geq\; & \overline{F}_R(t)\, g_W(S,t\,|\,x_0) + f_R(t)\int_{-\infty}^{S} \alpha_W(x,t\,|\,x_0)\,\overline{F}_J(S-x)\,\mathrm{d}x \\
& + \int_0^t \overline{F}_R(t-\theta)\,\mathrm{d}F_R(\theta)\int_{-\infty}^{S}\alpha_W(x,\theta\,|\,x_0)\int_{-\infty}^{S-x} g_W(S,t-\theta\,|\,x+u)\,\mathrm{d}F_J(u)\,\mathrm{d}x.
\end{aligned}
$$

Note that a tighter lower bound can be obtained by repeated substitutions of g_X in the last term of the right-hand-side of (13). However, this would include many terms involving progressively high-order integrals, unsuitable for computational purposes.

3.1 A special case Let us now study a special case of model (1). First of all, we assume that the jumps are separated by i.i.d. exponential random times R_i, i.e. restrict our attention to the case when $\{N(t)\}$ is a Poisson process whose parameter will be denoted by λ. In this case pdf $f(x, t\,|\,x_0)$ is solution of the following integro-differential equation (see Buonocore *et al.* [6] or Di Crescenzo *et al.* [9]):

$$
\frac{\partial f}{\partial t} = -\lambda f - \mu\frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} + \lambda\int_{\mathbb{R}} f(x-y,t\,|\,x_0)\,\mathrm{d}F_J(y),
$$

which clearly reduces to the Fokker-Plank equation of a Wiener process if $\lambda = 0$. Moreover, let us assume that the jumps have positive or negative constant amplitude. In other words, the random jumps J_i are distributed as

$$
(15)\qquad J_i = \begin{cases} a & \text{w.p. } \eta \\ -b & \text{w.p. } 1-\eta, \end{cases}
$$

with $0 < \eta < 1$, $a > 0$ and $b > 0$. Hence, $Y(t)$ can be expressed as

$$
Y(t) = a\,N_1(t) - b\,N_2(t), \qquad t > 0
$$

where $\{N_1(t)\}_{t\geq 0}$ and $\{N_2(t)\}_{t\geq 0}$ are independent Poisson processes with rate $\eta\lambda$ and $(1-\eta)\lambda$, respectively. ¿From (2), for $x \in \mathbb{R}$ and $t \geq 0$ we obtain

(16)
$$f(x,t\,|\,x_0) = \frac{e^{-\lambda t}}{\sqrt{2\pi\sigma^2 t}} \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{(\eta\lambda t)^j}{j!}\frac{((1-\eta)\lambda t)^k}{k!} \exp\left\{-\frac{(x+aj-bk-x_0-\mu t)^2}{2\sigma^2 t}\right\}.$$

Since under the present assumptions $\psi_J(s) = \eta\, e^{as} + (1-\eta)\, e^{-bs}$ and $\phi_N(u,t) = e^{-\lambda t(1-u)}$, from (3) the m.g.f. of $\{X(t)\}$ for $s \in \mathbb{R}$ becomes

$$E\left[e^{sX(t)}\,\Big|\, X(0)=x_0\right] = \exp\left\{-\lambda t\left[1-\eta\, e^{as} - (1-\eta)\, e^{-bs}\right] + (x_0+\mu t)s + \frac{\sigma^2 t}{2}s^2\right\}.$$

As already pointed out in Di Crescenzo *et al.* [7], from (4) and (5) we have

$$\begin{aligned} E[X(t)\,|\,X(0)=x_0] &= x_0+\mu t + [\eta a - (1-\eta)b]\lambda t,\\ \mathrm{Var}[X(t)\,|\,X(0)=x_0] &= \sigma^2 t + [\eta a^2 + (1-\eta)b^2]\lambda t. \end{aligned}$$

