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Microlocal and Spectral Analysis of Tensor Products of Pseudodifferential Operators

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Contents

Introduction

In this thesis we deal with the analysis of some microlocal properties of tensor products of pseudodifferential operators. More generally, we will also be concerned with the so-called class of bisingular pseudodifferential operators. In fact, calculi of bisingular pseudodifferential operators can be seen as a systematic approach for studying tensor products of pseudodifferential operators. Within the elliptic theory, a typical question would be the following. Given classical (or poly-homogeneous) pseudodifferential operators $A_j \in \mathrm{L}^{\mu}_{\mathrm{cl}}(M_1)$ and $B_j \in \mathrm{L}^{\nu}_{\mathrm{cl}}(M_2)$ for $j = 1, \dots, k$, $k \in \mathbb{N}$, on smooth manifolds M_1 and M_2, how can we characterize the existence of a parametrix, the Fredholm property or the invertibilty of the operator $A_1 \otimes B_1 + \cdots + A_k \otimes B_k$? Here, the tensor product $A \otimes B$ denotes an operator acting on functions defined on $M_1 \times M_2$ with the property that

$$(A \otimes B)(u \otimes v) = Au \otimes Bv, \qquad u \in \mathcal{C}^{\infty}(M_1),\ v \in \mathcal{C}^{\infty}(M_2),$$

where $(f \otimes g)(x, y) = f(x)g(y)$ for any two functions f and g on M_1 and M_2, respectively. Such tensor products, in general, do not define a classical pseudodifferential operator on $M_1 \times M_2$, hence the question cannot be answered using the standard pseudodifferential calculus only. Questions of this kind arose naturally, in particular, in the framework of the celebrated Atiyah-Singer index theorem. In fact, M. F. Atiyah and I. M. Singer in [AS68] were led to study systems of the form

$$A \boxtimes B = \begin{pmatrix} A \otimes 1 & -1 \otimes B^* \\ 1 \otimes B & A^* \otimes 1 \end{pmatrix},$$

acting on $\mathbb{C}^2$-valued functions over $M_1 \times M_2$, where both A and B are zero-order classical pseudodifferential operators on M_1 and M_2, respectively. Again, $A \boxtimes B$ is not a classical pseudodifferential operator on $M_1 \times M_2$. However, if both A and B are elliptic, then $A \boxtimes B$ is a Fredholm operator in $L^2(M_1 \times M_2, \mathbb{C}^2)$ with index $\operatorname{ind} A \boxtimes B = \operatorname{ind} A \cdot \operatorname{ind} B$.

Motivated by these phenomena, L. Rodino introduced in [Rod75] a pseudodifferential calculus of operators over a product of smooth, closed (i.e., compact and without boundary) manifolds $M_1 \times M_2$, containing such kinds of tensor product type operators. The elements of this calculus are defined as linear and continuous operators $A = \mathrm{Op}(a)$ whose symbol satisfies, in local product-type coordinates, for all multi-indices α_j, β_j, $j = 1, 2$, the estimates

$$|D^{\alpha_1}_{\xi_1} D^{\alpha_2}_{\xi_2} D^{\beta_1}_{x_1} D^{\beta_2}_{x_2} a(x_1, x_2, \xi_1, \xi_2)| \leq C_{\alpha_1,\alpha_2,\beta_1,\beta_2} \langle \xi_1 \rangle^{m_1 - |\alpha_1|} \langle \xi_2 \rangle^{m_2 - |\alpha_2|}.$$

As already pointed out, a fundamental example of a bisingular operator is the tensor product $A_1 \otimes A_2$ of two pseudodifferential operators, with symbols in the Hörmander class, $A_i \in \mathrm{L}^{m_i}(M_i)$, $i = 1, 2$, while more complex examples include the double Cauchy integral operator studied by F. Nicola and L. Rodino in [NR06]. With each symbol of a bisingular operator A we can associate two maps, namely

$$\sigma^1(A) : M_1 \times \mathbb{R}^{n_1} \to \mathrm{L}^{m_2}(M_2),$$
$$\sigma^2(A) : M_2 \times \mathbb{R}^{n_2} \to \mathrm{L}^{m_1}(M_1).$$

With these maps, bisingular calculus can be considered a calculus with operator-valued symbols, in the spirit of the theories developed by B.-W. Schulze, see, e.g., [Sch98]. In particular, ellipticity in the context of bisingular calculus refers to the invertibility, as pseudodifferential operators, of the two operator-valued principal symbols associated with each bisingular operator.

Another motivation for the study of bisingular operators derives from the study of the spectral asymptotics of the counting function of a pseudodifferential operator. Let P be a positive self-adjoint operator with compact resolvent, such that the spectrum is discrete and formed only by eigenvalues with finite multiplicity. Let $\{\lambda_j\}_{j\in\mathbb{N}} = \sigma(P)$ be the set of the eigenvalues counted with multiplicity. The counting function $N_P(\tau)$ is defined as

$$N_P(\tau) = \sum_{\lambda_j \in \sigma(P) \cap [0,\tau)} 1 = \sum_{\lambda_j < \tau} 1. \tag{0.1}$$

The Weyl's law describes the asymptotic expansion of the counting function $N_P(\tau)$, as τ goes to infinity. In the standard settings, it is well known that that the leading term of the asymptotic expansion of (0.1) depends on the dimension of the manifold, on the order of the operator and on its principal symbol, see, e.g., L. Hörmander [Hör85a] and M. A. Shubin [Shu01] for the classical theory in the case of closed manifolds and the Shubin calculus on $\mathbb{R}^n$, respectively. See also Y. Safarov and D. Vassiliev [SV97] and W. Arendt, R. Nittka, W. Peter and F. Steiner [ANPS09] for a detailed analysis of Weyl's law and several developments. Weyl's laws can be obtained in many other situations, see, e.g., U. Battisti and S. Coriasco [BC11], S. Coriasco and L. Maniccia [CM13], F. Nicola [Nic03] for *SG* operators (on manifolds with ends and $\mathbb{R}^n$), P. Boggiatto and F. Nicola [BN03] for anisotropic Shubin calculus, J. Gil and P. Loya [GL02] for conic manifolds, K. Datchev and S. Dyatlov [DD13] for asymptotically hyperbolic manifolds, S. Moroianu

