Econ Theory Bull (2015) 3:53–64
DOI 10.1007/s40505-015-0069-5

RESEARCH ARTICLE

Equilibrium selection through $\mathbf{p}_u$-dominance

Andrea Gallice

Received: 26 January 2015 / Accepted: 4 February 2015 / Published online: 15 February 2015

Abstract The paper introduces and discusses the concept of $\mathbf{p}_u$-dominance in the context of finite games in normal form. It then presents the $\mathbf{p}_u$-dominance criterion for equilibrium selection. The $\mathbf{p}_u$-dominance criterion is inspired by and closely related to the **p**-dominance criterion originally proposed by Morris et al. (Econometrica 63:145–157, 1995). However, there are games in which the two criteria lead to different refinements. We provide sufficient conditions under which equilibrium selection through $\mathbf{p}_u$-dominance is weakly finer (respectively, coarser) than equilibrium selection through **p**-dominance.

Keywords Equilibrium selection · Normal form games · $\mathbf{p}_u$-Dominance · **p**-Dominance

JEL Classification C72 · C73

1 Introduction

Multiplicity of equilibria is a feature that characterizes many strategic interactions. A typical example is a coordination game where multiple pure strategy Nash equilibria occur when players' actions match. Whenever multiple equilibria exist, equilibrium selection obviously becomes a key issue, both from a normative and a positive point of view (Kim 1996; Haruvy and Stahl 2007).

A. Gallice (✉)
Department of Economics and Statistics, University of Torino,
Corso Unione Sovietica 218bis, 10134 Turin, Italy
e-mail: andrea.gallice@unito.it

A. Gallice
Collegio Carlo Alberto, Moncalieri, Italy

In this paper, we introduce the concept of $\mathbf{p}_u$-dominance and use it as a criterion for equilibrium selection in the context of finite one-shot simultaneous games. In a nutshell, the $\mathbf{p}_u$-dominance criterion selects the equilibrium whose supporting actions turn out to be $\mathbf{p}_u$-dominant for the less stringent set of beliefs. More precisely, we say that an equilibrium is $\mathbf{p}_u$-dominant for the vector $\mathbf{p}_u = (p_1, \ldots, p_n)$ if, for every player $i \in \{1, \ldots, n\}$, i's equilibrium action best responds to any conjecture according to which any player $j \neq i$ plays his equilibrium action with probability of at least p_i and uniformly randomizes the remaining probability over his alternative actions.[1]

In terms of its "microfoundations", $\mathbf{p}_u$-dominance thus mimics the mental process according to which every agent evaluates the likelihood of an equilibrium by focusing on the probability of the actions that sustain it, while assuming a streamlined uniform distribution for what concerns the other players' alternative actions. Such an approach can be the result of bounded rationality or can stem from a conscious consideration of the player. In this respect, $\mathbf{p}_u$-dominance postulates a behavior that is consistent with the one that the so-called "wordly" archetype of players would implement (for experimental evidence about the incidence of this archetype of players see Stahl and Wilson 1995; Haruvy and Stahl 2007; Weizsäcker 2003): play best response to a prior based on a belief that a fraction of the population chooses the equilibrium strategy (so called naive-Nash type) while the remaining part uniformly randomizes over all strategies (level-0 type).

Notice also that, by construction, the $\mathbf{p}_u$-dominance criterion evaluates an equilibrium in light of a conjecture that assigns positive probability to the event that the opponents deviate and do not play their corresponding equilibrium action. As such, $\mathbf{p}_u$-dominance considers not only the profitability of playing a certain equilibrium action but also its riskiness. The $\mathbf{p}_u$-dominance criterion thus tackles the issue of equilibrium selection using the same intuition that underlies the well-known concepts of risk-dominance (Harsanyi and Selten 1988) and $\mathbf{p}$-dominance (Morris et al. 1995; Kajii and Morris 1997): if multiple equilibria exist and agents do not know which equilibrium will arise, they will coordinate their expectations on the one that better solves the trade-off between risk and return.

In the second part of the paper, we more thoroughly investigate the relationships that exist between $\mathbf{p}_u$-dominance, risk-dominance, and $\mathbf{p}$-dominance. We show that the $\mathbf{p}_u$-dominance criterion and the risk-dominance criterion coincide in any 2×2 games, but $\mathbf{p}_u$-dominance is a more general concept, as it can also be applied to games that have more than two players. We then compare $\mathbf{p}_u$-dominance and $\mathbf{p}$-dominance. The two concepts are clearly similar in their approach ($\mathbf{p}_u$-dominance is actually inspired by $\mathbf{p}$-dominance) but they may lead to different conclusions in terms of equilibrium selection. We show in fact that there exist games in which the $\mathbf{p}_u$-dominance criterion leads to a finer selection (i.e., it sensibly discriminates among equilibria that the $\mathbf{p}$-dominance criterion deems equal) as well as games in which the opposite holds true.

[1] The fact that the conjecture posits that players use a uniform distribution to randomize over the actions that do not belong to the profile under scrutiny explains the subscript "u" in the term "$\mathbf{p}_u$-dominance". This should not be confused with the concept of u-dominance Kojima (2006) that has been proposed as a criterion for equilibrium selection based on perfect foresight dynamics and that assumes that the number of opponents adopting a certain strategy follows a uniform distribution.

We then provide sufficient conditions that discriminate between the two cases and at the same time ensure that the $\mathbf{p}_u$-dominance criterion selects a unique equilibrium.

