HIGHER HAMMING WEIGHTS FOR LOCALLY RECOVERABLE CODES ON ALGEBRAIC CURVES

Edoardo Ballico *& Chiara Marcolla[1]

ABSTRACT

We study locally recoverable codes on algebraic curves. In the first part of the manuscript, we provide a bound on the generalized Hamming weight of these codes. In the second part, we propose a new family of algebraic geometric LRC codes, which are LRC codes from the Norm-Trace curve. Finally, using some properties of Hermitian codes, we improve the bounds on the distance proposed in [1] of some Hermitian LRC codes.

1 INTRODUCTION

The ν-th generalized Hamming weight $d_\nu(C)$ of a linear code C is the minimum support size of ν-dimensional subcodes of C. The sequence $d_1(C), \dots, d_k(C)$ of generalized Hamming weights was introduced by Wei [37] to characterize the performance of a linear code on the wire-tap channel of type II. Later, the GHWs of linear codes have been used in many other applications regarding the communications, as for bounding the covering radius of linear codes [15], in network coding [26], in the context of list decoding [7, 9], and finally for secure secret sharing [18]. Moreover, in [2] the authors show in which way an arbitrary linear code gives rise to a secret sharing scheme, in [16, 17] the connection between the trellis or state complexity of a code and its GHWs is found and in [4] the author proves the equivalence to the dimension/length profile of a code and its generalized Hamming weight. For these reasons, the GHWs (and their *extended* version, the *relative* generalized Hamming weights [21, 19]) play a central role in coding theory. In particular, generalized and relative generalized Hamming weights are studied for Reed-Muller codes [10, 23] and for codes constructed by using an algebraic curve [6]

* The first author is partially supported by MIUR and GNSAGA of INDAM (Italy).

* *Department of Mathematics, University of Trento, Italy.* Email address: ballico@science.unitn.it

[1] *Department of Mathematics, University of Turin, Italy.* Email address: chiara.marcolla@unito.it

as Goppa codes [24, 38], Hermitian codes [12, 25] and Castle codes [27].

In this paper, we provide a bound on the generalized Hamming weight of locally recoverable codes on the algebraic curves proposed in [1]. Moreover, we introduce a new family of algebraic geometric LRC codes and improve the bounds on the distance for some Hermitian LRC codes.

Locally recoverable codes were introduced in [8] and they have been significantly studied because of their applications in distributed and cloud storage systems [3, 13, 32, 34, 35]. We recall that a code $C \in (\mathbb{F}_q)^n$ has locality r if every symbol of a codeword c can be recovered from a subset of r other symbols of c.
In other words, we consider a finite field $K = \mathbb{F}_q$, where q is a power of a prime, and an $[n,k]$ code C over the field K, where $k = \log_q(|C|)$. For each $i \in \{1,\dots,n\}$ and each $a \in K$ set $C(i,a) = \{c \in C \mid c_i = a\}$. For each $I \subseteq \{1,\dots,n\}$ and each $S \subseteq C$ let S_I be the restriction of S to the coordinates in I.

Definition 1. Let C be an $[n,k]$ code over the field K, where $k = \log_q(|C|)$. Then C is said to have **all-symbol locality r** if for each $a \in \mathbb{F}_q$ and each $i \in \{1,\dots,n\}$ there is $I_i \subset \{1,\dots,n\} \setminus \{i\}$ with $|I_i| \leqslant r$, such that for $C_{I_i}(i,a) \cap C_{I_i}(i,a') = \emptyset$ for all $a \neq a'$. We use the notation (n,k,r) to refer to the parameters of this code.

Note that if we receive a codeword c correct except for an erasure at i, we can recover the codeword by looking at its coordinates in I_i. For this reason, I_i is called a *recovering set* for the symbol c_i.

Let C be an (n,k,r) code, then the distance of this code has to verify the bound proved in [28, 8] that is $d \leqslant n - k - \lceil k/r \rceil + 2$. The codes that achieve this bound with equality are called *optimal* LRC codes [32, 34, 35]. Note that when $r = k$, we obtain the Singleton bound, therefore optimal LRC codes with $r = k$ are MDS codes.

LAYOUT OF THE PAPER This paper is divided as follows. In Section 2 we recall the notions of algebraic geometric codes and the definition of algebraic geometric locally recoverable codes introduced in [1]. In Section 3 we provide a bound on the generalized Hamming weights of the latter codes. In Section 4 we propose a new family of algebraic geometric LRC codes, which are LRC codes from the Norm–Trace curve. Finally, in Section 5 we improve the bounds on the distance proposed in [1] for some Hermitian LRC codes, using some properties of the Hermitian codes.

