

Gelfand-Shilov type spaces through Hermite expansions

Todor Gramchev, Alexander Lecke, Stevan Pilipovic and Luigi Rodino

Abstract. Gelfand-Shilov spaces of the type $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ and $\Sigma_\alpha^\alpha(\mathbb{R}^d)$ can be realized as sequence spaces by means of the Hermite representation Theorem. In this article we show that for a function $f = \sum a_k \Psi_k \in \mathcal{S}_\beta^\beta(\mathbb{R}^d)$ (resp. $\Sigma_\beta^\beta(\mathbb{R}^d)$) in order that $\sum a_k \Psi_k \rightarrow f$ in $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ (resp. $\Sigma_\alpha^\alpha(\mathbb{R}^d)$), where $\frac{1}{2} \leq \alpha \leq \beta$ (resp. $1/2 < \alpha \leq \beta$), it follows that $\alpha = \beta$. Furthermore we characterize spaces of the type $(\mathcal{S}_\alpha^\alpha \otimes \mathcal{S}_\beta^\beta)(\mathbb{R}^{s+t})$ (resp. $(\Sigma_\alpha^\alpha \otimes \Sigma_\beta^\beta)(\mathbb{R}^{s+t})$) defined by Gelfand and Shilov through the estimates Hermite coefficients and, moreover, introduce a new spaces of Gelfand-Shilov type $S_\sigma^{\otimes, \sigma}(\mathbb{R}^n)$, $\sigma \geq 1/2$, and $\Sigma_\sigma^{\otimes, \sigma}(\mathbb{R}^n)$, $\sigma > 1/2$. All the spaces are compared.

Mathematics Subject Classification (2000). Primary 42A16, 46F05; Secondary 46F12, 35Sxx.

Keywords. Hermite-Expansion, Hermite functions, test function space, Gelfand-Shilov space.

1. Introduction

Gelfand-Shilov spaces $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$, $\mathcal{S}^\beta(\mathbb{R}^d)$ and $\mathcal{S}_\alpha^\beta(\mathbb{R}^d)$ and their generalisations, the Gelfand-Shilov spaces of Roumieu and Beuerling type $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ respectively $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$ are discussed in [2], [3], [4], [6], [7], [9], [10], [11] and [12]. In this paper we focus on the special cases $\mathcal{S}_\alpha^\beta(\mathbb{R}^d)$, resp., $\Sigma_\alpha^\beta(\mathbb{R}^d)$. We show that if the Hermite expansion $\sum a_k \Psi_k$ converges to f (here Ψ_k denote the d -dimensional Hermite functions and a_k the Hermite coefficients of f) in the sense of $\mathcal{S}_\alpha^\beta(\mathbb{R}^d)$ ($\alpha < \beta$), resp., $\Sigma_\alpha^\beta(\mathbb{R}^d)$ ($\alpha < \beta$), then it belongs to $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$, resp., $\Sigma_\alpha^\alpha(\mathbb{R}^d)$.

Furthermore we analyze intermediate spaces $(\mathcal{S}_\alpha^\alpha \otimes \mathcal{S}_\beta^\beta)(\mathbb{R}^{s+t})$ and $(\Sigma_\alpha^\alpha \otimes \Sigma_\beta^\beta)(\mathbb{R}^{s+t})$, introduced also by Gelfand and Shilov, through the estimates of Hermite coefficients. The elements of spaces of this type are functions f which behave in their first s components like a function in $\mathcal{S}_\alpha^\alpha(\mathbb{R}^s)$ (resp. $\Sigma_\alpha^\alpha(\mathbb{R}^s)$) and in their last t

components like a function in $\mathcal{S}_\beta^\beta(\mathbb{R}^t)$ (resp. $\sum_\beta^\beta(\mathbb{R}^t)$). In the last part of the paper we introduce one more class of Gelfand-Shilov type spaces $\mathcal{S}_\sigma^{\otimes, \sigma}(\mathbb{R}^n)$, $\sigma \geq 1/2$, and $\Sigma_\sigma^{\otimes, \sigma}(\mathbb{R}^n)$, $\sigma > 1/2$. These spaces were obtained through the iteration of Harmonic oscillators and are related to our study of Weyl formula for tensorised products of elliptic Shubin type operators (see [1]). We compare all the considered spaces through the estimates of Hermite coefficients.

1.1. Notation and basic notions

In this paper we use the following notation: Let $j, p, q \in \mathbb{N}_0^d$ and $\alpha, \beta \in \mathbb{R}_+$, we have $p^{p\alpha} = p_1^{p_1\alpha} \cdot \dots \cdot p_d^{p_d\alpha}$ and the analogues for $q^{q\beta}$. Similarly for $x \in \mathbb{R}^d$ we have $x^p = x_1^{p_1} \cdot \dots \cdot x_d^{p_d}$ and for $c \in \mathbb{R}$ we have $c^j = c^{j_1} \cdot \dots \cdot c^{j_d}$. In addition we write $\partial^q f = \frac{\partial^{|q|}}{\partial x_1^{q_1} \dots \partial x_d^{q_d}} f$ for $f \in \mathcal{C}^\infty(\mathbb{R}^d)$.

We want to recall a few definitions and facts corresponding to the spaces of type $\mathcal{S}_\alpha^\beta(\mathbb{R}^d)$ and $\sum_\alpha^\beta(\mathbb{R}^d)$.

The Gelfand-Shilov spaces are defined as follows, cf. [10, Theorem 2.6]:

Definition 1.1. Let $\alpha, \beta \in \mathbb{R}_+$, $p, q \in \mathbb{N}_0^d$ and assume that A, B, C are positive numbers.

