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Morphisms of 1-motives Defined by Line Bundles

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Let S be a normal base scheme. The aim of this paper is to study the line bundles on 1-motives defined over S. We first compute a *dévissage* of the Picard group of a 1-motive M according to the weight filtration of M. This dévissage allows us to associate, to each line bundle $\mathcal{L}$ on M, a linear morphism $\varphi_{\mathcal{L}} : M \to M^*$ from M to its Cartier dual. This yields a group homomorphism $\Phi : \mathrm{Pic}(M)/\mathrm{Pic}(S) \to \mathrm{Hom}(M, M^*)$. We also prove the *Theorem of the Cube* for 1-motives, which furnishes another construction of the group homomorphism $\Phi : \mathrm{Pic}(M)/\mathrm{Pic}(S) \to \mathrm{Hom}(M, M^*)$. Finally, we prove that these two independent constructions of linear morphisms $M \to M^*$ using line bundles on M coincide. However, the 1st construction, involving the dévissage of $\mathrm{Pic}(M)$, is more explicit and geometric and it furnishes the *motivic origin* of some linear morphisms between 1-motives. The 2nd construction, involving the Theorem of the Cube, is more abstract but also more enlightening.

1 Introduction

Let A be an abelian variety over a field k and let $A^* = \mathrm{Pic}^0_{A/k}$ be its dual. It is a classical fact that if L is a line bundle on A, then the morphism $\varphi_L : A \to A^*$, defined by $\varphi_L(a) = \mu_a^* L \otimes L^{-1}$, where $\mu_a : A \to A$ is the translation by a, is a group

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homomorphism. This is an easy consequence of the Theorem of the Square, which itself is a consequence of the Theorem of the Cube. We have then a functorial homomorphism $\Phi : \mathrm{Pic}(A) \to \mathrm{Hom}(A, A^*)$, which is a key result in the basic foundations of the theory of abelian varieties. In [8, Section 10] Deligne introduced the notion of 1-motives, which can be seen as a generalization of abelian schemes. Let S be a scheme. A 1-motive $M = (X, A, T, G, u)$ defined over S is a complex $[u : X \to G]$ of commutative S-group schemes concentrated in degree 0 and -1, where

- X is an S-group scheme that is locally for the étale topology a constant group scheme defined by a finitely generated free $\mathbb{Z}$-module,
- G is an extension of an abelian S-scheme A by an S-torus T,
- $u : X \to G$ is a morphism of S-group schemes.

A linear morphism of 1-motives is a morphism of complexes of S-group schemes. We will denote by

$$\mathrm{Hom}(M_1, M_2)$$

the group of linear morphisms from M_1 to M_2. In this paper we study line bundles on a 1-motive M and their relation to linear morphisms from M to its Cartier dual M^*.

Our aim is to answer the following natural questions:

(1) If M is a 1-motive over S, is it possible to construct a functorial homomorphism $\Phi : \mathrm{Pic}(M) \to \mathrm{Hom}(M, M^*)$ that extends the known one for abelian schemes?
(2) Is there an analog of the Theorem of the Cube for 1-motives?

We give a positive answer to both questions if the base scheme S is normal (for comments on what happens if the base scheme S is not normal, see Remark 7.4).

The notion of line bundle on a 1-motive M over S already implicitly exists in the literature. Actually, in [15, p. 64] Mumford introduced a natural notion of line bundles on an arbitrary S-stack $\mathcal{X}$ (see Definition 3.1). Since to any 1-motive M over S we can associate by [7, Section 1.4] a commutative group stack $\mathrm{st}(M)$, we can define the category $\mathrm{PIC}(M)$ of line bundles on M as the category of line bundles on $\mathrm{st}(M)$. The *Picard group of M*, denoted by $\mathrm{Pic}(M)$, is the group of isomorphism classes of line bundles on $\mathrm{st}(M)$ (see Definition 3.2).

The stack $\mathrm{st}(M)$ associated to a 1-motive $M = [X \overset{u}{\to} G]$ is isomorphic to the quotient stack $[G/X]$, where X acts on G by translations via u. Under this identification, the inclusion of 1-motives $\iota : G \to M$ corresponds to the projection map $G \to [G/X]$, which is étale and surjective. We can then describe line bundles on M as couples

$$(L, \delta),$$

where L is a line bundle on G and δ is a descent datum for L with respect to the covering $\iota : G \to [G/X]$ (see Section 3, after Lemma 3.3). Throughout this paper, we will use this description of line bundles on M, which amounts to say that a line bundle on a 1-motive M is a line bundle on G endowed with an action of X that is compatible with the translation action of X on G.

The main result of our paper is the following theorem, which generalizes to 1-motives the classical homomorphism $\Phi : \mathrm{Pic}(A) \to \mathrm{Hom}(A, A^*)$ for abelian varieties.

Theorem 1.1. Let M be a 1-motive defined over scheme S. Assume that the toric part of M is trivial or that S is normal. Then there is a functorial homomorphism

$$\Phi : \mathrm{Pic}(M)/\mathrm{Pic}(S) \longrightarrow \mathrm{Hom}(M, M^*). \tag{1.1}$$

We actually provide two independent constructions of Φ:

(1) The 1st construction, given in Section 5, is the most explicit and geometric one. It is based on the "*dévissage*" of the Picard group of M, computed in Section 4, and on the explicit functorial description of the Cartier dual M^* of M in terms of extensions given in [8, (10.2.11)].

(2) The 2nd construction, given in Sections 6 and 7, is more abstract but also more enlightening. It works for a category that is a bit larger than 1-motives (see 7.1) and it also provides the fact that Φ is a group homomorphism. This construction relies on the "*Theorem of the Cube for 1-motives*" (Theorem 7.1), a result that we think is of independent interest, and on the description of the Cartier dual of a 1-motive in terms of commutative group stacks.

In Proposition 7.3 we prove that these two constructions coincide.

Dévissage of the Picard group of M: 1-motives are endowed with a weight filtration W_* defined by $\mathrm{W}_0(M) = M, \mathrm{W}_{-1}(M) = G, \mathrm{W}_{-2}(M) = T, \mathrm{W}_j(M) = 0$ for each $j \leq -3$. This weight filtration allows us to "*dévisser*" the Picard group of M, which is our 2nd main result. We will first describe the Picard group of G in terms of $\mathrm{Pic}(A)$ and $\mathrm{Pic}(T)$ using the 1st short exact sequence $0 \to T \xrightarrow{i} G \xrightarrow{\pi} A \to 0$ given by W_*. Consider the morphism

$$\xi : \mathrm{Hom}(T, \mathbb{G}_m) \to \mathrm{Pic}(A)$$

defined as follows: for any morphism of S-group schemes $\alpha : T \to \mathbb{G}_m$, $\xi(\alpha)$ is the image of the class $[\alpha_* G]$ of the pushdown of G via α under the inclusion $\mathrm{Ext}^1(A, \mathbb{G}_m) \hookrightarrow H^1(A, \mathbb{G}_m) = \mathrm{Pic}(A)$. At the beginning of Section 4 we will show the following:

Proposition 1.2. Assume the base scheme S to be normal. The following sequence of groups is exact:

$$0 \longrightarrow \mathrm{Hom}(G, \mathbb{G}_m) \xrightarrow{i^*} \mathrm{Hom}(T, \mathbb{G}_m) \xrightarrow{\xi} \frac{\mathrm{Pic}(A)}{\mathrm{Pic}(S)} \xrightarrow{\pi^*} \frac{\mathrm{Pic}(G)}{\mathrm{Pic}(S)} \xrightarrow{i^*} \frac{\mathrm{Pic}(T)}{\mathrm{Pic}(S)}.$$

The 2nd short exact sequence $0 \to G \xrightarrow{\iota} M \xrightarrow{\beta} X[1] \to 0$ given by the weight filtration W_* of M induces by pullback the sequence $\mathrm{Pic}(X[1]) \xrightarrow{\beta^*} \mathrm{Pic}(M) \xrightarrow{\iota^*} \mathrm{Pic}(G)$, which is not exact as we will see in Example 4.3, but which is nevertheless interesting since the kernel of the homomorphism $\iota^* : \mathrm{Pic}(M) \to \mathrm{Pic}(G)$ fits in a long exact sequence. In fact, at the end of Section 4 we will prove the following:

Proposition 1.3. Assume the base scheme S to be reduced. Then the kernel K of the homomorphism $\iota^* : \mathrm{Pic}(M) \to \mathrm{Pic}(G)$ fits in an exact sequence

$$\mathrm{Hom}(G, \mathbb{G}_m) \xrightarrow{\circ u} \mathrm{Hom}(X, \mathbb{G}_m) \xrightarrow{\beta^*} K \xrightarrow{\Theta} \Lambda \xrightarrow{\Psi} \Sigma.$$

Note that the group $\mathrm{Hom}(X, \mathbb{G}_m)$ in the above sequence identifies in a natural way with $\mathrm{Pic}(X[1])/\mathrm{Pic}(S)$.

Here the group Λ is the subgroup of $\mathrm{Hom}(X, G^D)$, where $G^D = \underline{\mathrm{Hom}}(G, \mathbb{G}_m)$, consisting of those morphisms of S-group schemes that satisfy the equivalent conditions of Lemma 4.4, and Σ is a quotient of the group of symmetric bilinear morphisms $X \times_S X \to \mathbb{G}_m$ (see Definition 4.5 and (4.6) for the definitions of Λ, Σ, Ψ, and Θ). Remark that there is a natural identification of K with the kernel of $\mathrm{Pic}(M)/\mathrm{Pic}(S) \to \mathrm{Pic}(G)/\mathrm{Pic}(S)$ and so the map β^* in the above sequence is really the pullback along $\beta : M \to X[1]$.

Theorem of the Cube for 1-motives: In its classical form, the Theorem of the Cube asserts that for any line bundle L on an abelian variety, the associated line bundle $\theta(L)$ is trivial (see Section 6 for the definitions of $\theta(L)$ and $\theta_2(L)$). In [3] Breen proposed the following *reinforcement of the Theorem of the Cube*. A cubical structure on L is a section of $\theta(L)$ that satisfies some additional conditions so that $\theta_2(L)$ is endowed with a structure of symmetric biextension. A cubical line bundle is a line bundle endowed with a cubical structure. Then a commutative S-group scheme G is said to satisfy the (strengthened form of the) Theorem of the Cube if the forgetful functor

$$\mathrm{CUB}(G) \longrightarrow \mathrm{RLB}(G)$$

from the category $\mathrm{CUB}(G)$ of cubical line bundles on G to the category $\mathrm{RLB}(G)$ of rigidified line bundles on G is an equivalence of categories.

The notion of cubical structure introduced by Breen generalizes seamlessly to commutative group stacks (see Definition 6.1). In a very general context, in Theorem 6.2, we explain how a cubical line bundle $(\mathcal{L}, \tau)$ on a commutative group stack $\mathcal{G}$ defines an additive functor from $\mathcal{G}$ to its dual $D(\mathcal{G}) = \mathcal{H}om(\mathcal{G}, B\mathbb{G}_m)$:

$$\begin{array}{rccl} \varphi_{(\mathcal{L},\tau)}: & \mathcal{G} & \longrightarrow & D(\mathcal{G}) \\ & a & \longmapsto & \left(b \mapsto \mathcal{L}_{ab} \otimes \mathcal{L}_a^{-1} \otimes \mathcal{L}_b^{-1}\right). \end{array}$$

In Theorem 7.1 we show that over a normal base scheme, 1-motives satisfy the Theorem of the Cube in the above sense, which is our 3rd main result. Then Theorem 1.1 is an immediate corollary of Theorems 6.2 and 7.1. Remark that the quotient $\mathrm{Pic}(M)/\mathrm{Pic}(S)$ is isomorphic to the group of isomorphism classes of rigidified line bundles on M.

We finish observing that the construction of the morphism $\Phi(L, \delta) : M \to M^*$, with (L, δ) a line bundle on M, which we give in Section 5, is completely geometric and so it allows the computation of the Hodge, the De Rham, and the ℓ-adic realizations of $\Phi(L, \delta) : M \to M^*$, with their comparison isomorphisms. This furnishes the motivic origin of some linear morphisms between 1-motives and their Cartier duals (here *motivic* means coming from geometry—see [9]). In this setting, an ancestor of this paper is [1] where the 1st author defines the notion of biextensions of 1-motives and shows that such biextensions furnish bilinear morphisms between 1-motives in the Hodge, the De Rham, and the ℓ-adic realizations. Just as biextensions of 1-motives are the motivic origin of bilinear morphisms between 1-motives, line bundles on a 1-motive M are the motivic origin of some linear morphisms between M and its Cartier dual M^*. As observed in Remark 5.5 not all morphisms from M to M^* are defined by line bundles.

2 Notation

Let $\mathbf{S}$ be a site. For the definitions of $\mathbf{S}$-stacks and the related vocabulary we refer to [11]. By a stack we always mean a stack in groupoids. If $\mathcal{X}$ and $\mathcal{Y}$ are two $\mathbf{S}$-stacks, $\mathcal{H}om_{\mathbf{S}-stacks}(\mathcal{X}, \mathcal{Y})$ will be the $\mathbf{S}$-stack such that for any object U of $\mathbf{S}$, $\mathcal{H}om_{\mathbf{S}-stacks}(\mathcal{X}, \mathcal{Y})(U)$ is the category of morphisms of $\mathbf{S}$-stacks from $\mathcal{X}_{|U}$ to $\mathcal{Y}_{|U}$. If S is a scheme, an S-stack will be a stack for the *fppf* topology.

