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Generic Large Cardinals and Absoluteness

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Dottorato in Matematica – XXVII ciclo

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Anno Accademico 2015/2016

Settore Scientifico Disciplinare MAT/01

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INTRODUCTION

In the wake of early Gödel's Incompleteness Theorems, a large part of set theory has revolved around the study of the independence phenomena. The most significant boost in this field was due to the introduction of the forcing technique. Many independence results were found in different areas of mathematics since then, including among many the continuum hypothesis [32], Borel's conjecture [34], Kaplansky's conjecture [14], Whitehead problem [38], strong Fubini theorem [18], and the existence of outer automorphisms of the Calkin algebra [15].

A dual approach in set theory is to search for new axioms of mathematics in order to control the independence phenomena. These new axioms need to be justified, so that their mathematical properties interact with philosophical guidelines. An axiom candidate can be judged both from its *premises*, that is, whether the statement can be argued to be a "natural" extension of the accepted axioms of ZFC, and from its *consequences*, that is, whether the statement is able to entail a large number of desired properties in mathematics.

In this thesis we follow this approach, considering axioms in two of the most important families: forcing axioms (Chapters 3, 4) and large cardinals (Chapter 5), considering their effectiveness both on the premises and consequences side. While these chapters follow mostly independent paths, all of them fit in the same quest for natural and powerful axioms for mathematics, and are strongly dependent on the *iterated forcing* technique covered in Chapter 2. Basic notation and background topics are dealt with in Chapter 1.

In all the chapters of this thesis we shall focus on the boolean valued models approach to forcing. A partial order and its boolean completion can produce exactly the same consistency results, however:

- in a specific consistency proof the forcing notion we have in mind in order to obtain the desired result is given by a partial order and passing to its boolean completion may obscure our intuition on the nature of the problem and the combinatorial properties we wish our partial order to have;

- when the problem aims to find general properties of forcings which are shared by a wide class of partial orders, we believe that focusing on complete boolean algebras gives a more efficient way to handle the problem. In fact, the rich algebraic theory of complete boolean algebras can greatly simplify our calculations.

Since the focus of the thesis is on the general correlations between forcing, large cardinals, and forcing axioms, rather than on specific applications of the forcing method to prove the consistency of a given mathematical statement, we are naturally led to analyze forcing through an approach based on boolean valued models.

Iterated forcing is a procedure that describes transfinite applications of the forcing method, with a special attention to limit stages, and is one of the main tools for proving the consistency of forcing axioms. In Chapter 2 we give a detailed account of iterated forcing through boolean algebras, inspired by the algebraic approach of Donder and Fuchs [19], thus providing a solid background on which the rest of the thesis is built. All the material in this chapter is joint work with Matteo Viale and Silvia Steila.

Even though all the results in this chapter come from a well established part of the current development of set theory, the proofs are novel and (according to us) simpler and more elegant, due to a systematic presentation of the whole theory of iterated forcing in terms of algebraic constructions. In Chapter 3 we introduce the notion of weakly iterable forcing class and prove the preservation theorems for semiproper and stationary set preserving iterations (for the latter result, we assume the existence of class many supercompact cardinals). We believe there will be no problem in rearranging these techniques in order to cover also the cases of proper or ccc iterations.

Forcing axioms are set-theoretic principles that arise directly from the technique of forcing. It is a matter of fact that forcing is one of the most powerful tools to produce consistency results in set theory: forcing axioms turn it into a powerful instrument to prove theorems. This is done by showing that a statement ϕ follows from an extension T of ZFC if and only if T proves that ϕ is consistent by means of a certain type of forcing. These types of results are known in the literature as generic absoluteness results and have the general form of a completeness theorem for some $T \supseteq$ ZFC with respect to the semantic given by boolean valued models and first order calculus. More precisely generic absoluteness theorems fit within the following general framework:

> Assume T is an extension of ZFC, Θ is a family of first order formulas in the language of set theory and Γ is a certain class of forcing notions definable in T. Then the following are equivalent for a $\phi \in \Theta$ and $S \supseteq T$:
>
> 1. S proves ϕ.
> 2. S proves that there exists a forcing $\mathbb{B} \in \Gamma$ such that $\mathbb{B}$ forces ϕ and T jointly.
> 3. S proves that $\mathbb{B}$ forces ϕ for all forcings $\mathbb{B} \in \Gamma$ such that $\mathbb{B}$ forces T.

We say that a structure M definable in a theory T is generically invariant with respect to forcings in Γ and parameters in $X \subset M$ when the above situation occurs with Θ being the first order theory of M with parameters in X. A brief overview of the main known generic absoluteness results is the following:

- Shoenfield's absoluteness theorem is a generic absoluteness result for Θ the family of Σ^1_2-properties with real parameters, Γ the class of all forcings, $T =$ ZFC.
- The pioneering "modern" generic absoluteness results are Woodin's proofs of the invariance under set forcings of the first order theory of $L(\mathrm{ON}^\omega)$ with real parameters in ZFC + *class many Woodin cardinals which are limit of Woodin cardinals* [33, Thm. 3.1.2] and of the invariance under set forcings of the family of Σ^2_1-properties with real parameters in the theory ZFC + CH + *class many measurable Woodin cardinals* [33, Thm. 3.2.1].

  Further results pin down the exact large cardinal strength of the assertion that $L(\mathbb{R})$ is generically invariant with respect to certain classes of forcings (among others see [37]).
- The bounded forcing axiom $\mathrm{BFA}(\Gamma)$ is equivalent to the statement that generic absoluteness holds for $T =$ ZFC and Θ the class of Σ_1-formulas with parameters in $\mathcal{P}(\omega_1)$, as shown in [6].
- Recently, Hamkins and Johnstone [23] introduced the resurrection axioms $\mathrm{RA}(\Gamma)$ and Viale [42] showed that these axioms produce generic absoluteness for Θ the Σ_2-theory with parameters of $H_{\mathfrak{c}}$, $T = \mathrm{ZFC} + \mathrm{RA}(\Gamma)$, Γ any of the standard classes of forcings closed under two step iterations.
- Viale introduced the forcing axiom MM^{+++} (a natural strenghtening of MM) and proved that $L(\mathrm{ON}^{\omega_1})$ with parameters in $P(\omega_1)$ is generically invariant with respect to SSP forcings for

  $$T = \mathrm{ZFC} + \mathrm{MM}^{+++} + \text{there are class many superhuge cardinals.}$$

Motivated by the latter results as well as by the work of Tsaprounis [40], in Chapter 4 we introduce over the theory MK (i.e. the Morse-Kelley set theory with sets and classes) a new natural class of forcing axioms: the iterated resurrection axioms $\mathrm{RA}_\alpha(\Gamma)$ of increasing strength as α runs through the ordinals, with Γ a definable class of forcing notions. All the material in this chapter is joint work with Matteo Viale. We remark that for most classes Γ, $\mathrm{RA}_1(\Gamma)$ is substantially equal to $\neg$ CH plus the resurrection axiom $\mathrm{RA}(\Gamma)$ recently introduced by Hamkins and Johnstone in [23].

We are able to prove over MK the consistency relative to large cardinal axioms of the axioms $\mathrm{RA}_\alpha(\Gamma)$ for any class of forcing notions Γ which is *weakly iterable*. The latter is a property of classes of forcing notions which we introduce in Definition 3.3.3, and which holds for most standard classes such as locally ccc, Axiom-A, proper, semiproper, stationary set preserving (this latter class is weakly iterable only in the presence of sufficiently strong large cardinal axioms).

From the axioms $\mathrm{RA}_\alpha(\Gamma)$ with $\alpha \geq \omega$ we are able to prove the following generic absoluteness result over the theory MK.

Theorem. *Let V be a model of* MK, *Γ be a definable class of forcing notions, γ be the largest cardinal preserved by forcings in Γ. Assume $\mathrm{RA}_\omega(\Gamma)$ holds and $\mathbb{B} \in \Gamma$ forces $\mathrm{RA}_\omega(\Gamma)$. Then $H^V_{2^\gamma} \prec H^{V^\mathbb{B}}_{2^\gamma}$.*

This is a generic absoluteness result for $T = \mathrm{MK} + \mathrm{RA}_\omega(\Gamma)$ and Θ the first order theory of H_{2^γ} with parameters. We also prove that $\mathrm{RA}_\alpha(\Gamma)$ fits naturally within the hierarchy of the previously known results, that is, $\mathrm{MM}^{+++} \Rightarrow \mathrm{RA}_\alpha(\mathrm{SSP})$ and $\mathrm{RA}_\alpha(\Gamma) \Rightarrow \mathrm{RA}(\Gamma)$ for any Γ and for all $\alpha > 0$. Furthermore, the consistency strength of the axioms $\mathrm{RA}_\alpha(\Gamma)$ is below that of a Mahlo cardinal for all relevant Γ and for all α except for $\Gamma = \mathrm{SSP}$, in which case our upper bound is below a stationary limit of supercompact cardinals.

We remark that the present result cannot be formulated in ZFC alone since the iterated resurrection axioms $\mathrm{RA}_\alpha(\Gamma)$ are second-order statements. However, it is possible that some theory strictly weaker than MK (e.g., NBG together with a truth predicate) would suffice to carry out the arguments at hand. Notice that $\mathrm{RA}_n(\Gamma)$ can also be formulated by an equivalent first-order sentence for all $n < \omega$, and $\mathrm{RA}_\omega(\Gamma)$ can be formulated as the corresponding first-order axiom schema $\{\mathrm{RA}_n(\Gamma) : \ n < \omega\}$ if needed.

Altogether these results show the effectiveness of the axioms $\mathrm{RA}_\omega(\Gamma)$ both on the *premises* side (low consistency strength, natural generalization of well-known axioms) and on the *consequences* side (generic invariance of H_{2^γ}). However, the axioms $\mathrm{RA}_\omega(\Gamma)$ are pairwise incompatible for different choices of Γ thus making it difficult to support the adoption of a specific $\mathrm{RA}_\omega(\Gamma)$ as a natural axiom for set theory. Towards this aim, we remark the following two special cases.

- If we focus on forcing classes preserving ω_1, there is a unique largest class (the class of stationary set preserving posets SSP) which contains all the possible classes Γ for which the axiom $\mathrm{RA}_\omega(\Gamma)$ is consistent. Thus $\mathrm{RA}_\omega(\mathrm{SSP})$ gives the strongest form of generic absoluteness which can be instantiated by means of the iterated resurrection axioms for forcing classes preserving ω_1.

- If we consider the forcing classes $<\omega_\alpha$-closed for $\alpha \in \mathrm{ON}$, the corresponding resurrection axioms are all pairwise compatible. In fact, from a Mahlo cardinal is possible to obtain the consistency of $\mathrm{MK} + \mathrm{GCH} + \mathrm{RA}_\omega(<\kappa\text{-closed})$ for all cardinals κ simultaneously. This gives a very strong and uniform generic absoluteness result, that is, given any forcing $\mathbb{B}$ we have that $H_{2^\kappa} = H_{2^\kappa}^{\mathbb{B}}$ where κ is such that $\mathbb{B}$ is $<\kappa$-closed.

We remark that the case $\Gamma = <\omega_\alpha$-closed pushes the limits on which generic absoluteness can be obtained above the usual threshold 2^ω whenever $\alpha > 0$. Even though the class of $<\omega_\alpha$-closed forcing is narrow and (some notion of) ω_α-proper forcing would be preferable, our result highlights a strong connection of the theory of iterations with generic absoluteness. In fact, it shows that a preservation theorem for a forcing class Γ translates in a corresponding axiom yielding a generic absoluteness result for the same class Γ.

Compared to the generic absoluteness result obtained in [43], the present results for $\Gamma = \mathrm{SSP}$ are weaker since they regard the structure $H_{2^{\omega_1}}$ instead of $L(\mathrm{ON}^{\omega_1})$. On the other hand, the consistency of $\mathrm{RA}_\alpha(\Gamma)$ is obtained from (in most cases much) weaker large cardinal hypothesis and the results are more general since they also apply to interesting choices of $\Gamma \neq \mathrm{SP}, \mathrm{SSP}$. Moreover the arguments we employ to prove the consistency of $\mathrm{RA}_\alpha(\Gamma)$ are considerably simpler than the arguments developed in [43].

Finally, a byproduct of our results is that the theory $T = \mathrm{MK} + \mathrm{RA}_\omega(\mathrm{all})$ is consistent relative to the existence of a Mahlo cardinal and makes the theory of projective sets of reals generically invariant with respect to any forcing which preserves T. Notice that T is consistent with the failure of projective determinacy. This shows that the request of generic absoluteness for the projective sets of reals (with respect to forcings preserving T) is much weaker than Woodin's generic absoluteness for all sets of reals in $L(\mathbb{R})$ with respect to $T = \mathrm{ZFC} + $ *large cardinals*.

Large cardinals can be regarded as natural strenghtenings of the accepted principle of generation of new sets such as the *powerset* axiom and the *replacement* axiom. Furthermore, they are extremely useful in set theory: on the one hand they provide a fine scale to measure the consistency strength of a rich variety of combinatorial principles, on the other hand they also solve important questions within set theory. However, such cardinals are rarely used in common mathematics outside of set theory: for example, large parts of number theory and analysis can be formalized within $H_\mathfrak{c}$, and even if new subjects can push this limit beyond that point, it is uncommon for structures of inaccessible or larger size to be employed outside of set theory.

Generic large cardinal axioms try to address this point, and postulate the existence of elementary embeddings $j : V \to M$ with $M \subseteq V[G]$ a transitive class definable in a generic extension $V[G]$ of V. Contrary to the classical case one can consistently have generic large cardinal properties at cardinals as small as ω_1. Thus, generic large cardinal axioms are fit to produce consequences on small objects, and might be able to settle questions arising in domains of mathematics other than set theory, scoring both on the *premises* and *consequences* side. A detailed presentation of this approach can be found in [16].

Due to the *class* nature of the elementary embeddings involved in the definitions of large cardinals (both classical and generic), a key issue concerns the possibility to define (or derive) such embeddings from set-sized objects. The first natural candidates are ideals, although it turns out that they are not able to represent various relevant large cardinal properties. For this reason many extensions of the concept have been proposed, the most important of which are extenders (see among many [9, 29, 31]) and normal towers (see for example [11, 33, 43, 44]).

In Chapter 5 we introduce the notion of *$\mathcal{C}$-system of filters* (see Section 5.1). This concept is inspired by the well-known definitions of extenders and towers of normal ideals, generalizes both of them, and provides a common framework in which the standard properties of extenders and towers used to define classical or generic large cardinals can be expressed in an elegant and concise way. Using the new framework given by $\mathcal{C}$-system of filters we easily generalize to the setting of generic large cardinals well-known results about extenders and towers, providing shorter and modular proofs of several well-known facts regarding classical and generic large cardinals. Furthermore, we are able to examine closely the relationship between extenders and towers, and investigate when they are equivalent or not, both in the standard case and in the generic one (see Section 5.1.5). All the material in this chapter is joint work with Silvia Steila.

The second part of this chapter investigates some natural questions regarding generic large cardinals. In particular, we first examine the difference between having a generic large cardinal property *ideally* or *generically*, and study when a generic $\mathcal{C}$-system of ultrafilters is able to reproduce a given large cardinal property. Then we focus on *ideally* large cardinals, and study how the large cardinal properties are captured by the combinatorial structure of the $\mathcal{C}$-system of filters used to induce the embedding. In particular, we are able to characterize strongness-like properties via the notion of *antichain splitting*, and closure-like properties via the notion of *antichain guessing* (a generalization of the well-known concept of *presaturation* for normal towers). Finally, we investigate to what extent it is possible to collapse a generic large cardinal while preserving its properties.

We remark that the main original contribution of this chapter is to streamline the essential features common to a variety of arguments involving a notion of normality. Thus we shall assume that all results in this chapter without an explicit attribution are adaptations of well-known facts to the setting of $\mathcal{C}$-systems of filters.

Acknowledgments

Firstly, I would like to express my sincere gratitude to my advisor Prof. Matteo Viale for the continuous support of my Ph.D study and related research, for his patience, motivation and insightful comments. Besides my advisor, I would like to thank Silvia Steila who proved to be a perfect workmate and helped me in writing a significant part of the material from which this thesis is built. My sincere thanks to the whole logic group in Torino (and in particular Alessandro Andretta and Luca Motto Ros), which provided a perfect environment for learning and great opportunities for discussion.

I thank all the Ph.D students in Torino who shared with me many working days and proved to be very good friends, and the young set-theorists that I met throughout the European conferences and made all these meetings special. Special thanks to my family and closest friends who supported me during these years: without them it would have been difficult to finish this work. Last but not least, I thank Martina who put up with me during the busiest moments and encouraged me through the good times and bad.

Notation

As in common set-theoretic use, $\mathrm{trcl}(x)$, $\mathrm{rank}(x)$ denote respectively the transitive closure and the rank of a given set x. We denote by V_α the sets x such that $\mathrm{rank}(x) < \alpha$ and by H_κ the sets x such that $|\mathrm{trcl}(x)| < \kappa$. We use $\mathcal{P}(x)$, $[x]^\kappa$, $[x]^{<\kappa}$ to denote the powerset, the set of subsets of size κ and the ones of size less than κ. The notation $f : A \to B$ is improperly used to denote partial functions in $A \times B$, ${}^A B$ to denote the collection of all such (partial) functions, and $f[A]$ to denote the pointwise image of A through f. We denote by $\mathrm{id} : V \to V$ the identity map on V. We use $s^\frown t$ for sequence concatenation and $s^\frown x$ where x is not a sequence as a shorthand to $s^\frown\langle x\rangle$. We use $s \lhd t$ to denote that $t = s\restriction(|s| - 1)$. CH denote the continuum hypothesis and $\mathfrak{c}$ the continuum itself. We prefer the notation ω_α instead of $\aleph_\alpha$ for cardinals.

Let $\mathcal{L}^2$ be the language of set theory with two sorts of variables, one for sets and one for classes. We will work with models of set theory with sets and classes and with theories in the language $\mathcal{L}^2$ extending the Gödel-Bernays system NBG and in most cases also the Morse-Kelley system MK (see [32, Sec. II.10] for the axioms of NBG and [1] for MK). We will identify a model $N = \langle \mathrm{set}(N), \mathrm{class}(N) \rangle$ of NBG with its underlying collection of classes class(N): e.g. if κ is inaccessible and $N = \langle V_\kappa, V_{\kappa+1} \rangle$, we shall just denote N by $V_{\kappa+1}$. We recall that from the underlying collection class(N) we can reconstruct whether $x \in \mathrm{set}(N)$ via the formula $\exists y \in \mathrm{class}(N)\ x \in y$.

We will use $M \prec_n N$ to denote that $(M, \in)$ is a Σ_n-elementary substructure of $(N, \in)$. Given an elementary embedding $j : V \to M$, we use $\mathrm{crit}(j)$ to denote the critical point of j. We denote by $\mathrm{SkH}^M(X)$ the Skolem Hull of the set X in the structure M, and by $M[X]$ be the smallest model of ZFC containing M and X (even if X is not generic for any forcing). We say that $I \subseteq \mathcal{P}(X)$ is an *ideal* on X whenever it is closed under unions and subsets, and feel free to confuse an ideal with its dual filter when clear from the context. We denote the collection of I-positive sets by $I^+ = \mathcal{P}(X) \setminus I$.

We follow Jech's approach [27] to forcing via boolean valued models. The letters $\mathbb{B}$, $\mathbb{C}$, $\mathbb{D}, \dots$ are used for set sized complete boolean algebras, and $\mathbf{0}$, $\mathbf{1}$ denote their minimal and maximal element. We use $V^{\mathbb{B}}$ for the boolean valued model obtained from V and $\mathbb{B}$, $\dot{x}$ for the elements (names) of $V^{\mathbb{B}}$, $\check{x}$ for the canonical name for a set $x \in V$ in the boolean valued model $V^{\mathbb{B}}$, $[\![\phi]\!]_{\mathbb{B}}$ for the truth value of the formula ϕ. We shall sometimes confuse $\mathbb{B}$-names with their defining properties. For example, given a collection of $\mathbb{B}$-names $\{\dot{x}_\alpha :\ \alpha < \gamma\}$, we confuse $\{\dot{x}_\alpha :\ \alpha < \gamma\}$ with a $\mathbb{B}$-name $\dot{x}$ such that for all $\dot{y} \in V^{\mathbb{B}}$, $[\![\dot{y} \in \dot{x}]\!]_{\mathbb{B}} = [\![\exists \alpha < \check{\gamma}\ \ \dot{y} = \dot{x}_\alpha]\!]_{\mathbb{B}}$. When we believe this convention may be ambiguous we shall be explicitly more careful.

We use $\mathrm{Coll}(\kappa, {<}\lambda)$ for the Lévy collapse that generically adds a surjective function from κ to any $\gamma < \lambda$, $\mathrm{Add}(\kappa, \lambda)$ for the ${<}\kappa$-closed poset that generically adds λ many subsets to κ. We prefer the notation "X has the *${<}\kappa$-property*" for all properties that are defined in terms of $\forall \gamma < \kappa\ \phi(\gamma, X)$ for some formula ϕ. In all such cases we shall explicitly avoid the notation "*κ-property*" and use *${<}\kappa^+$-property* instead. In general we shall feel free to confuse a partial order $\mathbb{P}$ with its boolean completion $\mathrm{RO}(\mathbb{P})$ and a boolean algebra $\mathbb{B}$ with the partial order $\mathbb{B}^+$ given by its positive elements. When we believe that this convention may generate misunderstandings we shall be explicitly more careful.

When convenient we also use the generic filters approach to forcing. The letters G, H will be used for generic filters over V, $\dot{G}_{\mathbb{B}}$ denotes the canonical name for the generic filter for $\mathbb{B}$, $\mathrm{val}_G(\dot{x})$ the valuation map on names by the generic filter G, $V[G]$ the generic extension of V by G. Let ϕ be a formula in $\mathcal{L}^2$. We shall write $V^{\mathbb{B}} \models \phi$ to denote that ϕ holds in all generic extensions $V[G]$ with G generic for $\mathbb{B}$. We shall also write $H_{2^\gamma}^{\mathbb{B}} \prec H_{2^\gamma}^{\mathbb{C}}$ (and similarly $H_{2^\gamma} \prec H_{2^\gamma}^{\mathbb{B}}$) to denote that for all G V-generic for $\mathbb{B}$ and H $V[G]$-generic for $\mathbb{C}$, $H_{2^\gamma}^{V[G]} \prec H_{2^\gamma}^{V[G][H]}$.

ordinals	$\alpha, \beta, \xi, \zeta, \eta$
cardinals	$\gamma, \kappa, \lambda, \delta, \mu, \theta$
formulae	ϕ, ψ, φ
strategies	σ, Σ
forcing classes	Γ
domains in $\mathcal{C}$	a, b, c, d, e
other functions	f, g, h
immersions of boolean algebras	i
elementary embeddings	j
factor maps	k
finite ordinals	m, n
forcing conditions	p, q, r
sequences, threads in iteration systems	s, t
ultrapower functions	u, v
set elements	x, y, z, w
subsets	A, B, C, D, E
filters and ideals	F, I
generic filters	G, H, K
transitive models	M, N
stationary sets	S, T
open sets	U
other sets	X, Y, Z, W
families of sets	$\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$
iteration systems	$\mathcal{F}$
games	$\mathcal{G}$
systems of ultrafilters	$\mathcal{S}, \mathcal{E}, \mathcal{T}$
boolean algebras	$\mathbb{B}, \mathbb{C}, \mathbb{D}$
forcing posets	$\mathbb{P}, \mathbb{Q}, \mathbb{R}$
systems of filters	$\mathbb{S}, \mathbb{E}, \mathbb{T}$

Table 1: Summary of the naming conventions used throughout this thesis.

CHAPTER 1

BACKGROUNDS

In this chapter we present some unrelated topics that constitute the basis on which the remainder of the thesis is built. A reader acquainted with the following topics can safely skip this chapter or parts of it, although we recommend to browse through the content in order to fix the notation and conventions used.

Section 1.1 introduces the topic of second-order elementarity and determinacy of clopen class games. Section 1.2 reviews the main theory of generalized stationary sets. Section 1.3 gives a compact presentation of forcing via boolean valued models.

1.1 Second-order elementarity and class games

Let T be a theory in the language $\mathcal{L}^2$ extending NBG that provably holds in $V_{\delta+1}$ with δ inaccessible. Let $\Delta^1_1(T)$ denote the formulae in $\mathcal{L}^2$ with set parameters that are provably equivalent modulo T both to a Σ^1_1 formula (i. e. a formula with one existential class quantifier and set parameters) and to a Π^1_1 formula (with one universal class quantifier and set parameters). Let N, M with $\mathrm{set}(N) \subseteq \mathrm{set}(M)$ be models of T. We will write $N \prec_{\Delta^1_1(T)} M$ to denote that all $\Delta^1_1(T)$ formulae with set parameters in N holding in N also hold in M (and viceversa). We will write $N \equiv_{\Delta^1_1(T)} M$ for $N \prec_{\Delta^1_1(T)} M$ when $\mathrm{set}(N) = \mathrm{set}(M)$.

The $\Delta^1_1(T)$ formulae are interesting for their absolute behavior with respect to models of T with the same sets.

Proposition 1.1.1. *Let $N \subseteq M$ be models of a theory T extending* NBG *with the same sets. Then $N \equiv_{\Delta^1_1(T)} M$.*

Proof. Let $X = \mathrm{set}(N) = \mathrm{set}(M)$, and $\phi = \exists C\ \psi_1(C, \vec{p}) = \forall C\ \psi_2(C, \vec{p})$ be a $\Delta^1_1(T)$ formula, with $\vec{p} \in X$. Then,

$$\begin{aligned} N \models \phi &\Rightarrow \exists C \in N\ \langle X, C\rangle \models \psi_1(C, \vec{p}) \\ &\Rightarrow \exists C \in M\ \langle X, C\rangle \models \psi_1(C, \vec{p}) \Rightarrow \\ M \models \phi &\Rightarrow \forall C \in M\ \langle X, C\rangle \models \psi_2(C, \vec{p}) \\ &\Rightarrow \forall C \in N\ \langle X, C\rangle \models \psi_2(C, \vec{p}) \Rightarrow N \models \phi \end{aligned}$$

concluding the proof. □

Corollary 1.1.2. *Let T be a theory extending* NBG *that provably holds in $V_{\delta+1}$ with δ inaccessible. Let N, M be models of T with* $\mathrm{set}(N) = \mathrm{set}(M) = V_\delta$*, δ inaccessible. Then $N \equiv_{\Delta^1_1(T)} M$.*

Proof. Since δ is inaccessible, $V_{\delta+1}$ is a model of T containing both N, M hence by Lemma 1.1.1 $N \equiv_{\Delta^1_1(T)} V_{\delta+1} \equiv_{\Delta^1_1(T)} M$. □

Corollary 1.1.2 tells us that for any inaccessible δ the truth value of a $\Delta^1_1(T)$-formula does not depend on the choice of the particular T-model whose family of sets is V_δ. Thus we will focus only on NBG-models of the kind $V_{\delta+1}$.

In the remainder of this thesis we will need to prove that certain statements about class games are $\Delta^1_1(T)$ for a suitable theory T. In order to provide a definition of such a theory we need to introduce clopen games on class trees. Our reference text for the basic notions and properties of games is [30, Sec. 20.A].

We shall consider well-founded trees T as collections of finite sequences ordered by inclusion and closed under initial segments, such that there exists no infinite chain (totally ordered subset) in T. Given $s, t \in T$, let $s \lhd t$ denote that $t = s\restriction(|s|-1)$ (s is obtained extending t with one more element).

The clopen game on the well-founded tree T is a two-player game $\mathcal{G}^T$ defined as follows: Player I starts with some $s_0 \in T$ of length 1, then each player has to play a $s_{n+1} \lhd s_n$. The last player who can move wins the game. A winning strategy σ for Player I in $\mathcal{G}^T$ is a subtree $\sigma \subseteq T$ such that for all $s \in \sigma$ of even length $|s|$ there is exactly one $t \in \sigma$ with $t \lhd s$, and for every $s \in \sigma$ of odd length, every $t \lhd s$ is in σ. A winning strategy for Player II is defined interchanging odd with even in the above statement. A game $\mathcal{G}^T$ is determined if either one of the two players has a winning strategy.

We recall that there is correspondence between games $\mathcal{G}^T$ on a well-founded tree T and games on a pruned tree (as defined in [30]) whose winning condition is a clopen set. This justifies our terminology.

In the following we will be interested in the theory $T = \mathrm{NBG} + \mathrm{AD}(\Delta^0_1)$, where $\mathrm{AD}(\Delta^0_1)$ is the following axiom of determinacy for clopen class games:

Definition 1.1.3 ($\mathrm{AD}(\Delta^0_1)$)**.** $\mathcal{G}^T$ is determined for any well-founded class tree T.

Games $\mathcal{G}^T$ on well-founded set trees $T \subseteq V_\delta$ are determined in ZFC (see [30, Thm. 20.1]) and the corresponding strategies $\sigma \subseteq T$ are elements of $V_{\delta+1}$. Thus the theory $\mathrm{NBG} + \mathrm{AD}(\Delta^0_1)$ holds in any $V_{\delta+1}$ with δ inaccessible, and we can apply the results of this section to this theory. Moreover, a finer upper bound for $\mathrm{NBG} + \mathrm{AD}(\Delta^0_1)$ is given by the next proposition (we thank Alessandro Andretta for pointing this fact to us).

Proposition 1.1.4. *The Morse-Kelley theory* MK *(with the axiom of global choice) implies* $\mathrm{AD}(\Delta^0_1)$*.*

Proof. Since the recursion theorem on well-founded class trees holds in MK (see [1, Prop. 2]), we can follow the classical ZFC proof of determinacy for clopen games $\mathcal{G}^T$ on well-founded set trees T.

For any $s \in T$ the next moving player is I if $|s| \bmod 2 = 0$ and II otherwise. Define recursively a (class) map $w : T \to 2$ so that $w(s) = |s| \bmod 2$ iff there exists a $t \lhd s$ such that $w(t) = |s| \bmod 2$. Intuitively, we can think of the map w as assigning (coherently) to every position s in $\mathcal{G}^T$ a "supposedly winning" player $w(s)$ (I if $|s| \bmod 2 = 0$ and II otherwise).

Then we can use the map w and the axiom of global choice to define a winning strategy σ for Player I if $w(\emptyset) = 0$, and for Player II if $w(\emptyset) = 1$. Precisely, define $T_w = \{s \in T : w(s) = w(\emptyset)\}$ and $s^+ = \min_{\lessdot} \{t \in T_w : t \lhd s\}$ where $\lessdot$ is a well-order on V. Then,

$$\sigma = \left\{s \in T_w : \forall m < |s| \quad (m \bmod 2 = w(\emptyset)) \to \left(s\restriction(m+1) = (s\restriction m)^+\right)\right\}$$

is the desired strategy. □

In the remainder of this thesis we shall focus on extensions of the theory $T = \mathsf{NBG} + \mathsf{AD}(\Delta^0_1)$ and use Δ^1_1 as a shorthand for $\Delta^1_1(\mathsf{NBG} + \mathsf{AD}(\Delta^0_1))$. We remark that this theory is a natural strengthening of ZFC, since it asserts second order properties that are true for natural models of ZFC (models of the kind V_δ with δ inaccessible). Moreover, the theory has a reasonable consistency strength since it holds in all NBG models of the form $V_{\delta+1}$ with δ inaccessible and follows from MK.

It is not clear to us at the moment (and not relevant for the purpose of this thesis) whether the theory $\mathsf{NBG} + \mathsf{AD}(\Delta^0_1)$ is preserved by set forcing. Since we know that this is true for MK (see [1, Thm. 23]), the most convenient theory to present and apply the results of this section is MK. In the remainder of this thesis we shall feel free to implicitly assume MK when needed, while pointing out some passages where $\mathsf{AD}(\Delta^0_1)$ is essentially used.

1.2 Generalized stationarity

We now recall the main definitions and properties of generalized stationary sets. Full references on this subject can be found in [27], [33, Chp. 2], [44].

Definition 1.2.1. Let X be an uncountable set. A set C is a *club* on $\mathcal{P}(X)$ iff there is a function $f_C : [X]^{<\omega} \to X$ such that C is the set of elements of $\mathcal{P}(X)$ closed under f_C, i.e.

$$C = \left\{Y \in \mathcal{P}(X) : [f_C[Y]]^{<\omega} \subseteq Y\right\}$$

A set S is *stationary* on $\mathcal{P}(X)$ iff it intersects every club on $\mathcal{P}(X)$.

The reference to the support set X for clubs or stationary sets may be omitted, since every set S can be club or stationary only on $\bigcup S$. Examples of stationary sets are $\{X\}$, $\mathcal{P}(X) \setminus \{X\}$ and $[X]^\kappa$ for any $\kappa \leq |X|$ (the latter following the proof of the well-known downwards Löwhenheim-Skolem Theorem). Notice that every element of a club C must contain $f_C(\emptyset)$, a fixed element of X.

Given any first-order structure M, from the set M we can define a Skolem function $f_M : [M]^{<\omega} \to M$ (i.e., a function coding solutions for all existential first-order formulas over M). Then the set C of all elementary submodels of M contains a club (the one corresponding to f_M). Henceforth, every set S stationary on X must contain an elementary submodel of any first-order structure on X.

Definition 1.2.2. The *club filter* on X is

$$\mathrm{CF}_X = \{C \subset \mathcal{P}(X) : \ C \text{ contains a club}\},$$

and its dual ideal is

$$\mathrm{NS}_X = \{A \subset \mathcal{P}(X) : \ A \text{ not stationary}\},$$

the *non-stationary ideal* on X.

If $|X| = |Y|$, then $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ are isomorphic and so are CF_X and CF_Y (or NS_X and NS_Y). Thus we can suppose $X \in \mathrm{ON}$ or $X \supseteq \alpha$ for some $\alpha \in \mathrm{ON}$ if needed.

Lemma 1.2.3. *CF_X is a σ-complete filter on $\mathcal{P}(X)$, and the stationary sets are exactly the CF_X-positive sets.*

Proof. CF_X is closed under supersets by definition. Given a family of clubs C_i, $i < \omega$, let f_i be the function corresponding to the club C_i. Let $\pi : \omega \to \omega^2$ be a surjection, with components π_1 and π_2, such that $\pi_2(n) \leq n$. Define $g : X^{<\omega} \to X$ to be $g(s) = f_{\pi_1(|s|)}(s \restriction \pi_2(|s|))$. It is easy to verify that $C_g = \bigcap_{i<\omega} C_i$. □

The above result is optimal, that is, CF_X is never $<\omega_2$-complete unlike its well-known counterpart on cardinals. Let $A \subseteq X$ be such that $|A| = \omega_1$, and C_x be the club corresponding to $f_x : [X]^{<\omega} \to \{x\}$. Then $C = \bigcap_{x\in A} C_x = \{B \subseteq X : \ B \supseteq A\}$ is disjoint from the stationary set $[X]^\omega$, hence is not a club.

Definition 1.2.4. Given a family $\{S_x \subseteq \mathcal{P}(X) : \ x \in X\}$, the *diagonal union* of the family is $\nabla_{x\in X} S_x = \{A \in \mathcal{P}(X) : \ \exists x \in A \ A \in S_x\}$, and the *diagonal intersection* of the family is $\Delta_{x\in X} S_x = \{A \in \mathcal{P}(X) : \forall x \in A \ A \in S_x\}$.

Lemma 1.2.5 (Fodor)**.** *CF_X is closed under diagonal intersection and NS_X is closed under diagonal union.*

Proof. Given a family $\langle C_x : \ x \in X\rangle$ of clubs, with corresponding functions f_x, let $g(x^\frown s) = f_x(s)$. It is easy to verify that $C_g = \Delta_{x\in X} C_x$.

Notice that $(\Delta_{x\in X} S_x)^c = \nabla_{x\in X} S_x^c$, where A^c denotes $\mathcal{P}(X) \setminus A$. It follows that NS_X is closed under diagonal union. □

The above property is often applied by means of the equivalence in Lemma 1.2.7.

Definition 1.2.6. Let $I \subseteq \mathcal{P}(X)$ be an ideal. We say that I (or equivalently its dual filter) is normal if for any $S \in I^+$ and for any choice function $f : \ S \to X$ (i.e. such that $f(A) \in A$ for any $A \in S$) there exists $x \in X$ such that $\{A \in S : \ f(A) = x\} \in I^+$. We say that I (or equivalently its dual filter) is fine if for any $x \in X$ the set $\{A \subseteq X : \ x \notin A\}$ is in I.

Notice that the set $C_x = \{A \subseteq X : x \in A\}$ is the club given by the constant function $f_x : \mathcal{P}(X) \to \{x\}$, thus $\{A \subseteq X : x \notin A\}$ is non stationary.

Lemma 1.2.7. *Let $I \subseteq \mathcal{P}(X)$ be an ideal. Then the following are equivalent:*

1. *I is closed under diagonal unions;*

2. *I is normal.*

Proof. **1 ⇒ 2.** Assume by contradiction that $f: S \to X$ is a choice function on $S \in I^+$ such that $f^{-1}[\{x\}] = S_x \in I$ for every $x \in X$. Let $T = \nabla_{x \in X} S_x$ which is also in I since I is closed under diagonal unions. Notice that for every $A \in S$, $A \in S_{f(A)}$ and $f(A) \in A$ hence $A \in T$. Thus, $S \subseteq T \in I^+$, a contradiction.

2 ⇒ 1. Assume by contradiction that $\{S_x : x \in X\} \subseteq I$ is such that $T = \nabla_{x \in X} S_x$ is in I^+. Define

$$\begin{aligned} f : T &\to X \\ A &\mapsto x_A \end{aligned}$$

where $x_A \in A$ is such that $A \in S_{x_A}$. Since I is normal, there exists an x such that $f^{-1}[\{x\}] \in I^+$. Since $f^{-1}[\{x\}] \subseteq S_x \in I$, we have a contradiction. □

We are now able to prove that NS_X is the smallest normal fine ideal on X.

Lemma 1.2.8. *Let $I \subseteq \mathcal{P}(X)$ be a normal and fine ideal, F be its dual filter. Then:*

1. *I is ω-fine, that is, for every $A \in [X]^\omega$, $\{B \in \mathcal{P}(X) : B \supseteq A\} \in F$;*

2. *I is σ-closed;*

3. *I is ω-normal, that is, for any function $f : S \to [X]^{<\omega}$ such that $S \in I^+$ and $f(A) \in [A]^{<\omega}$ for all $A \in S$, there exists an $s \in [X]^{<\omega}$ such that $\{A \in S : f(A) = s\} \in I^+$;*

4. *$I \supseteq \mathrm{NS}_X$.*

Proof. 1. Let $A = \{x_n : n \in \omega\}$, $S = \{B \in \mathcal{P}(X) : B \not\supseteq A\}$ and assume by contradiction that $S \in I^+$. Let $T = S \cap C_{x_0}$ be in I^+. Define

$$\begin{aligned} f : T &\to X \\ B &\mapsto x_n, \end{aligned}$$

where n is such that $\{x_0, \ldots, x_n\} \subseteq B$ and $x_{n+1} \notin B$. Since f is a choice function and I is normal, let x_m be such that $f^{-1}[\{x_m\}] \in I^+$. Since I is fine, $f^{-1}[\{x_m\}] \cap C_{x_{m+1}} = \emptyset \in I^+$, a contradiction.

2. Let $A = \{x_n : n \in \omega\}$ be a subset of X, $\{S_n : n \in \omega\} \subseteq F$. Define

$$T_x = \begin{cases} S_n \cap C_x & \text{if } x = x_n; \\ C_x & \text{otherwise.} \end{cases}$$

Since I is normal, $T = \Delta_{x \in X} T_x$ is in F. Since I is ω-fine, $T' = T \cap \{B \subseteq X : A \subseteq B\}$ is in F and $T' \subseteq \bigcap_{n<\omega} S_n$.

3. Let $f : S \to [X]^{<\omega}$ be such that $\forall A \in S$, $f(A) \in [A]^{<\omega}$. Define $S_n = \{A \in S : |f(A)| = n\}$. Since $S \in I^+$ and I is σ-closed there exists n such that $S_n \in I^+$. Let T_m be defined by induction on $m \leq n$ as in the following. Put $T_0 = S_n$. Given T_m for $m < n$, define h_m as

$$\begin{aligned} f_m : T_m &\to X \\ A &\mapsto f(A)(m). \end{aligned}$$

Since I is normal, let x_m be such that $T_{m+1} = f_m^{-1}[\{x_m\}] \in I^+$. Then T_n is in I^+ and $f(A) = (x_0, \ldots, x_{n-1}) = t$ for all $A \in T_n$, witnessing ω-normality.

4. Assume by contradiction that $S \in I^+$ and $S \cap C_f = \emptyset$ for some $f : X^{<\omega} \to X$, that is, for any $A \in S$ there exists $t_A \in A^{<\omega}$ such that $f(t_A) \notin A$. Define

$$\begin{aligned} g : S &\to [X]^{<\omega} \\ A &\mapsto t_A \end{aligned}$$

By ω-normality there exists $t \in X^{<\omega}$ such that $T = g^{-1}[\{t\}] \in I^+$. Since $T \subseteq \{A \in S : \ f(t) \notin A\}$, $T \cap C_{f(t)} = \emptyset \in I^+$, a contradiction. □

The generalized notion of club and stationary set is closely related to the well-known one defined for subsets of cardinals.

Lemma 1.2.9. *$C \subseteq \omega_1$ is a club in the classical sense if and only if $C \cup \{\omega_1\}$ is a club in the generalized sense. Thus, $S \subseteq \omega_1$ is stationary in the classical sense if and only if it is stationary in the generalized sense.*

Proof. Let $C \subseteq \omega_1 + 1$ be a club in the generalized sense. Then C is closed: given any $\alpha = \sup \alpha_i$ with $f[\alpha_i]^{<\omega} \subseteq \alpha_i$, $f[\alpha]^{<\omega} = \bigcup_i f[\alpha_i]^{<\omega} \subseteq \bigcup_i \alpha_i = \alpha$. Furthermore, C is unbounded: given any $\beta_0 < \omega_1$, define a sequence β_i by taking $\beta_{i+1} = \sup f[\beta_i]^{<\omega}$. Then $\beta_\omega = \sup \beta_i \in C$.

Let now $C \subseteq \omega_1$ be a club in the classical sense. Let $C = \{x_\alpha : \ \alpha < \omega_1\}$ be an enumeration of the club. For every $\alpha < \omega_1$, let $\{y_i^\alpha : \ i < \omega\} \subseteq x_\alpha$ be a cofinal sequence in x_α (possibly constant), and $\{z_i^\alpha : \ i < \omega\} \subseteq \alpha$ be an enumeration of α. Define $f_C : [X]^{<\omega} \to X$ as

$$f_C(s) = \begin{cases} y_n^\alpha & \text{if } s = (x_\alpha)^n \text{ with } \alpha > 0; \\ z_n^\alpha & \text{if } s = x_0^\frown(\alpha)^n; \\ x_0 & \text{otherwise.} \end{cases}$$

The sequence z_i^α forces all closure points of f_C to be ordinals, while the sequence y_i^α forces the ordinal closure points of f_C being in C. □

Lemma 1.2.10. *If κ is a cardinal with cofinality at least ω_1, $C \subseteq \kappa$ contains a club in the classical sense if and only if $C \cup \{\kappa\}$ contains the ordinals of a club in the generalized sense. Thus $S \subseteq \kappa$ is stationary in the classical sense if and only if it is stationary in the generalized sense.*

Proof. If C is a club in the generalized sense, then $C \cap \kappa$ is closed and unbounded by the same reasoning of Lemma 1.2.9. Let now C be a club in the classical sense, and define $f: \kappa^{<\omega} \to \kappa$ to be $f(s) = \min\{\alpha \in C : \sup s < \alpha\}$. Then $C_f \cap \kappa$ is exactly the set of ordinals in $C \cup \{\kappa\}$ that are limits within C. □

Notice that if S is stationary in the generalized sense on ω_1, then $S \cap \omega_1$ is stationary (since $\omega_1 + 1$ is a club by Lemma 1.2.9), while this is not true for $\kappa > \omega_1$. In this case, $\mathcal{P}(\kappa) \setminus (\kappa + 1)$ is a stationary set: given any function f, the closure under f of $\{\omega_1\}$ is countable, hence not an ordinal.

The non-stationary ideals NS_X forms a coherent system varying $X \in V$, as shown in the following.

Lemma 1.2.11 (Lifting and Projection)**.** *Let $X \subseteq Y$ be uncountable sets. If S is stationary on $\mathcal{P}(X)$, then $S \uparrow Y = \{B \subseteq Y : B \cap X \in S\}$ is stationary. If S is stationary on $\mathcal{P}(Y)$, then $S \downarrow X = \{B \cap X : B \in S\}$ is stationary.*

Proof. For the first part, given any function $f: [X]^{<\omega} \to X$, extend it in any way to a function $g: [Y]^{<\omega} \to Y$. Since S is stationary, there exists a $B \in S$ closed under g, hence $B \cap X \in S \downarrow X$ is closed under f.

For the second part, fix an element $x \in X$. Given any function $f: [Y]^{<\omega} \to Y$, replace it with a function $g: [Y]^{<\omega} \to Y$ such that for any $A \subset Y$, $g[[A]^{<\omega}]$ contains $A \cup \{x\}$ and is closed under f. To achieve this, fix a surjection $\pi: \omega \to \omega^2$ (with projections π_1 and π_2) such that $\pi_2(n) \leq n$ for all n, and an enumeration $\langle t_i^n : i < \omega\rangle$ of all first-order terms with n variables, function symbols f_i for $i \leq n$ (that represent an i-ary application of f) and a constant x. The function g can now be defined as $g(s) = t_{\pi_1(|s|)}^{\pi_2(|s|)}(s \restriction \pi_2(|s|))$. Finally, let $h: [X]^{<\omega} \to X$ be defined by $h(s) = g(s)$ if $g(s) \in X$, and $h(s) = x$ otherwise. Since S is stationary, there exists a $B \in S$ with $h[[B]^{<\omega}] \subseteq B$, but $h[[B]^{<\omega}] = g[[B]^{<\omega}] \cap X$ (since x is always in $g[[B]^{<\omega}]$) and $g[[B]^{<\omega}] \supset B$, so actually $h[[B]^{<\omega}] = g[[B]^{<\omega}] \cap X = B \in S$. Then, $g[[B]^{<\omega}] \in S \uparrow Y$ and $g[[B]^{<\omega}]$ is closed under f (by definition of g). □

Following the same proof, a similar result holds for clubs. If C_f is club on $\mathcal{P}(X)$, then $C_f \uparrow Y = C_g$ where $g = f \cup \mathrm{id} \restriction (Y \setminus X)$. If C_f is club on $\mathcal{P}(Y)$ such that $\bigcap C_f$ intersects X in x, and g, h are defined as in the second part of Theorem 1.2.11, $C_f \downarrow X = C_h$ is club. If $\bigcap C_f$ is disjoint from X, $C_f \downarrow X$ is not a club, but is still true that it contains a club (namely, $\left(C_f \cap C_{\{x\}}\right) \downarrow X$ for any $x \in X$).

Theorem 1.2.12 (Ulam)**.** *Let κ be an infinite cardinal. Then for every stationary set $S \subseteq \kappa^+$, there exists a partition of S into κ^+ many disjoint stationary sets.*

Proof. For every $\beta \in [\kappa, \kappa^+)$, fix a bijection $\pi_\beta: \kappa \to \beta$. For $\xi < \kappa$, $\alpha < \kappa^+$, define $A_\alpha^\xi = \{\beta < \kappa^+ : \pi_\beta(\xi) = \alpha\}$ (notice that $\beta > \alpha$ when $\alpha \in \mathrm{ran}(\pi_\beta)$). These sets can be fit in a $(\kappa \times \kappa^+)$-matrix, called *Ulam Matrix*, where two sets in the same row or column are always disjoint. Moreover, every row is a partition of $\bigcup_{\alpha<\kappa^+} A_\alpha^\xi = \kappa^+$, and every column is a partition of $\bigcup_{\xi<\kappa} A_\alpha^\xi = \kappa^+ \setminus (\alpha + 1)$.

Let S be a stationary subset of κ^+. For every $\alpha < \kappa^+$, define $f_\alpha: S \setminus (\alpha+1) \to \kappa$ by $f_\alpha(\beta) = \xi$ if $\beta \in A_\alpha^\xi$. Since $\kappa^+ \setminus (\alpha + 1)$ is a club, every f_α is regressive on a stationary set, then by Fodor's Lemma 1.2.5 there exists a $\xi_\alpha < \kappa$ such that

$f_\alpha^{-1}[\{\xi_\alpha\}] = A_\alpha^{\xi_\alpha} \cap S$ is stationary. Define $g: \kappa^+ \to \kappa$ by $g(\alpha) = \xi_\alpha$, g is regressive on the stationary set $\kappa^+ \setminus \kappa$, again by Fodor's Lemma 1.2.5 let $\xi^* < \kappa$ be such that $g^{-1}[\{\xi^*\}] = T$ is stationary. Then, the row ξ^* of the Ulam Matrix intersects S in a stationary set for stationary many columns T. So S can be partitioned into $S \cap A_\alpha^{\xi^*}$ for $\alpha \in T \setminus \{\min(T)\}$, and $S \setminus \bigcup_{\alpha \in T \setminus \{\min(T)\}} A_\alpha^{\xi^*}$. □

In the proof of Theorem 1.2.12 we actually proved something more: the existence of a Ulam Matrix, i.e. a $\kappa \times \kappa^+$-matrix such that every stationary set $S \subseteq \kappa^+$ is compatible (i.e., has stationary intersection) with stationary many elements of a certain row.

Stationary sets are to be intended as *large* sets. Moreover, they cannot be too small even in literal sense.

Lemma 1.2.13. *Let $S \subseteq \mathcal{P}(X) \setminus \{X\}$ be such that $|S| < |X|$. Then S is non-stationary.*

Proof. Let $S = S_1 \cup S_2$, $S_1 = \{Y \in S : |Y| < |S|\}$, $S_2 = \{Y \in S : |Y| \geq |S|\}$. Since $|\bigcup S_1| \leq |S| \cdot |S| = |S| < |X|$, S_1 is non-stationary. We now prove that S_2 is non-stationary as well.

Fix an enumeration $S_2 = \{Y_\alpha : \alpha < \gamma\}$ with $\gamma = |S_2| < |X|$. For all $\alpha < \gamma$, define recursively $x_\alpha \in X \setminus Y_\alpha$, $y_\alpha \in Y_\alpha \setminus \{y_\beta : \beta < \alpha\}$. Such x_α exists since $\{X\} \notin S$, and such y_α exists since $|Y_\alpha| \geq |S| = \gamma > \alpha$. Let $f : [X]^{<\omega} \to X$ be such that $f(\{y_\alpha\}) = x_\alpha$, $f(s) = x_0$ otherwise. Thus $C_f \cap S_2 = \emptyset$, hence S_2 is non-stationary. □

As previously mentioned, $[X]^\kappa$ and $[X]^{<\kappa}$ for any $\kappa \leq |X|$ are the prototypical examples of stationary sets. This encourages to consider the notion of club and stationary set relative to them.

Definition 1.2.14. Let X be an uncountable set, $\kappa \leq |X|$ be a cardinal. A set C is a *club* on $[X]^\kappa$ (resp. $[X]^{<\kappa}$) iff there is a function $f_C : X^{<\omega} \to X$ such that C is the set of elements of $[X]^\kappa$ (resp. $[X]^{<\kappa}$) closed under f_C, i.e.

$$C = \left\{Y \in [X]^\kappa : f_C[Y]^{<\omega} \subseteq Y\right\}$$

A set S is *stationary* on $[X]^\kappa$ (respectively $[X]^{<\kappa}$) iff it intersects every club on $[X]^\kappa$ (respectively $[X]^{<\kappa}$).

As in the general case, the club sets on $[X]^\kappa$ (resp. $[X]^{<\kappa}$) form a normal σ-complete filter on $[X]^\kappa$ (resp. $[X]^{<\kappa}$). We can also state a restricted version of Lemma 1.2.11 to this setting.

Lemma 1.2.15 (Lifting and Projection II)**.** *Let $X \subseteq Y$ be uncountable sets, $\kappa \leq |X|$ be a cardinal. If C contains a club on $[Y]^\kappa$ (resp. $[Y]^{<\kappa}$), then $C \downarrow [X]^\kappa = (C \downarrow X) \cap [X]^\kappa$ (resp. $C \downarrow [X]^{<\kappa}$) contains a club on $[X]^\kappa$ (resp. $[X]^{<\kappa}$). If C contains a club on $[X]^{<\kappa}$, then $C \uparrow [Y]^{<\kappa} = (C \uparrow Y) \cap [Y]^{<\kappa}$ contains a club on $[Y]^{<\kappa}$.*

If S is stationary on $[Y]^{<\kappa}$, then $S \downarrow [X]^{<\kappa}$ is stationary on $[X]^{<\kappa}$. If S is stationary on $[X]^\kappa$ (resp. $[X]^{<\kappa}$), then $S \uparrow [Y]^\kappa$ is stationary on $[Y]^\kappa$ (resp. with $[Y]^{<\kappa}$).

The latter lemma is optimal, that is, the lifting $[X]^\kappa \uparrow [Y]^\kappa$ may not be a club on $[Y]^\kappa$ if $|X| < |Y|$. For example, such a set is not a club if $|Y|$ is a Completely Jónsson cardinal (see [33]) since in this case its complement $[Y]^\kappa \setminus ([X]^\kappa \uparrow [Y]^\kappa) = [X]^{<\kappa} \uparrow [Y]^\kappa$ is stationary.

1.3 Forcing and boolean valued models

We now give a brief account of forcing via boolean valued models. Our presentation is self-contained but extremely compact, more details on the material of this section can be found in [2, 8, 26, 27, 32].

Recall that $\mathbb{P}$ is a *poset* iff it is a set equipped with a partial order $\leq_{\mathbb{P}}$. Given $p, q \in \mathbb{P}$, we say that $p \parallel q$ (p is compatible with q) iff there exists an $r \leq p$, $r \leq q$; and $p \perp q$ (p is incompatible with q) otherwise.

A subset $D \subseteq \mathbb{P}$ is *dense* iff for every $p \in \mathbb{P}$ there is a $q \in D$, $q \leq p$ and is open iff for every $p \in D$, any $q \in \mathbb{P}$ such that $q \leq p$ is in D. We say that a subset $A \subseteq \mathbb{P}$ is an *antichain* iff $p \perp q$ for all $p, q \in A$ and that it is a *chain* iff it is totally ordered by $\leq_{\mathbb{P}}$. An antichain $A \subseteq \mathbb{P}$ is *maximal* iff every $p \in \mathbb{P}$ is compatible with some $q \in A$.

We say that $\mathbb{P}$ is a suborder of $\mathbb{Q}$ if $\mathbb{P} \subseteq \mathbb{Q}$ and the inclusion map of $\mathbb{P}$ into $\mathbb{Q}$ is preserving the order and the incompatibility relation. Furthermore, $\mathbb{P}$ is a complete suborder of $\mathbb{Q}$ if any maximal antichain in $\langle \mathbb{P}, \leq_{\mathbb{P}} \rangle$ remains such in $\langle \mathbb{Q}, \leq_{\mathbb{Q}} \rangle$.

Recall that a poset $\mathbb{P}$ is a *lattice* iff every $p, q \in \mathbb{P}$ have a least upper bound $p \vee q$ and a greater lower bound $p \wedge q$, and that $\mathbb{B}$ is a *boolean algebra* iff it is a complemented distributive lattice. A boolean algebra $\mathbb{B}$ is *complete* iff every subset $A \subset \mathbb{B}$ has a unique supremum $\vee A$ in $\mathbb{B}$.

Definition 1.3.1. Let $\mathbb{P}$ be a poset, p be in $\mathbb{P}$. Then

$$\mathbb{P} \restriction p = \{q \in \mathbb{P} : q \leq p\},$$

is the *restriction* of $\mathbb{P}$ below p. If $\mathbb{P} = \mathbb{B}$ is a boolean algebra, we define

$$\begin{array}{rccc} i_p : & \mathbb{B} & \longrightarrow & \mathbb{B} \restriction p \\ & q & \longmapsto & q \wedge p \end{array}$$

the *restriction map* from $\mathbb{B}$ to $\mathbb{B} \restriction p$.

We call *forcing notion* a poset (or complete boolean algebra) used for forcing, and *conditions* the elements of a forcing notion. Throughout the remainder of this section, let M be a transitive model of ZFC and $\mathbb{B}$ be a complete boolean algebra in M.

Definition 1.3.2. The class $M^{\mathbb{B}}$ of $\mathbb{B}$-names in M is

$$M^{\mathbb{B}} = \{\dot{x} \in M : \dot{x} : M^{\mathbb{B}} \to \mathbb{B} \text{ is a partial function}^1\}.$$

We let for the atomic formulas $x \in y$, $x \subseteq y$, $x = y$:

[1] This definition is a shorthand for a recursive definition by rank. We remark that in certain cases (for example in the definition of the name $\dot{\beta}$ in the proof of Lemma 3.4.8) it will be convenient to allow a name $\dot{y}$ to be a relation (as in Kunen's [32, Def. 2.5]); given a $\mathbb{B}$-name $\dot{y}$ according to Kunen's definition, the corresponding intended name $f_{\dot{y}}$ according to the above definition is given by $f_{\dot{y}}(f_{\dot{c}}) = \bigvee \{b : \langle \dot{c}, b \rangle \in \dot{y}\}$.

- $[\![\dot{x}_0 \in \dot{x}_1]\!]_{\mathbb{B}} = \bigvee \{[\![\dot{y} = \dot{x}_0]\!]_{\mathbb{B}} \wedge \dot{x}_0(\dot{y}) : \dot{y} \in \operatorname{dom}(\dot{x}_1)\}$,
- $[\![\dot{x}_0 \subseteq \dot{x}_1]\!]_{\mathbb{B}} = \bigwedge \{\neg \dot{x}_0(\dot{y}) \vee [\![\dot{y} \in \dot{x}_0]\!]_{\mathbb{B}} : \dot{y} \in \operatorname{dom}(\dot{x}_0)\}$,
- $[\![\dot{x}_0 = \dot{x}_1]\!]_{\mathbb{B}} = [\![\dot{x}_0 \subseteq \dot{x}_1]\!]_{\mathbb{B}} \wedge [\![\dot{x}_1 \subseteq \dot{x}_0]\!]_{\mathbb{B}}$.

and for general formulas $\phi(x_0, \ldots, x_n)$:

- $[\![\neg\phi]\!]_{\mathbb{B}} = \neg [\![\phi]\!]_{\mathbb{B}}$,
- $[\![\phi \wedge \psi]\!]_{\mathbb{B}} = [\![\phi]\!]_{\mathbb{B}} \wedge [\![\psi]\!]_{\mathbb{B}}$,
- $[\![\phi \vee \psi]\!]_{\mathbb{B}} = [\![\phi]\!]_{\mathbb{B}} \vee [\![\psi]\!]_{\mathbb{B}}$,
- $[\![\exists x \phi(x, \dot{x}_1, \ldots, \dot{x}_n)]\!]_{\mathbb{B}} = \bigvee \left\{[\![\phi(\dot{y}, \dot{x}_1, \ldots, \dot{x}_n)]\!]_{\mathbb{B}} : \ \dot{y} \in M^{\mathbb{B}}\right\}$.

$M^{\mathbb{B}}$ together with the interpretation of formulas $[\![\ldots]\!]_{\mathbb{B}}$ given above forms a *boolean valued model* of set theory. When clear from the context, we shall omit the index $\mathbb{B}$ from $[\![\ldots]\!]$. We also use $p \Vdash_{\mathbb{B}} \phi$ to denote that $p \leq [\![\phi]\!]_{\mathbb{B}}$.

Definition 1.3.3. Given $x \in M$, we define the canonical $\mathbb{B}$-name $\check{x}$ for x recursively by:

$$\check{x} = \{\langle \check{y}, \mathbf{1} \rangle : y \in x\},$$

and expand the language with an additional symbol $\check{M}$ for a boolean-valued subclass of $M^{\mathbb{B}}$ defined by:

$$[\![\dot{x} \in \check{M}]\!]_{\mathbb{B}} = \bigvee \{[\![\dot{x} = \check{y}]\!]_{\mathbb{B}} : \ y \in M\}.$$

Finally, we denote as $\dot{G}_{\mathbb{B}} \in M^{\mathbb{B}}$ the canonical name for a M-generic filter for $\mathbb{B}$, i.e. $\dot{G}_{\mathbb{B}} = \{\langle \check{p}, p \rangle : p \in \mathbb{B}\}$.

Lemma 1.3.4 (Mixing)**.** *Let $A \subseteq \mathbb{B}$ be an antichain in M and $\{\dot{x}_p : \ p \in A\}$ be a family of $\mathbb{B}$-names*[2] *indexed by A in M. Then there exists $\dot{y} \in M^{\mathbb{B}}$ such that $[\![\dot{y} = \dot{x}_p]\!]_{\mathbb{B}} \geq p$ for all $p \in A$.*

Proof. The property is witnessed by $\dot{y} = \bigcup_{p \in A} \{\langle \dot{z}, q \wedge p \rangle : \ \langle \dot{z}, q \rangle \in \dot{x}_p\}$. □

Lemma 1.3.5 (Fullness)**.** *For all formula $\phi(x, x_1, \ldots, x_n)$ and $\dot{x}_1, \ldots, \dot{x}_n \in M^{\mathbb{B}}$, there exists a $\dot{y} \in M^{\mathbb{B}}$ such that*

$$[\![\exists x \phi(x, \dot{x}_1, \ldots \dot{x}_n)]\!]_{\mathbb{B}} = [\![\phi(\dot{y}, \dot{x}_1, \ldots \dot{x}_n)]\!]_{\mathbb{B}}.$$

Proof. First notice that for any $\dot{y} \in M^{\mathbb{B}}$,

$$[\![\exists x \phi(x, \dot{x}_1, \ldots \dot{x}_n)]\!]_{\mathbb{B}} \geq [\![\phi(\dot{y}, \dot{x}_1, \ldots \dot{x}_n)]\!]_{\mathbb{B}}.$$

Set $B = \left\{[\![\phi(\dot{y}, \dot{x}_1, \ldots \dot{x}_n)]\!]_{\mathbb{B}} : \ \dot{y} \in M^{\mathbb{B}}\right\}$, and let B' be its downward closure. Since $q = [\![\exists x \phi(x, \dot{x}_1, \ldots \dot{x}_n)]\!]_{\mathbb{B}} = \bigvee B$, we can find an antichain $A \subseteq B'$ in M which is maximal below q. For all $p \in A$, let $\dot{x}_p$ be such that $[\![\phi(\dot{x}_p, \dot{x}_1, \ldots \dot{x}_n)]\!]_{\mathbb{B}} \geq p$, $\dot{y}$ be obtained from $\{\dot{x}_p : \ p \in A\}$ by mixing as in Lemma 1.3.4. Then

$$[\![\phi(\dot{y}, \dot{x}_1, \ldots \dot{x}_n)]\!]_{\mathbb{B}} \geq [\![\dot{y} = \dot{x}_p]\!] \wedge [\![\phi(\dot{x}_p, \dot{x}_1, \ldots \dot{x}_n)]\!]_{\mathbb{B}} \geq p \wedge p = p$$

for all $p \in A$, hence $[\![\phi(\dot{y}, \dot{x}_1, \ldots \dot{x}_n)]\!]_{\mathbb{B}} \geq \bigvee A = q = [\![\exists x \phi(x, \dot{x}_1, \ldots \dot{x}_n)]\!]_{\mathbb{B}}$. □

[2] Contrary to the convention used throughout this thesis, here we use the notation $\{\dot{x}_p : \ p \in A\}$ to denote a family of $\mathbb{B}$-names rather than a single $\mathbb{B}$-name for the whole collection.