[Mor08] for cusp manifolds, and many others.
Now consider $A = A_1 \otimes A_2$, where $A_j \in \mathrm{L}^{m_j}_{\mathrm{cl}}(M_j)$ is positive, self-adjoint and elliptic, for $j = 1, 2$, and M_1, M_2 are closed manifolds. Denoting by $\sigma(A_1) = \{\lambda_k\}_{k\in\mathbb{N}}$ and $\sigma(A_2) = \{\mu_i\}_{i\in\mathbb{N}}$, the eigenvalues of A_1 and A_2, counted with multiplicity, we easily obtain that

$$\sigma(A) = \{\lambda_k \cdot \mu_i\}_{(k,i)\in(\mathbb{N}\times\mathbb{N})}\,.$$

Therefore,

$$N_A(\tau) = \sum_{\rho\in\sigma(A)\cap[0,\tau)} 1 = \sum_{\lambda_k\cdot\mu_i<\tau} 1. \tag{0.2}$$

Counting functions of the type (0.2) allow to use a spectral approach to a prominent type of lattice problem, the so-called Dirichlet divisors problem. Let us suppose that the spectrum of both A_1 and A_2 in (0.2) is formed by all strictly positive natural numbers, all with multiplicity one. Then

$$N_A(\tau) = \sum_{n\cdot m<\tau} 1 = D(\tau).$$

The function $D(\tau)$ is called Dirichlet divisor summatory function and it is straightforward to check that it amounts to the number of points with integer coordinates which lie in the first quadrant below the hyperbola $xy = \tau$. In 1849, Dirichlet proved that

$$D(\tau) = \tau\log\tau + (2\gamma - 1)\tau + \mathcal{O}(\tau^{1/2}), \qquad \tau \to +\infty, \tag{0.3}$$

where γ is the Euler-Mascheroni constant. Several papers aimed at finding the sharp remainder in (0.3), see A. Ivić, E. Krätzel, M. Kühleitner and W. G. Nowak [IKKN06] for an overview on this type of problems. In [Har16], G. H. Hardy proved that $\mathcal{O}(\tau^{\frac{1}{4}})$ is a lower bound for the remainder in (0.3). It is conjectured that the sharp estimate in this case is $\mathcal{O}(\tau^{\frac{1}{4}+\epsilon})$ or, more precisely, $\mathcal{O}\left(\tau^{1/4}\log\tau\right)$. The best known result, due to M. N. Huxley [Hux03], is that the remainder is $\mathcal{O}(\tau^{\alpha}\,(\log\tau)^{\beta+1})$, where

$$\alpha = \frac{131}{416} \sim 0,3149\ldots \quad \beta = \frac{18627}{8320} \sim 2,2513\ldots.$$

In order to have a spectral interpretation of the Dirichlet divisor problem, U. Battisti, T. Gramchev, S. Pilipović and L. Rodino introduced in [BGPR13] a global bisingular calculus based on Shubin calculus. Then, the Hermite-type operator

$$H_j = \frac{1}{2}\left(-\partial^2_{x_j} + x_j^2\right) + \frac{1}{2}, \quad j = 1, 2,$$

was considered. Using Hermite polynomials, one can check that $\sigma(H_j) = \{n\}_{n\in\mathbb{N}^*}$, $j = 1, 2$, all with multiplicity one. Therefore $\sigma(H_1 \otimes H_2) = \{n \cdot m\}_{(n,m)\in(\mathbb{N}^*\times\mathbb{N}^*)}$ and

$$N_{H_1\otimes H_2}(\tau) = D(\tau).$$

This spectral meaning of the Dirichlet divisor problem was one of the main motivations of the papers by U. Battisti, T. Gramchev, S. Pilipović and L. Rodino [BGPR13] and by T. Gramchev, S. Pilipović, L. Rodino and J. Vindas [GPRV14]. For the connection between Dirichlet divisor problem and standard bisingular operators on the product of closed manifolds, see also U. Battisti [Bat12].
The thesis is organized as follows. In Chapter 1 we recall the main definitions and properties of tensor products of operators, and we review the bisingular calculus on the product of two closed manifolds. In Chapter 2 we study the microlocal properties of bisingular operators. To do this, we define a suitable wave front set for such operators, called the *bi-wave front set*, which is the union of three components,

$$\mathrm{WF}_{\mathrm{bi}}(u) = \mathrm{WF}^1_{\mathrm{bi}}(u) \cup \mathrm{WF}^2_{\mathrm{bi}}(u) \cup \mathrm{WF}^{12}_{\mathrm{bi}}(u),$$

$u \in \mathcal{D}'(M_1 \times M_2)$, M_1, M_2 closed manifolds. This definition is formulated using the calculus only, and is related to the classical Hörmander wave front set $\mathrm{WF}_{\mathrm{cl}}$, cfr. L. Hörmander [Hör83], via the following inclusion

$$\mathrm{WF}_{\mathrm{cl}}(u) \cap (\Omega_1 \times \Omega_2 \times (\mathbb{R}^{n_1} \setminus 0) \times (\mathbb{R}^{n_2} \setminus 0)) \subset \mathrm{WF}^{12}_{\mathrm{bi}}(u).$$

The following Theorem 1 is the main result of Chapter 2.

Theorem 1. Let A be a bisingular operator, $u \in \mathcal{D}'(\Omega_1 \times \Omega_2)$. Then,

$$\mathrm{WF}_{\mathrm{bi}}(Au) \subset \mathrm{WF}_{\mathrm{bi}}(u).$$

Theorem 1 shows the bi-wave front set is microlocal with respect to bisingular operators. Then, we define an appropriate notion of characteristic set for a bisingular operator A, given again as a union of three components, namely

$$\mathrm{Char}_{\mathrm{bi}}(A) := \mathrm{Char}^1_{\mathrm{bi}}(A) \cup \mathrm{Char}^2_{\mathrm{bi}}(A) \cup \mathrm{Char}^{12}_{\mathrm{bi}}(A).$$

With this notion, we prove a microellipticity result for the 1- and 2-components of the bi-wave front set, which is the content of the next Theorem 2.