1. The Gelfand-Shilov space of type $\mathcal{S}_{\alpha, A}(\mathbb{R}^d)$ is defined as follows:

$$\mathcal{S}_{\alpha, A}(\mathbb{R}^d) = \left\{ f \in \mathcal{C}^\infty(\mathbb{R}^d) : \forall q \exists C_q \text{ s.t. } \|x^p \partial^q f\|_{L^2(\mathbb{R}^d)} \leq C_q A^{|p|} p!^\alpha \text{ for all } p \right\}.$$

2. The Gelfand-Shilov space of type $\mathcal{S}^{\beta, B}(\mathbb{R}^d)$ is defined as follows:

$$\mathcal{S}^{\beta, B}(\mathbb{R}^d) = \left\{ f \in \mathcal{C}^\infty(\mathbb{R}^d) : \forall p \exists C_p \text{ s.t. } \|x^p \partial^q f\|_{L^2(\mathbb{R}^d)} \leq C_p B^{|q|} q!^\beta \text{ for all } q \right\}.$$

3. The Gelfand-Shilov space of type $\mathcal{S}_{\alpha, A}^{\beta, B}(\mathbb{R}^d)$ is defined as follows:

$$\mathcal{S}_{\alpha, A}^{\beta, B}(\mathbb{R}^d) = \left\{ f \in \mathcal{C}^\infty(\mathbb{R}^d) : \exists C \text{ s.t. } \forall p, q : \|x^p \partial^q f\|_{L^2(\mathbb{R}^d)} \leq C A^{|p|} p!^\alpha B^{|q|} q!^\beta \right\}.$$

Their inductive and projective limits are denoted by:

1. $\mathcal{S}_\alpha(\mathbb{R}^d) = \text{indlim}_A \mathcal{S}_{\alpha, A}(\mathbb{R}^d)$; $\sum_\alpha(\mathbb{R}^d) = \text{projlim}_A \mathcal{S}_{\alpha, A}(\mathbb{R}^d)$
2. $\mathcal{S}^\beta(\mathbb{R}^d) = \text{indlim}_B \mathcal{S}^{\beta, B}(\mathbb{R}^d)$; $\sum^\beta(\mathbb{R}^d) = \text{projlim}_B \mathcal{S}^{\beta, B}(\mathbb{R}^d)$
3. $\mathcal{S}_{\alpha, A}^\beta(\mathbb{R}^d) = \text{indlim}_{A, B} \mathcal{S}_{\alpha, A}^{\beta, B}(\mathbb{R}^d)$; $\sum_{\alpha, A}^\beta(\mathbb{R}^d) = \text{projlim}_{A, B} \mathcal{S}_{\alpha, A}^{\beta, B}(\mathbb{R}^d)$

These spaces are subspaces of the Schwartz space

$$\mathcal{S}(\mathbb{R}^d) = \left\{ f \in \mathcal{C}^\infty(\mathbb{R}^d) : \forall p, q : \|x^p \partial^q f\|_{L^2(\mathbb{R}^d)} < \infty \right\}.$$

Note that the space $\mathcal{S}_\alpha^\beta(\mathbb{R}^d)$ is nontrivial if $\alpha + \beta \geq 1$ (resp. $\sum_\alpha^\beta(\mathbb{R}^d)$ is nontrivial if $\alpha + \beta > 1$) and that holds $\mathcal{S}_\alpha^\beta(\mathbb{R}^d) \subseteq \mathcal{S}_{\alpha'}^{\beta'}(\mathbb{R}^d)$ and $\sum_\alpha^\beta(\mathbb{R}^d) \subseteq \sum_{\alpha'}^{\beta'}(\mathbb{R}^d)$ if $\alpha \leq \alpha'$ and $\beta \leq \beta'$.

The polynomials

$$H_k(t) = (-1)^k \exp(t^2) \left(\frac{d}{dt} \right)^k \exp(-t^2), \quad t \in \mathbb{R}, \quad k \in \mathbb{N}_0$$

are the Hermite polynomials. The one-dimensional Hermite functions ψ_k are given by

$$\psi_k(t) = \left(\pi^{\frac{1}{2}} 2^k k!\right)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}t^2\right) H_k(t), \quad t \in \mathbb{R}, \quad k \in \mathbb{N}_0.$$

The d -dimensional Hermite functions are

$$\Psi_m(x) = \prod_{j=1}^d \psi_{m_j}(x_j) \quad \text{with } x = (x_1, \dots, x_d) \in \mathbb{R}^d \text{ and } m \in \mathbb{N}_0^d.$$

For $f \in \mathcal{S}(\mathbb{R}^d)$ the Hermite coefficients are

$$(a_m)_{m \in \mathbb{N}_0^d} = \left((f, \Psi_m)_{L^2(\mathbb{R}^d)} \right)_{m \in \mathbb{N}_0^d}. \quad (1.1)$$

Lemma 1.2. *Let $\alpha, \beta \in \mathbb{R}_+$. Then the following holds:*

$$\mathcal{S}_\alpha(\mathbb{R}^d) \cap \mathcal{S}^\beta(\mathbb{R}^d) = \mathcal{S}_\alpha^\beta(\mathbb{R}^d)$$