A *commutative group* $\mathbf{S}$*-stack* is an $\mathbf{S}$-stack $\mathcal{G}$ endowed with a functor $+ : \mathcal{G} \times_{\mathbf{S}} \mathcal{G} \to \mathcal{G}, \quad (a,b) \mapsto a+b$, and two natural isomorphisms of associativity σ and of commutativity τ, such that for any object U of $\mathbf{S}$, $(\mathcal{G}(U), +, \sigma, \tau)$ is a strictly commutative Picard category. An *additive functor* $(F, \sum) : \mathcal{G}_1 \to \mathcal{G}_2$ between two commutative group S-stacks is a morphism of $\mathbf{S}$-stacks $F : \mathcal{G}_1 \to \mathcal{G}_2$ endowed with a natural isomorphism $\sum$: $F(a+b) \cong F(a)+F(b)$ (for all $a, b \in \mathcal{G}_1$) that is compatible with the natural isomorphisms σ and τ underlying $\mathcal{G}_1$ and $\mathcal{G}_2$. A *morphism of additive functors* $u : (F, \sum) \to (F', \sum')$ is an $\mathbf{S}$-morphism of Cartesian $\mathbf{S}$-functors (see [11, Chp. I 1.1]) that is compatible with the natural isomorphisms $\sum$ and $\sum'$ of F and F', respectively. For more information about commutative group stacks we refer to [7, Section 1.4] or [5].

Let $\mathcal{D}^{[-1,0]}(\mathbf{S})$ be the subcategory of the derived category of abelian sheaves on $\mathbf{S}$ consisting of complexes K such that $\mathrm{H}^i(K) = 0$ for $i \neq -1$ or 0. Denote by $\mathbf{Picard}(\mathbf{S})$ the category whose objects are commutative group stacks and whose arrows are isomorphism classes of additive functors. In [7, Section 1.4] Deligne constructs an equivalence of category

$$\mathrm{st} : \mathcal{D}^{[-1,0]}(\mathbf{S}) \longrightarrow \mathbf{Picard}(\mathbf{S}). \tag{2.1}$$

We denote by [ ] the inverse equivalence of st. Via this equivalence of categories to each 1-motive M is associated a commutative group S-stack $\mathrm{st}(M)$ and morphisms of 1-motives correspond to additive functors between the corresponding commutative group stacks.

We will denote by $B\mathbb{G}_m$ the classifying $\mathbf{S}$-stack of $\mathbb{G}_m$, that is, the commutative group S-stack such that for any object U of $\mathbf{S}$, $B\mathbb{G}_m(U)$ is the category of $\mathbb{G}_m$-torsors over U. Remark that $[B\mathbb{G}_m] = \mathbb{G}_m[1]$, where $\mathbb{G}_m[1]$ is the complex with the multiplicative sheaf $\mathbb{G}_m$ in degree -1. If $\mathcal{G}$ and $\mathcal{Q}$ are two commutative group stacks, $\mathcal{H}om(\mathcal{G}, \mathcal{Q})$ will be the commutative group S-stack such that for any object U of $\mathbf{S}$, $\mathcal{H}om(\mathcal{G}, \mathcal{Q})(U)$ is the category whose objects are additive functors from $\mathcal{G}_{|U}$ to $\mathcal{Q}_{|U}$ and whose arrows are morphisms of additive functors. We have that $[\mathcal{H}om(\mathcal{G}, \mathcal{Q})] = \tau_{\leq 0}\mathrm{RHom}\big([\mathcal{G}], [\mathcal{Q}]\big)$, where $\tau_{\leq 0}$ is the good truncation in degree 0. *The dual* $D(\mathcal{G})$ *of a commutative group stack* $\mathcal{G}$ is the commutative group stack $\mathcal{H}om(\mathcal{G}, B\mathbb{G}_m)$. In particular $[D(\mathcal{G})] = \tau_{\leq 0}\mathrm{RHom}\big([\mathcal{G}], \mathbb{G}_m[1]\big)$. Note that the Cartier duality of 1-motives coincides with the duality for commutative group stacks via the equivalence st, that is, $D(\mathrm{st}(M)) \simeq \mathrm{st}(M^*)$, where M^* is the Cartier dual of the 1-motive M (see [8, (10.2.11)]).

Let S be an arbitrary scheme. An *abelian* S*-scheme* A is an S-group scheme that is smooth, proper over S, and with connected fibers. An S*-torus* T is an S-group scheme that is locally isomorphic for the fpqc topology (equivalently for the étale topology) to an S-group scheme of type $\mathbb{G}_m^r$ (with r a nonnegative integer and $\mathbb{G}_m^0$ the trivial torus).

If G is an S-group scheme, we denote by G^D the S-group scheme $\underline{\mathrm{Hom}}(G, \mathbb{G}_m)$ of group homomorphisms from G to $\mathbb{G}_{\mathrm{m}}$. If T is an S-torus, then T^D is an S-group scheme that is locally for the étale topology a constant group scheme defined by a finitely generated free $\mathbb{Z}$-module.

3 Line Bundles on 1-motives

Let S be a scheme. The following definition is directly inspired from [15, p. 64].

Definition 3.1. Let $p : \mathcal{X} \to S$ be an S-stack.

1. A *line bundle* $\mathcal{L}$ on $\mathcal{X}$ consists of
   - for any S-scheme U and any object x of $\mathcal{X}(U)$, a line bundle $\mathcal{L}(x)$ on U;
   - for any arrow $f : y \to x$ in $\mathcal{X}$, an isomorphism $\mathcal{L}(f) : \mathcal{L}(y) \to p(f)^*\mathcal{L}(x)$ of line bundles on U verifying the following compatibility: if $f : y \to x$ and $g : z \to y$ are two arrows of $\mathcal{X}$, then $\mathcal{L}(f \circ g) = p(g)^*\mathcal{L}(f) \circ \mathcal{L}(g)$.
2. A *morphism* $F : \mathcal{L}_1 \to \mathcal{L}_2$ of line bundles over $\mathcal{X}$ consists of a morphism of line bundles $F(x) : \mathcal{L}_1(x) \to \mathcal{L}_2(x)$ for any S-scheme U and for any object x of $\mathcal{X}(U)$, such that $p(f)^*F(x) \circ \mathcal{L}_1(f) = \mathcal{L}_2(f) \circ F(y)$ for any arrow $f : y \to x$ in $\mathcal{X}$.

The usual tensor product of line bundles over schemes extends to stacks and allows us to define the tensor product $\mathcal{L}_1 \otimes \mathcal{L}_2$ of two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on the stack $\mathcal{X}$. This tensor product equips the set of isomorphism classes of line bundles on $\mathcal{X}$ with an abelian group law. Using the equivalence of categories [7, Section 1.4] between 1-motives and commutative group stacks, we can then define line bundles on 1-motives as follows:

Definition 3.2. Let M be a 1-motive defined over S.

1. The category $\mathrm{PIC}(M)$ of line bundles on M is the category of line bundles on $\mathrm{st}(M)$.
2. The *Picard group of M*, denoted by $\mathrm{Pic}(M)$, is the group of isomorphism classes of line bundles on $\mathrm{st}(M)$.

The following lemma will allow us to describe line bundles on a 1-motive $M = [X \overset{u}{\to} G]$ as line bundles on G endowed with an action of X that is compatible with the translation action of X on G.

Lemma 3.3. Let $\iota : \mathcal{X}_0 \to \mathcal{X}$ be a representable morphism of stacks over S. Assume that ι is faithfully flat and quasi-compact or locally of finite presentation. Then the category of line bundles on $\mathcal{X}$ is equivalent to the category of line bundles on $\mathcal{X}_0$ with descent data, that is, to the category whose objects are pairs $(\mathcal{L}, \delta)$, where $\mathcal{L}$ is a line bundle on $\mathcal{X}_0$ and $\delta : q_1^*\mathcal{L} \to q_2^*\mathcal{L}$ is an isomorphism such that, up to canonical isomorphisms, $p_{13}^*\delta = p_{23}^*\delta \circ p_{12}^*\delta$ (with the obvious notations for the projections $q_i : \mathcal{X}_0 \times_{\mathcal{X}} \mathcal{X}_0 \to \mathcal{X}_0$ and $p_{ij} : \mathcal{X}_0 \times_{\mathcal{X}} \mathcal{X}_0 \times_{\mathcal{X}} \mathcal{X}_0 \to \mathcal{X}_0 \times_{\mathcal{X}} \mathcal{X}_0$).

Proof. We have to prove that the pullback functor ι^* from the category of line bundles on $\mathcal{X}$ to the category of line bundles on $\mathcal{X}_0$ with descent data is an equivalence. The result is well known if $\mathcal{X}$ is algebraic, see [12, (13.5)]. Hence, for any S-scheme U and any morphism $x : U \to \mathcal{X}$, the statement is known for the morphism $\iota_U : \mathcal{X}_0 \times_{\mathcal{X}} U \to U$ obtained by base change. Since a line bundle on $\mathcal{X}$ is by definition a collection of line bundles on the various schemes U, the general case follows. ■

Let $M = [X \xrightarrow{u} G]$ be a 1-motive over scheme S. By [12, (3.4.3)] the associated commutative group stack $\mathrm{st}(M)$ is isomorphic to the quotient stack $[G/X]$ (where X acts on G via the given morphism $u : X \to G$). Note that in general it is not algebraic in the sense of [12] because it is not quasi-separated. However, the quotient map $\iota : G \to [G/X]$ is representable, étale, and surjective, and the above lemma applies. The fiber product $G \times_{[G/X]} G$ is isomorphic to $X \times_S G$. Via this identification, the projections $q_i : G \times_{[G/X]} G \to G$ (for $i = 1, 2$) correspond to the 2nd projection $p_2 : X \times_S G \to G$ and to the map $\mu : X \times_S G \to G$ given by the action $(x, g) \mapsto u(x)g$, respectively. We can further identify the fiber product $G \times_{[G/X]} G \times_{[G/X]} G$ with $X \times_S X \times_S G$ and the partial projections $p_{13}, p_{23}, p_{12} : G \times_{[G/X]} G \times_{[G/X]} G \to G \times_{[G/X]} G$ with the map $m_X \times \mathrm{id}_G : X \times_S X \times_S G \to X \times_S G$ where m_X denotes the group law of X, the map $\mathrm{id}_X \times \mu : X \times_S X \times_S G \to X \times_S G$, and the partial projection $p'_{23} : X \times_S X \times_S G \to X \times_S G$, respectively. Hence, by Lemma 3.3 the category of line bundles on M is equivalent to the category of couples

$$(L, \delta),$$

where L is a line bundle on G and δ is a descent datum for L with respect to $\iota : G \to [G/X]$. More explicitly, the descent datum δ is an isomorphism $\delta : p_2^*L \to \mu^*L$ of line bundles on $X \times_S G$ satisfying the cocycle condition

$$(m_X \times id_G)^*\delta = \big((\mathrm{id}_X \times \mu)^*\delta\big) \circ \big((p'_{23})^*\delta\big).$$

It is often convenient to describe line bundles in terms of "points". If g is a point of G, that is, a morphism $g : U \to G$ for some S-scheme U, we denote by L_g the line bundle g^*L on U. Then δ is given by a collection of isomorphisms

$$\delta_{x,g} : L_g \to L_{u(x)g}$$

for all points x of X and g of G, such that for all points x, y of X and g of G,

$$\delta_{x+y,g} = \delta_{x,u(y)g} \circ \delta_{y,g} \,. \tag{3.1}$$

With this description, the pullback functor ι^* maps a line bundle (L, δ) on M to L, that is, ι^* just forgets the descent datum. Note for further use that ι^* is faithful.

4 Dévissage of the Picard Group of a 1-motive

Let us first recall the following global version of Rosenlicht's Lemma from [17, Corollaire VII 1.2].

Lemma 4.1 (Rosenlicht). Let S be a reduced base scheme and let P be a flat S-group scheme locally of finite presentation. Assume that the maximal fibers of P are smooth and connected. Let $\lambda : P \to \mathbb{G}_m$ be a morphism of S-schemes. If $\lambda(1) = 1$, then λ is a group homomorphism.

(I) 1st dévissage coming from the short exact sequence $0 \to T \stackrel{i}{\to} G \stackrel{\pi}{\to} A \to 0$.

Proof of Proposition 1.2. By [13, Chp. I, Prop. 7.2.2], the category $\mathrm{CUB}(A)$ is equivalent to the category of pairs (L, s), where L is a cubical line bundle on G and s is a trivialization of i^*L in the category $\mathrm{CUB}(T)$. With this identification, the pullback functor $\pi^* : \mathrm{CUB}(A) \to \mathrm{CUB}(G)$ is the forgetful functor that maps a pair (L, s) to L. But since the base scheme is assumed to be normal, all these categories of cubical line bundles are equivalent to the categories of line bundles rigidified along the unit section [13, Chp. I, Prop. 2.6]. The group of isomorphism classes of rigidified line bundles on G is isomorphic to $\mathrm{Pic}(G)/\mathrm{Pic}(S)$, and similarly for A and T. Hence, the equivalence of categories [13, Chp. I, Prop. 7.2.2] induces the following exact sequence when we take the groups of isomorphism classes:

$$\mathrm{Aut}(\mathcal{O}_G) \xrightarrow{i^*} \mathrm{Aut}(i^*\mathcal{O}_G) \longrightarrow \mathrm{Pic}(A)/\mathrm{Pic}(S) \xrightarrow{\pi^*} \mathrm{Pic}(G)/\mathrm{Pic}(S) \xrightarrow{i^*} \mathrm{Pic}(T)/\mathrm{Pic}(S) \,, \tag{4.1}$$

where the automorphism groups on the left are the automorphism groups in the categories of rigidified line bundles on G and on T. An automorphism of $\mathcal{O}_G$ (rigidified) is an automorphism $\lambda : \mathcal{O}_G \to \mathcal{O}_G$ such that $e^*\lambda = \mathrm{id}$, where e is the unit section of G. Hence, the above group $\mathrm{Aut}(\mathcal{O}_G)$ identifies with the kernel of $e^* : \Gamma(G, \mathcal{O}_G^*) \to \Gamma(S, \mathcal{O}_S^*)$, that is, with the group of morphisms of schemes $\lambda : G \to \mathbb{G}_\mathrm{m}$ such that $\lambda(1) = 1$. Since S is reduced, this kernel is isomorphic to $\mathrm{Hom}(G, \mathbb{G}_m)$ by Lemma 4.1. Similarly, the group $\mathrm{Aut}(i^*\mathcal{O}_G)$ of automorphisms in the category of rigidified line bundles is isomorphic to $\mathrm{Hom}(T, \mathbb{G}_m)$. Moreover, since $\mathrm{Hom}(A, \mathbb{G}_m) = 0$ the 1st map i^* is injective. ■

Remark 4.2. (1) Over any base scheme S, by [13, Chp. I, Prop. 7.2.1] the category $\mathrm{CUB}(T)$ is isomorphic to the category of extensions of T by $\mathbb{G}_m$. Moreover, by [13, Chp. I, Remark 7.2.4], if we assume the base scheme S to be normal, or geometrically unibranched, or local Henselien, then the group $\mathrm{Ext}^1(T, \mathbb{G}_m)$ vanishes if the torus T is split.