Theorem 2. Let A be a bisingular operator, $u \in \mathcal{D}'(\Omega_1 \times \Omega_2)$. Then,

$$\mathrm{WF}^j_{\mathrm{bi}}(u) \subseteq \mathrm{Char}^j_{\mathrm{bi}}(A) \cup \mathrm{WF}^j_{\mathrm{bi}}(Au), \qquad j = 1, 2.$$

In Chapter 3 we study the spectral asymptotics of the tensor product of two pseudodifferential operators $A_1 \otimes A_2$. Theorem 3 below is the main result of Chapter 3.

Theorem 3. Let M_1, M_2 be two closed manifolds of dimension n_1, n_2, respectively. Let $A = A_1 \otimes A_2$, where $A_j \in \mathrm{L}^{m_j}_{\mathrm{cl}}(M_j)$, $m_j > 0$, $j = 1, 2$, are positive, self-adjoint, invertible operators, with $\dfrac{n_1}{m_1} > \dfrac{n_2}{m_2}$. Then, for $\tau \to +\infty$,

$$N_A(\tau) = \begin{cases} \dfrac{C_1}{n_1}\zeta\left(A_2, \dfrac{n_1}{m_1}\right)\tau^{\frac{n_1}{m_1}} + \mathcal{O}\left(\tau^{\frac{n_1-1}{m_1}}\right) & \text{if } \dfrac{n_2}{m_2} < \dfrac{n_1-1}{m_1}, \\ \dfrac{C_1}{n_1}\zeta\left(A_2, \dfrac{n_1}{m_1}\right)\tau^{\frac{n_1}{m_1}} + \mathcal{O}\left(\tau^{\frac{n_1-1}{m_1}}\log\tau\right) & \text{if } \dfrac{n_2}{m_2} = \dfrac{n_1-1}{m_1}, \\ \dfrac{C_1}{n_1}\zeta\left(A_2, \dfrac{n_1}{m_1}\right)\tau^{\frac{n_1}{m_1}} + \mathcal{O}\left(\tau^{\frac{n_2}{m_2}}\right) & \text{if } \dfrac{n_2}{m_2} > \dfrac{n_1-1}{m_1}, \end{cases}$$

where ζ is the spectral ζ-function and

$$C_1 = \frac{1}{(2\pi)^{n_1}} \iint_{\mathbb{S}^*M_1} \frac{d\theta_1 dx_1}{[a_{m_1}(x_1,\theta_1)]^{\frac{n_1}{m_1}}}.$$

Moreover, using spherical harmonics, we show that the estimates in Theorem 3 are sharp. A similar statement holds for the tensor product of Shubin operators. In the aforementioned paper [GPRV14], T. Gramchev, S. Pilipović, L. Rodino and J. Vindas studied the same class of operators, and proved a slightly weaker estimate. Namely, under the same assumptions of Theorem 3, they showed that

$$N_A(\tau) = \frac{C_1}{n_1}\zeta\left(A_2, \frac{n_1}{m_1}\right)\tau^{\frac{n_1}{m_1}} + \mathcal{O}(\tau^\delta), \qquad \tau \to +\infty,$$

where $\max\left\{\frac{n_1-1}{m_1}, \frac{n_2}{m_2}\right\} < \delta < \frac{n_1}{m_1}$.

In Chapter 4, we consider the global bisingular operators based on Shubin pseudodifferential operators, introduced by U. Battisti, T. Gramchev, S. Pilipović and L. Rodino in [BGPR13]. In particular, we investigate the relationship between ellipticity and Fredholm property for such operators. As a consequence of the existence of parametrices to elliptic operators, elliptic operators act as Fredholm operators in a certain scale of naturally associated L^2-Sobolev spaces. The main result of Chapter 4 is the reverse statement, given in the next Theorem 4.

Theorem 4. Let $A \in \mathrm{G}^{0,0}_{\mathrm{cl}}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$, i.e. a classical global bisingular operator, induce a Fredholm operator from $L^2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ to itself. Then, A is elliptic.

Loosely speaking, this means that the ellipticity condition used in the calculus is “optimal”. The relationship between Fredholm property and ellipticity, in a quite general context of “abstract” pseudodifferential operators, has been studied, e.g., by J. Seiler in [Sei12].

Notation

$x = (x_1, ..., x_n)$ will denote a point in $\mathbb{R}^n$. If $x, y \in \mathbb{R}^n$, then

$$\begin{aligned} x \cdot y &= x_1 y_1 + \cdots + x_n y_n \\ |x| &= (x \cdot x)^{1/2} = (x_1^2 + \cdots + x_n^2)^{1/2} \\ \langle x \rangle &= \sqrt{1 + |x|^2} \\ dx &= dx_1 \cdots dx_n \\ đx &= (2\pi)^{-n}\, dx = (2\pi)^{-n}\, dx_1 \cdots dx_n \end{aligned}$$

Given $\mathbb{N} = \mathbb{Z}_+ = \{0, 1, 2, ...\}$, we define a multi-index α as

$$\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}_+^n.$$

If α and β are both multi-indices, then

$$\begin{aligned} |\alpha| &= \alpha_1 + \cdots + \alpha_n, \\ \beta \leq \alpha &\Leftrightarrow \text{for all } i, 1 \leq i \leq n : \beta_i \leq \alpha_i \\ \alpha \pm \beta &= (\alpha_1 \pm \beta_1, \cdots, \alpha_n \pm \beta_n), && (\beta \leq \alpha \text{ in } \alpha - \beta), \\ \binom{\alpha}{\beta} &= \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n}, && (\beta \leq \alpha), \\ \partial^\alpha &= \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}. \end{aligned}$$