2 The concept of $\mathbf{p}_u$-dominance

Consider a generic one-shot simultaneous game $G = \langle N, (A_i)_{i \in N}, (u_i)_{i \in N} \rangle$ where $N = \{1, \ldots, n\}$ with $n \geq 2$ denotes the set of players, $A_i = \{a_1, \ldots, a_{k_i}\}$ with $k_i \geq 2$ is the action space of player $i \in N$, and $u_i : A \to \mathbb{R}$ is the payoff function with $A = \times_{j \in N} A_j$. As usual, a profile $a^* \in A$ is a Nash equilibrium of G if the relation $u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*)$ holds for every $i \in N$ and every $a_i \in A_i$.

Now take any action profile $\hat{a}$ and define the vector $\mathbf{p}_u(\hat{a}) = (p_1(\hat{a}), \ldots, p_n(\hat{a}))$. The vector $\mathbf{p}_u(\hat{a})$ assigns to every player $i \in N$ a specific conjecture about what every other agent will play (i.e., a probability distribution defined on every A_j for $j \neq i$). More precisely, $\mathbf{p}_u(\hat{a})$ postulates that each agent $i \in N$ believes that any other player $j \in N_{-i}$ will play action $\hat{a}_j$ with probability $p_i(\hat{a})$ and any other action $a_j \neq \hat{a}_j$ with probability $\frac{(1-p_i(\hat{a}))}{k_j - 1}$.

We then say that the action profile $\hat{a}$ is $\mathbf{p}_u$-dominant for $\mathbf{p}_u(\hat{a}) = (p_1(\hat{a}), \ldots, p_n(\hat{a}))$ if, for any player $i \in N$ and any $j \in N_{-i}$, action $\hat{a}_i$ is a best response to any probability distribution $\lambda(\hat{a}) \in \Delta(A_{-i})$ that assigns at least probability $p_i(\hat{a})$ to the event of j playing action $\hat{a}_j$ and lets j uniformly to randomize with the remaining probability over his alternative actions.

Definition 1 Action profile $\hat{a} \in A$ is $\mathbf{p}_u$-dominant with $\mathbf{p}_u(\hat{a}) = (p_1(\hat{a}), \ldots, p_n(\hat{a}))$ if for all $i \in N$, $a_i \neq \hat{a}_i$ and all $\lambda(\hat{a}) \in \Delta(A_{-i})$ with $\lambda(\hat{a}_j) \geq p_i(\hat{a})$ and $\lambda(a_j) = \frac{(1-\lambda(\hat{a}_j))}{k_j - 1}$ for all $a_j \neq \hat{a}_j$ and every $j \in N_{-i}$,

$$\sum_{a_{-i} \in A_{-i}} \lambda(a_{-i}) u_i(\hat{a}_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \lambda(a_{-i}) u_i(a_i, a_{-i}). \tag{1}$$

Some basic concepts in game theory can be formulated in terms of $\mathbf{p}_u$-dominance. For instance, an equilibrium in dominant strategies is $\mathbf{p}_u$-dominant with $\mathbf{p}_u(a^*) = (0, \ldots, 0)$. In contrast, there is no vector $\mathbf{p}_u$ for which a profile that contains dominated actions can turn out to be $\mathbf{p}_u$-dominant.

Focusing on Nash equilibria in pure strategies, every equilibrium is $\mathbf{p}_u$-dominant with $\mathbf{p}_u(a^*) = (1, \ldots, 1)$: notice in fact that with such a (degenerate) vector of probability distributions, the definition of $\mathbf{p}_u$-dominance boils down to the requirement that every action in the profile under scrutiny is a best response to the actions taken by the other players, i.e., the definition of a Nash equilibrium. However, an equilibrium a^* is in general $\mathbf{p}_u$-dominant also for some other vectors $\mathbf{p}_u(a^*) \leq (1, \ldots, 1)$. More precisely, if the equilibrium a^* is $\mathbf{p}_u$-dominant with $\mathbf{p}_u(a^*)$, then a^* is also $\mathbf{p}_u$-dominant for any $\mathbf{p}'_u(a^*) \geq \mathbf{p}_u(a^*)$.

Whenever multiple equilibria exist and the set of equilibria is given by $A^* = \{a^{*1}, \ldots, a^{*\Psi}\}$ with $\Psi \geq 2$, what characterizes a specific equilibrium $a^{*\psi} \in A^*$ is the smallest $\mathbf{p}_u(a^{*\psi})$ for which $a^{*\psi}$ turns out to be $\mathbf{p}_u$-dominant. This vector, which we indicate with $\bar{\mathbf{p}}_u(a^{*\psi}) = (\bar{p}_1(a^{*\psi}), \ldots, \bar{p}_n(a^{*\psi}))$, reports the smallest probabilities $\bar{p}_i(a^{*\psi})$ for which $a_i^{*\psi}$ is a (weak) best response to the associated conjecture $\bar{\lambda}(a^{*\psi})$ with $\bar{\lambda}(a_j^{*\psi}) = \bar{p}_i(a^{*\psi})$ and $\bar{\lambda}(a_j) = \frac{(1-\bar{p}_i(a^{*\psi}))}{k_j-1}$ for all $a_j \neq a_j^{*\psi}$ and $j \in N_{-i}$. As such, $\bar{p}_i(a^{*\psi})$ provides a measure of the riskiness of playing the equilibrium action $a_i^{*\psi}$ as well as a tool to identify the equilibrium upon which players' expectations should coordinate. In particular, and in the same spirit of the sufficient **p**-dominance condition introduced by Kajii and Morris (1997) for the identification of robust equilibria, the $\mathbf{p}_u$-dominance criterion selects the equilibrium (or the subset of equilibria, Sect. 3 will investigate the conditions under which uniqueness is ensured) $a^{**} \in A^*$ for which the following relation holds:

$$\sum_{i=1}^{n} \bar{p}_i(a^{**}) \leq \sum_{i=1}^{n} \bar{p}_i(a^{*\psi}) \quad \text{for any } a^{*\psi} \in A^* \tag{2}$$

In other words, the $\mathbf{p}_u$-dominance criterion selects the equilibrium whose supporting actions emerge as $\mathbf{p}_u$-dominant under the less stringent set of beliefs. Notice that if the game is symmetric requirement (2) holds if and only if $\bar{\mathbf{p}}_u(a^{**}) \leq \bar{\mathbf{p}}_u(a^{*\psi})$ for any $a^{*\psi} \in A^*$.