2 PRELIMINARY NOTIONS

2.1 Algebraic geometric codes

Let $K = \mathbb{F}_q$ be a finite field, where q is a power of a prime. Let $\mathcal{X}$ be a smooth projective absolutely irreducible nonsingular curve over K. We denote by $K(\mathcal{X})$ the rational functions field on $\mathcal{X}$. Let D be a divisor on the curve $\mathcal{X}$. We recall that the *Riemann-Roch space* associated to D is a vector space $\mathcal{L}(D)$ over K defined as

$$\mathcal{L}(D) = \{f \in K(\mathcal{X}) \mid (f) + D \geqslant 0\} \cup \{0\}.$$

where we denote by (f) the divisor of f.

Assume that $P_1, \dots, P_n$ are rational points on $\mathcal{X}$ and D is a divisor such that $D = P_1 + \dots + P_n$. Let G be some other divisor such that $supp(D) \cap supp(G) = \emptyset$. Then we can define the algebraic geometric code as follows:

Definition 2. The **algebraic geometric code** (or AG code) $C(D, G)$ associated with the divisors D and G is defined as

$$C(D, G) = \{(f(P_1), \dots, f(P_n)) \mid f \in \mathcal{L}(G)\} \subset K^n.$$

The dual $C^{\perp}(D, G)$ of $C(D, G)$ is an algebraic geometric code.

In other words an algebraic geometric code is the image of the evaluation map $\mathrm{Im}(ev_D) = C(D, G)$, where the *evaluation map* $ev_D : \mathcal{L}(G) \to K^n$ is given by

$$ev_D(f) = (f(P_1), \dots, f(P_n)) \in K^n.$$

Note that if $D = P_1 + \dots + P_n$ and we denote by $\mathcal{P} = \{P_1, \dots, P_n\}$ we can also indicate ev_D as $ev_{\mathcal{P}}$.

2.2 Algebraic geometric locally recoverable codes

In this section we consider the construction of algebraic geometric locally recoverable codes of [1].

Let $\mathcal{X}$ and $\mathcal{Y}$ be smooth projective absolutely irreducible curves over K. Let $g : \mathcal{X} \to \mathcal{Y}$ be a rational separable map of curves of degree $r + 1$. Since g is separable, then there exists a function $x \in K(\mathcal{X})$ such that $K(\mathcal{X}) = K(\mathcal{Y})(x)$ and that x satisfies the equation $x^{r+1} + b_r x^r + \dots + b_0 = 0$, where $b_i \in K(\mathcal{Y})$. The function x can be considered as a map $x : \mathcal{X} \to \mathbb{P}_K$. Let $h = \deg(x)$ be the degree of x.
We consider a subset $S = \{P_1, \dots, P_s\} \subset \mathcal{Y}(K)$ of $\mathbb{F}_q$-rational points of $\mathcal{Y}$, a divisor Q_∞ such that $supp(Q_\infty) \cap supp(S) = \emptyset$ and a positive divisor $D = tQ_\infty$. We denote by

$$\mathcal{A} = g^{-1}(S) = \{P_{ij}, \text{ where } i = 0, \dots, r,\ j = 1, \dots, s\} \subset \mathcal{X}(K),$$

where $g(P_{ij}) = P_i$ for all i, j and assume that b_i are functions in $\mathcal{L}(n_i Q_\infty)$ for some natural numbers n_i with $i = 1, \dots, r$.
Let $\{f_1, \dots, f_m\}$ be a basis of the Riemann-Roch space $\mathcal{L}(D)$. By the Riemann-Roch Theorem we have that $m \geqslant \deg(D) + 1 - g_{\mathcal{Y}}$, where $g_{\mathcal{Y}}$ is the genus of $\mathcal{Y}$.

From now on, we assume that $m = \deg(D) + 1 - g_{\mathcal{Y}}$, where $\deg(D) = t\ell$, and we consider the K-subspace V of $K(\mathcal{X})$ of dimension rm generated by

$$\mathcal{B} = \{f_j x^i,\ i = 0, \dots, r-1,\ j = 1, \dots, m\}.$$

We consider the evaluation map $ev_{\mathcal{A}} : V \to K^{(r+1)s}$. Then we have the following theorem.