(2) If L is a rigidified line bundle on G, the class of the line bundle i^*L in $\mathrm{Pic}(T)/\mathrm{Pic}(S)$ represents the obstruction to the fact that L comes from a rigidified line bundle over A. Since $\mathrm{Pic}(T)/\mathrm{Pic}(S) \simeq \mathrm{Ext}^1(T, \mathbb{G}_m)$ and since the tori underlying 1-motives are split locally for the étale topology, as a consequence of (1) of this Remark we have that if S is normal, there exists an étale and surjective morphism $S' \to S$ such that $(i^*L)_{|S'} = 0$, that is, after a base change to S', the rigidified line bundle L on G comes from A.

(II) 2nd dévissage coming from the exact sequence $0 \to G \overset{\iota}{\to} M \overset{\beta}{\to} X[1] \to 0$.

Let us describe more explicitly the maps $\iota^* : \mathrm{Pic}(M) \to \mathrm{Pic}(G)$ and $\beta^* : \mathrm{Pic}(X[1]) \to \mathrm{Pic}(M)$ in terms of line bundles with descent data. As explained in Section 3, we identify the category of line bundles on M with the category of couples

$$(L, \delta),$$

where L is a line bundle on G and δ is a descent datum for L with respect to the covering $\iota : G \to [G/X]$. Then the pullback functor ι^* maps a line bundle (L, δ) on M to L: $\iota^*(L, \delta) = L$.

If L is the trivial bundle $\mathcal{O}_G$, via the canonical isomorphism $p_2^*L \simeq \mu^*L$, a descent datum δ on L can be seen as a morphism of S-schemes $\delta : X \times_S G \to \mathbb{G}_m$, and the cocycle condition (3.1) on δ can be rewritten as follows: for any points x, y of X and g of G, we have the equation

$$\delta(x + y, g) = \delta(x, u(y)g).\delta(y, g)\,. \tag{4.2}$$

The category of line bundles on $X[1]$ is equivalent to the category of line bundles on S together with a descent datum with respect to the presentation $S \to [S/X]$. By [4, Example 5.3.7] we have that

$$\frac{\mathrm{Pic}(X[1])}{\mathrm{Pic}(S)} \simeq \mathrm{Hom}(X, \mathbb{G}_m).$$

Let us now describe the pullback morphism β^* in these terms. Unwinding the various definitions, it can be seen that given a character $\alpha : X \to \mathbb{G}_m$, the associated element $\beta^*\alpha \in \mathrm{Pic}(M)$ is the class of the line bundle $(\mathcal{O}_G, \delta_\alpha)$, where δ_α is the automorphism of $\mathcal{O}_{X\times_S G}$ corresponding to the morphism of S-schemes $\delta_\alpha : X \times_S G \to \mathbb{G}_m$, $(x, g) \mapsto \alpha(x)$:

$$\beta^*\alpha = [(\mathcal{O}_G, \delta_\alpha)].$$

Even if the composition $\iota^*\beta^*$ is trivial, the sequence $\mathrm{Pic}(X[1]) \to \mathrm{Pic}(M) \to \mathrm{Pic}(G)$ is not exact in general as shown in the following example. However, in the special case of 1-motives without toric part, this sequence is always exact (see Remark 4.9).

Example 4.3. Let S be any base scheme with $\mathrm{Pic}(S) = 0$. Let T be an S-torus, let $X = \mathbb{Z}$, and let $M = [u : X \to T]$ be a 1-motive with u the trivial morphism. Let $(\mathcal{O}_T, \delta)$ be a line bundle on M (using the above description) that is mapped to the neutral element of $\mathrm{Pic}(T)$. Note that since u is trivial the cocycle condition (4.2) here means that for any $g \in T(U)$, $\delta(., g)$ is a group homomorphism in the variable x.

The class of $(\mathcal{O}_T, \delta)$ is in the image of $\mathrm{Pic}(X[1])$ if and only if there is an $\alpha \in \mathrm{Hom}(X, \mathbb{G}_m)$ such that $(\mathcal{O}_T, \delta) \simeq (\mathcal{O}_T, \delta_\alpha)$. An isomorphism $(\mathcal{O}_T, \delta) \simeq (\mathcal{O}_T, \delta_\alpha)$ is an automorphism λ of $\mathcal{O}_T$ such that $\delta_\alpha \circ p_2^*\lambda = \mu^*\lambda \circ \delta$. But here $\mu = p_2$ (since u is trivial) and the group of automorphisms of $\mathcal{O}_{X\times_S T}$ is commutative. So $(\mathcal{O}_T, \delta)$ and $(\mathcal{O}_T, \delta_\alpha)$ are isomorphic if and only if $\delta = \delta_\alpha$. This proves that $(\mathcal{O}_T, \delta)$ is in the image of $\mathrm{Pic}(X[1])$ if and only if δ, seen as a morphism of S-schemes $\delta : X \times_S T \to \mathbb{G}_m$, is constant in the variable $g \in T$ (for the "if" part, we define α by $\alpha(x) = \delta(x, 1)$ and the cocycle condition on δ ensures that α is a group homomorphism). We will now construct a descent datum δ on $\mathcal{O}_T$ that is not constant in g and this will prove that the sequence $\mathrm{Pic}(X[1]) \to \mathrm{Pic}(M) \to \mathrm{Pic}(T)$ is not exact. Let $\lambda \in \mathrm{Hom}(T, \mathbb{G}_m)$ be a nontrivial homomorphism and define δ functorially by $\delta(n, g) = \lambda(g)^n$. This δ is a homomorphism in the variable n for any g and so it is indeed a descent datum, but it is nonconstant in g since λ is nonconstant. Hence, the corresponding line bundle $(\mathcal{O}_T, \delta)$ is not in the image of $\mathrm{Pic}(X[1])$.

Now we compute the kernel of $\iota^* : \mathrm{Pic}(M) \to \mathrm{Pic}(G)$. Let $G^D = \underline{\mathrm{Hom}}(G, \mathbb{G}_m)$ and $X^D = \underline{\mathrm{Hom}}(X, \mathbb{G}_m)$ be the Cartier duals of G and X, respectively.

Lemma 4.4. For a morphism of S-group schemes $\lambda : X \to G^D$, the following conditions are equivalent:

1. For any S-scheme U and any two points $x, y \in X(U)$, $\lambda(x)(u(y)) = \lambda(y)(u(x))$.
2. The diagram

$$\begin{array}{ccc} X & \xrightarrow{\lambda} & G^D \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u^D} \\ G & \xrightarrow[\lambda^D]{} & X^D, \end{array} \tag{4.3}$$

   where we have identified the term G in the bottom left-hand corner with its double dual $(G^D)^D$, and where u^D, λ^D are the morphisms of group schemes induced by u, λ by "taking the Cartier dual", commutes.

Proof. We just give the proof of (2). Put $f = u^D \circ \lambda : X \to X^D$. Identifying X with its double dual $(X^D)^D$, we have that $f^D = X \to X^D$ coincides with f and so $\lambda^D \circ u = (u^D \circ \lambda)^D = u^D \circ \lambda$. ■

We say that a morphism of S-schemes $\sigma : X \times_S X \to \mathbb{G}_m$ is *symmetric* if it satisfies the equation $\sigma(x, y) = \sigma(y, x)$. If $\alpha : X \to \mathbb{G}_m$ is a morphism of S-schemes, we denote by $\sigma_\alpha : X \times_S X \to \mathbb{G}_m$ the symmetric morphism given by $\sigma_\alpha(x, y) = \frac{\alpha(x+y)}{\alpha(x)\alpha(y)}$. Hence, α is a morphism of S-group schemes if and only if σ_α is trivial.

Definition 4.5.

1. We denote by Λ the subgroup of $\mathrm{Hom}(X, G^D)$ consisting of those morphisms of S-group schemes that satisfy the equivalent conditions of Lemma 4.4.
2. We denote by Σ the quotient of the group of symmetric bilinear morphisms $X \times_S X \to \mathbb{G}_m$ by the subgroup of morphisms of the form σ_α for some morphism of S-schemes $\alpha : X \to \mathbb{G}_m$.
3. We denote by $\Psi : \Lambda \to \Sigma$ the natural homomorphism that maps $\lambda \in \Lambda$ to the class of the function $(x, y) \mapsto \lambda(x)(u(y))$.

Remark 4.6. Note that, following [6, XIV, Sections 2–4] we can view Σ as a subgroup of the kernel of the natural morphism $\mathrm{Ext}^1(X, \mathbb{G}_\mathrm{m}) \to H^1(X, \mathbb{G}_\mathrm{m})$. Since the framework and statements of [6] are not exactly the same as ours, we briefly recall the construction here. If $\sigma : X \times_S X \to \mathbb{G}_\mathrm{m}$ is a symmetric bilinear morphism, let E_σ be the group scheme $\mathbb{G}_\mathrm{m} \times_S X$, where the group law is given by $(\gamma_1, x).(\gamma_2, y) := (\gamma_1\gamma_2\sigma(x, y), x + y)$. With the 2nd projection $\pi : E_\sigma \to X$ and the inclusion $i : \mathbb{G}_\mathrm{m} \to E_\sigma$ given by $i(\gamma) = (\gamma, 0)$, the group scheme E_σ is a commutative extension of X by $\mathbb{G}_\mathrm{m}$. Then a direct computation shows that $\sigma \mapsto E_\sigma$ induces an injective group homomorphism from Σ to $\mathrm{Ext}^1(X, \mathbb{G}_\mathrm{m})$. Since the projection $\pi : E_\sigma \to X$ has a section $x \mapsto (1, x)$, the $\mathbb{G}_\mathrm{m}$-torsor over X induced by E_σ is trivial, which proves that the image of Σ lies in the kernel of $\mathrm{Ext}^1(X, \mathbb{G}_\mathrm{m}) \to H^1(X, \mathbb{G}_\mathrm{m})$. Actually, if E is an extension of X by $\mathbb{G}_\mathrm{m}$, its class $[E] \in \mathrm{Ext}^1(X, \mathbb{G}_\mathrm{m})$ lies in Σ if and only if the projection $E \to X$ has a section $s : X \to E$ (only as a morphism of schemes, not of group schemes), which is of degree 2 in the language of [3] or [13], that is, such that $\theta_3(s) = 1$.

Remark 4.7. In particular, if X is split (i.e., $X \simeq \mathbb{Z}^r$ for some r) then $\Sigma = 0$ since the morphism $\mathrm{Ext}^1(X, \mathbb{G}_\mathrm{m}) \to H^1(X, \mathbb{G}_\mathrm{m})$ is injective.

For the rest of this Section, we assume that the base scheme S is reduced. Denote by K the kernel of the forgetful functor $\iota^* : \mathrm{Pic}(M) \to \mathrm{Pic}(G)$. This kernel is the group of classes of pairs $(\mathcal{O}_G, \delta)$, where δ is a descent datum on $\mathcal{O}_G$. Such a descent datum can be seen as a morphism of schemes $\delta : X \times_S G \to \mathbb{G}_m$ that satisfies the cocycle condition (4.2). Two pairs $(\mathcal{O}_G, \delta_1), (\mathcal{O}_G, \delta_2)$ are in the same class if and only if they are isomorphic in the category of line bundles on G equipped with a descent datum relative to $\iota : G \to M$, which means that there is a morphism of S-schemes $\nu : G \to \mathbb{G}_m$ such that $(\mu^*\nu).\delta_1 = \delta_2.p_2^*\nu$, where $\mu, p_2 : X \times_S G \to G$ are the action of X on G and the 2nd projection. The latter equation can be rewritten as $\nu(u(x)g)\delta_1(x, g) = \delta_2(x, g)\nu(g)$ for any $(x, g) \in X(U) \times G(U)$. Replacing ν with $g \mapsto \nu(g)/\nu(1)$, we may assume that $\nu(1) = 1$ so that ν is a group homomorphism by Rosenlicht's Lemma 4.1. The equation then becomes

$$\nu(u(x))\delta_1(x, g) = \delta_2(x, g). \tag{4.4}$$

The group law on K is given by $[(\mathcal{O}_G, \delta_1)].[(\mathcal{O}_G, \delta_2)] = [(\mathcal{O}_G, \delta_1.\delta_2)]$.

We will now construct a homomorphism $\Theta : K \to \Lambda$, where Λ was defined in Definition 4.5. Let $[(\mathcal{O}_G, \delta)]$ be a class in K where δ is a solution of (4.2). For any point x of X, consider the morphism of S-schemes

$$\lambda_\delta(x) : G \to \mathbb{G}_m, \ g \mapsto \frac{\delta(x, g)}{\delta(x, 1)}. \tag{4.5}$$

Since $\lambda_\delta(x)(1) = 1$, the morphism $\lambda_\delta(x)$ is actually a homomorphism by Lemma 4.1, hence a section of G^D. This construction is functorial and defines a morphism of S-schemes $\lambda_\delta : X \to G^D$. By (4.2), for any $x, y \in X$ and any $g \in G$ we have

$$\begin{aligned}
\lambda_\delta(x+y)(g) &= \frac{\delta(x+y,g)}{\delta(x+y,1)} \\
&= \frac{\delta(x,u(y)g)\delta(y,g)}{\delta(x,u(y))\delta(y,1)} \\
&= \frac{\delta(x,u(y)g)}{\delta(x,1)}.\frac{\delta(x,1)}{\delta(x,u(y))}.\frac{\delta(y,g)}{\delta(y,1)} \\
&= \frac{\lambda_\delta(x)(u(y)g)}{\lambda_\delta(x)(u(y))}.\lambda_\delta(y)(g) \\
&= \lambda_\delta(x)(g).\lambda_\delta(y)(g),
\end{aligned}$$

where the last equality follows from the fact that $\lambda_\delta(x)$ is a homomorphism. Hence, λ_δ is a morphism of S-group schemes. Moreover, by (4.2) for any $x, y \in X$ we have

$$\delta(x,u(y))\delta(y,1) = \delta(x+y,1) = \delta(y+x,1) = \delta(y,u(x))\delta(x,1).$$

Hence, $\lambda_\delta(x)(u(y)) = \lambda_\delta(y)(u(x))$ and so λ_δ belongs to Λ. Since λ_δ only depends on the class $[(\mathcal{O}_G,\delta)]$, this construction induces a well-defined homomorphism

$$\Theta : K \to \Lambda,\ [(\mathcal{O}_G,\delta)] \mapsto \lambda_\delta. \tag{4.6}$$

It is a homomorphism because $\lambda_{\delta_1\delta_2} = \lambda_{\delta_1}\lambda_{\delta_2}$.