If $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{Z}_+^n$, then

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

where, if $\alpha_j = 0$, we set $x_j^{\alpha_j} = 1$. We define the operator D^α, with $\alpha \in \mathbb{Z}_+^n$, as :

$$D^\alpha = i^{-|\alpha|}\, \partial^\alpha = \left(\frac{1}{i}\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{1}{i}\frac{\partial}{\partial x_n}\right)^{\alpha_n} = (-i\, \partial_1)^{\alpha_1} \cdots (-i\, \partial_n)^{\alpha_n}.$$

Let $u \in L^1(\mathbb{R}^n)$. The Fourier transform of u is

$$\hat{u}(\xi) = \mathcal{F}\{u\}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\, dx,$$

and the inverse Fourier transform of u is

$$\mathcal{F}^{-1}\{u\}(\xi) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} u(\xi)\, đ\xi = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} u(\xi)\, d\xi.$$

Let $f, g : X \to \mathbb{R}_+$. We write $f \prec g$ if there exists $C > 0$ such that

$$f(x) \leq C g(x)$$

for all $x \in X$. C is independent from x, but can possibly depend from various parameters which intervene in the definition of f and g. We write $f \sim g$ if $f \prec g$ and $g \prec f$.

Definition 0.0.1. Let A, B, C be three sets such that $C \subset A \times B$. Then,

$$C \circ B := \{a \in A : \text{ exists } b \in B : (a, b) \in C\},$$

that is, C is considered *as a relation acting on* B.

Let Ω be and open subset of $\mathbb{R}^n$. Then Ω_Δ is the diagonal in $\Omega \times \Omega$, defined as

$$\Omega_\Delta = \{(x, x) : x \in \Omega\}.$$

Let $\Gamma \subset \mathbb{R}^n \setminus 0$ be an open set containing x_0. Γ is called *conic* with respect to x_0 if $\lambda x_0 \in \Gamma$ for all $\lambda > 0$.
A manifold M is called *closed* if it is compact and without boundary.
We denote by $\mathbb{S}^n$ the n-dimensional sphere of radius 1, equipped with the metric induced by the standard Euclidean metric on $\mathbb{R}^{n+1}$, i.e.

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}.$$

Definition 0.0.2. Let $\Omega \subset \mathbb{R}^n$ an open set, and $m \in \mathbb{R}$. $\mathrm{S}^m(\Omega)$ is the set of all $a \in \mathcal{C}^\infty(\Omega \times \mathbb{R}^n)$ such that, for all multi-indices $\alpha, \beta \in \mathbb{Z}^n_+$ and for all compact subsets $K \subset \Omega$, there exists a constant $C_{\alpha,\beta,K} > 0$ such that

$$|D^\alpha_\xi D^\beta_x a(x, \xi)| \leq C_{\alpha,\beta,K} \langle\xi\rangle^{m-|\alpha|},$$

for all $x \in K,\ \xi \in \mathbb{R}^n$. An element of $\mathrm{S}^m(\Omega)$ is called a (Hörmander-type) symbol. A linear operator $A : \mathcal{C}^\infty_0(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n)$ is called a pseudodifferential operator if it can be written in the form

$$A(u)(x) = (\mathrm{Op}(a)[u])(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, a(x, \xi)\hat{u}(\xi)\, đ\xi,$$

where $a \in \mathrm{S}^m(\Omega)$ and $\hat{u}$ denotes the Fourier transform of u. $\mathrm{L}^m(\Omega)$ denotes the set of all pseudodifferential operators with symbol in $\mathrm{S}^m(\Omega)$. Moreover, we set

$$\mathrm{S}^\infty(\Omega) := \bigcup_m \mathrm{S}^m(\Omega),$$

$$\mathrm{S}^{-\infty}(\Omega) := \bigcap_m \mathrm{S}^m(\Omega),$$

and denote by $\mathrm{L}^{\infty}(\Omega)$, $\mathrm{L}^{-\infty}(\Omega)$, respectively, the corresponding class of operators. The operators in $\mathrm{L}^{-\infty}(\Omega)$ are called *smoothing operators*.

The class of operators in Definition 0.0.2 can be extended to operators on a closed manifold M. For all the details and properties of the Hörmander calculus of pseudodifferential operators we refer to [Hör85b, KG82]. Let $a, b \in \mathrm{S}^{\infty}(M)$. We write $a = b$ modulo $\mathrm{S}^{-\infty}$ if $a - b \in \mathrm{S}^{-\infty}(M)$. Let $A, B \in \mathrm{L}^{\infty}(M)$. We write $A = B$ modulo $\mathrm{L}^{-\infty}$ if $A - B \in \mathrm{L}^{-\infty}(M)$.

Chapter 1

Preliminaries

In this fist chapter we will give a brief survey on tensor products and bisingular operators.

1.1 Tensor products

We begin by recalling the well-known notion of tensor product for functions and distributions. First, we recall an important inequality which will be used throughout the whole thesis

Proposition 1.1.1 (Peetre's inequality). Let $x, y \in \mathbb{R}^n$. For any $s \in \mathbb{R}$ one has

$$\langle x \rangle^s \leq 2^{|s|} \langle x - y \rangle^{|s|} \langle y \rangle^s.$$