Definition 1.3.6. Let G be any ultrafilter on $\mathbb{B}$. For $\dot{x}, \dot{y} \in M^{\mathbb{B}}$ we let

- $[\dot{x}]_G = [\dot{y}]_G$ iff $[\![\dot{x} = \dot{y}]\!]_{\mathbb{B}} \in G$,
- $[\dot{x}]_G \in [\dot{y}]_G$ iff $[\![\dot{x} \in \dot{y}]\!]_{\mathbb{B}} \in G$,
- $M^{\mathbb{B}}/G = \left\{[\dot{x}]_G : \ \dot{x} \in M^{\mathbb{B}}\right\}$.

Theorem 1.3.7 (Łoś, [27, Lemma 14.14, Thm. 14.24])**.** *Let G be any ultrafilter on $\mathbb{B}$. Then*

1. $\langle M^{\mathbb{B}}/G, \in_G\rangle$ *is a model of* ZFC*,*
2. $\langle M^{\mathbb{B}}/G, \in_G\rangle$ *models* $\phi([\dot{x}_1]_G, \ldots, [\dot{x}_n]_G)$ *iff* $[\![\phi(\dot{x}_1, \ldots, \dot{x}_n)]\!]_{\mathbb{B}} \in G$.

Even though the above theorem holds for any ultrafilter G, $M^{\mathbb{B}}/G$ is commonly considered for M-generic filters G.

Definition 1.3.8. We say that G is an M-generic filter for $\mathbb{B}$ if it is an ultrafilter and for every $D \in M$ dense subset of $\mathbb{B}$, $G \cap D \cap M$ is not empty.

We remark that the above definition can be applied as stated even to non-transitive models $M \subseteq V$ of ZFC (with the natural $\in$-relation). However, for the remainder of this section we shall still assume that M is transitive.

Definition 1.3.9. Let G be an M-generic filter for $\mathbb{B}$. For any $\dot{x} \in M^{\mathbb{B}}$ we let

$$\operatorname{val}_G(\dot{x}) = \{\operatorname{val}_G(\dot{y}) : \ \exists p \in G \ \langle \dot{y}, p\rangle \in \dot{x}\}$$

and $M[G] = \left\{\operatorname{val}_G(\dot{x}) : \ \dot{x} \in M^{\mathbb{B}}\right\}$.

Theorem 1.3.10 (Cohen's Forcing, [27, Thm. 14.6])**.** *Let G be an M-generic filter for $\mathbb{B}$. Then:*

1. $\operatorname{val}_G[\check{M}] = M$*, that is,* $\left\{\operatorname{val}_G(\dot{x}) : \ [\![\dot{x} \in \check{M}]\!]_{\mathbb{B}} \in G\right\} = M$.
2. $M[G]$ *is isomorphic to* $M^{\mathbb{B}}/G$ *via the map which sends* $\operatorname{val}_G(\dot{x})$ *to* $[\dot{x}]_G$.
3. $M[G] \models \phi\left(\operatorname{val}_G(\dot{x}_1), \ldots, \operatorname{val}_G(\dot{x}_n)\right)$ *iff* $[\![\phi(\dot{x}_1, \ldots, \dot{x}_n)]\!]_{\mathbb{B}} \in G$.
4. $p \Vdash_{\mathbb{B}} \phi(\dot{x}_1, \ldots, \dot{x}_n)$ *iff* $M[G] \models \phi\left(\operatorname{val}_G(\dot{x}_1), \ldots, \operatorname{val}_G(\dot{x}_n)\right)$ *for all M-generic filters G for $\mathbb{B}$ such that $p \in G$.*
5. $M[G]$ *is the smallest transitive model of* ZFC *including M and containing G.*

Proposition 1.3.11. *Let G be an M-generic ultrafilter for $\mathbb{B}$. Then $\bigwedge A \in G$ for any $A \subset G$ which belongs to M.*

Proof. Suppose instead that $\bigwedge A \notin G$ and w.l.o.g. $\bigwedge A = \mathbf{0}$, and define

$$D = \{p \in \mathbb{B} : \ \exists q \in A \ \ p \wedge q = \mathbf{0}\}.$$

Since D is a dense subset of $\mathbb{B}$ in M, there exists a $p \in G \cap D$. Let $q \in A$ be such that $p \wedge q = \mathbf{0}$. Then $p, q \in G$ hence G is not a filter, a contradiction. □

CHAPTER 2

ITERATION SYSTEMS

This chapter present a compact and self contained development of the theory of iterated forcing. We shall pursue the approach to iterated forcing devised by Donder and Fuchs in [19], thus we shall present iterated forcing by means of directed system of complete and injective homomorphisms of complete boolean algebras. The results of this chapter will constitute the basis on which the remainder of this thesis is built. All the material in this chapter is joint work with Matteo Viale and Silvia Steila (see [45]).

In Section 2.1 we introduce regular homomorphisms between complete boolean algebras (i.e. injective complete homomorphisms) and their associated retractions. In Section 2.2 we present iterated forcing in the setting of complete boolean algebras, together with the definition of revised countable support iterations. In the final part of the section we study basic properties of complete iteration systems, in particular we set up sufficient conditions to establish when the direct limit of an iteration system of length λ is $<\lambda$-cc (this corresponds to the well known result of Baumgartner on direct limits of $<\lambda$-cc forcings), and when $\varinjlim \mathcal{F}$ and $\varprojlim \mathcal{F}$ do overlap for a given iteration system $\mathcal{F}$. In Section 2.3 we introduce the definition of two-step iteration following Jech's approach [27], and present the main properties of generic quotients for iteration systems.

2.1 Embeddings and retractions

We now introduce the notions of complete homomorphism and regular embedding and their basic properties.

Definition 2.1.1. Let $\mathbb{B}$, $\mathbb{C}$ be complete boolean algebras, $i : \mathbb{B} \to \mathbb{C}$ is a *complete homomorphism* iff it is an homomorphism that preserves arbitrary suprema. We say that i is a *regular embedding* iff it is an injective complete homomorphism of boolean algebras.

Definition 2.1.2. Let $i : \mathbb{B} \to \mathbb{C}$ be a complete homomorphism. We define

$$\ker(i) = \bigvee \{p \in \mathbb{B} : \ i(p) = \mathbf{0}\}$$

$$\mathrm{coker}(i) = \neg \ker(i)$$

We can always factor a complete homomorphism $i : \mathbb{B} \to \mathbb{C}$ as the restriction map from $\mathbb{B}$ to $\mathbb{B}\restriction \mathrm{coker}(i)$ (which we can trivially check to be a complete and surjective homomorphism) composed with the regular embedding $i\restriction \mathrm{coker}(i)$. This factorization allows to generalize easily many results on regular embeddings to results on complete homomorphisms.

Definition 2.1.3. Let $i : \mathbb{B} \to \mathbb{C}$ be a regular embedding, the *retraction* associated to i is the map

$$\begin{array}{rccl} \pi_i : & \mathbb{C} & \to & \mathbb{B} \\ & p & \mapsto & \bigwedge \{q \in \mathbb{B} : \ i(q) \geq p\} . \end{array}$$

Proposition 2.1.4. *Let $i : \mathbb{B} \to \mathbb{C}$ be a regular embedding, $p \in \mathbb{B}$, $q, r \in \mathbb{C}$ be arbitrary. Then,*

1. *$\pi_i \circ i(p) = p$ hence π_i is surjective;*
2. *$i \circ \pi_i(q) \geq q$ hence π_i maps $\mathbb{C}^+$ to $\mathbb{B}^+$;*
3. *π_i preserves joins, i.e. $\pi_i(\bigvee A) = \bigvee \pi_i[A]$ for all $A \subseteq \mathbb{C}$;*
4. *$i(p) = \bigvee \{q : \pi_i(q) \leq p\}$.*
5. *$\pi_i(q \wedge i(p)) = \pi_i(q) \wedge p = \bigvee \{\pi_i(r) : \ r \leq q, \pi_i(r) \leq p\}$;*
6. *π_i does not preserve neither meets nor complements whenever i is not surjective, but $\pi_i(r \wedge q) \leq \pi_i(r) \wedge \pi_i(q)$ and $\pi_i(\neg q) \geq \neg \pi_i(q)$;*

Proof. 1. Since i is injective,

$$\pi_i \circ i(p) = \bigwedge \{q \in \mathbb{B} : i(q) \geq i(p)\}) = \bigwedge \{q \in \mathbb{B} : q \geq p\}) = p,$$

thus π is surjective.

2. Suppose by contradiction that $q > \mathbf{0}$, $\pi_i(q) = \mathbf{0}$. Since

$$\begin{aligned} i \circ \pi_i(q) = i \left(\bigwedge \{p \in \mathbb{B} : i(p) \geq q\} \right) &= \bigwedge \{i(p) : p \in \mathbb{B},\, i(p) \geq q\} \\ &\geq \bigwedge \{r \in \mathbb{C} : r \geq q\} = q, \end{aligned}$$

we have that $\mathbf{0} = i \circ \pi(q) \geq q > \mathbf{0}$, a contradiction.

3. Let $A = \{q_\alpha : \ \alpha < \gamma\} \subseteq \mathbb{C}$. Thus for all $\beta < \gamma$,

$$\begin{aligned} \pi_i \left(\bigvee \{q_\alpha : \ \alpha < \gamma\} \right) &= \bigwedge \{p \in \mathbb{B} : i(p) \geq \bigvee \{q_\alpha : \ \alpha < \gamma\}\} \\ &\geq \bigwedge \{p \in \mathbb{B} : \ i(p) \geq q_\beta\} = \pi_i(q_\beta). \end{aligned}$$

and we obtain the first inequality: $\pi_i(\bigvee A) \geq \bigvee \pi_i[A]$.

Let $p = \bigvee \pi_i[A]$, so that $p \geq \pi_i(q_\alpha)$ for all $\alpha < \gamma$. Thus for all $\beta < \gamma$:

$$i(p) \geq i \circ \pi_i(q_\beta) \geq q_\beta.$$

In particular $i(p) \geq \bigvee\{q_\alpha : \ \alpha < \gamma\}$. By definition, π_i is increasing then

$$p = \pi_i(i(p)) \geq \pi\left(\bigvee\{q_\alpha : \ \alpha < \gamma\}\right),$$

that is, the second inequality $\bigvee \pi_i[A] \geq \pi_i(\bigvee A)$ holds.

4. Let $q \in \mathbb{C}$ be such that $\pi_i(q) \leq p$. Since i is order preserving, $q \leq i(\pi_i(q)) \leq i(p)$. Thus,

$$\bigvee\{q : \pi_i(q) \leq p\} \leq i(p).$$

In order to prove the other inequality; recall that $p = \pi_i(i(p))$. So,

$$i(p) \leq \bigvee\{q : \pi_i(q) \leq \pi_i(i(p))\} = \bigvee\{q : \pi_i(q) \leq p\}.$$

5. For $p \in \mathbb{B}, q \in \mathbb{C}$, the following three equations hold:

$$\pi_i(q \wedge i(p)) \vee \pi_i(q \wedge \neg i(p)) = \pi_i(q); \tag{2.1}$$
$$(\pi_i(q) \wedge p) \vee (\pi_i(q) \wedge \neg p) = \pi_i(q); \tag{2.2}$$
$$(\pi_i(q) \wedge p) \wedge (\pi_i(q) \wedge \neg p) = \mathbf{0}. \tag{2.3}$$

Furthermore, by definition of π_i we have:

$$\pi_i(q \wedge i(p)) \leq \pi_i(q) \wedge p; \tag{2.4}$$

$$\pi_i(q \wedge \neg i(p)) = \pi_i(q \wedge i(\neg p)) \leq \pi_i(q) \wedge \neg p. \tag{2.5}$$

By (2.4), (2.5), and (2.3) we get

$$\pi_i(q \wedge i(p)) \wedge \pi_i(q \wedge \neg i(p)) = (\pi_i(q) \wedge p) \wedge (\pi_i(q) \wedge \neg p) = \mathbf{0}.$$

Moreover, by (2.1) and (2.2),

$$\pi_i(q \wedge i(p)) \vee \pi_i(q \wedge \neg i(p)) = (\pi_i(q) \wedge p) \vee (\pi_i(q) \wedge \neg p).$$

We conclude that

$$\pi_i(q \wedge i(p)) = \pi_i(q) \wedge p \text{ and } \pi_i(q \wedge \neg i(p)) = \pi_i(q) \wedge \neg p.$$

6. Suppose that $i : \mathbb{B} \to \mathbb{C}$ is not surjective, and pick $q \in \mathbb{C} \setminus i[\mathbb{B}]$. Then $i(\pi_i(q)) \neq q$ hence $i(\pi_i(q)) > q$. Put $r = i(\pi_i(q)) \wedge \neg q > \mathbf{0}$. Then $\pi_i(r) > \mathbf{0}$ and

$$\pi_i(q) \vee \pi_i(r) = \pi_i(q \vee r) = \pi_i(i(\pi_i(q))) = \pi_i(q).$$

thus $\pi_i(r) \wedge \pi_i(q) = \pi_i(r) > \mathbf{0}$. Since $\pi_i(r \wedge q) = \pi_i(\mathbf{0}) = \mathbf{0}$, π_i does not preserve meets. Furthermore, π_i preserves joins thus it must be the case that it does not preserve complements. In addition, for all r, $q \in \mathbb{C}$,

$$\pi_i(r \wedge q) \leq \pi_i(r \wedge i(\pi_i(q))) = \pi_i(r) \wedge \pi_i(q)$$

and $\neg\pi_i(r) \leq \pi_i(\neg r)$, since

$$\neg\pi_i(r) \wedge \neg\pi_i(\neg r) = \neg(\pi_i(r) \vee \pi_i(\neg r)) = \neg(\pi_i(r \vee \neg r)) = \neg(\pi_i(\mathbf{1})) = \mathbf{0}. \quad \square$$

Complete homomorphisms and regular embeddings are the boolean algebraic counterpart of two-step iterations. This will be spelled out in detail in Section 2.3. We now outline the relation existing between generic extensions by $\mathbb{B}$ and $\mathbb{C}$ in case there is a complete homomorphism $i : \mathbb{B} \to \mathbb{C}$.

Lemma 2.1.5. *Let $i : \mathbb{B} \to \mathbb{C}$ be a regular embedding, $D \subset \mathbb{B}$, $E \subset \mathbb{C}$ be predense sets. Then $i[D]$ and $\pi_i[E]$ are predense (i.e. predense subsets are mapped into predense subsets). Moreover π_i maps V-generic filters to V-generic filters.*

Proof. First, let $p \in \mathbb{C}$ be arbitrary. Since D is predense, there exists $q \in D$ such that $q \wedge \pi(p) > \mathbf{0}$. Then by Property 2.1.4.(5) also $i(q) \wedge p > \mathbf{0}$ hence $i[D]$ is predense. Furthermore, let $p \in \mathbb{B}$ be arbitrary. Since E is predense, there exists $q \in E$ such that $q \wedge i(p) > \mathbf{0}$. Then by Property 2.1.4.(5) also $\pi_i(q) \wedge p > \mathbf{0}$ hence $\pi_i[E]$ is predense.

For the last point in the lemma, we first prove that $\pi_i[G]$ is a filter whenever G is a filter. Let p be in G, and suppose $q > \pi_i(p)$. Then by Property 2.1.4.(2) also $i(q) > i(\pi_i(p)) \geq p$, hence $i(q) \in G$ and $q \in \pi_i[G]$, proving that $\pi_i[G]$ is upward closed. Now suppose $p, q \in G$, then by Property 2.1.4.(6) we have that $\pi_i(p) \wedge \pi_i(q) \geq \pi_i(p \wedge q) \in \pi_i[G]$ since $p \wedge q \in G$. Combined with the fact that $\pi_i[G]$ is upward closed this concludes the proof that $\pi_i[G]$ is a filter.

Finally, let D be a predense subset of $\mathbb{B}$ and assume G is V-generic for $\mathbb{C}$. We have that $i[D]$ is predense hence $i[D] \cap G \neq \emptyset$ by V-genericity of G. Fix $p \in i[D] \cap G$, then $\pi_i(p) \in D \cap \pi_i[G]$ concluding the proof. □

Lemma 2.1.6. *Let $i : \mathbb{B} \to \mathbb{C}$ be an homomorphism of boolean algebras. Then i is a complete homomorphism iff for every V-generic filter G for $\mathbb{C}$, $i^{-1}[G]$ is a V-generic filter for $\mathbb{B}$.*

Proof. If i is a complete homomorphism and G is a V-generic filter, then $i^{-1}[G]$ is trivially a filter. Furthermore, given D dense subset of $\mathbb{B}$, $i[D]$ is predense so there exists a $p \in G \cap i[D]$, hence $i^{-1}(p) \in i^{-1}[G] \cap D$.

Conversely, suppose by contradiction that there exists an $A \subseteq \mathbb{B}$ such that $i(\bigvee A) \neq \bigvee i[A]$ (in particular, necessarily $i(\bigvee A) > \bigvee i[A]$). Let $p = i(\bigvee A) \setminus \bigvee i[A]$, G be a V-generic filter with $p \in G$. Then $i^{-1}[G] \cap A = \emptyset$ hence is not V-generic below $\bigvee A \in i^{-1}[G]$, a contradiction. □

Later in this chapter we shall use the following lemma to produce local versions of various results.

Lemma 2.1.7 (Restriction)**.** *Let $i : \mathbb{B} \to \mathbb{C}$ be a regular embedding, $q \in \mathbb{C}$, then*

$$\begin{array}{rccc} i_q : & \mathbb{B}\restriction\pi_i(q) & \to & \mathbb{C}\restriction q \\ & p & \mapsto & i(p) \wedge c \end{array}$$

is a regular embedding and its associated retraction is $\pi_{i_q} = \pi_i\restriction(\mathbb{C}\restriction q)$.

Proof. First suppose that $i_q(p) = \mathbf{0}$, then by Proposition 2.1.4.(5),

$$\mathbf{0} = \pi_i(i_q(p)) = \pi_i(i(p) \wedge q) = p \wedge \pi_i(q) = p$$

that ensures the regularity of i_q. Furthermore for any $r \leq q$,

$$\begin{aligned} \pi_{i_q}(r) &= \bigwedge \{p \leq \pi_i(q) : \ i(p) \wedge q \geq r\} \\ &= \bigwedge \{p \leq \pi_i(q) : \ i(p) \geq r\} = \pi_i(r), \end{aligned}$$

concluding the proof. □

2.1.1 Embeddings and boolean valued models

Complete homomorphisms of complete boolean algebras induce natural Δ_1-elementary maps between the corresponding boolean valued models.

Proposition 2.1.8. *Let $i : \mathbb{B} \to \mathbb{C}$ be a complete homomorphism, and define by recursion $\hat{\imath} : V^{\mathbb{B}} \to V^{\mathbb{C}}$ as*

$$\hat{\imath}(\dot{x}) = \{\langle \hat{\imath}(\dot{y}), i(\dot{x}(\dot{y}))\rangle : \ \dot{y} \in \mathrm{dom}(\dot{x})\}$$

Then the map $\hat{\imath}$ is Δ_1-elementary, i.e. for every Δ_1-formula ϕ,

$$i\left(\llbracket \phi(\dot{x}_1, \ldots, \dot{x}_n) \rrbracket_{\mathbb{B}}\right) = \llbracket \phi(\hat{\imath}(\dot{x}_1), \ldots, \hat{\imath}(\dot{x}_n)) \rrbracket_{\mathbb{C}}$$

Proof. We prove the result by induction on the complexity of ϕ. For atomic formulas ψ (either $x = y$ or $x \in y$), we proceed by further induction on the rank of $\dot{x}_1$, $\dot{x}_2$.

$$\begin{aligned} i\left(\llbracket \dot{x}_1 \in \dot{x}_2 \rrbracket_{\mathbb{B}}\right) &= i\left(\bigvee \{\dot{x}_2(\dot{y}) \wedge \llbracket \dot{x}_1 = \dot{y} \rrbracket_{\mathbb{B}} : \dot{y} \in \mathrm{dom}(\dot{x}_2)\}\right) \\ &= \bigvee \{i\left(\dot{x}_2(\dot{y})\right) \wedge i\left(\llbracket \dot{x}_1 = \dot{y} \rrbracket_{\mathbb{B}}\right) : \dot{y} \in \mathrm{dom}(\dot{x}_2)\} \\ &= \bigvee \{i\left(\dot{x}_2(\dot{y})\right) \wedge \llbracket \hat{\imath}(\dot{x}_1) = \hat{\imath}(\dot{y}) \rrbracket_{\mathbb{C}} : \dot{y} \in \mathrm{dom}(\dot{x}_2)\} \\ &= \llbracket \hat{\imath}(\dot{x}_1) \in \hat{\imath}(\dot{x}_2) \rrbracket_{\mathbb{C}} \\ i\left(\llbracket \dot{x}_1 \subseteq \dot{x}_2 \rrbracket_{\mathbb{B}}\right) &= i\left(\bigwedge \{\dot{x}_1(\dot{y}) \to \llbracket \dot{y} \in \dot{x}_2 \rrbracket_{\mathbb{B}} : \dot{y} \in \mathrm{dom}(\dot{x}_1)\}\right) \\ &= \bigwedge \{i\left(\dot{x}_1(\dot{y})\right) \to i\left(\llbracket \dot{y} \in \dot{x}_2 \rrbracket_{\mathbb{B}}\right) : \dot{y} \in \mathrm{dom}(\dot{x}_1)\} \\ &= \bigwedge \{i\left(\dot{x}_1(\dot{y})\right) \to \llbracket \hat{\imath}(\dot{y}) \in \hat{\imath}(\dot{x}_2) \rrbracket_{\mathbb{C}} : \dot{y} \in \mathrm{dom}(\dot{x}_1)\} \\ &= \llbracket \hat{\imath}(\dot{x}_1) \subseteq \hat{\imath}(\dot{x}_2) \rrbracket_{\mathbb{C}}. \end{aligned}$$

We used the inductive hypothesis in the last row of each case. Since $\llbracket \dot{x}_1 = \dot{x}_2 \rrbracket = \llbracket \dot{x}_1 \subseteq \dot{x}_2 \rrbracket \wedge \llbracket \dot{x}_2 \subseteq \dot{x}_1 \rrbracket$, the proof for ψ atomic is complete.

For ψ quantifier-free formula the proof is immediate since i is an embedding hence preserves $\vee$, $\neg$. Suppose now that $\psi = \exists x \in y\ \phi$ is a Δ_0-formula.

$$\begin{aligned} & i\left(\llbracket \exists x \in \dot{x}_1 \phi(x, \dot{x}_1, \ldots, \dot{x}_n) \rrbracket_{\mathbb{B}}\right) \\ &= \bigvee \{i\left(\dot{x}_1(\dot{y})\right) \wedge i\left(\llbracket \phi(\dot{y}, \dot{x}_1, \ldots, \dot{x}_n) \rrbracket_{\mathbb{B}}\right) : \dot{y} \in \mathrm{dom}(\dot{x}_1)\} \\ &= \bigvee \{i(\dot{x}_1(\dot{y})) \wedge \llbracket \phi\left(\hat{\imath}(\dot{y}), \hat{\imath}(\dot{x}_1), \ldots, \hat{\imath}(\dot{x}_n)\right) \rrbracket_{\mathbb{C}} : \dot{y} \in \mathrm{dom}(\dot{x}_1)\} \\ &= \llbracket \exists x \in \hat{\imath}(\dot{x}_1)\ \phi\left(x, \hat{\imath}(\dot{x}_1), \ldots, \hat{\imath}(\dot{x}_n)\right) \rrbracket_{\mathbb{C}} \end{aligned}$$

Furthermore, if $\psi = \exists x\ \phi$ is a Σ_1-formula, by the fullness lemma there exists a $\dot{y} \in V^{\mathbb{B}}$ such that $[\![\exists x \phi(x, \dot{x}_1, \ldots, \dot{x}_n)]\!]_{\mathbb{B}} = [\![\phi(\dot{y}, \dot{x}_1, \ldots, \dot{x}_n)]\!]_{\mathbb{B}}$ hence

$$\begin{aligned} i\left([\![\exists x \phi(x, \dot{x}_1, \ldots, \dot{x}_n)]\!]_{\mathbb{B}}\right) &= i\left([\![\phi(\dot{y}, \dot{x}_1, \ldots, \dot{x}_n)]\!]_{\mathbb{B}}\right) \\ &= [\![\phi\left(\hat{\imath}(\dot{y}), \hat{\imath}(\dot{x}_1), \ldots, \hat{\imath}(\dot{x}_n)\right)]\!]_{\mathbb{C}} \\ &\leq [\![\exists x \phi\left(x, \hat{\imath}(\dot{x}_1), \ldots, \hat{\imath}(\dot{x}_n)\right)]\!]_{\mathbb{C}} . \end{aligned}$$

Thus, if ϕ is a Δ_1-formula, either ϕ and $\neg\phi$ are Σ_1 hence the above inequality holds and also

$$\begin{aligned} i\left([\![\phi(\dot{x}_1, \ldots, \dot{x}_n)]\!]_{\mathbb{B}}\right) &= \neg i\left([\![\neg\phi(\dot{x}_1, \ldots, \dot{x}_n)]\!]_{\mathbb{B}}\right) \\ &\geq \neg\, [\![\neg\phi\left(\hat{\imath}(\dot{x}_1), \ldots, \hat{\imath}(\dot{x}_n)\right)]\!]_{\mathbb{C}} \\ &= [\![\phi\left(\hat{\imath}(\dot{x}_1), \ldots, \hat{\imath}(\dot{x}_n)\right)]\!]_{\mathbb{C}} , \end{aligned}$$

concluding the proof. □

2.2 Iteration systems

In this section we present iteration systems and some of their algebraic properties. We defer to later sections an analysis of their forcing properties. In order to develop the theory of iterations, from now on we shall consider only regular embeddings.

Definition 2.2.1. $\mathcal{F} = \{i_{\alpha\beta} : \mathbb{B}_\alpha \to \mathbb{B}_\beta : \alpha \leq \beta < \lambda\}$ is a *complete iteration system* of complete boolean algebras iff for all $\alpha \leq \beta \leq \xi < \lambda$:

1. $\mathbb{B}_\alpha$ is a complete boolean algebra and $i_{\alpha\alpha}$ is the identity on it;
2. $i_{\alpha\beta}$ is a regular embedding with associated retraction $\pi_{\alpha\beta}$;
3. $i_{\beta\xi} \circ i_{\alpha\beta} = i_{\alpha\xi}$.

If $\xi < \lambda$, we define $\mathcal{F}\restriction\xi = \{i_{\alpha\beta} : \alpha \leq \beta < \xi\}$.

Definition 2.2.2. Let $\mathcal{F}$ be a complete iteration system of length λ. Then:

- The *inverse limit* of the iteration is

$$\varprojlim \mathcal{F} = \left\{ s \in \prod_{\alpha<\lambda} \mathbb{B}_\alpha :\ \forall\alpha\forall\beta > \alpha\ \pi_{\alpha\beta}(s(\beta)) = s(\alpha) \right\}$$

  and its elements are called *threads*.

- The *direct limit* is

$$\varinjlim \mathcal{F} = \left\{ s \in \varprojlim \mathcal{F} :\ \exists\alpha\forall\beta > \alpha\ s(\beta) = i_{\alpha\beta}(s(\alpha)) \right\}$$

  and its elements are called *constant threads*. The support of a constant thread $\text{supp}(s)$ is the least α such that $i_{\alpha\beta} \circ s(\alpha) = s(\beta)$ for all $\beta \geq \alpha$.

- The *revised countable support limit* is

$$\lim_{\mathsf{rcs}} \mathcal{F} = \left\{ s \in \varprojlim \mathcal{F} :\ s \in \varinjlim \mathcal{F} \vee \exists\alpha\ s(\alpha) \Vdash_{\mathbb{B}_\alpha} \text{cof}(\check{\lambda}) = \check{\omega} \right\} .$$

Definition 2.2.3. Let $\mathcal{F} = \{i_{\alpha\beta} : \alpha \leq \beta < \lambda\}$ be an iteration system. We say that $\mathcal{F}$ is a *$<\kappa$-support iteration* iff $\mathbb{B}_\alpha = \varinjlim \mathcal{F}\restriction\alpha$ whenever $\mathrm{cof}(\alpha) \geq \kappa$, and $\mathbb{B}_\alpha = \varprojlim \mathcal{F}\restriction\alpha$ otherwise. We say that $\mathcal{F}$ is a *revised countable support* (RCS) iteration iff $\mathbb{B}_\alpha = \lim_{\mathsf{rcs}} \mathcal{F}\restriction\alpha$ for all $\alpha < \lambda$.

Every thread in $\varprojlim \mathcal{F}$ is completely determined by its tail, while every thread in $\varinjlim \mathcal{F}$ is entirely determined by the restriction to its support (and in particular by $s(\mathrm{supp}(s))$). Notice that $\varinjlim \mathcal{F} \subseteq \lim_{\mathsf{rcs}} \mathcal{F} \subseteq \varprojlim \mathcal{F}$ are partial orders with the order relation given by pointwise comparison of threads. Furthermore, if λ is singular and $f : \mathrm{cof}(\lambda) \to \lambda$ is an increasing cofinal map,we have the followings isomorphisms of partial orders:

$$\varinjlim \mathcal{F} \cong \varinjlim\{i_{f(\alpha)f(\beta)} : \alpha \leq \beta < \mathrm{cof}(\lambda)\};$$
$$\varprojlim \mathcal{F} \cong \varprojlim\{i_{f(\alpha)f(\beta)} : \alpha \leq \beta < \mathrm{cof}(\lambda)\};$$
$$\lim_{\mathsf{rcs}} \mathcal{F} \cong \lim_{\mathsf{rcs}}\{i_{f(\alpha)f(\beta)} : \alpha \leq \beta < \mathrm{cof}(\lambda)\}.$$

Thus we can always assume without loss of generality that λ is a regular cardinal.

The limits of an iteration system previously defined are closely related with the elements of the iteration system, as shown by the following definition and remarks.

Definition 2.2.4. Let $\mathcal{F} = \{i_{\alpha\beta} : \alpha \leq \beta < \lambda\}$ be an iteration system. For all $\alpha < \lambda$, we define $i_{\alpha\lambda}$ as

$$\begin{array}{rccl} i_{\alpha\lambda}: & \mathbb{B}_\alpha & \to & \varinjlim \mathcal{F} \\ & p & \mapsto & \langle \pi_{\beta\alpha}(p) : \beta < \alpha\rangle^\frown\langle i_{\alpha\beta}(p) : \alpha \leq \beta < \lambda\rangle \end{array}$$

and $\pi_{\alpha\lambda}$ as

$$\begin{array}{rccl} \pi_{\alpha\lambda}: & \varprojlim \mathcal{F} & \to & \mathbb{B}_\alpha \\ & s & \mapsto & s(\alpha) \end{array}$$

When it is clear from the context, we will denote $i_{\alpha\lambda}$ by i_α and $\pi_{\alpha\lambda}$ by π_α.

The maps $i_{\alpha\lambda}$ naturally extend the iteration system to λ, as the following diagram commutes:

$$\begin{array}{ccccc} \mathbb{B}_\alpha & & \xrightarrow{\quad i_{\alpha\lambda}\quad} & & \varinjlim \mathcal{F} \\ & {\scriptstyle i_{\alpha\beta}}\searrow & & \nearrow{\scriptstyle i_{\beta\lambda}} & \\ & & \mathbb{B}_\beta & & \end{array}$$

Furthermore, these maps can naturally be seen as regular embeddings of $\mathbb{B}_\alpha$ in any of $\mathrm{RO}(\varinjlim \mathcal{F})$, $\mathrm{RO}(\lim_{\mathsf{rcs}} \mathcal{F})$, $\mathrm{RO}(\varprojlim \mathcal{F})$. Moreover by Property 2.1.4.(3) in all three cases $\pi_{\alpha\lambda} = \pi_{i_{\alpha,\lambda}}\restriction\mathbb{P}$ where $\mathbb{P} = \varinjlim \mathcal{F}, \lim_{\mathsf{rcs}} \mathcal{F}, \varprojlim \mathcal{F}$.

2.2.1 Boolean algebra operations on iteration system limits

We can now equip the different limits of iteration systems with boolean algebras operations.

Definition 2.2.5. $\varinjlim \mathcal{F}$ inherits the structure of a boolean algebra with boolean operations defined as follows:

- $s_1 \wedge s_2$ is the unique thread s whose support β is the maximum of the supports of s_1 and s_2 and such that $s(\beta) = s_1(\beta) \wedge s_2(\beta)$,
- $\neg s$ is the unique thread t whose support β is the support of s and such that $t(\beta) = \neg s(\beta)$.

Definition 2.2.6. Let A be any subset of $\varprojlim \mathcal{F}$. We define the *pointwise supremum* of A as

$$\tilde{\bigvee} A = \langle \bigvee \{s(\alpha) : s \in A\} : \ \alpha < \lambda \rangle.$$

The previous definition makes sense since by Proposition 2.1.4.(3) $\tilde{\bigvee} A$ is a thread. It must be noted that if A is an infinite subset of $\varprojlim \mathcal{F}$, $\tilde{\bigvee} A$ might *not* be the least upper bound of A in $\mathrm{RO}(\varprojlim \mathcal{F})$. A sufficient condition on A for this to happen is given by Lemma 2.2.8 below.

If $s \in \varprojlim \mathcal{F}$ and $t \in \varinjlim \mathcal{F}$ we can check that $s \wedge t$, defined as the thread where eventually all coordinates α are the pointwise meet of $s(\alpha)$ and $t(\alpha)$, is the infimum of s and t in $\varprojlim \mathcal{F}$. There can be nonetheless two distinct incompatible threads $s, t \in \varprojlim \mathcal{F}$ such that $s(\alpha) \wedge t(\alpha) > \mathbf{0}$ for all $\alpha < \lambda$. Thus in general the pointwise meet of two threads could not even be a thread.

In general $\varinjlim \mathcal{F}$ is not complete and $\mathrm{RO}(\varinjlim \mathcal{F})$ cannot be identified with a complete subalgebra of $\mathrm{RO}(\varprojlim \mathcal{F})$ (i.e. $\varinjlim \mathcal{F}$ and $\varprojlim \mathcal{F}$ as forcing notions in general share little in common). However, $\mathrm{RO}(\varinjlim \mathcal{F})$ can be identified with a subalgebra of $\varprojlim \mathcal{F}$ that is complete (even though it is not a complete subalgebra), as shown in the following proposition.

Proposition 2.2.7. *Let $\mathcal{F} = \{i_{\alpha\beta} : \alpha \leq \beta < \lambda\}$ be an iteration system. Then* $\mathrm{RO}(\varinjlim \mathcal{F}) \simeq D = \left\{ s \in \varprojlim \mathcal{F} : \ s = \tilde{\bigvee} \left\{ t \in \varinjlim \mathcal{F} : \ t \leq s \right\} \right\}$.

Proof. The isomorphism associates to a regular open $U \in \mathrm{RO}(\varinjlim \mathcal{F})$ the thread $k(U) = \tilde{\bigvee} U$, with inverse $k^{-1}(s) = \left\{ t \in \varinjlim \mathcal{F} : \ t \leq s \right\}$.

First, we prove that $k^{-1} \circ k(U) = \left\{ t \in \varinjlim \mathcal{F} : \ t \leq \tilde{\bigvee} U \right\} = U$. Since $\tilde{\bigvee} U > \bigvee U$, it follows that $U \subseteq k^{-1} \circ k(U)$. Furthermore, since U is a regular open set, if $t \notin U$, there exists a $t' \leq t$ that is in the interior of the complement of U (i.e., $\forall t'' \leq t' \ t'' \notin U$). So suppose towards a contradiction that there exist a $t \leq \tilde{\bigvee} U$ as above (i.e., $\forall t' \leq t \ t' \notin U$). Let α be the support of t, so that $t(\alpha) \leq \bigvee \{s(\alpha) : \ s \in U\}$. Then, there exists an $s \in U$ such that $s(\alpha)$ is compatible with $t(\alpha)$, hence $s \wedge t > \mathbf{0}$ and is in U (since U is open). Since $s \wedge t \leq t$, this is a contradiction.

It follows that $k(U) \in D$ for every $U \in \mathrm{RO}(\varinjlim \mathcal{F})$. Moreover, $k^{-1}(s)$ is in $\mathrm{RO}(\varinjlim \mathcal{F})$ (i.e., is regular open). In fact, it is open and if $t \notin k^{-1}(s)$ then $t \not\leq s$ and this is witnessed by some $\alpha > \mathrm{supp}(t)$, so that $t(\alpha) \not\leq s(\alpha)$. Let $t' = i_\alpha(t(\alpha) \setminus s(\alpha)) > \mathbf{0}$, then for all $t'' \leq t'$, $t''(\alpha) \perp s(\alpha)$ hence $t'' \not\leq s$, thus $k^{-1}(s)$ is regular.

Furthermore, k^{-1} is the inverse map of k since we already verified that $k^{-1} \circ k(U) = U$ and for all $s \in D$, $k \circ k^{-1}(s) = s$ by definition of D. Finally, k and k^{-1} are order-preserving maps since $U_1 \subseteq U_2$ iff $\tilde{\bigvee} U_1 \leq \tilde{\bigvee} U_2$. $\square$

As noted before, the notion of supremum in $\varprojlim \mathcal{F}$ may not coincide with the notion of pointwise supremum. However, it is possible to give a sufficient condition for this to happen, as in the following proposition.

Proposition 2.2.8. *Let $\mathcal{F} = \{i_{\alpha\beta} : \alpha \leq \beta < \lambda\}$ be an iteration system and $A \subseteq \varprojlim \mathcal{F}$ be an antichain such that $\pi_{\alpha\lambda}[A]$ is an antichain for some $\alpha < \lambda$. Then $\tilde{\bigvee} A$ is the supremum of the elements of A in $\mathrm{RO}(\varprojlim \mathcal{F})$.*

Proof. Suppose by contradiction that $\bigvee A < \tilde{\bigvee} A$ in $\mathrm{RO}(\varprojlim \mathcal{F})$. Then there exists $t \in \varprojlim \mathcal{F}$ such that $\mathbf{0} < t \leq \neg \bigvee A \wedge \tilde{\bigvee} A$. Let $\alpha < \lambda$ be such that $\pi_{\alpha\lambda}[A]$ is an antichain and let $s \in A$ be such that $s(\alpha)$ is compatible with $t(\alpha)$. Such an s exists because $t(\alpha) \leq \bigvee \{s(\alpha) : s \in A\}$ and $t(\alpha) > \mathbf{0}$. We now prove that s and t are compatible. Consider

$$t' = \langle t(\beta) \wedge i_{\alpha\beta}\,(s(\alpha)) : \ \alpha \leq \beta < \lambda \rangle.$$

Then $t' \leq t$ and it is a thread in $\varprojlim \mathcal{F}$. In fact, since $i_{\alpha,\beta} = i_{\xi,\beta} \circ i_{\alpha,\xi}$ for each $\alpha \leq \xi \leq \beta < \lambda$,

$$\pi_{\xi,\beta}(t'(\beta)) = \pi_{\xi,\beta}(t(\beta) \wedge i_{\alpha,\beta} \circ s(\alpha)) = \pi_{\xi,\beta}(t(\beta)) \wedge i_{\alpha,\xi}(s(\alpha)) = t'(\xi).$$

It only remains to prove that $t'(\beta) \leq s(\beta)$ for each $\beta \geq \alpha$. Notice that $t'(\beta) \leq t(\beta) \leq \sup\{s'(\beta) : \ s' \in A\}$, $t'(\beta) > \mathbf{0}$ and $t'(\beta)$ is incompatible with $s'(\beta)$ for all $s' \neq s$ in A. Thus

$$t'(\beta) \leq \bigvee \{s'(\beta) : \ s' \in A\} \wedge \left(\neg \bigvee \{s'(\beta) : \ s' \in A, s' \neq s\}\right) = s(\beta)$$

for all $\beta \geq \alpha$, hence t and s are compatible. Since $s, t \in A$, we get a contradiction. □

2.2.2 Relation between inverse and direct limits

Even though in general $\varinjlim \mathcal{F}$ is different from $\varprojlim \mathcal{F}$, in certain cases they happen to coincide.

Lemma 2.2.9. *Let $\mathcal{F} = \{i_{\alpha\beta} : \ \alpha \leq \beta < \lambda\}$ be an iteration system such that $\varinjlim \mathcal{F}$ is $<\lambda$-cc*[1]*. Then $\varprojlim \mathcal{F} = \varinjlim \mathcal{F}$ is a complete boolean algebra.*

Proof. First, since every element of $\mathrm{RO}(\varinjlim \mathcal{F})$ is the supremum of an antichain in $\varinjlim \mathcal{F}$, and since $\varinjlim \mathcal{F}$ is $<\lambda$-cc and λ is regular, the supremum of such an antichain can be computed in some $\mathbb{B}_\alpha$ for $\alpha < \lambda$ hence $\mathrm{RO}(\varinjlim \mathcal{F}) = \varinjlim \mathcal{F}$.

Let s be in $\varprojlim \mathcal{F} \setminus \varinjlim \mathcal{F}$. Since s is a non-constant thread, for all $\alpha < \beta$ we have that $i_{\alpha\beta}(s(\alpha)) \geq s(\beta)$ and for all α there is an ordinal β_α such that $i_{\alpha\beta_\alpha}(s(\alpha)) > s(\beta_\alpha)$. By restricting to a subset of λ w.l.o.g. we can suppose that $s(\beta) < i_{\alpha\beta}(s(\alpha))$ for all $\beta > \alpha$. Hence $\{i_{\alpha\lambda}(s(\alpha)) : \alpha < \lambda\}$ is a strictly descending sequence of length λ of elements in $\varinjlim \mathcal{F}^+$. From a descending sequence we can always define an antichain in $\varinjlim \mathcal{F}$ setting $p_\alpha = i_{\alpha\lambda}(s(\alpha)) \wedge \neg i_{\alpha+1,\lambda}(s(\alpha+1))$. Since $\varinjlim \mathcal{F}$ is $<\lambda$-cc, this antichain has to be of size less than λ hence for coboundedly many α, $p_\alpha = \mathbf{0}$ hence $s(\alpha+1) = i_{\alpha,\alpha+1}(s(\alpha))$ and $s \in \varinjlim \mathcal{F}$, a contradiction. □

Theorem 2.2.10 (Baumgartner)**.** *Let $\mathcal{F} = \{i_{\alpha\beta} : \ \alpha \leq \beta < \lambda\}$ be an iteration system such that $\mathbb{B}_\alpha$ is $<\lambda$-cc for all α and $S = \left\{\alpha : \ \mathbb{B}_\alpha \cong \mathrm{RO}(\varinjlim \mathcal{F} \restriction \alpha)\right\}$ is stationary. Then $\varinjlim \mathcal{F}$ is $<\lambda$-cc.*

[1] We refer to Definition 3.1.1 for a definition of $<\lambda$-cc boolean algebra.

Proof. Suppose by contradiction that there exists an antichain $\langle s_\alpha : \alpha < \lambda \rangle$. Let $f : \lambda \to \lambda$ be such that $f(\alpha) > \alpha, \mathrm{supp}(s_\alpha)$. Let C be the club of closure points of f (i.e. such that for all $\alpha \in C$, $f[\alpha] \subseteq f(\alpha)$). Then we can define a regressive function

$$\begin{array}{rccl} g: & S & \to & \lambda \\ & \alpha & \mapsto & \min\left\{\mathrm{supp}(s) : \ s \in (\varinjlim \mathcal{F}\restriction\alpha)^+, s < s_\alpha(\alpha)\right\} \end{array}$$

and a corresponding function $h : S \to \varinjlim \mathcal{F}$ such that $\mathrm{supp}(h(\alpha)) = g(\alpha)$, $h(\alpha)(\alpha) < s_\alpha(\alpha)$. By Fodor's Lemma let $\xi \in \lambda$, $T \subset S$ be stationary such that $g[T] = \{\xi\}$.

Since $h[T \cap C]$ has size λ and $\mathbb{B}_\xi$ is $<\lambda$-cc, there are $\alpha, \beta \in T \cap C$ such that $h(\alpha) \wedge h(\beta) \geq p > \mathbf{0}$ for some p with $\mathrm{supp}(p) = \xi$. Moreover $s_\alpha(\alpha) > p(\alpha)$ holds and the support of p is below α, so that $s_\alpha \wedge p > \mathbf{0}$. Furthermore, $s_\beta(\beta) > p(\beta) \geq (s_\alpha \wedge p)(\beta)$ and the support of $s_\alpha \wedge p$ is below β, thus $s_\beta \wedge s_\alpha \wedge p > \mathbf{0}$ contradicting the hypothesis that $\langle s_\alpha : \alpha < \lambda \rangle$ is an antichain. □

Notice that for an iteration system $\mathcal{F}$ as above of regular length $\delta > \lambda$, $\varinjlim \mathcal{F}$ is also $<\lambda$-cc by an easier regularity argument.

2.3 Two-step iterations and generic quotients

In the first part of this section we define the two-step iteration $\mathbb{B} * \dot{\mathbb{C}}$ following [27, Chp. 16] and study the basic properties of the natural regular embedding of $\mathbb{B}$ into $\mathbb{B} * \dot{\mathbb{C}}$ where $\dot{\mathbb{C}}$ is a $\mathbb{B}$-name for a complete boolean algebra.

In the second part of this section we study the properties of generic quotients given by $\mathbb{B}$-names $\mathbb{C}/_{i[\dot{G}_\mathbb{B}]}$ where $i : \mathbb{B} \to \mathbb{C}$ is a complete homomorphism and show that if we have a commutative diagram of complete homomorphisms:

$$\begin{array}{ccc} \mathbb{B} & \xrightarrow{i_0} & \mathbb{C}_0 \\ & {\scriptstyle i_1}\searrow & \downarrow {\scriptstyle k} \\ & & \mathbb{C}_1 \end{array}$$

and G is a V-generic filter for $\mathbb{B}$, then the map defined by $k/_G([p]_{i_0[G]}) = [k(p)]_{i_1[G]}$ is a complete homomorphism in $V[G]$. We also show a converse of this property.

In the third part of this section we show that the two approaches are equivalent, that is, $i : \mathbb{B} \to \mathbb{C}$ is a complete homomorphism iff $\mathbb{C}$ is isomorphic to $\mathbb{B} * \mathbb{C}/_{i[\dot{G}_\mathbb{B}]}$ and prove a converse of the above factorization property when we start from $\mathbb{B}$-names for regular embeddings $\dot{k} : \dot{\mathbb{C}} \to \dot{\mathbb{D}}$.

Finally, in the last part we apply the above results to analyze generic quotients of iteration systems.

All the results in this section can be generalized to complete (not necessarily injective) homomorphisms i, by considering $i \restriction \mathrm{coker}(i)$, which is a regular embedding (see Definition 2.1.2).

2.3.1 Two-step iterations

We present two-step iterations following [27].

Definition 2.3.1. Let $\mathbb{B}$ be a complete boolean algebra, and $\dot{\mathbb{C}}$ be a $\mathbb{B}$-name for a complete boolean algebra. We denote by $\mathbb{B} * \dot{\mathbb{C}}$ the boolean algebra defined in V whose elements are the equivalence classes of $\mathbb{B}$-names for elements of $\dot{\mathbb{C}}$ (i.e. $\dot{p} \in V^{\mathbb{B}}$ such that $\left[\!\left[\dot{p} \in \dot{\mathbb{C}}\right]\!\right]_{\mathbb{B}} = \mathbf{1}$) modulo the equivalence relation:

$$\dot{p} \approx \dot{q} \;\Leftrightarrow\; [\![\dot{p} = \dot{q}]\!]_{\mathbb{B}} = \mathbf{1},$$

with the following operations:

$$[\dot{p}] \vee_{\mathbb{B}*\dot{\mathbb{C}}} [\dot{q}] = [\dot{r}] \iff \left[\!\left[\dot{r} = \dot{p} \vee_{\dot{\mathbb{C}}} \dot{q}\right]\!\right]_{\mathbb{B}} = \mathbf{1};$$
$$\neg_{\mathbb{B}*\dot{\mathbb{C}}}[\dot{p}] = [\dot{r}] \iff [\![\dot{r} = \neg_{\dot{\mathbb{C}}}\dot{p}]\!]_{\mathbb{B}} = \mathbf{1}.$$

Literally speaking our definition of $\mathbb{B} * \dot{\mathbb{C}}$ yields an object whose domain is a family of proper classes of $\mathbb{B}$-names. By means of Scott's trick we can arrange so that $\mathbb{B} * \dot{\mathbb{C}}$ is indeed a set. We leave the details to the reader.

Lemma 2.3.2. *Let $\mathbb{B}$ be a complete boolean algebra, and $\dot{\mathbb{C}}$ be a $\mathbb{B}$-name for a complete boolean algebra. Then $\mathbb{B} * \dot{\mathbb{C}}$ is a complete boolean algebra and the maps $i_{\mathbb{B}*\dot{\mathbb{C}}}$, $\pi_{\mathbb{B}*\dot{\mathbb{C}}}$ defined as*

$$\begin{array}{rccc} i_{\mathbb{B}*\dot{\mathbb{C}}}: & \mathbb{B} & \to & \mathbb{B}*\dot{\mathbb{C}} \\ & p & \mapsto & [\tau_p]_{\approx} \\ \pi_{\mathbb{B}*\dot{\mathbb{C}}}: & \mathbb{B}*\dot{\mathbb{C}} & \to & \mathbb{B} \\ & [\dot{p}]_{\approx} & \mapsto & [\![\dot{p} > \mathbf{0}]\!]_{\mathbb{B}} \end{array}$$

where $\tau_p \in V^{\mathbb{B}}$ is a $\mathbb{B}$-name for an element of $\dot{\mathbb{C}}$ such that $[\![\tau_p = \mathbf{1}]\!]_{\mathbb{B}} = p$ and $[\![\tau_p = \mathbf{0}]\!]_{\mathbb{B}} = \neg p$, are a regular embedding with its associated retraction.

Proof. We leave to the reader to verify that $\mathbb{B} * \dot{\mathbb{C}}$ is a boolean algebra. We can also check that

$$[\dot{p}] \leq [\dot{q}] \iff [\![\dot{p} \vee \dot{q} = \dot{q}]\!]_{\mathbb{B}} = \mathbf{1} \iff [\![\dot{p} \leq \dot{q}]\!]_{\mathbb{B}} = \mathbf{1}.$$

Observe that $\mathbb{B} * \dot{\mathbb{C}}$ is complete: if $\{[\dot{p}_\alpha] : \xi < \delta\} \subseteq \mathbb{B} * \dot{\mathbb{C}}$, let $\dot{q}$ be such that $[\![\dot{q} = \bigvee \{\dot{p}_\xi : \xi < \delta\}]\!]_{\mathbb{B}} = \mathbf{1}$. Then $[\dot{q}] \geq \bigvee \{[\dot{p}_\xi] : \xi < \delta\}$ since for all $\alpha < \delta$

$$\left[\!\left[\bigvee \{\dot{p}_\xi : \xi < \delta\} \geq \dot{p}_\alpha\right]\!\right]_{\mathbb{B}} = \mathbf{1}.$$

Moreover if $[\![\dot{r} \geq \dot{p}_\alpha]\!]_{\mathbb{B}} = \mathbf{1}$ for all $\alpha < \delta$, then

$$\bigwedge \{[\![\dot{r} \geq \dot{p}_\alpha]\!]_{\mathbb{B}} : \ \alpha < \delta\} = \mathbf{1}$$

thus $[\![\dot{r} \geq \dot{q}]\!]_{\mathbb{B}} = \mathbf{1}$, hence $[\dot{r}] \geq [\dot{q}]$, which gives that $[\dot{q}] = \bigvee \{[\dot{p}_\alpha] : \alpha < \delta\}$.

We now prove that $i_{\mathbb{B}*\dot{\mathbb{C}}}$ is a regular embedding and that $\pi_{\mathbb{B}*\dot{\mathbb{C}}}$ is its associated retraction.

- First of all, a standard application of the mixing lemma to the maximal antichain $\{p, \neg p\}$ and the family of $\mathbb{B}$-names $\{\dot{\mathbf{1}}, \dot{\mathbf{0}}\}$ shows that for each $p \in \mathbb{B}$ there exists a unique $[\tau_p] \in \mathbb{B} * \dot{\mathbb{C}}$ such that $[\![\tau_p = \mathbf{1}]\!]_{\mathbb{B}} = p$ and $[\![\tau_p = \mathbf{0}]\!]_{\mathbb{B}} = \neg p$.

- $i_{\mathbb{B}*\dot{\mathbb{C}}}$ preserves negation since $[\![\neg\tau_p = \tau_{\neg p}]\!]_{\mathbb{B}} = \mathbf{1}$.
- $i_{\mathbb{B}*\dot{\mathbb{C}}}$ preserves joins. Consider $\{p_\alpha \in \mathbb{B} : \alpha < \delta\}$. We have that

$$\left[\!\!\left[\bigvee \tau_{p_\alpha} = \mathbf{0}\right]\!\!\right]_{\mathbb{B}} = \bigwedge [\![\tau_{p_\alpha} = \mathbf{0}]\!]_{\mathbb{B}} = \bigwedge(\neg p_\alpha) = \neg(\bigvee p_\alpha).$$

Furthermore,

$$\begin{aligned}\left[\!\!\left[\bigvee \tau_{p_\alpha} = \mathbf{1}\right]\!\!\right]_{\mathbb{B}} \leq \left[\!\!\left[\bigvee \tau_{p_\alpha} > \mathbf{0}\right]\!\!\right]_{\mathbb{B}} &= \bigvee [\![\tau_{p_\alpha} > \mathbf{0}]\!]_{\mathbb{B}} \\ &= \bigvee [\![\tau_{p_\alpha} = \mathbf{1}]\!]_{\mathbb{B}} \leq \left[\!\!\left[\bigvee \tau_{p_\alpha} = \mathbf{1}\right]\!\!\right]_{\mathbb{B}};\end{aligned}$$

then $[\![\bigvee \tau_{p_\alpha} = \mathbf{1}]\!]_{\mathbb{B}} = \bigvee [\![\tau_{p_\alpha} = \mathbf{1}]\!]_{\mathbb{B}} = \bigvee p_\alpha$. Thus,

$$i_{\mathbb{B}*\dot{\mathbb{C}}}\left(\bigvee p_\alpha\right) = \left[\bigvee \tau_{p_\alpha}\right] = \bigvee \left[i_{\mathbb{B}*\dot{\mathbb{C}}}(p_\alpha)\right].$$

- $i_{\mathbb{B}*\dot{\mathbb{C}}}$ is regular since $i_{\mathbb{B}*\dot{\mathbb{C}}}(p) = i_{\mathbb{B}*\dot{\mathbb{C}}}(p') \Rightarrow p = [\![\tau_p = \mathbf{1}]\!]_{\mathbb{B}} = [\![\tau_{p'} = \mathbf{1}]\!]_{\mathbb{B}} = p'$.
- Finally, we show that $\pi_{i_{\mathbb{B}*\dot{\mathbb{C}}}}([\dot{p}]) = [\![\dot{p} > \mathbf{0}]\!]_{\mathbb{B}}$. By definition of retraction associated to $i_{\mathbb{B}*\dot{\mathbb{C}}}$,

$$\pi_{i_{\mathbb{B}*\dot{\mathbb{C}}}}([\dot{p}]) = \bigwedge\{q \in \mathbb{B} : i_{\mathbb{B}*\dot{\mathbb{C}}}(q) \geq [\dot{p}]\}.$$

If q is such that $i_{\mathbb{B}*\dot{\mathbb{C}}}(q) \geq [\dot{p}]$, then $[\![\tau_q \geq \dot{p}]\!]_{\mathbb{B}} = \mathbf{1}$ hence

$$q = [\![\tau_q = \mathbf{1}]\!]_{\mathbb{B}} = [\![\tau_q > \mathbf{0}]\!]_{\mathbb{B}} \geq [\![\dot{p} > \mathbf{0}]\!]_{\mathbb{B}} \wedge [\![\tau_q \geq \dot{p}]\!]_{\mathbb{B}} = [\![\dot{p} > \mathbf{0}]\!]_{\mathbb{B}},$$

so we have the first inequality $\pi_{i_{\mathbb{B}*\dot{\mathbb{C}}}}([\dot{p}]) \geq [\![\dot{p} > \mathbf{0}]\!]_{\mathbb{B}}$.

Let now $i_{\mathbb{B}*\dot{\mathbb{C}}}([\![\dot{p} > \mathbf{0}]\!]_{\mathbb{B}}) = [\tau]$. Then,

$$\neg [\![\dot{p} = \mathbf{0}]\!]_{\mathbb{B}} = [\![\dot{p} > \mathbf{0}]\!]_{\mathbb{B}} = [\![\tau = \mathbf{1}]\!]_{\mathbb{B}} \leq [\![\dot{p} \leq \tau]\!]_{\mathbb{B}}$$

and $[\![\dot{p} = \mathbf{0}]\!]_{\mathbb{B}} \leq [\![\dot{p} \leq \tau]\!]_{\mathbb{B}}$. If follows that $[\![\dot{p} \leq \tau]\!]_{\mathbb{B}} \geq \neg [\![\dot{p} = \mathbf{0}]\!]_{\mathbb{B}} \vee [\![\dot{p} = \mathbf{0}]\!]_{\mathbb{B}} = \mathbf{1}$, hence $[\dot{p}] \leq [\tau] = i_{\mathbb{B}*\dot{\mathbb{C}}}([\![\dot{p} > \mathbf{0}]\!]_{\mathbb{B}})$. Thus, $\pi_{i_{\mathbb{B}*\dot{\mathbb{C}}}}([\dot{p}]) \leq [\![\dot{p} > \mathbf{0}]\!]_{\mathbb{B}}$ as was to be shown. □

When clear from the context, we shall feel free to omit the subscripts in $i_{\mathbb{B}*\dot{\mathbb{C}}}$, $\pi_{\mathbb{B}*\dot{\mathbb{C}}}$. This definition is provably equivalent to Kunen's two-step iteration of posets [32], i.e. $\mathrm{RO}(\mathbb{P} * \dot{\mathbb{Q}})$ is isomorphic to $\mathrm{RO}(\mathbb{P}) * \mathrm{RO}(\dot{\mathbb{Q}})$.

Proposition 2.3.3. *$A = \{[\dot{p}_\alpha]_\approx : \alpha \in \lambda\}$ is a maximal antichain in $\mathbb{D} = \mathbb{B} * \dot{\mathbb{C}}$ iff*

$$\left[\!\!\left[\{\dot{p}_\alpha : \alpha \in \lambda\} \text{ is a maximal antichain in } \dot{\mathbb{C}}\right]\!\!\right]_{\mathbb{B}} = \mathbf{1}.$$

Proof. It is sufficient to observe the following:

$$[\![\dot{p}_\alpha \wedge \dot{p}_\beta = \dot{\mathbf{0}}]\!]_{\mathbb{B}} = \mathbf{1} \iff [\dot{p}_\alpha]_\approx \wedge [\dot{p}_\beta]_\approx = [\dot{\mathbf{0}}]_\approx,$$

$$\left[\!\!\left[\bigvee \dot{p}_\alpha = \dot{\mathbf{1}}\right]\!\!\right]_{\mathbb{B}} = \mathbf{1} \iff \bigvee [\dot{p}_\alpha]_\approx = \left[\bigvee \dot{p}_\alpha\right]_\approx = [\dot{\mathbf{1}}]_\approx.$$

□

2.3.2 Generic quotients

We now outline the main definition and properties of generic quotients.

Proposition 2.3.4. *Let $i : \mathbb{B} \to \mathbb{C}$ be a regular embedding of complete boolean algebras and G be a V-generic filter for $\mathbb{B}$. Then $\mathbb{C}/_G$, defined with abuse of notation as the quotient of $\mathbb{C}$ with the filter generated by $i[G]$, is a boolean algebra in $V[G]$.*

Proof. We have that

$$V[G] \models \mathbb{C} \text{ is a boolean algebra and } i[G] \text{ generates a filter on } \mathbb{C}.$$

Thus $\mathbb{C}/_G$ is a boolean algebra in $V[G]$ such that

- $[p] = [q]$ if and only if $p \triangle q = (p \setminus q) \vee (q \setminus p) \in i[G]^*$;
- $[p] \vee [q] = [p \vee q]$;
- $\neg [p] = [\neg p]$;

where $i[G]^*$ is the dual ideal of the filter $i[G]$. □

Lemma 2.3.5. *Let $i : \mathbb{B} \to \mathbb{C}$ be a regular embedding, $\dot{G}$ be the canonical name for a generic filter for $\mathbb{B}$ and $\dot{p}$ be a $\mathbb{B}$-name for an element of $\mathbb{C}/_{\dot{G}}$. Then there exists a unique $q \in \mathbb{C}$ such that $\left[\!\left[\dot{p} = [q]_{i[\dot{G}]}\right]\!\right]_{\mathbb{B}} = \mathbf{1}$.*

Proof. First, notice that the $\mathbb{B}$-name for the dual of the filter generated by $i[\dot{G}]$ is $\dot{I} = \{\langle q, \neg\pi_i(q)\rangle : q \in \mathbb{C}\}$.