Proof of Proposition 1.3. The morphism $\beta^* : \mathrm{Hom}(X,\mathbb{G}_m) \to K$ maps an $\alpha \in \mathrm{Hom}(X,\mathbb{G}_m)$ to the class $[(\mathcal{O}_G,\delta_\alpha)]$, where δ_α is defined by $\delta_\alpha(x,g) = \alpha(x)$. By the equality (4.4), $[(\mathcal{O}_G,\delta_\alpha)]$ is trivial if and only if there is a morphism of S-group schemes $\nu : G \to \mathbb{G}_m$ such that $\alpha = \nu \circ u$, which means that the sequence is exact in $\mathrm{Hom}(X,\mathbb{G}_m)$.

Now we check the exactness in K. Let $[(\mathcal{O}_G,\delta)]$ be a class in K. By (4.5) its image λ_δ under Θ is trivial if and only if δ satisfies the equation $\delta(x,1) = \delta(x,g)$ for any $x \in X$ and $g \in G$. If so, let $\alpha : X \to \mathbb{G}_m$ be the morphism of S-schemes defined by $\alpha(x) = \delta(x,1)$. Then by (4.2) α is a homomorphism, and we have $\delta = \delta_\alpha = \beta^*(\alpha)$, which proves the exactness in K.

It remains to prove the exactness in Λ. Let $\lambda \in \Lambda$. Assume that λ is in the image of K, that is, there is some solution δ of (4.2) such that $\lambda = \lambda_\delta$. Let $\alpha : X \to \mathbb{G}_m$ be the morphism of S-schemes defined by $\alpha(x) = \delta(x,1)$. Then for any $(x,g) \in X \times G$ we have $\delta(x,g) = \lambda(x)(g)\alpha(x)$. The bilinearity of λ and (4.2) yield $\lambda(x)(u(y)) = \frac{\alpha(x+y)}{\alpha(x)\alpha(y)}$. Hence, the image of λ in Σ is trivial. Conversely, assume that the image $\Psi(\lambda)$ is trivial in Σ, in other words there is a morphism of S-schemes $\alpha : X \to \mathbb{G}_m$ such that $\lambda(x)(u(y)) = \frac{\alpha(x+y)}{\alpha(x)\alpha(y)}$. Then we define δ by $\delta(x,g) = \lambda(x)(g)\alpha(x)$ and the same computations as above show that δ satisfies (4.2) and that $\lambda = \lambda_\delta$, which concludes the proof. ∎

If the lattice X underlying the 1-motive $M = [u : X \to G]$ is split then by Remark 4.7 the morphism $K \to \Lambda$ is surjective. Actually we can give an explicit section that depends on the choice of a $\mathbb{Z}$-basis for X as follows. Let $e_1, \dots, e_n$ be a $\mathbb{Z}$-basis of X. For $\lambda \in \Lambda$, let $\lambda_1, \dots, \lambda_l : G \to \mathbb{G}_{\mathrm{m}}$ be the images of $e_1, \dots, e_l$ under λ. We denote by δ_λ the morphism from $X \times_S G$ to $\mathbb{G}_m$ defined by

$$\delta_\lambda(x,g) = \lambda(x)(g) \prod_i \left(\lambda_i \circ u\left(\frac{n_i(n_i-1)}{2} e_i\right)\right) \prod_{1 \le i < j \le l} \lambda_i(u(e_j))^{n_i n_j} . \tag{4.7}$$

for any S-scheme U, any $x = \sum n_i e_i \in X(U)$, and any $g \in G(U)$.

Proposition 4.8. Let $M = [u : X \to G]$ be a 1-motive defined over a reduced base scheme S. Assume that the lattice X is split. With the above notations, the application $\lambda \mapsto [(\mathcal{O}_G, \delta_\lambda)]$ defines a section $s : \Lambda \to K$ of the homomorphism Θ defined in (4.6). In particular the group $\mathrm{Pic}(M)$ fits in the following exact sequence:

$$\mathrm{Hom}(G, \mathbb{G}_m) \longrightarrow \mathrm{Hom}(X, \mathbb{G}_m) \times \Lambda \longrightarrow \mathrm{Pic}(M) \xrightarrow{\iota^*} \mathrm{Pic}(G) . \tag{4.8}$$

Proof. A direct computation shows that δ_λ satisfies the Equation (4.2), hence it is a descent datum and s is well defined. From the definition of δ_λ, we see that $\delta_{\lambda.\lambda'} = \delta_\lambda.\delta_{\lambda'}$ hence s is a group homomorphism. Moreover, the quotient $\delta_\lambda(x,g)/\delta_\lambda(x,1)$ is equal to $\lambda(x)(g)$, which proves that $\Theta([(\mathcal{O}_G, \delta_\lambda)]) = \lambda$. The exact sequence (4.8) now follows from Proposition 1.3. ∎

Remark 4.9. Let $M = [v : X \to A]$ be a 1-motive without toric part. Since $\underline{\mathrm{Hom}}(A, \mathbb{G}_m) = 0$, the group Λ is trivial and so from Proposition 1.3, we obtain that $\beta^* : \mathrm{Hom}(X, \mathbb{G}_m) \to K$ is an isomorphism, that is, the short sequence defined by β^* and ι^*, $\mathrm{Pic}(X[1])/\mathrm{Pic}(S) \to \mathrm{Pic}(M)/\mathrm{Pic}(S) \to \mathrm{Pic}(A)/\mathrm{Pic}(S)$, is exact.

5 Construction of $\Phi : \mathrm{Pic}(M)/\mathrm{Pic}(S) \to \mathrm{Hom}(M, M^*)$ (1.1)

Using the dévissage of the Picard group of a 1-motive M, in this Section we construct the morphism $\Phi : \mathrm{Pic}(M)/\mathrm{Pic}(S) \to \mathrm{Hom}(M, M^*)$ of Theorem 1.1 in an explicit way.

We start proving the following lemma, which might be well known, but for which we were unable to find a convenient reference.

Lemma 5.1. Let S be a reduced base scheme. Consider the following commutative diagram of commutative S-group schemes:

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r, "i"] \arrow[d, "h"'] & G \arrow[r, "\pi"] \arrow[d, dotted, "u"] & A \arrow[r] \arrow[d, "v"] & 0 \\
0 \arrow[r] & T' \arrow[r, "i'"] & G' \arrow[r, "\pi'"] & A' \arrow[r] & 0,
\end{tikzcd}$$

where T, T' are tori, A, A' are abelian schemes, all the solid arrows are group homomorphisms, the rows are exact, and u is only assumed to be a morphism of schemes over S. Then,

1. u is a group homomorphism;
2. u is uniquely determined by h and v, that is, if u_1 and u_2 are two morphisms that make the whole diagram commutative, then $u_1 = u_2$; and
3. If $h = v = 0$, then $u = 0$.

Proof. Let us prove (3). Since $\pi' \circ u = 0$ the morphism u factorizes through a morphism of schemes $u' : G \to T'$. The question is local on S, and T' is locally isomorphic to $\mathbb{G}_\mathrm{m}^r$ for some integer r, hence we may assume that $T' = \mathbb{G}_\mathrm{m}$. Since $u' \circ i$ is trivial, in particular $u'(1) = 1$ and so by Rosenlicht's Lemma 4.1 u' is a group homomorphism. Now the result follows since $\mathrm{Hom}(A, \mathbb{G}_\mathrm{m}) = 0$.

Applying (3) with $u = u_1 - u_2$ we get (2). Now let us prove (1). It suffices to apply (2) with the exact sequence $0 \to T \times_S T \to G \times_S G \to A \times_S A \to 0$ and the morphisms $u_1, u_2 : G \times_S G \to G'$ defined by $u_1(x, y) = u(x + y)$ and $u_2(x, y) = u(x) + u(y)$. ■

Let S be a normal base scheme and let $M = [u : X \to G]$ be a 1-motive over S, where G fits in an extension $0 \to T \xrightarrow{i} G \xrightarrow{\pi} A \to 0$. We start recalling from [8, (10.2.11)] the description of the Cartier dual $M^* = [u' : T^D \to G']$ of M. Denote by $\overline{M}$ the 1-motive $M/W_{-2}M = [v : X \to A]$, where $v = \pi \circ u$. An extension of $\overline{M}$ by $\mathbb{G}_\mathrm{m}$ is a pair $(E, \widetilde{v})$, where

E is an extension of A by $\mathbb{G}_\mathrm{m}$ and $\widetilde{v}$ is a trivialization of v^*E:

$$\begin{tikzcd} & & & X \arrow[d, "v"] \arrow[dl, bend right, "\widetilde{v}"'] & \\ 0 \arrow[r] & \mathbb{G}_\mathrm{m} \arrow[r] & E \arrow[r] & A \arrow[r] & 0. \end{tikzcd}$$

Extensions of $\overline{M}$ by $\mathbb{G}_\mathrm{m}$ do not admit nontrivial automorphisms. The functor of isomorphism classes of such extensions is representable by a group scheme G', which is an extension of A^* by X^D:

$$0 \longrightarrow X^D \xrightarrow{i'} G' \xrightarrow{\pi'} A^* \longrightarrow 0$$

The 1-motive M is an extension of $\overline{M}$ by T. If $\tau : T \to \mathbb{G}_\mathrm{m}$ is a point of T^D, the pushdown τ_*M is an extension of $\overline{M}$ by $\mathbb{G}_\mathrm{m}$, that is, it is a point of G'. This defines a morphism $u' : T^D \to G'$ by $u'(\tau) = \tau_*M$ and by definition the Cartier dual of M is the 1-motive $M^* = [T^D \overset{u'}{\to} G']$.

Now, we start the construction of $\Phi : \mathrm{Pic}(M)/\mathrm{Pic}(S) \to \mathrm{Hom}(M, M^*)$. Let $(\mathcal{L}, \delta)$ be a line bundle on M, where $\mathcal{L}$ is a line bundle on G and δ is a descent datum on $\mathcal{L}$, that is, an isomorphism

$$\delta : p_2^*L \to \mu^*L$$

satisfying the cocycle condition (3.1) (see the end of Section 3). We have to construct a morphism $\Phi(\mathcal{L}, \delta) : M \to M^*$. The 1st dévissage of $\mathrm{Pic}(M)$ (see Proposition 1.2) furnishes the following exact sequence of groups:

$$\mathrm{Hom}(T, \mathbb{G}_m) \xrightarrow{\xi} \mathrm{Pic}(A)/\mathrm{Pic}(S) \xrightarrow{\pi^*} \mathrm{Pic}(G)/\mathrm{Pic}(S) \xrightarrow{i^*} \mathrm{Pic}(T)/\mathrm{Pic}(S).$$

By Remark 4.2 (2), since the tori underlying 1-motives are split locally for the étale topology, there exists an étale and surjective morphism $S' \to S$ such that $(i^*\mathcal{L})_{|_{S'}}$ is trivial, which means that

$$\mathcal{L}_{|_{S'}} = \pi^*L$$

for some line bundle $L \in \mathrm{Pic}(A_{|_{S'}})/\mathrm{Pic}(S')$. Below we will construct locally defined linear morphisms $\Phi((\mathcal{L}, \delta)_{|_{S'}}) : M_{|_{S'}} \to M^*_{|_{S'}}$ from $M_{|_{S'}}$ to its Cartier dual $M^*_{|_{S'}}$. Since these are induced by a global line bundle $(\mathcal{L}, \delta)$, they glue together and yield a linear morphism $\Phi(\mathcal{L}, \delta) : M \to M^*$ over S. Hence, it is not restrictive if we assume $S' = S$ and $\mathcal{L} = \pi^*L$ in order to simplify notation.

Via the classical homomorphism $\Phi_A : \mathrm{Pic}(A) \to \mathrm{Hom}(A, A^*)$, the line bundle L furnishes a morphism of S-group schemes

$$\varphi_L : A \longrightarrow A^*, \quad a \mapsto \left(\mu_a^* L\right) \otimes L^{-1},$$

where $\mu_a : A \to A$ is the translation by a. Let us check that $\varphi_L : A \to A^*$ does not depend on the choice of the line bundle L but only on its pullback $\mathcal{L} = \pi^* L$, in other words $\Phi_A \circ \xi = 0$. Let $\alpha \in \mathrm{Hom}(T, \mathbb{G}_m)$. By definition of ξ, $\xi(\alpha)$ is the image of the class $[\alpha_* G]$ under the inclusion $\mathrm{Ext}^1(A, \mathbb{G}_m) \hookrightarrow \mathrm{Pic}(A)$, that is, $\xi(\alpha)$ comes from $\mathrm{Ext}^1(A, \mathbb{G}_m)$. Hence, by [16, Prop. 1.8] $\Phi_A(\xi(\alpha)) = 0$.