We stress that the lower bound given in (14) can now be explicitly evaluated. Indeed, under the present assumptions for all $t>0$ we have:

$$g_X(S,t\,|\,x_0) \geq g_\ell(S,t\,|\,x_0),$$

with

(17)
$$\begin{aligned} g_\ell(S,t\,|\,x_0) \;:=\; e^{-\lambda t}\Bigg\{ & g_W(S,t\,|\,x_0) + \eta\lambda \int_{S-a}^{S} \alpha_W(x,t\,|\,x_0)\,\mathrm{d}x \\ &+\; \eta\lambda \int_0^t \mathrm{d}\theta \int_{-\infty}^{S-a} \alpha_W(x,\theta\,|\,x_0)\, g_W(S,t-\theta\,|\,x+a)\,\mathrm{d}x \\ &+\; (1-\eta)\lambda \int_0^t \mathrm{d}\theta \int_{-\infty}^{S} \alpha_W(x,\theta\,|\,x_0)\, g_W(S,t-\theta\,|\,x-b)\,\mathrm{d}x \Bigg\}, \end{aligned}$$

where $g_W(S,t\,|\,x_0)$ is given in (11) and

$$\begin{aligned} \int_{S-a}^{S} \alpha_W(x,t\,|\,x_0)\,\mathrm{d}x = {} & \Phi\left(\frac{S-x_0-\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{S-a-x_0-\mu t}{\sigma\sqrt{t}}\right) \\ & - \exp\left\{\frac{2\mu}{\sigma^2}(S-x_0)\right\}\left[\Phi\left(-\frac{S-x_0+\mu t}{\sigma\sqrt{t}}\right) - \Phi\left(-\frac{S+a-x_0+\mu t}{\sigma\sqrt{t}}\right)\right], \end{aligned}$$

with $\Phi(z)$ defined in (10). The right-hand-side of (17) can be numerically evaluated. Indeed, making use of identity

$$\int_0^\infty (z-\delta) e^{-(az^2+bz+\gamma)}\mathrm{d}z = \frac{e^{-\gamma}}{2a}\left\{1 - 2\sqrt{a\pi}\left(\frac{b}{2a}+\delta\right) e^{\frac{b^2}{4a}}\left[1-\Phi\left(\frac{b}{\sqrt{2a}}\right)\right]\right\}, \qquad a>0,$$

following from Eq. 7.4.2 of Abramowitz and Stegun [1], one obtains:

$$\begin{aligned} \int_{-\infty}^{S-m} \alpha_W(x,\theta\,|\,x_0)\, g_W(S,t-\theta\,|\,x+u)\,\mathrm{d}x \;=\; & \frac{1}{2\pi t}\sqrt{\frac{\theta}{t-\theta}} \exp\left\{-\frac{(S-u-x_0-\mu t)^2}{2\sigma^2 t}\right\} \\ & \times\; \left\{A_-(\theta,u) - \exp\left[-\frac{2(S-x_0)u}{\sigma^2 t}\right] A_+(\theta,u)\right\}, \end{aligned}$$