Uniqueness. Suppose that q_0, q_1 are such that $[\![\dot{p} = [q_k]_{\dot{I}}]\!]_{\mathbb{B}} = \mathbf{1}$ for $k < 2$. Then $[\![[q_0]_{\dot{I}} = [q_1]_{\dot{I}}]\!] = \mathbf{1}$ hence $\left[\!\left[q_0 \triangle q_1 \in \dot{I}\right]\!\right]_{\mathbb{B}} = \neg\pi_i(q_0 \triangle q_1) = \mathbf{1}$. This implies that $\pi_i(q_0 \triangle q_1) = \mathbf{0} \Rightarrow q_0 \triangle q_1 = \mathbf{0} \Rightarrow q_0 = q_1$.

Existence. Let $A \subset \mathbb{B}$ be a maximal antichain deciding the value of $\dot{p}$, and for every $r \in A$ let q_r be such that $r \Vdash_{\mathbb{B}} \dot{p} = [q_r]_{\dot{I}}$. Let $q \in \mathbb{C}$ be such that $q = \bigvee \{i(r) \wedge q_r : \ r \in A\}$, so that

$$[\![[q]_{\dot{I}} = [q_r]_{\dot{I}}]\!]_{\mathbb{B}} = \left[\!\left[q \triangle q_r \in \dot{I}\right]\!\right]_{\mathbb{B}} = \neg\pi_i(q \triangle q_r) \geq \neg\pi_i(i(\neg r)) = r$$

since $q \triangle q_r \leq \neg i(r) = i(\neg(r))$. Thus,

$$[\![\dot{p} = [q]_{\dot{I}}]\!]_{\mathbb{B}} \geq [\![\dot{p} = [q_r]_{\dot{I}}]\!]_{\mathbb{B}} \wedge [\![[q]_{\dot{I}} = [q_r]_{\dot{I}}]\!]_{\mathbb{B}} \geq r \wedge r = r.$$

The above inequality holds for any $r \in A$, so $[\![\dot{p} = [q]_{\dot{I}}]\!]_{\mathbb{B}} \geq \bigvee A = \mathbf{1}$ concluding the proof. □

Proposition 2.3.6. *Let $i : \mathbb{B} \to \mathbb{C}$ be a regular embedding of complete boolean algebras and G be a V-generic filter for $\mathbb{B}$. Then $\mathbb{C}/_G$ is a complete boolean algebra in $V[G]$.*

Proof. By Proposition 2.3.4, we only need to prove that $\mathbb{C}/_G$ is complete. Let $\{\dot{p}_\alpha : \alpha < \delta\} \in V$ be a set of $\mathbb{B}$ names for elements of $\mathbb{C}/_{\dot{G}}$. Then, by Lemma 2.3.5, for each $\alpha < \delta$ there exists $q_\alpha \in \mathbb{C}$ such that

$$\left[\!\!\left[\dot{p}_\alpha = [q_\alpha]_{i[\dot{G}]}\right]\!\!\right]_{\mathbb{B}} = \mathbf{1}.$$

We have that $\bigvee q_\alpha \in \mathbb{C}$, since $\mathbb{C}$ is complete. Let $r \in \mathbb{C}$ be such that $[r] \geq [q_\alpha]$ in $V[G]$ for all $\alpha < \delta$. Then,

$$\neg\pi(q_\alpha \wedge \neg r) = \left[\!\!\left[q_\alpha \wedge \neg r \in i[\dot{G}]^*\right]\!\!\right]_{\mathbb{B}} = [\![[r] \geq [q_\alpha]]\!]_{\mathbb{B}} \in G$$

So $\pi(q_\alpha \wedge \neg r) \notin G$ for all $\alpha < \delta$. In particular since $\{\pi(q_\alpha \wedge \neg r) : \alpha < \delta\} \in V$ is disjoint from G,

$$q = \bigvee\{\pi(q_\alpha \wedge \neg r) : \alpha < \delta\} = \pi(\neg r \wedge \bigvee\{q_\alpha : \alpha < \delta\}) \notin G.$$

Thus if $\pi(r) \in G$, $V[G] \models [r] \geq [\bigvee q_\alpha]$ while if $\pi(r) \notin G$, $\pi(\neg r) \in G$ and

$$\begin{aligned}\bigvee\{\pi(q_\alpha) : \alpha < \delta\} &= \bigvee\{\pi(q_\alpha \wedge \neg r) : \alpha < \delta\} \vee \bigvee\{\pi(q_\alpha \wedge r) : \alpha < \delta\} \\ &\leq q \vee \pi(r) \notin G,\end{aligned}$$

hence $[q_\alpha]$ and $[\bigvee\{q_\alpha : \alpha < \delta\}]$ are all equal to $\mathbf{0}$. In either cases $[\bigvee\{q_\alpha : \alpha < \delta\}]$ is the least upper bound of the family $\{[q_\alpha] : \alpha < \delta\}$ in $V[G]$. This shows that $V[G] \models \mathbb{C}/_G$ is complete for all V-generic filters G. □

The construction of generic quotients can be defined also for regular embeddings:

Proposition 2.3.7. *Let $\mathbb{B}$, $\mathbb{C}_0$, $\mathbb{C}_1$ be complete boolean algebras, and let G be a V-generic filter for $\mathbb{B}$. Let i_0, i_1, k form a commutative diagram of regular embeddings as in the following picture:*

$$\begin{array}{ccc} \mathbb{B} & \xrightarrow{i_0} & \mathbb{C}_0 \\ & {\scriptstyle i_1}\searrow & \downarrow{\scriptstyle k} \\ & & \mathbb{C}_1 \end{array}$$

Then $k/_G : \mathbb{C}_0/_G \to \mathbb{C}_1/_G$ defined by $k/_G([p]_{i_0[G]}) = [k(p)]_{i_1[G]}$ is a well-defined regular embedding of complete boolean algebras in $V[G]$ with associated retraction π such that $\pi([q]_{i_1[G]}) = [\pi_k(q)]_{i_0[G]}$.

Proof. By Proposition 2.3.6, $k/_G$ is a map between complete boolean algebras. We can also verify that $k/_G$ is:

well-defined. Suppose $[p]_{i_0[G]} = [q]_{i_0[G]}$. Then $p \triangle q \in i_0[G]^*$ and $k(p)\triangle k(q) = k(p\triangle q) \in i_1[G]^*$, so that $[k(p)]_{i_1[G]} = [k(q)]_{i_1[G]}$.

complete homomorphism of boolean algebras. By Proposition 2.3.6,

$$\begin{aligned}k/_G(\neg [p_\alpha]_{i_0[G]}) &= k/_G([\neg p_\alpha]_{i_0[G]}) = [k(\neg p_\alpha)]_{i_1[G]} \\ &= [\neg k(p_\alpha)]_{i_1[G]} = \neg [k(p_\alpha)]_{i_1[G]}\end{aligned}$$

and

$$k/_G\left(\bigvee [p_\alpha]_{i_0[G]}\right) = k/_G\left(\left[\bigvee p_\alpha\right]_{i_0[G]}\right) = \left[k\left(\bigvee p_\alpha\right)\right]_{i_1[G]}$$
$$= \left[\bigvee k(p_\alpha)\right]_{i_1[G]} = \bigvee [k(p_\alpha)]_{i_1[G]}.$$

injective. Let $p, q \in \mathbb{C}_0$ be such that $k/_G([p]_{i_0[G]}) = k/_G([q]_{i_0[G]})$. Then $k(p \triangle q) \in i_1[G]^*$ and there exists $r \notin G$ such that $k(p\triangle q) \leq i_1(r) = k(i_0(r))$. Since k is injective, $p \triangle q \in i_0[G]^*$.

projection is $\pi([q]_{i_1[G]}) = [\pi_k(q)]_{i_0[G]}$**.** For any $q \in \mathbb{C}_1$, by definition of projection

$$\pi([q]_{i_1[G]}) = \bigwedge\{[p]_{i_0[G]} \in \mathbb{C}_0/_G : \; k/_G([p]_{i_0[G]}) \geq [q]_{i_1[G]}\}.$$

Given any $p \in \mathbb{C}_0$,

$$\begin{aligned}\left[\!\left[k/_{\dot G}([p]_{i_0[\dot G]}) \geq [q]_{i_1[\dot G]}\right]\!\right]_{\mathbb{B}} &= \left[\!\left[[k(p)]_{i_1[\dot G]} \geq [q]_{i_1[\dot G]}\right]\!\right]_{\mathbb{B}} \\ &= \left[\!\left[q \wedge \neg k(p) \in i_1[\dot G]^*\right]\!\right]_{\mathbb{B}} \\ &= \neg\pi_{i_1}(q \wedge \neg k(p)) \\ &= \neg\pi_{i_0}(\pi_k(q \wedge k(\neg p))) = \neg\pi_{i_0}(\pi_k(q) \wedge \neg p)\end{aligned}$$

thus $k/_G([p]_{i_0[G]}) \geq [q]_{i_1[G]}$ iff $\pi_{i_0}(\pi_k(q) \wedge \neg p) \in G^*$ iff $\pi_k(q) \wedge \neg p \in i_0[G]^*$ iff $[p]_{i_0[G]} \geq [\pi_k(q)]_{i_0[G]}$. It follows that

$$\pi([q]_{i_1[G]}) = \bigwedge\left\{[p]_{i_0[G]} \in \mathbb{C}_0/_G : \; [p]_{i_0[G]} \geq [\pi_k(q)]_{i_0[G]}\right\} = [\pi_k(q)]_{i_0[G]}. \quad \square$$

2.3.3 Equivalence of two-step iterations and regular embeddings

We are now ready to prove that two-step iterations and regular embeddings capture the same concept.

Theorem 2.3.8. *Let $i : \mathbb{B} \to \mathbb{C}$ be a regular embedding of complete boolean algebras, $\dot G = \dot G_{\mathbb{B}}$ be the canonical name for the V-generic filter for $\mathbb{B}$. Then $\mathbb{B} * \mathbb{C}/_{i[\dot G]} \cong \mathbb{C}$.*

Proof. Define

$$\begin{aligned} i^* : \mathbb{C} &\to \mathbb{B} * \mathbb{C}/_{i[\dot G]} \\ p &\mapsto \left[[p]_{i[\dot G]}\right]_{\approx}. \end{aligned}$$

By Proposition 2.3.6 and definition of two-step iteration,

$$i^*(\neg p) = \left[[\neg p]_{i[\dot G]}\right]_{\approx} = \left[\neg [p]_{i[\dot G]}\right]_{\approx} = \neg\left[[p]_{i[\dot G]}\right]_{\approx} = \neg i^*(p),$$

$$i^*(\bigvee p_\alpha) = \left[[\bigvee p_\alpha]_{i[\dot G]}\right]_{\approx} = \left[\bigvee [p_\alpha]_{i[\dot G]}\right]_{\approx} = \bigvee\left[[p_\alpha]_{i[\dot G]}\right]_{\approx} = \bigvee i^*(p_\alpha).$$

Furthermore, given $\dot q$ a $\mathbb{B}$-name for an element of $\mathbb{C}/_{\dot G}$, by Lemma 2.3.5 there exists a unique $p \in \mathbb{C}$ such that $\left[\!\left[[p]_{i[\dot G]} = \dot q\right]\!\right]_{\mathbb{B}} = \mathbf{1}$, that is, $[[p]_{i[\dot G]}]_{\approx} = [\dot q]_{\approx}$. Thus i^* is a bijection. $\square$

Proposition 2.3.9. *Let $\dot{\mathbb{C}}_0$, $\dot{\mathbb{C}}_1$ be $\mathbb{B}$-names for complete boolean algebras, and let $\dot{k}$ be a $\mathbb{B}$-name for a regular embedding from $\dot{\mathbb{C}}_0$ to $\dot{\mathbb{C}}_1$. Then there is a regular embedding $i : \mathbb{B} * \dot{\mathbb{C}}_0 \to \mathbb{B} * \dot{\mathbb{C}}_1$ such that*

$$\left[\!\left[\dot{k} = i/_{\dot{G}_{\mathbb{B}}}\right]\!\right]_{\mathbb{B}} = \mathbf{1}.$$

Proof. Let $\dot{G} = \dot{G}_{\mathbb{B}}$. Define

$$\begin{aligned} i : \mathbb{B} * \dot{\mathbb{C}}_0 &\to \mathbb{B} * \dot{\mathbb{C}}_1 \\ [\dot{p}]_{\approx} &\mapsto [\dot{k}(\dot{p})]_{\approx}. \end{aligned}$$

Since $\dot{k}$ is a $\mathbb{B}$-name for a regular embedding with boolean value $\mathbf{1}$,

$$[\dot{p}]_{\approx} = [\dot{q}]_{\approx} \iff [\![\dot{p} = \dot{q}]\!]_{\mathbb{B}} = \mathbf{1} \iff \\ \left[\!\left[\dot{k}(\dot{p}) = \dot{k}(\dot{q})\right]\!\right]_{\mathbb{B}} = \mathbf{1} \iff \left[\dot{k}(\dot{p})\right]_{\approx} = \left[\dot{k}(\dot{q})\right]_{\approx}$$

hence i is well defined and injective. Furthermore, i is a complete homomorphism:

$$\begin{aligned} i\left(\neg [\dot{p}]_{\approx}\right) = i\left([\neg\dot{p}]_{\approx}\right) &= \left[\dot{k}\left(\neg\dot{p}\right)\right]_{\approx} \\ &= \left[\neg\dot{k}\left(\dot{p}\right)\right]_{\approx} = \neg\left[\dot{k}\left(\dot{p}\right)\right]_{\approx} = \neg i\left([\dot{p}]_{\approx}\right), \end{aligned}$$

$$\begin{aligned} i\left(\bigvee [\dot{p}_\alpha]_{\approx}\right) = i\left(\left[\bigvee \dot{p}_\alpha\right]_{\approx}\right) &= \left[\dot{k}\left(\bigvee \dot{p}_\alpha\right)\right]_{\approx} \\ = \left[\bigvee \dot{k}\left(\dot{p}_\alpha\right)\right]_{\approx} &= \bigvee \left[\dot{k}\left(\dot{p}_\alpha\right)\right]_{\approx} = \bigvee i\left([\dot{p}_\alpha]_{\approx}\right). \end{aligned}$$

Observe that the diagram

$$\begin{array}{ccc} \mathbb{B} & \xrightarrow{i_{\mathbb{B}*\dot{\mathbb{C}}_0}} & \mathbb{B} * \dot{\mathbb{C}}_0 \\ & \searrow^{i_{\mathbb{B}*\dot{\mathbb{C}}_1}} & \downarrow i \\ & & \mathbb{B} * \dot{\mathbb{C}}_1 \end{array}$$

commutes. Thus by Proposition 2.3.7,

$$\left[\!\left[i/_{\dot{G}}([[\dot{p}]_{\approx}]_{i_{\mathbb{B}*\dot{\mathbb{C}}_0}[\dot{G}]}) = [i\left([\dot{p}]_{\approx}\right)]_{i_{\mathbb{B}*\dot{\mathbb{C}}_1}[\dot{G}]} = \left[\left[\dot{k}\left(\dot{p}\right)\right]_{\approx}\right]_{i_{\mathbb{B}*\dot{\mathbb{C}}_1}[\dot{G}]} \right]\!\right]_{\mathbb{B}} = \mathbf{1}$$

hence $\dot{k} = i/_{\dot{G}}$. □

2.3.4 Generic quotients of iteration systems

The results on generic quotients of the previous sections generalize without much effort to iteration systems. In the following we outline how this occurs.

Lemma 2.3.10. *Let $\mathcal{F} = \{i_{\alpha\beta} : \mathbb{B}_\alpha \to \mathbb{B}_\beta : \alpha \leq \beta < \lambda\}$ be a complete iteration system of complete boolean algebras, G_ξ be a V-generic filter for $\mathbb{B}_\xi$. Then $\mathcal{F}/_{G_\xi} = \{i_{\alpha\beta}/_{G_\xi} : \xi < \alpha \leq \beta < \lambda\}$ is a complete iteration system in $V[G_\xi]$.*

Proof. Follows from Proposition 2.3.7. □

Lemma 2.3.11. *Let $\mathcal{F} = \{i_{\alpha\beta} : \mathbb{B}_\alpha \to \mathbb{B}_\beta : \alpha \leq \beta < \lambda\}$ be a complete iteration system of complete boolean algebras, $\dot{G}_\alpha$ be the canonical name for a generic filter for $\mathbb{B}_\alpha$ and $\dot{s}$ be a $\mathbb{B}_\alpha$-name for an element of $\varprojlim \mathcal{F}/_{\dot{G}_\alpha}$. Then there exists a unique $t \in \varprojlim \mathcal{F}$ such that $\left[\!\left[\dot{s} = \left[\check{t}\right]_{\dot{G}_\alpha}\right]\!\right]_{\mathbb{B}_\alpha} = \mathbf{1}$.*

Proof. We proceed applying Lemma 2.3.5 at every stage $\beta > \alpha$.

Existence. For every $\beta > \alpha$, by hypothesis $\dot{s}(\beta)$ is a name for an element of the quotient $\mathbb{B}_\beta/_{i_{\alpha\beta}[\dot{G}_\alpha]}$. Let $t(\beta)$ be the unique element of $\mathbb{B}_\beta$ such that $\left[\!\left[\dot{s}(\beta) = \left[\check{t}(\beta)\right]_{i_{\alpha\beta}[\dot{G}_\alpha]}\right]\!\right]_{\mathbb{B}_\alpha} = \mathbf{1}$. Then,

$$\begin{aligned}\left[\!\left[\dot{s} = \left[\check{t}\right]_{\dot{G}_\alpha}\right]\!\right]_{\mathbb{B}_\alpha} &= \left[\!\left[\forall\beta \in \lambda\ \dot{s}(\beta) = \left[\check{t}(\beta)\right]_{\dot{G}_\alpha}\right]\!\right]_{\mathbb{B}_\alpha} \\ &= \bigwedge \left\{ \left[\!\left[\dot{s}(\beta) = \left[\check{t}(\beta)\right]_{i_{\alpha\beta}[\dot{G}_\alpha]}\right]\!\right]_{\mathbb{B}_\alpha} : \beta \in \lambda \right\} = \bigwedge \mathbf{1} = \mathbf{1}.\end{aligned}$$

Furthermore, by Proposition 2.3.7 $\pi_{\alpha\beta}(t(\beta)) = t(\alpha)$ for all $\beta > \alpha$ hence t is a thread in $\varprojlim \mathcal{F}$.

Uniqueness. If t' is such that $\left[\!\left[\dot{s} = \left[\check{t}'\right]_{\dot{G}_\alpha}\right]\!\right]_{\mathbb{B}_\alpha} = \mathbf{1}$ then for every $\beta > \alpha$,

$$\left[\!\left[\dot{s}(\beta) = \left[\check{t}'(\beta)\right]_{i_{\alpha\beta}[\dot{G}_\alpha]}\right]\!\right]_{\mathbb{B}_\alpha} = \mathbf{1}.$$

Such an element is unique by Lemma 2.3.5, hence $t'(\beta) = t(\beta)$ defined above completing the proof. □

CHAPTER 3

FORCING AXIOMS

In the first part of this section we introduce the most common forcing classes, with a special focus on properness and semiproperness, and their corresponding forcing axioms. In the second part we introduce the definition of *weakly iterable* forcing class and outline its main properties. In this context will shall be able to explore how the forcing classes previously introduced behave with respect to the limit operations on iteration systems as introduced in Section 2.2. This will require to analyze them with respect to the following order.

Definition 3.0.1. Let Γ be a definable class of complete boolean algebras and let $\mathbb{B}, \mathbb{C}$ be complete boolean algebras.

We say that $\mathbb{B} \leq_\Gamma^* \mathbb{C}$ iff there is a complete homomorphism $i : \mathbb{C} \to \mathbb{B}$ such that the quotient algebra $\mathbb{B}/_{i[\dot{G}_\mathbb{C}]}$ is in Γ with boolean value $\mathbf{1}$.

We say that $\mathbb{B} \leq_\Gamma \mathbb{C}$ iff there is a *regular embedding* with the same properties as above.

Notice that we do not require neither $\mathbb{B}$ nor $\mathbb{C}$ to be in Γ when $\mathbb{B} \leq_\Gamma \mathbb{C}$. In the final part of this section, we shall prove that SP and SSP forcings are weakly iterable. The other relevant cases (ccc, Axiom-A, proper) are left to the reader, since the corresponding proofs are either simpler or given by straightforward modifications of the arguments presented.

Section 3.1 introduces the main examples of forcing classes and their basic properties. Section 3.2 introduces the corresponding forcing axioms, using the terminology developed in Chapter 2. In Section 3.3 we introduce the definition of weakly iterable forcing class, outline its main consequences and argue that most relevant forcing classes fall into the scope of this definition. In Section 3.4 we apply the machinery developed so far to prove the preservation of semiproperness through two step iterations and revised countable support iterations. The proof of the latter splits in three cases according to the cofinality of the length of the iteration system (ω, ω_1, bigger than ω_1) and mimics in this new setting the original proof of Shelah of these results. We then prove that stationary set preserving forcings are weakly iterable (provided there are class many supercompact cardinals), building on the results obtained for semiproper iterations.

3.1 Forcing classes

We list below the main forcing classes Γ we shall analyze throughout this thesis.

Definition 3.1.1. A boolean algebra $\mathbb{B}$ is *$<\kappa$-cc* (satisfies the *κ-chain condition*) iff every antichain has size less than κ. $\mathbb{B}$ is *locally* $<\kappa$-cc iff there exists a maximal antichain $A \subseteq \mathbb{B}$ such that $\mathbb{B}\restriction p$ is $<\kappa$-cc for all $p \in A$. $\mathbb{B}$ is *ccc* (locally ccc) iff it is $<\omega_1$-cc (locally $<\omega_1$-cc).

Definition 3.1.2. A poset $\mathbb{P}$ is $<\kappa$-closed iff every descending chain $\langle p_\alpha : \alpha < \gamma\rangle$ with $\gamma < \kappa$ has a lower bound in $\mathbb{P}$.

Definition 3.1.3. A boolean algebra $\mathbb{B}$ is *$<\kappa$-distributive* iff every family $\mathcal{D}$ of $< \kappa$ open dense sets in $\mathbb{B}$ has dense intersection $\bigcap \mathcal{D}$.

For a fixed κ, any $<\kappa$-cc non trivial boolean algebra $\mathbb{B}$ is not $<\kappa^+$-distributive and conversely. On the other hand, $<\kappa$-distributivity is strongly related to $<\kappa$-closure for posets: every $<\kappa$-closed poset has a $<\kappa$-distributive boolean completion, even though the converse does not hold. In addition, $<\kappa$-closed posets and $<\kappa$-distributive boolean algebras share their main forcing property (not adding subsets of V of size less than κ).

Definition 3.1.4. A boolean algebra $\mathbb{B}$ is *axiom-A* iff there exists a dense subset D of $\mathbb{P}$ and sequence $\langle \leq_n : n < \omega\rangle$ of partial orderings on D such that the following hold:

i. $p \leq_0 q \Rightarrow p \leq q$ and $p \leq_{n+1} q \Rightarrow p \leq_n q$ for all $p, q \in D$, $n < \omega$;

ii. if $\langle p_n : n < \omega\rangle$ is such that $p_0 \geq_0 p_1 \geq_1 p_2 \geq_2 \ldots$, then there exists a $q \in D$ such that $q \leq_n p_n$ for all $n < \omega$;

iii. given any $p \in D$, $n < \omega$ and $\dot\alpha$ name for an ordinal there exist a $q \leq_n p$ and a countable set A such that $q \Vdash_{\mathbb{B}} \dot\alpha \in \check{A}$.

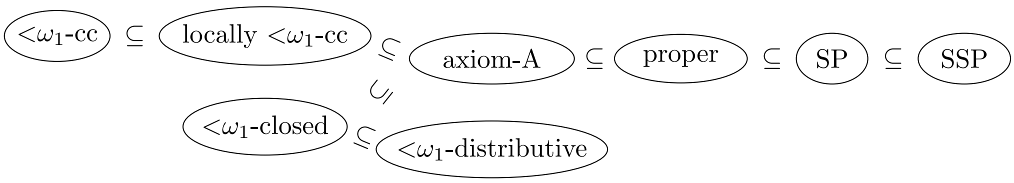
Definition 3.1.5. Let $\mathbb{B}$ be a boolean algebra, $\theta > 2^{|\mathbb{B}|}$ be a cardinal, $M \prec H_\theta$ such that $\mathbb{B} \in M$. We say that a condition $p \in \mathbb{B}$ is M-generic iff for every $\dot\alpha \in M \cap V^{\mathbb{B}}$ name for an ordinal, $p \Vdash_{\mathbb{B}} \dot\alpha \in \check{M}$. We say that a condition $p \in \mathbb{B}$ is M-semigeneric iff for every $\dot\alpha \in M \cap V^{\mathbb{B}}$ name for a *countable* ordinal, $p \Vdash_{\mathbb{B}} \dot\alpha \in \check{M}$.

Definition 3.1.6. A boolean algebra $\mathbb{B}$ is *proper* iff for every M as in the previous definition and $q \in M \cap \mathbb{B}$ there is an M-generic condition below q. Similarly, we say that $\mathbb{B}$ is *semiproper* (SP) iff for every M as in the previous definition and $q \in M \cap \mathbb{B}$ there is an M-semigeneric condition below q.

Equivalently, a boolean algebra $\mathbb{B}$ is proper iff it preserves stationary sets on $[\lambda]^\omega$ for any λ uncountable cardinal (see [27, Thm. 31.7]). If we require the latter property to hold only for $\lambda = \omega_1$, we obtain the following larger forcing class.

Definition 3.1.7. A boolean algebra $\mathbb{B}$ is *stationary set preserving* (in short, SSP) iff for every stationary set $S \subseteq \omega_1$, $\mathbf{1} \Vdash_{\mathbb{B}} \check{S}$ is stationary.

The following diagram summarizes the relations between the various forcing classes just introduced.



3.1.1 Algebraic formulation of properness and semiproperness

The notions of properness and semiproperness can also be stated in a more algebraic fashion. This will later be used throughout this chapter in order to simplify the treatment of semiproper iterations.

Definition 3.1.8. Let $\mathbb{B}$ be a complete boolean algebra, $M \prec H_\theta$ for some $\theta > 2^{|\mathbb{B}|}$, $\mathrm{PD}(\mathbb{B})$ be the collection of predense subsets of $\mathbb{B}$, and $\mathrm{PD}_{\omega_1}(\mathbb{B})$ be $\mathrm{PD}(\mathbb{B}) \cap [\mathbb{B}]^{\leq\omega_1}$.

The boolean value

$$\mathrm{g}(\mathbb{B}, M) = \bigwedge \left\{ \bigvee (D \cap M) : \ D \in \mathrm{PD}(\mathbb{B}) \cap M \right\}$$

is the *degree of genericity* of M with respect to $\mathbb{B}$. The boolean value

$$\mathrm{sg}(\mathbb{B}, M) = \bigwedge \left\{ \bigvee (D \cap M) : \ D \in \mathrm{PD}_{\omega_1}(\mathbb{B}) \cap M \right\}$$

is the *degree of semigenericity* of M with respect to $\mathbb{B}$.

The degree of semigenericity can be also calculated from maximal antichains, and behaves well with respect to the restriction operation.

Proposition 3.1.9. *Let $\mathbb{B}$, M be as in the previous definition, and let $\mathrm{A}(\mathbb{B}) \subseteq \mathrm{PD}(\mathbb{B})$, $\mathrm{A}_{\omega_1}(\mathbb{B}) \subseteq \mathrm{PD}_{\omega_1}(\mathbb{B})$ be the subcollections given by maximal antichains of $\mathbb{B}$. Then*

$$\mathrm{g}(\mathbb{B}, M) = \bigwedge \left\{ \bigvee (A \cap M) : A \in \mathrm{A}(\mathbb{B}) \cap M \right\}$$

and

$$\mathrm{sg}(\mathbb{B}, M) = \bigwedge \left\{ \bigvee (A \cap M) : A \in \mathrm{A}_{\omega_1}(\mathbb{B}) \cap M \right\}.$$

Proof. Since $\mathrm{A}(\mathbb{B}) \subseteq \mathrm{PD}(\mathbb{B})$, the inequality

$$\mathrm{g}(\mathbb{B}, M) \leq \bigwedge \left\{ \bigvee (A \cap M) : A \in \mathrm{A}(\mathbb{B}) \cap M \right\}$$

is trivial. Conversely, if $D = \{b_\alpha : \alpha < \lambda\} \in \mathrm{PD}(\mathbb{B}) \cap M$, define

$$A_D = \left\{ a_\alpha = b_\alpha \wedge \neg \bigvee \{b_\beta : \beta < \alpha\} : \alpha < \lambda \right\}.$$

By elementarity, since $D \in M$ also A_D is in M. It is straightforward to verify that A_D is an antichain, and since $\bigvee A_D = \bigvee D = \mathbf{1}$ it is also maximal. Moreover, since

$a_\alpha \leq b_\alpha$ we have that $\bigvee A_D \cap M \leq \bigvee D \cap M$. Thus, for any $D \in \mathrm{PD}(\mathbb{B}) \cap M$, we have that $\bigwedge \{\bigvee (A \cap M) : A \in \mathrm{A}(\mathbb{B}) \cap M\} \leq \bigvee D \cap M$ hence

$$\bigwedge \left\{ \bigvee (A \cap M) : A \in \mathrm{A}(\mathbb{B}) \cap M \right\} \leq \mathrm{g}(\mathbb{B}, M)$$

and the same can be proved also for $\mathrm{sg}(\mathbb{B}, M)$ following the same procedure. □

Proposition 3.1.10. *Let $\mathbb{B}$ be a complete boolean algebra and $M \prec H_\theta$ for some $\theta > 2^{|\mathbb{B}|}$. Then for all $p \in M \cap \mathbb{B}$,*

$$\mathrm{g}(\mathbb{B}{\restriction}p, M) = \mathrm{g}(\mathbb{B}, M) \wedge p$$

and

$$\mathrm{sg}(\mathbb{B}{\restriction}p, M) = \mathrm{sg}(\mathbb{B}, M) \wedge p.$$

Proof. Observe that if A is a maximal antichain in $\mathbb{B}$, then $A{\restriction}p = \{q \wedge p : q \in A\}$ is a maximal antichain in $\mathbb{B}{\restriction}p$. Moreover for each maximal antichain A_p in $\mathbb{B}{\restriction}p \cap M$, $A = A_p \cup \{\neg p\}$ is a maximal antichain in $\mathbb{B} \cap M$. Therefore

$$\mathrm{g}(\mathbb{B}, M) \wedge p = \bigwedge \bigvee (A \cap M) \wedge p = \bigwedge \bigvee ((A{\restriction}p) \cap M) = \mathrm{g}(\mathbb{B}{\restriction}p, M)$$

and the same holds for $\mathrm{sg}(\mathbb{B}, M)$. □

We are now ready to introduce the algebraic definition of semiproperness and properness for complete boolean algebras.

Theorem 3.1.11. *Let $\mathbb{B}$ be a complete boolean algebra, $\theta > 2^{|\mathbb{B}|}$ be a cardinal. Then $\mathbb{B}$ is proper iff for club many $M \prec H_\theta$, $\mathrm{g}(\mathbb{B}, M)$ is compatible with every p in $\mathbb{B} \cap M$, and $\mathbb{B}$ is semiproper iff for club many $M \prec H_\theta$, $\mathrm{sg}(\mathbb{B}, M)$ is compatible with every p in $\mathbb{B} \cap M$.*

Lemma 3.1.12. *Let $\mathbb{B}$ be a complete boolean algebra and fix $M \prec H_\theta$. Then*

$$\mathrm{g}(\mathbb{B}, M) = \bigvee \{p \in \mathbb{B} : \ p \text{ is } M\text{-generic}\}$$

and

$$\mathrm{sg}(\mathbb{B}, M) = \bigvee \{p \in \mathbb{B} : \ p \text{ is } M\text{-semigeneric}\}.$$

Proof. Given $A = \{p_\beta : \ \beta < \omega_1\} \in \mathrm{A}_{\omega_1}(\mathbb{B})$, define $\dot{\alpha}_A = \{\langle \check{\gamma}, p_\beta \rangle : \ \gamma < \beta < \omega_1\}$. Then we can easily check that

$$[\![\dot{\alpha}_A < \check{\omega}_1]\!]_{\mathbb{B}} = \bigvee \left\{ [\![\dot{\alpha}_A = \check{\beta}]\!]_{\mathbb{B}} : \ \beta < \omega_1 \right\} = \bigvee \{p_\beta : \ \beta < \omega_1\} = \mathbf{1}.$$

Conversely, given $\dot{\alpha} \in V^{\mathbb{B}} \cap M$ such that $[\![\dot{\alpha} < \check{\omega}_1]\!]_{\mathbb{B}} = \mathbf{1}$, define

$$A_{\dot{\alpha}} = \left\{ p_\beta = [\![\dot{\alpha} = \check{\beta}]\!]_{\mathbb{B}} : \ \beta < \omega_1 \right\}.$$

Then we can easily check that $A_{\dot{\alpha}} \in \mathrm{A}_{\omega_1}(\mathbb{B})$.

Suppose now that p is an M-semigeneric condition, and fix an arbitrary $A \in \mathrm{A}_{\omega_1}(\mathbb{B}) \cap M$. Then $\dot\alpha_A \in M$ and $[\![\dot\alpha_A < \check\omega_1]\!]_{\mathbb{B}} = \mathbf{1}$, hence

$$\begin{aligned} p \leq \left[\!\!\left[\dot\alpha_A < (M \check\cap \omega_1)\right]\!\!\right]_{\mathbb{B}} &= \bigvee \left\{ [\![\dot\alpha_A = \check\beta]\!]_{\mathbb{B}} :\ \beta \in M \cap \omega_1 \right\} \\ &= \bigvee \{p_\beta :\ \beta \in M \cap \omega_1\} = \bigvee A \cap M. \end{aligned}$$

It follows that $p \leq \bigwedge \{\bigvee (A \cap M) :\ A \in \mathrm{A}_{\omega_1}(\mathbb{B}) \cap M\} = \mathrm{sg}(\mathbb{B}, M)$, hence

$$\mathrm{sg}(\mathbb{B}, M) \geq \bigvee \{p \in \mathbb{B} :\ p \text{ is } M\text{-semigeneric}\}.$$

Finally, we show that $\mathrm{sg}(\mathbb{B}, M)$ is M-semigeneric itself. Fix an arbitrary $\dot\alpha \in V^{\mathbb{B}} \cap M$ such that $\mathbf{1} \Vdash_{\mathbb{B}} \dot\alpha < \check\omega_1$, and let $A_{\dot\alpha} \in \mathrm{A}_{\omega_1}(\mathbb{B})$ be as above. Since $\dot\alpha \in M$, also $A_{\dot\alpha} \in M$. Moreover,

$$\begin{aligned} \left[\!\!\left[\dot\alpha < (M \check\cap \omega_1)\right]\!\!\right]_{\mathbb{B}} &= \bigvee \left\{ [\![\dot\alpha = \check\beta]\!]_{\mathbb{B}} :\ \beta \in M \cap \omega_1 \right\} \\ &= \bigvee \{a_\beta :\ \beta \in M \cap \omega_1\} = \bigvee A_{\dot\alpha} \cap M \geq \mathrm{sg}(\mathbb{B}, M) \end{aligned}$$

and the same holds for $\mathrm{g}(\mathbb{B}, M)$. □

Proof of Theorem 3.1.11. First, suppose that $\mathbb{B}$ is semiproper and given $M \prec H_\theta$, $p \in \mathbb{B} \cap M$ let $q \leq p$ be an M-semigeneric condition. Then $q > \mathbf{0}$ and by Proposition 3.1.12, $q \leq \mathrm{sg}(\mathbb{B}, M)$, hence $\mathrm{sg}(\mathbb{B}, M) \wedge p \geq q > \mathbf{0}$.

Conversely, suppose that $\mathbb{B}$ is as in the hypothesis of the theorem, $M \prec H_\theta$, $p \in \mathbb{B} \cap M$. Then $\mathrm{sg}(\mathbb{B}, M) \wedge p = q > \mathbf{0}$. Since the set of M-semigeneric conditions is dense below $\mathrm{sg}(\mathbb{B}, M)$ (hence below q), we can find a $q' < q$ that is M-semigeneric.

The same reasoning holds also for $\mathrm{g}(\mathbb{B}, M)$ with trivial modifications. □

3.2 Forcing axioms

Forcing is well-known as a versatile tool for proving consistency results. The purpose of forcing axioms is to turn it into a powerful tool for proving (conditional) theorems. Let Γ denote a definable class of complete boolean algebras, κ be a cardinal.

Definition 3.2.1. The bounded forcing axiom $\mathrm{BFA}_\kappa(\Gamma)$ holds if for all $\mathbb{B} \in \Gamma$ and all families $\{D_\alpha : \alpha < \kappa\}$ of predense subsets of $\mathbb{B}$ of size at most κ, there is a filter $G \subset \mathbb{B}$ meeting all these sets.

Definition 3.2.2. The forcing axiom $\mathrm{FA}_\kappa(\Gamma)$ holds if for all $\mathbb{B} \in \Gamma$ and all families $\{D_\alpha : \alpha < \kappa\}$ of predense subsets of $\mathbb{B}$, there is a filter $G \subset \mathbb{B}$ meeting all these sets.

Definition 3.2.3. The forcing axiom $\mathrm{FA}^{++}_\kappa(\Gamma)$ holds if for all $\mathbb{B} \in \Gamma$ and all families $\{D_\alpha : \alpha < \kappa\}$ of predense subsets of $\mathbb{B}$ and all families $\{\dot S_\alpha : \alpha < \kappa\}$ of $\mathbb{B}$-names for stationary subsets of κ, there is a filter $G \subset \mathbb{B}$ meeting all these dense sets and evaluating each $\dot S_\alpha$ as a stationary subset of κ.

For $\kappa = \omega_1$, the forcing axioms are widely studied for many different classes Γ of complete boolean algebras. Thus, in the following we shall feel free to omit the index κ whenever $\kappa = \omega_1$. In particular, for the classes of posets:

$$\text{ccc} \subset \text{proper} \subset \text{SP} \subset \text{SSP}$$

the forcing axiom FA_{ω_1} is called respectively MA (Martin's Axiom), PFA (Proper Forcing Axiom), SPFA (Semiproper Forcing Axiom), MM (Martin's Maximum). The corresponding bounded versions are BMA, BPFA, BSPFA, BMM, while the strengthened versions are MA^{++}, PFA^{++}, SPFA^{++}, MM^{++}. BMA is provably equivalent to MA [32], while MM^{++} is a strengthening of MM [41]. These axioms also have distinct consistency strengths: for example, MA is consistent relative to ZFC [27, Thm. 16.13], BPFA and BSPFA^{++} are consistent relative to a reflecting cardinal [21], while BMM is consistent relative to ω-many Woodin cardinals [46], and MM^{++} is consistent relative to a supercompact cardinal (Theorem 3.4.15).

The class of SSP posets play a special role in the development of forcing axioms. In fact, MM is the strongest possible form of forcing axiom for ω_1.

Definition 3.2.4. $\mathbb{B}$ is *locally* SSP iff there exists a $p \in \mathbb{B}$ such that $\mathbb{B}\restriction p$ is SSP.

Proposition 3.2.5 (Shelah)**.** *If* $\mathbb{B}$ *is not locally* SSP *then* $\text{FA}_{\omega_1}(\mathbb{B})$ *is false.*

Proof. Assume that $\mathbb{B}$ is not locally SSP and let S be a stationary set on ω_1, $\dot{C} \in V^{\mathbb{B}}$ be such that $\mathbf{1} \Vdash_{\mathbb{B}} \dot{C} \subseteq \check{\omega}_1 \text{ club } \wedge \check{S} \cap \dot{C} = \check{\emptyset}$. Define:

$$\begin{aligned} D_\alpha &= \left\{ p \in \mathbb{B} : \ p \Vdash_{\mathbb{B}} \check{\alpha} \in \dot{C} \vee p \Vdash_{\mathbb{B}} \check{\alpha} \notin \dot{C} \right\} \\ E_\beta &= \left\{ p \in \mathbb{B} : \ p \Vdash_{\mathbb{B}} \check{\beta} \notin \dot{C} \Rightarrow \exists \gamma < \beta \ p \Vdash_{\mathbb{B}} \dot{C} \cap \check{\beta} \subseteq \check{\gamma} \right\} \\ F_\gamma &= \left\{ p \in \mathbb{B} : \ \exists \alpha > \gamma \ p \Vdash_{\mathbb{B}} \check{\alpha} \in \dot{C} \right\} \end{aligned}$$

Those sets are dense in $\mathbb{B}$ by Łoś theorem since $\dot{C}$ is forced to be a club. Suppose by contradiction that $\text{FA}_{\omega_1}(\mathbb{B})$ holds, and let G be a filter that intersects all the sets D_α, E_β, F_γ. Then the set $C = \left\{ \alpha < \omega_1 : \ \exists p \in G \ p \Vdash_{\mathbb{B}} \check{\alpha} \in \dot{C} \right\}$ is a club in V, so there is a $\beta \in S \cap C$. By definition of C, there exists a condition $q \in G$ such that $q \Vdash_{\mathbb{B}} \check{\beta} \in \dot{C}$, and $\beta \in S \Rightarrow q \Vdash_{\mathbb{B}} \beta \in \check{S} \cap \dot{C} \neq \check{\emptyset}$, a contradiction. □

Since FA(locally SSP) is easily seen to be equivalent to MM, Proposition 3.2.5 implies that MM is the strongest possible form of $\text{FA}_{\omega_1}(\Gamma)$ for some Γ.

The forcing axioms have many equivalent formulations. In Section 3.2.1 we shall see how many of them can be expressed as density properties of the corresponding forcing classes. Furthermore, they can also be formulated in terms of the existence of generic filters $G \in V$ for small set models $M \in V$ of ZFC.

Definition 3.2.6. We say that M is a κ*-model* iff $|M| = \kappa$, $\kappa \subseteq M$ and M models ZFC^- (i.e. ZFC without powerset and with collection and separation, as in [20]).

We remark that this definition of κ-model is weaker than other definitions for the same term present in literature (which often require also that $\kappa \in M$ and ${}^{<\kappa}M \subseteq M$).

Definition 3.2.7. Let $\mathbb{B}$ be a complete boolean algebra, M be a κ-model such that $\mathbb{B} \in M$. We say that G is an M-generic filter for $\mathbb{B}$ iff $G \cap D \cap M \neq \emptyset$ for all $D \in M$ dense subsets of $\mathbb{B}$. We say that G is an M-correct filter for $\mathbb{B}$ if it is M-generic and for every $\dot{S} \in M$ $\mathbb{B}$-name for a stationary subset of $P(\kappa)$, $\mathrm{val}_G(\dot{S})$ is stationary in V.

Proposition 3.2.8 ([46, Thm. 2.53])**.** *Let $\mathbb{B}$ be a complete boolean algebra with $\mathcal{P}(\mathbb{B}) \in H_\theta$. Then the following are equivalent:*

1. $\mathrm{FA}_\kappa(\mathbb{B})$,
2. *there exists a κ-model $M \prec H_\theta$ and a filter G such that $\mathbb{B} \in M$ and G is M-generic for $\mathbb{B}$,*
3. *there are stationary many κ-models $M \prec H_\theta$ and a filter G such that $\mathbb{B} \in M$ and G is M-generic for $\mathbb{B}$.*

Proof. **1** $\Rightarrow$ **2**. First, suppose that $\mathrm{FA}_\kappa(\mathbb{B})$ holds and let $M \prec H_\theta$ be a κ-model such that $\mathbb{B} \in M$. There are at most κ dense subsets of $\mathbb{B}$ in M, hence by $\mathrm{FA}_\kappa(\mathbb{B})$ there is a filter G meeting all those sets. However, G might not be M-generic since for some $D \in M$, the intersection $G \cap D$ might be disjoint from M. Define:

$$N = \left\{x \in H_\theta : \exists \tau \in M \cap V^{\mathbb{B}} \; \exists q \in G \; (q \Vdash_{\mathbb{B}} \tau = \check{x})\right\}$$

Clearly, N cointains M (hence contains κ), and the cardinality $|N| \leq \left|M \cap V^{\mathbb{B}}\right| = \kappa$ since every τ can be evaluated in an unique way by the elements of the filter G. To prove that $N \prec H_\theta$, let $\exists x \phi(x, a_1, \ldots, a_n)$ be any formula with parameters $a_1, \ldots, a_n \in N$ which holds in V. Let $\tau_i \in M^{\mathbb{B}}$, $q_i \in G$ be such that $q_i \Vdash_{\mathbb{B}} \tau_i = \check{a}_i$ for all $i < n$. Define $A_\phi = \{p \in \mathbb{B} : \; p \Vdash_{\mathbb{B}} \exists x \in V \; \phi(x, \tau_1, \ldots, \tau_n)\}$, this set is definable in M hence $A_\phi \in M$. Furthermore, $A_\phi \cap G$ is not empty since it contains any $q \in G$ below all q_i. By fullness in H_θ, we have that:

$$\begin{aligned}
&H_\theta \models \forall p \in A_\phi \; p \Vdash_{\mathbb{B}} \exists x \in V \; \phi(x, \tau_1, \ldots, \tau_n) \Rightarrow \\
&H_\theta \models \exists \tau \; \forall p \in A_\phi \; p \Vdash_{\mathbb{B}} \tau \in V \wedge \phi(\tau, \tau_1, \ldots, \tau_n) \Rightarrow \\
&M \models \exists \tau \; \forall p \in A_\phi \; p \Vdash_{\mathbb{B}} \tau \in V \wedge \phi(\tau, \tau_1, \ldots, \tau_n)
\end{aligned}$$

Fix such a τ, by elementarity the last formula holds also in H_θ and in particular for $q \in A_\phi$. Since the set $\{p \in \mathbb{B} : \; \exists x \in H_\theta \; p \Vdash_{\mathbb{B}} \check{x} = \tau\}$ is an open dense set definable in M, there is a $q' \in G$ below q belonging to this dense set, and an $a \in H_\theta$ such that $q' \Vdash_{\mathbb{B}} \tau = \check{a}$. Then q', τ witness that $a \in N$ hence the original formula $\exists x \phi(x, a_1, \ldots, a_n)$ holds in N.

Finally, we need to check that G is N-generic for $\mathbb{B}$. Let $D \in N$ be a dense subset of $\mathbb{B}$, and $\dot{D} \in M$ be such that $\mathbf{1} \Vdash_{\mathbb{B}} \dot{D}$ is dense $\wedge \, \dot{D} \in V$ and for some $q \in G$, $q \Vdash_{\mathbb{B}} \dot{D} = D$. Since $\mathbf{1} \Vdash_{\mathbb{B}} \dot{D} \cap \dot{G} \neq \emptyset$, by fullness lemma there exists a $\tau \in H_\theta$ such that $\mathbf{1} \Vdash_{\mathbb{B}} \tau \in \dot{D} \cap \dot{G}$, and by elementarity there is such a τ also in M. Let $q' \in G$ below q be deciding the value of τ, $q' \Vdash_{\mathbb{B}} \tau = \check{p}$. Since q' forces that $\check{p} \in \dot{G}$, it must be $q' \leq p$ so that $p \in G$ hence $p \in G \cap D \cap N$ is not empty.

2 $\Rightarrow$ **3**. Let $M \prec H_{\theta^+}$, G be as in the hypothesis of the proposition. Define:

$$S = \{N \prec H_\theta : \; \kappa \subset N \; \wedge \; |N| = \kappa \; \wedge \; \exists G \text{ filter } N\text{-generic}\}$$

Note that S is definable in M then $S \in M$. Furthermore, $M \cap H_\theta \prec H_\theta$ and $M \cap H_\theta$ is in S. Given any $C_f \in M$ club on H_θ, since $f \in M$ we have that $M \cap H_\theta \in C_f$. Then $V \vDash S \cap C_f \neq \emptyset$ and by elementarity the same holds for M. Thus, S is stationary in M and again by elementarity S is stationary also in V.

3 ⇒ 1. Fix a collection $\mathcal{D} = \langle D_\alpha : \alpha < \kappa \rangle$ of dense subsets of $\mathbb{B}$. Since the set of κ-models with a generic filter is stationary, we can find a κ-model M such that $\mathcal{D} \in M$ and there is an M-generic filter G for $\mathbb{B}$. Since $\kappa \subset M$ and $\mathcal{D}$ has size κ, $D_\alpha \in M$ for every $\alpha < \kappa$. Thus, the M-generic filter G will meet all dense sets in $\mathcal{D}$, verifying $\mathrm{FA}_\kappa(\mathbb{B})$ for this collection. □

Under some mild restrictions a similar result holds also for $\mathrm{FA}^{++}(\mathbb{B})$, as shown in the following proposition.

Proposition 3.2.9. *Let $\mathbb{B}$ be a complete boolean algebra with $|\mathcal{P}(\mathbb{B})|^+ \in H_\theta$. Then the following are equivalent:*

1. $\mathrm{FA}^{++}_{\omega_1}(\mathbb{B} * \mathrm{Coll}(\omega_1, (2^{|\mathbb{B}|})^+))$,
2. *there exists a ω_1-model $M \prec H_\theta$ and a filter G such that $\mathbb{B} \in M$ and G is M-correct for $\mathbb{B} * \mathrm{Coll}(\omega_1, (2^{|\mathbb{B}|})^+)$,*
3. *there are stationary many ω_1-models $M \prec H_\theta$ with a filter G such that $\mathbb{B} \in M$ and G is M-correct for $\mathbb{B} * \mathrm{Coll}(\omega_1, (2^{|\mathbb{B}|})^+)$.*

Proof. Along the same lines of the proof of Proposition 3.2.8 (see [46, Thm. 2.53]). □

3.2.1 Forcing axioms as density properties

Many of the common forcing axioms can be formulated in terms of density properties of their corresponding forcing classes Γ. This kind of formulation will be natural for the resurrection axioms that we shall consider in the rest of the chapter.

Theorem 3.2.10 (Bagaria, [6])**.** $\mathrm{BFA}(\Gamma)$ *is equivalent to the assertion that the class*

$$\left\{\mathbb{B} \in \Gamma : \quad H_{\omega_2} \prec_1 V^{\mathbb{B}}\right\}$$

is dense in $(\Gamma, \leq_{all})$.

We remark that the latter assertion is actually equivalent to requiring this class to coincide with the whole Γ (since Σ_1-formulas are always upwards absolute).

Under suitable large cardinal assumptions the unbounded versions of the forcing axioms can also be reformulated as density properties, but only for $\Gamma = \mathrm{SSP}$ (at least to our knowledge).

Theorem 3.2.11 (Woodin, [33])**.** *Assume there are class many Woodin cardinals. Then* MM *is equivalent to the assertion that the class*

$$\left\{\mathbb{B} \in \mathrm{SSP} : \quad \mathbb{B} \text{ is a presaturated tower forcing}^1\right\}$$

is dense in $(\mathrm{SSP}, \leq_{all})$.

[1]We refer the reader to [43, Def. 4.14] for a definition of presaturated tower and to [43, Def. 5.8] for a definition of strongly presaturated towers (see below).

Theorem 3.2.12 (Viale, [43]). *Assume there are class many Woodin cardinals, then* MM^{++} *is equivalent to the assertion that the above class is dense in* $(\mathrm{SSP}, \leq_{\mathrm{SSP}})$.

In this chapter we shall also refer to the following strengthening of MM^{++}, which is defined by a density property of the class SSP as follows:

Definition 3.2.13 (Viale, [43]). MM^{+++} is the assertion that the class

$$\{\mathbb{B} \in \mathrm{SSP} : \ \mathbb{B} \text{ is a strongly presaturated tower }^2\}$$

is dense in $(\mathrm{SSP}, \leq_{\mathrm{SSP}})$.

3.2.2 Resurrection axioms

The resurrection axiom, introduced by Hamkins and Johnstone in [23], can be concisely stated as a density property for the class partial order $\leq_\Gamma$ defined above.

Definition 3.2.14 (Hamkins, Johnstone [23]). The *resurrection axiom* $\mathrm{RA}(\Gamma)$ is the assertion that the class

$$\left\{\mathbb{B} \in \Gamma : \quad H_{\mathfrak{c}} \prec H_{\mathfrak{c}}^{\mathbb{B}}\right\}$$

is dense in $(\Gamma, \leq_\Gamma)$.

The *weak resurrection axiom* $\mathrm{wRA}(\Gamma)$ is the assertion that for all $\mathbb{B} \in \Gamma$, there exists a $\mathbb{C} \leq_{\mathrm{all}} \mathbb{B}$ such that $H_{\mathfrak{c}} \prec H_{\mathfrak{c}}^{\mathbb{C}}$.

Several variations of this axiom have been proposed.

Definition 3.2.15 (Hamkins, Johnstone [24]). The boldface resurrection axiom $\underset{\sim}{\mathrm{RA}}(\Gamma)$ asserts that for every $A \subseteq \mathfrak{c}$ the following class:

$$\left\{\mathbb{B} \in \Gamma : \ \exists A^* \subseteq \mathfrak{c}^{\mathbb{B}} \ \langle H_{\mathfrak{c}}, \in, A\rangle \prec \langle H_{\mathfrak{c}}^{\mathbb{B}}, \in, A^*\rangle\right\}$$

is dense in $(\Gamma, \leq_\Gamma)$.

Definition 3.2.16 (Tsaprounis [40]). The unbounded resurrection axiom $\mathrm{UR}(\Gamma)$ asserts that for every cardinal $\theta > \mathfrak{c}$ the following class:

$$\left\{\mathbb{B} \in \Gamma : \ \exists j : H_\theta \to H(j(\theta))^{\mathbb{B}} \text{ elementary } \wedge \mathrm{crit}(j) = \mathfrak{c} \wedge j(\mathfrak{c}) > \theta\right\}$$

is dense in $(\Gamma, \leq_\Gamma)$ below $\Gamma \cap H_\theta$.

3.3 Weak iterability

We are now ready to introduce the definition of *weakly iterable* class of forcing notions. Given a definable class Γ of forcing notions, let Γ^{lim} denote the (definable) class of complete iteration systems $\mathcal{F} = \{i_{\alpha\beta} : \mathbb{B}_\alpha \to \mathbb{B}_\beta : \ \alpha \leq \beta < \lambda\}$ such that $i_{\alpha\beta}$ witnesses that $\mathbb{B}_\beta \leq_\Gamma \mathbb{B}_\alpha$ for all $\alpha \leq \beta < \lambda$.

Definition 3.3.1. Let T be a theory extending ZFC, Γ a definable class of complete boolean algebras, $\Sigma : \Gamma^{\lim} \to \Gamma^{\lim}$ a definable class function, κ a definable cardinal. We say that an iteration system $\mathcal{F} \in \Gamma^{\lim}$ of length λ *follows* Σ if and only if for all β even[3], $\mathcal{F}\restriction(\beta+1) = \Sigma(\mathcal{F}\restriction\beta)$.

We say that Σ is a *weak iteration strategy* for Γ if and only if we can prove in T that for every $\mathcal{F}$ of length λ which follows Σ, $\Sigma(\mathcal{F})$ has length $\lambda+1$ and $\mathcal{F} = \Sigma(\mathcal{F})\restriction\lambda$.

We say that Σ is a *κ-weak iteration strategy* for Γ if in addition $\Sigma(\mathcal{F}) = \varinjlim \mathcal{F}$ whenever $\operatorname{cof}(\lambda) = \kappa$ or $\operatorname{cof}(\lambda) > \kappa$, $\operatorname{cof}(\lambda) > |\mathbb{B}|$ for all $\mathbb{B}$ in $\mathcal{F}$.

Definition 3.3.2. Let $\mathcal{B}$ be a collection of complete boolean algebras. We denote as $\prod \mathcal{B}$ the *lottery sum* of the algebras in $\mathcal{B}$, defined as the boolean algebra obtained by the cartesian product of the respective boolean algebras with pointwise operations.

The name *lottery sum* is justified by the intuition that forcing with $\prod \mathcal{B}$ corresponds with forcing with a "random" algebra in $\mathcal{B}$. In fact, since the set of $p \in \prod \mathcal{B}$ that are $\mathbf{1}$ in one component and $\mathbf{0}$ in all the others form a maximal antichain, every V-generic filter G for $\prod \mathcal{B}$ concentrates only on a specific $\mathbb{B} \in \mathcal{B}$ (determined by the generic).

Definition 3.3.3. Let T be a theory extending ZFC by a finite number of axioms, Γ a definable class of complete boolean algebras, $\Sigma : \Gamma^{\lim} \to \Gamma^{\lim}$ a definable class function, κ a definable cardinal.

We say that Γ is *κ-weakly iterable* through Σ iff we can prove in T that:

- Γ is closed under two-step iterations and set-sized lottery sums;
- Σ is a κ-weak iteration strategy for Γ;
- $\langle \Gamma, \Sigma \rangle$ as computed in V_α is equal to $\langle \Gamma \cap V_\alpha, \Sigma \cap V_\alpha \rangle$ whenever α is inaccessible and $V_\alpha \models T$[4].

We say that Γ is weakly iterable iff it is κ-weakly iterable for some κ.

We highlight that the latter definition (for a $T \supseteq$ ZFC) is not related to a specific model V of T, and requires that the above properties are provable in T hence hold for *every* T-model M: for example, if $T =$ ZFC they must hold in every V_κ where κ is inaccessible. We shall feel free to omit the reference to T when clear from the context, and in particular when $T =$ ZFC.

Many notable cases for Γ are ω_i-weakly iterable for some $i = 0, 1$: i.e, $\Gamma =$ all, locally ccc (using finite supports iteration strategy), σ-closed, axiom-A, proper (using countable supports iteration strategy). Furthermore, locally $<\kappa^+$-cc and $<\kappa$-closed are κ-weakly iterable (using $<\kappa$-sized supports iteration strategy).

In Section 3.4 we shall prove that semiproper forcings are ω_1-weakly iterable in ZFC and stationary set preserving forcings are ω_1-weakly iterable in the theory $T =$ ZFC + *"there exists a proper class of supercompact cardinals"*, thus concluding that most notable cases for Γ are weakly iterable. We leave all the other cases mentioned above to the interested reader (see [7] for further details), since the corresponding

[3] We remark that every limit ordinal is even.

[4] Note that since α is inaccessible, this statement is equivalent to $V_\alpha \models T \setminus \text{ZFC}$ which is finite.

proofs are either simpler or given by straightforward modifications of the arguments presented in Section 3.4. Contrary to the cases mentioned above, the strategies Σ involved in the iteration of SP and SSP forcings will have to make careful choices for $\Sigma(\mathcal{F})$ also when $\mathcal{F}$ has length α even successor ordinal.

The definition of weak iterability for a definable class of forcing notions Γ seem rather technical, and gets its motivation in providing the right conditions to carry out the *lottery iteration* $\mathbb{P}_\kappa^{\Gamma,f}$ with respect to a partial function $f : \kappa \to \kappa$ where κ is an inaccessible cardinal. The lottery iteration has been studied extensively by Hamkins [22] and is one of the main tools for proving the consistency of forcing axioms. We will employ these type of iterations in Section 3.4.3 and in Chapter 4.

Definition 3.3.4. Let Γ be weakly iterable through Σ and $f : \kappa \to \kappa$ be a partial function. Define $\mathcal{F}_\xi = \left\{ i_{\alpha\beta} : \mathbb{P}_\alpha^{\Gamma,f} \to \mathbb{P}_\beta^{\Gamma,f} : \alpha \leq \beta < \xi \right\}$ by recursion on $\xi \leq \kappa + 1$ as:

1. $\mathcal{F}_0 = \emptyset$ is the empty iteration system;
2. $\mathcal{F}_{\xi+1} = \Sigma(\mathcal{F}_\xi)$ if ξ is even;
3. $\mathcal{F}_{\xi+2}$ has $\mathbb{P}_{\xi+1}^{\Gamma,f} = \mathbb{P}_\xi^{\Gamma,f}$ if $\xi + 1$ is odd and $f(\xi)$ is undefined;
4. $\mathcal{F}_{\xi+2}$ has $\mathbb{P}_{\xi+1}^{\Gamma,f} = \mathbb{P}_\xi^{\Gamma,f} * \dot{\mathbb{C}}$ otherwise, where $\dot{\mathbb{C}}$ is a $\mathbb{P}_\xi^{\Gamma,f}$-name for the lottery sum (as computed in $V^{\mathbb{P}_\xi^{\Gamma,f}}$) of all complete boolean algebras in Γ of rank less than $f(\xi)$, i.e. a $\mathbb{P}_\xi^{\Gamma,f}$-name for $\prod \left(\Gamma \cap V_{f(\xi)}\right)$.

We say that $\mathbb{P}_\kappa^{\Gamma,f}$ is the lottery iteration of Γ relative to f.

Proposition 3.3.5. *Let $T \supseteq$ ZFC be a theory, Γ be γ-weakly iterable through Σ, $f : \kappa \to \kappa$ be a partial function with κ inaccessible cardinal such that $V_\kappa \models T$. Then:*

1. *$\mathbb{P}_\kappa^{\Gamma,f}$ exists and is in Γ;*
2. *$\mathbb{P}_\kappa^{\Gamma,f}$ is $<\kappa$-cc and for all $\alpha < \kappa$, $\mathbf{1} \Vdash_{\mathbb{P}_\kappa^{\Gamma,f}} 2^{\check{\alpha}} \leq \check{\kappa}$;*
3. *$\mathbb{P}_\kappa^{\Gamma,f}$ is definable within V_κ using the class parameter f;*
4. *Let $g : \lambda \to \lambda$ with λ inaccessible be such that $f = g \restriction \kappa$, $V_\lambda \models T$. Then $\mathbb{P}_\lambda^{\Gamma,g}$ absorbs every forcing in $\Gamma \cap V_{g(\kappa)}$ as computed in $V^{\mathbb{P}_\kappa^{\Gamma,f}}$. That is, for every $\dot{\mathbb{B}}$ in $(\Gamma \cap V_{g(\kappa)})^{V^{\mathbb{P}_\kappa^{\Gamma,f}}}$, there is a condition $p \in \mathbb{P}_\lambda^{\Gamma,g}$ such that $\mathbb{P}_\lambda^{\Gamma,g} \restriction p \leq_\Gamma \mathbb{P}_\kappa^{\Gamma,f} * \dot{\mathbb{B}}$.*

Proof. 1. Follows from Σ being a weak iteration strategy for $\alpha < \kappa$ even, and from Γ closed under two-step iterations and lottery sums for α odd.