Our next aim is to define a morphism $\widetilde{\varphi_L} : G \to G'$ that lifts φ_L. Before we recall briefly the isomorphism between $\underline{\mathrm{Ext}}^1(A, \mathbb{G}_m)$ and A^*: any extension of A by $\mathbb{G}_m$ is in particular a $\mathbb{G}_m$-torsor over A and therefore a line bundle over A, that is, a point of A^*; on the other hand, to any line bundle N over A we associate the sheaf E such that for any S-scheme T

$$E(T) = \left\{(a, \tau) \mid \quad a \in A(T),\ \tau : N_T \overset{\cong}{\to} \mu_a^* N_T\right\},$$

where N_T is the pullback of N to $A_T = A \times_S T$, which is in fact an extension of A by $\mathbb{G}_m$ (see [10, Section 2] for more details). Now let $g \in G(S)$. The line bundle $\varphi_L(\pi(g)) = \mu_{\pi(g)}^* L \otimes L^{-1}$ is a point of $A^*(S)$. We denote by $E_{\varphi_L(\pi(g))}$ the corresponding extension of A by $\mathbb{G}_\mathrm{m}$. As observed before, the extension $E_{\varphi_L(\pi(g))}$ has the following functorial description: $E_{\varphi_L(\pi(g))}(S)$ is the set of pairs (a, β), where $a \in A(S)$ and $\beta : \varphi_L(\pi(g)) \to \mu_a^* \varphi_L(\pi(g))$ is an isomorphism of line bundles over A. We define functorially

$$\begin{array}{rccl} \widetilde{\varphi_L} : & G & \longrightarrow & G' \\ & g & \longmapsto & \widetilde{\varphi_L}(g) = \left(E_{\varphi_L(\pi(g))}, \widetilde{v}_g\right), \end{array} \tag{5.1}$$

where the trivialization $\widetilde{v}_g : X \to E_{\varphi_L(\pi(g))}$ is defined by

$$\widetilde{v}_g(x) = (v(x), \varphi_{g,x}) \tag{5.2}$$

with $\varphi_{g,x} : \varphi_L(\pi(g)) \to \mu_{v(x)}^* \varphi_L(\pi(g))$ the isomorphism of line bundles on A given by the following lemma.

Lemma 5.2. With the above notation, there is a unique isomorphism $\varphi_{g,x} : \varphi_L(\pi(g)) \to \mu_{v(x)}^* \varphi_L(\pi(g))$ of line bundles on A such that $\pi^* \varphi_{g,x} : \mu_g^* \mathcal{L} \otimes \mathcal{L}^{-1} \to \mu_g^*(\mu_{u(x)}^* \mathcal{L}) \otimes (\mu_{u(x)}^* \mathcal{L})^{-1}$

is equal to $\mu_g^*\delta_x \otimes \delta_x^{-1}$, where $\delta_x : \mathcal{L} \to \mu_{u(x)}^*\mathcal{L}$ denotes the isomorphism $(x, \mathrm{id}_G)^*\delta$ of line bundles on G induced by the descent datum δ.

Proof. For any $x \in X(S)$ and $b \in G(S)$, let us denote by $\overline{\delta}_{x,b}$ the isomorphism $\mathcal{O}_S \to \mathcal{L}_{u(x)b} \otimes \mathcal{L}_b^{-1}$ induced by $\delta_{x,b}$ and by $\overline{\delta}_x : \mathcal{O}_G \to \mu_{u(x)}^*\mathcal{L} \otimes \mathcal{L}^{-1}$ the isomorphism induced by δ_x. Consider the line bundle $N = \mu_{\pi(g)}^*(\mu_{v(x)}^* L \otimes L^{-1}) \otimes (\mu_{v(x)}^* L \otimes L^{-1})^{-1}$ on A. In order to prove our Lemma it is enough to show that there is a unique isomorphism $\varphi : \mathcal{O}_A \to N$ such that $\pi^*\varphi = \mu_g^*\overline{\delta}_x \otimes \overline{\delta}_x^{-1}$.

By [13, Chp. I, Prop. 2.6, and 7.2.2] the pullback functor π^* induces an equivalence between the category of rigidified (at the origin) line bundles on A and the category of pairs $(\mathcal{N}, \alpha)$, where $\mathcal{N}$ is a rigidified line bundle on G and α is a trivialization of $i^*\mathcal{N}$ in the category of rigidified line bundles on T. The line bundle $\mathcal{O}_A$ is canonically rigidified at 1 and the line bundle N on A has a rigidification at 1 given by $\overline{\delta}_{x,g} \otimes \overline{\delta}_{x,1}^{-1}$. Hence, by the above equivalence of categories to prove the Lemma it suffices to prove that $\mu_g^*\overline{\delta}_x \otimes \overline{\delta}_x^{-1}$ is compatible with the trivializations of $i^*\pi^*\mathcal{O}_A$ and $i^*\pi^*N$. In other words, we have to prove that for any point t of T, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{O}_S \arrow[r, "\overline{\delta}_{x,gi(t)} \otimes \overline{\delta}_{x,i(t)}^{-1}"] \arrow[dd, "\overline{\delta}_{x,g} \otimes \overline{\delta}_{x,1}^{-1}"'] & (\mathcal{L}_{u(x)gi(t)} \otimes \mathcal{L}_{gi(t)}^{-1}) \otimes (\mathcal{L}_{u(x)i(t)} \otimes \mathcal{L}_{i(t)}^{-1})^{-1} \arrow[d, equal] \\
 & (L_{\pi(u(x)gi(t))} \otimes L_{\pi(gi(t))}^{-1}) \otimes (L_{\pi(u(x)i(t))} \otimes L_{\pi(i(t))}^{-1})^{-1} \arrow[d, equal] \\
(\mathcal{L}_{u(x)g} \otimes \mathcal{L}_g^{-1}) \otimes (\mathcal{L}_{u(x)} \otimes \mathcal{L}_1^{-1})^{-1} \arrow[r, equal] & (L_{\pi(u(x)g)} \otimes L_{\pi(g)}^{-1}) \otimes (L_{\pi(u(x))} \otimes L_1^{-1})^{-1}
\end{tikzcd}.$$

This diagram defines an automorphism of $\mathcal{O}_S$, hence an element of $\mathbb{G}_{\mathrm{m}}(S)$, and the diagram commutes if and only if this element is equal to $1 \in \mathbb{G}_{\mathrm{m}}(S)$. As g and t vary, these diagrams induce a morphism of schemes $\zeta : G \times_S T \to \mathbb{G}_{\mathrm{m}}$. If $t = 1$, the diagram obviously commutes, hence $\zeta(g, 1) = 1$ and by Rosenlicht's Lemma 4.1 $\zeta(g, .)$ is a group homomorphism $T \to \mathbb{G}_{\mathrm{m}}$. Then ζ corresponds to a morphism of schemes $G \to T^D$. Since G has connected fibers and T^D is a lattice, the latter morphism must be constant. But the diagram obviously commutes if $g = 1$, hence ζ is constant equal to 1 and the diagram commutes for all points g of G and t of T, as required. ■

Now $\widetilde{v}_g$ is well defined and the formula (5.1) defines a morphism of schemes $\widetilde{\varphi}_L : G \to G'$. If $g \in G(S)$, the image $\pi'(\widetilde{\varphi}_L(g))$ is the class in $A^*(S)$ of the extension $E_{\varphi_L(\pi(g))}$, that is, $\pi'(\widetilde{\varphi}_L(g)) = \varphi_L(\pi(g))$, and so the right square in the following diagram is commutative. We denote by $h : T \to X^D$ the unique morphism that makes the left square commutative:

$$\begin{tikzcd} 0 \arrow[r] & T \arrow[r, "i"] \arrow[d, dotted, "h"] & G \arrow[r, "\pi"] \arrow[d, "\widetilde{\varphi}_L"] & A \arrow[r] \arrow[d, "\varphi_L"] & 0 \\ 0 \arrow[r] & X^D \arrow[r, "i'"'] & G' \arrow[r, "\pi'"'] & A^* \arrow[r] & 0. \end{tikzcd}$$

Remark 5.3. We can give an explicit description of $h : T \to X^D$ in terms of $(\mathcal{L}, \delta)$ as follows. Let $t \in T(S)$ be a point of T. Then by definition $\widetilde{\varphi}_L(i(t)) = (E_{\varphi_L(\pi(i(t)))}, \widetilde{v}_{i(t)})$. Since $\pi(i(t)) = 1$ the extension $E_{\varphi_L(\pi(i(t)))}$ is trivial. The morphism $h(t) : X \to \mathbb{G}_{\mathrm{m}}$ is given by $\widetilde{v}_{i(t)}$. Let $x \in X(S)$. By definition $\widetilde{v}_{i(t)}(x) = (v(x), \varphi_{i(t),x})$. Since the line bundle $\varphi_L(1)$ is trivial, the isomorphism $\varphi_{i(t),x} : \varphi_L(1) \to \mu^*_{v(x)}\varphi_L(1)$ can be seen as a morphism of schemes $A \to \mathbb{G}_{\mathrm{m}}$, and $h(t)(x) \in \mathbb{G}_{\mathrm{m}}(S)$ is the (necessarily constant) value of this morphism. We may evaluate it at the origin of A and we see that $h(t)(x)$ is the point of $\mathbb{G}_{\mathrm{m}}$ that corresponds to the isomorphism of (canonically trivial) line bundles $\delta_{x,i(t)} \otimes \delta^{-1}_{x,1} : \mathcal{L}_{i(t)} \otimes \mathcal{L}^{-1}_1 \to \mathcal{L}_{u(x)i(t)} \otimes \mathcal{L}^{-1}_{u(x)}$.

It is clear from the above Remark that h does not depend on the choice of L. Moreover, since $h(1) = 1$, it follows from Rosenlicht's Lemma 4.1 that h is a group homomorphism. Then by Lemma 5.1 $\widetilde{\varphi}_L$ is also a group homomorphism, and it does not depend on the choice of the lifting L of $\mathcal{L}$ (since ϕ_L does not depend on this choice as we have already proved).

The following proposition proves that the pair $(h^D, \widetilde{\varphi}_L)$ is a morphism of 1-motives and so we can set

$$\begin{array}{rccl} \Phi : & \mathrm{Pic}(M)/\mathrm{Pic}(S) & \longrightarrow & \mathrm{Hom}(M, M^*) \\ & (\mathcal{L}, \delta) & \longmapsto & \Phi(\mathcal{L}, \delta) = (h^D, \widetilde{\varphi}_L). \end{array}$$

Proposition 5.4. Let $h^D : X \to T^D$ be the Cartier dual of h. Then the diagram

$$\begin{tikzcd} X \arrow[r, "h^D"] \arrow[d, "u"'] & T^D \arrow[d, "u'"] \\ G \arrow[r, "\widetilde{\varphi}_L"'] & G' \end{tikzcd}$$

is commutative. In other words, the pair $(h^D, \widetilde{\varphi}_L)$ is a morphism of 1-motives from M to M^*.

Proof. Let $x \in X(S)$. We have to prove that $u'(h^D(x)) = \widetilde{\varphi_L}(u(x))$. With the identification $X \simeq X^{DD}$, the morphism $h^D(x)$ is equal to $ev_x \circ h : T \to \mathbb{G}_{\mathrm{m}}$, where $ev_x : X^D \to \mathbb{G}_{\mathrm{m}}$ is the evaluation at x. Hence, by definition, $u'(h^D(x))$ is the extension of $\overline{M}$ by $\mathbb{G}_{\mathrm{m}}$ obtained from M by pushdown along the morphism $ev_x \circ h$.

$$u'\big(h^D(x)\big) = ev_{x*}h_*M. \tag{5.3}$$

Let $M_L = [\widetilde{\varphi_L} \circ u : X \to G']$ and $\overline{M_L} = M_L/W_{-2}M_L = [\varphi_L \circ v : X \to A^*]$. Consider the two morphisms of 1-motives $\varphi'_L = (id_X, \widetilde{\varphi_L}) : M \to M_L$ and $\overline{\varphi_L} = (id_X, \varphi_L) : \overline{M} \to \overline{M_L}$ that fit in the following diagram of extensions:

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r] \arrow[d, "h"] & M \arrow[r] \arrow[d, "\varphi'_L"] & \overline{M} \arrow[r] \arrow[d, "\overline{\varphi_L}"] & 0 \\
0 \arrow[r] & X^D \arrow[r] & M_L \arrow[r] & \overline{M_L} \arrow[r] & 0.
\end{tikzcd}$$

By [18, Chp. VII, (7), and (8)] the existence of φ'_L proves that h_*M and $\overline{\varphi_L}^*M_L$ are isomorphic as extensions of $\overline{M}$ by X^D. Combining this with (5.3) we get that

$$u'\big(h^D(x)\big) = ev_{x*}\overline{\varphi_L}^*M_L. \tag{5.4}$$

We can describe extensions of $\overline{M_L}$ by X^D in terms of pairs (E, ξ), where E is an extension of A^* by X^D and ξ is a trivialization of $(\varphi_L \circ v)^*E$. In these terms, the extension M_L corresponds to G' together with the morphism $\widetilde{\varphi_L} \circ u : X \to G'$. Hence, the extension $\overline{\varphi_L}^*M_L$ of $\overline{M}$ by X^D corresponds to the pair $(\varphi_L^*G', \overline{v})$, where the trivialization $\overline{v}$ is the morphism $X \to \varphi_L^*G'$ induced by $\widetilde{\varphi_L} \circ u$, with $\widetilde{\varphi_L}$ defined in (5.1):

$$\begin{tikzcd}
 & & & X \arrow[dl, bend right, "\overline{v}"'] \arrow[d, "v"] & \\
0 \arrow[r] & X^D \arrow[r] \arrow[d, equal] & \varphi_L^*G' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & A \arrow[r] \arrow[d, "\varphi_L"] & 0 \\
0 \arrow[r] & X^D \arrow[r, "i'"] & G' \arrow[r, "\pi'"] & A^* \arrow[r] & 0.
\end{tikzcd}$$

Set theoretically $\varphi_L^*G'(S) = G' \times_{A^*} A(S)$ consists of pairs $(a, (E_{\varphi_L(a)}, \widetilde{v}))$, where $a \in A(S)$ and $(E_{\varphi_L(a)}, \widetilde{v}) \in G'(S)$, with $\widetilde{v} : X \to E_{\varphi_L(a)}$ a trivialization of $v^*E_{\varphi_L(a)}$. The morphism

$\overline{v} : X \to \varphi_L^* G'$ is then defined by

$$\overline{v}(y) = \big(v(y), \big(E_{\varphi_L(v(y))}, \widetilde{v}_{u(y)}\big)\big)$$

for any point $y \in X(S)$, where $\widetilde{v}_{u(y)}$ is defined in Equation (5.2).