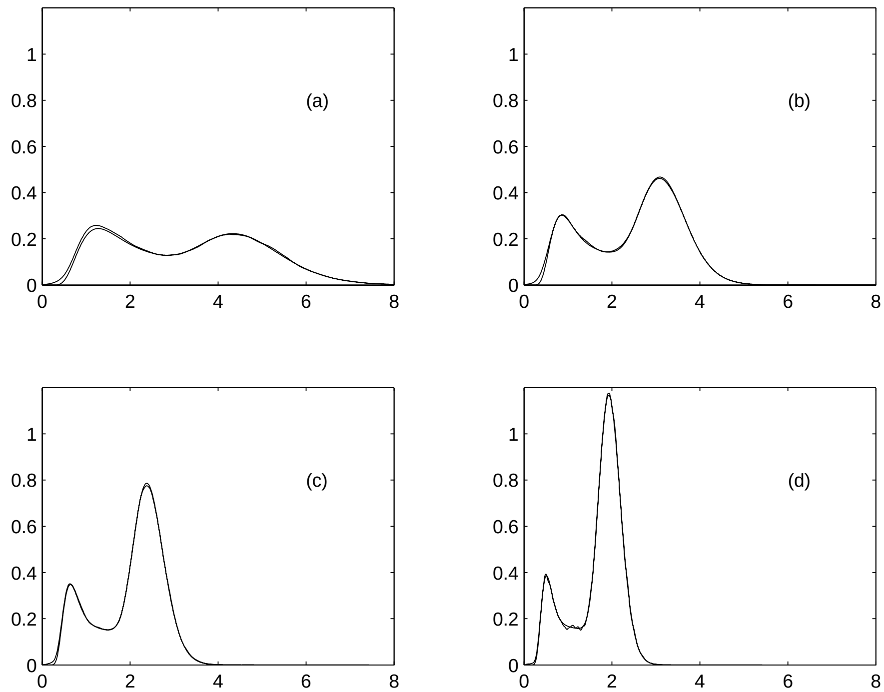

Figure 1: Estimated FCT density $\widehat{g}_X(S,t\,|\,x_0)$ for the Wiener process with constant Poisson-paced jumps for $a=b=3.75$, and lower bound $g_\ell(S,t\,|\,x_0)$, with $x_0=0$, $S=5$, $\lambda=0.2$, $\eta=1$, $\sigma^2=0.2$ and (a) $\mu=1.0$, (b) $\mu=1.5$, (c) $\mu=2.0$, (d) $\mu=2.5$.

where $m=\max\{u,0\}$ and

$$\begin{aligned} A_\pm(\theta,u) &= \exp\left\{-\frac{[(m-u)\theta\pm(S-x_0\pm m)(t-\theta)]^2}{2\sigma^2\theta t(t-\theta)}\right\} \\ &\qquad \mp\sqrt{\frac{2\pi(t-\theta)}{\sigma^2 t\theta}}\,(S-x_0\pm u)\left\{1-\Phi\left[\frac{(m-u)\theta\pm(S-x_0\pm m)(t-\theta)}{\sqrt{\sigma^2\theta t(t-\theta)}}\right]\right\}. \end{aligned}$$

Figures 1 and 2 show certain multimodal estimates of $g_X(S,t\,|\,x_0)$ obtained by means of the simulation procedure described in Di Crescenzo *et al.* [7], together with the lower bound given in (17). The latter appears to be very close to the estimated pdf in various cases; the goodness of the approximation is discussed in Section 5.

4 Lower bound for FCT cdf In this section we obtain a lower bound for the FCT distribution function (8). Similarly to the case of the FCT density we note that event $\{T_X\le t\}$, $t>0$, can be decomposed into three mutually exclusive events:
(i) the first jump occurs after t and the first crossing through S occurs in $(0,t]$ due to the diffusive component of $\{X(t)\}$;
(ii) the first jump occurs at time $\theta\in(0,t)$ and it causes the first crossing through S;
(iii) the first jump occurs at time $\theta\in(0,t)$ and it does not cause the first crossing, the diffusive component of $\{X(t)\}$ having not crossed the boundary in $(0,\theta)$; the first crossing finally occurs in $(\theta,t]$.