2. Since $\Sigma \cap V_\kappa$ is Σ as computed in V_κ, we can prove by induction on $\alpha < \kappa$ that $|\mathbb{B}_\alpha| < \kappa$. Furthermore, $\mathbb{P}_\kappa^{\Gamma,f} = \Sigma(\mathcal{F}_\kappa) = \varinjlim \mathcal{F}_\kappa$ since $\kappa = \operatorname{cof}(\kappa) > \gamma, |\mathbb{B}_\alpha|$ for all $\alpha < \kappa$ and Σ is a γ-weak iteration strategy, and the set

$$S = \left\{ \alpha < \kappa : \ \mathbb{P}_\alpha^{\Gamma,f} = \varinjlim \mathcal{F} \restriction \alpha \right\} \supseteq \{\alpha < \kappa : \ \operatorname{cof}(\alpha) = \gamma\}$$

is stationary. Thus by Theorem 2.2.10, $\mathbb{P}_\kappa^{\Gamma,f}$ is $<\kappa$-cc.

For the second part, given $\alpha < \kappa$ let $\dot{x}$ be a $\mathbb{P}^{\Gamma,f}_\kappa$-name for a subset of α. Then $\dot{x}$ is decided by $\alpha < \kappa$ antichains of size $<\kappa$, hence $\dot{x} = \hat{\imath}_\beta(\dot{y})$ for some $\dot{y} \in V^{\mathbb{P}^{\Gamma,f}_\beta}$, $\beta < \kappa$. Since $\left|\mathbb{P}^{\Gamma,f}_\beta\right| < \kappa$ and κ is inaccessible, there are less than κ-many names for subsets of α in $V^{\mathbb{P}^{\Gamma,f}_\beta}$. Thus there are at most κ-many names for subsets of α in $V^{\mathbb{P}^{\Gamma,f}_\kappa}$.

3. Follows since $\langle \Gamma \cap V_\kappa, \Sigma \cap V_\kappa \rangle$ is $\langle \Gamma, \Sigma \rangle$ as computed in V_κ.

4. Let $g : \lambda \to \lambda$ with λ inaccessible be such that $f = g \restriction \kappa$, $V_\lambda \models T$. Since $\langle \Gamma \cap V_\kappa, \Sigma \cap V_\kappa \rangle$ is $\langle \Gamma, \Sigma \rangle$ as computed in V_κ and the same holds for V_λ, $\mathbb{P}^{\Gamma,f}_\kappa$ is an initial part of $\mathbb{P}^{\Gamma,g}_\lambda$. Furthermore given any $\dot{\mathbb{B}}$ in $(\Gamma \cap V_{g(\kappa)})^{V^{\mathbb{P}^{\Gamma,f}_\kappa}}$, $\dot{\mathbb{B}}$ is a factor of $\dot{\mathbb{C}} = \prod \left(\Gamma \cap V_{f(\alpha)}\right)^{V^{\mathbb{P}^{\Gamma,f}_\kappa}}$ as defined in Definition 3.3.4.(4). Let $\dot{q} \in \dot{\mathbb{C}}$ be the condition choosing $\dot{\mathbb{B}}$ in the lottery sum (so that $\dot{q}$ is $\mathbf{1}$ in the component corresponding to $\dot{\mathbb{B}}$ and $\mathbf{0}$ in all the others), and let $p \in \mathbb{P}^{\Gamma,g}_\lambda$ be $p = i_\lambda([\dot{q}])$. Thus,
$$\mathbb{P}^{\Gamma,f}_\kappa * \dot{\mathbb{B}} \;=\; \mathbb{P}^{\Gamma,f}_\kappa * \dot{\mathbb{C}} \restriction \dot{q} \;=\; \mathbb{P}^{\Gamma,g}_{\kappa+1} \restriction [\dot{q}] \;\geq_\Gamma\; \mathbb{P}^{\Gamma,g}_\lambda \restriction p. \qquad \square$$

3.4 Semiproper and SSP iterations

We now prove that semiproper forcings are ω_1-weakly iterable and SSP forcings are ω_1-weakly iterable in the theory $T = \mathrm{ZFC} +$ *"there exists a proper class of supercompact cardinals"*. The treatment of semiproper forcings will be split in the two-step and limit cases, then we shall explore the SSP case building on the results previously obtained.

3.4.1 Semiproper two-step iterations

We recall here the algebraic definition of semiproperness, stated in a way that is more convenient to us.

Definition 3.4.1. Let $\mathbb{B}$ be a complete boolean algebra, S be a stationary set on H_θ with $\theta > 2^{|\mathbb{B}|}$. Then $\mathbb{B}$ is S-SP iff for club many $M \in S$, $\mathrm{sg}(\mathbb{B}, M) \wedge p > \mathbf{0}$ for all p in $\mathbb{B} \cap M$.

Similarly, $i : \mathbb{B} \to \mathbb{C}$ is S-SP iff $\mathbb{B}$ is S-SP and for club many $M \in S$,
$$\pi(q \wedge \mathrm{sg}(\mathbb{C}, M)) = \pi(q) \wedge \mathrm{sg}(\mathbb{B}, M).$$
for all q in $\mathbb{C} \cap M$.

Finally, an iteration system $\mathcal{F} = \{i_{\alpha\beta} : \alpha \leq \beta < \lambda\}$ is S-SP iff $i_{\alpha\beta}$ is S-SP for all $\alpha \leq \beta < \lambda$.

The previous definitions can be reformulated with a well-known trick in the following form.

Proposition 3.4.2. $\mathbb{B}$ *is* S-SP *iff for every* $\delta \gg \theta$ *regular,* $M \prec H_\delta$ *with* $\mathbb{B}, S \in M$ *and* $M \cap H_\theta \in S$ *we have that* $\mathrm{sg}(\mathbb{B}, M) \wedge p > \mathbf{0}$ *for every* $p \in \mathbb{B} \cap M$.

Similarly, $i : \mathbb{B} \to \mathbb{C}$ is S-SP iff $\mathbb{B}$ is S-SP and for every $\delta \gg \theta$ regular, $M \prec H_\delta$ with $i, S \in M$ and $M \cap H_\theta \in S$ we have that

$$\pi(q \wedge \text{sg}(\mathbb{C}, M)) = \pi(q) \wedge \text{sg}(\mathbb{B}, M).$$

for every $q \in \mathbb{C} \cap M$.

Proof. First, suppose that $\mathbb{B}$, $i : \mathbb{B} \to \mathbb{C}$ satisfy the above conditions. Then $C = \{M \cap H_\theta : M \prec H_\delta, \mathbb{B}, S \in M\}$ contains a club (since it is the projection of a club), and witnesses that $\mathbb{B}$, $i : \mathbb{B} \to \mathbb{C}$ are S-SP.

Conversely, suppose that $\mathbb{B}$, $i : \mathbb{B} \to \mathbb{C}$ are S-SP and fix $\delta \gg \theta$ regular and $M \prec H_\delta$ with $\mathbb{B}, S \in M$, $M \cap H_\theta \in S$. Since the sentence that $\mathbb{B}$, $i : \mathbb{B} \to \mathbb{C}$ are S-SP is entirely computable in H_δ and $M \prec H_\delta$, there exists a club $C \in M$ witnessing that $\mathbb{B}$, $i : \mathbb{B} \to \mathbb{C}$ are S-SP. Furthermore, M models that C is a club hence $M \cap H_\theta \in C$ and $\text{sg}(\mathbb{B}, M) \wedge p > \mathbf{0}$, $\pi(q \wedge \text{sg}(\mathbb{C}, M)) = \pi(q) \wedge \text{sg}(\mathbb{B}, M)$ hold for any $p \in \mathbb{B} \cap M$, $q \in \mathbb{C} \cap M$ since C witnesses that $\mathbb{B}$, $i : \mathbb{B} \to \mathbb{C}$ are S-SP and $M \cap H_\theta \in S \cap C$. □

Observe that if $i : \mathbb{B} \to \mathbb{C}$ is S-SP, then $\mathbb{C}$ is S-SP. As a matter of fact $q \in \mathbb{C} \cap M$ is such that $\text{sg}(\mathbb{C}, M) \wedge q = \mathbf{0}$ if and only if

$$\mathbf{0} = \pi(q \wedge \text{sg}(\mathbb{C}, M)) = \pi(q) \wedge \text{sg}(\mathbb{B}, M),$$

which contradicts the assumption that $\mathbb{B}$ is S-SP since $\pi(q) \in \mathbb{B} \cap M$.

The notion of being S-SP can change when we move to a generic extension: for example, S can be no longer stationary. In order to recover the “stationarity” in $V[G]$ of an S which is stationary in V, we are led to the following definition:

Definition 3.4.3. Let S be a subset of $\mathcal{P}(H_\theta)$, $\mathbb{B} \in H_\theta$ be a complete boolean algebra, G be a V-generic filter for $\mathbb{B}$. We define

$$S(G) = \{M[G] : \mathbb{B} \in M \in S\}.$$

Proposition 3.4.4. *Let S be a stationary set on H_θ, $\mathbb{B} \in H_\theta$ be a complete boolean algebra, G be a V-generic filter for $\mathbb{B}$. Then $S(G)$ is stationary in $V[G]$.*

Proof. Let $\dot{C} \in V^{\mathbb{B}}$ be a name for a club C on $\mathcal{P}(H_\theta)$, and let $M \prec H_{\theta^+}$ be such that $M \cap H_\theta \in S$, $\mathbb{B}, \dot{C} \in M$. Then $C \in M[G]$ and $M[G]$ models that C is a club, hence $M[G] \cap H_\theta \in C$. Furthermore, $M[G] \cap H_\theta = (M \cap H_\theta)[G]$ since $\mathbb{B}$ is $<\theta$-cc and $M[G] \cap H_\theta \in S(G) \cap C$. □

Proposition 3.4.5. *Let $\mathbb{B}$ be a S-SP complete boolean algebra, $\dot{\mathbb{C}}$ be such that*

$$\left[\!\left[\dot{\mathbb{C}} \text{ is } S(\dot{G})\text{-SP}\right]\!\right]_{\mathbb{B}} = \mathbf{1},$$

then $\mathbb{D} = \mathbb{B} * \dot{\mathbb{C}}$ and $i_{\mathbb{B} * \dot{\mathbb{C}}}$ are S-SP.

Proof. First, we verify that $i = i_{\mathbb{B} * \dot{\mathbb{C}}}$ is S-SP. Let $\dot{C}_1$ be the club that witnesses $\left[\!\left[\dot{\mathbb{C}} \text{ is } S(\dot{G})\text{-SP}\right]\!\right]_{\mathbb{B}} = \mathbf{1}$, and let M be such that $\dot{C}_1 \in M$. This guarantees that $V[G] \models M[G] \cap H_\theta^{V[G]} \in \text{val}(C_1, G)$.

First, we prove that $\pi(\mathrm{sg}(\mathbb{D}, M)) = \mathrm{sg}(\mathbb{B}, M)$. By Lemma 2.1.5, $\mathrm{sg}(\mathbb{D}, M) \leq i(\mathrm{sg}(\mathbb{B}, M))$ hence $\pi(\mathrm{sg}(\mathbb{D}, M)) \leq (\mathrm{sg}(\mathbb{B}, M))$. In order to prove the opposite inequality, suppose that G is a V-generic filter for $\mathbb{B}$ such that $\mathrm{sg}(\mathbb{B}, M) \in G$. By semiproperness of $\mathbb{B}$, $V[G] \vDash M \cap \omega_1 = M[G] \cap \omega_1$ and by Lemma 2.3.3, $V[G] \vDash [\mathrm{sg}(\mathbb{D}, M)]_{i[G]} = \mathrm{sg}(\mathbb{C}, M[G])$. If follows that

$$\left[\!\left[[\mathrm{sg}(\mathbb{D}, M)]_{i[\dot{G}]} = \mathrm{sg}(\dot{\mathbb{C}}, M[\dot{G}]) \right]\!\right]_{\mathbb{B}} \geq \mathrm{sg}(\mathbb{B}, M)$$

and

$$\mathrm{sg}(\mathbb{B}, M) \wedge \left[\!\left[[sg(\mathbb{D}, M)]_{i[\dot{G}]} > \dot{\mathbf{0}} \right]\!\right]_{\mathbb{B}} = \mathrm{sg}(\mathbb{B}, M) \wedge \left[\!\left[\mathrm{sg}(\dot{\mathbb{C}}, M[\dot{G}]) > \dot{\mathbf{0}} \right]\!\right]_{\mathbb{B}} = \mathrm{sg}(\mathbb{B}, M)$$

by semiproperness of $\dot{\mathbb{C}}$. Thus,

$$\pi(\mathrm{sg}(\mathbb{D}, M)) = \left[\!\left[[\mathrm{sg}(\mathbb{D}, M)]_{i[\dot{G}]} > \dot{\mathbf{0}} \right]\!\right]_{\mathbb{B}} \geq \mathrm{sg}(\mathbb{B}, M).$$

Finally, by Lemma 3.1.10 and 2.1.7 we can repeat the proof for $\mathbb{D}\restriction[\dot{p}]_{\approx} = \mathbb{B}\restriction\pi([\dot{p}]_{\approx}) * \dot{\mathbb{C}}\restriction\dot{p}$ (that is a two-step iteration of S-SP boolean algebras) and obtain that

$$\pi(\mathrm{sg}(\mathbb{D}, M) \wedge [\dot{p}]_{\approx}) = \mathrm{sg}(\mathbb{B}, M) \wedge \pi([\dot{p}]_{\approx})$$

hence i is S-SP. Since $\mathbb{B}$ is S-SP, it follows that $\mathbb{D}$ is also S-SP. □

Lemma 3.4.6. *Let $\mathbb{B}$, $\mathbb{C}_0$, $\mathbb{C}_1$ be S-SP complete boolean algebras, and let G be any V-generic filter for $\mathbb{B}$. Let i_0, i_1, k form the following commutative diagram of regular embeddings:*

$$\begin{array}{ccc} \mathbb{B} & \xrightarrow{i_0} & \mathbb{C}_0 \\ & {\scriptstyle i_1}\searrow & \downarrow{\scriptstyle k} \\ & & \mathbb{C}_1 \end{array}$$

Moreover assume that $\mathbb{C}_0/_{i_0[G]}$ is $S(G)$-SP and

$$\left[\!\left[\mathbb{C}_1/_{k[\dot{G}_{\mathbb{C}_0}]} \text{ is } S(\dot{G}_{\mathbb{C}_0})\text{-SP} \right]\!\right]_{\mathbb{C}_0} = \mathbf{1}.$$

Then in $V[G]$, $k/_G : \mathbb{C}_0/_G \to \mathbb{C}_1/_G$ is an $S(G)$-SP embedding.

Proof. Let K be any $V[G]$-generic filter for $\mathbb{C}_0/_{i_0[G]}$, and let

$$H = \{p \in \mathbb{C}_0 : \ [p]_{i_0[G]} \in K\}$$

so that

$$K = H/_{i_0[G]} = \{[p]_{i_0[G]} : \ p \in H\}.$$

Then H is V-generic for $\mathbb{C}_0$, $V[H] = V[G][K]$, $S(H) = S(G)(K)$. Furthermore, the latter equalities hold for any choice of K thus in $V[G]$ the map $k/_G : \mathbb{C}_0/_G \to \mathbb{C}_1/_G$ is such that

$$\left[\!\left[(\mathbb{C}_1/_G)/_{k/_G[\dot{G}_{\mathbb{C}_0/_G}]} \text{ is } S(G)(\dot{G}_{\mathbb{C}_0/_G})\text{- SP} \right]\!\right]_{\mathbb{C}_0/_G} = \mathbf{1}.$$

By Proposition 3.4.5, $k/_G$ is $S(G)$-SP in $V[G]$. □

3.4.2 Semiproper iteration systems

The limit case needs a slightly different approach depending on the length of the iteration. We shall start with some general lemmas, then we will proceed to examine the different cases one by one.

Proposition 3.4.7. *Let $\mathcal{F} = \{i_{\alpha\beta} : \alpha \leq \beta < \lambda\}$ be an S-SP iteration system, s be in $\varprojlim \mathcal{F}$. Then*

$$\mathcal{F}\restriction s = \{(i_{\alpha\beta})_{s(\beta)} : \mathbb{B}_\alpha\restriction s(\alpha) \to \mathbb{B}_\beta\restriction s(\beta) : \alpha \leq \beta < \lambda\}$$

is an S-SP iteration system and its associated retractions are the restriction of the original retractions.

Proof. By Lemma 2.1.7, $\mathcal{F}\restriction s$ is indeed an iteration system and its associated retractions are the restriction of the original retractions. By Proposition 3.1.10, $\mathrm{sg}(\mathbb{B}_\alpha\restriction s(\alpha)) = \mathrm{sg}(\mathbb{B}_\alpha) \wedge s(\alpha)$ for all $\alpha < \lambda$ thus for any $M \in S$ with $\mathcal{F}, s \in M$,

$$\pi_{\alpha\beta}(\mathrm{sg}(\mathbb{B}_\beta\restriction s(\beta)) = \pi_{\alpha\beta}(\mathrm{sg}(\mathbb{B}_\beta) \wedge s(\beta)) = \mathrm{sg}(\mathbb{B}_\alpha) \wedge s(\alpha) = \mathrm{sg}(\mathbb{B}_\alpha\restriction s(\alpha))$$

for all $\alpha, \beta \in M$, since in this case also $s(\alpha)$, $s(\beta)$ are in M and $i_{\alpha\beta}$ is S-SP. □

Lemma 3.4.8. *Let $\mathcal{F} = \{i_{\alpha\beta} : \mathbb{B}_\alpha \to \mathbb{B}_\beta : \alpha \leq \beta < \lambda\}$ be an RCS and S-SP iteration system with S stationary on $[H_\theta]^\omega$. Let M be in S, $t \in M$ be any condition in $\lim_{\mathsf{rcs}} \mathcal{F}$, $\dot\alpha \in M$ be a $\lim_{\mathsf{rcs}} \mathcal{F}$-name for a countable ordinal, $\zeta \in M$ be an ordinal smaller than λ. Then:*

- *if $\lim_{\mathsf{rcs}} \mathcal{F} = \varprojlim \mathcal{F}$, there exists a condition $t' \in \lim_{\mathsf{rcs}} \mathcal{F} \cap M$ below t with $t'(\zeta) = t(\zeta)$ such that $t' \wedge i_\zeta(\mathrm{sg}(\mathbb{B}_\zeta, M))$ forces that $\dot\alpha < M \cap \omega_1$;*
- *if $\lambda = \omega_1$, there exists a condition $t' \in \mathrm{RO}(\lim_{\mathsf{rcs}} \mathcal{F}) \cap M$ as above and such that $t' \wedge i_\zeta(\mathrm{sg}(\mathbb{B}_\zeta, M))$ is in $\varinjlim \mathcal{F}$ and has support contained in $M \cap \omega_1$.*

Proof. Let $D \in M$ be the set of conditions in $\lim_{\mathsf{rcs}} \mathcal{F}$ deciding the value of $\dot\alpha$ (D is open dense by the forcing theorem):

$$D = \{s \in \lim_{\mathsf{rcs}} \mathcal{F} : \exists \beta < \omega_1 \ s \Vdash_{\lim_{\mathsf{rcs}} \mathcal{F}} \dot\alpha = \check\beta\}.$$

Consider the set $\pi_\zeta[D\restriction t]$ (which is open dense below $t(\zeta)$ by Lemma 2.1.5) and fix $A \subseteq \mathbb{B}_\zeta$ a maximal antichain in M contained in it, so that $\bigvee A = t(\zeta)$. Let $f : A \to D\restriction t$ be a map in M such that $\pi_\zeta(f(p)) = p$ for every $p \in A$, and define $t' \in \varprojlim \mathcal{F} \cap M$ by $t' = \tilde{\bigvee} f[A]$. Observe that $t'(\zeta) = t(\zeta)$ by definition of pointwise supremum and $t' \leq t$ since $\tilde{\bigvee} f[A]$ is really the supremum of $f[A]$ in $\mathrm{RO}(\varprojlim \mathcal{F})$ by Lemma 2.2.8.

Furthermore, if $\lim_{\mathsf{rcs}} \mathcal{F} = \varprojlim \mathcal{F}$, t' is in $\lim_{\mathsf{rcs}} \mathcal{F}$ as required. If instead $\lambda = \omega_1$, $\lim_{\mathsf{rcs}} \mathcal{F} = \varinjlim \mathcal{F}$ and $t' \in \mathrm{RO}(\varinjlim \mathcal{F})$ as characterized in Proposition 2.2.7: in both cases, $t' \in \mathrm{RO}(\lim_{\mathsf{rcs}} \mathcal{F})$ is $\bigvee f[A]$. Then we can define a name[5] $\dot\beta \in V^{\mathbb{B}_\zeta} \cap M$ as:

$$\dot\beta = \left\{\langle \check\xi, p\rangle : \ p \in A, \ f(p) \Vdash_{\lim_{\mathsf{rcs}} \mathcal{F}} \dot\alpha > \check\xi\right\}$$

[5]Literally speaking this is not a $\mathbb{B}_\zeta$-name according to our definition. See the footnote below 1.3.2 to resolve this ambiguity.

so that for any $p \in A$, $p \Vdash \dot{\beta} = \check{\xi}$ iff $f(p) \Vdash \dot{\alpha} = \check{\xi}$. It follows that

$$\left[\!\left[\hat{\imath}_\zeta(\dot{\beta}) = \dot{\alpha}\right]\!\right]_{\lim_{\mathsf{rcs}} \mathcal{F}} \geq \bigvee f[A] = t'.$$

Moreover, $\mathrm{sg}(\mathbb{B}_\zeta, M) \leq \left[\!\left[\dot{\beta} < M \check{\cap} \omega_1\right]\!\right]_{\mathbb{B}_\zeta}$ and is compatible with $t'(\zeta) \in M$ (since $\mathbb{B}_\zeta$ is S-SP), so that $[\![\dot{\alpha} < M \check{\cap} \omega_1]\!]_{\lim_{\mathsf{rcs}} \mathcal{F}} \geq t' \wedge i_\zeta(\mathrm{sg}(\mathbb{B}_\zeta, M))$. If $\lim_{\mathsf{rcs}} \mathcal{F} = \varprojlim \mathcal{F}$, this concludes the proof.

If $\lambda = \omega_1$, we still need to prove that the support of $t' \wedge i_\zeta(\mathrm{sg}(\mathbb{B}_\zeta, M))$ is contained in $M \cap \omega_1$. Define a name $\dot{\eta} \in V^{\mathbb{B}_\zeta} \cap M$ for a countable ordinal setting:

$$\dot{\eta} = \left\{ \langle \check{\xi}, p \rangle : \ p \in A, \ \xi < \mathrm{supp}(f(p)) \right\}.$$

Notice that $\dot{\eta}$ is defined in such a way that for all $\beta < \omega_1$,

$$[\![\dot{\eta} = \check{\beta}]\!]_{\mathbb{B}_\zeta} = \bigvee \left\{ p \in A : \ \mathrm{supp}(f(p)) = \check{\beta} \right\}.$$

hence

$$\begin{aligned} i_\zeta([\![\dot{\eta} < \check{\beta}]\!]_{\mathbb{B}_\zeta}) \wedge t' &= i_\zeta(\bigvee \{p \in A : \ \mathrm{supp}(f(p)) < \beta\}) \wedge \bigvee \{f(p) : \ p \in A\} \\ &= \tilde{\bigvee} \{f(p) : \ p \in A \wedge \mathrm{supp}(f(p)) < \beta\}. \end{aligned}$$

Since $t' \wedge i_\zeta(\mathrm{sg}(\mathbb{B}_\zeta, M)) = \tilde{\bigvee} \{f(p) \wedge i_\zeta(\mathrm{sg}(\mathbb{B}_\zeta, M)) : \ p \in A\}$ and $\mathrm{sg}(\mathbb{B}_\zeta, M) \leq [\![\dot{\eta} < M \check{\cap} \omega_1]\!]_{\mathbb{B}_\zeta}$, we get that

$$\begin{aligned} t' \wedge i_\zeta(\mathrm{sg}(\mathbb{B}_\zeta, M)) &= t' \wedge i_\zeta(\mathrm{sg}(\mathbb{B}_\zeta, M)) \wedge i_\zeta([\![\dot{\eta} < M \cap \omega_1]\!]_{\mathbb{B}_\zeta}) \\ &= \tilde{\bigvee} \{f(p) : \ p \in A \wedge \mathrm{supp}(f(p)) < M \cap \omega_1\} \wedge i_\zeta(\mathrm{sg}(\mathbb{B}_\zeta, M)). \end{aligned}$$

It is now immediate to check that $t'' = \tilde{\bigvee} \{f(p) : \ p \in A \wedge \mathrm{supp}(f(p)) < M \cap \omega_1\}$ has support contained in $M \cap \omega_1$ as required. □

Lemma 3.4.9. *Let $\mathcal{F} = \{i_{nm} : n \leq m < \omega\}$ be an S-SP iteration system with S stationary on $[H_\theta]^\omega$. Then $\varprojlim \mathcal{F}$ and the corresponding $i_{n\omega}$ are S-SP.*

Proof. By Proposition 3.4.2, any countable $M \prec H_\delta$ with $\delta > \theta$, $M \cap H_\theta \in S$ and $\mathcal{F}, S \in M$ witnesses the semiproperness of every i_{nm}. We need to show that for every $s \in \varprojlim \mathcal{F} \cap M$ and $n < \omega$,

$$\pi_{n\omega}(\mathrm{sg}(\mathrm{RO}(\varprojlim \mathcal{F}), M) \wedge s) = \mathrm{sg}(\mathbb{B}_n, M) \wedge s(n)$$

this would also imply that $\mathrm{RO}(\varprojlim \mathcal{F})$is S-SP by the same reasoning of the proof of Lemma 3.4.5. Without loss of generality, we can assume that $n = 0$ and by Lemma 3.1.10 and 3.4.7 we can also assume that $s = \mathbf{1}$. Thus is sufficient to prove that

$$\pi_{0\omega}(\mathrm{sg}(\mathrm{RO}(\varprojlim \mathcal{F}), M)) = \mathrm{sg}(\mathbb{B}_0, M)$$

Let $\{\dot\alpha_n : n \in \omega\}$ be an enumeration of the $\varprojlim \mathcal{F}$-names in M for countable ordinals. Let $t_0 = \mathbf{1}$, t_{n+1} be obtained from t_n, $\dot\alpha_n$, n as in Lemma 3.4.8, so that

$$[\![\dot\alpha_n < M \check\cap \omega_1]\!]_{\varprojlim \mathcal{F}} \geq t_{n+1} \wedge i_n(\mathrm{sg}(\mathbb{B}_n, M))$$

Consider now the sequence $\bar t(n) = t_n(n) \wedge \mathrm{sg}(\mathbb{B}_n, M)$. This sequence is a thread since $i_{n,n+1}$ is S-SP and $t_n(n) \in M$ for every n, hence

$$\pi_{n,n+1}(\mathrm{sg}(\mathbb{B}_{n+1}, M) \wedge t_{n+1}(n+1)) = \mathrm{sg}(\mathbb{B}_n, M) \wedge \pi_{n,n+1}(t_{n+1}(n+1))$$

and $\pi_{n,n+1}(t_{n+1}(n+1)) = t_{n+1}(n) = t_n(n)$ by Lemma 3.4.8. Furthermore, for every $n \in \omega$, $\bar t \leq t_n$ since the sequence t_n is decreasing, and $\bar t \leq i_n(\mathrm{sg}(\mathbb{B}_n, M))$ since $\bar t(n) \leq \mathrm{sg}(\mathbb{B}_n, M)$. It follows that $\bar t$ forces that $\dot\alpha_n < M \check\cap \omega_1$ for every n, thus $\bar t \leq \mathrm{sg}(\mathrm{RO}(\varprojlim \mathcal{F}), M)$ by Lemma 3.1.12. Then,

$$\pi_0(\mathrm{sg}(\mathrm{RO}(\varprojlim \mathcal{F}), M)) \geq \bar t(0) = t_0(0) \wedge \mathrm{sg}(\mathbb{B}_0, M) = \mathrm{sg}(\mathbb{B}_0, M)$$

and the opposite inequality is trivial, completing the proof. □

Lemma 3.4.10. *Let $\mathcal{F} = \{i_{\alpha\beta} : \mathbb{B}_\alpha \to \mathbb{B}_\beta : \alpha \leq \beta < \omega_1\}$ be an* RCS *and S-*SP *iteration system with S stationary on $[H_\theta]^\omega$. Then $\varinjlim \mathcal{F}$ and the corresponding $i_{\alpha\omega_1}$ are S-*SP.

Proof. The proof follows the same pattern of the previous Lemma 3.4.9. By Proposition 3.4.2, any countable $M \prec H_\delta$ with $\delta > \theta$, $\mathcal{F}, S \in M$, $M \cap H_\theta \in S$, witnesses the semiproperness of every $i_{\alpha\beta}$ with $\alpha, \beta \in M \cap \omega_1$. As before, by Lemma 3.1.10 and 3.4.7 we only need to show that

$$\pi_0(\mathrm{sg}(\mathrm{RO}(\varinjlim \mathcal{F}), M)) \geq \mathrm{sg}(\mathbb{B}_0, M),$$

the other inequality being trivial. Let $\langle \zeta_n : n \in \omega \rangle$ be an increasing sequence of ordinals such that $\zeta_0 = 0$ and $\sup_n \zeta_n = \zeta = M \cap \omega_1$, and $\{\dot\alpha_n : n \in \omega\}$ be an enumeration of the $\varinjlim \mathcal{F}$-names in M for countable ordinals. Let $t_0 = \mathbf{1}$, t_{n+1} be obtained from t_n, $\dot\alpha_n$, ζ_n as in Lemma 3.4.8, so that

$$[\![\dot\alpha_n < M \check\cap \omega_1]\!]_{\varprojlim \mathcal{F}} \geq t_{n+1} \wedge i_{\zeta_n}(\mathrm{sg}(\mathbb{B}_{\zeta_n}, M)).$$

Consider now the sequence $\bar t(\zeta_n) = t_n(\zeta_n) \wedge \mathrm{sg}(\mathbb{B}_{\zeta_n}, M)$. This sequence induces a thread on $\mathcal{F}\restriction\zeta$, so that $\bar t \in \mathbb{B}_\zeta$ since $\mathcal{F}$ is an RCS iteration, ζ has countable cofinality and thus we can naturally identify $\varprojlim \mathcal{F}\restriction\zeta$ as a dense subset of $\mathbb{B}_\zeta$. Moreover we can also check that $i_\zeta(\bar t)$ is a thread in $\varinjlim \mathcal{F}$ with support ζ such that $i_\zeta(\bar t)(\alpha) = \bar t(\alpha)$ for all $\alpha < \zeta$. Since by Lemma 3.4.8,

$$\mathrm{supp}(t_{n+1} \wedge i_{\zeta_n}(\mathrm{sg}(\mathbb{B}_{\zeta_n}, M))) \leq \zeta,$$

the relation $i_\zeta(\bar t) \leq t_{n+1} \wedge i_{\zeta_n}(\mathrm{sg}(\mathbb{B}_{\zeta_n}, M))$ holds pointwise hence $i_\zeta(\bar t)$ forces that $\dot\alpha_n < M \check\cap \omega_1$ for every n. Thus, $i_\zeta(\bar t) \leq \mathrm{sg}(\mathrm{RO}(\varinjlim \mathcal{F}), M)$ by Lemma 3.1.12 and $\pi_0(\mathrm{sg}(\mathrm{RO}(\varprojlim \mathcal{F}), M)) \geq \bar t(0) = t_0(0) \wedge \mathrm{sg}(\mathbb{B}_0, M) = \mathrm{sg}(\mathbb{B}_0, M)$ as required. □

Lemma 3.4.11. *Let $\mathcal{F} = \{i_{\alpha\beta} : \mathbb{B}_\alpha \to \mathbb{B}_\beta : \alpha \leq \beta < \lambda\}$ be an* RCS *and S-*SP *iteration system with S stationary on $[H_\theta]^\omega$ such that $\varinjlim \mathcal{F}$ is $<\lambda$-cc. Then $\varinjlim \mathcal{F}$ and the corresponding $i_{\alpha\lambda}$ are S-*SP.

Proof. The proof follows the same pattern of the previous Lemmas 3.4.9 and 3.4.10. By Proposition 3.4.2, any countable $M \prec H_\delta$ with $\delta > \theta$, $\mathcal{F}, S \in M$, $M \cap H_\theta \in S$, witnesses the semiproperness of every $i_{\alpha\beta}$ with $\alpha, \beta \in M \cap \lambda$. By Lemma 3.1.10 and 3.4.7 we only need to show that

$$\pi_0(\mathrm{sg}(\mathrm{RO}(\varinjlim \mathcal{F}), M)) \geq \mathrm{sg}(\mathbb{B}_0, M).$$

Let $\langle \zeta_n : n \in \omega \rangle$ be an increasing sequence of ordinals such that $\zeta_0 = 0$ and $\sup_n \zeta_n = \zeta = \sup(M \cap \lambda)$, $\{\dot{\alpha}_n : n \in \omega\}$ be an enumeration of the $\varinjlim \mathcal{F}$-names in M for countable ordinals, and $t_0 = \mathbf{1}$ be in $\varinjlim \mathcal{F}$.

Since $\varinjlim \mathcal{F}$ is $<\lambda$-cc, by Theorem 2.2.9 we have that $\varinjlim \mathcal{F} = \lim_{\mathsf{rcs}} \mathcal{F} = \varprojlim \mathcal{F}$. Thus we can define t_{n+1} from t_n, $\dot{\alpha}_n$, ζ_n as in Lemma 3.4.8, so that $t_n \in \varinjlim \mathcal{F} \cap M$ and

$$[\![\dot{\alpha}_n < M \check{\cap} \omega_1]\!]_{\varinjlim \mathcal{F}} \geq t_{n+1} \wedge i_{\zeta_n}(\mathrm{sg}(\mathbb{B}_{\zeta_n}, M)).$$

Since t_n is in $\varinjlim \mathcal{F} \cap M$, M has to model t_n to be eventually constant hence $\mathrm{supp}(t_n) < \zeta$. Then the sequence $\bar{t}(\zeta_n) = t_n(\zeta_n) \wedge \mathrm{sg}(\mathbb{B}_{\zeta_n}, M)$ induces a thread on $\mathcal{F} \restriction \zeta$ (since $\bar{t} \in \mathbb{B}_\zeta = \mathrm{RO}(\varprojlim \mathcal{F} \restriction \zeta)$ by the countable cofinality of ζ) and $i_\zeta(\bar{t}) \leq t_{n+1} \wedge i_{\zeta_n}(\mathrm{sg}(\mathbb{B}_{\zeta_n}, M))$ for every n, so that $i_\zeta(\bar{t}) \leq \mathrm{sg}(\mathrm{RO}(\varinjlim \mathcal{F}), M)$ by Lemma 3.1.12 and $\pi_0(\mathrm{sg}(\mathrm{RO}(\varinjlim \mathcal{F}), M)) \geq \bar{t}(0) = t_0(0) \wedge \mathrm{sg}(\mathbb{B}_0, M) = \mathrm{sg}(\mathbb{B}_0, M)$ as required. □

Theorem 3.4.12. *Let $\mathcal{F} = \{i_{\alpha\beta} : \mathbb{B}_\alpha \to \mathbb{B}_\beta : \alpha \leq \beta < \lambda\}$ be an* RCS *and S-*SP *iteration system with S stationary on $[H_\theta]^\omega$, such that for all $\alpha < \beta < \lambda$,*

$$\left[\!\!\left[\mathbb{B}_\beta /_{i_{\alpha\beta}[\dot{G}_\alpha]} \text{ is } S(\dot{G}_\alpha)\text{-SP} \right]\!\!\right]_{\mathbb{B}_\alpha} = \mathbf{1}$$

*and for all α there is a $\beta > \alpha$ such that $\mathbf{1} \Vdash_{\mathbb{B}_\beta} |\mathbb{B}_\alpha| \leq \omega_1$. Then $\lim_{\mathsf{rcs}} \mathcal{F}$ and the corresponding $i_{\alpha\lambda}$ are S-*SP.

Proof. First, suppose that for all α we have that $|\mathbb{B}_\alpha| < \lambda$. Then, by Theorem 2.2.10, $\varinjlim \mathcal{F}$ is $<\lambda$-cc and $\lim_{\mathsf{rcs}} \mathcal{F} = \varinjlim \mathcal{F}$ hence by Lemma 3.4.11 we have the thesis.

Now suppose that there is an α such that $|\mathbb{B}_\alpha| \geq \lambda$. Then by hypothesis there is a $\beta > \alpha$ such that $\mathbf{1} \Vdash_{\mathbb{B}_\beta} |\mathbb{B}_\alpha| \leq \omega_1$, thus $\mathbf{1} \Vdash_{\mathbb{B}_\beta} \mathrm{cof}\, \lambda \leq \omega_1$. So by Lemma 3.4.6 $\mathcal{F}/_{\dot{G}_\beta}$ is a $\mathbb{B}_\beta$-name for an $S(\dot{G}_\beta)$-SP iteration system that is equivalent to a system of length ω or ω_1 hence its limit is $S(\dot{G}_\beta)$-SP by Lemma 3.4.9 or Lemma 3.4.10 applied in $V^{\mathbb{B}_\beta}$. Finally, $\lim_{\mathsf{rcs}} \mathcal{F}$ can always be factored as a two-step iteration of $\mathbb{B}_\beta$ and $\lim_{\mathsf{rcs}} \mathcal{F}/_{\dot{G}_\beta}$, hence by Proposition 3.4.5 we have the thesis. □

Corollary 3.4.13. *Semiproper forcings are ω_1-weakly iterable.*

Proof. It is easy to check that S-SP forcings are closed under lottery sums and in Section 3.4.1 we proved that they are closed under two step iterations. Consider the strategy $\Sigma : \Gamma^{\lim} \to \Gamma^{\lim}$ which extends an iteration system $\mathcal{F}$ with $\lim_{\mathsf{rcs}} \mathcal{F}$ if $\mathcal{F}$

has limit length, and with $\mathbb{B} * \mathrm{Coll}(\omega_1, |\mathbb{B}|)$ if $\mathcal{F}$ has successor length and terminates with $\mathbb{B}$. It is easy to check that $\langle \Gamma, \Sigma\rangle$ as computed in V_α where $V_\alpha \models$ ZFC is equal to $\langle \Gamma \cap V_\alpha, \Sigma \cap V_\alpha\rangle$. Finally, if $\mathcal{F} = \{i_{\alpha\beta} : \alpha \leq \beta < \lambda\}$ follows Σ it also satisfies the hypothesis of Theorem 3.4.12 hence $\Sigma(\mathcal{F}) = \lim_{\mathsf{rcs}} \mathcal{F}$ extends $\mathcal{F}$ to an S-SP iteration system of length $\lambda + 1$ whenever λ is limit. If instead λ is successor, $\Sigma(\mathcal{F})$ is S-SP by Proposition 3.4.5. □

3.4.3 Stationary set preserving iterations

In order to prove that SSP forcings are weakly iterable, we first prove the consistency of the semiproper forcing axiom SPFA from a supercompact cardinal. This axiom will play a crucial role in the iteration theorem for SSP forcings since it implies that SSP = SP (Shelah, [27, Thm. 37.10]). We will then be able to carry out the proof of an iteration theorem for SSP forcings in the theory $T =$ ZFC + *"there exists a proper class of supercompact cardinals"* along the lines of the iteration theorem for SP forcings.

Definition 3.4.14. A cardinal δ is *supercompact* and $f : \delta \to V_\delta$ is its *Menas function* iff for every δ cardinal there exists an elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$, $j(f)(\kappa) > \delta$ and ${}^{<\delta}M \subseteq M$.

Theorem 3.4.15 (Magidor, Foreman, Shelah)**.** *Let κ be supercompact with Menas function f. Then $\mathbb{P}_\kappa^{\mathrm{SP},f}$, the lottery iteration of* SP *forcings guided by f, forces* SPFA *and collapses κ to ω_2.*

Proof. Let $\dot{\mathbb{B}}$ be a $\mathbb{P}_\kappa^{\mathrm{SP},f}$-name for a semiproper forcing, $\dot{\mathcal{D}}$ be a $\mathbb{P}_\kappa^{\mathrm{SP},f}$-name for a collection of ω_1 many open dense sets of $\dot{\mathbb{B}}$. Let $j : V \to M$ be elementary such that $\mathrm{crit}(j) = \kappa$, $j(f)(\kappa) > \delta$ and ${}^{<\delta}M \subseteq M$ where $\delta > \mathrm{rank}(\dot{\mathbb{B}}), \left|\mathbb{P}_\kappa^{\mathrm{SP},f} * \dot{\mathbb{B}}\right|$.

Let G be V-generic for $\mathbb{P}_\kappa^{\mathrm{SP},f}$, H be $V[G]$-generic for $\mathrm{val}_G(\dot{\mathbb{B}})$. Since $j(\mathbb{P}_\kappa^{\mathrm{SP},f}) = \left(\mathbb{P}_{j(\kappa)}^{\mathrm{SP},j(f)}\right)^M$ and $\dot{\mathbb{B}}$ is in $(V_{j(f)(\kappa)})^M$, by Proposition 3.3.5 there is a $p \in j(\mathbb{P}_\kappa^{\mathrm{SP},f})$ such that $j(\mathbb{P}_\kappa^{\mathrm{SP},f}) \restriction p \leq_{\mathrm{SP}} \mathbb{P}_\kappa^{\mathrm{SP},f} * \dot{\mathbb{B}}$ hence we can find a further G' such that $G * H * G'$ is V-generic for $j(\mathbb{P}_\kappa^{\mathrm{SP},f})$.

Since $\mathrm{crit}(j) = \kappa$ and $\mathbb{P}_\kappa^{\mathrm{SP},f}$ is a direct limit of smaller forcings, $j[G] = G \subseteq G * H * G'$ thus we can extend j to an elementary map $\bar{\jmath}$:

$$\begin{array}{rlcl} \bar{\jmath}: & V[G] & \to & M[G * H * G'] \\ & \mathrm{val}_G(\dot{x}) & \mapsto & \mathrm{val}_{G*H*G'}(j(\dot{x})). \end{array}$$

Let $\mathbb{B} = \mathrm{val}_G(\dot{\mathbb{B}})$, $\mathcal{D} = \mathrm{val}_G(\dot{\mathcal{D}})$. Since H is a filter on $\mathbb{B}$ which meets every element of $\mathcal{D}$ and $\mathcal{D}$ has size $\omega_1 < \kappa$, $\bar{\jmath}(\mathcal{D}) = \bar{\jmath}[\mathcal{D}]$ and for any $\bar{\jmath}(A) \in \bar{\jmath}(\mathcal{D})$ there exists a $p \in H \cap A$ so that $\bar{\jmath}(p) \in j[H] \cap \bar{\jmath}(A)$. It follows that $\bar{\jmath}[H] \in V[G * H * G']$ is a filter on $\bar{\jmath}(\mathbb{B})$ which meets every element of $\bar{\jmath}(\mathcal{D})$.

Since $H = \mathrm{val}_{G*H*G'}(\dot{G}_{\mathbb{B}})$ and M is closed under $<\delta$-sequences with $\delta > |\mathbb{P}_\kappa^{\mathrm{SP},f} * \dot{\mathbb{B}}|$, $j[\dot{G}_{\mathbb{B}}] \in M$ hence $\bar{\jmath}[H] = \mathrm{val}_{G*H*G'}(j[\dot{G}_{\mathbb{B}}]) \in M[G * H * G']$. Thus,

$$M[G * H * G'] \models \exists F \subseteq \bar{\jmath}(\mathbb{B}) \text{ filter } \forall A \in \bar{\jmath}(\mathcal{D})\ F \cap A \neq \emptyset \Rightarrow$$
$$V[G] \models \exists F \subseteq \mathbb{B} \text{ filter } \forall A \in \mathcal{D}\ F \cap A \neq \emptyset$$

concluding the proof. □

We are now able to prove that SSP forcings are weakly iterable.

Lemma 3.4.16. *The class of* SSP *forcings is closed under two-step iterations and lottery sums.*

Proof. Let $\mathbb{B} * \dot{\mathbb{C}}$ be a two-step iteration of SSP forcings and $S \subseteq \omega_1$ be a stationary set in V. Since $\mathbb{B}$ is SSP, $\check{S}$ is a stationary subset of $\check{\omega_1}$ in $V^{\mathbb{B}}$. Since $\dot{\mathbb{C}}$ is SSP in $V^{\mathbb{B}}$, it follows that $\check{S}$ is a stationary subset of $\check{\omega_1}$ in $V^{\mathbb{B}*\dot{\mathbb{C}}}$ as well.

Let now $\mathbb{B} = \prod \{\mathbb{B}_\alpha : \alpha < \lambda\}$ be a lottery sum of SSP forcings, $S \subseteq \omega_1$ be a stationary set in V. Then the set of $p \in \prod \mathcal{B}$ that are $\mathbf{1}$ in one component and $\mathbf{0}$ in all the others form a maximal antichain in $\mathbb{B}$ which forces $\check{S}$ to be a stationary subset of $\check{\omega}_1$ in $V^{\mathbb{B}}$. □

Theorem 3.4.17. *Stationary set preserving forcings are ω_1-weakly iterable in $T =$* ZFC + "there exists a proper class of supercompact cardinals"*.*

Proof. By Lemma 3.4.16 we only need to show a definable iteration strategy for SSP. Let $\dot{\mathbb{P}}^{\mathbb{B}}$ denote the lottery sum of all the semiproper boolean algebras of minimal rank which force SPFA and collapse $|\mathbb{B}|$ to ω_1 *as computed in* $V^{\mathbb{B}}$. That is,

$$\dot{\mathbb{P}}^{\mathbb{B}} = \left(\prod \{\mathbb{C} \in \mathrm{SP} : \ \mathrm{rank}(\mathbb{C}) = \check{\alpha} \ \wedge \ \mathbf{1} \Vdash_{\mathbb{C}} \mathrm{SPFA} \wedge |\mathbb{B}| \leq \omega_1\}\right)^{V^{\mathbb{B}}}$$

where α is minimal such that the above set is not empty. Such an α exists by Theorem 3.4.15, together with the fact that any supercompact cardinal $\kappa > |\mathbb{B}|$ is supercompact in $V^{\mathbb{B}}$. Consider the following definable class $\Sigma : \mathrm{SSP}^{\mathrm{lim}} \to \mathrm{SSP}^{\mathrm{lim}}$.

- $\Sigma(\emptyset) = \dot{\mathbb{P}}^{\mathbf{2}}$, where $\mathbf{2}$ is the two-valued boolean algebra;
- $\Sigma(\{\mathbb{B}_\alpha : \ \alpha < \lambda + 1\}) = \mathbb{B}_\lambda * \dot{\mathbb{P}}^{\mathbb{B}_\lambda}$;
- $\Sigma(\{\mathbb{B}_\alpha : \ \alpha < \lambda\}) = \lim_{\mathsf{rcs}} \{\mathbb{B}_\alpha : \ \alpha < \lambda\}$ if λ is a limit ordinal.

It is easy to check that the above definition grants that for all κ with $V_\kappa \models T$, $\langle \mathrm{SSP} \cap V_\kappa, \Sigma \cap V_\kappa \rangle$ is $\langle \mathrm{SSP}, \Sigma \rangle$ as computed in V_κ. Thus we only need prove that Σ is indeed an iteration strategy.

Let $\mathcal{F} = \{\mathbb{B}_\alpha : \ \alpha < \lambda\}$ be an iteration system following Σ. We prove by induction on λ that $\mathcal{F}$ is an SP iteration system. If $\lambda = 0$, it follows since $\dot{\mathbb{P}}^{\mathbf{2}}$ is semiproper. If $\lambda = \alpha + 1$ is odd, it follows since $\mathbb{B}_\alpha$ forces SPFA hence SSP = SP thus $\mathbb{B}_{\alpha+1} \leq_{\mathrm{SSP}} \mathbb{B}_\alpha$ has to be semiproper. If $\lambda = \alpha + 2$ is even successor, it follows since $\dot{\mathbb{P}}^{\mathbb{B}_{\alpha+1}}$ is semiproper. If λ is limit, it follows by Theorem 3.4.12 since every $\mathbb{B}_\alpha * \dot{\mathbb{P}}^{\mathbb{B}_\alpha}$ collapses the size of $\mathbb{B}_\alpha$ to ω_1. □

CHAPTER 4

GENERIC ABSOLUTENESS

In this chapter we introduce the iterated resurrection axioms, which are forcing axioms with a relatively low consistency strength, and show that strong generic absoluteness results can be obtained from them. These axioms are variations of the *resurrection axioms* [23]. We shall assume from now on that the reader is acquainted with the content of Section 1.1 (second order elementarity and clopen class games), Chapter 2 (iterated forcing), Sections 3.1 - 3.3 (forcing axioms and weakly iterable forcing classes). All the material in this chapter is joint work with Matteo Viale [5].

Section 4.1 introduces the definition of the iterated resurrection axioms together with their basic properties, and proves the main Theorem 4.1.12. Section 4.2 develops the necessary technical devices for the consistency proofs of the axioms $\mathrm{RA}_\alpha(\Gamma)$. These proofs are carried out in Section 4.3 adapting the consistency proofs for the resurrection axioms introduced in [23] to this new setting of iterated resurrection. Section 4.4 outlines the main possible directions of further research on the topic.

4.1 Iterated resurrection and absoluteness

In this section we introduce the iterated resurrection axioms $\mathrm{RA}_\alpha(\Gamma)$ with Γ a suitable definable class of complete boolean algebras, and prove that the iterated resurrection axiom $\mathrm{RA}_\omega(\Gamma)$ gives generic absoluteness for the first order theory (with parameters) of H_{2^γ} for a certain cardinal γ which is computed in terms of the combinatorial properties of Γ. In particular, we aim to choose γ as high as possible while still being able to consistently prove the generic absoluteness of the theory of H_{2^γ} with respect to Γ. We are inspired by the resurrection axioms introduced by Hamkins and Johnstone in [23], which are formulated in similar terms when $\gamma = \omega$.

Definition 4.1.1. Let Γ be a definable class of complete boolean algebras closed under two step iterations. The *cardinal preservation degree* $\mathrm{cpd}(\Gamma)$ of Γ is the maximum cardinal κ such that every $\mathbb{B} \in \Gamma$ forces that every cardinal $\nu \leq \kappa$ is still a cardinal in $V^{\mathbb{B}}$. If all cardinals are preserved by Γ, we say that $\mathrm{cpd}(\Gamma) = \infty$.

The *distributivity degree* $\mathrm{dd}(\Gamma)$ of Γ is the maximum cardinal κ such that every $\mathbb{B} \in \Gamma$ is $<\kappa$-distributive.

We remark that the supremum of the cardinals preserved by Γ is preserved by Γ, and similarly for the property of being $<\kappa$ distributive. Furthermore, $\mathrm{dd}(\Gamma) \leq \mathrm{cpd}(\Gamma)$ and $\mathrm{dd}(\Gamma) \neq \infty$ whenever Γ is non trivial (i.e. it contains a boolean algebra that is not forcing equivalent to the trivial boolean algebra).

Definition 4.1.2. Let Γ be a definable class of complete boolean algebras. Then we let $\gamma = \gamma_\Gamma$ be:

- $\mathrm{cpd}(\Gamma)$ if $\mathrm{cpd}(\Gamma) < \infty$;
- $\mathrm{dd}(\Gamma)$ otherwise.

For example, $\gamma = \omega$ if Γ is among all, ccc, while for Axiom-A, proper, semiproper, SSP we have that $\gamma = \omega_1$ and for $<\kappa-$closed we have that $\gamma = \kappa$.

We aim to isolate an axiom $\mathrm{AX}(\Gamma)$ with the following properties:

1. assuming certain large cardinal axioms, the family of $\mathbb{B} \in \Gamma$ which force $\mathrm{AX}(\Gamma)$ is dense in $(\Gamma, \leq_\Gamma)$,
2. $\mathrm{AX}(\Gamma)$ gives generic aboluteness for the theory with parameters of H_{2^γ} with respect to all forcings in Γ which preserve $\mathrm{AX}(\Gamma)$.

Towards this aim remark the following:

- $\mathrm{cpd}(\Gamma)$ is the maximum possible cardinal γ for which an axiom $\mathrm{AX}(\Gamma)$ as above can grant generic absoluteness with respect to Γ of the theory of H_{2^γ} with parameters. To see this, let Γ be such that $\mathrm{cpd}(\Gamma) = \gamma < \infty$ and assume towards a contradiction that there is an axiom $\mathrm{AX}(\Gamma)$ yielding generic absoluteness with respect to Γ for the theory with parameters of $H_{2^{(\gamma^+)}}$.

  Assume that $\mathrm{AX}(\Gamma)$ holds in V. Since $\mathrm{cpd}(\Gamma) = \gamma$, there exists a $\mathbb{B} \in \Gamma$ which collapses γ^+. Let $\mathbb{C} \leq_\Gamma \mathbb{B}$ be obtained by (1) so that $\mathrm{AX}(\Gamma)$ holds in $V^\mathbb{C}$, and remark that γ^+ cannot be a cardinal in $V^\mathbb{C}$ as well. Then γ^+ is a cardinal in $H_{2^{(\gamma^+)}}$ and not in $H^\mathbb{C}_{2^{(\gamma^+)}}$, witnessing failure of generic absoluteness and contradicting (2).
- $\mathrm{dd}(\Gamma)$ is the least possible cardinal γ such that $\mathrm{AX}(\Gamma)$ is a non-trivial axiom asserting generic absoluteness for the theory of H_{γ^+} with parameters. In fact, $H_{\mathrm{dd}(\Gamma)}$ is never changed by forcings in Γ.

In the remainder of this chapter we shall see that the axiom $\mathrm{RA}_\omega(\Gamma)$ satisfies both of the above requirements and is consistent for a variety of forcing classes Γ. In particular, we shall prove the consistency of $\mathrm{RA}_\omega(\Gamma)$ for forcing classes which are weakly iterable (see Definition 3.3.3) and satisfy the following property.

Definition 4.1.3. Let Γ be a definable class of complete boolean algebras. We say that Γ is *well behaved* iff it is closed under two-step iterations and either:

- if $\mathrm{cpd}(\Gamma) < \infty$, for all $\kappa > \gamma$ there are densely many $\mathbb{B} \in \Gamma$ collapsing κ to γ;
- if $\mathrm{cpd}(\Gamma) = \infty$, for all $\kappa > \gamma$ there are densely many $\mathbb{B} \in \Gamma$ forcing that $2^\gamma > \kappa$.

The first requirement ensures that the cardinal preservation degree of Γ gives a uniform bound for the cardinals which are preserved by the forcings in Γ. The second requirement ensures that above the distributivity degree of Γ there are no further bounds on the possible values of 2^γ in extensions by forcings in Γ. We remark that this property is not significantly restrictive, as it holds for any forcing class Γ we defined in Section 3.1. Thus, throughout all of this chapter we shall always assume that Γ is a well behaved definable class of complete boolean algebras and $\gamma = \gamma_\Gamma$ is given by Definition 4.1.2.

In order to prove the consistency of $\mathrm{RA}_\alpha(\Gamma)$, we shall also need to assume that Γ is weakly iterable (Definition 3.3.3). Even though this hypothesis will not be needed for most of the results presented in this section, we shall sometimes assume it when convenient to carry out the proofs.

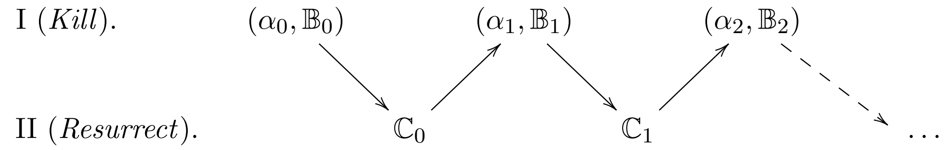
4.1.1 Resurrection game

Motivated by Hamkins and Johnstone's [23], as well as by Tsaprounis' [40], we introduce the following new class games and corresponding forcing axioms.

Definition 4.1.4. The Γ-weak resurrection game $\mathcal{G}^{\mathrm{wRA}}$ is as follows. Player I (*Kill*) plays couples $(\alpha_n, \mathbb{B}_n)$ where α_n is an ordinal such that $\alpha_{n+1} < \alpha_n$ and $\mathbb{B}_n$ is such that $\mathbb{B}_{n+1} \leq_\Gamma \mathbb{C}_n$. Player II (*Resurrect*) plays boolean algebras $\mathbb{C}_n$ such that $H_{2^\gamma}^{\mathbb{C}_n} \prec H_{2^\gamma}^{\mathbb{C}_{n+1}}$ and $\mathbb{C}_n \leq_{\mathrm{all}} \mathbb{B}_n$. The last player who can move wins.

The Γ-resurrection game $\mathcal{G}^{\mathrm{RA}}$ is the same game as $\mathcal{G}^{\mathrm{wRA}}$ with the additional requirement for Player II (*Resurrect*) to play so that $\mathbb{C}_n \leq_\Gamma \mathbb{B}_n$ for all n.

The Γ-strong resurrection game $\mathcal{G}^{\mathrm{RFA}^{++}}$ is the same game as $\mathcal{G}^{\mathrm{RA}}$ with the further requirement for Player II (*Resurrect*) to play so that $V^{\mathbb{C}_n} \models \mathrm{FA}_\gamma^{++}(\Gamma)$ for all n.



Definition 4.1.5. The *α-weak resurrection axiom* $\mathrm{wRA}_\alpha(\Gamma)$ is the assertion that Player II (*Resurrect*) wins the Γ-weak resurrection game after $\langle(\alpha, \mathbf{2}), \mathbf{2}\rangle$[1].

The *α-resurrection axiom* $\mathrm{RA}_\alpha(\Gamma)$ is the assertion that Player II (*Resurrect*) wins the Γ-resurrection game after $\langle(\alpha, \mathbf{2}), \mathbf{2}\rangle$.

The *α-strong resurrection axiom* $\mathrm{RFA}_\alpha^{++}(\Gamma)$ is the assertion that Player II (*Resurrect*) wins the Γ-strong resurrection game after $\langle(\alpha, \mathbf{2}), \mathbf{2}\rangle$.

We say that $\mathrm{wRA}_{\mathrm{ON}}(\Gamma)$ (respectively $\mathrm{RA}_{\mathrm{ON}}(\Gamma)$, $\mathrm{RFA}_{\mathrm{ON}}^{++}(\Gamma)$) holds iff the corresponding axioms hold for all $\alpha \in \mathrm{ON}$.

Note that all these games are clopen class games, since Player I plays descending sequences of ordinals. Since we require Γ to be (first-order) definable, the corresponding axioms asserting the existence of a class winning strategy for Player II in the relevant game (equivalently, that all strategies in the relevant game are

[1] With $\mathbf{2}$ we denote the two-valued boolean algebra $\{\mathbf{0}, \mathbf{1}\}$.

not winning for Player I) are Δ^1_1-statements (in set parameters) over the theory $\mathrm{NBG} + \mathrm{AD}(\Delta^0_1)$.

A posteriori these axioms can also be formulated in recursive terms, as we will see in the next proposition. However, this type of formulation cannot be directly used as a definition in $\mathcal{L}^2$. We will come back in more details to this delicate point after the proof of the next proposition.

Proposition 4.1.6 ($\mathrm{AD}(\Delta^0_1)$)**.** $\mathrm{wRA}_\alpha(\Gamma)$ *holds iff for all* $\beta < \alpha$ *and* $\mathbb{B} \in \Gamma$ *there is a* $\mathbb{C} \leq_{all} \mathbb{B}$ *such that* $H_{2^\gamma} \prec H^{\mathbb{C}}_{2^\gamma}$ *and* $V^{\mathbb{C}} \models \mathrm{wRA}_\beta(\Gamma)$*.*

$\mathrm{RA}_\alpha(\Gamma)$ *holds iff the same holds with* $\mathbb{C} \leq_\Gamma \mathbb{B}$*; or equivalently for all* $\beta < \alpha$ *the class*

$$\left\{\mathbb{B} \in \Gamma : \quad H_{2^\gamma} \prec H^{\mathbb{B}}_{2^\gamma} \wedge V^{\mathbb{B}} \models \mathrm{RA}_\beta(\Gamma)\right\}$$

is dense in $(\Gamma, \leq_\Gamma)$*.*

Similarly, $\mathrm{RFA}^{++}_\alpha(\Gamma)$ *holds iff for all* $\beta < \alpha$ *the class*

$$\left\{\mathbb{B} \in \Gamma : \quad H_{2^\gamma} \prec H^{\mathbb{B}}_{2^\gamma} \wedge V^{\mathbb{B}} \models \mathrm{FA}^{++}_\gamma(\Gamma), \mathrm{RFA}^{++}_\beta(\Gamma)\right\}$$

is dense in $(\Gamma, \leq_\Gamma)$*.*

Proof. First we prove that for any $\mathbb{C}$, $\mathrm{wRA}_\beta(\Gamma)$ holds in $V^{\mathbb{C}}$ iff Player II (*Resurrect*) wins $\mathcal{G}^{\mathrm{wRA}}$ in V after $\langle(\beta, \mathbb{C}), \mathbb{C}\rangle$. Let G be V-generic for $\mathbb{C}$ and $\mathbb{D} \leq_{\mathrm{all}} \mathbb{C}$ with witnessing map $i : \mathbb{C} \to \mathbb{D}$ (i being the identity for $\mathbb{D} = \mathbb{C}$), and define $\mathbb{D}_G := \mathbb{D}/_{i[G]}$ in $V[G]$. Let s be in $\mathcal{G}^{\mathrm{wRA}}$ extending $\langle(\beta, \mathbb{C}), \mathbb{C}\rangle$ and define s_G as the sequence obtained by substituting every boolean algebra $\mathbb{D}$ appearing in s with $\mathbb{D}_G$. By $\mathrm{AD}(\Delta^0_1)$, let σ be a winning strategy for player I or II in $\mathcal{G}^{\mathrm{wRA}}$ after $\langle(\beta, \mathbb{C}), \mathbb{C}\rangle$. Then $\sigma_G = \{s_G : s \in \sigma\}$ is a winning strategy for the same player in the corresponding game in $V[G]$ after $\langle(\beta, \mathbf{2}), \mathbf{2}\rangle$. It follows that $\mathcal{G}^{\mathrm{wRA}}\restriction\langle(\beta, \mathbf{2}), \mathbf{2}\rangle$ is determined in $V[G]$ and Player II (*Resurrect*) wins iff he wins the corresponding game $\mathcal{G}^{\mathrm{wRA}}\restriction\langle(\beta, \mathbb{C}), \mathbb{C}\rangle$ in V.