This result was obtained by Kashpirovskii [5] and later by other authors, cf [2]. In addition it holds $\sum_\alpha^\beta(\mathbb{R}^d) = \sum_\alpha(\mathbb{R}^d) \cap \sum^\beta(\mathbb{R}^d)$ cf [2, Theorem 7.2.2]. We define the spaces of the Hermite coefficients of $\mathcal{S}_\alpha^\beta(\mathbb{R}^d)$ (resp. $\sum_\alpha^\beta(\mathbb{R}^d)$) as follows:

$$\mathbf{s}_\alpha^\beta = \left\{ (a_n)_{n \in \mathbb{N}_0^d} \text{ there exists } t > 0 : \sum_{n \in \mathbb{N}_0^d} |a_n|^2 \exp(t|n|^{\frac{1}{2\alpha}}) < \infty \right\},$$

$$\tilde{\mathbf{s}}_\alpha^\beta = \left\{ (a_n)_{n \in \mathbb{N}_0^d} \text{ for all } t > 0 : \sum_{n \in \mathbb{N}_0^d} |a_n|^2 \exp(t|n|^{\frac{1}{2\alpha}}) < \infty \right\}.$$

The following lemma was proven in [2] and [7].

Lemma 1.3. *Let $\alpha \geq \frac{1}{2}$ (resp. $\alpha > \frac{1}{2}$). The mapping between $\mathcal{S}_\alpha^\beta(\mathbb{R}^d)$ (resp. $\sum_\alpha^\beta(\mathbb{R}^d)$) and the space of the Hermite coefficients, denoted \mathbf{s}_α^β (resp. $\tilde{\mathbf{s}}_\alpha^\beta$), $f = \sum_{m \in \mathbb{N}_0^d} a_m \Psi_m \rightarrow (a_m)_{m \in \mathbb{N}_0^d}$, is a topological isomorphism.*

2. The Hermite representation of $\mathcal{S}_\alpha^\beta(\mathbb{R}^d)$ and $\sum_\alpha^\beta(\mathbb{R}^d)$

Let $\alpha, \beta \in \mathbb{R}_+$ and $\frac{1}{2} \leq \alpha < \beta$ (resp. $\frac{1}{2} < \alpha < \beta$). Let X the set of all functions $f \in \mathcal{S}_\alpha^\beta(\mathbb{R}^d)$ (resp. $\sum_\alpha^\beta(\mathbb{R}^d)$) s.t. their series expansion $f = \sum_{k \in \mathbb{N}_0^d} a_k \Psi_k$ within $\mathcal{S}_\beta^\beta(\mathbb{R}^d)$ (resp. $\sum_\beta^\beta(\mathbb{R}^d)$) converges in the sense of the topology of $\mathcal{S}_\alpha^\beta(\mathbb{R}^d)$ (resp. $\sum_\alpha^\beta(\mathbb{R}^d)$).

Theorem 2.1. *If for every $f = \sum a_k \Psi_k \in X$, there exist positive constants C_f, c_f and s_f s.t.*

$$|a_k| \leq C_f \exp(-c_f |k|^{s_f}), \text{ for } k \in \mathbb{N}_0^d \text{ or resp.} \quad (2.1)$$

$$|a_k| \leq C_f \exp(-t |k|^{s_f}), \text{ for every } t > 0, \text{ for } k \in \mathbb{N}_0^d \quad (2.2)$$

then (2.1) (resp. (2.2)) is true for

$$s_f = \frac{1}{2\alpha}.$$

In particular, this implies $X = \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ (resp. $X = \sum_\alpha^\alpha(\mathbb{R}^d)$).

Proof. For the sake of simplicity we will only prove the case of $X \subset \mathcal{S}_\alpha^\beta(\mathbb{R}^d)$ and $d = 1$. But the proof in the case $X \subset \sum_\alpha^\beta(\mathbb{R}^d)$ is analogous just like the cases of higher dimensions.

Let $f \in X$, i.e. f has a convergent Hermite expansion in $\mathcal{S}_\alpha^\beta(\mathbb{R})$. This implies by lemma 1.2, that f has a convergent Hermite expansion in the spaces $\mathcal{S}_\alpha(\mathbb{R})$ and $\mathcal{S}^\beta(\mathbb{R})$. Then the sequence $\left(\sum_{k=0}^N a_k \psi_k\right)_{N \in \mathbb{N}}$ of partial sums is Cauchy in $\mathcal{S}_\alpha(\mathbb{R})$ and $\mathcal{S}^\beta(\mathbb{R})$, i.e. for $A > 0$:

- a) $\frac{\|x^p \sum_{k=N}^M a_k \psi_k\|_{L^2(\mathbb{R})}}{A^p p!^\alpha} \rightarrow 0$ for all p if $N, M \rightarrow \infty$ and
- b) $\frac{\|\partial^q \sum_{k=N}^M a_k \psi_k\|_{L^2(\mathbb{R})}}{A^q q!^\beta} \rightarrow 0$ for all q if $N, M \rightarrow \infty$.

It is known (cf. [8, 1.1]) that $\partial^q \psi_k = (ix)^q \mathcal{F} \psi_k = (ix)^q (-i)^k \psi_k$, where \mathcal{F} is the Fourier transform. Thus we have

$$\|\partial^q \sum_{k=N}^M a_k \psi_k\|_{L^2(\mathbb{R})} = \sum_{k=N}^M \|a_k x^q \psi_k\|_{L^2(\mathbb{R})}.$$