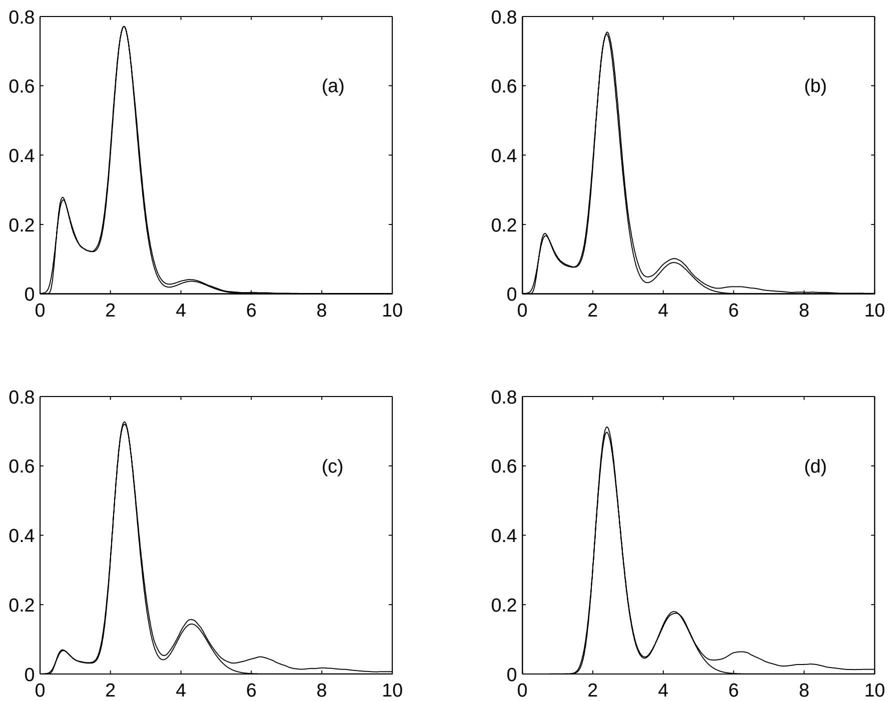

Figure 2: Estimated FCT density $\widehat{g}_X(S,t\,|\,x_0)$ for the Wiener process with constant Poisson-paced jumps, and lower bound $g_\ell(S,t\,|\,x_0)$, for $\mu = 2.0$ and (a) $\eta = 0.8$, (b) $\eta = 0.5$, (c) $\eta = 0.2$, (d) $\eta = 0$. All other parameters are chosen as in Figure 1.

The following equation thus holds for all $t > 0$:

$$(18)\quad G_X(S,t\,|\,x_0) = \overline{F}_R(t)\,G_W(S,t\,|\,x_0) + \int_0^t \mathrm{d}F_R(\vartheta)\int_{-\infty}^S \alpha_W(x,\vartheta\,|\,x_0)\,\overline{F}_J(S-x)\,\mathrm{d}x + \int_0^t \mathrm{d}F_R(\vartheta)\int_{-\infty}^S \alpha_W(x,\vartheta\,|\,x_0)\,\mathrm{d}x\int_{-\infty}^{S-x} G_X(S,t-\vartheta\,|\,x+u)\,\mathrm{d}F_J(u).$$

Hence, after setting

$$(19)\quad B_X(S,t\,|\,x_0) := \overline{F}_R(t)\,G_W(S,t\,|\,x_0) + \int_0^t \mathrm{d}F_R(\vartheta)\int_{-\infty}^S \alpha_W(x,\vartheta\,|\,x_0)\,\overline{F}_J(S-x)\,\mathrm{d}x + \int_0^t \overline{F}_R(t-\vartheta)\,\mathrm{d}F_R(\vartheta)\int_{-\infty}^S \alpha_W(x,\vartheta\,|\,x_0)\,\mathrm{d}x\int_{-\infty}^{S-x} G_W(S,t-\vartheta\,|\,x+u)\,\mathrm{d}F_J(u)$$

it is not hard to see that from (18) the following lower bound for the FCT distribution function is obtained:

$$(20)\qquad G_X(S,t\,|\,x_0) \ge B_X(S,t\,|\,x_0), \qquad t > 0.$$

4.1 An improved bound Bound (20) holds in the case of generally distributed jumps J_i and renewals R_i. By making use of a method already adopted in the proof of Theorem 3.3 of Di Crescenzo and Pellerey [10], hereafter we see how such bound can be improved by specifying the distributions F_J and F_R. Let us assume that

(i) the random times separating consecutive jumps are exponentially distributed, with $\overline{F}_R(t) = e^{-\lambda t}$, $t \geq 0$, and $\lambda > 0$, and that

(ii) the jumps have fixed amplitude a.s., with $F_J(x) = \mathbf{1}_{\{x \geq a\}}$ and $a > 0$, which corresponds to assumption (15) with $b = 0$ and $\eta = 1$.

In the following we shall stochastically compare $\{X(t)\}$ with another Wiener process with jumps $\{\widetilde{X}_n(t)\}_{t \geq 0}$, which is driven by the same diffusive component of $\{X(t)\}$. We formally have