As an example, consider the following 3×3 coordination game where the action space of player $i \in \{A, B, C\}$ is given by $A_i = \{H, M, L\}$. Player A chooses the row, player B chooses the column, and player C chooses the matrix. In each cell payoffs appear in the order u_A, u_B and u_C.

$a_C = H$

	$a_B = H$	$a_B = M$	$a_B = L$
$a_A = H$	3, 3, 3	2, 0, 2	0, 1, 2
$a_A = M$	0, 2, 2	0, 0, 2	2, 3, 1
$a_A = L$	1, 2, 0	3, 2, 1	1, 1, 1

$a_C = M$

	$a_B = H$	$a_B = M$	$a_B = L$
$a_A = H$	2, 2, 0	2, 0, 0	1, 3, 2
$a_A = M$	0, 2, 0	3, 3, 3	2, 1, 2
$a_A = L$	3, 1, 2	1, 2, 2	0, 0, 1

$a_C = L$

	$a_B = H$	$a_B = M$	$a_B = L$
$a_A = H$	2, 0, 1	1, 2, 3	1, 1, 1
$a_A = M$	2, 1, 3	2, 2, 1	1, 0, 0
$a_A = L$	1, 1, 1	0, 1, 0	3, 3, 3

The game has three Nash equilibria: $a^{*1} = (H, H, H)$, $a^{*2} = (M, M, M)$, and $a^{*3} = (L, L, L)$. These equilibria are Pareto equivalent such that the Pareto dominance criterion (i.e., the criterion that selects the Pareto superior equilibrium with the argument that this is the outcome upon which agents' expectations should converge) does not refine the set A^*.

The $\mathbf{p}_u$-dominance criterion selects instead a specific equilibrium. To find it, one needs to compute for every equilibrium $a^{*\psi}$ with $\psi \in \{1, 2, 3\}$ the functions $E(a_i^{*\psi}|\lambda(a^{*\psi}))$, i.e., the expected payoff of action $a_i^{*\psi}$ conditional on player i's conjecture $\lambda(a^{*\psi})$ that postulates that each opponent $j \in N_{-i}$ plays action $a_j^{*\psi}$ with probability $\lambda(a_j^{*\psi}) = p_i$ and each of his alternative actions $a_j \neq a_j^{*\psi}$ with probability $\lambda(a_j) = \frac{1-p_i}{2}$. Then, by imposing the conditions $E(a_i^{*\psi}|\lambda(a^{*\psi})) \geq E(a_i|\lambda(a^{*\psi}))$ for any $a_i \neq a_i^{*\psi}$, we find the components of the vector $\bar{\mathbf{p}}_u$ which is the smallest vector for which the equilibrium $a^{*\psi}$ is $\mathbf{p}_u$-dominant. Finally, we apply the $\mathbf{p}_u$-dominance criterion and select the equilibrium characterized by the smallest $\bar{\mathbf{p}}_u$.

For instance, starting from the equilibrium a^{*1} and focusing without loss of generality on player A (the game is symmetric), we have the following: $E(H|\lambda(a^{*1})) = \frac{5}{4}p_i^2 + \frac{1}{2}p_i + \frac{5}{4}$, $E(M|\lambda(a^{*1})) = -2p_i + 2$, and $E(L|\lambda(a^{*1})) = -2p_i^2 + 2p_i + 1$. Therefore, the equilibrium action H dominates action M for any $p_i \geq 0.265$ and action L for any $p_i \geq 0$. Given that similar relations also hold for players B and C, a^{*1} is $\mathbf{p}_u$-dominant with $\bar{\mathbf{p}}_u(a^{*1}) = (0.265, 0.265, 0.265)$. Similar computations show that a^{*2} is $\mathbf{p}_u$-dominant with $\bar{\mathbf{p}}_u(a^{*2}) = (0.547, 0.547, 0.547)$ while a^{*3} is $\mathbf{p}_u$-dominant with $\bar{\mathbf{p}}_u(a^{*3}) = (0.771, 0.771, 0.771)$. The $\mathbf{p}_u$-dominance criterion thus unambiguously selects equilibrium $a^{*1} = (H, H, H)$.

3 $\mathbf{p}_u$-dominance, risk-dominance and p-dominance

In this section, we investigate the relationships that exist between the $\mathbf{p}_u$-dominance criterion and the risk-dominance (Harsanyi and Selten 1988) and **p**-dominance (Morris et al. 1995; Kajii and Morris 1997) criteria. In the course of the analysis, we also explore how $\mathbf{p}_u$-dominance relates to mixed strategies equilibria.

Proposition 1 *In any* 2×2 *coordination game, the* $\mathbf{p}_u$*-dominance criterion always selects the risk-dominant equilibrium.*

Proof Consider a generic 2×2 coordination game with $i \in \{A, B\}$ and $A_i = \{H, L\}$ with $a > c$, $d > b$, $e > f$ and $h > g$.

