A natural sequent calculus for Lewis' logic of counterfactuals

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Abstract. The logic $\mathbb{V}$ is the basic logic of counterfactuals in the family of Lewis' systems. It is characterized by the whole class of so-called sphere models. We propose a new sequent calculus for this logic. Our calculus takes as primitive Lewis' connective of comparative plausibility $\preceq$: a formula $A \preceq B$ intuitively means that A is at least as plausible as B, so that a conditional $A \Rightarrow B$ can be defined as A is impossible or $A \wedge \neg B$ is less plausible than A. As a difference with previous attempts, our calculus is standard in the sense that each connective is handled by a finite number of rules with a fixed and finite number of premises. Moreover our calculus is "internal", in the sense that each sequent can be directly translated into a formula of the language. The peculiarity of our calculus is that sequents contain a special kind of structures, called *blocks*, which encode a finite combination of $\preceq$. We show that the calculus is terminating, whence it provides a decision procedure for the logic $\mathbb{V}$.

1 Introduction

In the recent history of conditional logics the work by Lewis [16] has a prominent place (among others [5, 18, 13, 11]). He proposed a formalization of conditional logics in order to represent a kind of hypothetical reasoning (if A were the case then B), that cannot be captured by classical logic with material implication. More precisely, the original motivation by Lewis was to formalize *counterfactual sentences*, i.e. conditionals of the form "if A were the case then B would be the case", where A is false. But independently from counterfactual reasoning, conditional logics have found then an interest also in several fields of artificial intelligence and knowledge representation. Just to mention a few: they have been used to reason about prototypical properties [8] and to model belief change [11, 9]. Moreover, conditional logics can provide an axiomatic foundation of nonmonotonic reasoning [4, 12], here a conditional $A \Rightarrow B$ is read as "in normal circumstances if A then B". Finally, a kind of (multi)-conditional logics [2, 3] have been used to formalize epistemic change in a multi-agent setting and in epistemic "games", each conditional operator expresses the "conditional beliefs" of an agent.

In this paper we concentrate on the logic $\mathbb{V}$ of counterfactual reasoning studied by Lewis. This logic is characterized by possible world models structured by a system of spheres. Intuitively, each world is equipped with a set of nested sets of worlds: inner sets represent "most plausible worlds" from the point of view of the given world and worlds belonging only to outer sets represent less plausible worlds. In other words, each sphere represent a degree of plausibility. The (rough) intuition involving the truth condition of a counterfactual $A \Rightarrow B$ at a world x is that B is true at the most plausible worlds where A is true, whenever there are worlds satisfying A. But Lewis is reluctant to assume that *most plausible worlds A exist* (whenever there are A-worlds), for philosophical reasons. He calls this assumption the Limit Assumption and he formulates his semantics in more general terms which do need this assumption (see below). The sphere semantics is the strongest semantics for conditional logics, in the sense that it characterizes only a subset of relatively strong systems; there are weaker (and more abstract) semantics such as the selection function semantics which characterize a wider range of systems [18].

From the point of view of proof-theory and automated deduction, conditional logics do not have a state of the art comparable with, say, the one of modal logics, where there are well-established alternative calculi, whose proof-theoretical and computational properties are well-understood. This is partially due to the mentioned lack of a unifying semantics. Similarly to modal logics and other extensions/alternative to classical logics two types of calculi have been studied: *external* calculi which make use of labels and relations on them to import the semantics into the syntax, and *internal* calculi which stay within the language, so that a "configuration" (sequent, tableaux node...) can be directly interpreted as a formula of the language. Limiting our account to Lewis' counterfactual logics, some external calculi have been proposed in [10] which presents modular labeled calculi for preferential logic PCL and its extensions, this family includes all counterfactual logics by Lewis. Internal calculi have been proposed by Gent [7] and by de Swart [6] for Lewis' logic $\mathbb{VC}$ and neighbours. These calculi manipulate sets of formulas and provide a decision procedure, although they comprise an *infinite set of rules and rules with a variable number of premises.* Finally in [15] the authors provide internal calculi for Lewis' conditional logic $\mathbb{V}$ and some extensions. Their calculi are formulated for a language comprising the comparative plausibility connective, the strong and the weak conditional operator. Both conditional operators can be defined in terms of the *comparative similarity* connective. These calculi are actually an extension of Gent's and de Swart's ones and they comprise an infinite set of rules with a variable number of premises. We mention also a seminal work by Lamarre [13] who proposed a tableaux calculus for Lewis' logic, but it is actually a model building procedure rather than a calculus made of deductive rules.