Proof. The triangular inequality gives

$$(1 + |x|) \leq (1 + |x - y| + |y|) \leq (1 + |x - y|)(1 + |y|),$$

so that

$$\langle x \rangle^2 \leq (1 + |x|)^2 \leq (1 + |x - y|)^2 (1 + |y|)^2.$$

On the other hand,

$$(1 + |y|)^2 \leq (1 + |y|)^2 + (1 - |y|)^2 = 2\langle y \rangle^2,$$

and estimating $(1 + |x - y|)^2$ in the same way, we thus get

$$\langle x \rangle^2 \leq 2^2 \langle x - y \rangle^2 \langle y \rangle^2.$$

If $s = 0$, the claim is obvious. If $s > 0$, we obtain our claim by raising the previous inequality to the power $\frac{s}{2}$. If $s < 0$, we exchange x and y to get

$$\langle y \rangle^{-s} \leq 2^{-s} \langle y - x \rangle^{-s} \langle x \rangle^{-s},$$

which can be rewritten

$$\langle x\rangle^s \le 2^{-s}\langle x-y\rangle^{-s}\langle y\rangle^s.$$

□

Definition 1.1.2. Let Ω_i, be an open subset of $\mathbb{R}^{n_i}$, and $f_i \in \mathcal{C}(\Omega_i)$, $i = 1, 2$. Then, the function $f_1 \otimes f_2 \in \mathcal{C}(\Omega_1 \times \Omega_2)$, $\Omega_1 \times \Omega_2 \subset \mathbb{R}^{n_1+n_2}$ defined by

$$(f_1 \otimes f_2)(x_1, x_2) := f_1(x_1)f_2(x_2),$$

$x_i \in \Omega_i$, $i = 1, 2$, is called the tensor product of f_1 and f_2.

To extend the previous definition to distributions, it is enough to observe that $f_1 \otimes f_2 \in \mathcal{C}(\Omega_1 \times \Omega_2)$, $\varphi_i \in \mathcal{C}_0^\infty(\Omega_i)$, $i = 1, 2$, imply

$$\int (f_1 \otimes f_2)(\varphi_1 \otimes \varphi_2)\, dx_1\, dx_2 = \left(\int u_1\varphi_1\, dx_1\right) \cdot \left(\int u_2\varphi_2\, dx_2\right).$$

We have the following

Theorem 1.1.3. Let $u_i \in \mathcal{D}'(\Omega_i)$, $i = 1, 2$. Then, there exists a unique $u \in \mathcal{D}'(\Omega_1 \times \Omega_2)$, denoted by $u_1 \otimes u_2$, such that

1. $(u_1 \otimes u_2)(\varphi_1 \otimes \varphi_2) = u_1(\varphi_1)u_2(\varphi_2)$, for all $\varphi_i \in \mathcal{C}_0^\infty(\Omega_i)$, $i = 1, 2$.
2. $(u_1 \otimes u_2)(\varphi) = u_1(u_2(\varphi(x_1, \cdot))) = u_2(u_1(\varphi(\cdot, x_2)))$, for all $\varphi \in \mathcal{C}_0^\infty(\Omega_1 \times \Omega_2)$.

The distribution $u_1 \otimes u_2$ is called the tensor product of u_1 and u_2. If u_i is in $\mathcal{E}'(\Omega_i)$, $i = 1, 2$, the same formulae are valid for $\varphi_i \in \mathcal{C}^\infty(\Omega_i)$, $i = 1, 2$, $\varphi \in \mathcal{C}^\infty(\Omega_1 \times \Omega_2)$.

Every function $f \in \mathcal{C}(\Omega_1 \times \Omega_2)$ defines an integral operator $A_f : \mathcal{C}_0(\Omega_2) \to \mathcal{C}(\Omega_1)$ by

$$A_f\, \varphi(x_1) := \int f(x_1, x_2)\varphi(x_2)\, dx_2 \quad \varphi \in \mathcal{C}_0(\Omega_2), x_1 \in \Omega_1.$$

This result can be extended to arbitrary distributions via the following Theorem:

Theorem 1.1.4 (The Schwartz kernel theorem). Every distribution $K_A \in \mathcal{D}'(\Omega_1 \times \Omega_2)$ defines a continuous linear map $A : \mathcal{C}_0^\infty(\Omega_2) \to \mathcal{D}'(\Omega_1)$ by

$$\langle A\varphi, \psi\rangle := \langle K_A, \psi \otimes \varphi\rangle \quad \psi \in \mathcal{C}_0^\infty(\Omega_1), \varphi \in \mathcal{C}_0^\infty(\Omega_2). \tag{1.1}$$

Conversely, for every such linear map A, there is one and only one distribution K_A such that (1.1) is valid. K_A is called the *(Schwartz) kernel* of A.

Proposition 1.1.5. Let $K_A \in \mathcal{D}'(\Omega_1 \times \Omega_2)$ be the kernel of of A. Then

$$\operatorname{supp} Au \subset \operatorname{supp} K_A \circ \operatorname{supp} u, \quad u \in \mathcal{C}_0^\infty(\Omega_2),$$

where $\operatorname{supp} K_A \subset \Omega_1 \times \Omega_2$ is considered as a relation acting on $\operatorname{supp} u \subset \Omega_2$.

Proof. From Definition 0.0.1, we have

$$\operatorname{supp} K_A \circ \operatorname{supp} u = \{x_1 \in \Omega_1 : \text{ exists } x_2 \in \operatorname{supp} u : (x_1, x_2) \in \operatorname{supp} K_A\}.$$

Notice that this is a closed set, since $\operatorname{supp} u$ is compact. Now assume that $x_1 \notin \operatorname{supp} K_A \circ \operatorname{supp} u$. Then we can find a neighbourhood V of x_1 such that

$$V \cap (\operatorname{supp} K_A \circ \operatorname{supp} u) = \emptyset.$$

If $v \in \mathcal{C}_0^\infty(V)$, then

$$\operatorname{supp}(v \otimes u) \cap \operatorname{supp} K_A = \emptyset,$$

which proves $\langle Au, v\rangle = 0$ by Theorem 1.1.4, i.e. $Au = 0$ in V, which is our claim. □

Theorem 1.1.6. Let $K_A \in \mathcal{C}^\infty(\Omega_1 \times \Omega_2)$. Then, the map A defined by (1.1) has a continuous extension from $\mathcal{E}'(\Omega_2)$ to $\mathcal{C}^\infty(\Omega_1)$ defined by

$$Au(x_1) := u(K_A(x_1 \cdot)), \quad u \in \mathcal{E}'(\Omega_1)\, x_1 \in \Omega_1.$$

Conversely, every continuous linear map $A : \mathcal{E}'(\Omega_2) \to \mathcal{C}^\infty(\Omega_1)$ is defined in this way by a kernel $K_A \in \mathcal{C}^\infty(\Omega_1 \times \Omega_2)$. We now give some examples.