	$a_B = H$	$a_B = L$
$a_A = H$	a, e	b, f
$a_A = L$	c, g	d, h

The equilibrium $a^{*1} = (H, H)$ is $\mathbf{p}_u$-dominant with $\bar{\mathbf{p}}_u(a^{*1}) = \left(\frac{d-b}{a-c+d-b}, \frac{h-g}{e-f+h-g}\right)$, while $a^{*2} = (L, L)$ is $\mathbf{p}_u$-dominant with $\bar{\mathbf{p}}_u(a^{*2}) = \left(\frac{a-c}{a-c+d-b}, \frac{e-f}{e-f+h-g}\right)$. Therefore, the $\mathbf{p}_u$-dominant criterion selects a^{*1} if $\frac{d-b}{a-c+d-b} < \frac{a-c}{a-c+d-b}$ and $\frac{h-g}{e-f+h-g} < \frac{e-f}{e-f+h-g}$, i.e., if $a - c > d - b$ and $e - f > h - g$. But if both conditions are valid then $(a - c)(e - f) > (d - b)(h - g)$, i.e., a^{*1} is the risk-dominant equilibrium because it is the equilibrium characterized by the highest product of the deviation losses. Similarly, if the $\mathbf{p}_u$-dominant criterion selects a^{*2}, then it must be the case that $(d - b)(h - g) > (a - c)(e - f)$ which means that a^{*2} is the risk-dominant equilibrium. □

The proof of Proposition 1 provides an example of a relation that holds more generally in any 2×2 game with two equilibria in pure strategies: if an equilibrium $a^{*\psi} \in \{a^{*1}, a^{*2}\}$ is $\mathbf{p}_u$-dominant with $\bar{\mathbf{p}}_u(a^{*\psi}) = \left(\bar{p}_1(a^{*\psi}), \bar{p}_2(a^{*\psi})\right)$, then $\bar{p}_i(a^{*\psi}) = q_j$ where q_j is the probability that defines the mixed-strategy equilibrium of the game: $\left(q_1 a_1^{*\psi} + (1 - q_1)\, a_1, q_2 a_2^{*\psi} + (1 - q_2)\, a_2\right)$ with $a_i \neq a_i^{*\psi}$. The intuition is the following: in a mixed-strategy equilibrium, agents randomize in such a way as to make the other player indifferent about what to play. This means that if player i attaches probability $\bar{p}_i(a^{*\psi}) = q_j$ to the event of j playing action $a_j^{*\psi}$ [and thus probability $\left(1 - \bar{p}_i(a^{*\psi})\right) = \left(1 - q_j\right)$ to the event of j playing a_j given that $k_j - 1 = 1$], then both actions in A_i are best responses for player i. However, for any $q_j' > q_j$, action $a_i^{*\psi}$ becomes player i's unique best response. It follows that the equilibrium $a^{*\psi}$ is $\mathbf{p}_u$-dominant with $\bar{\mathbf{p}}_u(a^{*\psi}) = \left(\bar{p}_1(a^{*\psi}), \bar{p}_2(a^{*\psi})\right)$ where $\bar{p}_1(a^{*\psi}) = q_2$ and $\bar{p}_2(a^{*\psi}) = q_1$.

We now move to a comparison between the $\mathbf{p}_u$-dominance criterion and the $\mathbf{p}$-dominance criterion. As defined in Morris et al. (1995) for the case with two players and then extended by Kajii and Morris (1997) to the many players case, an equilibrium a^* is $\mathbf{p}$-dominant with $\mathbf{p}(a^*) = (p_1(a^*), \ldots, p_n(a^*))$ if, for any agent $i \in N$, action a_i^* is a best response to any probability distribution $\lambda(a^*) \in \Delta(A_{-i})$ such that $\lambda(a_j^*) \geq p_i(a^*)$ for any $j \neq i$.[2] In other words, action a_i^* is $\mathbf{p}$-dominant if it maximizes player i's expected payoff whenever i thinks that each one of the other players will play with probability not smaller than $p_i(a^*)$, his component of the equilibrium profile. The difference with respect to $\mathbf{p}_u$-dominance is that $\mathbf{p}$-dominance does not require the remaining probability $(1 - \lambda(a_j^*))$ to follow any particular distribution over the alternative actions $a_j \neq a_j^*$. While at first sight $\mathbf{p}$-dominance may seem a more general concept with respect to $\mathbf{p}_u$-dominance, notice that every Nash equilibrium is always simultaneously $\mathbf{p}$-dominant for some $\mathbf{p}$ as well as $\mathbf{p}_u$-dominant for some $\mathbf{p}_u$. What may differ across the two concepts are the smallest probability vectors $\bar{\mathbf{p}}\,(a^*)$ and $\bar{\mathbf{p}}_u\,(a^*)$ for which the equilibrium is dominant.

[2] Tercieux (2006a, b) further extends the concept of $\mathbf{p}$-dominance by introducing the notion of $\mathbf{p}$-best response set: a set profile $S = (S_1, \ldots, S_n)$ is a $\mathbf{p}$-best response set if for every i the set S_i contains an action that best responds to any conjecture that assigns probability of at least p_i to the event that other players select their action from S_{-i}.

Proposition 2 *Let a^* be a pure Nash equilibrium of the game G. Then a^* is* **p**-*dominant with $\bar{\mathbf{p}}(a^*)$ as well as $\mathbf{p}_u$-dominant with $\bar{\mathbf{p}}_u(a^*)$ and the relation $\bar{\mathbf{p}}_u(a^*) \leq \bar{\mathbf{p}}(a^*)$ holds.*

Proof In the appendix. □

Proposition 2 clarifies the relationship that exists between the $\bar{\mathbf{p}}(a^*)$ and $\bar{\mathbf{p}}_u(a^*)$ vectors that characterize any Nash equilibrium. However, one thing is the definition and the computation of these vectors: a different issue is how **p**-dominance and $\mathbf{p}_u$-dominance perform in terms of equilibrium selection. The two criteria in fact do not always lead to the same solution. More precisely, there exist games in which the $\mathbf{p}_u$-dominance criterion discriminates among equilibria that on the contrary the **p**-dominance criterion deems equivalent (see Example 1 below) as well as games in which the opposite holds true (see Example 2 later).