In this paper we tackle the problem of providing a standard proof-theory for Lewis' logic $\mathbb{V}$ in the form of internal calculi. By "standard" we mean that we aim to obtain analytic sequent calculi where each connective is handled by a finite number of rules with a fixed and finite number of premises. As a preliminary result, we propose a new internal calculus for Lewis' logic $\mathbb{V}$. This is the most general logic of Lewis' family and it is complete with respect the whole class of sphere models (moreover, its unnested fragment essentially coincide with KLM rational logic **R** [14]). Our calculus takes as primitive Lewis' comparative plausibility connective $\preceq$: a formula $A \preceq B$ means, intuitively, that A is at least as plausible as B, so that a conditional $A \Rightarrow B$ can be defined as A is impossible or $A \wedge \neg B$ is less plausible than A[3]. As a difference with previous

[3] This definition avoids the Limit Assumption, in the sense that it works also for models where *at least* a sphere containing A worlds does not necessarily exist.

attempts, our calculus comprises *structured* sequents containing *blocks*, where a block is a new syntactic structure encoding a finite combination of $\preceq$. In other words, we introduce a new modal operator (but still definable in the logic) which encodes finite combinations of $\preceq$. This is the main ingredient to obtain a standard and internal calculus for $\mathbb{V}$. We show that the calculus is terminating whence it provides a decision procedure. In further research we shall study its complexity and we shall study how to extend it to stronger logics of Lewis' family.

2 Lewis' logic $\mathbb{V}$

We consider a propositional language $\mathcal{L}$ generated from a set of propositional variables $Varprop$ and boolean connectives plus two special connectives $\preceq$ (comparative plausibility) and $\Rightarrow$ (conditional). A formula $A \preceq B$ is read as "A is at least as plausible as B". The semantics is defined in terms of sphere models, we take the definition by Lewis without the limit assumption.

Definition 1. *A model $\mathcal{M}$ has the form $\langle W, \$, [\,]\rangle$, where W is a non-empty set whose elements are called worlds, $[\,] : Varprop \longrightarrow Pow(W)$ is the propositional evaluation, and $\$: W \longrightarrow Pow(Pow(W))$. We write $\$_x$ for the value of the function $\$$ for $x \in W$, and we denote the elements of $\$_x$ by $\alpha, \beta\ldots$. Models have the following property: $\forall \alpha, \beta \in \$_x \; \alpha \subseteq \beta \vee \beta \subseteq \alpha$.*

The truth definition is the usual one for boolean cases, for the additional connectives we have: (i) $x \in [A \preceq B]$ iff $\forall \alpha \in \$_x$ if $\alpha \cap [B] \neq \emptyset$ then $\alpha \cap [A] \neq \emptyset$ (ii) $x \in [A \Rightarrow B]$ iff either $\forall \alpha \in \$_x \; \alpha \cap [A] = \emptyset$ or there is $\alpha \in \$_x$, such that $\alpha \cap [A] \neq \emptyset$ and $\alpha \cap [A \wedge \neg B] = \emptyset$.

The semantic notions, satisfiability and validity are defined as usual.