Example 1.1.7. The kernel of the identity map $I : \mathcal{C}_0^\infty(\Omega) \to \mathcal{C}_0^\infty(\Omega)$, Ω open subset in $\mathbb{R}^n$, is the distribution

$$\langle K_I, \varphi\rangle = \int \varphi(x, x)\, dx, \quad \varphi \in \mathcal{C}_0^\infty(\Omega \times \Omega), \tag{1.2}$$

with support in Ω_Δ, the diagonal in $\Omega \times \Omega$.

Example 1.1.8. Let $f : \Omega_1 \to \Omega_2$ be a continuous map and $A\psi := \psi \circ f$, $\psi \in \mathcal{C}_0^\infty(\Omega_2)$. Then the kernel is given by

$$\langle K_A, \varphi\rangle = \int \varphi(x, f(x))\, dx, \quad \varphi \in \mathcal{C}_0^\infty(\Omega \times \Omega), \tag{1.3}$$

with support in the graph of f.

Example 1.1.9. Let A be a pseudodifferential operator in $\mathrm{L}^m(\Omega)$, Ω open subset in $\mathbb{R}^n$. Then, the kernel K_A is the distribution

$$K_A(x, y) = \int e^{i(x-y)\cdot\xi}\, a(x, \xi)\, đ\xi,$$

where $a \in \mathrm{S}^m(\Omega)$ is the symbol of A.

The kernel K_A from example 1.1.9 has one important property, given in the next result.

Proposition 1.1.10. Let $A \in \mathrm{L}^m(\Omega)$, and Ω_Δ the diagonal in $\Omega \times \Omega$. Then $K_A \in \mathcal{C}^\infty((\Omega \times \Omega) \setminus \Omega_\Delta)$.

Remark 1.1.11. From Example 1.1.9 and Proposition 1.1.10 it follows that operators in $\mathrm{L}^{-\infty}(M)$, where M is a closed manifold, have a $\mathcal{C}^\infty$ kernel.

Definition 1.1.12. A map $f : B \to C$ is called *proper* if, for every compact $K \subset C$, $f^{-1}(K) \subset B$ is a compact set. An operator $A \in \mathrm{L}^m(\Omega)$, Ω open subset in $\mathbb{R}^n$, is *properly supported* if both its standard projections π_1, π_2 : $\mathrm{supp}(K_A) \to \Omega$ are proper maps.

We conclude the section with an important property:

Proposition 1.1.13. Let A_i be a boundend linear operator with spectrum $\sigma(A_i)$, $i = 1, 2$. Then $\sigma(A_1 \otimes A_2) = \sigma(A_1)\sigma(A_2)$.

1.2 Bisingular operators

In the previous Section we recalled the basic properties of tensor products of operators. We now study a more general class of operators on the product of two closed manifolds, originally introduced by L. Rodino in [Rod75], which includes tensor products as a special case. Here, Ω_i, $i = 1, 2$, denote open domains of $\mathbb{R}^{n_i}$.

Definition 1.2.1. $\mathrm{S}^{m_1,m_2}(\Omega_1, \Omega_2)$ is the set of all $a \in \mathcal{C}^\infty(\Omega_1 \times \Omega_2 \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ such that, for all multi-indices $\alpha_i, \beta_i \in \mathbb{Z}^n_+$ and for all compact subsets $K_i \subset \Omega_i$, $i = 1, 2$, there exists a constant $C_{\alpha_1,\alpha_2,\beta_1,\beta_2,K_1,K_2} > 0$ such that

$$|D^{\alpha_1}_{\xi_1} D^{\alpha_2}_{\xi_2} D^{\beta_1}_{x_1} D^{\beta_2}_{x_2} a(x_1, x_2, \xi_1, \xi_2)| \leq C_{\alpha_1,\alpha_2,\beta_1,\beta_2,K_1,K_2} \langle \xi_1 \rangle^{m_1-|\alpha_1|} \langle \xi_2 \rangle^{m_2-|\alpha_2|},$$

for all $x_i \in K_i$, $\xi_i \in \mathbb{R}^{n_i}$, $i = 1, 2$. An element of $\mathrm{S}^{m_1,m_2}(\Omega_1, \Omega_2)$ is called a bi-symbol.

Definition 1.2.2. A linear operator $A : \mathcal{C}^\infty_0(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \to \mathcal{C}^\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is called a bisingular operator if it can be written in the form

$$\begin{aligned} A(u)(x_1, x_2) &= (\mathrm{Op}(a)[u])(x_1, x_2) \\ &= \frac{1}{(2\pi)^{n_1+n_2}} \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} e^{i(x_1 \cdot \xi_1 + x_2 \cdot \xi_2)}\, a(x_1, x_2, \xi_1, \xi_2) \hat{u}(\xi_1, \xi_2)\, d\xi_1 d\xi_2, \end{aligned}$$

where $a \in \mathrm{S}^{m_1,m_2}(\Omega_1, \Omega_2)$ and $\hat{u}$ denotes the Fourier transform of u.
Here and in the sequel we denote by the same letters the elements of $\mathcal{C}^\infty_0(\Omega_1 \times \Omega_2)$ and their trivial extension outside $\Omega_1 \times \Omega_2$ to elements of $\mathcal{C}^\infty_0(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.