Theorem 1. *The linear space* $C(D, g) = \mathrm{Span}_{K^{(r+1)s}}\langle ev_{\mathcal{A}}(\mathcal{B})\rangle$ *is an* (n, k, r) *algebraic geometric LRC code with parameters*

$$\begin{aligned} n &= (r+1)s \\ k &= rm \geqslant r(t\ell + 1 - g_{\mathcal{Y}}) \\ d &\geqslant n - t\ell(r+1) - (r-1)h. \end{aligned}$$

Proof. See Theorem 3.1 of [1]. □

The AG LRC codes have an additional property. They are LRC codes (n, k, r) with $(r+1) \mid n$ and $r \mid k$. The set $\{1, \ldots, n\}$ can be divided into $n/(r+1)$ disjoint subsets U_j for $1 \leqslant j \leqslant s$ with the same cardinality $r+1$. For each i the set $I_i \subseteq \{1, \ldots, n\} \setminus \{i\}$ is the complement of i in the element of the partition U_j containing j, i.e. for all $i, j \in \{1, \ldots, n\}$ either $I_i = I_j$ or $I_i \cap I_j = \emptyset$.
Moreover, they have also the following nice property. Fix $w \in (K)^n$ and denote by $w_{U_j} = \{w_\iota$, for any $\iota \in U_j\}$. Suppose we receive all the symbols in U_j. There is a simple linear parity test on the $r+1$ symbols of U_j such that if this parity check fails we know that at least one of the symbols in U_j is wrong. If we are guaranteed (or we assume) that at most one of the symbols in U_j is wrong and the parity check is OK, then all the symbols in U_j are correct. Moreover we can recover an erased symbol w_ι, with $\iota \in U_j$ using a polynomial interpolation through the points of the recovering set w_{U_j}.

3 GENERALIZED HAMMING WEIGHTS OF AG LRC CODES

Let K be a field and let $\mathcal{X}$ be a smooth and geometrically connected curve of genus $g \geqslant 2$ defined over the field K. We also assume $\mathcal{X}(K) \neq \emptyset$. We recall the following definitions:

Definition 3 ([29], [30])**.** The K-**gonality** $\gamma_K(\mathcal{X})$ of $\mathcal{X}$ over a field K is the smallest possible degree of a dominant rational map $\mathcal{X} \to \mathbb{P}^1_K$. For any field extension L of K, we define also the L-**gonality** $\gamma_L(\mathcal{X})$ of $\mathcal{X}$ as the gonality of the base extension $\mathcal{X}_L = \mathcal{X} \times_K L$. It is an invariant of the function field $L(\mathcal{X})$ of $\mathcal{X}_L$.

Moreover, for each integer $i > 0$, the i-*th gonality* $\gamma_{i,L}(\mathcal{X})$ of $\mathcal{X}$ is the minimal degree z such that there is $R \in \mathrm{Pic}^z(\mathcal{X})(L)$ with $h^0(R) \geqslant i+1$. The sequence $\gamma_{i,\overline{K}}(\mathcal{X})$ is the usual *gonality sequence* [20]. Moreover, the integer $\gamma_{1,K}(\mathcal{X}) = \gamma_K(\mathcal{X})$ is the K-gonality of $\mathcal{X}$.

Let $K = \mathbb{F}_q$ a finite field with q elements. Let $C \subset K^n$ be a linear $[n, k]$ code over K. We recall that the *support* of C is defined as follows

$$\mathrm{supp}(C) = \{i \mid c_i \neq 0 \text{ for some } c \in C\}.$$

So $\sharp\mathrm{supp}(C)$ is the number of nonzero columns in a generator matrix for C. Moreover, for any $1 \leqslant \nu \leqslant k$, the ν-*th generalized Hamming weight* of C [14, §7.10], [36, §1.1] is defined by

$$d_\nu(C) = \min\{\sharp\mathrm{supp}(\mathcal{D}) \mid \mathcal{D} \text{ is a linear subcode of } C \text{ with } \dim(\mathcal{D}) = \nu\}.$$

In other words, for any integer $1 \leqslant \nu \leqslant k$, $d_\nu(C)$ is the ν-th minimum support weights, i.e. the minimal integer t such that there are an $[n, \nu]$ subcode $\mathcal{D}$ of C and a subset $S \subset \{1, \dots, n\}$ such that $\sharp(S) = t$ and each codeword of $\mathcal{D}$ has zero coordinates outside S. The sequence $d_1(C), \dots, d_k(C)$ of generalized Hamming weights (also called *weight hierarchy* of C) is strictly increasing (see Theorem 7.10.1 of [14]). Note that $d_1(C)$ is the minimum distance of the code C.