Therefore we only have to consider case a). Note, by a) with $M = N$, for every $\varepsilon > 0$ there exists $N_0(\varepsilon)$ such that:

$$\frac{\|x^p a_N \psi_N\|_{L^2(\mathbb{R})}}{A^p p!^\alpha} < \varepsilon \text{ for } N \geq N_0(\varepsilon) \quad (2.3)$$

uniformly in $p \in \mathbb{N}_0$. We use the well known fact ([10, eq. 1.8]), that

$$x \psi_k(x) = \sqrt{\frac{k}{2}} \psi_{k-1}(x) + \sqrt{\frac{k+1}{2}} \psi_{k+1}(x)$$

and by the use of L^2 -norm and Parseval's formula, we obtain by induction that for all $k, p \in \mathbb{N}_0$

$$\|x^p \psi_k\|_{L^2(\mathbb{R})} \geq C^{p+1} k^{\frac{p}{2}} \quad (2.4)$$

for suitable $C > 0$. Now suppose that the assertion is not true which implies that there exists a function $f = \sum_{k=0}^\infty a_k \psi_k \in X$ such that, for a subsequence

$(a_{k_j})_{k_j \in \mathbb{N}_0}$ of $(a_k)_{k \in \mathbb{N}_0}$, it holds

$$|a_{k_j}| = C_f \exp\left(-c_f r(k_j) k_j^{\frac{1}{2\alpha}}\right) \text{ with } j \in \mathbb{N}_0$$

where $(r(k_j))_{k_j \in \mathbb{N}_0}$ is a sequence of positive numbers not bounded from below by a $c > 0$. Reformulating the quoted facts, if (2.1) does not hold, then there exists a function $f = \sum_{k=0}^{\infty} a_k \psi_k \in X$ such that

$$|a_k| = C_f \exp\left(-c_f r(k) k^{\frac{1}{2\alpha}}\right) \text{ with } k \in \mathbb{N}_0 \quad (2.5)$$

and

$$r(k) \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Therefore, (2.3) and (2.4) give, for large enough N ,

$$\varepsilon > \frac{\|x^p a_N \psi_N\|_{L^2(\mathbb{R})}}{A^p p!^\alpha} \geq \frac{C_f \exp(-c_f r(N) N^{1/2\alpha}) C^p N^{p/2}}{A^p p!^\alpha} \quad (2.6)$$

uniformly in $p \in \mathbb{N}_0$. Now we use the inequality

$$\sup_{p \in \mathbb{N}_0} \left(\frac{C^2 N^{1/2}}{A p!^{\alpha/p}} \right)^p \geq \exp(H N^{\frac{1}{2\alpha}}) \quad (2.7)$$

which holds for suitable $H > 0$. Thus (2.6) and (2.7) imply that for N large enough

$$\varepsilon > C_f \exp\left(\left(-c_f r(N) + H\right) N^{\frac{1}{2\alpha}}\right)$$

and this is not true since $(-c_f r(N) + H) N^{\frac{1}{2\alpha}} \rightarrow \infty$ $N \rightarrow \infty$. This completes the proof. \square

3. Tensorised Gelfand-Shilov spaces

In the following we will use the following notation. Let $s, t \in \mathbb{N}$, s. t. $s + t = d$. Therefore we write for $x \in \mathbb{R}^{s+t} = \mathbb{R}^d$: $x = (x^1, x^2) = (x_1, \dots, x_s, x_{s+1}, \dots, x_{s+t})$ and similarly for $p \in \mathbb{N}_0^{s+t}$: $p = (p^1, p^2) = (p_1, \dots, p_s, p_{s+1}, \dots, p_{s+t})$. In addition we put $m, k, q \in \mathbb{N}_0^d$ accordingly.

As a consequence of inequalities proved by Kahspirovskij, Pilipovic and others we have the following proposition.

Proposition 3.1. *Let $x \in \mathbb{R}^d$ and $k, m \in \mathbb{N}_0^d$*

1.

$$\begin{aligned}
x^m \Psi_k(x) &= \prod_{i=1}^d x_i^{m_i} \psi_{k_i}(x_i) \\
&= \prod_{i=1}^{d-1} 2^{-m_d} x_i^{m_i} \sum_{r_d=0}^{m_d} \psi_{k_i}(x_i) c_{r_d, m_d}^{k_d} \psi_{k_d - m_d + 2 + 2r_d}(x_d) \\
&= 2^{-m} \sum_{\substack{r_i=0, \\ i=1, \dots, d}}^{m_i} \prod_{i=0}^d c_{r_i, m_i}^{k_i} \psi_{k_i - m_i + 2 + 2r_i}(x_i)
\end{aligned}$$

where, for $j = 1, \dots, d$

$$|c_{r_j, m_j}^{k_j}| \leq \binom{m_j}{r_j} [(2k_j + 1)^{m_j/2} + m_j^{m_j/2}].$$

2. Put $\mathcal{R}_i^{j_i} = (x_i^2 - \partial_i^2)^{j_i}$, $i = 1, \dots, d$. Then for $f \in C^\infty(\mathbb{R}^d)$,

$$\begin{aligned}
\mathcal{R}_1^{j_1} \dots \mathcal{R}_d^{j_d} f &= \mathcal{R}_1^{j_1} \dots \mathcal{R}_{d-1}^{j_{d-1}} \sum_{\substack{p_d + q_d = 2k_d, \\ k_d \leq j_d}} C_{p_d, q_d}^{j_d} x_d^{p_d} \partial_d^{q_d} f(x_1, \dots, x_d) \\
&= \sum_{\substack{p_i + q_i = 2k_i, k_i \leq j_i, \\ i=1, \dots, d}} \prod_{i=1}^d C_{p_i, q_i}^{j_i} x_i^{p_i} \partial_i^{q_i} f(x_1, \dots, x_d),
\end{aligned}$$