Example 1 Consider the following game:

	$a_B = H$	$a_B = M$	$a_B = L$
$a_A = H$	4,4	2,3	0,0
$a_A = M$	3,2	0,0	2,0
$a_A = L$	0,0	0,2	4,4

The game has two Nash equilibria: $a^{*1} = (H, H)$ and $a^{*2} = (L, L)$. Both equilibria are **p**-dominant with $\bar{\mathbf{p}}\left(a^{*1}\right) = \bar{\mathbf{p}}\left(a^{*2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$. The **p**-dominant criterion thus does not refine the set of equilibria. On the contrary, a^{*1} is $\mathbf{p}_u$-dominant with $\bar{\mathbf{p}}_u\left(a^{*1}\right) = \left(\frac{1}{5}, \frac{1}{5}\right)$ while a^{*2} is $\mathbf{p}_u$-dominant with $\bar{\mathbf{p}}_u\left(a^{*1}\right) = \left(\frac{3}{7}, \frac{3}{7}\right)$. Therefore, the $\mathbf{p}_u$-dominance criterion unambiguously selects the equilibrium $a^{*1} = (H, H)$.

In the game of Example 1, equilibrium selection through $\mathbf{p}_u$-dominance is thus more effective with respect to a selection based on **p**-dominance. And indeed, given the overall structure of the game, the selection of $a^{*1} = (H, H)$ seems a sensible prediction: a^{*1} and a^{*2} are in fact Pareto equivalent but action $a_i = H$ is more rewarding/less risky.

More in general the intuition as to why in some games the $\mathbf{p}_u$-dominance criterion manages to discriminate among equilibria that the **p**-dominance criterion judges as equal is the following. The $\mathbf{p}_u$-dominance conjecture assigns to every player a belief system that attaches positive probability to the occurrence of any action that belongs to the action space of the opponents. On the contrary, the **p**-dominance conjecture does not. The **p**-dominance conjecture stems in fact from a worst-case scenario analysis that requires the equilibrium action under scrutiny (a_i^*) to dominate any alternative action ($a_i \neq a_i^*$) even under the most unfavorable probability distribution. This distribution is the one that allocates all the available probability $(1 - \bar{p}_i(a^*))$ to the occurrence of the action(s) $\tilde{a}_j$ for any $j \neq i$ where $\tilde{a}_j$ is the action that supports the profile for which the payoff difference $u_i(a_i, \tilde{a}_{-i}) - u_i(a_i^*, \tilde{a}_{-i})$ is maximal. As such the **p**-dominance conjecture attaches zero probability to the event that an opponent plays action $a_j \neq a'_j$

with $a'_j \in \{a^*_j, \tilde{a}_j\}$. The conjecture thus disregards some of the opponents' actions and therefore it ignores the associated payoffs.[3] In other words, **p**-dominance captures less information with respect to $\mathbf{p}_u$-dominance. However, while in general this is a positive feature of $\mathbf{p}_u$-dominance as it means that the $\mathbf{p}_u$-dominance criterion better captures the inherent structure of the strategic situation under study, the consequences of this fact in terms of equilibrium selection can go in both directions. In fact, and as it has already been mentioned, there also exist games in which the **p**-dominance criterion selects a unique equilibrium while the $\mathbf{p}_u$-dominance criterion does not.

Example 2 Consider the following game:

	$a_B = H$	$a_B = M$	$a_B = L$
$a_A = H$	4,4	2,2	0,0
$a_A = M$	2,2	0,0	1,3
$a_A = L$	0,0	3,1	3,3

The game has two Nash equilibria: $a^{*1} = (H, H)$ and $a^{*2} = (L, L)$. The a^{*1} equilibrium is **p**-dominant with $\bar{\mathbf{p}}\left(a^{*1}\right) = \left(\frac{3}{7}, \frac{3}{7}\right)$ while the a^{*2} equilibrium is **p**-dominant with $\bar{\mathbf{p}}\left(a^{*2}\right) = \left(\frac{4}{7}, \frac{4}{7}\right)$. The **p**-dominant criterion thus selects the equilibrium $a^{*1} = (H, H)$. Both equilibria are also $\mathbf{p}_u$ -dominant with $\bar{\mathbf{p}}_u\left(a^{*1}\right) = \bar{\mathbf{p}}_u\left(a^{*2}\right) = \left(\frac{1}{3}, \frac{1}{3}\right)$ such that in this case the $\mathbf{p}_u$-dominance criterion does refine the set of equilibria.

In what follows we focus on symmetric games and provide a sufficient condition that ensures that the $\mathbf{p}_u$-dominance criterion selects a unique equilibrium. Therefore, the same condition also guarantees that equilibrium selection through $\mathbf{p}_u$-dominance is weakly finer than a selection through **p**-dominance. As an intermediary result, Lemma 1 shows that whenever the $\mathbf{p}_u$-dominance criterion selects multiple equilibria then it must be the case that all the selected equilibria are $\mathbf{p}_u$-dominant with $\bar{\mathbf{p}}_u = \left(\frac{1}{k}, \ldots, \frac{1}{k}\right)$ where $k = \left|A_j\right|$ for any $j \in N$ is the number of actions at player j's disposal. Proposition 3 then specifies the above-mentioned sufficient condition.