Now we will construct a morphism $q : \varphi_L^* G' \to E_{\varphi_L(v(x))}$ that fits in the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & X^D \arrow[r] \arrow[d, "ev_x"] & \varphi_L^* G' \arrow[r] \arrow[d, "q"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{G}_{\mathrm{m}} \arrow[r] & E_{\varphi_L(v(x))} \arrow[r] & A \arrow[r] & 0.
\end{tikzcd} \tag{5.5}$$

This will allow us to identify the pushdown $ev_{x*}\varphi_L^* G'$ with $E_{\varphi_L(v(x))}$ and the extension $ev_{x*}\overline{\varphi_L}^* M_L$ of $\overline{M}$ by $\mathbb{G}_{\mathrm{m}}$ then corresponds to the pair $(E_{\varphi_L(v(x))}, q \circ \overline{v})$. The construction of q is as follows. Let $(a, (E_{\varphi_L(a)}, \widetilde{v}))$ be an element of $\varphi_L^* G'(S)$, that is, $a \in A(S)$ and $(E_{\varphi_L(a)}, \widetilde{v}) \in G'(S)$, with $\widetilde{v} : X \to E_{\varphi_L(a)}$ an A-morphism. In particular we have a point $\widetilde{v}(x) \in E_{\varphi_L(a)}(S)$ above $v(x)$, hence an isomorphism of line bundles $\beta : \varphi_L(a) \to \mu_{v(x)}^* \varphi_L(a)$. The latter isomorphism corresponds to a trivialization $\mathcal{O}_A \simeq \mu_{v(x)+a}^* L \otimes \mu_{v(x)}^* L^{-1} \otimes \mu_a^* L^{-1} \otimes L$. Via the symmetry isomorphism, this in turn induces a trivialization of $\mu_{v(x)+a}^* L \otimes \mu_a^* L^{-1} \otimes \mu_{v(x)}^* L^{-1} \otimes L$, hence an isomorphism of line bundles $\beta' : \varphi_L(v(x)) \to \mu_a^* \varphi_L(v(x))$. We define q by

$$q(a, \big(E_{\varphi_L(a)}, \widetilde{v}\big)\big) := (a, \beta')$$

with the above notation. In the diagram (5.5), it is obvious that the right-hand square commutes. To prove that the left square also commutes, we observe that both morphisms from X^D to $E_{\varphi_L(v(x))}$ map an element $\alpha : X \to \mathbb{G}_{\mathrm{m}}$ to the pair $(1, \alpha(x))$, where $1 \in A(S)$ is the unit of A and $\alpha(x) \in \mathbb{G}_{\mathrm{m}}(S)$ is seen as an automorphism of the line bundle $\varphi_L(v(x))$. Now it follows from Lemma 5.1 that q is automatically a group homomorphism.

We have proved that $u'(h^D(x))$ corresponds to the pair $(E_{\varphi_L(v(x))}, q \circ \overline{v})$. On the other hand, by definition of $\widetilde{\varphi}_L$, the extension $\widetilde{\varphi}_L(u(x))$ corresponds to the pair $(E_{\varphi_L(v(x))}, \widetilde{v}_{u(x)})$. Hence, to conclude the proof, it remains to prove that $q \circ \overline{v} = \widetilde{v}_{u(x)}$. Let $y \in X(S)$ be a point of X and let us prove that $q(\overline{v}(y)) = \widetilde{v}_{u(x)}(y)$. Unwinding the definitions of q, $\overline{v}$, and $\widetilde{v}_{u(x)}$, we have to prove that the isomorphism $\varphi_{u(x),y} : \varphi_L(v(x)) \to \mu_{v(y)}^* \varphi_L(v(x))$ (see Lemma 5.2) is equal to the isomorphism β' induced by $\varphi_{u(y),x} : \varphi_L(v(y)) \to \mu_{v(x)}^* \varphi_L(v(y))$ via the symmetry isomorphism as explained in the previous paragraph (with $a = v(y)$). Since π^* is faithful on the category of line bundles, it suffices to check the equality after applying π^*. In other words we have to prove that the descent datum δ on $\mathcal{L}$ satisfies the following condition; $\mu^*_{u(x)}\delta_y \otimes \delta_y^{-1}$ should be equal to the isomorphism induced by $\mu^*_{u(y)}\delta_x \otimes \delta_x^{-1}$ through the symmetry isomorphism. But this is a consequence of the cocycle condition (3.1) on the descent datum δ (use it both for δ_{x+y} and δ_{y+x}). ■

This concludes the proof of Theorem 1.1. We do not prove here that Φ : $\mathrm{Pic}(M)/\mathrm{Pic}(S) \to \mathrm{Hom}(M, M^*)$ (1.1) is a group homomorphism; this will follow from Corollary 7.2, where we give a 2nd construction of Φ, and from the comparison Theorem 7.3.

We finish this section by giving another interesting construction of the morphism $\Phi : \mathrm{Pic}(M)/\mathrm{Pic}(S) \to \mathrm{Hom}(M, M^*)$ in the special case of Kummer 1-motives, that is, 1-motives without abelian part. This construction, which is based on the 2nd dévissage of the Picard group of M, involves only the group Λ introduced in Definition 4.5.

Let $M = [u : X \to T]$ be a Kummer 1-motive over a reduced scheme S. In this case $M^* = [u^D : T^D \to X^D]$ and a morphism from M to M^* is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "u"'] & T^D \arrow[d, "u^D"] \\ T \arrow[r, "h"'] & X^D. \end{tikzcd}$$

By Definition 4.5, Λ is a subgroup of $\mathrm{Hom}(M, M^*)$; an element $\lambda \in \Lambda$ defines the morphism $M \to M^*$ given by $\lambda : X \to T^D$ and $\lambda^D : T \to X^D$.

From Proposition 1.3, we know that the kernel K of $\iota^* : \mathrm{Pic}(M) \to \mathrm{Pic}(T)$ fits in the exact sequence

$$\mathrm{Hom}(T, \mathbb{G}_m) \xrightarrow{\circ u} \mathrm{Hom}(X, \mathbb{G}_m) \xrightarrow{\beta^*} K \xrightarrow{\Theta} \Lambda \xrightarrow{\Psi} \Sigma.$$

Then, locally on S, the morphism $\Phi : \mathrm{Pic}(M) \to \mathrm{Hom}(M, M^*)$ coincides with Θ in the following sense. Let $\mathcal{L}$ be a line bundle on M. By Remark 4.2 (2), since the tori underlying 1-motives are split locally for the étale topology, there exists an étale and surjective morphism $S' \to S$ such that $(\iota^*\mathcal{L})_{|S'}$ is trivial, which means that $\mathcal{L}_{|S'} \in K$. Then $\Phi(\mathcal{L}_{|S'})$ is equal to $\Theta(\mathcal{L}_{|S'})$ via the inclusion $\Lambda \subset \mathrm{Hom}(M, M^*)$.

Remark 5.5. The homomorphism $\Phi : \mathrm{Pic}(M)/\mathrm{Pic}(S) \to \mathrm{Hom}(M, M^*)$ is far from being surjective. For example, let $M = [X \xrightarrow{u} T]$ with $X = \mathbb{Z}$, $T = \mathbb{G}_\mathrm{m}$ and u the trivial morphism. Then $\mathrm{Hom}(M, M^*)$ identifies with $\mathrm{Hom}(X, X)^2 \simeq \mathbb{Z}^2$ and by Proposition 4.8, the group $\mathrm{Pic}(M)/\mathrm{Pic}(S)$ identifies with $\mathrm{Hom}(X, \mathbb{G}_\mathrm{m}) \times \Lambda \simeq \mathbb{G}_\mathrm{m}(S) \times \mathbb{Z}$. The morphism $\Phi : \mathbb{G}_\mathrm{m}(S) \times \mathbb{Z} \to \mathbb{Z}^2$ is given by $(\gamma, n) \mapsto (n, n)$.

6 Linear Morphisms Defined by Cubical Line Bundles

In this Section we first give the definition and basic properties of cubical structure on a line bundle over a commutative group stack $\mathcal{G}$. Then we explain how a cubical line bundle on $\mathcal{G}$, that is, a line bundle on $\mathcal{G}$ endowed with a cubical structure, defines an additive functor $\mathcal{G} \to D(\mathcal{G})$ from $\mathcal{G}$ to its dual.

Let $\mathcal{G}$ be a commutative group stack over S, whose group law $(a, b) \mapsto ab$ will be denoted multiplicatively. We denote by $\mathcal{G}^3$ the commutative group stack $\mathcal{G} \times_S \mathcal{G} \times_S \mathcal{G}$. Following [13, Chp. I, 2.4] we define a functor from the category of line bundles on $\mathcal{G}$ to the category of line bundles on $\mathcal{G}^3$

$$\theta : \mathrm{PIC}(\mathcal{G}) \longrightarrow \mathrm{PIC}\big(\mathcal{G}^3\big)$$

with

$$\theta(\mathcal{L}) = m_{123}^*\mathcal{L} \otimes \big(m_{12}^*\mathcal{L}\big)^{-1} \otimes \big(m_{13}^*\mathcal{L}\big)^{-1} \otimes \big(m_{23}^*\mathcal{L}\big)^{-1} \otimes m_1^*\mathcal{L} \otimes m_2^*\mathcal{L} \otimes m_3^*\mathcal{L},$$

where for $I = \{i_1, \dots, i_l\} \subset \{1, 2, 3\}$, $m_{i_1 \dots i_l}$ denotes the additive functor $\mathcal{G}^3 \to \mathcal{G}$ given by $(a_1, a_2, a_3) \mapsto a_{i_1} \dots a_{i_l}$. (Our $\theta(\mathcal{L})$ is denoted by $\theta_3(\mathcal{L})$ in [13].) In terms of points the above definition becomes

$$\theta(\mathcal{L})_{a_1,a_2,a_3} = \mathcal{L}_{a_1a_2a_3} \otimes \big(\mathcal{L}_{a_1a_2}\big)^{-1} \otimes \big(\mathcal{L}_{a_1a_3}\big)^{-1} \otimes \big(\mathcal{L}_{a_2a_3}\big)^{-1} \otimes \mathcal{L}_{a_1} \otimes \mathcal{L}_{a_2} \otimes \mathcal{L}_{a_3} \tag{6.1}$$

for any $(a_1, a_2, a_3) \in \mathcal{G}^3$. As in [13, Chp. I, (2.4.2)] the symmetric group $\mathfrak{S}_3$ of permutations acts on $\theta(\mathcal{L})$, that is, for $(a_1, a_2, a_3) \in \mathcal{G}^3$ and for $\sigma \in \mathfrak{S}_3$ there is a natural isomorphism

$$p^\sigma_{a_1,a_2,a_3} : \theta(\mathcal{L})_{a_1,a_2,a_3} \xrightarrow{\sim} \theta(\mathcal{L})_{a_{\sigma(1)},a_{\sigma(2)},a_{\sigma(3)}}. \tag{6.2}$$

Moreover, as in [13, Chp. I, (2.4.4)], $\theta(\mathcal{L})$ is endowed with cocycle isomorphisms; for $a, b, c, d \in \mathcal{G}$ one of these cocycle isomorphisms is

$$coc_{a,b,c,d} : \theta(\mathcal{L})_{ab,c,d} \otimes \theta(\mathcal{L})_{a,b,d} \xrightarrow{\sim} \theta(\mathcal{L})_{a,bc,d} \otimes \theta(\mathcal{L})_{b,c,d}\,, \tag{6.3}$$

the others are obtained from this one by permutation.

Definition 6.1. Let $\mathcal{L}$ be a line bundle on $\mathcal{G}$. A *cubical structure on* $\mathcal{L}$ is an isomorphism $\tau : \mathcal{O}_{\mathcal{G}^3} \to \theta(\mathcal{L})$ of line bundles over $\mathcal{G}^3$ that is compatible with the isomorphisms (6.2) and (6.3). In other words,

(i) For any $\sigma \in \mathfrak{S}_3$ and any $(a_1, a_2, a_3) \in \mathcal{G}^3$, $\tau_{a_{\sigma(1)},a_{\sigma(2)},a_{\sigma(3)}} = p^{\sigma}_{a_1,a_2,a_3} \circ \tau_{a_1,a_2,a_3}$.

(ii) For any $a, b, c, d \in \mathcal{G}$, $\tau_{a,bc,d} \otimes \tau_{b,c,d} = coc_{a,b,c,d} \circ (\tau_{ab,c,d} \otimes \tau_{a,b,d})$.

A *cubical line bundle on* $\mathcal{G}$ is a pair $(\mathcal{L}, \tau)$, where $\mathcal{L}$ is a line bundle on $\mathcal{G}$ and τ is a cubical structure on $\mathcal{L}$. A *morphism of cubical line bundles* $(\mathcal{L}, \tau) \to (\mathcal{L}', \tau')$ is a morphism $f : \mathcal{L} \to \mathcal{L}'$ of line bundles on $\mathcal{G}$ such that $\tau' = \theta(f) \circ \tau$.

We denote by $\mathrm{CUB}(\mathcal{G})$ the category of cubical line bundles on $\mathcal{G}$, and by $\mathrm{CUB}^1(\mathcal{G})$ the group of isomorphism classes of cubical line bundles on $\mathcal{G}$.

Let $\mathcal{C}ub(\mathcal{G})$ be the stack of cubical line bundles on $\mathcal{G}$, that is, for any S-scheme U, $\mathcal{C}ub(\mathcal{G})(U)$ is the category of cubical line bundles on $\mathcal{G} \times_S U$. If $(\mathcal{L}, \tau)$ and $(\mathcal{L}', \tau')$ are two cubical line bundles on $\mathcal{G}$, then τ and τ' induce a canonical cubical structure on the line bundle $\mathcal{L} \otimes \mathcal{L}'$ and we denote by $(\mathcal{L}, \tau) \otimes (\mathcal{L}', \tau')$ the resulting cubical line bundle. The operation $\otimes$ endows $\mathcal{C}ub(G)$ with a structure of commutative group stack.