Let us consider $\mathcal{X}$ and $\mathcal{Y}$ smooth projective absolutely irreducible curves over K and let $g : \mathcal{X} \to \mathcal{Y}$ be a rational separable map of curves of degree $r+1$. Moreover we take $r, t, Q_\infty, f_1, \dots, f_m$ and $\mathcal{A} = g^{-1}(S)$ defined as Section 2.2. So we can construct an (n, k, r) algebraic geometric LRC code C as in Theorem 1. For this code we have the following:

Theorem 2. *Let C be an (n, k, r) algebraic geometric LRC code as in Theorem 1. For every integer $\nu \geqslant 2$ we have that*

$$d_\nu(C) \geqslant n - t\ell(r+1) - (r-1)h + \gamma_{\nu-1,K}(\mathcal{X}).$$

Proof. Take a ν-dimensional linear subspace $\mathcal{D}$ of C and call

$$E \subseteq \{P_{ij} \mid i = 0, \dots r,\ j = 1, \dots, s\},$$

the set of common zeros of all elements of $\mathcal{D}$. Since $n - d_\nu(C) = \sharp(E)$, we have to prove that $t\ell(r+1) + (r-1)h - \sharp(E) \geqslant \gamma_{\nu-1,K}(X)$. Fix $u \in \mathcal{D} \setminus \{0\}$ and let F_u denote the zeros of u. Note that F_u is contained in the set $\{P_{ij} \mid i = 0, \dots r, j = 1, \dots, s\}$ by the definition of the code C. We have $F_u \supseteq E$. By the definition of the integers t, ℓ and $h := \deg(x)$, we have $\sharp(F_u) \leqslant t\ell(r+1) + (r-1)h$. The divisors $F_u - E$, $u \in \mathcal{D} \setminus \{0\}$ form a family of linearly equivalent non-negative divisors, each of them defined over K. Since $\dim(\mathcal{D}) = \nu$, the definition of $\gamma_{\nu-1,\overline{K}}(\mathcal{X})$ gives $\sharp(F_u) - \sharp(E) \geqslant \gamma_{\nu-1,K}(\mathcal{X})$. This inequality for a single $u \in \mathcal{D} \setminus \{0\}$ proves the theorem. □

See Remark 1 for an application of Theorem 2.

4 LRC CODES FROM NORM–TRACE CURVE

In this section we propose a new family of Algebraic Geometric LRC codes, that is, a LRC codes from the Norm–Trace curve. Moreover, we compute the $\mathbb{F}_{q^u}$-gonality of the Norm-Trace curve.

Let $K = \mathbb{F}_{q^u}$ be a finite field, where q is a power of a prime. We consider the *norm* $\mathrm{N}_{\mathbb{F}_q}^{\mathbb{F}_{q^u}}$ and the *trace* $\mathrm{Tr}_{\mathbb{F}_q}^{\mathbb{F}_{q^u}}$, two functions from $\mathbb{F}_{q^u}$ to $\mathbb{F}_q$ defined as

$$\mathrm{N}_{\mathbb{F}_q}^{\mathbb{F}_{q^u}}(x) = x^{1+q+\dots+q^{u-1}} \quad \text{and} \quad \mathrm{Tr}_{\mathbb{F}_q}^{\mathbb{F}_{q^u}}(x) = x + x^q + \dots + x^{q^{u-1}}.$$

The *Norm-Trace curve* χ is the curve defined over K by the following affine equation

$$\mathrm{N}_{\mathbb{F}_q}^{\mathbb{F}_{q^u}}(x) = \mathrm{Tr}_{\mathbb{F}_q}^{\mathbb{F}_{q^u}}(y),$$

that is,

$$x^{(q^u-1)/(q-1)} = y^{q^{u-1}} + y^{q^{u-2}} + \ldots + y \text{ where } x, y \in K \tag{1}$$

The Norm-Trace curve χ has exactly $n = q^{2u-1}$ K-rational affine points (see Appendix A of [5]), that we denote by $\mathcal{P}_\chi = \{P_1, \ldots, P_n\}$. The genus of χ is $g = \frac{1}{2}(q^{u-1} - 1)(\frac{q^u-1}{q-1} - 1)$. Note that if we consider $u = 2$, we obtain the Hermitian curve.

Starting from the Norm–Trace curve, we have two different ways to construct Norm–Trace LRC codes.

PROJECTION ON x We have to construct a q^u-ary (n, k, r) LRC codes. We consider the natural projection $g(x, y) = x$. Then the degree of g is $q^{u-1} = r + 1$ and the degree of y is $h = 1 + q + \cdots + q^{u-1}$.
To construct the codes we consider $S = \mathbb{F}_{q^u}$ and $D = tQ_\infty$ for some $t \geqslant 1$. Then, using a construction of Theorem 1 we find the parameters for these Norm–Trace LRC codes.