Furthermore we observe that whenever there is a winning strategy σ for player I or II in $\mathcal{G}^{\mathrm{wRA}}$ and $s = \langle(\alpha, \mathbf{2}), \mathbf{2}, (\beta, \mathbb{B}), \mathbb{C}\rangle$ is in σ, we can define a winning strategy for the same player $\sigma_s = \left\{\langle(\beta, \mathbb{C}), \mathbb{C}\rangle^\frown u : s^\frown u \in \sigma\right\}$ in the game $\mathcal{G}^{\mathrm{wRA}}\restriction\langle(\beta, \mathbb{C}), \mathbb{C}\rangle$.

Now we can prove the proposition: suppose that $\mathrm{wRA}_\alpha(\Gamma)$ holds, and fix $\beta < \alpha$, $\mathbb{B} \in \Gamma$. Let σ be a winning strategy for Player II (*Resurrect*) in $\mathcal{G}^{\mathrm{wRA}}$ after $s_0 = \langle(\alpha, \mathbf{2}), \mathbf{2}\rangle$. Then $s_1 = s_0^\frown(\beta, \mathbb{B}) \in \sigma$ and there is exactly one $s_2 \lhd s_1$[2] in σ, $s_2 = s_1^\frown\mathbb{C}$ with $\mathbb{C} \leq_{\mathrm{all}} \mathbb{B}$ and $H^{\mathbf{2}}_{2^\gamma} = H_{2^\gamma} \prec H^{\mathbb{C}}_{2^\gamma}$. Moreover, σ_{s_2} is a winning strategy for Player II (*Resurrect*) in $\mathcal{G}^{\mathrm{wRA}}$ after $\langle(\beta, \mathbb{C}), \mathbb{C}\rangle$, thus $\mathrm{wRA}_\beta(\Gamma)$ holds in $V^{\mathbb{C}}$.

Conversely, suppose that for all $\beta < \alpha$, $\mathbb{B} \in \Gamma$ there is a $\mathbb{C}$ such that $\mathrm{wRA}_\beta(\Gamma)$ holds in $V^{\mathbb{C}}$ and $H_{2^\gamma} \prec H^{\mathbb{C}}_{2^\gamma}$. Assume towards a contradiction that $\mathrm{wRA}_\alpha(\Gamma)$ fails. Then by $\mathrm{AD}(\Delta^0_1)$ Player I (*Kill*) has a winning strategy σ in $\mathcal{G}^{\mathrm{wRA}}$ after $s_0 = \langle(\alpha, \mathbf{2}), \mathbf{2}\rangle$, and there is exactly one $s_1 = s_0^\frown(\beta, \mathbb{B}) \in \sigma$. Let $\mathbb{C}$ be such that $\mathrm{wRA}_\beta(\Gamma)$ holds in $V^{\mathbb{C}}$, $H_{2^\gamma} \prec H^{\mathbb{C}}_{2^\gamma}$. Then $s_2 = s_1^\frown\mathbb{C}$ is a valid move (hence is in σ) and by the first part of this proof, since $V^{\mathbb{C}} \models \mathrm{wRA}_\beta(\Gamma)$, Player II (*Resurrect*) wins the game $\mathcal{G}^{\mathrm{wRA}}\restriction\langle(\beta, \mathbb{C}), \mathbb{C}\rangle$. Since σ_{s_2} is a winning strategy for Player I (*Kill*) in the same game, we get a contradiction.

Similar arguments yield the thesis also for $\mathrm{RA}_\alpha(\Gamma)$, $\mathrm{RFA}^{++}_\alpha(\Gamma)$. $\square$

[2]We remark that $s_2 \lhd s_1$ iff $s_1 = s_2\restriction(|s_2| - 1)$.

Remark 4.1.7. Assume $\langle V, \mathcal{C}, \in\rangle$ is a model of MK. Let $\phi_0(x, y)$ be the formula

$$\phi_0(x, y) \;\equiv\; x,\, y \text{ are complete boolean algebras and } H^x_{2^\gamma} \prec H^y_{2^\gamma}$$

and for all $n < \omega$, let $\phi_{n+1}(x, y)$ be the formula

$$\phi_{n+1}(x, y) \;\equiv\; \phi_0(x, y) \;\wedge (\forall z \leq_\Gamma y\; \exists w \leq_\Gamma z\; \phi_n(y, w)) \,.$$

Then for all $n < \omega$ the assertion $\phi_n(\mathbf{2}, \mathbf{2})$ is equivalent to $\mathrm{RA}_n(\Gamma)$ in $\langle V, \mathcal{C}, \in\rangle$ and it is a formula with no class quantifier. In particular we get that

$$\langle V, \mathcal{C}, \in\rangle \models \mathrm{RA}_n(\Gamma) \quad\Longleftrightarrow\quad \langle V, \in\rangle \models \phi_n(\mathbf{2}, \mathbf{2}).$$

hence the formulae $\phi_n(\mathbf{2}, \mathbf{2})$ can be used as a first order formulation of the axioms $\mathrm{RA}_n(\Gamma)$ expressible in ZFC with no sort for class variables. However, if

$$\langle V, \mathcal{C}, \in\rangle \models \mathrm{RA}_\omega(\Gamma)$$

we can infer that for all $n < \omega$, $\langle V, \in\rangle \models \phi_n(\mathbf{2}, \mathbf{2})$ but it is not at all clear whether we can express in the structure $\langle V, \in\rangle$ that $\mathrm{RA}_\omega(\Gamma)$ holds. In fact, the simplest strategy to express this property of V would require us to perform an infinite conjunction of the formulae $\phi_n(\mathbf{2}, \mathbf{2})$ for all $n < \omega$, thus getting out of first order syntax.

This problem can be overcome in models of MK appealing to the class-game formulation of these axioms.

From now on, we will stick to the recursive formulation of the α-resurrection axioms given by the latter proposition in order to prove the main results by induction on α. The same will be done with the subsequent Definitions 4.2.1, 4.2.6, 4.2.9 of other class games. Note that by Proposition 4.1.6, $\mathrm{wRA}_0(\Gamma)$, $\mathrm{RA}_0(\Gamma)$ hold vacuously true for any Γ. Thus $\mathrm{wRA}_1(\Gamma)$, $\mathrm{RA}_1(\Gamma)$ imply the non-iterated formulations of resurrection axioms given in [23].

The different forcing axioms we have just introduced are connected by the following implications:

- if $\beta < \alpha$, $\mathrm{wRA}_\alpha(\Gamma) \Rightarrow \mathrm{wRA}_\beta(\Gamma)$ (same with RA, RFA^{++}),
- if $\Gamma_1 \subseteq \Gamma_2$ and their associated $\gamma_1 \leq \gamma_2$, $\mathrm{wRA}(\Gamma_2) \Rightarrow \mathrm{wRA}(\Gamma_1)$,
- $\mathrm{RFA}^{++}_\alpha(\Gamma) \Rightarrow \mathrm{RA}_\alpha(\Gamma) \Rightarrow \mathrm{wRA}_\alpha(\Gamma)$ whenever Γ is closed under two step iterations: for the latter implication notice that the winning strategy σ for II in $\mathcal{G}^{\mathrm{RA}}$ starting from $\langle(\alpha, \mathbf{2}), \mathbf{2}\rangle$ can also be used in $\mathcal{G}^{\mathrm{wRA}}$ and will force I to play always a $\mathbb{B}_n$ in Γ. In particular σ will remain a winning strategy also in $\mathcal{G}^{\mathrm{wRA}}$.

We shall be mainly interested in $\mathrm{RA}_\alpha(\Gamma)$, even though $\mathrm{wRA}_\alpha(\Gamma)$ will be convenient to state theorems in a modular form (thanks to its monotonic behavior with respect to Γ), and $\mathrm{RFA}^{++}_\alpha(\Gamma)$ will be convenient to handle the $\Gamma = \mathrm{SSP}$ case.

Some implications can be drawn between iterated resurrection axiom and the usual forcing axioms, as shown in the following theorems.

Theorem 4.1.8. $\mathrm{wRA}_1(\Gamma)$ *implies* $H_{2^\gamma} \prec_1 V^{\mathbb{B}}$ *for all* $\mathbb{B} \in \Gamma$ *and* $\mathrm{BFA}_\kappa(\Gamma)$ *for all* $\kappa < 2^\gamma$.

Proof. Let $\mathbb{B}$ be any boolean algebra in Γ, and $\mathbb{C} \leq_{\text{all}} \mathbb{B}$ be such that $H_{2^\gamma} \prec H_{2^\gamma}^{\mathbb{C}}$, hence $H_{2^\gamma} \prec H_{2^\gamma}^{\mathbb{C}} \prec_1 V^{\mathbb{C}}$ by Levy's absoluteness. Let $\phi = \exists x \psi(x)$ be a Σ_1 formula. If ϕ holds in H_{2^γ}, it trivially holds in $V^{\mathbb{B}}$ since Σ_1 formulas are upwards absolute. If ϕ holds in $V^{\mathbb{B}}$, it holds in $V^{\mathbb{C}}$ as well hence in H_{2^γ}, concluding the first part.

Let now $\mathbb{B}$ be in Γ and $\mathcal{D}$ be a family of κ many predense subsets of $\mathbb{B}$ of size at most κ. Let $\mathbb{B}' \subseteq \mathbb{B}$ be boolean algebra finitely generated by $\bigcup \mathcal{D}$ in $\mathbb{B}$, so that $|\mathbb{B}'| \leq \kappa$. Without loss of generality we can assume that both $\mathbb{B}'$ and $\mathcal{D}$ are in $H_{\kappa^+} \subseteq H_{2^\gamma}$ by replacing $\mathbb{B}$ with an isomorphic copy if necessary. Let G be a V-generic filter for $\mathbb{B}$. Then G meets every predense in $\mathcal{D}$, that is,

$$V[G] \models \exists F \subseteq \mathbb{B}' \text{ filter } \wedge \forall A \in \mathcal{D}\ F \cap A \neq \emptyset$$

and since $H_{2^\gamma} \prec_1 V^{\mathbb{B}}$, H_{2^γ} has to model the same completing the proof. □

Theorem 4.1.9. *Assume there are class many super almost huge cardinals. Then* MM^{+++} *implies* $\mathrm{RFA}^{++}_{\mathrm{ON}}(\mathrm{SSP})$.

Proof. Recall that $\gamma_{\mathrm{SSP}} = \omega_1$ and MM implies that $2^\omega = 2^{\omega_1} = \omega_2$. We prove that MM^{+++} implies $\mathrm{RFA}^{++}_\alpha(\mathrm{SSP})$ by induction on α. For $\alpha = 0$ it follows since $\mathrm{MM}^{+++} \Rightarrow \mathrm{MM}^{++}$, suppose now that $\alpha > 0$ and the thesis holds for all $\beta < \alpha$.

Let A be the class of all super almost huge cardinals in V. Let $\mathbb{U}^{\mathrm{SSP}}_\delta$ be the forcing whose condition are the SSP-cbas in $\mathrm{SSP} \cap V_\delta$ ordered by $\leq_{\mathrm{SSP}}$. Since A is a proper class, by [43, Thm. 3.5, Lemma 3.12] the class $\{\mathbb{U}^{\mathrm{SSP}}_\delta : \delta \in A\}$ is predense in $(\mathrm{SSP}, \leq_{\mathrm{SSP}})$. Moreover $H_{2^{\omega_1}} = H_{\omega_2} \prec H_{\omega_2}^{\mathbb{U}^{\mathrm{SSP}}_\delta}$ for all $\delta \in A$, since by [43, Lemma 5.19] $\mathbb{U}^{\mathrm{SSP}}_\delta$ is forcing equivalent to a (strongly) presaturated tower for any such δ. Finally, by [43, Cor. 5.20], every such $\mathbb{U}^{\mathrm{SSP}}_\delta$ forces MM^{+++} and preserves that there are class many super almost huge cardinals (since large cardinals are indestructible by small forcings). It follows by inductive hypothesis that every such $\mathbb{U}^{\mathrm{SSP}}_\delta$ forces $\mathrm{RFA}^{++}_\beta(\mathrm{SSP})$ for any $\beta < \alpha$ as well, hence $\mathrm{RFA}^{++}_\alpha(\mathrm{SSP})$ holds in V. □

It is also interesting to examine the consequences of the iterated resurrection axioms on the cardinal arithmetic.

Proposition 4.1.10. *If* $\mathrm{cpd}(\Gamma) = \gamma < \infty$, $\mathrm{wRA}_1(\Gamma)$ *implies that* $2^\gamma = \gamma^+$.

Proof. Suppose by contradiction that $2^\gamma > \kappa = \gamma^+$. By definition of cpd, there exists a $\mathbb{B} \in \Gamma$ such that κ is collapsed in $V^{\mathbb{B}}$. Let $\mathbb{C} \leq_{\text{all}} \mathbb{B}$ be obtained from $\mathrm{wRA}_1(\Gamma)$ so that $H_{2^\gamma} \prec H_{2^\gamma}^{\mathbb{C}}$. Since κ is a cardinal in $H_{2^\gamma}^V$ and it is not a cardinal in $H_{2^\gamma}^{V^{\mathbb{C}}}$, this is a contradiction. Thus $2^\gamma \leq \gamma^+$, concluding the proof. □

The latter proposition suggests that formulating the resurrection axioms in terms of the theory of H_{2^γ} instead of that of H_{γ^+} might be misleading, since in most cases they turn out to be the same. However, this is not the case whenever $\mathrm{cpd}(\Gamma) = \infty$.

Proposition 4.1.11. *If* $\mathrm{cpd}(\Gamma) = \infty$, *then* $\mathrm{wRA}_1(\Gamma)$ *implies that* 2^γ *is a limit cardinal.*

Proof. Since $\mathrm{cpd}(\Gamma)$ is infinite, $\gamma = \mathrm{dd}(\Gamma)$. Suppose by contradiction that $2^\gamma = \kappa^+$ is a successor cardinal. Since Γ is well behaved, there exists a $\mathbb{B} \in \Gamma$ such that $2^\gamma > (\kappa^+)^V$ in $V^{\mathbb{B}}$. Then by $\mathrm{wRA}_1(\Gamma)$ we can find a $\mathbb{C} \leq_\Gamma \mathbb{B}$ such that $H_{2^\gamma}^V \prec H_{2^\gamma}^{V^{\mathbb{C}}}$. Since κ is the maximum cardinal in H_{2^γ} and not in $H_{2^\gamma}^{\mathbb{C}}$, we get a contradiction. □

We remark that similar results were obtained by Hamkins and Johnstone from their formulation of the resurrection axiom $\mathrm{RA}(\Gamma)$ (see Definition 3.2.14). In fact, $\mathrm{RA}_1(\Gamma)$ seem to overlap with $\mathrm{RA}(\Gamma)$ whenever $2^\gamma = \mathfrak{c}$. However, even in this case there are some subtle differences. In particular, let Γ be such that $\gamma = \omega_1$, $2^{\omega_1} = 2^\omega$ and $\mathrm{Add}(\omega_1, 1)$ is in Γ (e.g. Γ is among axiom-A, proper, semiproper, SSP). Then:

- $\mathrm{Add}(\omega_1, 1)$ preserves $\mathrm{RA}(\Gamma)$. Assume that $\mathrm{RA}(\Gamma)$ holds in V and let G be V-generic for $\mathrm{Add}(\omega_1, 1)$ so that $2^\omega = \omega_1$ in $V[G]$ and $H_{\mathfrak{c}}^{V[G]} = H_{\omega_1}^{V[G]} = H_{\omega_1}^V$. Then, any $\mathbb{B} \in \Gamma$ in $V[G]$ which resurrect the theory of $H_{\mathfrak{c}}^V$ will also resurrect the theory of $H_{\mathfrak{c}}^{V[G]} = H_{\omega_1}^{V[G]} = H_{\omega_1}^V$. Thus $\mathbb{B} * \mathrm{Add}(\omega_1, 1)$ will resurrect the theory of $H_{\mathfrak{c}}^{V[G]}$ and there are densely many such forcing since there are densely many $\mathbb{B}$ resurrecting the theory of $H_{\mathfrak{c}}^V$ in V.
- There is no reason to expect that $\mathrm{Add}(\omega_1, 1)$ preserves $\mathrm{RA}_1(\Gamma)$. In fact, in this latter case we want to resurrect the theory of $H_{2^{\omega_1}}$ as computed in $V[G]$ and $H_{2^{\omega_1}}^{V[G]} \supseteq H_{\mathfrak{c}^+}^{V[G]} \supseteq H_{\mathfrak{c}^+}^V$, and the theory of $H_{\mathfrak{c}^+}^V$ is not at all controlled by $\mathrm{RA}_1(\Gamma)$.

Thus, it is not transparent under which conditions on Γ the axiom $\mathrm{RA}_1(\Gamma)$ can be compatible with the GCH, unlike its non-iterated counterpart $\mathrm{RA}(\Gamma)$ which is compatible with CH whenever Γ contains an $<\omega_1$-distributive boolean algebra forcing CH.

4.1.2 Resurrection axioms and generic absoluteness

The main motivation for the iterated resurrection axioms can be found in the following result:

Theorem 4.1.12. *Suppose $n \in \omega$, $\mathrm{RA}_n(\Gamma)$ holds and $\mathbb{B} \in \Gamma$ forces $\mathrm{RA}_n(\Gamma)$. Then $H_{2^\gamma} \prec_n H_{2^\gamma}^{\mathbb{B}}$.*

Proof. We proceed by induction on n. Since $2^\gamma \leq (2^\gamma)^{V^{\mathbb{B}}}$, $H_{2^\gamma} \subseteq H_{2^\gamma}^{\mathbb{B}}$ and the thesis holds for $n = 0$ by the fact that all transitive structures $M \subset N$ are Σ_0-elementary. Suppose now that $n > 0$, and fix G V-generic for $\mathbb{B}$. By Proposition 4.2.2 and $\mathrm{RA}_n(\Gamma)$, let $\mathbb{C} \in V[G]$ be such that whenever H is $V[G]$-generic for $\mathbb{C}$, $V[G * H] \models \mathrm{RA}_{n-1}(\Gamma)$ and $H_{2^\gamma}^V \prec H_{2^\gamma}^{V[G*H]}$. Then we have the following diagram:

$$\begin{array}{ccc} H_{2^\gamma}^V & \xrightarrow{\Sigma_\omega} & H_{2^\gamma}^{V[G*H]} \\ & {\scriptstyle \Sigma_{n-1}} \searrow \quad \nearrow {\scriptstyle \Sigma_{n-1}} & \\ & H_{2^\gamma}^{V[G]} & \end{array}$$

obtained by inductive hypothesis applied both on V, $V[G]$ and on $V[G]$, $V[G * H]$ since in all those classes $\mathrm{RA}_{n-1}(\Gamma)$ holds.

Let $\phi = \exists x \psi(x)$ be any Σ_n formula with parameters in $H_{2^\gamma}^V$. First suppose that ϕ holds in V, and fix $\bar{x} \in V$ such that $\psi(\bar{x})$ holds. Since $H_{2^\gamma}^V \prec_{n-1} H_{2^\gamma}^{V[G]}$ and ψ is Π_{n-1}, it follows that $\psi(\bar{x})$ holds in $V[G]$ hence so does ϕ. Now suppose that ϕ holds in $V[G]$ as witnessed by $\bar{x} \in V[G]$. Since $H_{2^\gamma}^{V[G]} \prec_{n-1} H_{2^\gamma}^{V[G*H]}$ it follows that $\psi(\bar{x})$ holds in $V[G*H]$, hence so does ϕ. Since $H_{2^\gamma}^{V} \prec H_{2^\gamma}^{V[G*H]}$, the formula ϕ holds also in V concluding the proof. □

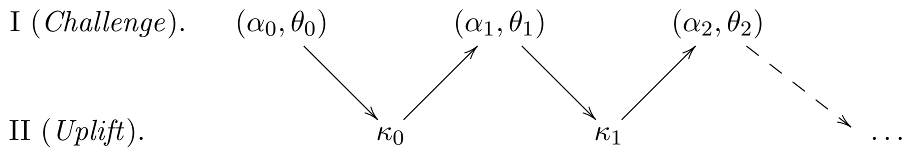
Corollary 4.1.13. *Assume* $\mathrm{RA}_\omega(\Gamma)$ *holds and* $\mathbb{B} \in \Gamma$ *forces* $\mathrm{RA}_\omega(\Gamma)$. *Then* $H_{2^\gamma} \prec H_{2^\gamma}^{\mathbb{B}}$.

A similar absoluteness result for $L(\mathrm{ON}^{\omega_1})$ concerning SSP forcings in the MM^{+++} setting was previously obtained in [43]. Corollary 4.1.13 for $\Gamma = \mathrm{SSP}$ provides a weaker statement (since it concerns the smaller model H_{ω_2}). However, it is a more general result since it holds also for $\Gamma \neq \mathrm{SSP}$, and the consistency of $\mathrm{RA}_\omega(\Gamma)$ follows from much weaker large cardinal assumptions than the principle MM^{+++} needed to prove absoluteness for $L(\mathrm{ON}^{\omega_1})$. Furthermore, the above result can be applied to classes Γ such that the corresponding γ is above ω_1 thus providing generic absoluteness for the (partially) larger model H_{2^γ} with $2^\gamma \geq \omega_3$.

4.2 Uplifting cardinals and definable Menas functions

In order to measure the consistency strength of the α-resurrection axiom, we need to introduce the following large cardinal notion:

Definition 4.2.1. The *uplifting game* $\mathcal{G}^{\mathrm{UP}}$ is as follows. Player I (*challenge*) plays couples of ordinals (α_n, θ_n) such that $\alpha_{n+1} < \alpha_n$. Player II (*uplift*) plays regular cardinals κ_n such that $V_{\kappa_n+1} \prec_{\Delta^1_1} V_{\kappa_{n+1}+1}$, $\kappa_n \geq \theta_n$. The last player who can move wins.



We say that κ is (α)-uplifting iff Player II (*Uplift*) wins the uplifting game after $\langle(\alpha, 0), \kappa\rangle$. We say that κ is (ON)-uplifting iff it is (α)-uplifting for all $\alpha \in \mathrm{ON}$.

The uplifting game is a clopen class game, since Player I (*Challenge*) plays a descending sequence of ordinals. It follows that upliftingness is a Δ^1_1-property. As for the iterated resurrection axioms, we can give a formulation in recursive terms.

Proposition 4.2.2 ($\mathrm{AD}(\Delta^0_1)$)**.** κ *is* (α)*-uplifting iff it is regular and for all* $\beta < \alpha$, $\theta > \kappa$ *there is a* $\lambda > \theta$ *that is* (β)*-uplifting and* $V_{\kappa+1} \prec_{\Delta^1_1} V_{\lambda+1}$.

Proof. Follows the one of Proposition 4.1.6. Let σ be a winning strategy for Player I or II in $\mathcal{G}^{\mathrm{UP}}$, $s = \langle(\alpha, 0), \kappa, (\beta, \theta), \lambda\rangle$ be in σ. Then $\sigma_s = \left\{\langle(\beta, 0), \lambda\rangle^\frown u : \ s^\frown u \in \sigma\right\}$ is a winning strategy for the same player in $\mathcal{G}^{\mathrm{UP}} \restriction \langle(\beta, 0), \lambda\rangle$. Then we can follow step by step the last part of the proof of Proposition 4.1.6 using the reduction $\sigma \to \sigma_s$ defined above. □

Note that replacing “regular” with “inaccessible” in Definition 4.2.1 would produce no difference for $\alpha > 0$, since no successor cardinal κ can satisfy $V_\kappa \prec V_\lambda$ with $\lambda > \kappa$. However, the formulation hereby chosen for Definition 4.2.1 is more convenient for the proofs in Section 4.3.

We remark that the definition of (0)-uplifting cardinal overlaps with that of a regular cardinal. Although similar, the class of (1)-uplifting cardinals does not coincide with the class of *uplifting cardinals* as defined by Johnstone and Hamkins in [23] (see Definition 4.3.11 below). In fact, being (1)-uplifting is a stronger property than being *uplifting*, since the family of Δ^1_1-formulae for the MK-model $V_{\kappa+1}$ strictly includes all formulae in the language of set theory with parameters in V_κ. However the consistency strength of (ON)-uplifting cardinals is close to that of uplifting cardinals as defined by Johnstone and Hamkins. This is shown in Proposition 4.2.4 below.

The key reason which led us to introduce the notion of (α)-uplifting cardinal as a natural strengthening of the Hamkins and Johnstone original notion of upliftingness is to be found in Lemma 4.2.5 below which states a very nice reflection property of (α)-uplifting cardinal which we cannot predicate for the recursive strengthenings of Hamkins and Johnstone notion of upliftingness (see Def. 4.3.11). These reflection properties are a key ingredient in our proof of the consistency of the iterated resurrection axioms. We shall come back to these issues in more details in Section 4.3.

4.2.1 Consistency strength of (α)-uplifting cardinals

Lemma 4.2.3. *Assume δ is Mahlo. Then there are stationarily many inaccessible $\kappa < \delta$ such that $V_{\kappa+1} \prec_{\Delta^1_1} V_{\delta+1}$.*

Proof. Let $C \subseteq \delta$ be a club, M_0 be the Skolem hull of $\{C\}$ in $V_{\delta+1}$. Define a sequence $\langle M_\alpha, \kappa_\alpha : \alpha < \delta \rangle$ where $\kappa_\alpha = \max(\alpha, \operatorname{rank}(M_\alpha \cap V_\delta))$, $M_{\alpha+1} \prec V_{\delta+1}$ is obtained by Lowenheim-Skolem Theorem from $M_\alpha \cup V_{\kappa_\alpha}$, and $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for α limit ordinal.

Since $M_\alpha < \delta$ implies that $\kappa_\alpha < \delta$ and $|M_{\alpha+1}| = |M_\alpha \cup V_{\kappa_\alpha}| < \delta$, by induction on α we have that $|M_\alpha|, \kappa_\alpha < \delta$ for all α. Furthermore, the sequence $\langle \kappa_\alpha : \alpha < \delta \rangle$ is a club on δ, which is Mahlo, thus we can find an $\bar{\alpha} < \delta$ limit such that $\kappa_{\bar{\alpha}}$ is inaccessible.

Since $\bar{\alpha}$ is limit, $M_{\bar{\alpha}} \cap V_\delta = V_{\kappa_\alpha}$. Since $V_{\delta+1} \models C$ is a club and $M_{\bar{\alpha}} \prec V_{\delta+1}$, $C \in M_{\bar{\alpha}}$, also $M_{\bar{\alpha}} \models C$ is a club hence $\kappa_{\bar{\alpha}}$ is a limit point of C. Thus, $\kappa_{\bar{\alpha}}$ is an inaccessible cardinal in C and by Corollary 1.1.2 $V_{\kappa_{\bar{\alpha}}+1} \equiv_{\Delta^1_1(T)} M_{\bar{\alpha}} \prec V_{\delta+1}$, concluding the proof. □

Proposition 4.2.4. *Assume δ is Mahlo. Then $V_{\delta+1}$ models* MK + *there are class many* (ON)*-uplifting cardinals.*

Proof. Since δ is inaccessible, $V_{\delta+1}$ models MK. Furthermore,

$$S = \left\{ \kappa < \delta : \ \kappa \text{ inaccessible } \wedge \ V_{\kappa+1} \prec_{\Delta^1_1} V_{\delta+1} \right\}$$

is stationary by Lemma 4.2.3. We prove that every element of S is (α)-uplifting in $V_{\delta+1}$ by induction on $\alpha < \delta$.

First, every element of S is (0)-uplifting by definition. Suppose now that every element of S is (β)-uplifting for every $\beta < \alpha$, and let κ be in S. Since S is unbounded, for every $\beta < \alpha$, $\theta > \kappa$ in V_δ there is a $\lambda \in S$, $\lambda > \theta$. Such λ is (β)-uplifting by inductive hypothesis and $V_{\kappa+1}, V_{\lambda+1} \prec_{\Delta^1_1} V_{\delta+1}$ implies that $V_{\kappa+1} \prec_{\Delta^1_1} V_{\lambda+1}$. We can now use Property 4.2.2 to conclude. □

As shown in [23, Thm. 11], if there is an uplifting cardinal then there is a transitive model of ZFC + ON is Mahlo. So the existence of an (ON)-uplifting cardinal is in consistency strength strictly between the existence of a Mahlo cardinal and the scheme "ON is Mahlo". We take these bounds to be rather close together and low in the large cardinal hierarchy.

4.2.2 Reflection properties of (α)-uplifting cardinals

The following proposition outlines a key reflection property of (α)-uplifting cardinals:

Lemma 4.2.5. *Let κ be an (α)-uplifting cardinal with $\alpha < \kappa$, and $\delta < \kappa$ be an ordinal. Then $(\delta$ is (α)-uplifting$)^{V_\kappa}$ iff it is (α)-uplifting.*

Proof. Let $\phi(\alpha)$ be the statement of this theorem, i.e:

$$\forall \kappa > \alpha \ (\alpha)\text{-uplifting} \ \forall \delta < \kappa \ \Big((\delta \text{ is } (\alpha)\text{-uplifting})^{V_\kappa} \Leftrightarrow \delta \text{ is } (\alpha)\text{-uplifting} \Big)$$

We shall prove $\phi(\alpha)$ by induction on α using the recursive formulation given by Proposition 4.2.2. For $\alpha = 0$ it is easily verified, suppose now that $\alpha > 0$.

For the forward direction, suppose that $(\delta$ is (α)-uplifting$)^{V_\kappa}$, and let $\beta < \alpha$, $\theta > \delta$ be ordinals. Let $\lambda > \theta$ be a (β)-uplifting cardinal with $V_{\kappa+1} \prec_{\Delta^1_1} V_{\lambda+1}$, so that $(\delta$ is (α)-uplifting$)^{V_\lambda}$ since upliftingness is a Δ^1_1-property under $\mathrm{AD}(\Delta^0_1)$ (and $\mathrm{AD}(\Delta^0_1)$ holds at inaccessible cardinals). Then there is a $\nu > \theta$ in V_λ with $V_{\delta+1} \prec_{\Delta^1_1} V_{\nu+1}$ and $(\nu$ is (β)-uplifting$)^{V_\lambda}$. By inductive hypothesis, since $\beta < \alpha$ and λ is (β)-uplifting in V, also ν is (β)-uplifting in V concluding this part.

Conversely, suppose that δ is (α)-uplifting in V and let $\beta < \alpha$, $\theta > \delta$ be ordinals in V_κ. Let $\nu > \theta$ be a (β)-uplifting cardinal such that $V_{\delta+1} \prec_{\Delta^1_1} V_{\nu+1}$, and let $\lambda > \nu$ be a (β)-uplifting cardinal such that $V_{\kappa+1} \prec_{\Delta^1_1} V_{\lambda+1}$. By inductive hypothesis, since $\beta < \alpha$ and ν, λ are (β)-uplifting in V, $(\nu$ is (β)-uplifting$)^{V_\lambda}$, thus

$$V_{\lambda+1} \models \exists \nu > \theta \ V_{\delta+1} \prec_{\Delta^1_1} V_{\nu+1} \wedge \nu \text{ is } (\beta)\text{-uplifting}.$$

By Δ^1_1-elementarity, $V_{\kappa+1}$ models the same concluding the proof. □

4.2.3 Variations and strengthenings of (α)-upliftingness

To achieve a consistency result also for RFA^{++}, we need to introduce the following definition:

Definition 4.2.6. The *uplifting for supercompacts game* $\mathcal{G}^{\mathrm{UP}^{++}}$ is the same game as $\mathcal{G}^{\mathrm{UP}}$ (see Definition 4.2.1) with the additional requirement that Player II (*Uplift*) has to play supercompact cardinals.

We say that κ is (α)-uplifting for supercompacts iff it is supercompact and Player II (*Uplift*) wins the uplifting for supercompacts game after $\langle(\alpha, 0), \kappa\rangle$. We say that κ is (ON)-uplifting for supercompacts iff it is (α)-uplifting for supercompacts for all $\alpha \in \mathrm{ON}$.

Proposition 4.2.7 (AD(Δ^0_1))**.** *κ is (α)-uplifting for supercompacts iff it is supercompact and for all $\beta < \alpha$, $\theta > \kappa$ there is a $\lambda > \theta$ that is (β)-uplifting for supercompacts and $V_{\kappa+1} \prec_{\Delta^1_1} V_{\lambda+1}$.*

Proof. Follows the one of Proposition 4.2.2. □

A bound for this large cardinal notion can be obtained in a completely similar way to Proposition 4.2.4:

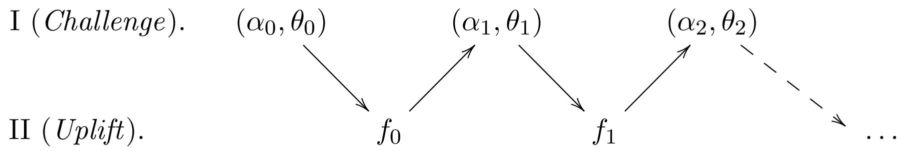
Proposition 4.2.8. *There are class many* (ON)*-uplifting for supercompacts cardinals consistently relative to an inaccessible cardinal which is a stationary limit of supercompact cardinals.*

4.2.4 Menas functions for uplifting cardinals

As previously mentioned, to obtain the consistency results at hand we shall use a lottery iteration relative to a fast-growing function $f : \kappa \to \kappa$ for a sufficiently large cardinal κ. The exact notion of fast-growth we will need is given by the *Menas* property schema introduced in [36] and developed by Hamkins for several different cardinal notions in [23, 22].

We remark that it is always possible to define Menas functions for cardinals that have a Laver function, while it is also possible to define such functions for some cardinals that don't have a Laver function. Moreover, from Menas functions we can obtain many of the interesting consequences given by Laver functions.

Definition 4.2.9. The *Menas uplifting game* $\mathcal{G}^{M-\mathrm{UP}}$ is as follows. Player I (*Challenge*) plays couples of ordinals (α_n, θ_n) such that $\alpha_{n+1} < \alpha_n$. Player II (*Uplift*) plays partial functions $f_n : \kappa_n \to \kappa_n$ with κ_n regular cardinal such that[3] $\langle V_{\kappa_n+1}, f_n\rangle \prec_{\Delta^1_1} \langle V_{\kappa_{n+1}+1}, f_{n+1}\rangle$ and $f_{n+1}(\kappa_n) \geq \theta_{n+1}$. The last player who can move wins.



We say that a partial function $f : \kappa \to \kappa$ is Menas for (α)-uplifting iff Player II (*Uplift*) wins the Menas uplifting game after $\langle(\alpha, 0), f\rangle$. We say that f is Menas for (ON)-uplifting iff it is Menas for (α)-uplifting for all $\alpha \in \mathrm{ON}$.

Proposition 4.2.10 (AD(Δ^0_1))**.** *A partial function $f : \kappa \to \kappa$ is Menas for (α)-uplifting iff κ is regular and for all $\beta < \alpha$, $\theta > \kappa$ there is a Menas for (β)-uplifting function $g : \lambda \to \lambda$ with $g(\kappa) > \theta$ and $\langle V_{\kappa+1}, f\rangle \prec_{\Delta^1_1} \langle V_{\lambda+1}, g\rangle$.*

[3] $\langle M, C\rangle \prec_{\Delta^1_1} \langle N, D\rangle$ iff $\mathrm{set}(M) \subseteq \mathrm{set}(N)$ and for all Δ^1_1-properties $\phi(\vec{x}, Y)$ with $\vec{x}$ a tuple of set variables and Y a class variable, $\langle M, C\rangle \models \phi(\vec{a}, C)$ iff $\langle N, D\rangle \models \phi(\vec{a}, D)$.

Proof. Follows the one of Proposition 4.2.2. □

We can now prove the existence of definable Menas functions for (α)-uplifting cardinals:

Proposition 4.2.11. *If κ is (α)-uplifting, then there is a definable Menas function for (α)-uplifting on κ.*

Proof. We shall prove by induction on α that whenever κ is (α)-uplifting, such a function is given by the following definition (failure of upliftingness function) relativized to $V_{\kappa+1}$:

$$f(\xi) = \sup \Big\{ \nu : \ V_{\xi+1} \prec_{\Delta^1_1} V_{\nu+1} \ \wedge \\ \exists \beta \ \nu \text{ is } (\beta)\text{-uplifting} \ \wedge \ \xi \text{ is not } (\beta+1)\text{-uplifting} \}$$

Note that $f(\xi)$ is undefined only if ξ is (ON)-uplifting (in the domain of f), since otherwise ξ would be an element considered in the supremum. Thus $f(\xi) \geq \xi$ when it is defined.

If κ is (0)-uplifting any function $g : \kappa \to \kappa$ is Menas, in particular our f. Now suppose that κ is (α)-uplifting with $\alpha > 0$ and let $\beta < \alpha$, $\theta > \kappa$ be ordinals. Let $\nu > \theta$ be a (β)-uplifting cardinal such that $V_{\kappa+1} \prec_{\Delta^1_1} V_{\nu+1}$, and let λ be the least (β)-uplifting cardinal bigger than ν such that $V_{\kappa+1} \prec_{\Delta^1_1} V_{\lambda+1}$.

Thus no $\nu' \in (\nu, \lambda)$ with $V_{\kappa+1} \prec_{\Delta^1_1} V_{\nu'+1}$ can be (β)-uplifting in V, hence by Lemma 4.2.5 neither in V_λ. It follows that κ cannot be ($\beta+1$)-uplifting in V_λ, while again by Lemma 4.2.5 it is (β)-uplifting in V_λ. Then any ν' considered in calculating $f^{V_{\lambda+1}}(\kappa)$ must be witnessed by a $\beta' \geq \beta$ (since κ is ($\beta'+1$)-uplifting in $V_{\lambda+1}$ for any $\beta' < \beta$) hence be (β)-uplifting.

It follows that $f^{V_{\lambda+1}}(\kappa) = \nu$ and $\langle V_{\kappa+1}, f^{V_{\kappa+1}} \rangle \prec_{\Delta^1_1} \langle V_{\lambda+1}, f^{V_{\lambda+1}} \rangle$ since f is Δ^1_1-definable, concluding the proof. □

Note that in the proof of this Lemma we used in key steps the reflection properties of (α)-uplifting cardinals given by Lemma 4.2.5.

4.3 Consistency strength

The results of this section expand on the ones already present in [23] and [43]. In the previous section we outlined the large cardinal properties we shall need for our consistency proofs. Now we shall apply the machinery developed by Hamkins and Johnstone in their proof of the consistency of RA(Γ) for various classes of Γ, and show that with minor adjustments their techniques will yield the desired consistency results for $\mathrm{RA}_\alpha(\Gamma)$ for weakly iterable Γ when applied to lottery preparation forcings guided by suitable Menas functions. To prove the consistency of RFA^{++}_α(SSP) we will instead employ the technology introduced by the Viale in [43]. For this reason we shall feel free to sketch some of the proofs leaving to the reader to check the details, that can be developed in analogy to what is done in [23] and [43].

4.3.1 Upper bounds

Let Γ be a weakly iterable class of forcing notions. In this section we shall prove that $\mathbb{P}_\kappa^{\Gamma,f}$, the lottery iteration of Γ relative to a function $f : \kappa \to \kappa$, forces $\mathrm{RA}_\alpha(\Gamma)$ whenever f is Menas for (α)-uplifting. In order to prove this result, we will need to ensure that $\mathbb{P}_\kappa^{\Gamma,f}$ "behaves well" as a class forcing with respect to V_κ.

There are two possible approaches. In the first one, we can consider $\langle V_\kappa, f\rangle$ as a ZFC model extended with an additional unary predicate for f, so that $\mathbb{P}_\kappa^{\Gamma,f}$ can be handled as a definable class forcing. Thus we can proceed following step by step the analogous argument carried out in [23]. In the second one, we can consider the MK model $V_{\kappa+1}$ and expand on the results in [1] to prove that $\mathbb{P}_\kappa^{\Gamma,f}$ preserve enough of MK and behaves well with respect to elementarity. Even though the second approach is more general and natural in some sense, the first approach is considerably simpler. Thus we will follow the argument in [23] and give the following definition.

Definition 4.3.1. Let $\langle M, C\rangle$ be a model of ZFC expanded with an additional class predicate C, and $\mathbb{P}$ be a class partial order definable in $\langle M, C\rangle$.

An $\langle M, C\rangle$-generic filter G for $\mathbb{P}$ is a filter meeting all dense subclasses of $\mathbb{P}$ which are definable in $\langle M, C\rangle$ with parameters.

$\mathbb{P}$ is *nice for forcing* in $\langle M, C\rangle$ if the forcing relation $\Vdash_\mathbb{P}$ is definable in $\langle M, C\rangle$ and the forcing theorem holds, i.e. for every first-order formula ϕ with parameters in $\langle M, C\rangle$ and every $\langle M, C\rangle$-generic filter G for $\mathbb{P}$,

$$\langle M[G], C\rangle \models \phi \ \Leftrightarrow\ \exists p \in G\ \langle M, C\rangle \models (p \Vdash_\mathbb{P} \phi)\,.$$

Note that the definition is interesting only when $\mathbb{P} \notin M$ is a class forcing with respect to M. The next lemmas provide a sufficient condition for being nice for forcing in H_λ.

Lemma 4.3.2. *Let λ be a regular cardinal in V and $\mathbb{P} \subseteq H_\lambda$ be a partial order preserving the regularity of λ. Assume G is V-generic for $\mathbb{P}$. Then*

$$H_\lambda[G] = \{\mathrm{val}_G(\dot{x}) : \dot{x} \in V^\mathbb{P} \cap H_\lambda\} = H_\lambda^{V[G]}.$$

The above Lemma is rather standard but we sketch a proof since we cannot find a precise reference for it.

Proof. Since every element of H_λ with λ regular is coded with a bounded subset of λ, and $\mathbb{P}$ preserves the regularity of λ, we can assume that every $\mathbb{P}$-name for an element of $H_\lambda^{V[G]}$ is coded by a $\mathbb{P}$-name for a function $\dot{f} : \lambda \to 2$ such that $\dot{f}$ is allowed to assume the value 1 only on a bounded subset of λ. In particular we let for any such $\dot{f}$,

$$D_{\dot{f}} = \left\{p \in \mathbb{P} :\ \exists\alpha\ p \Vdash \dot{f}^{-1}[\{1\}] \subseteq \alpha\right\}$$

and for all $\xi < \lambda$,

$$E_{\xi,\dot{f}} = \left\{p \in \mathbb{P} :\ \exists i < 2\ \dot{f}(\xi) = i\right\}$$

Notice that $p \in D_{\dot{f}}$ as witnessed by α implies that $p \in E_{\xi,\dot{f}}$ for all $\xi \geq \alpha$. In particular to decide the values of $\dot{f}$ below such conditions p we just need to consider the dense sets $E_{\xi,\dot{f}}$ for $\xi < \alpha$.

Let $p \in \mathbb{P}$ be arbitrary, $A_\xi \subseteq E_{\xi,\dot{f}} \cap D_{\dot{f}}$ be maximal antichains. Since $\mathbb{P}$ preserves the regularity of λ, it is $<\lambda$-presaturated (see Definition 5.2.5) hence we can find $q \leq p$ such that $q \in D_{\dot{f}}$ as witnessed by α and

$$B_\xi = \{r \in A_\xi : r \text{ is compatible with } q\}$$

has size less than λ for all $\xi < \alpha$. We can now use these antichains B_ξ to cook up a name $\dot{g}_q \in H_\lambda \cap V^{\mathbb{P}}$ such that q forces that $\dot{f} = \dot{g}_q$. By standard density arguments, the thesis follows. □

Lemma 4.3.3. *Let $\mathbb{P} \subseteq H_\kappa$ be a partial order preserving the regularity of κ that is definable in $\langle H_\kappa, C\rangle$. Then $\mathbb{P}$ is nice for forcing in $\langle H_\kappa, C\rangle$.*

Proof. Since $\mathbb{P}$ is a definable class in $\langle H_\kappa, C\rangle$, the corresponding forcing relation $\Vdash_\kappa$ given by formulas with parameters in $H_\kappa \cap V^{\mathbb{P}}$ and whose quantifiers range only over the $\mathbb{P}$-names in H_κ is clearly definable in $\langle H_\kappa, C\rangle$. Moreover, we can prove by induction on ϕ that this relation coincides with the forcing relation as calculated in V, i.e. $p \Vdash \phi^{\dot{H}_\kappa}$ iff $(p \Vdash_\kappa \phi)^{H_\kappa}$. The case ϕ atomic follows from absoluteness of Δ_1 formulas and the case of propositional connectives is easily handled, so we focus on the case $\phi = \exists x \psi(x)$. Using the previous Lemma 4.3.2,

$$\begin{aligned} p \Vdash \phi^{\dot{H}_\kappa} &\Leftrightarrow \left\{q \leq p : \ \exists \tau \in V^{\mathbb{P}} \ q \Vdash \psi(\tau)^{\dot{H}_\kappa} \wedge \tau \in \dot{H}_\kappa\right\} \text{ open dense} \\ &\Leftrightarrow \left\{q \leq p : \ \exists \sigma_q \in H_\kappa \cap V^{\mathbb{P}} \ q \Vdash \psi(\sigma_q)^{\dot{H}_\kappa}\right\} \text{ open dense} \\ &\Leftrightarrow (p \Vdash_\kappa \phi)^{H_\kappa} \end{aligned}$$

since the intersection of two open dense sets is open dense. It follows that

$$\begin{aligned} H_\kappa[G] \models \phi \Leftrightarrow & H_\kappa^{V[G]} \models \phi \Leftrightarrow V[G] \models \phi^{H_\kappa} \Leftrightarrow \\ & \exists p \in G \ p \Vdash \phi^{\dot{H}_\kappa} \Leftrightarrow \exists p \in G \ H_\kappa \models p \Vdash_\kappa \phi \end{aligned}$$

concluding the proof. □

Lemma 4.3.4 (Lifting Lemma, [23, Lemma 17])**.** *Let $\langle M, C\rangle \prec \langle M', C'\rangle$ be models of* ZFC *expanded with additional class predicates C and C'. Let $\mathbb{P}$ be a definable class poset in $\langle M, C\rangle$ that is nice for forcing. Let $\mathbb{P}'$ be defined by the same formula in $\langle M', C'\rangle$ (obtained replacing C with C') and suppose that $\mathbb{P}'$ is also nice for forcing.*

Then for any G $\langle M, C\rangle$-generic for $\mathbb{P}$ and G' $\langle M', C'\rangle$-generic for $\mathbb{P}'$ such that $G' \cap M = G$, we have that $\langle M[G], C, G\rangle \prec \langle M'[G'], C', G'\rangle$.

We remark that since $\mathbb{P}_\kappa^{\Gamma,f}$ is definable in $\langle V_\kappa, f\rangle$, the above results are applicable to this kind of iteration (even though the $\mathbb{P}_\kappa^{\Gamma,f}$ we will be interested with are non-definable classes in V_κ alone).

Theorem 4.3.5. *Let Γ be weakly iterable in* ZFC*. Then* $\mathrm{RA}_\alpha(\Gamma)$ *is consistent relative to* MK+ *the existence of an (α)-uplifting cardinal.*

Proof. The proof follows the one of [23, Thm. 18]. Let V be the standard model of MK. We prove by induction on α that $\mathbb{P}_\kappa = \mathbb{P}_\kappa^{\Gamma,f}$, the lottery iteration of Γ relative to a function $f : \kappa \to \kappa$, forces $\mathrm{RA}_\alpha(\Gamma)$ whenever f is Menas for (α)-uplifting.

By Lemma 4.2.11, the existence of such an f follows from the existence of an (α)-uplifting cardinal, giving the desired result.

Since $\mathrm{RA}_0(\Gamma)$ holds vacuously true, the thesis holds for $\alpha = 0$. Suppose now that $\alpha > 0$. Let $\dot{\mathbb{Q}} \in V^{\mathbb{P}_\kappa}$ be a name for a forcing in Γ, $\beta < \alpha$ be an ordinal. Using the Menas property for f, let $g : \lambda \to \lambda$ be such that $\langle V_{\kappa+1}, f\rangle \prec_{\Delta^1_1} \langle V_{\lambda+1}, g\rangle$, $g(\kappa) \geq \mathrm{rank}(\dot{\mathbb{Q}})$ and g is a Menas for (β)-uplifting function on λ. Let $\mathbb{P}_\lambda = \mathbb{P}^{\Gamma,g}_\lambda$ be the lottery iteration of Γ relative to g.

Since $g \restriction \kappa = f$, by Proposition 3.3.5 we have that:

- $\mathbb{P}_\kappa$ is $<\kappa$-cc and is definable in $\langle V_\kappa, f\rangle$ thus by Lemma 4.3.3 is nice for forcing. Similarly, $\mathbb{P}_\lambda$ is $<\lambda$-cc and definable in $\langle V_\lambda, g\rangle$ thus nice for forcing.
- $\mathbb{P}_\kappa$ forces $2^\gamma \leq \kappa$ and $\mathbb{P}_\lambda$ forces $2^\gamma \leq \lambda$.
- Since $g(\kappa) > \mathrm{rank}(\dot{\mathbb{Q}})$ and $\dot{\mathbb{Q}}$ is in $\Gamma^{V^{\mathbb{P}_\kappa}}$, $\mathbb{P}_\lambda \restriction p \leq_\Gamma \mathbb{P}_\kappa * \dot{\mathbb{Q}}$ for a certain $p \in \mathbb{P}_\lambda$.

Furthermore, by inductive hypothesis $\mathbb{P}_\lambda$ forces $\mathrm{RA}_\beta(\Gamma)$. Thus, we only need to prove that $(H_{2^\gamma})^{V^{\mathbb{P}_\kappa}} \prec (H_{2^\gamma})^{V^{\mathbb{P}_\lambda}}$. The thesis will then follow by Proposition 4.1.6, since $\mathbb{P}_\lambda \restriction p$ would be a legal (and winning) move in $\mathcal{G}^{\mathrm{RA}}$ after $\mathbb{P}_\kappa * \dot{\mathbb{Q}}$.

Let G be any V-generic filter for $\mathbb{P}_\kappa$, H be a $V[G]$-generic filter for $\mathrm{val}_G(\dot{\mathbb{Q}})$. Since $g(\kappa) \geq \mathrm{rank}(\dot{\mathbb{Q}})$, $\dot{\mathbb{Q}}$ is one of the elements of the lottery sum considered at stage $\kappa+1$ so that $G * H$ is V-generic for $\mathbb{P}_\lambda \restriction (\kappa+1)$. Let G' be $V[G*H]$-generic for $\mathbb{P}_\lambda$. Since $\mathbb{P}_\kappa$, $\mathbb{P}_\lambda$ are nice for forcing in the respective models and $\langle H_\kappa, f\rangle \prec \langle H_\lambda, f\rangle$, we can apply Lemma 4.3.4 to obtain that $H_\kappa[G] \prec H_\lambda[G * H * G']$. Furthermore by Lemma 4.3.2,

$$H_\kappa^{V[G]} = H_\kappa[G] \prec H_\lambda[G * H * G'] = H_\lambda^{V[G*H*G']}$$

Since $2^\gamma \leq \kappa$ in $V[G]$ and $2^\gamma \leq \lambda$ in $V[G * H * G']$, we can restrict the above elementarity obtaining that $H_{2^\gamma}^{V[G]} \prec H_{2^\gamma}^{V[G*H*G']}$ and concluding the proof. □

We also mention the following interesting cases, where generic absoluteness can be obtained at the level of any cardinal κ.

Corollary 4.3.6. *Generic absoluteness for the theory of H_{2^κ} and $<\kappa$-closed forcing follows from* $\mathrm{RA}_\omega(<\kappa\text{-closed})$ *and is consistent relative to an (ω)-uplifting cardinal.*

In general, the axioms $\mathrm{RA}_\omega(\Gamma)$ for different choices of Γ are pairwise incompatible. However, the axioms $\mathrm{RA}_\omega(<\kappa\text{-closed})$ for different values of κ are all compatible together, as shown in the following result.

Lemma 4.3.7. $\mathrm{RA}_\alpha(<\kappa\text{-closed})$ *is preserved under $<\kappa^+$-closed forcing.*

Proof. Assume $\mathrm{RA}_\alpha(<\kappa\text{-closed})$ holds in V, and $\mathbb{B}$ is a $<\kappa^+$-closed forcing. We prove that $\mathrm{RA}_\alpha(<\kappa\text{-closed})$ holds in $V^{\mathbb{B}}$ by induction on α and appealing to the formulation given by Proposition 4.1.6.

Given $\beta < \alpha$ and $\mathbb{C} \leq_{\Gamma_\kappa} \mathbb{B}$ where $\Gamma_\kappa = <\kappa$-closed, notice that $\mathbb{B}$ is $<\kappa^+$-closed thus $<\kappa$-closed hence so is $\mathbb{C}$. Then by $\mathrm{RA}_\alpha(<\kappa\text{-closed})$ in V we can find a $\mathbb{D} \leq_{\Gamma_\kappa} \mathbb{C}$ such that $\mathrm{RA}_\beta(<\kappa\text{-closed})$ holds in $V^{\mathbb{D}}$ and $H_{2^\kappa} \prec H_{2^\kappa}^{\mathbb{D}}$. Since $\mathbb{B}$ is $<\kappa^+$-closed and $2^\kappa = \kappa^+$ by Proposition 4.1.10, $H_{2^\kappa} = H_{2^\kappa}^{\mathbb{B}}$ hence $H_{2^\kappa}^{\mathbb{B}} \prec H_{2^\kappa}^{\mathbb{D}}$ witnessing that $\mathrm{RA}_\alpha(<\kappa\text{-closed})$ holds in $V^{\mathbb{B}}$. □

The last lemma directly implies that the axioms $\mathrm{RA}_\alpha(<\omega_\alpha\text{-closed})$ are pairwise compatible for any two ordinals α, β. Moreover, we can also show that these axioms are all simultaneously compatible together.

Definition 4.3.8. Let $\mathbb{B}$ be a complete boolean algebra. Then its closure degree $\mathrm{cd}(\mathbb{B})$ is the largest cardinal κ such that there exists a dense subset $D \subseteq \mathbb{B}$ which is a $<\kappa$-closed poset.

Theorem 4.3.9. $\mathrm{RA}_\omega(<\kappa\text{-}closed)$ *for all cardinals* κ *simultaneously is consistent relative to a Mahlo cardinal.*

Proof. Let δ be a Mahlo cardinal, and let

$$S = \left\{\kappa < \delta : \ \kappa \text{ inaccessible} \ \wedge \ V_{\kappa+1} \prec_{\Delta^1_1} V_{\delta+1}\right\}$$

be stationary by Lemma 4.2.3. Fix an increasing enumeration $\langle \kappa_\alpha : \ \alpha < \delta \rangle$ of a subset of S which is discontinuous at the limits, and let $\langle f_\alpha : \kappa_\alpha \to \kappa_\alpha : \ \alpha < \delta \rangle$ be a sequence of corresponding Menas for (ω)-uplifting functions.

Define the following forcing iteration[4] $\mathcal{F} = \langle \mathbb{B}_\alpha : \ \alpha < \delta \rangle$:

1. $\mathbb{B}_0 = \mathbf{2}$ the trivial boolean algebra;
2. $\mathbb{B}_{\alpha+1} = \mathbb{B}_\alpha * \dot{\mathbb{P}}_{\kappa_\alpha}^{\Gamma_{\omega_\alpha}, f_\alpha}$ where $\dot{\mathbb{P}}_{\kappa_\alpha}^{\Gamma_{\omega_\alpha}, f_\alpha}$ is a $\mathbb{B}_\alpha$-name for the lottery iteration of length κ_α guided by f_α of $<\omega_\alpha$-closed forcings;
3. $\mathbb{B}_\alpha = \varinjlim \{\mathbb{B}_\beta : \ \beta < \alpha\}$ whenever α is inaccessible;
4. $\mathbb{B}_\alpha = \varprojlim \{\mathbb{B}_\beta : \ \beta < \alpha\}$ otherwise.

By Theorem 4.3.5, for any $\alpha < \delta$ $\mathbb{B}_{\alpha+1}$ forces that $\mathrm{RA}_\omega(<\omega_\alpha\text{-closed})$ holds in $V_\delta^{\mathbb{B}_{\alpha+1}}$ thus $2^{\omega_\alpha} = \omega_{\alpha+1}$ by Proposition 4.1.10. We now prove by induction on $\alpha \leq \delta$ that for every $\beta < \alpha$, $\mathbb{B}_\alpha/_{i_{\beta\alpha}[\dot{G}_\beta]}$ is a $<\omega_\beta$-closed forcing.

Remark that given any $<\gamma$-closed iteration $\mathcal{F} = \langle \mathbb{C}_\alpha : \alpha < \lambda \rangle$ (i.e. an $\mathcal{F}$ such that $\left[\!\left[\mathrm{cd}(\mathbb{C}_\beta/_{i_{\alpha\beta}[\dot{G}_\alpha]}) \geq \gamma\right]\!\right]_{\mathbb{C}_\alpha} = \mathbf{1}$ for all $\alpha \leq \beta < \lambda$), we have that $\varprojlim \mathcal{F}$ is always $<\gamma$-closed and $\varinjlim \mathcal{F}$ is $<\gamma$-closed if $\mathrm{cof}(\lambda) > \gamma$. The proof splits in three cases:

- If $\alpha = \xi + 1$ is successor, the thesis holds since $\dot{\mathbb{C}} = \mathbb{B}_\xi/_{i_{\beta\xi}[\dot{G}_\beta]}$ is $<\omega_\beta$-closed by inductive hypothesis and $\mathbb{B}_\alpha/_{i_{\beta\alpha}[\dot{G}_\beta]}$ is the two-step iteration $\dot{\mathbb{C}} * \dot{\mathbb{P}}_{\kappa_\xi}^{\Gamma_{\omega_\xi}, f_\xi}$ of $<\omega_\beta$-closed forcings.
- If α is inaccessible, by inductive hypothesis $(\mathcal{F}\restriction\alpha)/_{\dot{G}_\beta}$ is a $<\omega_\beta$-closed iteration system of regular length $\alpha > \omega_\beta$. It follows that its direct limit $\varinjlim(\mathcal{F}\restriction\alpha)/_{\dot{G}_\beta} = \mathbb{B}_\alpha/_{\dot{G}_\beta}$ is $<\omega_\beta$-closed as well.

[4] We do not specify here the embeddings corresponding to the iteration system, since they can be easily derived from the standard embeddings defined in Sections 2.2 and 2.3.

- If α is a limit ordinal which is not inaccessible, there exists a $\xi < \alpha$, $\xi > \beta$ such that $\omega_\xi > \mathrm{cof}(\alpha)$. By inductive hypothesis, $\dot{\mathbb{C}} = \mathbb{B}_\xi/_{i_{\beta\xi}[\dot{G}_\beta]}$ is $<\omega_\beta$-closed. Furthermore, by inductive hypothesis $(\mathcal{F}\restriction\alpha)/_{\dot{G}_\xi}$ is a $<\omega_\beta$-closed iteration system of length $\mathrm{cof}(\alpha) < \omega_\beta$. It follows that its inverse limit $\varprojlim(\mathcal{F}\restriction\alpha)/_{\dot{G}_\xi} = \mathbb{B}_\alpha/_{\dot{G}_\xi} = \dot{\mathbb{D}}$ is $<\omega_\beta$-closed as well. Finally, $\mathbb{B}_\alpha/_{\dot{G}_\beta}$ can be split as the two step iteration $\dot{\mathbb{C}} * \dot{\mathbb{D}}$ thus it is $<\omega_\beta$-closed, concluding the induction.

Since $\mathbb{B}_{\alpha+1}$ forces that $2^{\omega_\alpha} = \omega_{\alpha+1}$ hold in $V_\delta^{\mathbb{B}_{\alpha+1}}$, and $\mathbb{B}_\delta/_{i_{\alpha+1,\delta}[\dot{G}_{\alpha+1}]}$ is $<\omega_{\alpha+1} = <2^{\omega_\alpha}$-closed, in $V^{\mathbb{B}_\delta}$ the GCH holds up to δ which is a limit cardinal. Furthermore, the inaccessible cardinals below δ are stationary in δ hence in $\mathcal{F}$ the direct limit is taken stationarily often and by Theorem 2.2.10 $\mathbb{B}_\delta$ is $<\delta$-cc. Thus, δ is inaccessible in $V^{\mathbb{B}_\delta}$ and $V_\delta[G] = (V_\delta)^{V[G]}$ is a model of MK for any G a V-generic filter for $\mathbb{B}_\delta$.

We are now able to prove that $\mathrm{RA}_\omega(<\kappa\text{-closed})$ holds in $V_\delta^{\mathbb{B}_\delta}$ for all $\kappa < \delta$. Since $\mathbb{B}_\delta/_{i_{\alpha\delta}[\dot{G}_\alpha]}$ is a class forcing for any $\alpha < \delta$, we cannot directly apply Lemma 4.3.7. However, its proof can be adapted to work also in this setting.