$$\widetilde{X}_n(t) = W(t) + \widetilde{Y}_n(t), \qquad t \geq 0, \tag{21}$$

where $\{W(t)\}$ and $\{\widetilde{Y}_n(t)\}$ are independent stochastic processes, $\{W(t)\}$ is the same process appearing in the right-hand-side of (1), $\{\widetilde{Y}_n(t)\}$ is a jump process such that $\widetilde{Y}_n(0) = 0$ and $\widetilde{Y}_n(t) = \sum_{i=1}^{\widetilde{N}(t)} \widetilde{J}_i$, $t > 0$, with $\widetilde{Y}_n(t) = 0$ when $\widetilde{N}(t) = 0$. Moreover, $\widetilde{J}_1, \widetilde{J}_2, \ldots$ are real-valued i.i.d. r.v.'s degenerating at na, with $a > 0$ and n a fixed positive integer, i.e. possessing cdf $F_{\widetilde{J}}(x) = \mathbf{1}_{\{x \geq na\}}$. Furthermore, $\{\widetilde{N}(t)\}_{t \geq 0}$ is a counting process independent of $\{\widetilde{J}_1, \widetilde{J}_2, \ldots\}$ and characterized by i.i.d. Erlang-distributed renewal times $\widetilde{R}_1, \widetilde{R}_2, \ldots$ having survival function $\overline{F}_{\widetilde{R}}(t) = e^{-\lambda t} \sum_{j=0}^{n-1} \frac{(\lambda t)^j}{j!}$, $t \geq 0$, with $\lambda > 0$. We stress that processes $\{X(t)\}$ and $\{\widetilde{X}_n(t)\}$ share the two parameters λ and a.

Making use of a customary tecnique based on the constructions of "clone" processes (see, for instance, Theorem 3.3 of Di Crescenzo *et al.* [8]) and recalling (1) and (21), it is not hard to prove that

$$\widetilde{X}_n(t) \leq_{st} X(t) \qquad \text{for all } n \geq 1, \tag{22}$$

where $\leq_{st}$ denotes the usual stochastic order (see Section 1.A of Shaked and Shanthikumar [20]). An immediate consequence of (22) is that the FCT's of those processes are stochastically ordered too, i.e.

$$T_X \leq_{st} T_{\widetilde{X}_n} \qquad \text{for all } n \geq 1. \tag{23}$$

In other terms,

$$G_X(S,t\,|\,x_0) \geq G_{\widetilde{X}_n}(S,t\,|\,x_0) \qquad \text{for all } n \geq 1 \text{ and } t \geq 0, \tag{24}$$

where $G_{\widetilde{X}_n}(S,t\,|\,x_0)$ is the FCT cdf of $\{\widetilde{X}_n(t)\}$. Similarly to (20) the following bound holds for all $n \geq 1$:

$$G_{\widetilde{X}_n}(S,t\,|\,x_0) \geq B_{\widetilde{X}_n}(S,t\,|\,x_0), \qquad t > 0, \tag{25}$$

where

$$\begin{aligned} B_{\widetilde{X}_n}(S,t\,|\,x_0) \;:=\; & e^{-\lambda t} \sum_{j=0}^{n-1} \frac{(\lambda t)^j}{j!} \, G_W(S,t\,|\,x_0) \\ & + \int_0^t \lambda e^{-\lambda\vartheta} \frac{(\lambda\vartheta)^{n-1}}{(n-1)!} \mathrm{d}\vartheta \int_{S-na}^{S} \alpha_W(x,\vartheta\,|\,x_0)\,\mathrm{d}x \\ & + \int_0^t e^{-\lambda(t-\vartheta)} \sum_{j=0}^{n-1} \frac{[\lambda(t-\vartheta)]^j}{j!} \lambda e^{-\lambda\vartheta} \frac{(\lambda\vartheta)^{n-1}}{(n-1)!} \mathrm{d}\vartheta \\ & \times \int_{-\infty}^{S-na} \alpha_W(x,\vartheta\,|\,x_0)\, G_W(S,t-\vartheta\,|\,x+na)\,\mathrm{d}x, \end{aligned} \tag{26}$$