Proposition 1. *A family of Norm–Trace LRC codes has the following parameters:*

$$n = q^{2u-1}, \quad k = mr = (t+1)(q^{u-1} - 1)$$

and

$$d \geqslant n - tq^{u-1} - (q^{u-1} - 1)(1 + q + \cdots + q^{u-1}).$$

PROJECTION ON y We have to construct a q^u-ary (n, k, r) LRC codes. We consider the other natural projection $g'(x, y) = y$. Then $\deg(g') = 1 + q + \cdots + q^{u-1} = r + 1$. In this case we take $S = \mathbb{F}_{q^u} \backslash M$, where

$$M = \{a \in \mathbb{F}_{q^u} \mid a^{q^{u-1}} + a^{q^{u-2}} + \ldots + a = 0\},$$

so $r = q + \cdots + q^{u-1}$ and $h = \deg(x) = q^{u-1}$. Then, using Theorem 1 we have the following

Proposition 2. *A family of Norm–Trace LRC codes has the following parameters:*

$$n = q^{2u-1} - q^{u-1}, \quad k = mr = (t+1)(q + \cdots + q^{u-1})$$

and

$$d \geqslant n - tq^{u-1} - (q + \cdots + q^{u-1}) - q^{u-1}(q^{u-1} + \cdots + q - 1).$$

For the Norm–Trace curve χ we are able to find the K-gonality of χ.

Lemma 1. *Let χ be a Norm–Trace curve defined over $\mathbb{F}_{q^u}$, where $u \geqslant 2$. We have $\gamma_{1,\mathbb{F}_{q^u}}(\chi) = q^{u-1}$.*

Proof. The linear projection onto the x axis has degree q^{u-1} and it is defined over $\mathbb{F}_q$ and hence over $\mathbb{F}_{q^u}$. Thus $\gamma_{1,\mathbb{F}_{q^u}}(\chi) \leqslant q^{u-1}$. Denote by $z = \gamma_{1,\mathbb{F}_{q^u}}(\chi)$ and assume that $z \leqslant q^{u-1} - 1$. By the definition of K-gonality, there is a non-constant morphism $w : \chi \to \mathbb{P}^1$ with $\deg(w) = z$ and defined over $\mathbb{F}_{q^u}$. Since $w(\chi(\mathbb{F}_{q^u})) \subseteq \mathbb{P}^1(\mathbb{F}_{q^u})$, we get $\sharp(\chi(\mathbb{F}_{q^u})) \leqslant z(q^u + 1) \leqslant (q^{u-1} - 1)(q^u + 1)$, that is a contradiction. $\square$

Remark 1. By Lemma 1, we can apply Theorem 2 to the Norm–Trace curve. In fact, we can consider the gonality sequence over K of χ to get a lower bound on the second generalized Hamming weight of the two families of Norm–Trace LRC codes:

- Let $t \geqslant 1$ and let C be a $(q^{2u-1}, (t+1)(q^{u-1}-1), q^{u-1}-1)$ Norm–Trace LRC code. Then we have
$$d_2(C) \geqslant q^{2u-1} + q^{u-1} - tq^{u-1} - (q^{u-1}-1)(1+q+\cdots+q^{u-1}).$$
- Let $t \geqslant 1$ and let C be a Norm–Trace LRC code with parameters $(q^{2u-1} - q^{u-1}, (t+1)(q+\cdots+q^{u-1}), q+\cdots+q^{u-1})$. Then we have
$$d_2(C) \geqslant q^{2u-1} - (t-1)q^{u-1} - (1+q^{u-1})(q+\cdots+q^{u-1}).$$

5 HERMITIAN LRC CODES

In this section we improve the bound on the distance of Hermitian LRC codes proposed in [1] using some properties of *Hermitian codes* which are a special case of algebraic geometric codes.

5.1 Hermitian codes

Let us consider $K = \mathbb{F}_{q^2}$ a finite field with q^2 elements. The *Hermitian curve* $\mathcal{H}$ is defined over K by the affine equation

$$x^{q+1} = y^q + y \text{ where } x, y \in K. \tag{2}$$

This curve has genus $g = \frac{q(q-1)}{2}$ and has q^3+1 points of degree one, namely a pole Q_∞ and $n = q^3$ rational affine points, denoted by $\mathcal{P}_{\mathcal{H}} = \{P_1, \dots, P_n\}$ [31].

Definition 4. Let $m \in \mathbb{N}$ such that $0 \leqslant m \leqslant q^3+q^2-q-2$. Then the **Hermitian code** $C(m,q)$ is the code $C(D, mQ_\infty)$ where

$$D = \sum_{\alpha^{q+1}=\beta^q+\beta} P_{\alpha,\beta}$$

is the sum of all places of degree one (except Q_∞, that is a point at infinity) of the Hermitian function field $K(\mathcal{H})$.