where for $i = 1, \dots, d$

$$|C_{p_i, q_i}^{j_i}| \leq 10^{j_i} j_i^{j_i - \frac{p_i + q_i}{2}}.$$

Definition 3.2. We define the tensored spaces $(\mathcal{S}_\alpha^\alpha \otimes \mathcal{S}_\beta^\beta)(\mathbb{R}^{s+t})$ and $(\Sigma_\alpha^\alpha \otimes \Sigma_\beta^\beta)(\mathbb{R}^{s+t})$ as follows:

$$\begin{aligned}
(\mathcal{S}_\alpha^\alpha \otimes \mathcal{S}_\beta^\beta)(\mathbb{R}^{s+t}) &:= \left\{ f \in \mathcal{S}(\mathbb{R}^d) : \exists A, B, C \text{ s.t. } \forall p, q \right. \\
&\quad \left. \|x^{1^{p_1}} x^{2^{p_2}} \partial_{x_1}^{q_1} \partial_{x_2}^{q_2} f\|_{L^2(\mathbb{R}^d)} \leq CA^{|p|} p^{1!^\alpha} q^{1!^\alpha} B^{|q|} p^{2!^\beta} q^{2!^\beta} \right\} \\
(\Sigma_\alpha^\alpha \otimes \Sigma_\beta^\beta)(\mathbb{R}^{s+t}) &:= \left\{ f \in \mathcal{S}(\mathbb{R}^d) : \exists C \forall h \text{ s.t. } \forall p, q \right. \\
&\quad \left. \|x^{1^{p_1}} x^{2^{p_2}} \partial_{x_1}^{q_1} \partial_{x_2}^{q_2} f\|_{L^2(\mathbb{R}^d)} \leq Ch^{|p|+|q|} p^{1!^\alpha} q^{1!^\alpha} p^{2!^\beta} q^{2!^\beta} \right\}
\end{aligned}$$

The next proposition is formulated in the simple form so that it can be used for the characterization of new Gelfand-Shilov type spaces.

Theorem 3.3. Let $1/2 \leq \nu < \mu$ and $s, t \in \mathbb{N}_0$ such that $s + t = d$.

1. Let $f \in C^\infty(\mathbb{R}^{s+t})$. If for some $A > 0$ and some $C > 0$ (resp. for every $A > 0$ there exists $C > 0$) such that

$$\|x^{1^{p_1}} x^{2^{p_2}} \partial_{x_1}^{q_1} \partial_{x_2}^{q_2} f(x^1, x^2)\|_{L^2} \leq CA^{|p||q|} p^{1!^\nu} p^{2!^\mu} q^{1!^\nu} q^{2!^\mu}, \quad (3.1)$$

then $f \in L^2(\mathbb{R}^{s+t})$, $f(x) = \sum_{k \in \mathbb{N}_0^{s+t}} a_{(k^1, k^2)} \psi_{k^1}(x^1) \psi_{k^2}(x^2)$, $x \in \mathbb{R}^{s+t}$, and there exist constants $C > 0$ and $\delta > 0$ (respectively, for every $\delta > 0$ there exists a $C > 0$) such that

$$|a_k| \leq C \exp\left(-\delta(|k^1|^{1/(2\nu)} + |k^2|^{1/(2\mu)})\right), \quad k \in \mathbb{N}_0^{s+t}. \quad (3.2)$$

2. If $f \in L^2(\mathbb{R}^{s+t})$, $f(x) = \sum_{k \in \mathbb{N}_0^{s+t}} a_{(k^1, k^2)} \psi_{k^1}(x^1) \psi_{k^2}(x^2)$, $x \in \mathbb{R}^{s+t}$, satisfies (3.2) for some $C > 0$ and $\delta > 0$ (respectively, for every $\delta > 0$ there exists $C > 0$), then $f \in C^\infty(\mathbb{R}^{s+t})$ and (3.1) holds with some $A > 0$ and $C > 0$ (respectively, for every $A > 0$ there exists $C > 0$).

Proof. Assume (3.1) to hold as well as that $j = (j^1, j^2) \in \mathbb{N}_0^{s+t}$. We have

$$\mathcal{R}_1^{j^1} \dots \mathcal{R}_d^{j^d} f = \sum_{k \in \mathbb{N}_0^d} a_k \mathcal{R}_1^{j^1} \dots \mathcal{R}_d^{j^d} \psi_{k_1} \dots \psi_{k_d} = \sum_{k \in \mathbb{N}_0^d} a_k \prod_{i=1}^d (2k_i + 1)^{j_i} \psi_{k_i}.$$

There exists a constant $C > 0$ such that

$$\begin{aligned} \|\mathcal{R}_1^{j^1} \dots \mathcal{R}_d^{j^d} f\|_{L^2} &= \sum_{\substack{p_i + q_i = 2k_i, k_i \leq j_i, \\ i=1, \dots, d}} \prod_{i=1}^d C_{p_i, q_i}^{j_i} \|x_i^{p_i} \partial^{q_i} f\|_{L^2(\mathbb{R}^d)} \\ &\leq C 10^j \sum_{\substack{p_i + q_i = 2k_i, k_i \leq j_i, \\ i=1, \dots, d}} \prod_{i=1}^d j_i^{j_i - k_i} p_i!^\nu q_i!^\nu p_i^{2\mu} q_i^{2\mu} l^k \\ &\leq C 10^j \sum_{\substack{p_i + q_i = 2k_i, k_i \leq j_i, \\ i=1, \dots, d}} \prod_{i=1}^d j_i^{j_i - k_i} k_i!^{2\nu} k_i^{2\mu} (2lr)^k \frac{1}{2^k} \\ &\leq C (20lr)^j \sum_{\substack{p_i + q_i = 2k_i, k_i \leq j_i, \\ i=1, \dots, d}} \prod_{i=1}^d j_i^{j_i - k_i} \frac{k_i!^{2\nu} k_i^{2\mu}}{j_i!^{2\nu} j_i^{2\mu}} j_i^{1!^{2\nu}} j_i^{2!^{2\mu}} \frac{1}{2^k}. \end{aligned}$$