Lemma 1 *Let $A^* = \left\{a^{*1}, \ldots, a^{*\Psi}\right\}$ with $\Psi \geq 2$ be the set of Nash equilibria of a symmetric game G with k actions for each player. If the $\mathbf{p}_u$-dominance criterion selects the set of equilibria $A^{**} = \left\{a^{**1}, \ldots, a^{**\Phi}\right\}$ with $\Phi \geq 2$ and $A^{**} \subseteq A^*$ then $\bar{\mathbf{p}}_u(a^{**\phi}) = \left(\frac{1}{k}, \ldots, \frac{1}{k}\right)$ for any $a^{**\phi} \in A^{**}$.*

Proof In the appendix. □

Proposition 3 *Let $A^* = \left\{a^{*1}, \ldots, a^{*\Psi}\right\}$ with $\Psi \geq 2$ be the set of Nash equilibria of a symmetric game G with k actions for each player. A sufficient condition*

[3] Notice that the actions that the **p**-dominance conjecture disregards are not necessarily dominated (see for instance action M in the game in Example 1). Notice also that the $\mathbf{p}_u$-dominance conjecture suffers instead from the opposite problem as it assigns positive probability to the event that the opponents play strictly dominated actions (if these exist). Still, this shortcoming can be easily fixed by applying the $\mathbf{p}_u$-dominance criterion only after the elimination of all strictly dominated actions.

for the $\mathbf{p}_u$*-dominance criterion to select a unique equilibrium* $a^{**} \in A^*$ *is that* $\sum_{a_{-i} \in A_{-i}} u_i(a_i^{*\psi}, a_{-i}) \neq \sum_{a_{-i} \in A_{-i}} u_i(a_i^{*\xi}, a_{-i})$ *for any* $a_i^{*\psi}$ *and* $a_i^{*\xi} \neq a_i^{*\psi}$.

Proof In the appendix. □

4 Conclusions

This paper introduced the concept of $\mathbf{p}_u$-dominance and proposed the $\mathbf{p}_u$-dominance criterion as a tool to refine multiple equilibria in normal form games. The criterion selects the equilibrium (or the set of equilibria) whose supporting actions are $\mathbf{p}_u$-dominant under the less stringent set of beliefs. The intuition for such a choice is that the selected equilibrium is the one that better solves the trade-off between the profitability of the equilibrium outcome and the riskiness of playing the supporting action in case some of the opponents deviate. The paper also explored the tight connections that exist between $\mathbf{p}_u$-dominance and $\mathbf{p}$-dominance (Morris et al. 1995; Kajii and Morris 1997). The two concepts are clearly similar in their premises as well as in their scope but we feel $\mathbf{p}_u$-dominance presents some advantages. First, $\mathbf{p}_u$-dominance relies on a conjecture that is more in line with the actual behavior of a relevant fraction inexperienced players (so called "wordly" archetype). Second, it is more easily computable. In particular, one does not need to identify the most unfavorable distribution that defines $\mathbf{p}$-dominance, a task that may turn out to be not so trivial especially in games that involve more than two players and/or actions. Finally, $\mathbf{p}_u$-dominance more fully captures the inherent structure of the game under study as it takes into account any possible deviation of the other players.

Appendix

Proof of Proposition 2

If a^* is a Nash equilibrium of G, then there exists at least one vector for which a^* is $\mathbf{p}$-dominant. In particular, a^* is certainly $\mathbf{p}$-dominant for $\mathbf{p}(a^*) = (1, \ldots, 1)$. It follows that the vector $\bar{\mathbf{p}}(a^*) = (\bar{p}_1(a^*), \ldots, \bar{p}_n(a^*))$ also exists where, for any i and any $j \neq i$, $\bar{p}_i(a^*)$ is the smallest probability for which action a_i^* best responds to any conjecture $\lambda(a^*) \in \Delta(A_{-i})$ according to which $\lambda(a_j^*) \geq \bar{p}_i(a^*)$ while the remaining probability $(1 - \lambda(a_j^*))$ can follow any distribution on actions $a_j \neq a_j^*$. The conjecture thus includes the situation in which $\lambda(a_j) = \frac{(1-\lambda(a_j^*))}{k_j - 1}$ for any $j \neq i$, $a_j \neq a_j^*$ and $k_j = |A_j|$. Therefore, if a^* is $\mathbf{p}$-dominant with $\bar{\mathbf{p}}(a^*)$ then a^* is also $\mathbf{p}_u$-dominant with $\mathbf{p}_u(a^*) = \bar{\mathbf{p}}(a^*)$. Notice that if $k_j = 2$ for any $j \in N$, then $\bar{\mathbf{p}}_u(a^*) = \bar{\mathbf{p}}(a^*)$ as both $\mathbf{p}$-dominance and $\mathbf{p}_u$-dominance assign probability $(1 - \lambda(a_j^*))$ to the event of j playing action $a_j \neq a_j^*$. Now consider the case in which $k_j > 2$ for some j. Assume there exists at least one action $a_i \in A_i$ that is not (weakly or strictly) dominated by a_i^*.[4] Let $\tilde{a}(a_i) = (a_i, \tilde{a}_{-i})$ be the action profile that supports the (non-necessarily

[4] If an undominated action does not exist then a^* is $\mathbf{p}_u$-dominant as well $\mathbf{p}$-dominant with $\bar{\mathbf{p}}_u(a^*) = \bar{\mathbf{p}}(a^*) = (0, \ldots, 0)$.