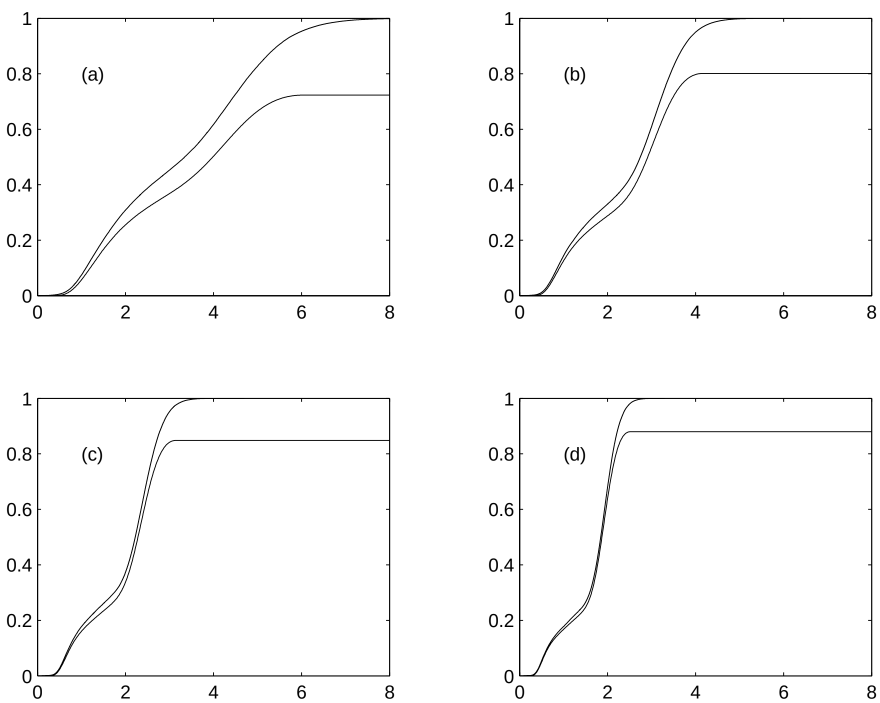

Figure 3: Estimated FCT distribution function $\widehat{G}_X(S,t\,|\,x_0)$ and lower bound $G_\ell(S,t\,|\,x_0)$ for the same cases treated in Figure 1.

with G_W and α_W defined in (9) and (12), respectively. In conclusion, by (24) and (25) we obtain

$$G_X(S,t\,|\,x_0) \geq \sup_{n\geq 1} B_{\widetilde{X}_n}(S,t\,|\,x_0), \qquad t \geq 0. \tag{27}$$

Since (20) provides a bound of type $G_X(S,t\,|\,x_0) \geq B_{\widetilde{X}_1}(S,t\,|\,x_0)$, Eq. (27) yields a better bound. Furthermore, the right-hand-side of (27) is not necessarily a distribution function, so that we can improve the bound as follows:

$$G_X(S,t\,|\,x_0) \geq G_\ell(S,t\,|\,x_0) \equiv \max_{0\leq\tau\leq t} \sup_{n\geq 1} B_{\widetilde{X}_n}(S,\tau\,|\,x_0), \qquad t \geq 0. \tag{28}$$

Figure 3 shows estimates of FCT distribution function $\widehat{G}_X(S,t\,|\,x_0)$ obtained from simulations perfomed by means of the procedure given in Di Crescenzo *et al.* [7], together with the respective lower bound (28). We stress that the lower bounds given in Figure 3 shows a case in which, for all $\tau \geq 0$

$$\sup_{n\geq 1} B_{\widetilde{X}_n}(S,\tau\,|\,x_0) = B_{\widetilde{X}_1}(S,\tau\,|\,x_0). \tag{29}$$

Note that (29) holds because relation $S - na > 0$ is satisfied only if $n = 1$. Instead, the angular points appearing in the lower bound on the right of Figure 4 show a case in which

$$\sup_{n\geq 1} B_{\widetilde{X}_n}(S,\tau\,|\,x_0) = B_{\widetilde{X}_k}(S,\tau\,|\,x_0),$$

for different values of k as τ varies.

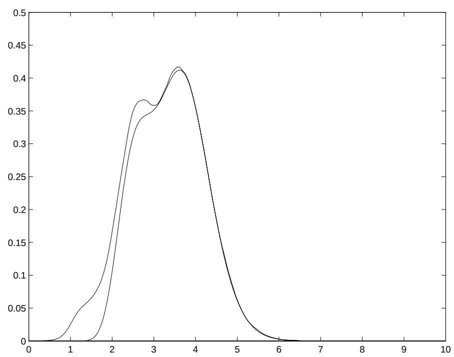

Figure 4: On the left: Estimated FCT density $\widehat{g}_X(S,t\,|\,x_0)$ and lower bound $g_\ell(S,t\,|\,x_0)$ for $a = 2$, $b = 0$, $x_0 = 0$, $S = 6$, $\lambda = 0.2$, $\eta = 1$, $\mu = 1.5$ and $\sigma^2 = 0.2$. On the right: The corresponding estimated distribution function $\widehat{G}_X(S,t\,|\,x_0)$ and lower bound $G_\ell(S,t\,|\,x_0)$.