This implies, using

$$\left(\sum_{k \in \mathbb{N}_0^d} |a_k|^2 \prod_{i=1}^d (2k_i + 1)^{2j_i} \psi_{k_i} \right)^{1/2} = \|\mathcal{R}_1^{j^1} \dots \mathcal{R}_d^{j^d} f\|_{L^2}$$

and with new constants

$$|a_k| \prod_{i=1}^d (2k_i + 1)^{j_i} \leq C c^j j_i!^{2\nu} j_i^{2\mu}.$$

Thus, with suitable $C > 0$ and $\delta > 0$,

$$|a_k| \leq C e^{-\delta(|k^1|^{1/(2\nu)} + |k^2|^{1/(2\mu)})}.$$

b) Now, assume (3.2) and let $m = (m^1, m^2) \in \mathbb{N}_0^{s+t}$. We have

$$\begin{aligned} \|x^{1m^1} x^{2m^2} f\|_{L^2} &= \left\| \sum_{k \in \mathbb{N}_0^d} a_k x^{1m^1} x^{2m^2} \psi_{k^1} \psi_{k^2} \right\|_{L^2} \\ &\leq 2^{-m^1/2 - m^2/2}. \end{aligned}$$

Therefore we get with $\binom{m}{k} = \prod_{i=1}^d \binom{m_i}{k_i}$

$$\begin{aligned} \sum_{k \in \mathbb{N}_0^d} |a_k| \left(\sum_{\substack{j_i \leq m_i, \\ i=1, \dots, d}} \binom{m}{j} [(2k_1 + 1)^{m_1/2} + m_1^{m_1/2}] \cdot \dots \cdot [(2k_d + 1)^{m_d/2} + m_d^{m_d/2}] \right) \\ \leq 2^{m/2} \left(\sum_{k \in \mathbb{N}_0^d} |a_k|^2 C_k^2 \right)^{1/2} \left(\sum_{k \in \mathbb{N}_0^d} C_k^{-2} \bar{C}_k^2 \right)^{1/2}, \end{aligned}$$

where $C_k = \exp\left(\delta \left(k^{1/(2\nu)} + k^{2/(2\mu)}\right)\right)$ and $\bar{C}_k = \prod_{i=1}^d [(2k_i + 1)^{m_i/2} + m_i^{m_i/2}]$.

There exists a constant \mathcal{C} such that

$$\begin{aligned} \|x^{1m^1} x^{2m^2} f\|_{L^2} &\leq \mathcal{C} 2^{\frac{m}{2}} \left[\sum_{k \in \mathbb{N}_0^d} C_k^{-2} \bar{C}_k^2 \right]^{1/2} \\ &\leq \mathcal{C} 2^{\frac{m}{2}} \bar{C} \left[\sum_{k \in \mathbb{N}_0^d} \exp\left(-\delta \left((2k^1 + 1)^{1/(2\nu)} + (2k^2 + 1)^{1/(2\mu)}\right)\right) \right]^{\frac{1}{2}} \\ &\leq \mathcal{C} 2^{\frac{m}{2}} m^1!^\nu m^2!^\mu \tilde{C} + \frac{m^{1m^1/2} m^{2m^2/2}}{m^1!^\nu m^2!^\mu} m^1!^\nu m^2!^\mu, \end{aligned}$$

where $\bar{C} = \prod_{i=1}^d C_i$ and

$$\bar{C}_i := \begin{cases} \sup_{k_i \in \mathbb{N}_0} ((2k_i + 1)^{m_i/2} + m_i^{m_i/2}) e^{-\frac{1}{2}\delta(2k_i+1)^{1/(2\nu)}} & \text{for } i = 1, \dots, s \\ \sup_{k_i \in \mathbb{N}_0} ((2k_i + 1)^{m_i/2} + m_i^{m_i/2}) e^{-\frac{1}{2}\delta(2k_i+1)^{1/(2\mu)}} & \text{for } i = s + 1, \dots, t \end{cases}$$

and $\tilde{C} = \prod_{i=1}^d \tilde{C}_i$ with

$$\tilde{C}_i := \begin{cases} \sup_{k_i \in \mathbb{N}_0} \frac{(2k_i+1)^{m_i/2} e^{-\frac{1}{2}\delta(2k_i+1)^{1/(2\nu)}}}{m_i!^\nu} & \text{for } i = 1, \dots, s \\ \sup_{k_i \in \mathbb{N}_0} \frac{(2k_i+1)^{m_i/2} e^{-\frac{1}{2}\delta(2k_i+1)^{1/(2\mu)}}}{m_i!^\mu} & \text{for } i = s + 1, \dots, t \end{cases}$$

We conclude that

$$\|x^{1m^1} x^{2m^2} f\|_{L^2} \leq \mathcal{C} 2^{m/2} m^1!^\nu m^2!^\mu. \quad (3.3)$$

By the Fourier transformation, we have

$$\|\partial_{x^1}^{m^1} \partial_{x^2}^{m^2} f\|_{L^2} \leq \mathcal{C} 2^{m/2} m^1!^\nu m^2!^\mu. \quad (3.4)$$