For the ease of reading we introduce the following conventions and abbreviations: we write $x \models A$, where the model is understood instead of $x \in [A]$. Moreover given $\alpha \in \$_x$, we use the following notations:

$\alpha \models^{\forall} A$ if $\alpha \subseteq [A]$, i.e. $\forall y \in \alpha$ it holds $y \models A$
$\alpha \models^{\exists} A$ if $\alpha \cap [A] \neq \emptyset$, i.e. $\exists y \in \alpha$ such that $y \models A$

Observe that with this notation, the truths conditions for $\preceq$ and $\Rightarrow$ become:

- $x \models A \preceq B$ iff $\forall \alpha \in \$_x$ either $\alpha \models^{\forall} \neg B$ or $\alpha \models^{\exists} A$
- $x \models A \Rightarrow B$ iff either $\forall \alpha \in \$_x \; \alpha \models^{\forall} \neg A$ or there is $\beta \in \$_x$ such that $\beta \models^{\exists} A$ and $\beta \models^{\forall} A \rightarrow B$.

It can be observed that the two connectives $\preceq$ and $\Rightarrow$ are interdefinable, in particular:

$$A \Rightarrow B \equiv (\bot \preceq A) \vee \neg(A \wedge \neg B \preceq A)$$

Also the $\preceq$ connective can be defined in terms of the conditional $\Rightarrow$ as follows:

$$A \preceq B \equiv (A \vee B) \Rightarrow \bot \vee \neg((A \vee B) \Rightarrow \neg A)$$

The logic $\mathbb{V}$ can be axiomatized taking as primitive the connective $\preceq$ and the axioms are the following [16]:

- classical axioms and rules
- if $B \rightarrow (A_1 \vee \ldots \vee A_n)$ then $(A_1 \preceq B) \vee \ldots \vee (A_n \preceq B)$
- $(A \preceq B) \vee (B \preceq A)$
- $(A \preceq B) \wedge (B \preceq C) \rightarrow (A \preceq C)$
- $A \Rightarrow B \equiv (\bot \preceq A) \vee \neg(A \wedge \neg B \preceq A)$

3 An internal sequent calculus for $\mathbb{V}$

In this section we present $\mathcal{I}^{\mathsf{V}}$, a structured calculus for Lewis' conditional logic $\mathbb{V}$ introduced in the previous section. In addition to ordinary formulas, sequents contains also *blocks* of the form:

$$[A_1, \ldots, A_m \lhd B_1, \ldots, B_n]$$

where each A_i, B_j are formulas. The interpretation is the following:

$$x \models [A_1, \ldots, A_m \lhd B_1, \ldots, B_n]$$

if and only if, $\forall \alpha \in \$_x$:

- either $\alpha \models^{\forall} \neg B_j$ for some j, or
- $\alpha \models^{\exists} A_i$ for some i.

A *sequent* Γ is a multiset $G_1, \ldots G_k$, where each G_i is either a formula or a block. A sequent $\Gamma = G_1, \ldots G_k$, is valid if for every model $\mathcal{M} = \langle W, \$, [\,]\rangle$, for every world $x \in W$, it holds that $x \models G_1 \vee \ldots \vee G_k$. The calculus $\mathcal{I}^{\mathsf{V}}$ comprises the following axiom and rules:

- Standard Axioms: (i) $\Gamma, \top$ $\quad (ii)$ $\Gamma, \neg\bot$ $\quad (iii)$ $\Gamma, P, \neg P$
- Standard external rules of sequent calculi for boolean connectives
- $(\preceq^{+})$

$$\frac{\Gamma, [A \lhd B]}{\Gamma, A \preceq B}\ (\preceq +)$$

- $(\preceq^{-})$

$$\frac{\Gamma, \neg(A \preceq B), [B, \Sigma \lhd \Pi] \quad \Gamma, \neg(A \preceq B), [\Sigma \lhd \Pi, A]}{\Gamma, \neg(A \preceq B), [\Sigma \lhd \Pi]}\ (\preceq^{-})$$

- $(\Rightarrow^{+})$

$$\frac{\Gamma, [\bot \lhd A], \neg(A \wedge \neg B \preceq A)}{\Gamma, A \Rightarrow B}\ (\Rightarrow^{+})$$

- $(\Rightarrow^{-})$

$$\frac{\Gamma, \neg(\bot \preceq A) \qquad \Gamma, [A \wedge \neg B \lhd A]}{\Gamma, \neg(A \Rightarrow B)}\ (\Rightarrow^{-})$$