5 Remarks on computational aspects Let us now point out some computational features related to the evaluation of bound (28). The last integral in (26), i.e.

$$\int_{-\infty}^{S-na} \alpha_W(x,\vartheta\,|\,x_0)\, G_W(S,t-\vartheta\,|\,x+na)\,\mathrm{d}x,$$

can be expressed in terms of integrals of the form

$$I = \int_0^{+\infty} e^{-a(x-b)^2}\,\Phi(cx+d)\,\mathrm{d}x, \tag{30}$$

with $a > 0$, with $b, c, d \in \mathbb{R}$ and Φ given in (10). Unfortunately no closed forms are available for (30), so that we have been forced to numerically evaluate it by splitting the integration domain as follows:

$$I = \int_0^{b} e^{-a(x-b)^2}\,\Phi(cx+d)\,\mathrm{d}x + \int_b^{+\infty} e^{-a(x-b)^2}\,\Phi(cx+d)\,\mathrm{d}x.$$

Then, transforming the second integral we obtain:

$$I = \int_0^{b} e^{-a(x-b)^2}\,\Phi(cx+d)\,\mathrm{d}x + \frac{1}{2\sqrt{a}}\int_0^{+\infty} x^{1/2}e^{-x}\,\Phi\left(c\sqrt{\frac{x}{a}} + c\,b + d\right)\mathrm{d}x. \tag{31}$$

A Gauss-Legendre quadrature rule has been applied to the first integral by choosing 16 abscissas, after an *ad hoc* transformation of the integration domain in $[-1, 1]$, whereas a generalized Gauss-Laguerre quadrature rule has been applied to the second integral by choosing 16 abscissas. Note that the latter quadrature formula is exact for weighting functions as $w(x) = x^\beta e^{-x}$, with $\beta \in \mathbb{R}$, which includes the case of the second integral in (31).

In order to evaluate the goodness of lower bound (17) we shall now adopt the following "measure of closeness":

$$\mathcal{E} = h\sum_{j=1}^{n}\left[\widehat{g}_X(S,jh\,|\,x_0) - g_\ell(S,jh\,|\,x_0)\right], \tag{32}$$

$\mu \setminus \eta$	1	0.8	0.5	0.2	0.0
1.0	3.9358e-02	1.1393e-01	2.4054e-01	3.6791e-01	4.5211e-01
1.5	2.1059e-02	6.5103e-02	1.4572e-01	2.2688e-01	2.8409e-01
2.0	1.7266e-02	4.4515e-02	9.6393e-02	1.5005e-01	1.9656e-01
2.5	1.7283e-02	3.4975e-02	7.0264e-02	1.1009e-01	1.4486e-01

Table 1: Values of $\mathcal{E}$ for various choices of μ and η.

where h is a discretization step and n is the minimum integer such that

$$h \sum_{j=1}^{n} \widehat{g}_X(S, jh \,|\, x_0) \geq 0.99.$$

Differently stated, (32) measures how the bound g_ℓ is close to $\widehat{g}_X$ by the multiples of h up to the 0.99-percentile. For $h = 0.01$, Table 1 shows values of $\mathcal{E}$, where $\widehat{g}_X$ is an estimate of g_X, whereas g_ℓ is the lower bound given in (17). It is evident that $\mathcal{E}$ increases if μ decreases or if η decreases.

We finally point out that throughout the paper the estimated functions $\widehat{g}_X$ and $\widehat{G}_X$ have been obtained by use of 10^6 simulated FCT's. The estimated pdf $\widehat{g}_X$ has been built by adopting an Epanechnikov kernel estimator, while $\widehat{G}_X$ derives by use of a Kaplan-Meier estimator.

Acknowledgments This work has been performed under partial support by MIUR (PRIN 2005), by G.N.C.S.-INdAM and by Campania Region.

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