unique) outcome for which $u_i(a_i, a_{-i}) - u_i(a_i^*, a_{-i}) > 0$ is maximal. Given that $u_i(a_i, a_{-i}^*) - u_i(a_i^*, a_{-i}^*) \leq 0$ (a^* is a Nash equilibrium), it must be the case that in $\tilde{a}(a_i)$ there is at least one player $j \neq i$ that plays $\tilde{a}_j \neq a_j^*$. The equilibrium a^* is $\mathbf{p}$-dominant with $\bar{\mathbf{p}}(a^*) = (\bar{p}_1(a^*), \ldots, \bar{p}_n(a^*))$ if a_i^* dominates any a_i even under the specific conjecture that assigns probability $(1 - \bar{p}_i(a^*))$ to the event of j playing $\tilde{a}_j \neq a_j^*$. On the other hand, a^* is $\mathbf{p}_u$-dominant with $\bar{\mathbf{p}}_u(a^*) = \left(\bar{p}_1'(a^*), \ldots, \bar{p}_n'(a^*)\right)$ if a_i^* dominates any a_i under the conjecture that assigns probability $\left(\frac{1-\bar{p}_i'(a^*)}{k_j-1}\right)$ to the event of j playing $\tilde{a}_j \neq a_j^*$. Equilibrium a^* cannot be $\mathbf{p}$-dominant with $\mathbf{p}(a^*) = \bar{\mathbf{p}}_u(a^*)$ given that $(1 - \bar{p}_i'(a^*)) > \left(\frac{1-\bar{p}_i'(a^*)}{k_j-1}\right)$ and the conjecture would thus assign too much probability to the occurrence of the profile $\tilde{a}(a_i)$. It then must be the case that $\bar{p}_i'(a^*) < \bar{p}_i(a^*)$. We can thus conclude that $\bar{\mathbf{p}}_u(a^*) \leq \bar{\mathbf{p}}(a^*)$.

Proof of Lemma 1

Without loss of generality, assume the $\mathbf{p}_u$-dominance criterion selects two equilibria, i.e., let $A^{**} = \{a^{**1}, a^{**2}\}$. By expression (2) this means:

$$\sum_{i=1}^{n} \bar{p}_i(a^{**1}) = \sum_{i=1}^{n} \bar{p}_i(a^{**2}) \tag{3}$$

The fact that G is symmetric implies that (3) reduces to $\bar{\mathbf{p}}_u(a^{**1}) = \bar{\mathbf{p}}_u(a^{**2})$, i.e., $\bar{p}_i(a^{**1}) = \bar{p}_i(a^{**2})$ for any $i \in N$. Now define the conjecture $\bar{\lambda}(a^{**\phi})$ with $\phi \in \{1, 2\}$ such that $\bar{\lambda}(a_j^{**\phi}) = \bar{p}_i(a^{**\phi})$ and $\bar{\lambda}(a_j) = \frac{(1-\bar{p}_i(a^{**\phi}))}{k-1}$ for all $a_j \neq a_j^{**\phi}$ and for $j \in N_{-i}$. Let $\lambda^{(\phi)}(a_{-i}) = \times_{j \in N_{-1}} \bar{\lambda}(a^{**\phi})$. With the probability distribution $\lambda^{(\phi)}$, any player is by construction indifferent between playing action $a_i^{**\phi}$ and action $a_i^{**\xi}$ with $\xi \in \{1, 2\}$ and $\xi \neq \phi$. This implies that the following two conditions must simultaneously hold:

$$\sum_{a_{-i} \in A_{-i}} \lambda^{(1)}(a_{-i})\, u_i\left(a_i^{**1}, a_{-i}\right) = \sum_{a_{-i} \in A_{-i}} \lambda^{(1)}(a_{-i})\, u_i\left(a_i^{**2}, a_{-i}\right) \tag{4}$$

$$\sum_{a_{-i} \in A_{-i}} \lambda^{(2)}(a_{-i})\, u_i\left(a_i^{**2}, a_{-i}\right) = \sum_{a_{-i} \in A_{-i}} \lambda^{(2)}(a_{-i})\, u_i\left(a_i^{**1}, a_{-i}\right) \tag{5}$$

Therefore, the following condition also holds:

$$\begin{aligned} &\sum_{a_{-i} \in A_{-i}} \lambda^{(1)}(a_{-i})\, u_i\left(a_i^{**1}, a_{-i}\right) - \sum_{a_{-i} \in A_{-i}} \lambda^{(1)}(a_{-i})\, u_i\left(a_i^{**2}, a_{-i}\right) \\ &\quad = \sum_{a_{-i} \in A_{-i}} \lambda^{(2)}(a_{-i})\, u_i\left(a_i^{**1}, a_{-i}\right) - \sum_{a_{-i} \in A_{-i}} \lambda^{(2)}(a_{-i})\, u_i\left(a_i^{**2}, a_{-i}\right) \end{aligned} \tag{6}$$

Now notice that, given that by assumption $A^{**} = \{a^{**1}, a^{**2}\}$ then it must be the case that $\bar{p}_i(a^{**1}) = \bar{p}_i(a^{**2})$ for any $i \in N$. Therefore, $\frac{(1-\bar{p}_i(a^{**1}))}{k-1} = \frac{(1-\bar{p}_i(a^{**2}))}{k-1}$ for any $i \in N$. Expression (6) thus boils down to:

$$\left(\left(\bar{p}_i(a^{**1})\right)^{n-1} - \left(\frac{(1-\bar{p}_i\left(a^{**1}\right))}{k-1}\right)^{n-1}\right)\left(u_i\left(a_i^{**1}, a_{-i}^{**1}\right) - u_i\left(a_i^{**2}, a_{-i}^{**1}\right)\right)$$
$$= \left(\left(\bar{p}_i(a^{**1})\right)^{n-1} - \left(\frac{(1-\bar{p}_i\left(a^{**1}\right))}{k-1}\right)^{n-1}\right)\left(u_i\left(a_i^{**1}, a_{-i}^{**2}\right) - u_i\left(a_i^{**2}, a_{-i}^{**2}\right)\right) \tag{7}$$