We prove that $\mathrm{RA}_n(<\omega_\alpha\text{-closed})$ holds in $V_\delta^{\mathbb{B}_\delta}$ by induction on $n < \omega$ and appealing to the formulation given by Proposition 4.1.6. Let $\dot{\mathbb{C}}$ be a $\mathbb{B}_\delta$-name for a $<\omega_\alpha$-closed boolean algebra in $V_\delta^{\mathbb{B}_\delta}$. Since $\mathbb{B}_\delta$ is $<\delta$-cc and δ is inaccessible in $V^{\mathbb{B}_\delta}$, there exists a $\xi < \delta$, $\xi > \omega_{\alpha+1}$ such that $\dot{\mathbb{C}}$ is indeed a $\mathbb{B}_\xi$-name. Furthermore, $\dot{\mathbb{C}}$ has to be $<\omega_\alpha$-closed in $V^{\mathbb{B}_\xi}$ as well. Since by Lemma 4.3.7 $\mathrm{RA}_\omega(<\omega_\alpha\text{-closed})$ holds in $V_\delta^{\mathbb{B}_\xi}$, there exists a further $\dot{\mathbb{D}} \leq_{\Gamma_{\omega_\alpha}} \dot{\mathbb{C}}$ such that $H_{2^{\omega_\alpha}}^{\mathbb{B}_\xi} \prec H_{2^{\omega_\alpha}}^{\mathbb{B}_\xi * \dot{\mathbb{D}}}$. Since $\mathbb{B}_\delta/_{i_{\xi\delta}[\dot{G}_\xi]}$ is $<2^{\omega_\alpha}$-closed, $\dot{\mathbb{D}}$ is $<\omega_\alpha$-closed in $V^{\mathbb{B}_\delta}$ as well and

$$H_{2^{\omega_\alpha}}^{\mathbb{B}_\delta} = H_{2^{\omega_\alpha}}^{\mathbb{B}_\xi} \prec H_{2^{\omega_\alpha}}^{\mathbb{B}_\xi * \dot{\mathbb{D}}} = H_{2^{\omega_\alpha}}^{\mathbb{B}_\delta * \dot{\mathbb{D}}}$$

concluding the proof. □

In Theorem 4.3.5 one notable case was excluded, i.e. $\Gamma = \mathrm{SSP}$ since SSP forcings are not weakly iterable in ZFC. The best known upper bound for the consistency strength of RA(SSP) is given in [40, Thm. 3.1] where it is shown that in the presence of class many Woodin cardinals MM^{++} implies RA(SSP) (according to Hamkins and Johnstone's terminology), although it is not clear how this result can be generalized to the axioms $\mathrm{RA}_\alpha(\mathrm{SSP})$ we introduced.

Assuming the existence of an (α)-uplifting cardinal κ which is a limit of supercompact cardinals, we can obtain the consistency of $\mathrm{RA}_\alpha(\mathrm{SSP})$ since in this case V_κ models that there are class many supercompact cardinals and under such assumption the class SSP is weakly iterable. We remark that an (α)-uplifting cardinal which is a limit of supercompact cardinals has lower consistency strength than a Mahlo cardinal which is a stationary limit of supercompact cardinals.

We know that MM^{++} can be forced by a semiproper iteration of length a supercompact cardinal to get SPFA and SP = SSP. For $\mathrm{RA}_\alpha(\mathrm{SSP})$ a similar idea can be applied iteratively, in order to ensure that after each resurrection the equality SP = SSP still holds. This led to the definition of $\mathrm{RFA}_\alpha^{++}(\Gamma)$ and (α)-uplifting for supercompacts cardinal. In fact, we can now prove that the stronger axiom $\mathrm{RFA}_\alpha^{++}(\mathrm{SSP})$ can be obtained from an (α)-uplifting for supercompacts cardinal.

Notice that we cannot directly generalize the previous result in order to obtain $\mathrm{RFA}^{++}_{\alpha}(\mathrm{SSP})$. In fact, a function f that is Menas for (α)-uplifting for supercompacts defined in analogy to the one in Lemma 4.2.11, might fail to produce a lottery iteration $\mathbb{P}^{\mathrm{SSP},f}_{\kappa}$ that forces MM^{++}. Thus, in order to prove a consistency upper bound for $\mathrm{RFA}^{++}_{\alpha}(\mathrm{SSP})$ we will change slightly the approach, replacing the lottery iteration with the category forcing $\mathbb{U}^{\mathrm{SSP}}_{\kappa}$ introduced in [43].

Theorem 4.3.10. $\mathrm{RFA}^{++}_{\alpha}(\mathrm{SSP})$ *is consistent relative to an* (α)*-uplifting for supercompacts cardinal.*

Proof. We prove by induction on α that $\mathbb{U}^{\mathrm{SSP}}_{\kappa}$, the category forcing of height κ, forces $\mathrm{RFA}^{++}_{\alpha}(\mathrm{SSP})$ whenever κ is (α)-uplifting for supercompacts. Since $\mathrm{RFA}^{++}_{0}(\mathrm{SSP})$ holds vacuously true, the thesis holds for $\alpha = 0$.

Suppose now that $\alpha > 0$. Let $\dot{\mathbb{Q}} \in V^{\mathbb{U}^{\mathrm{SSP}}_{\kappa}}$ be a name for a forcing in SSP, $\beta < \alpha$ be an ordinal. Since κ is (α)-uplifting for supercompacts, let $\lambda > \mathrm{rank}(\dot{\mathbb{Q}})$ be a (β)-uplifting for supercompacts cardinal such that $V_{\kappa+1} \prec_{\Delta^1_1} V_{\lambda+1}$. By inductive hypothesis, $\mathbb{U}^{\mathrm{SSP}}_{\lambda}$ forces $\mathrm{RFA}^{++}_{\beta}(\mathrm{SSP})$, and by [43, Thm. 3.5.2] it forces also MM^{++}.

Let G be any V-generic filter for $\mathbb{U}^{\mathrm{SSP}}_{\kappa}$, H be a $V[G]$-generic filter for $\mathrm{val}_G(\dot{\mathbb{Q}})$. Since $\lambda \geq \mathrm{rank}(\dot{\mathbb{Q}})$, $\mathbb{U}^{\mathrm{SSP}}_{\kappa} * \dot{\mathbb{Q}}$ is in $\mathbb{U}^{\mathrm{SSP}}_{\lambda}$ hence by [43, Lemma 3.12, Lemma 3.22] there is a G' such that $G * H * G'$ is generic for $\mathbb{U}^{\mathrm{SSP}}_{\lambda}$ with $\mathbb{U}^{\mathrm{SSP}}_{\kappa} * \dot{\mathbb{Q}} \in G * H * G'$.

By [43, Lemma 3.19], $\mathbb{U}^{\mathrm{SSP}}_{\kappa}$ preserves the regularity of κ, hence by Lemma 4.3.3 $\mathbb{U}^{\mathrm{SSP}}_{\kappa}$ is nice for forcing in H_κ. Then we can apply Lemma 4.3.4 to obtain $H_\kappa[G] \prec H_\lambda[G * H * G']$.

Furthermore, since $\mathbb{U}^{\mathrm{SSP}}_{\kappa}$ preserves the regularity of κ and $\mathbb{U}^{\mathrm{SSP}}_{\lambda}$ preserves the regularity of λ, we have that $H_\kappa[G] = H^{V[G]}_\kappa$ and $H_\lambda[G * H * G'] = H^{V[G*H*G']}_\lambda$. Finally, by [43, Thm. 3.23] $\kappa = \omega_2 = 2^\omega$ in $V[G]$ and $\lambda = \omega_2 = 2^\omega$ in $V[G * H * G']$, so that $H^{V[G]}_{2^\omega} \prec H^{V[G*H*G']}_{2^\omega}$ concluding the proof. □

4.3.2 Lower bounds

A lower bound for the axioms $\mathrm{RA}_\alpha(\Gamma)$ can be quickly found noticing that $\mathrm{RA}_\alpha(\Gamma)$ implies the resurrection axiom $\mathrm{RA}(\Gamma)$ introduced by Hamkins and Johnstone for any $\alpha \geq 1$, and lower bounds for the latter with several choices of Γ were already given in [23]. For a more detailed analysis, we need the following definition:

Definition 4.3.11. The HJ*-uplifting game* $\mathcal{G}^{\mathrm{HJ}}$ is the weakening of game $\mathcal{G}^{\mathrm{UP}}$ in Definition 4.2.1 where Player II (*uplift*) is allowed to play regular cardinals κ_n such that $V_{\kappa_n} \prec V_{\kappa_{n+1}}$ (instead of requiring Δ^1_1-elementarity).

We say that κ is HJ(α)-uplifting iff Player II (*Uplift*) wins the HJ-uplifting game after $\langle(\alpha, 0), \kappa\rangle$.

Proposition 4.3.12 ($\mathrm{AD}(\Delta^0_1)$)**.** κ *is* HJ(α)*-uplifting iff it is regular and for all* $\beta < \alpha$, $\theta > \kappa$ *there is a* $\lambda > \theta$ *that is* HJ(β)*-uplifting and such that* $V_\kappa \prec V_\lambda$.

Proof. Follows the one of Proposition 4.2.2. □

The HJ(1)-uplifting cardinals are exactly the uplifting cardinals introduced by Hamkins and Johnstone in [23], by a reasoning similar to the one shown in Remark

4.1.7. Notice that if κ is (α)-uplifting according to our definition it is also HJ(α)-uplifting. Since Definition 4.3.11 and Proposition 4.3.12 are based on a second order formalization of set theory such as the one provided by MK, we need to translate all the notions involved in the proof of [23, Thm. 16] to the MK setting.

In particular, we need to define a natural expansion of L into a model of MK. If the starting model of MK is of the kind $V_{\kappa+1}$ with κ inaccessible, such expansion is naturally given by considering $\mathcal{C}_L = (V_{\kappa+1})^L = \mathcal{P}(L_\kappa) \cap L$. By [32, Thm. II.6.23], $\mathcal{C}_L = \mathcal{P}(L_\kappa) \cap L_{\kappa^+}$ and every $x \in L_{\kappa^+}$ has $|\text{trcl}(x)| \leq \kappa$. This suggests a natural strategy to define the canonical constructible model of the theory MK.

Definition 4.3.13. Let $\langle V, \mathcal{C} \rangle$ be a (transitive) model of MK.

Given a well-founded extensional class relation $E \subseteq X^2$ in $\mathcal{C}$ and $x \in X$, we define $\text{trcl}_E(x)$ as the subclass of X that represents the transitive closure of x in the well-founded extensional structure $\langle X, E \rangle$, identifying E with the membership relation on X (it can be easily shown under the above assumptions that $\text{trcl}_E(x) \in \mathcal{C}$). Let $E{\restriction}x$ denote $E \cap \text{trcl}_E(x)^2$.

We say that $\langle Y, F \rangle$ is a definable powerset of $\langle X, E \rangle$ if $F \subseteq Y^2$ is a well-founded extensional relation, $X \subseteq Y$, $F \cap X^2 = E$ and for all $y \in Y$ there is some formula ϕ in the language of set theory and some $\vec{a} \in X^{<\omega}$ such that for all $z \in Y$:

$$z \; F \; y \;\Leftrightarrow\; z \in X \;\wedge\; \langle X, E \rangle \models \phi(z, \vec{a}) \tag{4.1}$$

and conversely for all formulae ϕ in the language of set theory and $\vec{a} \in X^{<\omega}$ there is some $y \in Y$ such that (4.1) holds.

Definition 4.3.14. Let $\langle V, \mathcal{C} \rangle$ be a (transitive) model of MK. Given a well-founded extensional class relation $E \subseteq X^2$ and a class $A \subseteq X$ all in $\mathcal{C}$, we say that $\langle X, E \rangle$ is a *constructible initial segment* (c.i.s.) of length A iff:

- $L \subseteq X$ and $E \cap L^2 = \in$,
- $E' = E{\restriction}A$ is a well-order on A with a maximum and such that the sequence $\langle \{L_\alpha : \alpha \in \text{ON}\}, \in \rangle$ is an initial segment of $\langle A, E' \rangle$,
- if $y \in A$ is the successor of x in E', then $\langle y, E{\restriction}y \rangle$ is a definable powerset of $\langle x, E{\restriction}x \rangle$,
- if $y \in A$ is limit in E', then $E{\restriction}y$ is the union of $E{\restriction}x$ for $x \; E' \; y$ with $x \in A$,
- if $y \in A$ is the maximum of E', then $x \; E \; y$ for every $x \in X$.

Define $L^+ = \langle L, \mathcal{C}_L \rangle$, where $\mathcal{C}_L$ is the collection of classes in $\mathcal{C}$ defined by:

$$\begin{aligned} \mathcal{C}_L = \{ C \subseteq L : \exists X, E \in \mathcal{C} \text{ with } \langle X, E \rangle \text{ c.i.s. } \exists x \in X \text{ such that} \\ \forall y \in L \quad y \in C \leftrightarrow y \; E \; x \}. \end{aligned}$$

Note that the structure $\langle X, E \rangle$ in the previous definition can be thought as representing the hyper-class L_α for some α a meta-ordinal of size ON, while the class A which is the length of the c.i.s $\langle X, E \rangle$ can be thought as the collection of representatives in E of hyper-classes L_β for $\beta < \alpha$. Such c.i.s. can be proved to exist for arbitrary meta-ordinal length, and can be proved to be unique and absolute (modulo isomorphisms) for a fixed length by an adaptation of the usual ZFC arguments for L_α with α ordinal.

Moreover, by adapting the proof of [32, Thm. II.6.23] we can prove that MK holds in $L^+ = \langle L, \mathcal{C}_L \rangle$, that L^+ is absolute between models of MK with the same ordinals, and hence that L^+ is the canonical extension of L to a model $\langle V, \mathcal{C} \rangle$ of MK. We are now able to prove the following theorems.

Theorem 4.3.15. *Assume that* Γ *is weakly iterable and* $\mathrm{RA}_1(\Gamma)$ *implies that* 2^γ *is regular. Then* $\mathrm{RA}_\alpha(\Gamma)$ *implies that* $(2^\gamma)^V$ *is* $\mathrm{HJ}(\alpha)$*-uplifting in* L^+.

Proof. Since L^+ is a model of MK, $\mathrm{AD}(\Delta^0_1)$ holds in L^+ and $\mathrm{set}(L^+) = L$ hence we can resort to the recursive formulation given by Proposition 4.3.12 relativized to L. We proceed by induction on α. For $\alpha = 0$, $\kappa = 2^\gamma$ is regular in V hence in L, concluding the proof.

Assume now that $\alpha > 0$ and let $\beta < \alpha$, $\theta > \kappa$ be arbitrary. If $\mathrm{cpd}(\Gamma) = \gamma < \infty$, let $\mathbb{B} \in \Gamma$ be such that θ^+ is collapsed to γ. If $\mathrm{cpd}(\Gamma) = \infty$ and $\mathrm{dd}(\Gamma) = \gamma$, let $\mathbb{B} \in \Gamma$ be adding θ^+ subsets of γ. Thus in both cases $\mathbb{B}$ forces that 2^γ is bigger than θ. Let $\mathbb{C} \leq_\Gamma \mathbb{B}$ be such that $H_{2^\gamma} \prec H^{\mathbb{C}}_{2^\gamma}$ and $\mathrm{RA}_\beta(\Gamma)$ holds in $V^{\mathbb{C}}$. Then $\lambda = (2^\gamma)^{\mathbb{C}}$ is bigger than θ, $H_\kappa \prec H^{\mathbb{C}}_\lambda$, and by inductive hypothesis λ is $\mathrm{HJ}(\beta)$-uplifting in L^+.

Restrict the latter elementarity to L obtaining that $(H_\kappa)^L \prec (H_\lambda)^L$. It follows that κ, λ cannot be successor cardinals in L, hence they need to be inaccessible obtaining that $(H_\kappa)^L = (V_\kappa)^L$ and the same for λ, concluding the proof. □

We remark that the hypothesis that $\mathrm{wRA}_1(\Gamma)$ implies that 2^γ is regular is true whenever $\gamma = \mathrm{cpd}(\Gamma) < \infty$ by Proposition 4.1.10.

There might be a gap between the lower and upper bounds showed in this section, due to the fact that the Δ^1_1-elementarity between V_κ and V_λ is not exploited to give additional properties between the corresponding H_{2^γ} in the forcing extensions. Nonetheless these bounds are very close together (since both are between the axioms scheme "ON is Mahlo" and "there exists a Mahlo cardinal"), so we claim these results to be quite satisfactory.

4.4 Conclusions and open problems

In this chapter we showed how strong generic absoluteness results can be obtained from forcing axioms of relatively low consistency strength. This was achieved via a *iterated* version of a *base axiom*. Thus it is natural to ask whether this same procedure can be carried out for different *base axioms* and to inquire on the properties that such *base axioms* might have. This gives rise to the following.

Question 4.4.1. What is the consistency strength of $\underset{\sim}{\mathrm{RA}}_\omega(\Gamma)$, and what degree of generic absoluteness can entail?

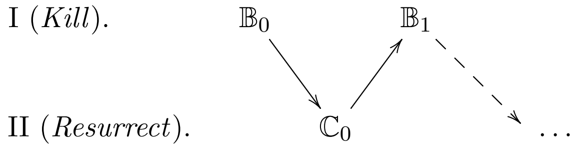
This is currently a work in progress [3]. Whereas the consistency strength in this case seems to be quite mild, it is not clear whether it is possible to infer a stronger generic absoluteness result from $\underset{\sim}{\mathrm{RA}}_\omega(\Gamma)$ than the one we can obtain from $\mathrm{RA}_\omega(\Gamma)$.

Question 4.4.2. What is the consistency strength of $\mathrm{UR}_\alpha(\Gamma)$, and what degree of generic absoluteness can entail?

We suspect an I_3 cardinal to be needed for the consistency of $\mathrm{UR}_\alpha(\Gamma)$. On the other hand, it is likely that $\mathrm{UR}_\alpha(\Gamma)$ would be able to extend the generic absoluteness we have for H_{2^ω} to all initial segments of $L(\mathrm{ON}^{\omega_1})$. This would give an intermediate axiom in between MM^{+++} (see [43]) and $\mathrm{RA}_\omega(\mathrm{SSP})$.

In this chapter we focused on the ω-iterated axiom $\mathrm{RA}_\omega(\Gamma)$. Another possible direction of research is considering whether a larger amount of iteration might affect the axioms significantly. This not only regards $\mathrm{RA}_\alpha(\Gamma)$ for $\alpha > \omega$, but also a stronger $\mathrm{RA}_\infty(\Gamma)$, which can be defined as follows.

Definition 4.4.3. The long Γ-resurrection game $\mathcal{G}^{\mathrm{RA}_\infty}$ is the game of length ω defined as follows. Player I (*Kill*) plays boolean algebras $\mathbb{B}_{n+1}$ such that $\mathbb{B}_{n+1} \leq_\Gamma \mathbb{C}_n$. Player II (*Resurrect*) plays boolean algebras $\mathbb{C}_n$ such that $H_{2^\gamma}^{\mathbb{C}_n} \prec H_{2^\gamma}^{\mathbb{C}_{n+1}}$ and $\mathbb{C}_n \leq_\Gamma \mathbb{B}_n$. Player II looses if it cannot move, and wins if the game reaches full length.



$\mathrm{RA}_\infty(\Gamma)$ is the assertion that Player II (*Resurrect*) wins the long Γ-resurrection game after $\langle \mathbf{2}, \mathbf{2} \rangle$.

Since the game $\mathcal{G}^{\mathrm{RA}_\infty}$ is open, it is still determined in MK and a theory similar to that of $\mathcal{G}^{\mathrm{RA}}$ can be carried out. Furthermore, a similar definition of (∞)-uplifting cardinal can be outlined and proved to follow from a Mahlo cardinal, while an easy adaptation of the results in this chapter can show that $\mathrm{RA}_\infty(\Gamma)$ is consistent relative to a Menas for (∞)-uplifting cardinal. However, it is not clear how to obtain a Menas for (∞)-uplifting cardinal from an (∞)-uplifting cardinal.

Question 4.4.4. What is the consistency strength of $\mathrm{RA}_\infty(\Gamma)$? Does $\mathrm{RA}_\infty(\Gamma)$ or $\mathrm{RA}_{\mathrm{ON}}(\Gamma)$ entail stronger generic absoluteness properties than $\mathrm{RA}_\omega(\Gamma)$?

Furthermore, we mentioned after Proposition 4.1.11 that even though $\mathrm{RA}(\Gamma)$ as defined by Hamkins and Johnstone is compatible with CH for most classes Γ, their proof fails for $\mathrm{RA}_1(\Gamma)$ and it is not at all clear whether it can be recovered by other means. This leads to the following question.

Question 4.4.5. For which classes Γ can $\mathrm{RA}_\omega(\Gamma)$ be compatible with the GCH?

In Theorem 4.3.15 we were not able to prove that $\mathrm{RA}_\omega(\Gamma)$ implies that 2^γ is regular when $\gamma = \mathrm{dd}(\Gamma)$ (hence when any forcing in Γ preserves all cardinals). This implies also that a precise lower bound for $\mathrm{RA}_\omega(\Gamma)$ is missing in this case. However, we know that $\mathrm{RA}_\omega(\Gamma)$ implies $\mathrm{BFA}_\kappa(\Gamma)$ for all $\kappa \in [\gamma, 2^\gamma)$, which in the case $\Gamma = \mathrm{ccc}$ (the only case for $\mathrm{cpd}(\Gamma) = \infty$ we considered) gives that $2^\omega = 2^\kappa$ for all such κ and hence that 2^γ is regular. It is therefore natural to investigate on the following question.

Question 4.4.6. Does $\mathrm{RA}_\omega(\Gamma)$ implies that 2^γ is regular?

Finally, in this chapter we developed the axiom $\mathrm{RA}_\omega(\Gamma)$ and we showed that it gives generic absoluteness for the theory of H_{2^γ} and forcings in Γ. However, the unique example with $2^\gamma > 2^\omega$ is given by choosing $\Gamma = {<}\gamma$-closed for some $\gamma > \omega$. This is a very narrow forcing class among the ones which preserve γ.

This situation is due to the lack of fully satisfactory results regarding forcing iterations preserving cardinals $\gamma > \omega_1$. However, the arguments presented in this chapter show that the two topics are strongly connected, so that an iteration theorem for (a certain notion of) γ-proper forcings immediately entails a generic absoluteness theorem for the same class of forcings and the theory of H_{2^γ}. Conversely, a counterexample to generic absoluteness for H_{2^γ} with respect to (a certain notion of) γ-proper forcings prevents the possibility of having an iteration theorem for the same class of forcings. This naturally leads us to the last open problem.

Question 4.4.7. Can $\mathrm{RA}_\alpha(\Gamma)$ be consistent for $\mathrm{cpd}(\Gamma) = \gamma > \omega_1$, $\Gamma \supsetneq {<}\gamma$-closed? That is, are there weakly iterable $\Gamma \supsetneq {<}\gamma$-closed which preserve γ?

CHAPTER 5

SYSTEMS OF FILTERS AND GENERIC LARGE CARDINALS

In this chapter we introduce the notion of *$\mathcal{C}$-system of filters*, a concept generalizing the well-known definitions of both extenders and towers. In this framework we investigate when extenders and towers happen to be equivalent, together with some natural questions regarding definability of generic large cardinals. In particular, we consider the difference between having a generic large cardinal property *ideally* or *generically* and study how the large cardinal properties of an embedding reflect into the structure of the derived $\mathcal{C}$-system of filters. All the material in this chapter is joint work with Silvia Steila [4]. We remark that the main contribution of this chapter is given by the definition of $\mathcal{C}$-system of filters, and that almost all mentioned results are adaptations of well-known arguments to this new setting.

Section 5.1 introduces the concept of $\mathcal{C}$-system of filters and develops their general theory. Section 5.2 addresses some issues regarding generic large cardinals, using the machinery previously developed. Section 5.3 gives an example showing that the towers in V induced from a generic embedding could have very little in common with the forcing initially used to originate them.

5.1 Systems of filters

In this section we present the definition and main properties of $\mathcal{C}$-systems of filters. This notion has both classical extenders [29, 31], ideal extenders (recently introduced by Claverie in [9]) and towers [11, 33, 43] as special cases, and it is able to generalize and subsume most of the standard results about extenders and towers. We recall here the standard definitions of $\langle \kappa, \lambda \rangle$-extender (see e.g. [31]) and tower of height λ (see e.g. [43]) in the form that is more convenient to us.

Given $a, b \in [\lambda]^{<\omega}$ such that[1] $b = \{\alpha_0, \ldots, \alpha_n\} \supseteq a = \{\alpha_{i_0}, \ldots, \alpha_{i_m}\}$ and $s = \{s_0, \ldots, s_n\}$, let $\pi_{ba}(s) = \{s_{i_0}, \ldots, s_{i_m}\}$.

[1]Here and in the following we assume that finite sets of ordinals are always implicitly ordered by the natural ordering on the ordinals.

Definition 5.1.1. $\mathbb{E} = \{F_a : a \in [\lambda]^{<\omega}\}$ is a standard $\langle\kappa, \lambda\rangle$-extender with supports $\langle \kappa_a : a \in [\lambda]^{<\omega}\rangle$ iff the following holds.

1. (Filter property) For all $a \in [\lambda]^{<\omega}$, F_a is a $<\kappa$-complete filter on $[\kappa_a]^{|a|}$ and κ_a is the least ξ such that $[\xi]^{|a|} \in F_a$;
2. (Compatibility) if $a \subseteq b \in [\lambda]^{<\omega}$ then
   (a) $\kappa_a \leq \kappa_b$;
   (b) if $\max(a) = \max(b)$, then $\kappa_a = \kappa_b$;
   (c) $A \in F_a$ iff $\pi_{ba}^{-1}[A] \in F_b$;
3. (Uniformity) $\kappa_{\{\kappa\}} = \kappa$;
4. (Normality) Assume that $a \in [\lambda]^{<\omega}$, $A \in I_a^+$ where I_a is the dual of F_a, $u : A \to \kappa_a$, $i \in |a|$ are such that $u(s) \in s_i$ for all $s \in A$. Then there exist $\beta \in a_i$, $b \supseteq a \cup \{\beta\}$ and $B \leq_{\mathbb{E}} A$ (i.e. such that $\pi_{ba}^{-1}[A] \supseteq B$) with $B \in I_b^+$ such that for all $s \in B$, $u(\pi_{ba}(s)) = s_j$, where $b_j = \beta$.

Definition 5.1.2. $\mathbb{T} = \{F_a : a \in V_\lambda\}$ is a standard tower of height λ iff the following holds.

1. (Filter property) For all $a \in V_\lambda$, F_a is a non trivial filter on $\mathcal{P}(a)$;
2. (Compatibility) For all $a \subseteq b$, $A \in F_a$ iff $A \uparrow b = \{X \subseteq b : X \cap a \in A\} \in F_b$;
3. (Fineness) For all $a \in V_\lambda$ and $x \in a$ we have $\{X \subseteq a : x \in X\} \in F_a$;
4. (Normality) Given $A \in I_a^+$, $u : A \to V$ such that $u(X) \in X$ for any $X \in A$, there exist $b \supseteq a$, $B \in I_b^+$ with $B \leq_{\mathbb{T}} A$ (i.e. such that $A \uparrow b \supseteq B$), and a fixed y such that $u(X \cap a) = y$ for all $X \in B$.

5.1.1 Main definitions

Throughout this section let V denote a transitive model of ZFC.

Definition 5.1.3. We say that a set $\mathcal{C} \in V$ is a *directed set of domains* iff the following holds:

1. *(Ideal property)* $\mathcal{C}$ is closed under subsets and unions;
2. *(Transitivity)* $\bigcup \mathcal{C}$ is transitive, i.e. for every $y \in x \in a \in \mathcal{C}$ we have $y \in \bigcup \mathcal{C}$ (or, equivalently in presence of the ideal property, $\{y\} \in \mathcal{C}$).

We say that $\mathcal{C}$ has length λ iff $\operatorname{rank}(\mathcal{C}) = \lambda$, and that $\mathcal{C}$ is $<\gamma$-directed iff it is closed under unions of size $<\gamma$ in V.

Example 5.1.4. In the case of extenders, $\mathcal{C}$ will be $[\lambda]^{<\omega}$, while for towers it will be V_λ. The first is absolute between transitive models of ZFC, while the latter is $<\lambda$-directed whenever λ is regular. These two different properties entail most of the differences in behaviour of these two objects.

Definition 5.1.5. Let $\mathcal{C} \in V$ be a directed set of domains. Given a domain $a \in \mathcal{C}$, we define O_a as the set of functions

$$O_a = \{\pi_M \restriction (a \cap M) : \ M \subseteq \mathrm{trcl}(a),\, M \in V \text{ extensional}\}$$

where π_M is the Mostowski collapse map of M. If $a \subseteq b$, we define the *standard projection* $\pi_{ba} : O_b \to O_a$ by $\pi_{ba}(f) = f \restriction a$.

We shall sometimes denote π_{ba} by π_a and π_{ba}^{-1} by π_b^{-1} when convenient. Notice that every $f \in O_b$ is $\in$-preserving, and that $\pi_{ba}(f) = f \restriction a \in O_a$ for any $a \subseteq b$, so that π_{ba} is everywhere defined. From now on we shall focus on filters on the boolean algebra $\mathcal{P}^V(O_a)$ for $a \in \mathcal{C}$ and $\mathcal{C} \in V$ a directed set of domains.

Example 5.1.6. In the case of extenders, any $f \in O_a$ will be an increasing function from the sequence $a \in [\lambda]^{<\omega}$ to smaller ordinals. O_a can be put in correspondence with the domain $\kappa_a^{|a|}$ of a standard extender via the mapping $f \mapsto \mathrm{ran}(f)$, π_{ba} will correspond in the new setting to the usual notion of projection for extenders.

In the case of towers, any $f \in O_a$ with a transitive will be the collapsing map of a $M \subseteq a$. In this case O_a can be put in correspondence with the classical domain $\mathcal{P}^V(a)$ via the mapping $f \mapsto \mathrm{dom}(f)$, and π_{ba} will correspond to the usual notion of projection for towers.

A complete proof of the above mentioned equivalences can be found in Section 5.1.2.

Definition 5.1.7. Define $x \trianglelefteq y$ as $x \in y \vee x = y$. We say that $u : O_a \to V$ is *regressive* on $A \subseteq O_a$ iff for all $f \in A$, $u(f) \trianglelefteq f(x_f)$ for some $x_f \in \mathrm{dom}(f)$. We say that u is *guessed* on $B \subseteq O_b$, $b \supseteq a$ iff there is a fixed $y \in b$ such that for all $f \in B$, $u(\pi_{ba}(f)) = f(y)$.

Definition 5.1.8. Let $V \subseteq W$ be transitive models of ZFC and $\mathcal{C} \in V$ be a directed set of domains. We say that $\mathbb{S} = \{F_a : \ a \in \mathcal{C}\} \in W$ is a $\mathcal{C}$-system of V-filters, and we equivalently denote $\mathbb{S}$ also by $\{I_a : \ a \in \mathcal{C}\}$ where I_a is the dual ideal of F_a, iff the following holds:

1. *(Filter property)* for all $a \in \mathcal{C}$, F_a is a non-trivial filter on the boolean algebra $\mathcal{P}^V(O_a)$;
2. *(Fineness)* for all $a \in \mathcal{C}$ and $x \in a$, $\{f \in O_a : \ x \in \mathrm{dom}(f)\} \in F_a$;
3. *(Compatibility)* for all $a \subseteq b$ in $\mathcal{C}$ and $A \subseteq O_a$, $A \in F_a \iff \pi_{ba}^{-1}[A] \in F_b$;
4. *(Normality)* every function $u : A \to V$ in V that is regressive on a set $A \in I_a^+$ for some $a \in \mathcal{C}$ is guessed on a set $B \in I_b^+$ for some $b \in \mathcal{C}$ such that $B \subseteq \pi_{ba}^{-1}[A]$;

We say that $\mathcal{S}$ is a $\mathcal{C}$-system of V-ultrafilters if in addition:

5. *(Ultrafilter)* for all $a \in \mathcal{C}$, F_a is an ultrafilter on $\mathcal{P}^V(O_a)$.

We shall feel free to drop the reference to V when clear from the context, hence denote the $\mathcal{C}$-systems of V-filters as $\mathcal{C}$-systems of filters. When we believe that this convention may generate misunderstandings we shall be explicitly more careful. To clearly distinguish $\mathcal{C}$-systems of filters from $\mathcal{C}$-systems of ultrafilters, in the following we shall use $\mathbb{S}$, $\mathbb{E}$, $\mathbb{T}$ for the first and $\mathcal{S}$, $\mathcal{E}$, $\mathcal{T}$ for the latter.

Definition 5.1.9. Let $\mathbb{S}$ be a $\mathcal{C}$-system of filters, a be in $\mathcal{C}$. We say that κ_a is the *support* of a iff it is the minimum α such that $O_a \cap {}^aV_\alpha \in F_a$. We say that $\mathbb{S}$ is a $\langle\kappa, \lambda\rangle$-system of filters if and only if:

- it has length λ and $\kappa \subseteq \bigcup \mathcal{C}$,
- $F_{\{\gamma\}}$ is principal generated by $\mathrm{id} \restriction \{\gamma\}$ whenever $\gamma < \kappa$,
- $\kappa_a \leq \kappa$ whenever $a \in V_{\kappa+2}$.

Notice that $\kappa_a \leq \mathrm{rank}(a)$, and $\kappa_a = \mathrm{rank}(a)$ when F_a is principal as in the above definition. In particular, $\kappa_{\{\gamma\}} = \gamma + 1$ in this case. The definition of $\mathcal{C}$-system of filters entails several other properties commonly required for coherent systems of filters.

Proposition 5.1.10. *Let $\mathbb{S}$ be a $\mathcal{C}$-system of filters. Then $dF_a = \{\mathrm{dom}[A] : A \in F_a\}$ is a normal and fine filter on $\mathcal{P}(a)$, for any a in $\mathcal{C}$ infinite. In particular if a is uncountable,* $\mathrm{dom}[A]$ *is stationary for all $A \in F_a$.*

Proof. Filter property and fineness follow directly from restricting the corresponding points in Definition 5.1.8 to $\mathrm{dom}[O_a]$. We now focus on normality. Let $u : D \to a$ where $D = \mathrm{dom}[A]$ be such that $u(X) \in X$ for all $X \in D$ (i.e. $X = \mathrm{dom}(f)$ for some f in A). Then we can define $v : A \to V$ as $v(f) = f(u(\mathrm{dom}(f)))$. Let $B \in I_b^+$, $y \in b$ be such that $v(\pi_{ba}(f)) = f(u(\mathrm{dom}(\pi_{ba}(f)))) = f(y)$ for all $f \in B$ by normality. Since every $f \in B$ is injective, $u(\mathrm{dom}(\pi_{ba}(f))) = y$ for all $f \in B$ hence u is constant on $\mathrm{dom}[B] \in dI_a^+$. By Lemma 1.2.8 if a is uncountable we conclude that $\mathrm{dom}[A]$ is stationary for any $A \in F_a$. □

Proposition 5.1.11. *Let $\mathbb{S}$ be a $\mathcal{C}$-system of filters, a, b be in $\mathcal{C}$. Then* $\mathrm{rank}(a) \leq \mathrm{rank}(b)$ *implies that $\kappa_a \leq \kappa_b$, i.e. the supports depend (monotonically) only on the ranks of the domains.*

Proposition 5.1.12. *Let $\mathbb{S}$ be a $\langle\kappa, \lambda\rangle$-system of filters, a be in $\mathcal{C}$. Then F_a is $<\kappa$-complete.*

We defer the proof of the last two propositions to Section 5.1.4 (just before Proposition 5.1.30) for our convenience. We are now ready to introduce the main practical examples of $\mathcal{C}$-system of filters.

Definition 5.1.13. Let $V \subseteq W$ be transitive models of ZFC.

$\mathbb{E} \in W$ is an *ideal extender* on V iff it is a $[\lambda]^{<\omega}$-system of filters on V for some λ. $\mathcal{E} \in W$ is an *extender* on V iff it is a $[\lambda]^{<\omega}$-system of ultrafilters on V.

$\mathbb{E} \in W$ is an *ideal γ-extender* on V iff it is a $([\lambda]^{<\gamma})^V$-system of filters for some λ. $\mathcal{E}$ is a *γ-extender* on V iff it is a $([\lambda]^{<\gamma})^V$-system of ultrafilters on V.

$\mathbb{T} \in W$ is an *ideal tower* iff it is a V_λ-system of filters for some λ. $\mathcal{T} \in W$ is a *tower* iff it is a V_λ-system of ultrafilters.

The above definitions of extender and tower can be proven equivalent to the classical ones presented at the beginning of this section (see also [31, 43]) via the mappings $\mathrm{ran}_a : O_a \to [\kappa_a]^{|a|}$, $f \mapsto \mathrm{ran}(f)$ (for extenders) and $\mathrm{dom}_a : O_a \to \mathcal{P}(a)$, $f \mapsto \mathrm{dom}(f)$ (for towers). Furthermore, $\langle \kappa_a : a \in \mathcal{C} \rangle$ correspond to the supports of long extenders as defined in [31]. A detailed account of this correspondence is given in Section 5.1.2.

Given a $\mathcal{C}$-system of V-filters $\mathbb{S}$, we can define a preorder $\leq_{\mathbb{S}}$ on the collection $\mathbb{S}^+ = \{A : \exists a \in \mathcal{C}\ A \in I_a^+\}$ as in the following.

Definition 5.1.14. Given $A \in I_a^+$, $B \in I_b^+$ we say that $A \leq_{\mathbb{S}} B$ iff $\pi_{ca}^{-1}[A] \leq_{I_c} \pi_{cb}^{-1}[B]$ where $c = a \cup b$, and $A =_{\mathbb{S}} B$ iff $A \leq_{\mathbb{S}} B$ and $B \leq_{\mathbb{S}} A$.

Consider the quotient $\mathbb{S}^+/=_{\mathbb{S}}$. With an abuse of notation for $p, q \in \mathbb{S}^+/=_{\mathbb{S}}$, we let $p \leq_{\mathbb{S}} q$ iff $A \leq_{\mathbb{S}} B$ for any (some) $A \in p$, $B \in q$. The partial order $\langle \mathbb{S}^+/=_{\mathbb{S}}, \leq_{\mathbb{S}} \rangle$ is a boolean algebra which is the limit of a directed system of boolean algebras, and can be used as a forcing notion in order to turn $\mathbb{S}$ into a system of ultrafilters. This process will be described in Section 5.1.3.

Proposition 5.1.15. *Let $\mathcal{C}$ be a $<\gamma$-directed set of domains, $\mathbb{S}$ be a $\mathcal{C}$-system of filters. Then $\langle \mathbb{S}^+/=_{\mathbb{S}}, \leq_{\mathbb{S}} \rangle$ forms a $<\gamma$-closed boolean algebra.*

Proof. Let $\mathcal{A} = \langle A_\alpha : \alpha < \mu \rangle \subseteq \mathbb{S}^+$ be such that $\mu < \gamma$ with $A_\alpha \in I_{a_\alpha}^+$. Since $\mathcal{C}$ is $<\gamma$-directed, there is a domain $a \in \mathcal{C}$ with $|a| \geq \mu$ such that $a_\alpha \subseteq a$ for all $\alpha < \mu$. Fix $\langle x_\alpha : \alpha < \mu \rangle$ a (partial) enumeration of a, and define

$$B = \left\{ f \in O_a : \forall \alpha < \mu\ x_\alpha \in \mathrm{dom}(f) \Rightarrow f \in \pi_a^{-1}[A_\alpha] \right\}.$$

First, $B <_{\mathbb{S}} A_\alpha$ for all $\alpha < \mu$ by fineness, since $\{f \in B : x_\alpha \in \mathrm{dom}(f)\} \subseteq \pi_a^{-1}[A_\alpha]$. Suppose now by contradiction that for some $c \supseteq a$, $C \in I_c^+$ is such that $C \leq_{\mathbb{S}} A_\alpha$ for all $\alpha < \mu$ and $C \cap \pi_c^{-1}[B] = \emptyset$. Then for any $f \in C$ we can find an $\alpha_f < \mu$ such that $x_{\alpha_f} \in \mathrm{dom}(f)$ and $f \notin \pi_c^{-1}[A_{\alpha_f}]$. Define

$$\begin{array}{rccc} u : & C & \longrightarrow & V \\ & f & \longmapsto & f(x_{\alpha_f}) \end{array}$$

By normality we can find a single $\bar{\alpha}$ and a $d \supseteq c \cup \{\bar{\alpha}\}$ such that

$$D = \left\{ f \in \pi_d^{-1}[C] : u(\pi_d(f)) = f(x_{\bar{\alpha}}) \right\} \in I_d^+.$$

Thus $D \cap A_{\bar{\alpha}} = \emptyset$ and $D \leq_{\mathbb{S}} C \leq_{\mathbb{S}} A_{\bar{\alpha}}$, a contradiction. □

5.1.2 Standard extenders and towers as $\mathcal{C}$-systems of filters

Extenders. We now compare the definition of $\langle \kappa, \lambda \rangle$-extender just introduced (Definition 5.1.13) with the definition of standard $\langle \kappa, \lambda \rangle$-extender (Definition 5.1.1).

Let $\mathbb{E}$ be a $\langle \kappa, \lambda \rangle$-extender with supports $\langle \kappa_a : a \in [\lambda]^{<\omega} \rangle$ according to Definition 5.1.13. Notice that given any $a \in [\lambda]^{<\omega}$, the collection

$$O'_a = \{ f \in O_a : \mathrm{dom}(f) = a \wedge \mathrm{ran}(f) \subseteq \kappa_a \}$$

is in F_a by fineness and definition of κ_a. Consider the injective map $\operatorname{ran}_a : O'_a \to [\kappa_a]^{|a|}$, which maps F_a into a corresponding filter F'_a on $[\kappa_a]^{|a|}$ that is the closure under supersets of $\{\operatorname{ran}_a[A \cap O'_a] : A \in F_a\}$. Notice that many sequences $s \in [\kappa_a]^{|a|}$ cannot be obtained as the range of Mostowski collapse maps, e.g. $s = \{\beta, \beta+2\}$ whenever a is of the kind $\{\alpha, \alpha+1\}$.

Let us denote with π'_{ba} the projection map from F'_b to F'_a in the standard case. Notice that for any $a \subseteq b \in [\lambda]^{<\omega}$ and $f \in O_b$, $\operatorname{ran}_a(\pi_{ba}(f)) = \pi'_{ba}(\operatorname{ran}_b(f))$. Define $\mathbb{E}' = \left\{F'_a : a \in [\lambda]^{<\omega}\right\}$. We claim that $\mathbb{E}'$ is a $\langle\kappa, \lambda\rangle$-extender with respect to the standard definition whenever $\mathbb{E}$ is a $\langle\kappa, \lambda\rangle$-extender.

Proposition 5.1.16. *If $\mathbb{E}$ is a $\langle\kappa, \lambda\rangle$-extender then $\mathbb{E}'$ is a standard $\langle\kappa, \lambda\rangle$-extender.*

Proof.

1. (Filter property) It follows since F'_a is an injective image of $F_a \restriction O'_a$.
2. (Compatibility)
   - (a-b) Follow by Proposition 5.1.11, since $\operatorname{rank}(a)$ depends only on $\max(a)$.
   - (c) By compatibility of $\mathbb{E}$, it follows: $A' = \operatorname{ran}_a[A] \in F'_a$ iff $A \in F_a$ iff $\pi_{ba}^{-1}[A] \in F_b$ iff $\operatorname{ran}_b[\pi_{ba}^{-1}[A]] = {\pi'}_{ba}^{-1}[A] \in F'_b$.
3. (Uniformity) By definition of $\langle\kappa, \lambda\rangle$-system of filters.
4. (Normality) Given $a \in [\lambda]^{<\omega}$, $A' = \operatorname{ran}_a[A] \in {I'}_a^{+}$, $u : A' \to \kappa_a$, $i < |a|$ such that $u(s) \in s(i)$ for all $s \in A'$, let $\alpha = a_i$. Define

$$\begin{aligned} v : \ A &\longrightarrow V \\ f &\longmapsto u(\operatorname{ran}_a(f)). \end{aligned}$$

   Since $s_i = f(\alpha)$, v is regressive. By normality of $\mathbb{E}$, there exist β and $B \subseteq \pi_{ba}^{-1}[A]$, where $\beta \in b \supseteq a$ such that for all $f \in B$, $v(\pi_{ba}(f)) = f(\beta)$. Since $\pi_{ba}(f) \in A$, $f(\beta) = v(\pi_{ba}(f)) \in f(\alpha)$. Since f is $\in$-preserving, $\beta \in \alpha$ and $B' = \operatorname{ran}_b[B]$ witnesses normality of $\mathbb{E}'$. □

On the other hand, given a standard $\langle\kappa, \lambda\rangle$-extender $\mathbb{E}'$ we can define a collection of corresponding filters F_a on O'_a for any $a \in [\lambda]^{<\omega}$. This can be achieved since $\operatorname{ran}_a[O'_a] \in F'_a$ for any $a \in [\lambda]^{<\omega}$ and standard $\langle\kappa, \lambda\rangle$-extender $\mathbb{E}'$[2]. Let $\mathbb{E}$ consists of the closure of F_a under supersets in O_a, for any $a \in [\lambda]^{<\omega}$. Then we can show the following.

Proposition 5.1.17. *If $\mathbb{E}'$ is a standard $\langle\kappa, \lambda\rangle$-extender then $\mathbb{E}$ is a $\langle\kappa, \lambda\rangle$-extender.*

Proof.

1. (Filter property) Follows directly from the filter property of $\mathbb{E}'$.
2. (Compatibility) By compatibility of $\mathbb{E}'$ and unfolding definitions, $A \in F_a$ iff $A \cap O'_a \in F_a$ iff $\operatorname{ran}_a[A \cap O'_a] \in F'_a$ iff $\operatorname{ran}_b[\pi_{ba}^{-1}[A \cap O'_a]] \in F'_b$ iff $\pi_{ba}^{-1}[A \cap O'_a] \in F_b$ iff $\pi_{ba}^{-1}[A] \in F_b$.
3. (Fineness) For any $x \in a$, $\{f \in O_a : x \in \operatorname{dom}(f)\} \supseteq O'_a \in F_a$.

[2] This fact can be proved directly using the corresponding version of Proposition 5.1.28 (i.e. $A \in F'_a$ iff $a \in j(A)$) and Łoś Theorem 5.1.26 for standard extenders.

4. (Normality) Assume that $A \in I_a^+$ and that $u : A \to V$ is regressive. By definition of regressive function we have that $A = A_0 \cup A_1$, where

$$A_0 = \{f \in A : \exists x \in \mathrm{dom}_a(f)(u(f) = f(x))\} ;$$

$$A_1 = \{f \in A : \exists x \in \mathrm{dom}_a(f)(u(f) \in f(x))\} .$$

We have two cases.

- If $A_0 \in I_a^+$ there exists a fixed $x \in a$ such that $B = \{f \in A : u(f) = f(x)\}$ is in I_a^+ since a is finite. Hence B and x witness normality for $\mathbb{E}$.
- Otherwise, if $A_0 \in I_a$, we have that $A_1 \in I_a^+$. Since a is finite, there exists $x = a_i \in a$ such that $A^* = \{f \in A : u(f) \in f(x)\} \in I_a^+$. Let $A' = \mathrm{ran}_a[A^* \cap O'_a]$. Let $v : A' \to \kappa_a$ be such that $v(s) = u(f)$ for any $s = \mathrm{ran}_a(f)$, so that $v(s) \in s_i$. By normality of $\mathbb{E}'$ there exist $\beta \in a_i$ and $B' <_{\mathbb{E}'} A'$ with $B' = \mathrm{ran}_b[B] \in I'^+_b$ for $b = a \cup \{\beta\}$ such that for all $s \in B'$, $v(\pi'_{ba}(s)) = s_j$, where $b_j = \beta$. Hence for any $f \in B$, $u(\pi_{ba}(f)) = f(\beta)$. □

Towers. We now compare the definition of tower just introduced (Def. 5.1.13) with the definition of standard tower (Def. 5.1.2). Let $\mathbb{T}$ be a tower of length λ. Notice that the whole tower can be induced from the filters F_a where a is a transitive set. Furthermore, whenever a is transitive the map $\mathrm{dom}_a : O_a \to \mathcal{P}(a)$ is a bijection. In fact, any $f \in O_a$ with $\mathrm{dom}(f) = X$ has to be $f = \pi_X$. Thus we can map any F_a with a transitive into an isomorphic filter $F'_a = \{\mathrm{dom}_a[A] : A \in F_a\}$ on $\mathcal{P}(a)$. Define $\mathbb{T}' = \{F'_a : a \in V_\lambda\}$, then we can prove the following.

Proposition 5.1.18. *If $\mathbb{T}$ is a tower of length λ then $\mathbb{T}'$ is a standard tower of length λ.*

Proof. 1. (Filter property) It follows since F'_a is isomorphic to F_a.

2. (Compatibility) Due to compatibility of $\mathbb{T}$, $A' = \mathrm{dom}_a[A] \in F'_a$ iff $A \in F_a$ iff $\pi_{ba}^{-1}[A] \in F_b$ iff $\mathrm{dom}_b[\pi_{ba}^{-1}[A]] = \{X \subseteq b : X \cap a \in A'\} \in F'_b$.
3. (Fineness) Follows directly from fineness of $\mathbb{T}$.
4. (Normality) Let $A' = \mathrm{dom}_a[A] \in I'^+_a$, $u : A' \to V$ be such that $u(X) \in X$ for all $X \in A'$. Since $u(\mathrm{dom}_a(f)) \in \mathrm{dom}_a(f)$, we can define

$$\begin{array}{rccl} v: & A & \longrightarrow & V \\ & f & \longmapsto & f(u(\mathrm{dom}_a(f))). \end{array}$$

that is regressive on A. Thus by normality of $\mathbb{T}$ there exist $a \subseteq b \in V_\lambda$, $B \subseteq \pi_{ba}^{-1}[A]$ and $y \in b$ such that $B \in I_b^+$ and for any $f \in B$, $v(\pi_{ba}(f)) = f(u(\mathrm{dom}(f) \cap a)) = f(y)$. Since f is injective, $u(\mathrm{dom}(f) \cap a) = y$, hence $B' = \mathrm{dom}_b[B]$ and y witness normality of $\mathbb{T}'$. □

Proposition 5.1.19. *If $\mathbb{T}'$ is a standard tower of length λ then $\mathbb{T}$ is a tower of length λ.*

Proof. 1. (Filter property) It follows since F'_a is isomorphic to F_a.

2. (Compatibility) By compatibility of $\mathbb{T}'$, $A \in F_a$ iff $\mathrm{dom}_a[A] \in F'_a$ iff
$$\{X \subseteq b : X \cap a \in \mathrm{dom}_a[A]\} = \mathrm{dom}_b[\pi_{ba}^{-1}[A]] \in F'_b$$
iff $\pi_{ba}^{-1}[A] \in F_b$.

3. (Fineness) Follows directly from fineness of $\mathbb{T}'$.

4. (Normality) Assume that $A \in I_a^+$ and that $u : A \to V$ is regressive, i.e. for all $f \in A$ there exists $x_f \in \mathrm{dom}(f)$ such that $u(f) \trianglelefteq f(x_f)$. Since $f = \pi_{\mathrm{dom}(f)}$ and a is transitive, $u(f) \in f(x_f)$ implies that there is an $y_f \in x_f \in \mathrm{dom}(f)$ such that $u(f) = f(y_f)$. Thus we can assume without loss of generality that x_f is such that $u(f) = f(x_f)$ and define
$$\begin{array}{rccc} v: & \mathrm{dom}_a[A] & \longrightarrow & V \\ & \mathrm{dom}_a(f) & \longmapsto & x_f. \end{array}$$
Then $v(X) \in X$ for all $X \in \mathrm{dom}_a[A]$ and we can apply normality of $\mathbb{T}'$ to find a $\mathrm{dom}_b[B] <_{\mathbb{T}} \mathrm{dom}_a[A]$, $B \in I_b^+$ and a fixed y such that for any $\mathrm{dom}_b(f) \in \mathrm{dom}_b[B]$ $v(\mathrm{dom}_b(f) \cap a) = v(\mathrm{dom}_a(\pi_{ba}(f))) = x_{\pi_{ba}(f)} = y$. Thus for any $f \in B$, $u(\pi_{ba}(f)) = f(y)$. □

5.1.3 Systems of filters in V and generic systems of ultrafilters

In this section we shall focus on ideal extenders and ideal towers in V, and their relationship with the corresponding generic systems of ultrafilters. This relation will expand from the following bidirectional procedure, mapping a V-ultrafilter in a generic extension with an ideal in V and viceversa. Full references on this procedure can be found in [16].

Definition 5.1.20. Let $\dot{F}$ be a $\mathbb{B}$-name for an ultrafilter on $\mathcal{P}^V(X)$. Let $\mathbf{I}(\dot{F}) \in V$ be the ideal on $\mathcal{P}^V(X)$ defined by:
$$\mathbf{I}(\dot{F}) = \left\{ Y \subset X : \left[\!\left[\check{Y} \in \dot{F} \right]\!\right]_{\mathbb{B}} = \mathbf{0} \right\}$$

Conversely, let I be an ideal in V on $\mathcal{P}(X)$ and consider the poset $\mathbb{C} = \mathcal{P}(X)/I$. Let $\dot{\mathbf{F}}(I)$ be the $\mathbb{C}$-name for the V-generic ultrafilter for $\mathcal{P}^V(X)$ defined by:
$$\dot{\mathbf{F}}(I) = \left\{ \langle \check{Y}, [Y]_I \rangle : \ Y \subseteq X \right\}$$

Notice that $\mathbf{I}(\dot{\mathbf{F}}(I)) = I$, while the $\mathbb{B}$-name $\dot{F}$ and the $\mathbb{C}$-name $\dot{\mathbf{F}}(\mathbf{I}(\dot{F}))$ might be totally unrelated (since $\mathbb{C} = \mathcal{P}(X)/\mathbf{I}(\dot{F})$ does not necessarily embeds completely into $\mathbb{B}$). We refer to Theorem 5.3.12 and subsequent corollary for an example of this behavior.

Definition 5.1.21. Let $\dot{F}$ be a $\mathbb{B}$-name for an ultrafilter on $\mathcal{P}^V(X)$. Set $\mathbb{C} = \mathcal{P}(X)/\mathbf{I}(\dot{F})$. The *immersion* map $i_{\dot{F}}$ is defined as follows:
$$\begin{array}{rccc} i_{\dot{F}}: & \mathbb{C} & \longrightarrow & \mathbb{B} \\ & [A]_{\mathbf{I}(\dot{F})} & \longmapsto & \left[\!\left[\check{A} \in \dot{F} \right]\!\right]_{\mathbb{B}} \end{array}$$

Proposition 5.1.22. *Let $\dot{F}$, $i_{\dot{F}}$ be as in the previous definition. Then $i_{\dot{F}}$ is a (not necessarily complete) morphism of boolean algebras.*

Proof. By definition of $\mathbf{I}(\dot{F})$, the morphism is well-defined. Since $\dot{F}$ is a $\mathbb{B}$-name for a filter $i_{\dot{F}}$ preserves the order of boolean algebras, and since $\dot{F}$ satisfies the ultrafilter property it also preserves complementation. □

The above can be immediately extended to systems of filters, by means of the following.

Definition 5.1.23. Let $\dot{\mathcal{S}} = \langle \dot{F}_a : a \in \mathcal{C} \rangle$ be a $\mathbb{B}$-name for a $\mathcal{C}$-system of ultrafilters. Then $\mathbf{I}(\dot{\mathcal{S}}) = \langle I_a = \mathbf{I}(\dot{F}_a) : a \in \mathcal{C} \rangle$ is the corresponding system of filters in V. Conversely, let $\mathbb{S} = \langle I_a : a \in \mathcal{C} \rangle$ be a $\mathcal{C}$-system of filters in V. Then $\dot{\mathbf{F}}(\mathbb{S}) = \langle \dot{F}_a = \dot{\mathbf{F}}(I_a) : a \in \mathcal{C} \rangle$ is the corresponding $\mathbb{S}$-name for a system of ultrafilters.

Proposition 5.1.24. *Let $\dot{\mathcal{S}}$ be a $\mathbb{B}$-name for a $\mathcal{C}$-system of ultrafilters. Then $\mathbf{I}(\dot{\mathcal{S}})$ is a $\mathcal{C}$-system of filters in V. Conversely, let $\mathbb{S}$ be a $\mathcal{C}$-system of filters in V. Then $\dot{\mathbf{F}}(\mathbb{S})$ is the canonical $\langle \mathbb{S}^+, <_{\mathbb{S}} \rangle$-name for the V-generic filter on $\mathbb{S}$ and defines a $\mathcal{C}$-system of V-ultrafilters.*

Proof. $\mathbf{I}(\dot{\mathcal{S}})$ and $\dot{\mathbf{F}}(\mathbb{S})$ satisfy the following properties.

1. *(Filter and ultrafilter property)* Left to the reader.
2. *(Fineness)* Fix $a \in \mathcal{C}$.

   $\mathbf{I}(\dot{\mathcal{S}})$: Since F_a is fine and $\dot{\mathbf{F}}(I_a)$ contains F_a with boolean value $\mathbf{1}$, the latter is also fine.

   $\dot{\mathbf{F}}(\mathbb{S})$: Let $A = \{f \in O_a : x \notin \mathrm{dom}(f)\}$ be obtained from any $x \in a$. Since $\dot{F}_a$ is fine with boolean value $\mathbf{1}$, $\left[\!\left[A \in \dot{F}_a \right]\!\right] = \mathbf{0}$ hence $A \in \mathbf{I}(\dot{F}_a)$.
3. *(Compatibility)* Fix $a \subseteq b$ in $\mathcal{C}$, $A \subseteq O_a$.

   $\mathbf{I}(\dot{\mathcal{S}})$: $A \in \mathbf{I}(\dot{F}_a) \iff \left[\!\left[\check{A} \in \dot{F}_a \right]\!\right] = \mathbf{0}_{\mathbb{B}} = \left[\!\left[\pi_{ba}^{-1}[\check{A}] \in \dot{F}_b \right]\!\right] \iff \pi_{ba}^{-1}[A] \in \mathbf{I}(\dot{F}_b)$.

   $\dot{\mathbf{F}}(\mathbb{S})$: $\left[\!\left[\check{A} \in \dot{\mathbf{F}}(I_a) \right]\!\right] = [A]_{\mathbb{S}} = \left[\pi_{ba}^{-1}[A]\right]_{\mathbb{S}} = \left[\!\left[\pi_{ba}^{-1}[\check{A}] \in \dot{\mathbf{F}}(I_a) \right]\!\right]$.
4. *(Normality)* Let $u : O_a \to V$ in V be regressive on A.

   $\mathbf{I}(\dot{\mathcal{S}})$: Suppose that $A \in \mathbf{I}(\dot{\mathcal{S}})^+$ (i.e. $\left[\!\left[\check{A} \in \dot{F}_a \right]\!\right] = p > \mathbf{0}$). Then $p \Vdash \check{A} \in \dot{F}_a$ implies that

   $$p \Vdash \exists X <_{\dot{\mathcal{S}}} \check{A}\ X \in \dot{\mathcal{S}}^+\ \exists y \in \check{b}\ \forall f \in X\ \check{u}(\pi_{ba}(f)) = f(y)$$

   Thus by the forcing theorem there is a $q < p$, $q > \mathbf{0}$ and fixed $B \subseteq \pi_{ba}^{-1}[A]$, $y \in b$ such that q forces the above formula with the quantified X replaced by B. Then $\left[\!\left[\check{B} \in \dot{\mathcal{S}}^+ \right]\!\right] \geq q > \mathbf{0} \Rightarrow B \in \mathbf{I}(\dot{\mathcal{S}})^+$, and $\forall f \in B\ u(\pi_{ba}(f)) = f(y)$ holds true in V.

   $\dot{\mathbf{F}}(\mathbb{S})$: Consider the system $\dot{\mathbf{F}}(\mathbb{S})$. Suppose that $A \in I_a^+$. Given any $C \leq_{\mathbb{S}} A$ in I_c^+, we can find $B \leq_{\mathbb{S}} C$ in $\mathbb{S}^+$ witnessing the normality of $\mathbb{S}$ for the regressive map on C defined by $h \mapsto u(h \restriction a)$. We conclude that there are densely many B below A such that $\exists y \in b\, \forall f \in B\; u(\pi_{ba}(f)) = f(y)$, hence

$$\left[\!\left[\exists B <_{\dot{\mathbf{F}}(\mathbb{S})} \check{A}\; B \in \dot{\mathbf{F}}(\mathbb{S})^+\; \exists y \in \check{b}\; \forall f \in B\; \check{u}(\pi_{ba}(f)) = f(y)\right]\!\right] \geq [A]_{\mathbb{S}}$$

and $[A]_{\mathbb{S}} = \left[\!\left[\check{A} \in \dot{\mathbf{F}}(\mathbb{S})^+\right]\!\right]$. □

As already noticed for single filters, the maps $\mathbf{I}$ and $\dot{\mathbf{F}}$ are not inverse of each other and $\dot{\mathbf{F}}(\mathbf{I}(\dot{\mathcal{S}}))$ might differ from $\dot{\mathcal{S}}$.

5.1.4 Embedding derived from a system of ultrafilters

We now introduce a notion of *ultrapower* induced by a $\mathcal{C}$-system of V-ultrafilters $\mathcal{S}$. Notice that the results of the last section allows to translate any result about $\mathcal{C}$-systems of V-ultrafilters to a result on $\mathcal{C}$-systems of filters in V, by simply considering the $\mathcal{C}$-system of V-ultrafilters $\dot{\mathbf{F}}(\mathbb{S})$.

Definition 5.1.25. Let $V \subseteq W$ be transitive models of ZFC and $\mathcal{S} \in W$ be a $\mathcal{C}$-system of V-ultrafilters. Let

$$U_{\mathcal{S}} = \{u : O_a \to V \;:\; a \in \mathcal{C},\; u \in V\}.$$

Define the relations

$$u =_{\mathcal{S}} v \iff \{f \in O_c :\; u(\pi_{ca}(f)) = v(\pi_{cb}(f))\} \in F_c$$
$$u \in_{\mathcal{S}} v \iff \{f \in O_c :\; u(\pi_{ca}(f)) \in v(\pi_{cb}(f))\} \in F_c$$

where $O_a = \mathrm{dom}(u)$, $O_b = \mathrm{dom}(v)$, $c = a \cup b$. The ultrapower of V by $\mathcal{S}$ is $\mathrm{Ult}(V, \mathcal{S}) = \langle U_{\mathcal{S}} / =_{\mathcal{S}},\; \in_{\mathcal{S}}\rangle$.