By Lemma 6.4.4. of [33] we have that

$$\mathcal{B}_{m,q} = \{x^i y^j \mid qi + (q+1)j \leqslant m,\ 0 \leqslant i \leqslant q^2-1,\ 0 \leqslant j \leqslant q-1\},$$

forms a basis of $\mathcal{L}(mQ_\infty)$. For this reason, the Hermitian code $C(m,q)$ could be seen as $\mathrm{Span}_{\mathbb{F}_{q^2}}\langle ev_{\mathcal{P}_{\mathcal{H}}}(\mathcal{B}_{m,q})\rangle$. Moreover, the dual of $C(m,q)$ denoted by $C(m_\perp, q) = C^\perp(m,q)$ is again an Hermitian code and it is well known (Proposition 8.3.2 of [33]) that the degree m of the divisor has the following relation with respect to $m_\perp$:

$$m_\perp = n + 2g - 2 - m. \tag{3}$$

The Hermitian codes can be divided in four phases [11], any of them having specific explicit formulas linking their dimension and their distance [22]. In particular we are interested in the first and the last phase of Hermitian codes, which are:

I PHASE: $0 \leqslant m_\perp \leqslant q^2 - 2$. Then we have $m_\perp = aq + b$ where $0 \leqslant b \leqslant a \leqslant q - 1$ and $b \neq q - 1$. In this case, the distance is

$$\begin{cases} d = a + 1 & \text{if} \quad a > b \\ d = a + 2 & \text{if} \quad a = b. \end{cases} \tag{4}$$

IV PHASE: $n - 1 \leqslant m_\perp \leqslant n + 2g - 2$. In this case $m_\perp = n + 2g - 2 - aq - b$ where a, b are integers such that $0 \leqslant b \leqslant a \leqslant q - 2$ and the distance is

$$d = n - aq - b. \tag{5}$$

5.2 Bound on distance of Hermitian LRC codes

Let $K = \mathbb{F}_{q^2}$ be a finite field, where q is a power of a prime. Let $\mathcal{X} = \mathcal{H}$ be the Hermitian curve with affine equation as in (2). We recall that this curve has q^3 $\mathbb{F}_{q^2}$-rational affine points plus one at infinity, that we denoted by Q_∞.
We consider two of the three constructions of Hermitian LRC codes proposed in [1] and we improve the bound on distance of Hermitian LRC codes using properties of Hermitian codes. In particular, if we find an Hermitian code $C(m, q) = C_{Her}$ such that $C_{LRC} \subset C_{Her}$, then we have $d_{LRC} \geqslant d_{Her}$.

PROJECTION ON X By Proposition 4 of [1], we have a family of (n, k, r) Hermitian LRC codes with $r = q - 1$, length $n = q^3$, dimension $k = (t - 1)(q - 1)$ and distance $d \geqslant n - tq - (q - 2)(q + 1)$. Moreover, for these codes, $S = K$, $D = tQ_\infty$ for some $1 \leqslant t \leqslant q^2 - 1$ and the basis for the vector space V is

$$\mathcal{B} = \{x^j y^i \mid j = 0, \dots, t, \ i = 0, \dots, q - 2\}. \tag{6}$$

Using the Hermitian codes, we improve the bound on the distance for any integer t, such that $q^2 - q + 1 \leqslant t \leqslant q^2 - 1$.
To find an Hermitian code $C(m, q) = C_{Her}$ such that $C_{LRC} \subset C_{Her}$, we have to compute the set $\mathcal{B}_{m,q}$, that is, we have to find m. After that, to compute the distance of $C(m, q)$ we use (4) and (5).
We consider the first Hermitian phase: $0 \leqslant m_\perp \leqslant q^2 - 2$, that is, $q^2 - q + 1 \leqslant t \leqslant q^2 - 1$.
For this phase $m_\perp = aq + b$, where $0 \leqslant b \leqslant a \leqslant q - 1$ and the distance of the Hermitian code is either $d = a + 1$ if $a > b$ or $d = a + 2$ if $a = b$. By (6), m must be equal to $m = qt + (q + 1)(q - 2)$ and by (3) we have that $m_\perp = n + 2g - 2 - m = q(q^2 - t)$. So $b = 0$ and $a = q^2 - t$ and the distance of the Hermitian code is $d_{Her} = a + 1 = q^2 - t + 1$, since $a > b$. This implies that

$$d_{LRC} \geqslant q^2 - t + 1, \ \text{for any } t \geqslant q^2 - q + 1. \tag{7}$$

Note that (7) improves the bound on the distance proposed in Proposition 4 of [1] since

$$q^2 - t + 1 > q^3 - tq - (q-2)(q+1) \iff t(q-1) > q(q-1)^2 + 1 \iff t > q^2 - q.$$

We just proved the following:

Proposition 3. *Let* $q^2 - q + 1 \leqslant t \leqslant q^2 - 1$*. It is possible to construct a family of* (n, k, r) *Hermitian LRC codes* $\{C_t\}_{q^2-q+1 \leqslant t \leqslant q^2-1}$ *with the following parameters:*

$$n = q^3,\ k = (t-1)(q-1),\ r = q-1 \text{ and } d \geqslant q^2 - t + 1.$$

TWO RECOVERING SETS In [1] the authors propose an Hermitian code with two recovering sets of size $r_1 = q - 1$ and $r_2 = q$, denoted by LRC(2). They consider

$$L = \mathrm{Span}\{x^i y^j,\ i = 0, \dots, q-2,\ j = 0, \dots, q-1\}$$

and a linear code C obtained by evaluating the functions in L at the points of $B = g^{-1}(\mathbb{F}_{q^2} \setminus M)$, where $g(x, y) = x$ and $M = \{a \in \mathbb{F}_q \mid a^q + a = 0\}$. So $|B| = q^3 - q$. By Proposition 4.3 of [1], the LRC(2) code has length $n = (q^2 - 1)q$, dimension $k = (q-1)q$ and distance

$$d \geqslant (q+1)(q^2 - 3q + 3) = q^3 - 2q^2 + 3. \tag{8}$$

As before, we improve the bound on the distance using Hermitian codes that contains the LRC(2) code. To do this we have to find $m_\perp$. By L, we have that $m = q(q-1) + (q+1)(q-2)$ so we are in the fourth phase of Hermitian codes because $m_\perp = n + 2g - 2 - m = q^3 - q^2 + q$. In this case $d_{Her} = m_\perp - 2g + 2 = q^3 + 2q + 2$. Since $|B| = q^3 - q$, we have that

$$d_{LRC} \geqslant d_{Her} - q = q^3 + q + 2. \tag{9}$$

Note that this bound improves bound (8). We just proved the following proposition:

Proposition 4. *Let* C *be a linear code obtained by evaluating the functions in* L *at the points of* B*. Then* C *has the following parameters:*

$$n = (q^2 - 1)q,\ k = (q-1)q,\ r_1 = q-1,\ r_2 = q \text{ and } d \geqslant q^3 + q + 2.$$

ACKNOWLEDGEMENTS

The authors would like to thank the anonymous referees for their comments.

REFERENCES

[1] A. Barg, I. Tamo, and S. Vlăduţ. Locally recoverable codes on algebraic curves. *arXiv preprint arXiv:1501.04904*, 2015.

[2] H. Chen, R. Cramer, S. Goldwasser, R. De Haan, and V. Vaikuntanathan. Secure computation from random error correcting codes. In *Advances in Cryptology-EUROCRYPT 2007*, pages 291–310. Springer, 2007.

[3] M. Forbes and S. Yekhanin. On the locality of codeword symbols in non-linear codes. *Discrete Mathematics*, 324:78–84, 2014.

[4] G. D. Forney. Dimension/length profiles and trellis complexity of linear block codes. *Information Theory, IEEE Transactions on*, 40(6):1741–1752, 1994.

[5] O. Geil. On codes from Norm-Trace curves. *Finite Fields Appl.*, 9:351–371, 2003.

[6] O. Geil, S. Martin, R. Matsumoto, D. Ruano, and Y. Luo. Relative generalized Hamming weights of one-point algebraic geometric codes. *Information Theory, IEEE Transactions on*, 60(10):5938–5949, 2014.

[7] P. Gopalan, V. Guruswami, and P. Raghavendra. List decoding tensor products and interleaved codes. *SIAM Journal on Computing*, 40(5):1432–1462, 2011.

[8] P. Gopalan, C. Huang, H. Simitci, and S. Yekhanin. On the locality of codeword symbols. *Information Theory, IEEE Transactions on*, 58(11):6925–6934, 2012.

[9] V. Guruswami. List decoding from erasures: Bounds and code constructions. *Information Theory, IEEE Transactions on*, 49(11):2826–2833, 2003.

[10] P. Heijnen and R. Pellikaan. Generalized Hamming weights of q-ary Reed-Muller codes. In *IEEE Trans. Inform. Theory*. Citeseer, 1998.

[11] T. Høholdt, J. H. van Lint, and R. Pellikaan. Algebraic geometry of codes. In V. S. Pless and W. Huffman, editors, *Handbook of coding theory, Vol. I, II*, pages 871–961. North-Holland, 1998.

[12] M. Homma and S. J. Kim. The second generalized hamming weight for two-point codes on a Hermitian curve. *Designs, Codes and Cryptography*, 50(1):1–40, 2009.

[13] C. Huang, M. Chen, and J. Li. Pyramid codes: Flexible schemes to trade space for access efficiency in reliable data storage systems. *ACM Transactions on Storage (TOS)*, 9(1):3, 2013.