The fact that a^{**1} and a^{**2} are Nash equilibria implies that $(u_i(a_i^{**1}, a_{-i}^{**1}) - u_i(a_i^{**2}, a_{-i}^{**1})) \geq 0$ and $(u_i(a_i^{**1}, a_{-i}^{**2}) - u_i(a_i^{**2}, a_{-i}^{**2})) \leq 0$ with at least one strict inequality if at least one equilibrium is strict. Then, for expression (7) to always hold [notice that both $\bar{p}_i(a^{**1})$ and $\left(\frac{1-\bar{p}_i(a^{**1})}{k-1}\right)$ are non-negative and cannot be simultaneously equal to zero], it must be the case that $\bar{p}_i(a^{**1}) = \left(\frac{1-\bar{p}_i(a^{**1})}{k-1}\right)$. But since $\bar{p}_i(a^{**1}) = \bar{p}_i(a^{**2})$ then the following result holds:

$$\bar{p}_i(a^{**\phi}) = \left(\frac{1-\bar{p}_i(a^{**\phi})}{k-1}\right) \quad \text{for } \phi \in \{1, 2\} \tag{8}$$

which implies $\bar{p}_i(a^{**\phi}) = \frac{1}{k}$ for any $\phi \in \{1, 2\}$. Given that the choice of $|A^{**}| = 2$ was made without loss of generality as one can replicate the passages above for any possible couple of equilibria that belong to A^{**}, the more general result that $\bar{\mathbf{p}}_u(a^{**\phi}) = \left(\frac{1}{k}, \ldots, \frac{1}{k}\right)$ for any $a^{**\phi} \in A^{**} = \left\{a^{**1}, \ldots, a^{**\Phi}\right\}$ and $\Phi \geq 2$ easily follows.

Proof of Proposition 3

Assume the $\mathbf{p}_u$-dominance criterion selects multiple equilibria. Without loss of generality, let $A^{**} = \left\{a^{**1}, a^{**2}\right\}$ be the set of selected equilibria. Then:

$$\sum_{a_{-i}\in A_{-i}} \lambda^{(1)}(a_{-i})\, u_i\left(a_i^{**1}, a_{-i}\right) = \sum_{a_{-i}\in A_{-i}} \lambda^{(1)}(a_{-i})\, u_i\left(a_i^{**2}, a_{-i}\right) \tag{9}$$

where $\lambda^{(1)}(a_{-i}) = \times_{j\in N_{-1}} \lambda^1\left(a_j\right)$ and $\lambda^1\left(a_j\right) = \bar{p}_i(a^{**1})$ if $a_j = a_j^{**1}$ while $\lambda^1\left(a_j\right) = \frac{(1-\bar{p}_i(a^{**1}))}{k-1}$ if $a_j \neq a_j^{**1}$ for any $j \in N_{-i}$. But given that $a^{**1} \in A^{**}$ and $|A^{**}| > 1$, Lemma 1 implies that $\bar{p}_i(a^{**1}) = \frac{(1-\bar{p}_i(a^{**1}))}{k-1} = \frac{1}{k}$. Therefore, (9) becomes:

$$\left(\frac{1}{k}\right)^{n-1} \sum_{a_{-i} \in A_{-i}} u_i(a_i^{**1}, a_{-i}) = \left(\frac{1}{k}\right)^{n-1} \sum_{a_{-i} \in A_{-i}} u_i(a_i^{**2}, a_{-i}) \tag{10}$$

which necessarily requires:

$$\sum_{a_{-i} \in A_{-i}} u_i(a_i^{**1}, a_{-i}) = \sum_{a_{-i} \in A_{-i}} u_i(a_i^{**2}, a_{-i}). \tag{11}$$

It follows that if the above condition does not hold then a^{**1} and a^{**1} cannot simultaneously belong to A^{**}. Therefore, a sufficient condition for the $\mathbf{p}_u$-dominance criterion to select a unique equilibrium is that the sum of an agent's individual payoffs across any equilibrium action differs. More formally:

$$\sum_{a_{-i} \in A_{-i}} u_i(a_i^{*\psi}, a_{-i}) \neq \sum_{a_{-i} \in A_{-i}} u_i(a_i^{*\xi}, a_{-i}) \quad \text{for any } a_i^{*\psi}, a_i^{*\xi} \in A^* \text{ and } a_i^{*\psi} \neq a_i^{*\xi}$$

References

Harsanyi, J., Selten, R.: A General Theory of Equilibrium Selection in Games. MIT Press, Cambridge (1988)

Haruvy, E., Stahl, D.O.: Equilibrium selection and bounded rationality in symmetric normal-form games. J. Econ. Behav. Org. **62**, 98–119 (2007)

Kajii, A., Morris, S.: The robustness of equilibria to incomplete information. Econometrica **65**, 1283–1309 (1997)

Kim, Y.: Equilibrium selection in n-person coordination games. Games Econ. Behav. **15**, 203–227 (1996)

Kojima, F.: Risk-dominance and perfect foresight dynamics in n-player games. J. Econ. Theory **128**, 255–273 (2006)

Morris, S., Rob, R., Shin, H.S.: p -Dominance and belief potential. Econometrica **63**, 145–157 (1995)

Stahl, D.O., Wilson, P.W.: On players' models of other players: theory and experimental evidence. Games Econ. Behav. **10**, 218–254 (1995)

Tercieux, O.: p-Best response set. J. Econ. Theory **131**, 45–70 (2006a)

Tercieux, O.: p-Best response set and the robustness of equilibria to incomplete information. Games Econ. Behav. **56**, 371–384 (2006b)

Weizsäcker, G.: Ignoring the rationality of others: evidence from experimental normal-form games. Games Econ. Behav. **44**, 145–171 (2003)