We leave to the reader to check that the latter definition is well-posed. From now on, we identify the well-founded part of the ultrapower with its Mostowski collapse.

Theorem 5.1.26 (Łoś). *Let $\phi(x_1, \ldots, x_n)$ be a formula and let $u_1, \ldots, u_n \in U_{\mathcal{S}}$. Then $\mathrm{Ult}(V, \mathcal{S}) \models \phi([u_1]_{\mathcal{S}}, \ldots, [u_n]_{\mathcal{S}})$ if and only if*

$$\{f \in O_b :\; \phi(u_1(\pi_{ba_1}(f)), \ldots, u_n(\pi_{ba_n}(f)))\} = A \in F_b$$

where $O_{a_i} = \mathrm{dom}(u_i)$ for $i = 1 \ldots n$, $b = \bigcup a_i$.

Proof. We proceed by induction on ϕ. The case ϕ atomic follows directly from the definition of ultrapower, and the case for $\neg$ and $\vee$ is easily handled, so we focus on the case $\phi(\vec{x}) = \exists y\; \psi(y, \vec{x})$. First, suppose that $A \in F_b$ and define $v : O_b \to V$ so that $\psi(v(f), u_1(\pi_{ba_1}(f)), \ldots, u_n(\pi_{ba_n}(f))$ holds if $f \in A$, $v(f) = \emptyset$ otherwise. Then $\mathrm{Ult}(V, \mathcal{S}) \models \psi([v]_{\mathcal{S}}, [u_1]_{\mathcal{S}}, \ldots, [u_n]_{\mathcal{S}})$ by inductive hypothesis and the thesis follows.

Conversely, suppose that $\mathrm{Ult}(V, \mathcal{S}) \models \exists y\; \psi(y, [u_1]_{\mathcal{S}}, \ldots, [u_n]_{\mathcal{S}})$, and fix $v \in U_{\mathcal{S}}$ such that $\mathrm{Ult}(V, \mathcal{S}) \models \psi([v]_{\mathcal{S}}, [u_1]_{\mathcal{S}}, \ldots, [u_n]_{\mathcal{S}})$, $v : O_c \to V$ with $c \supseteq b$. Then, by inductive hypothesis,

$$\{f \in O_c :\; \psi(v(f), u_1(\pi_{ca_1}(f)), \ldots, u_n(\pi_{ca_n}(f)))\} = B \in F_c,$$

hence $\pi_{cb}^{-1}[A] \supseteq B$ is also in F_c and by coherence $A \in F_b$. □

As in common model-theoretic use, define $j_{\mathcal{S}} : V \to \mathrm{Ult}(V, \mathcal{S})$ by $j_{\mathcal{S}}(x) = [c_x]_{\mathcal{S}}$ where $c_x : O_\emptyset \to \{x\}$. From the last theorem it follows that the map $j_{\mathcal{S}}$ is elementary. Notice that the proof of the last theorem does not use neither *fineness* nor *normality* of the system of ultrafilters. However these properties allows us to study the elements of the ultrapower by means of the following proposition.

Proposition 5.1.27. *Let $\mathcal{S}$ be a $\mathcal{C}$-system of ultrafilters, $j : V \to M = \mathrm{Ult}(V, \mathcal{S})$ be the derived embedding. Then,*

1. $[c_x]_{\mathcal{S}} = j(x)$ *for any* $x \in V$*;*
2. $[\mathrm{proj}_x]_{\mathcal{S}} = x$ *for any* $x \in \bigcup \mathcal{C}$*, where*

$$\begin{array}{rccc} \mathrm{proj}_x : & O_{\{x\}} & \longrightarrow & V \\ & f & \longmapsto & f(x) \end{array}$$

3. $[\mathrm{ran}_a]_{\mathcal{S}} = a$ *for any* $a \in \mathcal{C}$*;*
4. $[\mathrm{dom}_a]_{\mathcal{S}} = j[a]$ *for any* $a \in \mathcal{C}$*;*
5. $[\mathrm{id}_a]_{\mathcal{S}} = (j \restriction a)^{-1}$ *for any* $a \in \mathcal{C}$*.*

Proof. 1. Follows from the definition of j.

2. By induction on $\mathrm{rank}(x)$. Fix $x \in \bigcup \mathcal{C}$. If $y \in x$, then $y = [\mathrm{proj}_y]_{\mathcal{S}} \in [\mathrm{proj}_x]_{\mathcal{S}}$ since all f in $O_{\{x,y\}}$ are $\in$-preserving. Conversely, assume that $x \in a$ and $[u : O_a \to V]_{\mathcal{S}} \in [\mathrm{proj}_x]_{\mathcal{S}}$. By Łoś's Theorem, u is regressive on $A = \{f \in O_a : u(f) \in f(x)\} \in F_a$ thus there exist $y \in b \supseteq a$ and $B \subseteq \pi_b^{-1}[A]$ in F_b such that $u(\pi_a(f)) = f(y)$ for all $f \in B$. Since $f(y) = u(\pi_a(f)) \in f(x)$ and any $f \in B$ is $\in$-preserving, it follows that $y \in x$. Finally, by Łoś's Theorem and inductive hypothesis $[u]_{\mathcal{S}} = [\mathrm{proj}_y]_{\mathcal{S}} = y \in x$.

3. Fix $a \in \mathcal{C}$. If $y = [\mathrm{proj}_y]_{\mathcal{S}} \in a$, by fineness

$$\{f \in O_a : f(y) \in \mathrm{ran}_a(f)\} = \{f \in O_a : y \in \mathrm{dom}(f)\} \in F_a.$$

thus $y = [\mathrm{proj}_y]_{\mathcal{S}} \in [\mathrm{ran}_a]_{\mathcal{S}}$ by Łoś's Theorem. Conversely, assume that $u : O_b \to V$ is such that $[u]_{\mathcal{S}} \in [\mathrm{ran}_a]_{\mathcal{S}}$, $b \supseteq a$. By Łoś's Theorem, u is regressive on

$$A = \{f \in O_b : u(f) \in \mathrm{ran}_a(\pi_a(f)) = f[a]\} \in F_b$$

thus by normality there exist $y \in c \supseteq b$ and $B \subseteq \pi_c^{-1}[A]$ such that $u(\pi_b(f)) = f(y)$ for all $f \in B$. Since $B \subseteq \pi_c^{-1}[A]$, $f(y) = u(\pi_b(f)) = f(x)$ for some $x \in a$. Since f is injective, $x = y \in a$ and $[u]_{\mathcal{S}} = [\mathrm{proj}_y]_{\mathcal{S}} = y$ by Łoś's Theorem.

4. Fix $a \in \mathcal{C}$. If $x \in a$, by fineness $\{f \in O_a : x \in \mathrm{dom}_a(f)\} \in F_a$ hence $j(x) = [c_x]_{\mathcal{S}} \in [\mathrm{dom}_a]_{\mathcal{S}}$. Conversely, assume $[u : O_b \to V]_{\mathcal{S}} \in [\mathrm{dom}_a]_{\mathcal{S}}$ with $b \supseteq a$. By Łoś's Theorem, $A = \{f \in O_b : u(f) \in \mathrm{dom}_a(\pi_a(f))\} \in F_b$ and we can define

$$\begin{array}{rccc} v : & A & \longrightarrow & V \\ & f & \longmapsto & f(u(f)). \end{array}$$

that is regressive on A. Then by normality there exist $y \in c \supseteq b$ and $B \subseteq \pi_c^{-1}[A]$ such that $v(\pi_b(f)) = f(u(\pi_b(f))) = f(y)$ for all $f \in B$. Since f is injective, $u(\pi_b(f)) = y$ hence y is in $\mathrm{dom}_a(\pi_a(f)) = \mathrm{dom}(f) \cap a$. Thus by Łoś's Theorem $[u]_{\mathcal{S}} = [c_y]_{\mathcal{S}} = j(y)$.

5. Follows from points 3 and 4, together with the observation that $[\mathrm{id}_a]_{\mathcal{S}}$ has to be an $\in$-preserving function by Łoś's Theorem and $(j \restriction a)^{-1}$ is the only such function with domain $j[a]$ and range a. $\square$

These canonical representatives can be used in order to prove many general properties of $\mathcal{C}$-system of filters and of the induced ultrapowers. In particular, we shall use them to prove Propositions 5.1.11 and 5.1.12, and other related properties.

Proposition 5.1.28. *Let $\mathcal{S}$ be a $\mathcal{C}$-system of ultrafilters, $A \subseteq O_a$ be such that $a \in \mathcal{C}$. Then $A \in F_a$ if and only if $(j_{\mathcal{S}} \restriction a)^{-1} \in j_{\mathcal{S}}(A)$.*

Proof. By Łoś's Theorem, we have $A = \{f \in O_a : \ f = \mathrm{id}_a(f) \in A\} \in F_a$ if and only if $(j_{\mathcal{S}} \restriction a)^{-1} = [\mathrm{id}_a]_{\mathcal{S}} \in [c_A]_{\mathcal{S}} = j_{\mathcal{S}}(A)$. $\square$

Lemma 5.1.29. *Let $\mathbb{S}$ be a $\mathcal{C}$-system of filters and $a \in \mathcal{C}$. Then κ_a is the minimum α such that $\left[\!\left[j_{\dot{\mathbf{F}}(\mathbb{S})}(\check{\alpha}) \geq \mathrm{rank}(\check{a}) \right]\!\right]_{\mathbb{S}} = \mathbf{1}$.*

Proof. Let j be the elementary embedding derived from $\dot{\mathbf{F}}(\mathbb{S})$ in a generic extension by $\mathbb{S}$. Notice that $O_a \cap {}^aV_\alpha \in F_a$ is equivalent by Proposition 5.1.28 to

$$(j \restriction a)^{-1} \in j(O_a \cap {}^aV_\alpha) = j(O_a) \cap {}^{j(a)}V_{j(\alpha)}$$

which is in turn equivalent to $a \subseteq V_{j(\alpha)}$ i.e. $j(\alpha) \geq \mathrm{rank}(a)$. Since this holds in all generic extensions by $\mathbb{S}$, we are done. $\square$

Proof of Proposition 5.1.11. By the previous proposition κ_a is the minimum α such that $\mathbf{1} \Vdash_{\mathbb{S}} j(\alpha) \geq \mathrm{rank}(a)$, hence it depends (monotonically) only on $\mathrm{rank}(a)$. The conclusion of Proposition 5.1.11 follows. $\square$

Proposition 5.1.30. *Let $\mathbb{S}$ be a $\mathcal{C}$-system of filters. Then κ is the critical point of $j = j_{\dot{\mathbf{F}}(\mathbb{S})}$ with boolean value $\mathbf{1}$ iff $\mathbb{S}$ is a $\langle \kappa, \lambda \rangle$-system of filters.*

Proof. Suppose that κ is the critical point of j with boolean value $\mathbf{1}$. If $\gamma < \kappa$, $A \in F_{\{\gamma\}}$ iff $\mathbf{1} \Vdash_{\mathbb{S}} (j \restriction \{\gamma\})^{-1} = j(\mathrm{id} \restriction \{\gamma\}) \in j(A)$ iff $\mathrm{id} \restriction \{\gamma\} \in A$. Thus $F_{\{\gamma\}}$ is principal generated by $\mathrm{id} \restriction \{\gamma\}$. If $a \in \mathcal{C} \cap V_{\kappa+2}$, $\mathrm{rank}(a) \leq \kappa + 1 \leq j_{\dot{\mathbf{F}}(\mathbb{S})}(\kappa)$ with boolean value $\mathbf{1}$, thus $\kappa_a \leq \kappa$ by Lemma 5.1.29.

Conversely, suppose that $\{\mathrm{id} \restriction \{\gamma\}\} \in F_{\{\gamma\}}$ for $\gamma < \kappa$, and $\kappa_a \leq \kappa$ for $a \in V_{\kappa+2}$. If there is an $A \in \mathbb{S}^+$ forcing that j has no critical point or has critical point bigger than κ, $\kappa_a = \mathrm{rank}(a) > \kappa$ for $a \in V_{\kappa+2} \setminus V_{\kappa+1}$, a contradiction. If there is a $B \in \mathbb{S}^+$ forcing that j has critical point γ smaller than κ, $F_{\{\gamma\}}$ cannot be principal generated by $\mathrm{id} \restriction \{\gamma\}$, again a contradiction. $\square$

Proof of Proposition 5.1.12. Let $\mathbb{S}$ be a $\langle\kappa,\lambda\rangle$-system of filters, a be in $\mathcal{C}$, j be derived from $\dot{\mathbf{F}}(\mathbb{S})$. We need to prove that F_a is $<\kappa$-complete for all a. Suppose that $\mathcal{A} \subseteq F_a$ is such that $|\mathcal{A}| < \kappa$. Hence by Proposition 5.1.30, $j(A) = j[A]$. Then $\bigcap \mathcal{A} \in F_a$ iff

$$\mathbf{1} \Vdash_{\mathbb{S}} (j{\restriction}a)^{-1} \in j(\bigcap \mathcal{A}) = \bigcap j(\mathcal{A}) = \bigcap j[\mathcal{A}]$$

which is true since $A \in F_a \Rightarrow \mathbf{1} \Vdash_{\mathbb{S}} (j{\restriction}a)^{-1} \in j(A)$ for all $A \in \mathcal{A}$. □

The ultrapower $\mathrm{Ult}(V,\mathcal{S})$ happens to be the direct limit of the directed system of ultrapowers $\langle \mathrm{Ult}(V,F_a) : a \in \mathcal{C}\rangle$ with the following factor maps:

$$\begin{array}{rccc} k_{ab}: & \mathrm{Ult}(V,F_a) & \longrightarrow & \mathrm{Ult}(V,F_b) \\ & [u]_{F_a} & \longmapsto & [u \circ \pi_{ba}]_{F_b} \end{array}$$

$$\begin{array}{rccc} k_{a}: & \mathrm{Ult}(V,F_a) & \longrightarrow & \mathrm{Ult}(V,\mathcal{S}) \\ & [u]_{F_a} & \longmapsto & [u]_{\mathcal{S}} \end{array}$$

The ultrapower $\mathrm{Ult}(V,\mathcal{S})$ is also the direct limit of the ultrapowers given by the restrictions of $\mathcal{S}$, as shown in the following.

Definition 5.1.31. Let $\mathbb{S}$ be a $\mathcal{C}$-system of filters of length λ, $\alpha < \lambda$ be an ordinal. The restriction of $\mathbb{S}$ to α is the $(\mathcal{C} \cap V_\alpha)$-system of filters $\mathbb{S}{\restriction}\alpha = \{F_a : a \in \mathcal{C} \cap V_\alpha\}$. Moreover, if $\mathcal{S}$ is a $\mathcal{C}$-system of ultrafilters the corresponding factor map is

$$\begin{array}{rccc} k_{\alpha}: & \mathrm{Ult}(V,\mathcal{S}{\restriction}\alpha) & \longrightarrow & \mathrm{Ult}(V,\mathcal{S}) \\ & [u]_{\mathcal{S}{\restriction}\alpha} & \longmapsto & [u]_{\mathcal{S}}. \end{array}$$

Proposition 5.1.32. *Let $\mathcal{S}$ be a $\mathcal{C}$-system of ultrafilters of length λ, $\alpha < \lambda$ be an ordinal. Then*

1. *k_α is elementary;*
2. *$k_\alpha \circ j_{\mathcal{S}{\restriction}\alpha} = j_{\mathcal{S}}$;*
3. *$k_\alpha {\restriction} \bigcup(\mathcal{C} \cap V_\alpha) = \mathrm{id} {\restriction} \bigcup(\mathcal{C} \cap V_\alpha)$, hence $\mathrm{crit}(k_\alpha) \geq \alpha$.*

Proof. A particular case of Proposition 5.1.36 to follow. □

5.1.5 System of ultrafilters derived from an embedding

We now present the definitions and main properties of $\mathcal{C}$-system of ultrafilters derived from a generic elementary embedding. With abuse of notation, we denote as generic elementary embedding any map $j : V \to M$ which is elementary and such that $M \subseteq W$ for some $W \supseteq V$. In the following we shall assume that j is a definable class in W. However, we believe that it should be possible to adapt the present results to non-definable j, provided we are working in a strong enough set theory with sets and classes (e.g. MK). We also provide a comparison between derived $\mathcal{C}$-systems of ultrafilters for different choices of $\mathcal{C}$.

Let $\mathcal{S}$ be a $\mathcal{C}$-system of ultrafilters, $A \subseteq O_a$ be such that $a \in \mathcal{C}$. Then by Proposition 5.1.28,

$$A \in F_a \iff (j_{\mathcal{S}}{\restriction}a)^{-1} \in j_{\mathcal{S}}(A)$$

and this relation actually provides a definition of $\mathcal{S}$ from $j_{\mathcal{S}}$. This justifies the following definition.

Definition 5.1.33. Let $V \subseteq W$ be transitive models of ZFC. Let $j : V \to M \subseteq W$ be a generic elementary embedding definable in W, $\mathcal{C} \in V$ be a directed set of domains such that for any $a \in \mathcal{C}$, $(j \restriction a)^{-1} \in M$. The *$\mathcal{C}$-system of ultrafilters derived from j* is $\mathcal{S} = \langle F_a : \ a \in \mathcal{C} \rangle$ such that:

$$F_a = \left\{ A \subseteq O_a : (j \restriction a)^{-1} \in j(A) \right\}.$$

Definition 5.1.33 combined with Proposition 5.1.28 guarantees that for a given a $\mathcal{C}$-system of ultrafilters $\mathcal{S}$, the $\mathcal{C}$-system of ultrafilters derived from $j_{\mathcal{S}}$ is $\mathcal{S}$ itself. We now show that the definition is meaningful for any embedding j.

Proposition 5.1.34. *Let j, $\mathcal{C}$, $\mathcal{S}$ be as in the definition above. Then $\mathcal{S}$ is a $\mathcal{C}$-system of V-ultrafilters.*

Proof.
1. *(Filter and ultrafilter property)* Fix $a \in \mathcal{C}$ and assume that $A, B \in F_a$. Then $(j \restriction a)^{-1} \in j(A) \cap j(B) = j(A \cap B)$. Moreover if $C \subseteq O_a$ and $A \subseteq C$, then $(j \restriction a)^{-1} \in j(A) \subseteq j(C)$. Finally, if $(j \restriction a)^{-1} \notin j(A)$ we have that $(j \restriction a)^{-1} \in j(O_a) \setminus j(A) = j(O_a \setminus A)$.

2. *(Fineness)* Fix $x \in a$ so that $j(x) \in j[a]$. Then $j(x) \in \mathrm{dom}((j \restriction a)^{-1})$ hence we have $\{f \in O_a : x \in \mathrm{dom}(f)\} \in F_a$ by definition of F_a.

3. *(Compatibility)* Assume that $a \subseteq b \in \mathcal{C}$ and $A \subseteq O_a$. Then

$$(j \restriction b)^{-1} \in j(\pi_{ba}^{-1}[A]) = \left\{ f \in O_{j(b)} : \pi_{j(a)}(f) \in j(A) \right\}$$

if and only if $(j \restriction a)^{-1} = \pi_{j(a)}((j \restriction b)^{-1}) \in j(A)$.

4. *(Normality)* Let $u : A \to V$ be regressive on $A \in F_a$ and in V. By elementarity,

$$M \models \forall f \in j(A) \ \exists x \in \mathrm{dom}(f) \ j(u)(f) \trianglelefteq f(x)$$

Since $(j \restriction a)^{-1} \in j(A)$, there exists $x \in j[a]$ with $j(u)((j \restriction a)^{-1}) \trianglelefteq (j \restriction a)^{-1}(x)$. Define $y = j(u)((j \restriction a)^{-1})$, and put $b = a \cup \{y\}$. Note that $y \trianglelefteq j^{-1}(x) \in a$ hence by transitivity of $\bigcup \mathcal{C}$, $\{y\} \in \mathcal{C}$. Define $B = \{f \in O_b : u(\pi_{ba}(f)) = f(y)\}$. Then $(j \restriction b)^{-1} \in j(B)$, i.e. $B \in F_b$, since

$$(j \restriction b)^{-1}(j(y)) = y = j(u)((j \restriction a)^{-1}) = j(u)(\pi_{j(a)}((j \restriction b)^{-1})).$$ □

Given a $\mathcal{C}$-system of ultrafilters $\mathcal{S}$ derived from a generic embedding j, we can factor out the embedding j through $j_{\mathcal{S}}$.

Definition 5.1.35. Let $j : V \to M \subseteq W$ be a generic elementary embedding, $\mathcal{C} \in V$ be a directed set of domains of length λ, $\mathcal{S}$ be the $\mathcal{C}$-system of ultrafilters derived from j. Then

$$\begin{array}{rccl} k: & \mathrm{Ult}(V, \mathcal{S}) & \longrightarrow & M \\ & [u : O_a \to V]_{\mathcal{S}} & \longmapsto & j(u)((j \restriction a)^{-1}) \end{array}$$

is the *factor map associated to $\mathcal{S}$*.

Proposition 5.1.36. *Let j, $\mathcal{C}$, λ, $\mathcal{S}$, k be as in the previous definition. Then*

1. *k is elementary;*

2. *$k \circ j_{\mathcal{S}} = j$;*

3. *$k \restriction \bigcup \mathcal{C} = \mathrm{id} \restriction \bigcup \mathcal{C}$ hence $\mathrm{crit}(k) \geq \lambda$;*

4. *if $\lambda = j(\gamma)$ for some γ, then $\mathrm{crit}(k) > \lambda$.*

Proof. 1. Let $\phi(x_1, \ldots, x_n)$ be a formula, and for any $i \in n$ let $u_i : O_{a_i} \to V$, $a_i \in \mathcal{C}$. Put $b = \bigcup \{a_i : 1 \leq i \leq n\}$. Then $\mathrm{Ult}(V, \mathcal{S}) \models \phi([u_1]_{\mathcal{S}}, \ldots, [u_n]_{\mathcal{S}})$ if and only if (by Łoś Theorem)

$$B = \{f \in O_b : \phi(u_1(\pi_{a_1}(f)), \ldots, u_n(\pi_{a_n}(f)))\} \in F_b.$$

if and only if $(j \restriction b)^{-1} \in j(B)$ (by definition of F_b) i.e.

$$M \models \phi(j(u_1)(\pi_{j(a_1)}(j \restriction b)^{-1}), \ldots, j(u_n)(\pi_{j(a_n)}(j \restriction b)^{-1}))$$

if and only if (by definition of $\pi_{j(a_i)}$)

$$M \models \phi(j(u_1)((j \restriction a_i)^{-1}), \ldots, j(u_n)((j \restriction a_n)^{-1})).$$

i.e. $M \models \phi(k([u_1]_{\mathcal{S}}), \ldots, k([u_n]_{\mathcal{S}})$.

2. For any $x \in V$,

$$k(j_{\mathcal{S}}(x)) = k([c_x]_{\mathcal{S}}) = j(c_x)(\emptyset) = c_{j(x)}(\emptyset) = j(x).$$

3. Let $x \in \bigcup \mathcal{C}$. Then by Proposition 5.1.27 for some $a \in \mathcal{C}$ with $x \in a$,

$$k(x) = k([\mathrm{proj}_x]_{\mathcal{S}}) = j(\mathrm{proj}_x)((j \restriction a)^{-1}) = j^{-1}(j(x)) = x.$$

4. If $\lambda = j(\gamma)$, the following diagram commutes:

$$\begin{array}{ccccc} V & \xrightarrow{\;j\;} & M & \subseteq & W \\ & \searrow^{j_{\mathcal{S}}} & \uparrow k & & \cup\! \mid \\ & & \mathrm{Ult}(V, \mathcal{S}) & \subseteq & V[\mathcal{S}] \end{array}$$

Thus $\mathrm{crit}(k) \geq j(\gamma)$ and $k \circ j_{\mathcal{S}}(\gamma) = j(\gamma)$. Therefore $j(\gamma) \in \mathrm{ran}(k)$ hence $j(\gamma)$ cannot be the critical point of k, showing that the above inequality is strict[3]. □

Observe that in Definition 5.1.33 $(j \restriction a)^{-1} \notin M$ would imply that the derived filter F_a is empty. Thus, depending on the choice of $\mathcal{C}$, there can be a limit on the maximal length attainable for a $\mathcal{C}$-system of ultrafilter derived from j. If $\mathcal{C} = [\lambda]^{<\omega}$, $(j \restriction a)^{-1}$ is always in M thus there is no limit on the length of the extenders derived from j. If $\mathcal{C} = V_\lambda$, the maximal length is the minimal λ such that $j[V_\lambda] \notin M$. These bounds are relevant, as shown in the following proposition.

[3]Remark that in the above diagram $V[\mathcal{S}]$ is the smallest transitive model N of ZFC such that $V, \{\mathcal{S}\} \subseteq N \subseteq W$.

Proposition 5.1.37. *Let $\mathcal{T} \in W$ be a tower of length a limit ordinal λ, $j : V \to M \subseteq W$ be the derived embedding. Then the tallest tower derivable from j is $\mathcal{T}$.*

Proof. Since dom_{V_α} represents $j[V_\alpha]$ for all $\alpha < \lambda$, $j[V_\alpha] \in M$ for all $\alpha < \lambda$ and we only need to prove that $j[V_\lambda] \notin M$. Suppose by contradiction that $u : O_a \to V$ is such that $[u]_{\mathcal{T}} = j[V_\lambda]$. Let $\alpha < \lambda$ be such that $a \in V_\alpha$, and let $v : O_a \to V$ be such that $v(x) = u(x) \cap V_{\alpha+1}$. Thus $[v]_{\mathcal{T}} = j[V_\lambda] \cap j(V_{\alpha+1}) = j[V_{\alpha+1}]$, and by Łoś Theorem

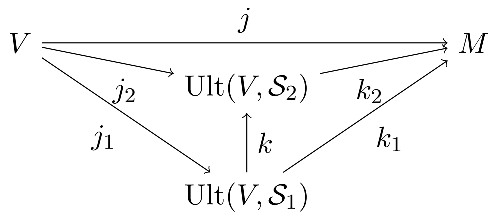
$$A = \left\{ f \in O_{V_{\alpha+1}} : \ v(\pi_a(f)) = \mathrm{dom}(f) \right\} \in F_{V_{\alpha+1}}$$

Since $|\mathrm{dom}[A]| \leq |\mathrm{ran}(v)| \leq |O_a| < |V_{\alpha+1}|$, $\mathrm{dom}[A]$ is a non-stationary subset of $V_{\alpha+1}$ by Lemma 1.2.13 contradicting Proposition 5.1.10. □

We now consider the relationship between different $\mathcal{C}$-systems of ultrafilters derived from a single j.

Proposition 5.1.38. *Let $j : V \to M \subseteq W$ be a generic elementary embedding definable in W, $\mathcal{C}_1 \subseteq \mathcal{C}_2$ be directed sets of domains in V, $\mathcal{S}_n$ be the $\mathcal{C}_n$-system of V-ultrafilters derived from j for $n = 1, 2$. Then* $\mathrm{Ult}(V, \mathcal{S}_2)$ *can be factored into* $\mathrm{Ult}(V, \mathcal{S}_1)$, *and* $\mathrm{crit}(k_1) \leq \mathrm{crit}(k_2)$ *where k_1, k_2 are the corresponding factor maps.*

Proof. We are in the following situation:



where k is defined as

$$\begin{array}{rccc} k : & \mathrm{Ult}(V, \mathcal{S}_1) & \longrightarrow & \mathrm{Ult}(V, \mathcal{S}_2) \\ & [u]_{\mathcal{S}_1} & \longmapsto & [u]_{\mathcal{S}_2} \end{array}$$

Observe that j_1, j_2 and k commute. Moreover given $u : O_a \to V$ with $a \in \mathcal{C}_1$,

$$k_2 \circ k([u]_{\mathcal{S}_1}) = j(u)\left((j \restriction a)^{-1}\right) = k_1([u]_{\mathcal{S}_1})$$

therefore the diagram commutes. Since k_1 and k_2 are elementary, k has to be elementary as well and $\mathrm{crit}(k_1) \leq \mathrm{crit}(k_2)$. □

Notice that the last proposition can be applied whenever $\mathcal{S}_1$ is an extender and $\mathcal{S}_2$ is a tower, both of the same length λ and derived from the same generic elementary embedding $j : V \to M \subseteq W$. It is also possible for a "thinner" system of filters (i.e. an extender) to factor out a "fatter" one.

Definition 5.1.39. Let F be an ultrafilter. We denote by $\mathrm{non}(F)$ the minimum of $|A|$ for $A \in F$. Let $\mathcal{S}$ be a $\mathcal{C}$-system of ultrafilters. We denote by $\mathrm{non}(\mathcal{S})$ the supremum of $\mathrm{non}(F_a) + 1$ for $a \in \mathcal{C}$.

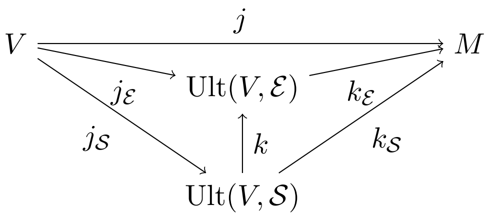
If the length of $\mathcal{S}$ is a limit ordinal λ, $\mathrm{non}(\mathcal{S})$ is bounded by $\beth_\lambda$. If $\mathcal{E}$ is a γ-extender of regular length $\lambda > \gamma$, $\mathrm{non}(\mathcal{S})$ is also bounded by $2^{<\lambda} + 1$.

Theorem 5.1.40. *Let $\mathcal{C}$ be a directed set of domains. Let $j : V \to M \subseteq W$ be a generic elementary embedding definable in W, $\mathcal{S}$ be the $\mathcal{C}$-system of filters derived from j, $\mathcal{E}$ be the extender of length $\lambda \supseteq j[\mathrm{non}(\mathcal{S})]$ derived from j. Then* $\mathrm{Ult}(V, \mathcal{E})$ *can be factored into* $\mathrm{Ult}(V, \mathcal{S})$*, and* $\mathrm{crit}(k_\mathcal{S}) \leq \mathrm{crit}(k_\mathcal{E})$.

Proof. Let $\rho_a : [\mathrm{non}(F_a)]^1 \to O_a$ be an enumeration of an $A \in F_a$ of minimum cardinality, so that $(j{\restriction}a)^{-1} \in j(A) = \mathrm{ran}(j(\rho_a))$. Let k be defined by

$$\begin{array}{rrcl} k: & \mathrm{Ult}(V,\mathcal{S}) & \longrightarrow & \mathrm{Ult}(V,\mathcal{E}) \\ & [u : O_a \to V]_\mathcal{S} & \longmapsto & [u \circ \rho_a \circ \mathrm{ran}_{\{\beta\}}]_\mathcal{E} \end{array}$$

where $\beta < j(\mathrm{non}(F_a)) \leq \lambda$ is such that $j(\rho_a)(\{\beta\}) = (j{\restriction}a)^{-1}$. We are in the following situation:



Observe that $j_\mathcal{S}$, $j_\mathcal{E}$ and k commute. Moreover given $u : O_a \to V$ with $a \in \mathcal{C}$,

$$\begin{aligned} k_\mathcal{E} \circ k([u]_\mathcal{S}) &= j(u \circ \rho_a \circ \mathrm{ran}_{\{\beta\}}) \left((j{\restriction}\{\beta\})^{-1} \right) \\ &= j(u \circ \rho_a)(\{\beta\}) \\ &= j(u) \left((j{\restriction}a)^{-1} \right) = k_\mathcal{S}([u]_\mathcal{S}) \end{aligned}$$

therefore the diagram commutes. Since $k_\mathcal{S}$ and $k_\mathcal{E}$ are elementary, k has to be elementary as well and $\mathrm{crit}(k_\mathcal{S}) \leq \mathrm{crit}(k_\mathcal{E})$. □

The last proposition with $j = j_\mathcal{S}$ shows that from any $\mathcal{C}$-system of filters $\mathcal{S}$ can be derived an extender $\mathcal{E}$ of sufficient length such that $\mathrm{Ult}(V, \mathcal{S}) = \mathrm{Ult}(V, \mathcal{E})$. The derived extender $\mathcal{E}$ might have the same length as $\mathcal{S}$, e.g. when $\lambda = \beth_\lambda$ and $j[\lambda] \subseteq \lambda$. In particular, this happens in the notable case when $\mathcal{S}$ is the full stationary tower of length λ a Woodin cardinal.

5.2 Generic large cardinals

Generic large cardinal embeddings are analogous to classical large cardinal embeddings. The difference between the former and the latter is that the former is definable in some forcing extension of V and not in V itself as the latter. An exhaustive survey on this topic is given in [16]. Most of the large cardinal properties commonly considered can be built from the following basic blocks.

Definition 5.2.1. Let $V \subseteq W$ be transitive models of ZFC. Let $j : V \to M \subseteq W$ be a generic elementary embedding with critical point κ. We say that

- j is γ-tall iff $j(\kappa) \geq \gamma$[4];
- j is γ-strong iff $V_\gamma^W \subseteq M$;
- j is $<\gamma$-closed iff ${}^{<\gamma}M \subseteq M$ from the point of view of W.

Notice that the definition of a large cardinal property through the existence of an embedding j with (some version of) the above properties *is not a first-order statement*, since it quantifies over a *class* object. In the theory of large cardinals in V, this problem is overcome by showing that an extender $\mathcal{E}$ of sufficient length is able to capture all the aforementioned properties of j in $j_\mathcal{E}$. For generic large cardinals the same can be done with some additional limitations, as shown in Section 5.2.1.

In contrast with the classical case, this process requires the use of $\mathcal{C}$-systems of ultrafilters *in some generic extension.* However, it would be a desirable property to be able to obtain a description of such generic elementary embeddings from objects living in V. Natural intuition suggest the feasibility of this option (see e.g. [33]). We thus introduce the following definition schema (already suggested in [9]).

Definition 5.2.2 (Claverie)**.** Let P be a large cardinal property of an elementary embedding[5], κ be a cardinal. We say that κ has property P iff there exists an elementary embedding $j : V \to M \subseteq V$ with critical point κ and satisfying property P.

We say that κ has *generically* property P iff there exists a forcing extension $V[G]$ and an elementary embedding $j : V \to M \subseteq V[G]$ definable in $V[G]$ and with critical point κ satisfying property P.

We say that κ has *ideally* property P iff there exist a $\mathcal{C}$-system of filters $\mathbb{S}$ in V such that the corresponding generic ultrapower embedding $j_{\dot{\mathbf{F}}(\mathbb{S})}$ satisfies property P in the corresponding generic extension.

Observe that for any κ, $P(\kappa) \Rightarrow$ ideally $P(\kappa) \Rightarrow$ generically $P(\kappa)$. On the other side, it is not clear whether generically $P(\kappa) \Rightarrow$ ideally $P(\kappa)$ as pointed out in [9, 10]. In Section 5.3 an example is given suggesting that the natural procedure of inducing a $\mathcal{C}$-system of filters in V from a generic elementary embedding might fail to preserve large cardinal properties, thus giving some hints against the equivalence of these two concepts.

Furthermore, having ideally property P can be much weaker than having property P in V: e.g. the consistency of an ideally I_1 cardinal follows from the consistency of a Woodin cardinal [33]. Nonetheless, upper bounds on the consistency of generic large cardinals similar to those for classical large cardinals can be proven (see [25, 39]), e.g. the inconsistency of a set-generic Reinhardt cardinal. Since from a stationary tower of height a Woodin cardinal we can obtain a *class*-generic Reinhardt cardinal, it is clear that the strength of a generic large cardinal very much depends on the nature of the forcing allowed to obtained it. In fact, the strength of a generic

[4]We remark that the present definition of γ-tall for an embedding does not coincide with the classical notion of γ-tall for cardinals, which is witnessed (in the present terms) by a *γ-tall and κ-closed* embedding with critical point κ. We believe that the present definition is more convenient to our purposes since it avoids overlapping of concepts and simplifies the corresponding combinatorial version for $\mathcal{C}$-systems of filters.

[5]More precisely, we can assume that P is a first-order property in the class parameter j.

large cardinal hypothesis depends on the interaction of *three parameters*, as outlined in [16]: the size of the critical point, the closure properties of the embedding, and the nature of the forcing used to define it. We shall not expand on the impact of the nature of forcing, while we shall spend some time on the *size* of the critical point. In this setting, the trivial observation that $P(\kappa) \Rightarrow$ ideally $P(\kappa) \Rightarrow$ generically $P(\kappa)$ is not really satisfying, since we are interested in the consistence of *small* cardinals κ having ideally (or generically) property P. However, it is sometimes possible to collapse a large cardinal in order to obtain a small generic large cardinal. Examples of positive results on this side can be found in [9, 10, 17, 28, 35], we present and generalize some of them in Section 5.2.2.

Notice that having ideally property P is inherently a statement on the structure of the relevant $\mathcal{C}$-system of filters $\mathbb{S}$ in question. In Section 5.2.3 we provide a characterization of these properties as combinatorial statements on $\mathbb{S}$.

Since having a generic large cardinal property is possibly weaker than having the same property in V, two large cardinal properties which are inequivalent for classical large cardinals may turn out to be equivalent for their generic counterparts. In Section 5.2.4 we show some examples of embeddings separating different generic large cardinal properties. These examples are an application of the techniques introduced throughout all this section.

5.2.1 Deriving large cardinal properties from generic systems of filters

All over this section G is V-generic for some forcing $\mathbb{B}$ and $j : V \to M \subseteq V[G]$ is a generic elementary embedding definable in $V[G]$ with some large cardinal property P and critical point κ. We aim to approximate j via a suitable $\mathcal{C}$-system of V-ultrafilters $\mathcal{S}$ in $V[G]$ (with $\mathcal{C} \in V$) closely enough so as to preserve the large cardinal property in question.

Proposition 5.2.3. *Let j be γ-tall, $\mathcal{C} \in V$ be a directed set of domains with $\lambda \subseteq \bigcup \mathcal{C}$, $\mathcal{S}$ be the $\mathcal{C}$-system of ultrafilters of length $\lambda \geq j(\kappa)$ derived from j. Then $j_\mathcal{S}$ is γ-tall.*

Proof. By Proposition 5.1.36, $\mathrm{crit}(k) > j(\kappa)$ hence $j_\mathcal{S}(\kappa) = k(j_\mathcal{S}(\kappa)) = j(\kappa) > \gamma$. □

Proposition 5.2.4. *Let j be γ-strong and λ be such that either $\lambda > \gamma$ or $\lambda = j(\mu) = \gamma$ for some μ. Let $\mathcal{C} \in V$ be a directed set of domains with $\lambda \subseteq \bigcup \mathcal{C}$, and $\mathcal{S}$ be the $\mathcal{C}$-system of ultrafilters derived from j. Then $j_\mathcal{S}$ is γ-strong; i.e. $V_\gamma^{\mathrm{Ult}(V,\mathcal{S})} = V_\gamma^{V[\mathcal{S}]} = V_\gamma^{V[G]}$.*

Proof. By Proposition 5.1.36 we have that $\mathrm{crit}(k) > \gamma$. Thus

$$V_\gamma^{\mathrm{Ult}(V,\mathcal{S})} = k(V_\gamma^{\mathrm{Ult}(V,\mathcal{S})}) = V_\gamma^M = V_\gamma^{V[G]}.$$

Furthermore, since $V_\gamma^{\mathrm{Ult}(V,\mathcal{S})} \subseteq V_\gamma^{V[\mathcal{S}]} \subseteq V_\gamma^{V[G]}$ they must all be equal[6]. □

While tallness and strongness are easily handled, in order to ensure preservation of closure we need some additional technical effort.

[6] Once again $V[\mathcal{S}]$ is the minimal transitive model N of ZFC such that $V, \{\mathcal{S}\} \subseteq N \subseteq V[G]$.

Definition 5.2.5. A boolean algebra $\mathbb{B}$ is $<\lambda$-presaturated is for any $\gamma < \lambda$ and family $\mathcal{A} = \langle A_\alpha : \alpha < \gamma \rangle$ of maximal antichains of size λ, there are densely many $p \in \mathbb{B}^+$ such that

$$\forall \alpha < \gamma \ \ |\{a \in A_\gamma : \ a \wedge p > 0\}| < \lambda.$$

Proposition 5.2.6. *Let λ be a regular cardinal. A boolean algebra $\mathbb{B}$ is $<\lambda$-presaturated if and only if it preserves the regularity of λ.*

Lemma 5.2.7. *Let $\mathcal{S}$ be a $\mathcal{C}$-system of ultrafilters in $V[G]$ and $N = \mathrm{Ult}(V, \mathcal{S})$ be such that:*

- *$V[G]$ is a $<\lambda^+$-cc forcing extension for some λ regular in $V[G]$;*
- *$V_\lambda^{V[G]} = V_\lambda^N$ and $\mathcal{C}$ has length at least λ;*
- *N is closed for $<\lambda$-sequences in V.*

Then N is closed for $<\lambda$-sequences in $V[G]$.

Proof. Let $\dot{s}$ be the name for a sequence of length $\gamma < \lambda$ of elements of N. Since λ is regular in $V[G]$, the forcing $\mathbb{C}$ which defines $V[G]$ is $<\lambda$-presaturated. Moreover, $\mathbb{C}$ is $<\lambda^+$-cc hence for any $\alpha < \gamma$ there are at most λ-many possibilities for $\dot{s}(\alpha)$. Therefore we can apply presaturation and find a condition $p \in G$ such that

$$p \Vdash \dot{s} = \left\{ \langle \langle \alpha, [u^\alpha_\beta]_{\dot{\mathcal{S}}} \rangle, q^\alpha_\beta \rangle : \alpha < \gamma, \beta < \mu \right\},$$

for some $\mu < \lambda$.

Let $\langle x_\alpha : \ \alpha < \lambda \rangle$ be a (partial) enumeration of $\mathcal{C} \cap V_\lambda$. Define $t : \gamma \times \mu \to V$ so that $t(\alpha, \beta) = \langle u^\alpha_\beta, \mathrm{ran}_{\{x_\alpha\}}, \mathrm{ran}_{\{x_\beta\}} \rangle$ is a sequence in V. Since N is closed for sequences in V, the sequence represented by t is in N; i.e.

$$X = \left\{ \langle [u^\alpha_\beta]_{\mathcal{S}}, \langle \{x_\alpha\}, \{x_\beta\} \rangle \rangle : \ \alpha < \gamma, \beta < \delta \right\} \in N.$$

Moreover, $Y \in V[G]$ where $Y = \left\{ \langle \{x_\alpha\}, \{x_\beta\} \rangle : \ q^\alpha_\beta \in G \right\}$. Since $Y \in V_\lambda^{V[G]} = V_\lambda^N$, inside N we can define $\mathrm{val}_G(\dot{s}) = \left\{ \langle \alpha, [u^\alpha_\beta]_{\mathcal{S}} \rangle \in N : \ \exists y \in Y \ \langle u^\alpha_\beta, y \rangle \in X \right\}$. □

Theorem 5.2.8. *Let j be $<\lambda$-closed with λ regular cardinal, $V[G]$ be a $<\lambda^+$-cc forcing extension. Let $\mathcal{C} \in V$ be a $<\lambda$-directed set of domains in V, let $\mathcal{S}$ be the $\mathcal{C}$-system of filters derived from j. Then $j_{\mathcal{S}}$ is $<\lambda$-closed.*

Proof. Let $u_\alpha : O_{a_\alpha} \to V$ for $\alpha < \gamma$ be a sequence of length $\gamma < \lambda$ in V of elements of $\mathrm{Ult}(V, \mathcal{S})$. Since $\mathcal{C}$ is $<\lambda$-directed, there is a $b \supseteq \bigcup \{a_\alpha : \alpha < \gamma\} \in \mathcal{C}$ such that $|b| \geq \gamma$. Let $\langle x_\alpha : \ \alpha < \gamma \rangle$ be a (partial) enumeration of b. Define

$$\begin{aligned} v : O_b &\to V \\ f &\mapsto \{u_\alpha(f) : \ x_\alpha \in \mathrm{dom}(f)\} \end{aligned}$$

so that by fineness and normality $[v]_{\mathcal{S}} = \{[u_\alpha]_{\mathcal{S}} : \ \alpha < \gamma\}$. Thus $\mathrm{Ult}(V, \mathcal{S})$ is closed under $<\lambda$-sequences in V and is λ-strong by Proposition 5.2.4. We can apply Lemma 5.2.7 to obtain that $j_{\mathcal{S}}$ is $<\lambda$-closed. □

Note that since in the hypothesis of the previous theorem j is $<\lambda$-closed, it is always possible to derive a system of filters with a $<\lambda$-directed set of domains. In particular, it is possible to derive towers of length λ and λ-extenders of any length. Thus the only significant limitation is the hypothesis that $V[G]$ is $<\lambda^+$-cc where λ is the amount of closure required for M. However, this hypothesis is satisfied in most classical examples of generic elementary embeddings with high degrees of closure, as e.g. the full stationary tower of length a Woodin cardinal.

It is also possible to ensure the same closure properties with non-directed system of filters, as e.g. extenders. This can be done by means of Theorem 5.1.40 and the following remarks.

5.2.2 Consistency of small generic large cardinals

In this section we shall prove that in most cases the assertion that a small cardinal (e.g. ω_1) has generically or ideally property P consistently follows from the existence of any such cardinal (Corollary 5.2.13). Similar results were proved independently in [28, 35] and echoed by [9]; we generalize them to $\mathcal{C}$-system of filters, obtaining a simpler proof.

In the following we shall need to lift embeddings and systems of filters in forcing extensions. We refer to [13, Chp. 9] for a complete treatment of the topic. Recall that if $j : V \to M$ is an elementary embedding and $\mathbb{B} \in V$ is a boolean algebra, j is also an elementary embedding of the boolean valued model $V^{\mathbb{B}}$ into $M^{j(\mathbb{B})}$. Furthermore, j can be lifted to the generic extensions $V[G]$ and $M[H]$ where G is V-generic for $\mathbb{B}$ and H is $j(\mathbb{B})$-generic for M whenever $j[G] \subseteq H$.

For sake of simplicity, we shall focus on the boolean valued models approach and avoid explicit use of generic filters. This will be convenient to handle several different forcing notions at the same time. All the proofs will then be carried out in V using names and explicitly mentioning in which boolean valued model $V^{\mathbb{B}}$ every sentence is to be interpreted. We recall the following definition of two-step iteration.

Definition (2.3.1)**.** Let $\mathbb{B}$ be a complete boolean algebra, and $\dot{\mathbb{C}}$ be a $\mathbb{B}$-name for a complete boolean algebra. We denote by $\mathbb{B} * \dot{\mathbb{C}}$ the boolean algebra defined in V whose elements are the equivalence classes of $\mathbb{B}$-names for elements of $\dot{\mathbb{C}}$ (i.e. $\dot{a} \in V^{\mathbb{B}}$ such that $\left[\!\left[\dot{a} \in \dot{\mathbb{C}}\right]\!\right]_{\mathbb{B}} = \mathbf{1}$) modulo the equivalence relation $\dot{a} \approx \dot{b} \Leftrightarrow \left[\!\left[\dot{a} = \dot{b}\right]\!\right]_{\mathbb{B}} = \mathbf{1}$.

We refer to [44, 45] for further details on two-step iterations and iterated forcing.

Definition 5.2.9. Let $\mathbb{S}$ be a $\mathcal{C}$-system of filters, $\mathbb{C}$ be a cBa. Then $\mathbb{S}^{\mathbb{C}} = \left\{F_a^{\mathbb{C}} : a \in \mathcal{C}\right\}$ where $F_a^{\mathbb{C}} = \left\{A \subseteq (O_a)^{V^{\mathbb{C}}} : \exists B \in \check{F}_a \ A \supseteq B\right\}$.

We remark that the following theorem is built on the previous results by Kakuda and Magidor [28, 35] for single ideals and by Claverie [9] for ideal extenders.

Theorem 5.2.10. *Let $j : V \to M \subseteq V^{\mathbb{B}}$ be elementary with critical point κ, and $\mathbb{C} \in V$ be a $<\kappa$-cc cBa. Then $\mathbb{B} * j(\mathbb{C})$ factors into $\mathbb{C}$, and the embedding j lifts to $j^{\mathbb{C}} : V^{\mathbb{C}} \to M^{j(\mathbb{C})}$.*

$$\begin{array}{ccccc} V & \xrightarrow{\;j\;} & M & \subseteq & V^{\mathbb{B}} \\ \cap & & \cap & & \cap \\ V^{\mathbb{C}} & \xrightarrow[j^{\mathbb{C}}]{} & M^{j(\mathbb{C})} & \subseteq & V^{\mathbb{B}*j(\mathbb{C})} \end{array}$$

Furthermore, if $\mathbb{B} = \mathbb{S} = \langle F_a : \ a \in \mathcal{C} \rangle$ *is a* $\langle \kappa, \lambda \rangle$*-system of filters and* $j = j_{\dot{\mathbf{F}}(\mathbb{S})}$*, then* $\mathbb{C} * \mathbb{S}^{\mathbb{C}}$ *is isomorphic to* $\mathbb{S} * j(\mathbb{C})$ *and* $j^{\mathbb{C}}$ *is the embedding induced by* $\mathbb{S}^{\mathbb{C}}$*.*

$$\begin{array}{ccccccc} V & \xrightarrow{\;j_{\dot{\mathbf{F}}(\mathbb{S})}\;} & M & \subseteq & V^{\mathbb{S}} & & \\ \cap & & \cap & & \cap & & \\ V^{\mathbb{C}} & \xrightarrow[j_{\dot{\mathbf{F}}(\mathbb{S}^{\mathbb{C}})}]{} & M^{j(\mathbb{C})} & \subseteq & V^{\mathbb{S}*j(\mathbb{C})} & = & V^{\mathbb{C}*\mathbb{S}^{\mathbb{C}}} \end{array}$$

Proof. For the first part, consider the embedding:

$$\begin{array}{rccc} i_1 : & \mathbb{C} & \longrightarrow & \mathbb{B} * j(\mathbb{C}) \\ & p & \longmapsto & j(p) \end{array}$$

By elementarity of j, i_1 must preserve $\leq, \perp$. Given any maximal antichain $\mathcal{A}$, $\mathbb{C}$ is $<\kappa$-cc hence $j[\mathcal{A}] = j(\mathcal{A})$ which is maximal again by elementarity of j. Then i_1 is a complete embedding hence $\mathbb{B} * j(\mathbb{C})$ is a forcing extension of $\mathbb{C}$. Thus we can lift j to a generic elementary embedding $j^{\mathbb{C}}$.

For the second part, consider the embedding:

$$\begin{array}{rccc} i_2 : & \mathbb{C} * \mathbb{S}^{\mathbb{C}} & \longrightarrow & \mathbb{S} * j(\mathbb{C}) \\ & \dot{A} \subseteq O_a & \longmapsto & \left[\!\left[[\mathrm{id}_a]_{\dot{\mathbf{F}}(\mathbb{S})} \in j(\dot{A}) \right]\!\right]_{\mathbb{S}*j(\mathbb{C})} \end{array}$$

This map is well-defined since the set of $\dot{A} \in \mathbb{C} * \mathbb{S}^{\mathbb{C}}$ such that $\dot{A} \subsetneq O_a$ for some fixed $a \in \mathcal{C}$ is dense in $\mathbb{C} * \mathbb{S}^{\mathbb{C}}$. Suppose now that $\dot{A} \leq_{\mathbb{C}*\mathbb{S}^{\mathbb{C}}} \dot{B}$ with $\dot{B} \subseteq O_b$, $b \in \mathcal{C}$, $c = a \cup b$. Then,

$$\begin{aligned} &\mathbf{1} \Vdash_{\mathbb{C}} \left(\pi_c^{-1}[\dot{A}] \setminus \pi_c^{-1}[\dot{B}] \right) \in I_c^{\mathbb{C}} \quad \Rightarrow \\ &\mathbf{1} \Vdash_{\mathbb{C}} \exists C \in I_c \ \left(\pi_c^{-1}[\dot{A}] \setminus \pi_c^{-1}[\dot{B}] \right) \subseteq C \end{aligned}$$

and we can find a maximal antichain $\mathcal{A} \subseteq \mathbb{C}$ such that $p \Vdash_{\mathbb{C}} \left(\pi_c^{-1}[\dot{A}] \setminus \pi_c^{-1}[\dot{B}] \right) \subseteq \check{C}_p$ for every $p \in \mathcal{A}$ and corresponding $C_p \in I_c \Rightarrow \mathbf{1} \Vdash_{\mathbb{C}} [\mathrm{id}_c]_{\dot{\mathbf{F}}(\mathbb{S})} \notin j(C_p)$. Thus by elementarity of j, for all $p \in \mathcal{A}$ we have that

$$j(p) \Vdash_{j(\mathbb{C})} \left(\pi_c^{-1}[j(\dot{A})] \setminus \pi_c^{-1}[j(\dot{B})] \right) \subseteq j(\check{C}_p) \not\ni [\mathrm{id}_c]_{\dot{\mathbf{F}}(\mathbb{S})}$$

and since $j[\mathcal{A}]$ is maximal in $j(\mathbb{C})$,

$$\begin{aligned} &\mathbf{1} \Vdash_{j(\mathbb{C})} [\mathrm{id}_{a \cup b}]_{\dot{\mathbf{F}}(\mathbb{S})} \notin \left(\pi_c^{-1}[j(\dot{A})] \setminus \pi_c^{-1}[j(\dot{B})] \right) \quad \Rightarrow \\ &\mathbf{1} \Vdash_{j(\mathbb{C})} [\mathrm{id}_b]_{\dot{\mathbf{F}}(\mathbb{S})} \in j(\dot{B}) \vee [\mathrm{id}_a]_{\dot{\mathbf{F}}(\mathbb{S})} \notin j(\dot{A}) \quad \Rightarrow \\ &i_2(\dot{B}) \vee \neg i_2(\dot{A}) = \mathbf{1} \Rightarrow i_2(\dot{A}) \leq i_2(\dot{B}) \end{aligned}$$

Thus i_2 preserves $\leq$. Preservation of $\perp$ is easily verified by a similar argument, replacing everywhere $\dot{A} \setminus \dot{B}$ with $\dot{A} \cap \dot{B}$.

We still need to prove that i_2 has a dense image. Fix $[u^p]_{\dot{\mathbf{F}}(\mathbb{S})} \in \mathbb{S} * j(\mathbb{C})$, so that $u^p : A \to \mathbb{C}$, $A \in I_a^+$, $a \in \mathcal{C}$. Let $\dot{B} = \left\{ x \in \check{A} : \ \check{u}^p(x) \in \dot{G}_{\mathbb{C}} \right\}$ be in $V^{\mathbb{C}}$. Then,

$$\begin{aligned} i_2(\dot{B}) &= \left[\!\left[[\mathrm{id}_a]_{\dot{\mathbf{F}}(\mathbb{S})} \in j(\check{A}) \ \wedge \ j(\check{u}^p)([\mathrm{id}_a]_{\dot{\mathbf{F}}(\mathbb{S})}) \in j(\dot{G}_{\mathbb{C}}) \right]\!\right]_{\mathbb{S}*j(\mathbb{C})} \\ &= \left[\!\left[\check{A} \in \dot{\mathbf{F}}(\mathbb{S}) \ \wedge \ [\check{u}^p]_{\dot{\mathbf{F}}(\mathbb{S})} \in \dot{G}_{j(\mathbb{C})} \right]\!\right]_{\mathbb{S}*j(\mathbb{C})} = [u^p]_{\dot{\mathbf{F}}(\mathbb{S})} \end{aligned}$$

hence $V^{\mathbb{S}*j(\mathbb{C})} = V^{\mathbb{C}*\mathbb{S}^{\mathbb{C}}}$ is the forcing extension of $V^{\mathbb{C}}$ by $\mathbb{S}^{\mathbb{C}}$.

Finally, we prove that $j^{\mathbb{C}}$ is the generic ultrapower embedding derived from $\mathbb{S}^{\mathbb{C}}$. We can directly verify that $\mathbb{S}^{\mathbb{C}}$ satisfies filter property, fineness and compatibility. This is sufficient to define an ultrapower $N = \mathrm{Ult}(V^{\mathbb{C}}, \dot{\mathbf{F}}(\mathbb{S}^{\mathbb{C}}))$ and prove Łoś Theorem for it. The elements of N are represented by $\mathbb{C}$-names for functions $\dot{v} : O_a^{V^{\mathbb{C}}} \to V^{\mathbb{C}}$. Since $F_a^{\mathbb{C}}$ concentrates on O_a^V for all $a \in \mathcal{C}$, we can assume that $\dot{v} : \check{O}_a \to V^{\mathbb{C}}$. Furthermore, we can replace $\dot{v}$ by a function $u : O_a \to V^{\mathbb{C}}$ in V mapping $f \in O_a$ to a name for $\dot{v}(\check{f})$. These functions can then represent both all elements of N and all elements of $M^{j(\mathbb{C})}$. Furthermore, N and $M^{j(\mathbb{C})}$ must give the same interpretation to them. In fact, given $u_n : O_a \to V^{\mathbb{C}}$ for $n = 1, 2$ and $\dot{A} \in \left(I_a^{\mathbb{C}}\right)^+$:

$$\begin{aligned} &\dot{A} \Vdash_{\mathbb{C}*\mathbb{S}^{\mathbb{C}}} [u_1]_{\dot{\mathbf{F}}(\mathbb{S}^{\mathbb{C}})} = [u_2]_{\dot{\mathbf{F}}(\mathbb{S}^{\mathbb{C}})} \iff \\ &\mathbf{1} \Vdash_{\mathbb{C}} \left\{ f \in \dot{A} : \ u_1(f) \neq u_2(f) \right\} \in I_a^{\mathbb{C}} \iff \\ &\exists B \in I_a \ \mathbf{1} \Vdash_{\mathbb{C}} \forall f \in \dot{A} \setminus \check{B} \ \dot{v}_1(f) = \dot{v}_2(f) \iff \\ &\exists B \in F_a \ \mathbf{1} \Vdash_{\mathbb{C}} \forall f \in \dot{A} \cap \check{B} \ \dot{v}_1(f) = \dot{v}_2(f) \iff \\ &\exists B \in F_a \ \forall f \in B \ \mathbf{1} \Vdash_{\mathbb{C}} f \in \dot{A} \to u_1(f) = u_2(f) \iff \\ &\left\{ f \in O_a : \ \mathbf{1} \Vdash_{\mathbb{C}} f \in \dot{A} \to u_1(f) = u_2(f) \right\} \in F_a \iff \\ &\mathbf{1} \Vdash_{\mathbb{S}*j(\mathbb{C})} [\mathrm{id}_a]_{\dot{\mathbf{F}}(\mathbb{S})} \in j(\dot{A}) \to [u_1]_{\dot{\mathbf{F}}(\mathbb{S})} = [u_2]_{\dot{\mathbf{F}}(\mathbb{S})} \iff \\ &i_2(\dot{A}) = \left[\!\left[[\mathrm{id}_a]_{\dot{\mathbf{F}}(\mathbb{S})} \in j(\dot{A}) \right]\!\right]_{\mathbb{S}*j(\mathbb{C})} \Vdash_{\mathbb{S}*j(\mathbb{C})} [u_1]_{\dot{\mathbf{F}}(\mathbb{S})} = [u_2]_{\dot{\mathbf{F}}(\mathbb{S})} \end{aligned}$$

and the above reasoning works also replacing $=$ with $\in$. The second passage uses essentially that $\mathbb{C}$ is $<\kappa$-cc and $\mathbb{S}$ is a $\langle \kappa, \lambda \rangle$-system of filters. In fact, in this setting given $\dot{A} \in I_a^{\mathbb{C}}$ there are less than κ possibilities for a $B \in I_a$, $p \Vdash \check{B} \supseteq (\dot{A} \cap \check{O}_a)$, hence we can find a single such B by $<\kappa$-completeness of I_a (see Proposition 5.1.12). ☐

Corollary 5.2.11. *Let $\mathbb{S}$ be a $\langle \kappa, \lambda \rangle$-system of filters, $\mathbb{C}$ be a $<\kappa$-cc cBa. Then $\mathbb{S}^{\mathbb{C}}$ is a $\mathcal{C}$-system of filters.*

Proof. Since $\mathbb{S}^{\mathbb{C}}$ is the $\mathcal{C}_{\mathbb{S}}$-system of filters in V derived from $j^{\mathbb{C}}_{\dot{\mathbf{F}}(\mathbb{S})}$, it is a $\mathcal{C}$-system of filters by Propositions 5.1.34 and 5.1.24. ☐

Proposition 5.2.12. *Let $j : V \to M \subseteq V^{\mathbb{B}}$ be elementary with critical point κ, $\gamma < \kappa$ be a cardinal, and $j^{\mathbb{C}} : V^{\mathbb{C}} \to M^{j(\mathbb{C})}$ be obtained from j and $\mathbb{C} = \mathrm{Coll}(\gamma, <\kappa)$. Suppose that $j(\kappa)$ is regular in $V^{\mathbb{B}}$.*

If j is $<\delta$-closed with $\delta \geq j(\kappa)$, then $j^{\mathbb{C}}$ is $<\delta$-closed. If j is δ-strong with $\delta \geq j(\kappa)$, then $j^{\mathbb{C}}$ is δ-strong.

Proof. Since $j(\kappa)$ is regular in $V^{\mathbb{B}}$, $\mathrm{Coll}(\gamma, <j(\kappa))$ is $<j(\kappa)$-cc in $V^{\mathbb{B}}$. Moreover, the order on the Lévy collapse is absolute between transitive models thus $j(\mathbb{C}) = \mathrm{Coll}(\gamma, <j(\kappa))^M$ is a suborder of $\mathrm{Coll}(\gamma, <j(\kappa))$. Hence $j(\mathbb{C})$ is also $<j(\kappa)$-cc in $V^{\mathbb{B}}$.

First, suppose that j is $<\delta$-closed and let σ be a $j(\mathbb{C})$-name for a sequence of ordinals of size $\mu < \delta$. Since $\sigma(i)$ for $i < \mu$ is decided by an antichain of size less than $j(\kappa)$, the whole σ is coded by a subset of M of size less than $\delta + j(\kappa) = \delta$. Thus $\sigma \in M$ hence is evaluation is in $M^{j(\mathbb{C})}$.