As in [13, Chp. I, 2.3] we also have a functor from the category of line bundles on $\mathcal{G}$ to the category of line bundles on $\mathcal{G}^2$

$$\theta_2 : \mathrm{PIC}(\mathcal{G}) \longrightarrow \mathrm{PIC}\big(\mathcal{G}^2\big)$$

defined by

$$\theta_2(\mathcal{L})_{a,b} = \mathcal{L}_{ab} \otimes \mathcal{L}_a^{-1} \otimes \mathcal{L}_b^{-1}$$

for all $\mathcal{L} \in \mathrm{PIC}(\mathcal{G})$ and all $(a, b) \in \mathcal{G}^2$. This line bundle $\theta_2(\mathcal{L})$ furnishes a morphism of stacks

$$\begin{aligned} \varphi_{\mathcal{L}} : \mathcal{G} &\longrightarrow \mathcal{H}om_{S-stacks}(\mathcal{G}, B\mathbb{G}_{\mathrm{m}}) \\ a &\longmapsto \big(\varphi_{\mathcal{L}}(a) : b \mapsto \varphi_{\mathcal{L}}(a)(b) = \theta_2(\mathcal{L})_{a,b}\big). \end{aligned}$$

It is possible to recover $\theta(\mathcal{L})$ from $\theta_2(\mathcal{L})$ via the following two canonical isomorphisms:

$$\theta_2(\mathcal{L})_{ab,c} \otimes \theta_2(\mathcal{L})^{-1}_{a,c} \otimes \theta_2(\mathcal{L})^{-1}_{b,c} \simeq \theta(\mathcal{L})_{a,b,c} \simeq \theta_2(\mathcal{L})_{a,bc} \otimes \theta_2(\mathcal{L})^{-1}_{a,b} \otimes \theta_2(\mathcal{L})^{-1}_{a,c}.$$

Now let τ be a cubical structure on $\mathcal{L}$. Through the above two isomorphisms, τ induces two isomorphisms of line bundles (thought of as partial composition laws on $\theta_2(\mathcal{L})$):

$$\tau^1_{a,b,c} : \theta_2(\mathcal{L})_{a,c} \otimes \theta_2(\mathcal{L})_{b,c} \to \theta_2(\mathcal{L})_{ab,c}$$
$$\tau^2_{a,b,c} : \theta_2(\mathcal{L})_{a,b} \otimes \theta_2(\mathcal{L})_{a,c} \to \theta_2(\mathcal{L})_{a,bc}\,.$$

Generalizing [13, Chp. I, 2.5] to line bundles on stacks, the conditions (i) and (ii) on τ imply that the two composition laws τ^1 and τ^2 are structures of symmetric biextension of $(\mathcal{G},\mathcal{G})$ by $\mathbb{G}_\mathrm{m}$ on the $\mathbb{G}_m$-torsor $\theta_2(\mathcal{L})$ (see [2, Definition 5.1] for the notion of biextension of commutative group stacks). In particular, the isomorphism τ^2 provides for all points a, b, c of $\mathcal{G}$ a functorial isomorphism

$$\left(\tau^2_{a,b,c}\right)^{-1} : \varphi_{\mathcal{L}}(a)(bc) \to \varphi_{\mathcal{L}}(a)(b).\varphi_{\mathcal{L}}(a)(c).$$

The commutativity and associativity conditions that τ^2 satisfies (see for instance the diagrams (1.1.3) and (1.1.5) p.2 in [3]) imply that this isomorphism is compatible with the commutativity and associativity isomorphisms of $\mathcal{G}$ and $B\mathbb{G}_\mathrm{m}$. Hence, $\varphi_{\mathcal{L}}(a)$, equipped with this isomorphism, is an additive functor from $\mathcal{G}$ to $B\mathbb{G}_\mathrm{m}$, that is, $\varphi_{\mathcal{L}}(a)$ is a point of $D(\mathcal{G}) = \mathcal{H}om(\mathcal{G}, B\mathbb{G}_m)$. This defines a morphism of stacks

$$\varphi_{\mathcal{L}} : \mathcal{G} \longrightarrow D(\mathcal{G}).$$

The isomorphism $(\tau^1)^{-1}$ defines a functorial isomorphism from $\varphi_{\mathcal{L}}(ab)$ to $\varphi_{\mathcal{L}}(a).\varphi_{\mathcal{L}}(b)$ hence it endows $\varphi_{\mathcal{L}}$ with the structure of an additive functor. The required compatibility conditions are given by the commutativity and associativity conditions on τ^1 and by the compatibility of τ^1 and τ^2 with each other (see [3], diagrams (1.1.4), (1.1.5), and (1.1.6)). From now on we denote by $\varphi_{(\mathcal{L},\tau)}$ the resulting additive functor from $\mathcal{G}$ to $D(\mathcal{G})$.

If $\alpha : (\mathcal{L}, \tau) \to (\mathcal{L}', \tau')$ is an isomorphism of cubical line bundles, the isomorphism $\theta_2(\alpha) : \theta_2(\mathcal{L}) \to \theta_2(\mathcal{L}')$ provides an isomorphism of functors from $\varphi_{(\mathcal{L},\tau)}$ to $\varphi_{(\mathcal{L}',\tau')}$. Since α is compatible with the cubical structures τ and τ', it follows that the latter isomorphism of functors is compatible with the additive structures of $\varphi_{(\mathcal{L},\tau)}$ and $\varphi_{(\mathcal{L}',\tau')}$, in other words it is an isomorphism of additive functors, that is, it is an isomorphism in $\mathcal{H}om(\mathcal{G}, D(\mathcal{G}))$. This way the construction $(\mathcal{L}, \tau) \mapsto \varphi_{(\mathcal{L},\tau)}$ is functorial and we get a morphism of stacks from $\mathcal{C}ub(\mathcal{G})$ to $\mathcal{H}om(\mathcal{G}, D(\mathcal{G}))$. Lastly, if $(\mathcal{L}, \tau)$ and $(\mathcal{L}', \tau')$ are two cubical line bundles, the canonical isomorphism $\theta_2(\mathcal{L} \otimes \mathcal{L}') \simeq \theta_2(\mathcal{L}) \otimes \theta_2(\mathcal{L}')$ ([13, Chp. I, 2.2.1]) induces an isomorphism of functors from $\varphi_{(\mathcal{L},\tau)\otimes(\mathcal{L}',\tau')}$ to $\varphi_{(\mathcal{L},\tau)}.\varphi_{(\mathcal{L}',\tau')}$, which is compatible with the commutativity and associativity isomorphisms. Summing up, we have proved the following theorem.

Theorem 6.2. Let $\mathcal{G}$ be a commutative group S-stack.

1. Let $(\mathcal{L},\tau)$ be a cubical line bundle on $\mathcal{G}$. Then there is a natural additive functor $\varphi_{(\mathcal{L},\tau)}:\mathcal{G}\to D(\mathcal{G})$, given by the formula

$$\begin{array}{rccl} \varphi_{(\mathcal{L},\tau)}: & \mathcal{G} & \longrightarrow & D(\mathcal{G}) \\ & a & \longmapsto & \big(b\mapsto \theta_2(\mathcal{L})_{a,b}=\mathcal{L}_{ab}\otimes\mathcal{L}_a^{-1}\otimes\mathcal{L}_b^{-1}\big). \end{array}$$

2. The above construction induces an additive functor

$$\begin{array}{rccl} \varphi: & \mathcal{C}ub(\mathcal{G}) & \longrightarrow & \mathcal{H}om(\mathcal{G},D(\mathcal{G})) \\ & (\mathcal{L},\tau) & \longmapsto & \varphi_{(\mathcal{L},\tau)}. \end{array}$$

Remark 6.3. If a is a point of $\mathcal{G}$, the morphism $\varphi_{(\mathcal{L},\tau)}(a):\mathcal{G}\to B\mathbb{G}_{\mathrm{m}}$ corresponds to the line bundle $(\mu_a^*\mathcal{L})\otimes(f^*a^*\mathcal{L})^{-1}\otimes\mathcal{L}^{-1}$ on $\mathcal{G}$, where $\mu_a:\mathcal{G}\to\mathcal{G}$ is the translation by a and $f:\mathcal{G}\to S$ is the structural morphism. In particular, if $\mathcal{G}$ is an abelian S-scheme A, then $\varphi_{(\mathcal{L},\tau)}$ coincides with the classical morphism $\varphi_{\mathcal{L}}:A\to A^*$ defined by $\varphi_{\mathcal{L}}(a)=(\mu_a^*\mathcal{L})\otimes\mathcal{L}^{-1}$. By [16, VIII Prop. 1.8] $\varphi_{\mathcal{L}}=0$ if and only if $\mathcal{L}\in\mathrm{Pic}^0(A)$, hence φ factorizes through the Néron–Severi group $NS(A)$ and induces $\overline{\varphi}:NS(A)\to\mathrm{Hom}(A,A^*)$.

7 The Theorem of the Cube for 1-motives

If $\mathcal{G}$ is a commutative group stack with neutral object e, we denote by $\mathrm{RLB}(\mathcal{G})$ the category of line bundles on $\mathcal{G}$ *rigidified along* e, that is, the category of pairs $(\mathcal{L},\xi)$, where $\mathcal{L}$ is a line bundle on $\mathcal{G}$ and $\xi:\mathcal{O}_S\to e^*\mathcal{L}$ is an isomorphism of line bundles.

Theorem 7.1 (Theorem of the cube for 1-motives). Let S be a scheme. Let $[X\overset{u}{\to}G]$ be a complex of commutative S-group schemes. Assume that one of the following holds:

1. G is an abelian scheme.
2. S is normal, $X\times_S X$ is reduced, G is smooth with connected fibers, and the maximal fibers of G are multiple extensions of abelian varieties, tori (not necessarily split), and groups $\mathbb{G}_a$.

Let $\mathcal{M} = \mathrm{st}([X \stackrel{u}{\to} G])$ be the commutative group stack associated to the above complex via the equivalence of categories (2.1). Then the forgetful functor

$$\mathrm{CUB}(\mathcal{M}) \longrightarrow \mathrm{RLB}(\mathcal{M})$$

is an equivalence of categories.

Proof. In the sequel, the group laws of $\mathcal{M}$ and G are denoted multiplicatively while the one of X is denoted additively. We denote by $\iota : G \to \mathcal{M}$ the canonical projection and by 1 the unit section of G. Then $\iota \circ 1 : S \to \mathcal{M}$ is a neutral section of $\mathcal{M}$ and will also be denoted by 1.

By (6.1) for any line bundle $\mathcal{L}$ on $\mathcal{M}$, there is a canonical isomorphism $\theta(\mathcal{L})_{1,1,1} \simeq \mathcal{L}_1$, where $\mathcal{L}_1$ is the line bundle $1^*\mathcal{L}$ on S. Hence, a cubical structure $\tau : \mathcal{O}_{\mathcal{M}^3} \to \theta(\mathcal{L})$ on $\mathcal{L}$ induces a natural rigidification of $\mathcal{L}$ along the unit section that we still denote by $\tau_{1,1,1} : \mathcal{O}_S \to \mathcal{L}_1$ (by a slight abuse of notation). The operation $(\mathcal{L}, \tau) \mapsto (\mathcal{L}, \tau_{1,1,1})$ defines a functor $\mathrm{CUB}(\mathcal{M}) \to \mathrm{RLB}(\mathcal{M})$, which is the abovementioned forgetful functor. By [13, Chp. I, 2.6] we already know that G satisfies the theorem of the cube, that is, that the forgetful functor $\mathrm{CUB}(G) \to \mathrm{RLB}(G)$ is an equivalence of categories.

Let us prove that $\mathrm{CUB}(\mathcal{M}) \to \mathrm{RLB}(\mathcal{M})$ is fully faithful. Let $(\mathcal{L}, \tau)$ and $(\mathcal{L}', \tau')$ be two cubical line bundles on $\mathcal{M}$ and let $f : \mathcal{L} \to \mathcal{L}'$ be a morphism in $\mathrm{RLB}(\mathcal{M})$, that is, a morphism that is compatible with the rigidifications $\tau_{1,1,1}$ and $\tau'_{1,1,1}$. We have to prove that f is compatible with τ and τ', that is, that $\tau' = \theta(f) \circ \tau$. Since the functor ι^* from the category of line bundles on $\mathcal{M}$ to the category of line bundles on G is faithful, this is equivalent to $\iota^*\tau' = \iota^*\theta(f) \circ \iota^*\tau$. But, up to canonical isomorphisms, $\iota^*\theta(f)$ identifies with $\theta(\iota^* f)$. Moreover, by assumption on f, $\tau'_{1,1,1} = f_1 \circ \tau_{1,1,1}$, hence $(\iota^*\tau')_{1,1,1} = (\iota^* f)_1 \circ (\iota^*\tau)_{1,1,1}$. This means that $\iota^* f : \iota^*\mathcal{L} \to \iota^*\mathcal{L}'$ is compatible with the rigidifications induced by the cubical structures $\iota^*\tau$ and $\iota^*\tau'$ on $\iota^*\mathcal{L}$ and $\iota^*\mathcal{L}'$. By the theorem of the cube for G, this implies the desired equality $\iota^*\tau' = \theta(\iota^* f) \circ \iota^*\tau$.