Let $p, q \in \mathbb{N}_0^d$. Then

$$\begin{aligned} \|x^p \partial^q f\|_{L^2}^2 &= (x^p \partial^q f, x^p \partial^q f)_{L^2} = |(\partial^q (x^{2p} \partial^q f), f)_{L^2}| \\ &\leq \left| \sum_{\substack{\kappa \in \mathbb{N}_0^d \\ \kappa_i \leq \gamma_i}} \binom{q}{\kappa} \frac{(2p)!}{(2p - \kappa)!} (x^{2p - \kappa} f^{(2q - \kappa)}, f)_{L^2} \right| \\ &\leq \sum_{\substack{\kappa \in \mathbb{N}_0^d \\ \kappa_i \leq \gamma_i}} \binom{q}{\kappa} \binom{2p}{\kappa} \kappa! \|x^{2p - \kappa} f\|_{L^2} \|\partial^{2q - \kappa} f\|_{L^2}, \end{aligned}$$

where $\gamma_i := \min\{q_i, 2p_i\}$, $i = 1, \dots, d$.

This implies that (3.1) holds. \square

Corollary 3.4. Let $s, t \in \mathbb{N}_0^d$ s.t. $s + t = d$, and $1/2 \leq \alpha < \beta$ in the case $\mathcal{S}_\alpha^\beta(\mathbb{R}^d)$ (resp. $1/2 < \alpha < \beta$ in the case $\Sigma_\alpha^\beta(\mathbb{R}^d)$) then it holds

$$\begin{aligned} \Sigma_\alpha^\alpha(\mathbb{R}^d) &\subset (\Sigma_\alpha^\alpha \otimes \Sigma_\beta^\beta)(\mathbb{R}^{s+t}) \subset \Sigma_\beta^\beta(\mathbb{R}^d), \quad \Sigma_\alpha^\alpha(\mathbb{R}^d) \subset (\Sigma_\beta^\beta \otimes \Sigma_\alpha^\alpha)(\mathbb{R}^{s+t}) \subset \Sigma_\beta^\beta(\mathbb{R}^d), \\ \mathcal{S}_\alpha^\alpha(\mathbb{R}^d) &\subset (\mathcal{S}_\alpha^\alpha \otimes \mathcal{S}_\beta^\beta)(\mathbb{R}^{s+t}) \subset \mathcal{S}_\beta^\beta(\mathbb{R}^d) \quad \text{and} \quad \mathcal{S}_\alpha^\alpha(\mathbb{R}^d) \subset (\mathcal{S}_\beta^\beta \otimes \mathcal{S}_\alpha^\alpha)(\mathbb{R}^{s+t}) \subset \mathcal{S}_\beta^\beta(\mathbb{R}^d). \end{aligned}$$

The inclusions are strict and continuous.

Proof. The inclusions are obviously continuous. The statement that the inclusions are strict follows straightforward from Theorems 2.1 and 3.3. \square

Moreover it is clear that $\Sigma_\alpha^\beta(\mathbb{R}^{s+t})$ is not a subset of $(\Sigma_\alpha^\alpha \otimes \Sigma_\beta^\beta)(\mathbb{R}^{s+t})$ and that the opposite inclusion also does not hold; the same is true for $\mathcal{S}_\alpha^\beta(\mathbb{R}^{s+t})$ and $(\mathcal{S}_\alpha^\alpha \otimes \mathcal{S}_\beta^\beta)(\mathbb{R}^{s+t})$.

4. Gelfand–Shilov type spaces related to the tensorised harmonic oscillators on \mathbb{R}^d

We introduce one more class of Gevrey Gelfand-Shilov-type spaces.

$$S_{\sigma}^{\otimes, \sigma}(\mathbb{R}^d) = \text{indlim}_{\delta \rightarrow 0} S_{\sigma}^{\otimes, \sigma}(\mathbb{R}^d; \delta) \quad \text{and} \quad \Sigma_{\sigma}^{\otimes, \sigma} = \text{projlim}_{\delta \rightarrow \infty} S_{\sigma}^{\otimes, \sigma}(\mathbb{R}^d; \delta)$$

where

$$\begin{aligned} S_{\sigma}^{\otimes, \sigma}(\mathbb{R}^d; \delta) &= \{f \in \mathcal{S}(\mathbb{R}^d) : \mathbf{I}f\|_{\otimes; \sigma, \delta} < \infty\} \quad \text{with} \\ \mathbf{I}f\|_{\otimes; \sigma, \delta} &= \sum_{m \in \mathbb{N}_0^d} |a_m|^2 \exp(2\delta((m_1 + 1) \dots (m_d + 1))^{1/(2\sigma d)}) \end{aligned}$$

where the $(a_m)_{m \in \mathbb{N}_0^d}$ are the Hermite coefficients of f (cf. (1.1)).

Let

$$\vec{b} = (b_1, \dots, b_d) \in \mathbb{R}^d, \quad b_j > -1, \quad j = 1, \dots, d. \quad (4.1)$$

Define

$$\mathcal{H}_b^- = (-\partial_{x_1}^2 + x_1^2 + b_1) \dots (-\partial_{x_d}^2 + x_d^2 + b_d) = \otimes_{j=1}^d (-\partial_{x_j}^2 + x_j^2 + b_j) \quad (4.2)$$

By the arguments as in Theorem 3.3 one can prove the next theorem.

Theorem 4.1. *Under the hypotheses given above the following conditions are equivalent:*

1. $f \in \mathcal{S}_\sigma^{\otimes, \sigma}(\mathbb{R}^d)$, $\sigma \geq 1/2$, resp., $f \in \Sigma_\sigma^{\otimes, \sigma}(\mathbb{R}^d)$, $\sigma > 1/2$.
2. there exists $A > 0$, resp., for every $A > 0$,

$$\left\| \mathcal{H}_b^r f \right\| \leq A^{r+1} r!^{2d\sigma}, \quad r \in \mathbb{N}_0,$$

where $\sigma \geq 1/2$, resp., $\sigma > 1/2$.