Suppose now that j is δ-strong and let σ be a $j(\mathbb{C})$-name for a subset of $\mu < \delta$. Then σ is coded by a subset of M of size less than $\delta + j(\kappa) = \delta$ as before, hence σ is in M and its evaluation in $M^{j(\mathbb{C})}$. □

Corollary 5.2.13. *Let P be a property among (n)-huge, almost (n)-huge (for $n > 0$), α-superstrong (for $\alpha > \kappa$), (n)-superstrong (for $n > 1$).*

If κ is generically (resp. ideally) P, then it is so after $\mathrm{Coll}(\gamma, <\kappa)$ *for any $\gamma < \kappa$. Thus the existence of a generically (resp. ideally) P cardinal is equiconsistent with ω_1 being such a cardinal.*

Note that the previous corollary applies only to generically and ideally P: the existence of a large cardinal with property P in V is usually stronger than ω_1 being generically P. Due to the fact that a generically superstrong cardinal does not guarantee that $j(\kappa)$ is regular in $V^{\mathbb{B}}$, the previous result does not apply to superstrong cardinals. We recall that the case of a strong cardinal was already treated in [9, Corollary 4.14], which showed the following.

Theorem 5.2.14. *The existence of a strong cardinal is equiconsistent with ω_1 being ideally strong.*

As in Proposition 5.2.12, it is possible to prove that forcing with $\mathrm{Coll}(\gamma, <\kappa)$ with κ a strong cardinal preserves the ideally strongness of κ. However, starting with an ideally strong cardinal would not suffice in this case. In order to get a $j^{\mathbb{C}}$ with strength γ we need an embedding $j : V \to M \subseteq V^{\mathbb{B}}$ with enough strength so as to contain in M a name for $V_\gamma^{\mathbb{B}}$. Although, since the complexity of such a name depends on $\mathbb{B}$, and $\mathbb{B}$ depends on the amount of strength that we wish to achieve, there is no hope to sort out this circular reference. On the other hand, a generically strong cardinal is preserved under *Cohen* forcing under some assumptions [10].

Notice that the previous corollary does not apply also to generically supercompact cardinals. However, this is not surprising since $\kappa = \gamma^+$ being generically supercompact is equivalent to being generically almost huge: in fact, if $j : V \to M \subseteq V[G]$ is a γ-closed embedding obtained by γ-supercompactness, it is also almost huge since $j(\kappa) = (\gamma^+)^{V[G]}$. Thus such a preservation theorem for supercompactness would in turn imply the equiconsistency of generically supercompactness and generically almost hugeness, which is not expected to hold. However, if we restrict the class of forcing to *proper* forcings, is possible to obtain a similar preservation theorem [17].

5.2.3 Combinatorial equivalents of ideally large cardinal properties

The *ideal* properties of cardinals given in Definition 5.2.2 are inherently properties of a $\mathcal{C}$-system of filters, it is therefore interesting to reformulate them in purely combinatorial terms. In this section we review the main results on this topic present in literature, adapted to the paradigm introduced in Section 5.1; and we integrate them with a characterization of strongness that, to our knowledge, is not yet present in literature.

Critical point and tallness

In order to express any large cardinal property, we need to be able to identify the critical point of an embedding $j_{\dot{\mathbf{F}}(\mathbb{S})}$ derived from some $\mathcal{C}$-system of filters $\mathbb{S}$.

Definition 5.2.15. Let $\mathbb{S}$ be a $\mathcal{C}$-system of filters in V. The *completeness* of $\mathbb{S}$ is the minimum of the completeness of F_a for $a \in \mathcal{C}$, i.e. the unique cardinal κ such that every F_a is $<\kappa$-complete and there is an F_a that is not $<\kappa^+$-complete.

We say that $\mathbb{S}$ has *densely* completeness κ iff it has completeness κ and there are densely many $B \in \mathbb{S}^+$ disproving $<\kappa^+$-completeness (i.e. that are the union of κ sets in the relevant ideal).

Proposition 5.2.16. *Let $\mathbb{S}$ be a $\mathcal{C}$-system of filters in V. Then the following are equivalent:*

1. *the ultrapower map $\dot{k} = j_{\dot{\mathbf{F}}(\mathbb{S})}$ has critical point κ with boolean value* $\mathbf{1}$*;*
2. *$\mathbb{S}$ is a $\langle \kappa, \lambda \rangle$-system of filters;*
3. *$\mathbb{S}$ has densely completeness κ;*

Moreover, if $\kappa \in \mathcal{C}$ then the statements above are also equivalent to

4. $\{\mathrm{id} \restriction \alpha : \ \alpha < \kappa\} \in F_\kappa$.

Proof. (1) $\Leftrightarrow$ (2): has already been proved in Proposition 5.1.30.

(2) $\Rightarrow$ (3): By Proposition 5.1.12, we know that F_a is $<\kappa$-complete for all $a \in \mathcal{C}$. Let $\dot{u}$ be a name for a function representing κ in $\mathrm{Ult}(V, \dot{\mathbf{F}}(\mathbb{S}))$. Then there are densely many $B \in I_b^+$ deciding that $\dot{u} = \check{v}$, for some $v : B \to \kappa$. Since

$$\left[\!\left[[v]_{\dot{\mathbf{F}}(\mathbb{S})} \neq \alpha = [\mathrm{dom}_\alpha]_{\dot{\mathbf{F}}(\mathbb{S})} \right]\!\right]_{\mathbb{S}} \geq B$$

for all $\alpha < \kappa$, $B_\alpha = B \wedge v^{-1}[\{\alpha\}] \in I_b$ for any such B hence $B = \bigcup_{\alpha<\kappa} B_\alpha$ disproves $<\kappa^+$-completeness.

(3) $\Rightarrow$ (1): We prove by induction on $\alpha < \kappa$ that $\mathbf{1} \Vdash_{\mathbb{S}} j(\check{\alpha}) = \check{\alpha}$. Let $u : A \to \alpha$ with $A \in I_a^+$ be representing an ordinal smaller than $j(\alpha)$ in the ultrapower, and let $A_\beta = u^{-1}[\{\beta\}]$ for $\beta < \alpha$. Since $A = \bigcup_{\beta<\alpha} A_\beta$ and $\mathbb{S}$ is $<\kappa$-complete, the conditions A_β form a maximal antichain below A hence $[u]_{\dot{\mathbf{F}}(\mathbb{S})}$ is forced to represent some $\beta < \alpha$. Furthermore, there are densely many $B \in I_b^+$ that are a union of κ-many sets $B_\alpha \in I_b$. From any one of them we can build a function $u : B \to \kappa$, $u(f) = \alpha_f$

where $f \in B_{\alpha_f}$, so that B forces that $[u]_{\dot{\mathbf{F}}(\mathbb{S})} < j(\kappa)$ and $[u]_{\dot{\mathbf{F}}(\mathbb{S})} > \alpha$ for all $\alpha < \kappa$. Thus $B \Vdash_{\mathbb{S}} j(\kappa) > \kappa$ for densely many B.

Assume now that $\kappa \in \mathcal{C}$. Then (1) $\Leftrightarrow$ (4) follows from Proposition 5.1.27 and Łoś theorem, since $\{\mathrm{id} \restriction \alpha : \ \alpha < \kappa\}$ is equal to

$$\bigwedge_{\alpha<\kappa} \left[\!\!\left[[\mathrm{ran}_\alpha]_{\dot{\mathbf{F}}(\mathbb{S})} = j(\alpha) \right]\!\!\right]_{\mathbb{S}} \wedge \left[\!\!\left[[\mathrm{ran}_\kappa]_{\dot{\mathbf{F}}(\mathbb{S})} < j(\kappa) \right]\!\!\right]_{\mathbb{S}} = [\![j[\kappa] = \kappa \wedge j(\kappa) > \kappa]\!]_{\mathbb{S}}. \qquad \square$$

A similar approach can apply also to tallness-related properties.

Proposition 5.2.17. *Let $\mathbb{S}$ be a $\langle \kappa, \lambda \rangle$-system of filters in V. The ultrapower map $j = j_{\dot{\mathbf{F}}(\mathbb{S})}$ is γ-tall for $\gamma < \lambda$ iff $\left\{ f \in O_{\{x\}} : \mathrm{rank}(f(x)) \leq \kappa \right\} \in F_{\{x\}}$ for some $x \in \bigcup \mathcal{C}$ with $\mathrm{rank}(x) = \gamma$.*

Proof. By Proposition 5.1.27 and Łoś theorem the above set is equal to

$$\left[\!\!\left[\gamma = \mathrm{rank}(x) = \mathrm{rank}([\mathrm{proj}_x]_{\dot{\mathbf{F}}(\mathbb{S})}) \leq j(\kappa) \right]\!\!\right]_{\mathbb{S}}. \qquad \square$$

Measurability

We say that a cardinal is measurable iff there is an elementary embedding $j : V \to M$ with critical point κ such that the image is well-founded. Its generic counterpart can be characterized for $\mathcal{C}$-systems of filters by means of the following definition.

Definition 5.2.18. Let $\mathbb{S}$ be a $\mathcal{C}$-system of filters in V. We say that $\mathbb{S}$ is *precipitous* iff for every $B \in \mathbb{S}^+$ and sequence $\langle \mathcal{A}_\alpha : \ \alpha < \omega \rangle \in V$ of maximal antichains in $<_{\mathbb{S}}$ below B, there are $\bar{A}_\alpha \in \mathcal{A}_\alpha$, $\bar{A}_\alpha \in I^+_{\bar{a}_\alpha}$ and $h : \bigcup_\alpha \bar{a}_\alpha \to V$ such that $\pi_{\bar{a}_\alpha}(h) \in \bar{A}_\alpha$ for all $\alpha < \omega$.

This definition is equivalent to [9, Def. 4.4.*ii*] for ideal extenders, and to $<\omega$-closure for extenders in V (see [31]), while being applicable also to other systems of filters. The results relating these definitions with well-foundedness are subsumed in the following.

Theorem 5.2.19. *Let $\mathbb{S}$ be a $\mathcal{C}$-system of filters in V. The ultrapower map $j = j_{\dot{\mathbf{F}}(\mathbb{S})}$ is well-founded iff $\mathbb{S}$ is precipitous.*

Proof. First, suppose that $\mathbb{S}$ is precipitous and assume by contradiction that B forces the ultrapower to be ill-founded. Let $\langle \dot{u}_n : \ n < \omega \rangle$ be $\mathbb{S}$-names for functions $\dot{u}_n : O_{\dot{a}_n} \to V$ in $U_{\mathbb{S}}$ such that $\left[\!\!\left[[\dot{u}_{n+1}]_{\dot{\mathbf{F}}(\mathbb{S})} \in [\dot{u}_n]_{\dot{\mathbf{F}}(\mathbb{S})} \right]\!\!\right]_{\mathbb{S}} \geq B$. Define $\dot{b}_n = \bigcup \{\dot{a}_m : m \leq n\}$, $\dot{B}_0 = O_{\dot{b}_0}$, and

$$\dot{B}_{n+1} = \left\{ x \in O_{\dot{b}_{n+1}} : \ \dot{u}_{n+1}(\pi_{\dot{a}_{n+1}}(x)) \in \dot{u}_n(\pi_{\dot{a}_n}(x)) \right\}$$

so that $\left[\!\!\left[\dot{B}_n \in \dot{\mathbf{F}}(\mathbb{S}) \right]\!\!\right]_{\mathbb{S}} \geq B$. Fix $n < \omega$. By the forcing theorem there is a dense set of A in $\mathbb{S}$ below B deciding the values of $\dot{u}_n$, $\dot{B}_n$; and every such $A \Vdash \dot{B}_n = \check{B}_n$ must force that $\check{B}_n \in \dot{\mathbf{F}}(\mathbb{S})$ hence satisfy $A <_{\mathbb{S}} B_n$. It follows that the set of $A \in I^+_a$ deciding $\dot{u}_n$, $\dot{B}_n$ and with the additional property that every such A satisfy $a \supseteq b_n$,

$A \subseteq \pi_a^{-1}[B_n]$, is also dense below B. Let $\mathcal{A}_n$ be a maximal antichain below B in this set.

Let $\bar{A}_n$, $\bar{a}_n$, $h : \bar{a} = \cup_n \bar{a}_n$ be obtained from $\langle \mathcal{A}_n : n < \omega \rangle$ by precipitousness of $\mathbb{S}$. Let also u_n, B_n be such that $\bar{A}_n \Vdash \dot{u}_n = \check{u}_n \wedge \dot{B}_n = \check{B}_n$. Then $\pi_{\bar{a}\bar{a}_n}(h) \in \bar{A}_n \subseteq \pi_{\bar{a}_n}^{-1}[B_n]$ and $\pi_{\bar{a}b_n}(h) \in B_n$ for all $n < \omega$. Thus, $u_{n+1}(\pi_{a_{n+1}}(h)) \in u_n(\pi_{a_n}(h))$ is an infinite descending chain in V, a contradiction.

Suppose now that $\mathbb{S}$ is not precipitous, and fix B, $\langle \mathcal{A}_n : n < \omega \rangle$ witnessing it. Define a tree T of height ω consisting of couples of sequences $\langle \mathcal{B}, f \rangle$ such that $\mathcal{B}_n \in \mathcal{A}_n$ for all $n < |\mathcal{B}| < \omega$ and $f \in \bigwedge \mathcal{B}$, ordered by member-wise inclusion. Since $\langle \mathcal{A}_n : n < \omega \rangle$ contradicts precipitousness, the tree T has no infinite chain and we can define a rank-like function $r : T \to \mathrm{ON}$ by well-founded recursion on T as $r(x) = \bigcup \{ r(y) + 1 : y <_T x \}$. Notice that $y <_T x$ implies $r(y) < r(x)$.

Let $\mathcal{B}$ be as above, and define $u_{\mathcal{B}} : \bigwedge \mathcal{B} \to V$ by $u_{\mathcal{B}}(f) = r(\langle \mathcal{B}, f \rangle)$. Let $\dot{u}_n$ be the $\mathbb{S}$-name defined by $\dot{u}_n = \{ \langle \check{u}_{\mathcal{B}}, \bigwedge \mathcal{B} \rangle : \mathcal{B} \in \Pi_{m \leq n} \mathcal{A}_m \}$. Then any $\mathcal{B} \in \Pi_{m \leq n+1} \mathcal{A}_m$ forces $\dot{u}_{n+1}$ to be $u_{\mathcal{B}}$, $\dot{u}_n$ to be $u_{\mathcal{B} \restriction n}$, and $\dot{u}_{n+1} \in \dot{u}_n$ since for all $f \in \bigwedge \mathcal{B}$,

$$u_{\mathcal{B}}(f) = r(\langle \mathcal{B}, f \rangle) < r(\langle \mathcal{B} \restriction n, f' \rangle) = u_{\mathcal{B} \restriction n}(f')$$

where $f' = f \restriction \mathrm{dom}(\bigwedge (\mathcal{B} \restriction n))$. Since $\{ \bigwedge \mathcal{B} : \mathcal{B} \in \Pi_{m \leq n+1} \mathcal{A}_m \}$ forms a maximal antichain below B for every i, B forces that $\langle \dot{u}_n : n < \omega \rangle$ is a name for an ill-founded chain. □

Strongness

In this section we cover large cardinal properties defined in terms of the existence of elementary embeddings $j : V \to M \subseteq V[G]$ with certain degree of strongness (i.e. such that $V_\gamma^{V[G]} \subseteq M$ for some appropriate γ). Main examples of such properties are strongness, superstrongness and variants of them. We now present a criterion to characterize γ-strongness for an elementary embedding $j_{\dot{\mathbf{F}}(\mathbb{S})}$, which can in turn be applied in order to characterize all of the aforementioned large cardinal properties. To our knowledge, there is no equivalent version of the content of this section in the classical tower or extender setting.

Definition 5.2.20. Let $\mathbb{S}$ be a $\mathcal{C}$-system of filters, $\mathcal{A}_0 \cup \mathcal{A}_1$ be an antichain in $\mathbb{S}^+$. We say that $\langle \mathcal{A}_0, \mathcal{A}_1 \rangle$ *is split by* $\mathbb{S}$ iff there exist a $b \in \mathcal{C}$ and B_0, B_1 disjoint in $\mathcal{P}(O_b)$ such that $A \leq_{\mathbb{S}} B_n$ for all $A \in \mathcal{A}_n$, $n < 2$.

We say that a family of antichains $\langle \mathcal{A}_{\alpha 0} \cup \mathcal{A}_{\alpha 1} : \alpha < \mu \rangle$ is *simultaneously* split by $\mathbb{S}$ iff there is a single $b \in \mathcal{C}$ witnessing splitting for all of them.

Definition 5.2.21. Let $\mathbb{S}$ be a $\mathcal{C}$-system of filters. We say that $\mathbb{S}$ is *$<\gamma$-splitting* iff for all sequences $\langle \mathcal{A}_{\alpha 0} \cup \mathcal{A}_{\alpha 1} : \alpha < \mu \rangle$ of maximal antichains with $\mu < \gamma$, there are densely many $B \in \mathbb{S}^+$ such that the antichains $\langle \mathcal{A}_{\alpha 0} \restriction B, \mathcal{A}_{\alpha 1} \restriction B \rangle$ for $\alpha < \mu$ are simultaneously split by $\mathbb{S}$.

Theorem 5.2.22 (A., Viale). *Let $\mathbb{S}$ be a $<\gamma$-directed $\mathcal{C}$-system of filters. Then the ultrapower* $\mathrm{Ult}(V, \dot{\mathbf{F}}(\mathbb{S}))$ *contains $\mathcal{P}^{V^{\mathbb{S}}}(\mu)$ for all $\mu < \gamma$ iff $\mathbb{S}$ is $<\gamma$-splitting.*

Proof. Let $a \in \mathcal{C}$, $u_\alpha : O_a \to \mathrm{ON}$ be such that $\left[\!\left[[\check{u}_\alpha]_{\dot{\mathbf{F}}(\mathbb{S})} = \check{\alpha} \right]\!\right]_{\mathbb{S}} = \mathbf{1}$ for all $\alpha < \gamma$. First, suppose that $\mathbb{S}$ is $<\gamma$-splitting and let $\dot{X}$ be a name for a subset of $\mu < \gamma$.

Let $\mathcal{A}_{\alpha 0} \cup \mathcal{A}_{\alpha 1}$ for $\alpha < \mu$ be a maximal antichain deciding whether $\check{\alpha} \in \dot{X}$ and $\mathcal{S}$ be generic for $\mathbb{S}$. By $<\gamma$-splitting let $B \in \mathcal{S}$ be such that $a \subseteq b \in \mathcal{C}$, $B \subseteq O_b$ and $\langle \mathcal{A}_{\alpha 0} \restriction B, \mathcal{A}_{\alpha 1} \restriction B \rangle$ is split by $\mathbb{S}$ in $B_{\alpha 0}, B_{\alpha 1}$ partitioning B for all $\alpha < \mu$. Then we can define

$$\begin{array}{rccl} v: & B & \longrightarrow & \mathcal{P}(\mathrm{ON}) \\ & f & \longmapsto & \{u_\alpha(\pi_{ba}(f)) : f \in B_{\alpha 1},\ \alpha \in \mu\} \end{array}$$

Then B forces that $[v]_{\dot{\mathbf{F}}(\mathbb{S})} = \dot{X}$, and $B \in \mathcal{S}$ so $\mathrm{val}(\dot{X}, \mathcal{S}) = [v]_{\mathcal{S}}$ is in $\mathrm{Ult}(V, \mathcal{S})$.

Suppose now that $\mathrm{Ult}(V, \dot{\mathbf{F}}(\mathbb{S}))$ contains $\mathcal{P}^{V[\dot{\mathbf{F}}(\mathbb{S})]}(\mu)$ for all $\mu < \gamma$, and let $\langle \mathcal{A}_{\alpha 0} \cup \mathcal{A}_{\alpha 1} :\ \alpha < \mu \rangle$ be maximal antichains with $\mu < \gamma$. Let $\dot{X} = \{\langle \check{\alpha}, A \rangle :\ A \in \mathcal{A}_{\alpha 1}\}$ be the corresponding name for a subset of μ, and let B, $v : B \to \mathcal{P}(\mathrm{ON})$ be such that $B \Vdash [v]_{\dot{\mathbf{F}}(\mathbb{S})} = \dot{X}$. Let $B_{\alpha 0} = \{f \in B :\ u_\alpha(\pi_{ba}(f)) \in v(f)\}$, $B_{\alpha 1} = B \setminus B_{\alpha 0}$. Then $\langle \mathcal{A}_{\alpha 0} \restriction B, \mathcal{A}_{\alpha 1} \restriction B \rangle$ is split by $B_{\alpha 0}, B_{\alpha 1}$ partitioning B for all $\alpha < \mu$. $\square$

Corollary 5.2.23. *Let γ be a limit ordinal, $\mathbb{S}$ be a $<\beth_\gamma$-directed $\mathcal{C}$-system of filters. Then the ultrapower* $\mathrm{Ult}(V, \dot{\mathbf{F}}(\mathbb{S}))$ *is γ-strong iff $\mathbb{S}$ is $<\beth_\gamma$-splitting.*

Proof. If follows by Theorem 5.2.22, together with the observation that in every ZFC model there is a bijection between elements of V_γ and subsets of $\beth_\alpha$ for $\alpha < \gamma$. Such bijection codes $x \in V_\gamma$ as the transitive collapse of a relation on $|\mathrm{trcl}(x)| \leq \beth_\alpha$ for some $\alpha < \gamma$, which in turn is coded by a subset of $\beth_\alpha$. $\square$

Closure

In this section we cover large cardinal properties defined in terms of the existence of elementary embeddings $j : V \to M \subseteq V[G]$ with certain degree of closure (i.e. such that ${}^{<\gamma}M \subseteq M$ for some appropriate γ). Main examples of such properties are supercompactness, hugeness and variants of them. We now present a criterion to characterize $<\gamma$-closure for an elementary embedding $j_{\dot{\mathbf{F}}(\mathbb{S})}$, which can in turn be applied in order to characterize all of the aforementioned large cardinal properties.

Definition 5.2.24. Let $\mathbb{S}$ be a $\mathcal{C}$-system of filters, $\mathcal{A} = \{A_\alpha :\ \alpha < \delta\}$ be an antichain in $\mathbb{S}^+$. We say that $\mathcal{A}$ *is guessed by* $\mathbb{S}$ iff there exist a $b \in \mathcal{C}$ and $\mathcal{B} = \{B_\alpha :\ \alpha < \delta\}$ antichain in $\mathcal{P}(O_b)$ such that $A_\alpha =_{\mathbb{S}} B_\alpha$ for all $\alpha < \delta$.

We say that a family of antichains $\langle \mathcal{A}_\alpha :\ \alpha < \mu \rangle$ is *simultaneously* guessed by $\mathbb{S}$ iff there is a single $b \in \mathcal{C}$ witnessing guessing for all of them.

Definition 5.2.25. Let $\mathbb{S}$ be a $\mathcal{C}$-system of filters. We say that $\mathbb{S}$ is $<\gamma$-*guessing* iff for all sequences $\langle \mathcal{A}_\alpha :\ \alpha < \mu \rangle$ of maximal antichains with $\mu < \gamma$, there are densely many $B \in \mathbb{S}^+$ such that the antichains $\mathcal{A}_\alpha \restriction B$ for $\alpha < \mu$ are simultaneously guessed by $\mathbb{S}$.

Notice that if an antichain is guessed by $\mathbb{S}$, every partition of it is split by $\mathbb{S}$. It follows that $<\gamma$-guessing implies $<\gamma$-splitting. Furthermore, if $\mathbb{T}$ is a tower of inaccessible length λ, $<\lambda$-guessing as defined above is equivalent to $<\lambda$-presaturation for the boolean algebra $\langle \mathbb{T}^+, \leq_{\mathbb{T}} \rangle$.

Theorem 5.2.26. *Let λ be an inaccessible cardinal, $\mathbb{S}$ be a $<\lambda$-directed $\mathcal{C}$-system of filters of length λ, $\gamma < \lambda$ be a cardinal. Then the ultrapower* $\mathrm{Ult}(V, \dot{\mathbf{F}}(\mathbb{S}))$ *is $<\gamma$-closed iff $\mathbb{S}$ is $<\gamma$-guessing.*

Proof. Let $a \in \mathcal{C}$, $u_\alpha : O_a \to \mathrm{ON}$ be such that $\left[\!\left[[\check{u}_\alpha]_{\dot{\mathbf{F}}(\mathbb{S})} = \check{\alpha} \right]\!\right]_{\mathbb{S}} = \mathbf{1}$ for all $\alpha < \gamma$. First, suppose that $\mathbb{S}$ is $<\gamma$-guessing and let $\dot{s}$ be a name for a sequence $\dot{s} : \mu \to \mathrm{Ult}(V, \dot{\mathbf{F}}(\mathbb{S}))$ for some $\mu < \gamma$. Let $\mathcal{A}_\alpha$ for $\alpha < \mu$ be a maximal antichain deciding the value of $\dot{s}(\check{\alpha})$, so that given any $A \in \mathcal{A}_\alpha$, $A \Vdash \dot{s}(\check{\alpha}) = [\check{v}_A]_{\dot{\mathbf{F}}(\mathbb{S})}$ for some $v_A : O_{a_A} \to V$. Let $\mathcal{S}$ be generic for $\mathbb{S}$. Then by $<\gamma$-guessing there is a $B \in \mathcal{S}$ such that $\mathcal{A}_\alpha \restriction B$ is guessed by $\mathbb{S}$ in $\mathcal{B}_\alpha \subseteq \mathcal{P}(O_b)$ for all $\alpha < \mu$. Furthermore, there can be only $|b| < \lambda$ elements $A \in \mathcal{A}_\alpha$ such that the corresponding $A' \in \mathcal{B}_\alpha$ is not empty. Since $\mathbb{S}$ is $<\lambda$-directed, there is a single $c \in \mathcal{C}$, $c \supseteq a, b$, such that $a_A \subseteq c$ for any $A \in \mathcal{A}_\alpha$ that is guessed in an $A' \neq \emptyset$.

Let B^f_α denote the unique element of $\mathcal{B}_\alpha$ such that $\pi_b(f) \in B^f_\alpha$, and $v^f_\alpha : O_{a^x_\alpha} \to V$ be such that $B^f_\alpha \Vdash \dot{s}(\check{\alpha}) = [\check{v}^f_\alpha]_{\dot{\mathbf{F}}(\mathbb{S})}$. Then for any $\alpha < \mu$ we can define

$$\begin{array}{rccl} v'_\alpha : & O_c & \longrightarrow & V \\ & f & \longmapsto & v^f_\alpha(\pi_{a^f_\alpha}(f)) \end{array}$$

so that $B \Vdash [\check{s}]_{\dot{\mathbf{F}}(\mathbb{S})}(\check{\alpha}) = [\check{v}'_\alpha]_{\dot{\mathbf{F}}(\mathbb{S})}$. Since all the v'_α have the same domain O_c, we can glue them together forming a single function

$$\begin{array}{rccl} v' : & O_c & \longrightarrow & V \\ & f & \longmapsto & \{\langle u_i(\pi_a(f)), v'_\alpha(f)\rangle : \ \alpha < u_\mu(\pi_a(f))\} \end{array}$$

Then B forces that $[v']_{\dot{\mathbf{F}}(\mathbb{S})} = \dot{s}$, and $B \in \mathcal{S}$ so $\mathrm{val}(\dot{s}, \mathcal{S}) = [v']_{\mathcal{S}}$ is in $\mathrm{Ult}(V, \mathcal{S})$.

Suppose now that $\mathbb{S}$ is $<\gamma$-closed and let $\langle \mathcal{A}_\alpha : \ \alpha < \mu \rangle$, $\mathcal{A}_\alpha = \langle A_{\alpha\beta} : \ \beta < \xi_\alpha \rangle$ be as in the definition of $<\gamma$-guessing. Let $\dot{f}$ be a name for a sequence $\dot{s} : j_{\dot{\mathbf{F}}(\mathbb{S})}[\mu] \to \mathrm{ON}$ such that $\left[\!\left[\dot{s}(j_{\dot{\mathbf{F}}(\mathbb{S})}(\check{\alpha})) = j_{\dot{\mathbf{F}}(\mathbb{S})}(\check{\beta}) \right]\!\right]_{\mathbb{S}} = [A_{\alpha\beta}]_{\mathbb{S}}$. Since $j_{\dot{\mathbf{F}}(\mathbb{S})}$ is $<\gamma$-closed, we can find densely many $B \subseteq O_b$ for $b \in \mathcal{C}$, $v : B \to \mathrm{ON}^\mu$ such that $B \Vdash \dot{s} = [\check{v}]_{\dot{\mathbf{F}}(\mathbb{S})}$. Then given any $\alpha < \mu$, $\beta < \xi_i$ we can define $B_{\alpha\beta} = \{f \in B : \ v(f)(\alpha) = \beta\}$ witnessing guessing for $\langle \mathcal{A}_\alpha \restriction B : \ \alpha < \mu \rangle$. $\square$

The above result gives a good characterization of $<\gamma$-closure for ideal towers, since ideal towers $\mathbb{T}$ of inaccessible height λ are always $<\lambda$-directed. On the other hand, this result does not apply to ideal extenders since their associated system of domains is never $<\omega_1$-directed. Since it is not known whether there is such a characterization of $<\gamma$-closure for extenders in V, we cannot expect to have one in the more general case of ideal extenders.

We can also determine an upper bound to the amount of closure that a system of filters might possibly have.

Definition 5.2.27. Let $\mathbb{S}$ be a $\mathcal{C}$-system of filters of length λ, $\alpha < \lambda$ be an ordinal. We say that $\mathbb{S} \restriction \alpha$ *does not express* $\mathbb{S}$ iff $\mathbf{1} \Vdash_{\mathbb{S}} M \supsetneq M_\alpha$ where $M = \mathrm{Ult}(V, \dot{\mathbf{F}}(\mathbb{S}))$, $M_\alpha = \mathrm{Ult}(V, \dot{\mathbf{F}}(\mathbb{S} \restriction \alpha))$.

Notice that $\mathbf{1} \Vdash_{\mathbb{S}} M \supsetneq M_\alpha$ is equivalent to $\mathbf{1} \Vdash_{\mathbb{S}} M \supsetneq k_\alpha[M_\alpha]$. In fact, $M = M_\alpha$ implies that $k_\alpha = \mathrm{id} \restriction M$ by Kunen's inconsistency, and $M = k_\alpha[M_\alpha]$ implies that k_α has no critical point thus is the identity.

Theorem 5.2.28. *Let $\mathbb{S}$ be a $\mathcal{C}$-system of filters of length λ such that $\mathbb{S} \restriction \alpha$ does not express $\mathbb{S}$ for any $\alpha < \lambda$. Then $M = \mathrm{Ult}(V, \dot{\mathbf{F}}(\mathbb{S}))$ is not closed under* $\mathrm{cof}(\lambda)$-*sequences.*

Proof. Let $\langle \xi_\alpha : \alpha < \gamma = \mathrm{cof}(\lambda) \rangle$ be a cofinal sequence in λ. For all $\alpha < \gamma$, let $\dot{u}_\alpha$ be a name for an element of $M \setminus k_{\xi_\alpha}[M_{\xi_\alpha}]$, $M_{\xi_\alpha} = \mathrm{Ult}(V, \dot{\mathbf{F}}(\mathbb{S} \restriction \xi_\alpha))$. Let $\dot{s}$ be a name such that

$$\left[\!\left[\dot{s}(j_{\dot{\mathbf{F}}(\mathbb{S})}(\alpha)) = [\dot{u}_\alpha]_{\dot{\mathbf{F}}(\mathbb{S})} \right]\!\right]_{\mathbb{S}} = \mathbf{1}$$

for all $\alpha < \gamma$, i.e. $\dot{s}$ is a name for a γ-sequence of elements of M indexed by $j_{\dot{\mathbf{F}}(\mathbb{S})}[\gamma]$.

Suppose by contradiction that M is closed under γ-sequences, so that $\mathbf{1} \Vdash_{\mathbb{S}} \dot{s} \in \dot{M}$, then there is an $A \in \mathbb{S}^+$, $v : A \to V$ such that $A \Vdash_{\mathbb{S}} \dot{s} = [\check{v}]_{\dot{\mathbf{F}}(\mathbb{S})}$. Let $\bar{\alpha} < \gamma$ be such that $A \in (\mathbb{S} \restriction \xi_{\bar{\alpha}})^+$. Let $v' : A \to V$ be such that $v'(f) = v(f)(\bar{\alpha})$ if $\bar{\alpha} \in \mathrm{dom}(f)$. By fineness $A \Vdash_{\mathbb{S}} [v']_{\dot{\mathbf{F}}(\mathbb{S})} = [\dot{u}_{\bar{\alpha}}]_{\dot{\mathbf{F}}(\mathbb{S})}$ for all $\alpha < \gamma$. Thus $A \Vdash_{\mathbb{S}} [\dot{u}_{\bar{\alpha}}]_{\dot{\mathbf{F}}(\mathbb{S})} \in k_{\xi_{\bar{\alpha}}}[M_{\xi_{\bar{\alpha}}}]$, a contradiction. □

The situation described in the previous theorem occurs in several cases, as shown by the following proposition.

Proposition 5.2.29. *Let $\mathbb{T}$ be an ideal tower of height λ limit ordinal, $\alpha < \lambda$ be an ordinal. Then $\mathbb{T} \restriction \alpha$ does not express $\mathbb{T}$.*

Proof. Suppose by contradiction that there is an $A \in \mathbb{T}^+$ such that $A \Vdash_{\mathbb{T}} M \supseteq k_\alpha[M_\alpha]$. Then in particular $A \Vdash_{\mathbb{T}} [\mathrm{id}_{V_{\alpha+1}}]_{\dot{\mathbf{F}}(\mathbb{T})} \in k_\alpha[M_\alpha]$ hence let $A' \in \mathbb{T}^+$, $A' \leq A$ be such that $A' \Vdash_{\mathbb{T}} [\mathrm{id}_{V_{\alpha+1}}]_{\dot{\mathbf{F}}(\mathbb{T})} = [u]_{\dot{\mathbf{F}}(\mathbb{T})}$ for some $u : O_a \to V$, $a \in \mathcal{C} \restriction \alpha$. Thus by Łoś Theorem,

$$B = \left\{ f \in O_{V_{\alpha+1}} : f = u(\pi_a(f)) \right\} \in \mathbb{T}^+$$

Since $|B| \leq |\mathrm{ran}(u)| \leq |O_a| \leq |\beth_\alpha| < |V_{\alpha+1}|$, $\mathrm{dom}[B]$ is a non-stationary subset of $V_{\alpha+1}$ by Lemma 1.2.13 contradicting Proposition 5.1.10. □

Lemma 5.2.30. *Let $\mathbb{S}$ be a $\mathcal{C}$-system of filters of length λ, γ be a cardinal such that $|a| \leq \gamma$ for all $a \in \mathcal{C}$. Then $j_{\dot{\mathbf{F}}(\mathbb{S})}(\gamma) < ((\beth_\lambda \cdot 2^\gamma)^+)^V$.*

Proof. Consider the set U of functions $u : a \to \gamma$ for some $a \in \mathcal{C}$. The total number of such functions is bounded by

$$|\mathcal{C}| \cdot \gamma^{\sup_{a \in \mathcal{C}} |a|} \leq \beth_\lambda \cdot \gamma^\gamma = \beth_\lambda \cdot 2^\gamma = \delta$$

Let $U = \langle u_\alpha : \alpha < \delta \rangle$, $\mathcal{A}_\alpha$ be the maximal antichain in $\mathbb{S}^+$ deciding the value of $[u_\alpha]_{\dot{\mathbf{F}}(\mathbb{S})}$, and let $X_\alpha = \left\{ \xi : \exists A \in \mathcal{A}_\alpha \ A \Vdash [\check{u}_\alpha]_{\dot{\mathbf{F}}(\mathbb{S})} = \check{\xi} \right\}$, $X = \bigcup_{\alpha < \delta} X_\alpha$. Since $|X_\alpha| \leq |\mathcal{A}_\alpha| \leq \beth_\lambda$, we have that $|X| \leq \delta \cdot \beth_\lambda = \delta$. Let now $\dot{v}$ be such that $[\dot{v}]_{\dot{\mathbf{F}}(\mathbb{S})} < j_{\dot{\mathbf{F}}(\mathbb{S})}(\gamma)$. Then there is a dense set of $A \in \mathbb{S}^+$ such that $A \Vdash [\dot{v}]_{\dot{\mathbf{F}}(\mathbb{S})} = [\check{u}_\alpha]_{\dot{\mathbf{F}}(\mathbb{S})} \Rightarrow A \Vdash [\dot{v}]_{\dot{\mathbf{F}}(\mathbb{S})} \in \check{X}$. Thus $j_{\dot{\mathbf{F}}(\mathbb{S})}(\gamma) \subseteq X$ (actually, $j_{\dot{\mathbf{F}}(\mathbb{S})}(\gamma) = X$) and $|X| \leq \delta$, hence $j_{\dot{\mathbf{F}}(\mathbb{S})}(\gamma) < (\delta^+)^V$. □

Proposition 5.2.31. *Let $\mathbb{E}$ be a $\langle \kappa, \lambda \rangle$-ideal extender such that $\lambda = \beth_\lambda$. Suppose that $\mathbf{1} \Vdash j_{\dot{\mathbf{F}}(\mathbb{E})}(\gamma) \geq \lambda$ for some $\gamma < \lambda$. Then $\mathbb{E} \restriction \alpha$ does not express $\mathbb{E}$ for any $\alpha < \lambda$.*

Proof. Since $j_{\dot{\mathbf{F}}(\mathbb{E})}(\gamma) \geq \lambda$, $\kappa_{\{\alpha\}} \leq \gamma$ for any $\alpha < \lambda$ hence we can apply Lemma 5.2.30 to obtain $j_{\dot{\mathbf{F}}(\mathbb{E} \restriction \alpha)}(\gamma) < ((\beth_\alpha \cdot 2^\gamma)^+)^V$ which is smaller than λ since $\alpha, \gamma < \lambda$ and λ is a $\beth$-fixed point. It follows that the critical point of $k_\alpha : M_\alpha \to M$ is at most $j_{\dot{\mathbf{F}}(\mathbb{E} \restriction \alpha)}(\gamma)$ and in particular $k_\alpha[M_\alpha] \neq M$. □

We remark that the conditions of the previous proposition are often fulfilled. In particular they hold whenever $\mathbb{E}$ is the $\langle\kappa,\lambda\rangle$-ideal extender derived from an embedding j and the length λ is a $\beth$-fixed point but not a j-fixed point.

5.2.4 Distinction between generic large cardinal properties

Let $j : V \to M \subseteq M[G]$ be a generic elementary embedding with critical point κ. In this section we provide examples separating the following generic large cardinals notions at a successor cardinal $\kappa = \gamma^+$.

- j is almost superstrong if $V^M_{j(\kappa)} \prec V^{V[G]}_{\gamma^+}$;
- j is superstrong if it is $j(\kappa)$-strong;
- j is almost huge if it is $<j(\kappa)$-closed.

These examples will all be obtained by collapsing with $\mathbb{C} = \mathrm{Coll}(\gamma, <\kappa)$ a suitable large cardinal embedding in V, so that by Theorem 5.2.10 a generic large cardinal embedding $j^{\mathbb{C}}$ is obtained with the desired properties.

Proposition 5.2.32. *Let κ be a 2-superstrong cardinal. Then there is a generic elementary embedding on $\kappa = \gamma^+$ that is almost superstrong and not superstrong.*

Proof. Let j be a 2-superstrong embedding with critical point κ, and let $\mathcal{E}$ be the $\langle\kappa, j(\kappa)\rangle$-extender derived from j. Since V models that $\mathcal{E}$ is a superstrong $\langle\kappa, j(\kappa)\rangle$-extender, by elementarity M models that $j(\mathcal{E})$ is a superstrong $\langle j(\kappa), j^2(\kappa)\rangle$-extender. Thus

$$\begin{array}{rccl} j_{j(\mathcal{E})}: & M & \longrightarrow & N = \mathrm{Ult}(M, j(\mathcal{E})) \supseteq M_{j^2(\kappa)} = V_{j^2(\kappa)} \\ & j(\kappa) & \longmapsto & j^2(\kappa) \end{array}$$

Since $M \subseteq V$, also $\mathrm{Ult}(V, j(\mathcal{E})) \supseteq \mathrm{Ult}(M, j(\mathcal{E})) \supseteq V_{j^2(\kappa)}$ hence $j(\kappa)$ is superstrong as witnessed by $j(\mathcal{E})$ also in V.

Consider now $\mathbb{C} = \mathrm{Coll}(\gamma, <\kappa)$, j_1 induced by $\mathbb{C}$ and $j_{\mathcal{E}}$, j_2 induced by $j(\mathbb{C})$ and $j_{j(\mathcal{E})}$. By Proposition 5.2.12, j_1 and j_2 are still superstrong and we get the following diagram, where all the inclusions are superstrong:

$$\begin{array}{lccccc} j_0: & V^{\mathbb{C}} & \longrightarrow & M^{j(\mathbb{C})} & & \\ & & & \text{\rotatebox{90}{$\subseteq$}} & & \\ & & j_1: & V^{j(\mathbb{C})} & \longrightarrow & N^{j^2(\mathbb{C})} \\ & & & & & \text{\rotatebox{90}{$\subseteq$}} \\ & & & & & V^{j^2(\mathbb{C})} \end{array}$$

Thus j_0 considered as a generic elementary embedding in $V^{j^2(\mathbb{C})}$ is almost superstrong:

$$M^{j(\mathbb{C})}_{j(\kappa)} = V^{j(\mathbb{C})}_{j(\kappa)} \prec N^{j^2(\mathbb{C})}_{j^2(\kappa)} = V^{j^2(\mathbb{C})}_{j^2(\kappa)} = V^{j^2(\mathbb{C})}_{\gamma^+}$$

but not superstrong, since $V^{j(\mathbb{C})}_{\gamma+1}$ has cardinality $\delta \in (\gamma, j^2(\kappa))$ in $V^{j(\mathbb{C})}$ hence in $V^{j^2(\mathbb{C})}$ its cardinality is collapsed to γ and bijection between γ and $V^{j(\mathbb{C})}_{\gamma+1}$ is added. $\square$

Proposition 5.2.33. *Let κ be a 2-huge cardinal. Then there is a generic elementary embedding on $\kappa = \gamma^+$ that is superstrong and not $<\omega$-closed.*

Proof. Let j be a 2-huge embedding in V with critical point κ. Then we can derive a $\langle \kappa, j(\kappa) + \omega \rangle$-tower $\mathbb{T}$ from j, so that $j_{\mathbb{T}}$ is still $\kappa + \omega$-superstrong but by Theorem 5.2.28 is not closed under ω-sequences.

Let $j^{\mathbb{C}} : V^{\mathbb{C}} \to M^{j(\mathbb{C})}$ be derived from $j_{\mathbb{T}}$ as in Theorem 5.2.10. Since $j_{\mathbb{T}}$ is $\kappa+\omega$-superstrong, by Proposition 5.2.12 $j^{\mathbb{C}}$ is still $\kappa+\omega$-superstrong (hence superstrong). Moreover $j^{\mathbb{C}}$ is not $<\omega$-closed. In fact given any $A \in {}^{\omega}M \setminus M$, $j(\mathbb{C})$ cannot add A since it is a set of size ω and $\mathbb{C}$ is closed under ω-sequences. □

5.3 Derived ideal towers in V failing preservation of large cardinal properties

In this section we shall present an example of a tower $\mathbb{T}$ (of maximal length) derived from a generically superstrong embedding $j : V \to M \subset V^{\mathbb{B}}$, such that the natural embedding of $\mathbb{T}$ into $\mathbb{B}$ is densely incomplete (even though $\mathbb{T}$ could still induce a superstrong embedding). The construction will be built on Foreman and Woodin's work on self-genericity for towers (see e.g. [16, Sec. 9.4]). The superstrong embedding j will be obtained as the restriction of the huge embedding induced by a full stationary tower of length a Woodin cardinal.

Definition 5.3.1. We say that $\mathbb{T}_\delta = \langle \mathrm{NS}_a : a \in V_\delta \rangle$ is the *full stationary tower* of length δ. We denote by $\mathbb{T}_\delta \restriction A$ with $A \in \mathbb{T}_\delta$ the tower of ideals $\mathrm{NS}_a \restriction A = \{B \in \mathrm{NS}_a : A \wedge B \text{ is stationary}\}$.

We say that $\mathbb{T}^\gamma_\delta = \langle \mathrm{NS}^\gamma_a : a \in V_\delta \rangle$ is the *γ-stationary tower* of length δ where $\mathrm{NS}^\gamma_a = \{A : A \cap [a]^{<\gamma} \in \mathrm{NS}_a\}$.

By the results of Appendix 5.1.2, we can identify $\mathbb{T}_\delta$ with a $\mathcal{C}$-system of filters. Filter property and fineness can be directly verified, compatibility follows from Lemma 1.2.11 and normality from Lemma 1.2.5. If δ attains sufficient large cardinal properties, these towers provably induce embeddings with strong closure properties.

Theorem 5.3.2 (Woodin [33, Thm. 2.5.8])**.** *Let δ be a Woodin cardinal, $j : V \to M \subseteq V[G]$ be induced by $\mathbb{T}_\delta$. Then j is $<\delta$-closed.*

We shall follow the following procedure:

- carefully choose a $T \in \mathrm{NS}^+_{V_\lambda}$ such that $T \Vdash_{\mathbb{T}_\delta} j(\mathrm{crit}(j)) \leq \lambda < \delta$;
- consider the huge embedding $j : V \to M \subseteq V[G]$ induced by $\mathbb{T}_\delta \restriction T$;
- consider the reduced superstrong embedding j_λ induced by $\dot{\mathbf{F}}(\mathbb{T}_\delta \restriction T) \restriction \lambda$;
- derive a tower $\mathbb{T}^\gamma_\lambda \restriction S'$ from j_λ (of maximal height λ by Proposition 5.1.37);
- show that the natural immersion of $\mathbb{T}^\gamma_\lambda \restriction S'$ into $\mathbb{T}_\delta \restriction T$ is densely incomplete.

Definition 5.3.3. Let $\lambda < \delta$ be ordinals, $M \subseteq V_\delta$, and

$$G_M = \left\{ A \in \mathbb{T}_\lambda \cap M : \ M \cap \bigcup A \in A \right\}$$

We say that G_M is M-generic for $\mathbb{T}_\lambda$ iff for every $\mathcal{A} \in M$ predense subset of $\mathbb{T}_\lambda$, $\mathcal{A} \cap G_M \neq \emptyset$[7].

We aim to prove that

$$S = \left\{ M \in [V_{\lambda+2}]^{<\gamma} : \ G_M \text{ not } M\text{-generic for } \mathbb{T}_\lambda \right\}$$

is stationary and compatible with every $A \in \mathbb{T}_\lambda^\gamma$, for appropriately chosen γ, λ. This will be done by showing that S has the *end-extension property.*

Definition 5.3.4. We say that M is a *rank initial segment* of N iff $N \cap V_{\text{rank}(M)} = M$. We say that $N \prec V_\delta$ is a λ-*end-extension* of $M \prec V_\delta$ iff $N \supseteq M$ and $M \cap V_\lambda$ is a rank initial segment of $N \cap V_\lambda$. We say that an end-extension is *proper* iff $N \cap V_\lambda \supsetneq M \cap V_\lambda$, and that is sup-preserving iff $\sup(N \cap \alpha) = \sup(M \cap \alpha)$ for all $\alpha \in M$ with $\operatorname{cof}(\alpha) > \lambda$.

We say that $A \subseteq [V_\delta]^{<\gamma}$ has the λ-*end-extension property* iff for club many $M \in [V_{\delta+2}]^{<\gamma}$ there is $N \in A \uparrow V_{\delta+2}$ that is a λ-end-extension of M.

Proposition 5.3.5. *Suppose that $A \subseteq [V_\delta]^{<\gamma}$ has the λ-end-extension property. Then it is stationary and compatible with every $B \in \mathbb{T}_\lambda^\gamma$.*

Proof. Fix $B \in \mathbb{T}_\lambda^\gamma$ and C_f any club on V_δ. We prove that $A \wedge B \wedge C_f \neq \emptyset$. Let D be the club witnessing the end-extension property of A, and let $M \prec V_{\delta+2}$ be such that $A, B, f \in M$ and $M \in B \wedge D$. Let N be a λ-end extension of M in $A \uparrow V_{\delta+2}$. Then $N \cap V_\lambda \in A \wedge B \wedge C_f$ as required. □

Using a measurable cardinal, it is possible to produce end-extensions satisfying many requirements. The main building block is the following.

Theorem 5.3.6. *Let $M \prec V_\delta$ be such that $|M| < \lambda \in M$ with λ a measurable cardinal. Then there exists an N proper sup-preserving λ-end-extension of M such that $|N| = |M|$.*

Proof. Let $F \in M$ be a $<\lambda$-complete ultrafilter on λ, so that $\bigcap(F \cap M) \in F$ hence it is unbounded. Let γ be in $\bigcap(F \cap M) \setminus \sup(M \cap \lambda)$, and let

$$N = \{ f(\gamma) : \ f : \lambda \to V_\delta \wedge f \in M \}.$$

Then N is a λ-end-extension of M as shown in [33, Lemma 1.1.21], and it is proper since $\gamma = \operatorname{id}(\gamma)$ and $\operatorname{id} \restriction \lambda \in M$ thus $\gamma \in N$.

We now prove that it is also sup-preserving. Suppose by contradiction that there is an $\alpha \in M$, $\operatorname{cof}(\alpha) > \lambda$ and $\xi \in N \cap \alpha$ such that $\xi \geq \sup(M \cap \alpha)$. Let $f : \lambda \to \alpha$ in M be such that $f(\gamma) = \xi$. Since V_δ models that f is bounded below α, so does M hence let $\beta = \sup(\operatorname{ran}(f)) < \alpha$ be in M. Then $\beta \in (M \cap \alpha) \setminus \xi$, a contradiction. □

[7]Notice that $G_M \subseteq M$ hence $G_M = G_M \cap M$ and the present definition coincides with the usual notion of M-genericity.

End-extensions are powerful in capturing the following type of stationary sets.

Definition 5.3.7. Let $A \subseteq \lambda$ be stationary in the classical sense. Then,

$$\bar{A} = \{M \prec V_\lambda : \ \sup(M \cap \lambda) \in A\}$$

is the corresponding generalized stationary set on V_λ.

Lemma 5.3.8. *Let $M \prec V_\delta$ be such that $|M| < \gamma < \lambda \in M$ with λ measurable cardinal, $A \subseteq \{\alpha < \lambda : \ \mathrm{cof}(\alpha) < \gamma\}$ be stationary in the classical sense. Then there exists an N sup-preserving λ-end-extension of M such that $|N| < \gamma$ and $N \in \bar{A} \uparrow V_\delta$.*

Proof. Define M_α for any $\alpha < \lambda$ as follows.

- $M_0 = M$
- Let $M_{\alpha+1} \supseteq M_\alpha$ be the witness provided by Theorem 5.3.6 for M, δ, λ. Hence $\sup(M_{\alpha+1} \cap \lambda) > \sup(M_\alpha \cap \lambda)$, $M_{\alpha+1} \cap \eta = M_\alpha \cap \eta = M \cap \eta$ for $\eta = \sup(M \cap \lambda)$ and all sups of cofinality greater than λ are preserved.
- If α is a limit ordinal, $M_\alpha = \bigcup \{M_\beta : \ \beta < \alpha\}$.

Therefore $C = \{\sup(M_\alpha \cap \lambda) : \ \alpha < \lambda\}$ is a club such that there exists $\bar{\alpha} > \gamma$ with $\sup(M_{\bar{\alpha}} \cap \lambda) \in A$. Let $X \subseteq M_{\bar{\alpha}} \cap \lambda$ be minimal and cofinal. Hence $|X| < \gamma$. Define $N = \mathrm{SkH}^{M_{\bar{\alpha}}}(M \cup X)$. Then $N \prec M_{\bar{\alpha}} \prec V_\delta$. Hence N is a sup-preserving λ-end-extension in $\bar{A} \uparrow V_\delta$ such that $|N| < \gamma$. □

In presence of a limit of measurable cardinals, this process can be iterated in order to capture *simultaneously* a sequence of stationary sets.

Lemma 5.3.9. *Let $s = \langle \lambda_\alpha : \ \alpha < \xi \rangle$ be an increasing sequence of measurable cardinals such that $s \cap \lim(s) = \emptyset$ and $\alpha < \lambda_\alpha$ for all $\alpha < \xi$. Let $M \prec V_\delta$ be such that $|M| < \gamma < \lambda_0$ and $s \in M$. Let $A_\alpha \subseteq \{\beta < \lambda_\alpha : \ \mathrm{cof}(\beta) < \gamma\}$ be stationary in the classical sense for all $\alpha < \xi$. Then there exists a λ_0-end-extension N of M such that $|N| < \gamma$ and $N \in \left(\triangle_{\alpha<\xi} \bar{A}_\alpha\right) \uparrow V_\delta$.*

Proof. For any $i \in \omega$ define M^i by induction over i. Put $M^0 = M$. Let $M^i \cap \xi = \left\{\alpha^i_\beta : \ \beta < \nu_i\right\}$ where $\nu_i = \mathrm{otp}(M^i \cap \xi) < \gamma$. Let $M^{i+1} = \bigcup \left\{M^i_\beta : \ \beta < \nu_i\right\}$, where M^i_β are defined by induction as follows.

- $M^i_0 = M^i$.
- If $M^i_\beta \notin \bar{A}_{\alpha^i_\beta} \uparrow V_\delta$, define $M^i_{\beta+1} \supseteq M^i_\beta$ given by Lemma 5.3.8 applied to $A_{\alpha^i_\beta}$. Otherwise let $M^i_{\beta+1} = M^i_\beta$.
- If β is a limit ordinal, $M^i_\beta = \bigcup \left\{M^i_j : \ j < \beta\right\}$.

Let $M^\omega = \bigcup \left\{M^i : \ i < \omega\right\}$. Hence for any $\nu \in M^\omega \cap \gamma$, there exists i such that $\nu \in M^i$. Thus there exists β such that $\nu = \alpha^i_\beta$. It follows that $\sup(M^{i+1} \cap \lambda_\nu) \in A_\nu$, and by Lemma 5.3.8 for any $j > i$, $\sup(M^j \cap \lambda_\nu) = \sup(M^{i+1} \cap \lambda_\nu)$. Hence it holds also for $j = \omega$. □

Let now δ be a Woodin cardinal, $\kappa < \delta$ be regular. Let $\mu < \delta$ be such that

$$S' = \{X \subseteq \mu : \ X \cap \kappa \in \kappa \wedge \mathrm{otp}(X) \geq \kappa\}$$

is stationary. Notice that S' forces the critical point of j to be κ and $j(\kappa) \leq \mu$. Let $\lambda \in (\mu, \delta)$ be a limit of measurable cardinals, and $s = \langle \lambda_\alpha : \ \alpha < \xi \rangle$ be a cofinal sequence of measurable cardinals as in the hypothesis of the previous lemma and such that $\lambda_0 > \mu$. Let $\gamma \in (\mu, \lambda_0)$ be a cardinal, let A_α, B_α be disjoint stationary subsets of $\{\beta < \lambda_\alpha : \ \mathrm{cof}(\beta) < \gamma\}$ for all $\alpha < \xi$ by Ulam's Theorem 1.2.12, and let $C_\alpha = \{M \prec V_{\lambda_\alpha} : \ |M| \geq \gamma\}$ for $\alpha < \xi$.

Lemma 5.3.10. *The set $\mathcal{A}_\alpha = \left\{\bar{A}_\beta, C_\beta : \ \beta \in [\alpha, \xi)\right\}$ is predense in $\mathbb{T}_\lambda$ for all $\alpha < \xi$.*

Proof. Let $D \in \mathbb{T}_\lambda$ be arbitrary, and $\beta \in [\alpha, \xi)$ be such that $D \in V_{\lambda_\beta}$. Partition D in D_1, D_2 such that $D_1 = \{M \in D : \ |M| < \gamma\}$, $D_2 = D \setminus D_1$. At least one among D_1, D_2 is stationary. If D_2 is stationary it is compatible with C_β (since $D_2 < C_\beta$), otherwise D_1 is compatible with $\bar{A}_\beta$. In fact, given any club C_f on V_{λ_β} and $M \prec V_\lambda$ such that $|M| < \gamma$, $D_1, \bar{A}_\beta, f \in M$, $M \in D_1 \uparrow V_\lambda$, by Lemma 5.3.8 we can find an $N \in \bar{A}_\beta \uparrow V_\lambda$ so that $N \cap V_{\lambda_\beta} \in D_1 \wedge \bar{A}_\beta \wedge C_f$. □

Theorem 5.3.11. *Let γ, s, λ, δ be as above. Then,*

$$S = \left\{M \in [V_{\lambda+2}]^{<\gamma} : \ G_M \text{ not } M\text{-generic for } \mathbb{T}_\lambda\right\}$$

is stationary and compatible with any $A \in \mathbb{T}^\gamma_\lambda$.

Proof. We show that S has the λ_α-end-extension property for any $\alpha < \xi$, the thesis will then follow by Proposition 5.3.5. Let $M \prec V_{\lambda+4}$ be such that $|M| < \gamma$, $\mathcal{A}_\alpha, s \in M$. Let N be a λ_α-end-extension of M in $\left(\Delta_{\beta\in[\alpha,\xi)} \bar{B}_\beta\right) \uparrow V_{\lambda+4}$ by Lemma 5.3.9 (considering the sequence s after α). Then G_N is not N-generic for $\mathbb{T}_\lambda$, since $\beta \in N \cap [\alpha, \xi) \Rightarrow N \in \bar{B}_\beta \uparrow V_{\lambda+4} \Rightarrow N \notin \bar{A}_\beta \uparrow V_{\lambda+4} \Rightarrow \bar{A}_\beta \notin G_N$ hence $\mathcal{A}_\alpha \cap G_N = \emptyset$. □

Let j be the generic elementary embedding derived from $\mathbb{T}_\delta \restriction T$, $T = S \wedge S'$, and consider the reduced embedding j_λ derived from $\dot{\mathbf{F}}(\mathbb{T}_\delta \restriction T) \cap V_\lambda$. By Theorem 5.3.11 the elements of $\mathbb{T}^\gamma_\lambda$ which are compatible with T are exactly the elements of $\mathbb{T}^\gamma_\lambda \restriction S'$. Furthermore, no element $A \subseteq \mathcal{P}(X)$ of $\mathbb{T}_\lambda$ which concentrates on $[X]^{\geq\gamma}$ can be compatible with S (thus with T). It follows that the tallest tower derivable from j_λ is $\mathbb{T}^\gamma_\lambda \restriction S'$, and the corresponding embedding $i : \mathbb{T}^\gamma_\lambda \restriction S' \to \mathbb{T}_\delta \restriction T$ is the inclusion map.

Theorem 5.3.12. *The embedding $i : \mathbb{T}^\gamma_\lambda \restriction S' \to \mathbb{T}_\delta \restriction T$ is densely incomplete.*

Proof. Let $B \in \mathbb{T}^\gamma_\lambda \restriction S'$ be arbitrary. Since $T \wedge B$ is stationary, consider the function

$$\begin{array}{rccl} f: & T \wedge B & \longrightarrow & V_{\lambda+2} \\ & M & \longmapsto & \mathcal{A} \in M \text{ such that } G_M \cap \mathcal{A} = \emptyset \end{array}$$

By Fodor's Lemma 1.2.5 there is a stationary set $C \subseteq T \wedge B$ such that $f[C] = \{\mathcal{A}\}$. Then $\mathcal{A} \restriction B$ is maximal below B in $\mathbb{T}^\gamma_\lambda \restriction S'$, but is not maximal in $\mathbb{T}_\delta \restriction T$ as witnessed by C. □

Corollary 5.3.13. *Let δ be a Woodin cardinal. Then for any $\kappa \in [\omega_1, \delta)$ there is a generically superstrong embedding j with critical point κ such that the tallest tower derivable from j embeds in the original forcing in a densely incomplete way.*

5.4 Conclusions and open problems

In the last chapter we investigated some topics related to the definability of generic large cardinal properties. We gave a unified treatment of extenders and towers, and some partial results on how generic large cardinal embeddings are induced by set-sized objects. However, many questions remain open. We list them according to the ordering of sections of this chapter.

Question 5.4.1. Is $(j_{\mathcal{S}}\restriction a)^{-1}$ the unique element of $\bigcap j_{\mathcal{S}}[F_a]$?

Question 5.4.2. Assume that j is a $<\gamma$-closed embedding in $V[G]$, with G V-generic for $\mathbb{B}$. Can this be witnessed by a generic γ-extender of sufficient length *independently of the chain condition satisfied by* $\mathbb{B}$?

In Theorem 5.1.40, we showed that a generic extender can have enough expressive power to approximate any other generic $\mathcal{C}$-system of ultrafilters. This observation suggests the following question.

Question 5.4.3. Assume κ has ideally property P, can this be witnessed by an extender in the ground model V?

Since κ has ideally property P, there is a $\mathcal{C}$-system of filters $\mathbb{S}$ in V which witness property P for κ. However, in general $\mathbb{S}$ might not be an extender (e.g. a tower or a γ-extender). We already know that a generic extender $\mathcal{E}$ is able to fully approximate $\mathbb{S}$, is this possible also for an ideal extender $\mathbb{E}$ in V? We believe that this question could be an important cornerstone to uncover the following.

Question 5.4.4. Is having ideally property P equivalent to having generically property P?

A final question arises from inspecting the machinery used in Section 5.3.

Question 5.4.5 (Cox)**.** What is the consistency strength of self-genericity with respect to towers?

More precisely, let $\mathbb{T}$ be a tower of inaccessible height λ, $S^{\mathbb{T}}$ be the set of self-generic $M \prec H_{\lambda^+}$ for $\mathbb{T}$, and consider the following two versions of self-genericity (inspired by the analogous definitions for ideals introduced in [12]):

- StatCatch($\mathbb{T}$) abbreviates the statement: $S^{\mathbb{T}}$ is stationary;
- ProjectiveCatch($\mathbb{T}$) abbreviates the statement: $S^{\mathbb{T}}$ is stationary and compatible with all $A \in \mathbb{T}$.

What is the consistency strength of these properties holding for a tower $\mathbb{T}$ with critical point ω_1 (or ω_2)? A similar problem was inspected in [12] for single ideals, but it is not at all clear whether a similar argument can be applied to towers.

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