$L^{m_1,m_2}(\Omega_1, \Omega_2)$ denotes the set of all bisingular operators with bi-symbol in $S^{m_1,m_2}(\Omega_1, \Omega_2)$. Moreover, we set

$$S^{\infty,\infty}(\Omega_1, \Omega_2) := \bigcup_{m_1,m_2} S^{m_1,m_2}(\Omega_1, \Omega_2),$$
$$S^{-\infty,-\infty}(\Omega_1, \Omega_2) := \bigcap_{m_1,m_2} S^{m_1,m_2}(\Omega_1, \Omega_2),$$

and denote by $L^{\infty,\infty}(\Omega_1, \Omega_2)$, $L^{-\infty,-\infty}(\Omega_1, \Omega_2)$, respectively, the corresponding class of operators. The operators in $L^{-\infty,-\infty}(\Omega_1, \Omega_2)$ are called *smoothing operators.*

A simple and fundamental example of a bisingular operator is the tensor product $A_1 \otimes A_2$ of two pseudodifferential operators, with symbols in the Hörmander class, $A_i \in L^{m_i}(\Omega_i)$, $i = 1, 2$, while more complex examples include the vector-tensor product $A_1 \boxtimes A_2$ studied in [Rod75], and the double Cauchy integral operator studied in [NR06].
We associate with every $a \in S^{m_1,m_2}(\Omega_1, \Omega_2)$ the two maps

$$A^1 : \Omega_1 \times \mathbb{R}^{n_1} \to L^{m_2}(\Omega_2) : (x_1, \xi_1) \mapsto a(x_1, x_2, \xi_1, D_2),$$
$$A^2 : \Omega_2 \times \mathbb{R}^{n_2} \to L^{m_1}(\Omega_1) : (x_2, \xi_2) \mapsto a(x_1, x_2, D_1, \xi_2).$$

For $a \in S^{m_1,m_2}(\Omega_1, \Omega_2)$, $b \in S^{p_1,p_2}(\Omega_1, \Omega_2)$ we also set, for fixed (x_1, ξ_1), (x_2, ξ_2), respectively,

$$a \circ_1 b(x_1, x_2, \xi_1, D_2) := (A^1 \circ B^1)(x_1, x_2, \xi_1, D_2) \in L^{m_2+p_2}(\Omega_2),$$
$$a \circ_2 b(x_1, x_2, D_1, \xi_2) := (A^2 \circ B^2)(x_1, x_2, D_1, \xi_2) \in L^{m_1+p_1}(\Omega_1).$$

Remark 1.2.3. In view of the definitions of $a \circ_1 b$ and $a \circ_2 b$, the bisingular calculus can also be considered a form of calculus with vector-valued symbols. General vector-valued calculi have been deeply studied, for example, by B.-W. Schulze in [Sch98].

Definition 1.2.4. Let $a \in S^{m_1,m_2}(\Omega_1, \Omega_2)$. Then, a has a homogeneous principal bi-symbol if

i) there exists $a_{m_1;\cdot} \in S^{m_1,m_2}(\Omega_1, \Omega_2)$ such that

$$a_{m_1;\cdot}(x_1, x_2, t\xi_1, \xi_2) = t^{m_1} a_{m_1;\cdot}(x_1, x_2, \xi_1, \xi_2),\ \forall x_1, x_2, \xi_2,\ \forall |\xi_1| > 1, t > 0,$$
$$a - \psi_1(\xi_1) a_{m_1;\cdot} \in S^{m_1-1,m_2},$$

where ψ_1 is an 0-excision function; moreover, $a_{m_1;\cdot}(x_1, x_2, \xi_1, D_2)$ belongs to $L^{m_2}_{cl}(\Omega_2)$[1];

[1] Being a classical symbol on Ω_2, it admits an asymptotic expansion in homogeneous terms with respect to the ξ_2 variable.

ii) there exists $a_{\cdot;m_2} \in \mathrm{S}^{m_1,m_2}(\Omega_1,\Omega_2)$ such that

$$\begin{aligned}&a_{\cdot;m_2}(x_1,x_2,\xi_1,t\xi_2) = t^{m_2} a_{\cdot;m_2}(x_1,x_2,\xi_1,\xi_2),\ \forall x_1,x_2,\xi_1,\ \forall|\xi_2|>1, t>0,\\ &\quad a-\psi_2(\xi_2)a_{\cdot;m_2} \in \mathrm{S}^{m_1,m_2-1},\end{aligned}$$

where ψ_2 is an 0-excision function; moreover, $a_{\cdot;m_2}(x_1,x_2,D_1,\xi_2)$ belongs to $\mathrm{L}^{m_1}_{\mathrm{cl}}(\Omega_1)$;

iii) the symbols $a_{m_1;\cdot}$ and $a_{\cdot;m_2}$ have the same leading term, that is there exists $a_{m_1;m_2}$ such that

$$\begin{aligned}a_{m_1;\cdot} - \psi_2(\xi_2)a_{m_1;m_2} &\in \mathrm{S}^{m_1,m_2-1}(\Omega_1,\Omega_2),\\ a_{\cdot;m_2} - \psi_1(\xi_1)a_{m_1;m_2} &\in \mathrm{S}^{m_1-1,m_2}(\Omega_1,\Omega_2),\end{aligned}$$

and

$$a - \psi_1 a_{m_1;\cdot} - \psi_2 a_{\cdot;m_2} + \psi_1\psi_2 a_{m_1;m_2} \in \mathrm{S}^{m_1-1,m_2-1}(\Omega_1,\Omega_2).$$

The bi-symbols which admit a full bi-homogeneous expansion in ξ_1 and ξ_2 are called *classical bi-symbols*, their class is denoted by $\mathrm{S}^{m_1,m_2}_{\mathrm{cl}}(\Omega_1,\Omega_2)$, and the corresponding operator class by $\mathrm{L}^{m_1,m_2}_{\mathrm{cl}}(\Omega_1,\Omega_2)$. For more details on the classical bi-symbols, we refer to [NR06].