Now let us prove that $\mathrm{CUB}(\mathcal{M}) \to \mathrm{RLB}(\mathcal{M})$ is essentially surjective. As observed at the end of Section 3, a line bundle $\mathcal{L}$ on $\mathcal{M}$ is a pair (L, δ), where $L = \iota^*\mathcal{L}$ is a line bundle on G and $\delta : p_2^* L \to \mu^* L$ is a descent datum for L. Let $\xi : \mathcal{O}_S \to \mathcal{L}_1$ be a rigidification of $\mathcal{L}$ along the unit section of $\mathcal{M}$. Via the canonical isomorphism $\mathcal{L}_1 \simeq L_1$, ξ is also a rigidification of L along the unit section of G. By the theorem of the cube for G, there is a cubical structure $\tau : \mathcal{O}_{G^3} \to \theta(L)$ that induces ξ, that is, such that $\tau_{1,1,1} = \xi$. We want to construct a cubical structure $\overline{\tau} : \mathcal{O}_{\mathcal{M}^3} \to \theta(\mathcal{L})$ that induces ξ. The group stack $\mathcal{M}^3$ is canonically isomorphic to the quotient stack $[G^3/X^3]$ with the action of X^3 on G^3 by translations via $u^3 : X^3 \to G^3$. As for $\mathcal{M}$, we identify the category of line bundles on $\mathcal{M}^3$ with the category of line bundles on G^3 equipped with a descent datum. The line bundle $\mathcal{O}_{\mathcal{M}^3}$ corresponds to $\mathcal{O}_{G^3}$ equipped with the canonical isomorphism $p_2^*\mathcal{O}_{G^3} \to \mu^*\mathcal{O}_{G^3}$ (where $p_2, \mu : X^3 \times_S G^3 \to G^3$ denote the 2nd projection and the action by translation, respectively). The line bundle $\theta(\mathcal{L})$ on $\mathcal{M}^3$ corresponds to the line bundle $\theta(L)$ on G^3 equipped with the descent datum $p_2^*\theta(L) \simeq \theta(p_2^*L) \stackrel{\theta(\delta)}{\to} \theta(\mu^*L) \simeq \mu^*\theta(L)$ that by a slight abuse we denote by $\theta(\delta)$. In terms of points, $\theta(\delta)$ can be described as follows: for any points $x = (x_1, x_2, x_3)$ of X^3 and $a = (a_1, a_2, a_3)$ of G^3,

$$\theta(\delta)_{x,a} : \theta(L)_a \to \theta(L)_{u^3(x)a} \tag{7.1}$$

is equal to $\delta_{x_1+x_2+x_3,a_1a_2a_3} \otimes \delta^{-1}_{x_1+x_2,a_1a_2} \otimes \delta^{-1}_{x_1+x_3,a_1a_3} \otimes \delta^{-1}_{x_2+x_3,a_2a_3} \otimes \delta_{x_1,a_1} \otimes \delta_{x_2,a_2} \otimes \delta_{x_3,a_3}$.

We claim that the following diagram of line bundles on $X^3 \times_S G^3$ commutes:

$$\begin{tikzcd} p_2^*\mathcal{O}_{G^3} \arrow[r, "can."] \arrow[d, "p_2^*\tau"'] & \mu^*\mathcal{O}_{G^3} \arrow[d, "\mu^*\tau"] \\ p_2^*\theta(L) \arrow[r, "\theta(\delta)"'] & \mu^*\theta(L)\,. \end{tikzcd} \tag{7.2}$$

The proof of this claim will be the main part of the proof. It is equivalent to saying that for any points x of X^3 and a of G^3, we have $\theta(\delta)_{x,a} \circ \tau_a = \tau_{u^3(x)a}$. For any S-scheme U, we identify $\mathrm{Aut}(\mathcal{O}_U)$ with $\mathbb{G}_\mathrm{m}(U)$ and this allows us to define a morphism of S-schemes

$$\begin{array}{rrcl} \lambda : & X^3 \times_S G^3 & \longrightarrow & \mathbb{G}_\mathrm{m} \\ & (x, a) & \longmapsto & \tau^{-1}_{u^3(x)a} \circ \theta(\delta)_{x,a} \circ \tau_a\,. \end{array}$$

Now to prove the claim we have to prove that λ is constant equal to 1.

By (3.1), the following diagram commutes:

$$\begin{tikzcd} \theta(L)_a \arrow[rr, "\theta(\delta)_{x+x',a}"] \arrow[dr, "\theta(\delta)_{x',a}"'] & & \theta(L)_{u^3(x)u^3(x')a}\,. \\ & \theta(L)_{u^3(x')a} \arrow[ur, "\theta(\delta)_{x,u^3(x')}"'] & \end{tikzcd}$$

It follows that for any $x, x' \in X^3$ and any $a \in G^3$ we have the equation

$$\lambda(x + x', a) = \lambda\big(x, u^3(x')a\big).\lambda(x', a). \tag{7.3}$$

For any $x \in X^3$, $a \in G^3$, and any permutation $\sigma \in \mathfrak{S}_3$, by the condition (i) of Definition 6.1, the left and right triangles in the following diagram commute (where for $a = (a_1, a_2, a_3)$ we write $a^\sigma = (a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$)

$$\begin{tikzcd}
& \theta(L)_a \arrow[r, "\theta(\delta)_{x,a}"] \arrow[dd, "p_a^\sigma"'] & \theta(L)_{u^3(x)a} \arrow[dd, "p^\sigma_{u^3(x)a}"] & \\
\mathcal{O}_U \arrow[ur, "\tau_a"] \arrow[dr, "\tau_{a^\sigma}"'] & & & \mathcal{O}_U . \arrow[ul, "\tau_{u^3(x)a}"'] \arrow[dl, "\tau_{u^3(x^\sigma)a^\sigma}"] \\
& \theta(L)_{a^\sigma} \arrow[r, "\theta(\delta)_{x^\sigma,a^\sigma}"'] & \theta(L)_{u^3(x^\sigma)a^\sigma} &
\end{tikzcd}$$

The central square also commutes by construction of the canonical isomorphism p_a^σ and of $\theta(\delta)$. Hence,

$$\lambda\big(x^\sigma, a^\sigma\big) = \lambda(x, a). \tag{7.4}$$

Now let us choose $x \in X^3$ and $a \in G^3$ such that $x_3 = 0$ and $a_3 = 1$. From the above description (7.1) of $\theta(\delta)$ we see that, via the canonical isomorphisms $\theta(L)_a \simeq \theta(L)_{1,1,1}$ and $\theta(L)_{u^3(x)a} \simeq \theta(L)_{1,1,1}$, the isomorphism $\theta(\delta)_{x,a}$ is just the identity of $\theta(L)_{1,1,1}$. Moreover, as in [13, Chp. I, 2.5.3], from condition (ii) of Definition 6.1 it follows that $\tau_a = \tau_{u^3(x)a} = \tau_{1,1,1}$. Using (7.4), we get

$$\lambda(x, a) = 1 \tag{7.5}$$

as soon as there is an index i such that $x_i = 0$ and $a_i = 1$. In particular, if $x_i = 0$ for some i, we have $\lambda(x, 1) = 1$. Hence, Lemma 4.1, applied to the S-group scheme G^3, implies that λ is a group homomorphism in the variable a, that is, for any $x \in X^3$ such that some x_i is zero, and for any $a, a' \in G^3$ we have

$$\lambda(x, aa') = \lambda(x, a).\lambda(x, a'). \tag{7.6}$$

(Actually Rosenlicht only applies when the base scheme S is reduced. But we apply it for the "universal" point $(\mathrm{id}_{X \times_S X}, 0) \in X^3(U)$, where the base scheme $U = X \times_S X$ is reduced, and the general case follows.) In particular for $x = (x_1, 0, 0) \in X^3$ and for any $a = (a_1, a_2, a_3) \in G^3$, using (7.6) and (7.5) we get

$$\lambda(x, a) = \lambda(x, (a_1, a_2, 1))\lambda(x, (1, 1, a_3)) = 1.$$

By (7.4) this proves that $\lambda(x, a) = 1$ as soon as two of the x_i's are zero and finally using (7.3) this proves that λ is constant equal to 1. This finishes the proof of the claim.

Now, the commutativity of (7.2) means that τ is an isomorphism in the category of line bundles on G^3 equipped with descent data. Hence, it corresponds to an isomorphism $\overline{\tau} : \mathcal{O}_{\mathcal{M}^3} \to \theta(\mathcal{L})$. Moreover, the condition (i) (resp. (ii)) of Definition 6.1 can be expressed by the commutativity of some diagrams of line bundles over $\mathcal{M}^3$ (resp. $\mathcal{M}^4$). Since the functor ι^* is faithful, the fact that τ satisfies the conditions (i) and (ii) of Definition 6.1 implies that $\overline{\tau}$ itself satisfies these two conditions. Hence, $\overline{\tau}$ is a cubical structure on $\mathcal{L}$. From $\tau_{1,1,1} = \xi$ it follows that $\overline{\tau}_{1,1,1} = \xi$ and this concludes the proof of the theorem. ∎

Corollary 7.2. With the notation and assumptions of Theorem 7.1, there is a functorial group homomorphism $\Phi' : \mathrm{Pic}(\mathcal{M})/\mathrm{Pic}(S) \to \mathrm{Hom}(\mathcal{M}, D(\mathcal{M}))$.

Proof. Since $\mathrm{Pic}(\mathcal{M})/\mathrm{Pic}(S)$ is isomorphic to the group of isomorphism classes of rigidified line bundles on $\mathcal{M}$, this is an immediate consequence of Theorems 6.2 and 7.1. ∎

Theorem 7.3. Let M be a 1-motive defined over scheme S. Assume that the base scheme S is normal. The morphism Φ' defined above coincides with the morphism $\Phi : \mathrm{Pic}(M)/\mathrm{Pic}(S) \to \mathrm{Hom}(M, M^*)$ constructed in Section 5.

Proof. Let $(\mathcal{L}, \delta)$ be a line bundle on M. We want to prove that $\Phi(\mathcal{L}, \delta) = \Phi'(\mathcal{L}, \delta)$. The question is local on S hence as in Section 5 we may assume that the line bundle $\mathcal{L}$ on G is induced by a line bundle L on A, that is, $\mathcal{L} = \pi^* L$. To prove the theorem it suffices to prove that the morphisms $A \to A^*$, $X \to T^D$, and $T \to X^D$ induced by $\Phi'(\mathcal{L}, \delta)$ are equal to φ_L, h^D and h of Section 5, respectively.

The Cartier dual of G as a 1-motive is $G^* = [T^D \xrightarrow{v'} A^*]$ and $\mathrm{Hom}(G, G^*) = \mathrm{Hom}(A, A^*)$. By functoriality of Φ', the morphisms $\iota : G \to M$ and $\pi : G \to A$ induce a commutative diagram:

$$\begin{tikzcd}
\mathrm{Pic}(M) \arrow[r, "\iota^*"] \arrow[d, "\Phi'"'] & \mathrm{Pic}(G) \arrow[d, "\Phi'_G"] & \mathrm{Pic}(A) \arrow[l, "\pi^*"'] \arrow[d, "\Phi'_A"] \\
\mathrm{Hom}(M,M^*) \arrow[r] & \mathrm{Hom}(G,G^*) & \mathrm{Hom}(A,A^*) \arrow[l, "\sim"']
\end{tikzcd}.$$

The morphism $A \to A^*$ induced by $\Phi'(\mathcal{L},\delta)$ is the image of $\Phi'(\mathcal{L},\delta)$ under the bottom horizontal map of this diagram. Hence, it is equal to $\Phi'_A(L)$, which is equal to φ_L by Remark 6.3.

Now let us prove that the morphism $\xi : T \to X^D$ induced by $\Phi'(\mathcal{L},\delta)$ is equal to h. To this end we consider the action of $\Phi'(\mathcal{L},\delta)$ on the objects of $\mathrm{st}(M)$. Let $t \in T(S)$ be a point of T. Its image $i(t) \in G(S)$ induces an object of the stack $\mathrm{st}(M)$ still denoted by $i(t)$, and by definition $\Phi'(\mathcal{L},\delta)(i(t))$ is the morphism from $\mathrm{st}(M)$ to $B\mathbb{G}_\mathrm{m}$ that maps an object b to $\theta_2(\mathcal{L})_{i(t),b}$. To get the induced morphism from X to $\mathbb{G}_\mathrm{m}$ it suffices to consider the action of $\Phi'(\mathcal{L},\delta)(i(t))$ on the arrows of the stack $\mathrm{st}(M)$. If $b_1, b_2 \in G(S)$ and if $x \in X(S)$ is an arrow from b_1 to b_2 in $\mathrm{st}(M)$ (i.e., $u(x) = b_2 - b_1$) then $\Phi'(\mathcal{L},\delta)(i(t))$ maps this arrow to the induced isomorphism from $\theta_2(\mathcal{L})_{i(t),b_1}$ to $\theta_2(\mathcal{L})_{i(t),b_2}$. The induced element $\xi(t)(x) \in \mathbb{G}_\mathrm{m}(S)$ does not depend on the choice of the source b_1 hence we may choose $b_1 = 1$ and $\xi(t)(x)$ is the point of $\mathbb{G}_\mathrm{m}(S)$ induced by the isomorphism $\theta_2(\mathcal{L})_{i(t),1} \to \theta_2(\mathcal{L})_{i(t),u(x)}$ induced by δ. The latter is $\delta_{x,i(t)} \otimes \delta^{-1}_{0,i(t)} \otimes \delta^{-1}_{x,1}$. But, by the cocycle condition (3.1), $\delta_{0,i(t)}$ is the identity, hence this corresponds to the description of h given in Remark 5.3.

To prove that $\Phi'(\mathcal{L},\delta)$ induces h^D from X to T^D we have to consider its action on the arrows of $\mathrm{st}(M)$. The argument is very similar to the above one and left to the reader. ■

Remark 7.4. The hypothesis of normalness on S is essential in order to identify the categories of cubical line bundles with the categories of line bundles rigidified along the unit section, even on a torus. See [13, Chp. I, Example 2.6.1] for a counterexample. Hence, if the base scheme S is not normal, we only have the functorial homomorphism $\mathrm{CUB}^1(M) \to \mathrm{Hom}(M,M^*)$ given by Theorem 6.2. The morphism $\mathrm{CUB}^1(M) \to \mathrm{Pic}(M)/\mathrm{Pic}(S)$ induced by the forgetful functor $\mathrm{CUB}(M) \to \mathrm{RLB}(M)$ is neither injective nor surjective in general. If S is reduced, we can prove that the forgetful functor is fully faithful, hence $\mathrm{CUB}^1(M) \to \mathrm{Pic}(M)/\mathrm{Pic}(S)$ is injective. This inclusion is an isomorphism if the base scheme S is normal.

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