- (Communication)

$$\frac{\Gamma, [\Sigma_1 \lhd \Pi_1, \Pi_2], [\Sigma_1, \Sigma_2 \lhd \Pi_2] \quad \Gamma, [\Sigma_2 \lhd \Pi_1, \Pi_2], [\Sigma_1, \Sigma_2 \lhd \Pi_1]}{\Gamma, [\Sigma_1 \lhd \Pi_1], [\Sigma_2 \lhd \Pi_2]}\ (Com)$$

– $(Jump)$

$$\frac{\neg B_i, \Sigma}{\Gamma, [\Sigma \lhd B_1, \ldots, B_n]}\ (Jump)$$

Some remark on the rules: the rule $(\preceq^+)$ just introduces the block structure, showing that $\lhd$ is a generalization of $\preceq$; $(\preceq^-)$ prescribes case analysis and contribute to expand the blocks; the rules $(\Rightarrow^+)$ and $(\Rightarrow^-)$ just apply the definition of $\Rightarrow$ in terms of $\preceq$. The (Com) rule is directly motivated by the *nesting* of spheres, which means a *linear order* on sphere inclusion; this rule is very similar to the homonymous one used in *hypersequent* calculi for handling truth in linearly ordered structures [1, 17].

As usual, given a formula $G \in \mathcal{L}$, in order to check whether G is valid we look for a derivation of G in the calculus $\mathcal{I}^{\mathsf{V}}$. Given a sequent Γ, we say that Γ is derivable in $\mathcal{I}^{\mathsf{V}}$ if it admits a *derivation*. A derivation of Γ is a tree where: the root is Γ; a leaf is an instance of standard axioms; a non-leaf node is (an instance of) the conclusion of a rule having (an instance of) the premises of the rule as parents.

Here below we show a derivation of $(A \preceq B) \vee (B \preceq A)$:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\neg A, A}{[A \lhd B, A], [A, B \lhd A]}\ (Jump) \quad \cfrac{\neg B, B}{[B \lhd B, A], [A, B \lhd B]}\ (Jump)}{[A \lhd B], [B \lhd A]}\ (Com)}{[A \lhd B], B \preceq A}\ (\preceq^+)}{A \preceq B, B \preceq A}\ (\preceq^+)}{(A \preceq B) \vee (B \preceq A)}\ (\vee^+)$$

It can be shown that the calculus $\mathcal{I}^{\mathsf{V}}$ is sound, complete and terminating if rules are applied without redundancy[4]:

Theorem 1. *Given a sequent Γ, Γ is derivable if and only if it is valid. Given a sequent Γ, any non-redundant derivation-tree of Γ is finite.*

4 Conclusions

In this paper we begin a proof-theoretical investigation of Lewis' logics of counterfactuals characterized by the sphere-model semantics. We have presented a simple, analytic calculus $\mathcal{I}^{\mathsf{V}}$ for logic $\mathbb{V}$, the most general logic characterized by the sphere-model semantics. The calculus is *standard*, that is to say it contains a finite a number of rules with a fixed number of premisses and *internal* in the sense that each sequent denotes a formula of $\mathbb{V}$. The novel ingredient of $\mathcal{I}^{\mathsf{V}}$ is that sequents are structured objects containing blocks, where a block is a structure or a sort of n-ary modality encoding a finite combination of formulas with the connective $\preceq$. The calculus $\mathcal{I}^{\mathsf{V}}$ ensures termination, and therefore it provides a decision procedure for $\mathbb{V}$.

[4] Detailed proofs are confined in the accompanying report [19].

In future research, we aim at extending our approach to all the other conditional logics of the Lewis' family, in particular we aim at focusing on the logics $\mathbb{VT}$, $\mathbb{VW}$ and $\mathbb{VC}$. Moreover, we shall study the complexity of the calculus $\mathcal{I}^{\mathsf{V}}$ with the hope of obtaining optimal calculi.

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