3. if $f = \sum_{k=0}^{\infty} a_k \Psi_k(x)$, there exist $C, \varepsilon > 0$, resp. for every $\varepsilon > 0$ there exists $C > 0$ such that

$$|a_k| \leq C \exp(-\varepsilon(k \log^{-(d-1)}(1+k))^{1/(2d\sigma)}),$$

where $\sigma \geq 1/2$, resp., $\sigma > 1/2$.

Example 4.2. With $\sigma = 1/2$ we obtain $\phi = \sum_{k \in \mathbb{N}_0^d} a_k \Psi_k \in \mathcal{S}_{1/2}^{\otimes, 1/2}(\mathbb{R}^d)$, if and only if $|a_k| \leq C \exp(-\delta k^{1/d}(\log(1+k))^{-1+1/d})$, $k \in \mathbb{N}_0$, for some $C > 0, \delta > 0$.

It is natural to see the relations of the space $\mathcal{S}_\sigma^{\otimes, \sigma}(\mathbb{R}^d)$, resp., $\Sigma_\sigma^{\otimes, \sigma}(\mathbb{R}^d)$ and $\mathcal{S}_\sigma^\sigma(\mathbb{R}^d)$, resp., $\Sigma_\sigma^\sigma(\mathbb{R}^d)$ $\sigma > 0$, $\sigma \geq 1/2$, resp., $\sigma > 1/2$.

Since

$$((2k_1 + 1) \dots (2k_d + 1))^{1/d} \leq 2(k_1 + \dots + k_d + 1),$$

it follows

Theorem 4.3. *Let $d \geq 2$ and $\sigma \geq 1/2$, resp. $\sigma > 1/2$. Then*

$$\mathcal{S}_\sigma^\sigma(\mathbb{R}^d) \hookrightarrow \mathcal{S}_\sigma^{\otimes, \sigma}(\mathbb{R}^d) \hookrightarrow \mathcal{S}_{2\sigma}^{2\sigma}(\mathbb{R}^d),$$

resp.,

$$\Sigma_\sigma^\sigma(\mathbb{R}^d) \hookrightarrow \Sigma_\sigma^{\otimes, \sigma}(\mathbb{R}^d) \hookrightarrow \Sigma_{2\sigma}^{2\sigma}(\mathbb{R}^d).$$

The inclusions are strict.

Remark 4.4. The relation with the Gelfand-Shilov tensor-type spaces are clear.

References

1. U. Batisti, T. Gramchev, S. Pilipovic, and L. Rodino, *Globally bisingular elliptic operators*, New Developments in Pseudo-Differential Operators, Operator Theory: Advances and Applications, vol. 228, Springer, Basel, 2013, pp. 21–38.
2. R. D. Carmichael, A. Kaminski, and S. Pilipovic, *Boundary values and convolution in ultradistribution spaces*, World Scientific, London, 2007.
3. I. M. Gelfand and G. E. Shilov, *Generalized functions*, vol. 2, Academic Press, Orlando, 1968.

4. T. Gramchev, S. Pilipovic, and L. Rodino, *Classes of degenerate elliptic operators in Gelfand-Shilov spaces*, New Developments in Pseudo-Differential Operators, Operator Theory: Advances and Applications, vol. 189, Birkhäuser Basel, 2009, pp. 15–31.
5. A.I. Kashpirovskii, *Equality of the spaces \mathcal{S}_α^β and $\mathcal{S}_\alpha \cap \mathcal{S}^\beta$* , Functional Anal. Appl. (1980), no. 14, 129.
6. Michael Langenbruch, *Hermite functions and weighted spaces of generalized functions*, manuscripta mathematica **119** (2006), 269–285.
7. Z. Lozanov-Crvenkovic and D. Perisic, *Hermite expansions of elements of Gelfand-Shilov spaces in quasianalytic and non quasianalytic case*, Novi Sad J. Math **37** (2007), no. 2, 129–147.
8. M. Reed and B. Simon, *Methods of modern mathematical physics*, vol. 2, Academic Press, New York, London, 1975.
9. M. Soloviev, *Star product algebras of test functions*, Theor. Math. Phys. **153** (2007), no. 1, 1351–1363.
10. C. A. M. Van Berkel, *Integral transformations and spaces of type \mathcal{S}* , Ph.D. thesis, Eindhoven University of Technology, 1992.
11. S. J. L. Van Eijndhoven, *Functional analytic characterizations of the Gelfand-Shilov spaces $\mathcal{S}_\alpha^\beta(\mathbb{R})$* , Proc. Kon. Nederl. Akad. Wetensch. A90 (1987), 133–144.
12. J. Wloka, *Grundräume und verallgemeinerte Funktionen*, Springer, Berlin, Heidelberg, 1969.

Todor Gramchev

Dipartimento di Matematica, Università di Cagliari, Via Ospedale 72, 09124 Cagliari, Italia

e-mail: todor@unica.it

Alexander Lecke

Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria

e-mail: alexander.lecke@univie.ac.at

Stevan Pilipovic

Faculty of Sciences and Mathematics, University of Novi Sad, Trg D. Obradovica 4, 21000, Novi Sad, Serbia

e-mail: pilipovic@dmi.uns.ac.rs

Luigi Rodino

Dipartimento di Matematica, Università di Torino, Via Carlo Alberto 10, 10123 Torino, Italia

e-mail: luigi.rodino@unito.it