[14] W. C. Huffman and V. Pless. *Fundamentals of error-correcting codes*. Cambridge university press, 2003.

[15] H. Janwa and A. K. Lal. On generalized Hamming weights and the covering radius of linear codes. In *Applied algebra, algebraic algorithms and error-correcting codes*, pages 347–356. Springer, 2007.

[16] T. Kasami, T. Takata, T. Fujiwara, and S. Lin. On complexity of trellis structure of linear block codes. *Information Theory, IEEE Transactions on*, 39(3):1057–1064, 1993.

[17] T. Kasami, T. Takata, T. Fujiwara, and S. Lin. On the optimum bit orders with respect to the state complexity of trellis diagrams for binary linear codes. *Information Theory, IEEE Transactions on*, 39(1):242–245, 1993.

[18] J. Kurihara and T. Uyematsu. Strongly-secure secret sharing based on linear codes can be characterized by generalized Hamming weight. In *Communication, Control, and Computing (Allerton), 2011 49th Annual Allerton Conference on*, pages 951–957. IEEE, 2011.

[19] J. Kurihara, T. Uyematsu, and R. Matsumoto. Secret sharing schemes based on linear codes can be precisely characterized by the relative generalized hamming weight. *IEICE Transactions on Fundamentals of Electronics, Comications and Computer Sciences*, 95(11):2067–2075, 2012.

[20] H. Lange and G. Martens. On the gonality sequence of an algebraic curve. *Manuscripta mathematica*, 137(3-4):457–473, 2012.

[21] Y. Luo, C. Mitrpant, A. J. H. Vinck, and K. Chen. Some new characters on the wire-tap channel of type II. *Information Theory, IEEE Transactions on*, 51(3):1222–1229, 2005.

[22] C. Marcolla. *On structure and decoding of Hermitian codes*. PhD thesis, University of Trento, 2013.

[23] S. Martin and O. Geil. Relative generalized Hamming weights of q-ary Reed-Muller codes. *arXiv preprint arXiv:1407.6185*, 2014.

[24] C. Munuera. On the generalized Hamming weights of geometric Goppa codes. *IEEE transactions on information theory*, 40(6):2092–2099, 1994.

[25] C. Munuera and D. Ramirez. The second and third generalized Hamming weights of Hermitian codes. *Information Theory, IEEE Transactions on*, 45(2):709–712, 1999.

[26] C.-K. Ngai, R. W. Yeung, and Z. Zhang. Network generalized hamming weight. *Information Theory, IEEE Transactions on*, 57(2):1136–1143, 2011.

[27] W. Olaya-León and C. Granados-Pinzón. The second generalized hamming weight of certain Castle codes. *Designs, Codes and Cryptography*, 76(1):81–87, 2015.

[28] D. S. Papailiopoulos and A. G. Dimakis. Locally repairable codes. *Information Theory, IEEE Transactions on*, 60(10):5843–5855, 2014.

[29] R. Pellikaan. On the gonality of curves, abundant codes and decoding. In *Coding theory and algebraic geometry*, pages 132–144. Springer, 1992.

[30] B. Poonen. Gonality of modular curves in characteristic p. *Mathematical Research Letters*, 14(4):691–701, 2007.

[31] H. G. Ruck and H. Stichtenoth. A characterization of Hermitian function fields over finite fields. *Journal fur die Reine und Angewandte Mathematik*, 457:185–188, 1994.

[32] N. Silberstein, A. S. Rawat, O. O. Koyluoglu, and S. Vishwanath. Optimal locally repairable codes via rank-metric codes. In *Information Theory Proceedings (ISIT), 2013 IEEE International Symposium on*, pages 1819–1823. IEEE, 2013.

[33] H. Stichtenoth. *Algebraic function fields and codes*. Universitext. Springer-Verlag, Berlin, 1993.

[34] I. Tamo and A. Barg. A family of optimal locally recoverable codes. *Information Theory, IEEE Transactions on*, 60(8):4661–4676, 2014.

[35] I. Tamo, D. S. Papailiopoulos, and A. G. Dimakis. Optimal locally repairable codes and connections to matroid theory. In *Information Theory Proceedings (ISIT), 2013 IEEE International Symposium on*, pages 1814–1818. IEEE, 2013.

[36] M. Tsfasman, S. Vlădut, and D. Nogin. *Algebraic geometric codes: basic notions*, volume 139. American Mathematical Soc., 1990.

[37] V. K. Wei. Generalized Hamming weights for linear codes. *Information Theory, IEEE Transactions on*, 37(5):1412–1418, 1991.

[38] K. Yang, P. V. Kumar, and H. Stichtenoth. On the weight hierarchy of geometric Goppa codes. *Information Theory, IEEE Transactions on*, 40(3):913–920, 1994.