The previous Definition 1.2.4 implies that, given $A \in \mathrm{L}^{m_1,m_2}_{\mathrm{cl}}(\Omega_1,\Omega_2)$, we can introduce maps σ^1, σ^2, σ^{12} as

$$\begin{aligned}\sigma^1(A) &: T^*\Omega_1\setminus 0 \to \mathrm{L}^{m_2}_{\mathrm{cl}}(\Omega_2) : (x_1,\xi_1)\mapsto a_{m_1;\cdot}(x_1,x_2,\xi_1,D_2),\\ \sigma^2(A) &: T^*\Omega_2\setminus 0 \to \mathrm{L}^{m_1}_{\mathrm{cl}}(\Omega_1) : (x_2,\xi_2)\mapsto a_{\cdot;m_2}(x_1,x_2,D_1,\xi_2),\\ \sigma^{12}(A) &: (T^*\Omega_1\setminus 0)\times(T^*\Omega_2\setminus 0) \to \mathbb{C} : (x_1,x_2,\xi_1,\xi_2)\mapsto a_{m_1;m_2}(x_1,x_2,\xi_1,\xi_2).\end{aligned}$$

In this way, denoting by $\sigma(P)(x,\xi)$ the principal symbol of a pseudodifferential operator P, we have

$$\begin{aligned}\sigma(\sigma^1(A)(x_1,\xi_1))(x_2,\xi_2) &= \sigma(\sigma^2(A)(x_2,\xi_2))(x_1,\xi_1)\\ &= \sigma^{12}(A)(x_1,x_2,\xi_1,\xi_2) = a_{m_1;m_2}(x_1,x_2,\xi_1,\xi_2).\end{aligned}$$

We call the couple $(\sigma^1(A),\sigma^2(A))$ the *principal bi-symbol* of A.
In the sequel, we only consider bisingular operators on the product of two closed manifolds Ω_1,Ω_2. They are defined as above in local coordinates. For such operators, there exists a notion of ellipticity, called bi-ellipticity (for more details, see again [Rod75]).

Definition 1.2.5. Let $A \in \mathrm{L}^{m_1,m_2}_{\mathrm{cl}}(\Omega_1,\Omega_2)$. We say that A is bi-elliptic if

i) $\sigma^{12}(A)(v_1,v_2) \neq 0$ for all $(v_1,v_2)\in(T^*\Omega_1\setminus 0)\times(T^*\Omega_2\setminus 0)$;

ii) $\sigma^1(A)(v_1)$ is invertible as an operator in $\mathrm{L}^{m_2}_{\mathrm{cl}}(\Omega_2)$ for all $v_1 \in T^*\Omega_1 \setminus 0$, with inverse in $\mathrm{L}^{-m_2}_{\mathrm{cl}}(\Omega_2)$;

iii) $\sigma^2(A)(v_2)$ is invertible as an operator in $\mathrm{L}^{m_1}_{\mathrm{cl}}(\Omega_1)$ for all $v_2 \in T^*\Omega_2 \setminus 0$, with inverse in $\mathrm{L}^{-m_1}_{\mathrm{cl}}(\Omega_1)$.

Basic examples of classical bi-singular operators are described below.

Example 1.2.6. Consider the differential operator

$$A = \sum_{\substack{|\beta_1|\le m_1\\ |\beta_2|\le m_2}} c_{\beta_1,\beta_2}(x_1,x_2) D_1^{\beta_1} D_2^{\beta_2},$$

where $\beta_1, \beta_2 \in \mathcal{C}^\infty(\Omega_1 \times \Omega_2)$. In this case

$$\sigma^1(A)(x_1,\xi_1) = \sum_{\substack{|\beta_1|= m_1\\ |\beta_2|\le m_2}} c_{\beta_1,\beta_2}(x_1,x_2) \xi_1^{\beta_1} D_2^{\beta_2}, \tag{1.4}$$

$$\sigma^2(A)(x_2,\xi_2) = \sum_{\substack{|\beta_1|\le m_1\\ |\beta_2|= m_2}} c_{\beta_1,\beta_2}(x_1,x_2) D_1^{\beta_1} \xi_2^{\beta_2}. \tag{1.5}$$

A full bi-homogeneous expansion is given by

$$\tilde{\sigma}^{i,j}(A)(x_1,x_2,\xi_1,\xi_2) = \sum_{\substack{|\beta_1|= i\\ |\beta_2|= j}} c_{\beta_1,\beta_2}(x_1,x_2) \xi_1^{\beta_1} \xi_2^{\beta_2}.$$

The bi-ellipticity of A is given by the condition $\sigma^{12}(A) = \tilde{\sigma}^{m_1,m_2}(A)(v_1,v_2) \neq 0$ for all $(v_1,v_2) \in (T^*\Omega_1\setminus 0)\times(T^*\Omega_2\setminus 0)$ and the invertibility of the two maps (1.4) and (1.5). We may give a global meaning to A on a product of closed manifolds $\Omega_1 \times \Omega_2$ by taking, for example, $\Omega_j = \mathbb{T}_j$, the n_j-dimensional torus, $j = 1, 2$, and x_j angular coordinates on $\mathbb{T}_j$.

Example 1.2.7. A simple example of a bisingular operator which is not a pseudodifferential operator is

$$A = (-\Delta_1 + I)^{-1} \otimes (-\Delta_2 + I)^{-1},$$

where $-\Delta_i$ is the usual Laplacian on Ω_i, $i = 1, 2$. To see this, we notice that

$$K_{(-\Delta_1+I)^{-1}}(x_1,y_1) = \int e^{i(x_1-y_1)\cdot\xi_1} \frac{1}{1+|\xi_1|^2}\, đ\xi.$$

Therefore, we can find $\bar{x}_1, \bar{y}_1, \bar{x}_1 \neq \bar{y}_1$, such that $K_{(-\Delta_1+I)^{-1}}(\bar{x}_1,\bar{y}_1) \neq 0$. Since

$$K_A(x_1,y_1,x_2,y_2) = K_{(-\Delta_1+I)^{-1}}(x_1,y_1) \otimes K_{(-\Delta_2+I)^{-1}}(x_2,y_2),$$

we have that

$$(\bar{x}_1, \bar{y}_1, x_2, x_2) \in \operatorname{singsupp} K_A,$$

for every $x_2 \in \Omega_2$, but $(\bar{x}_1, \bar{y}_1, x_2, x_2) \notin (\Omega_1 \times \Omega_2)_\Delta$. Thus, according to Proposition 1.1.10, A cannot be a pseudodifferential operator.

However, notice that all pseudodifferential operators of order zero or lower, in particular those corresponding to cut-offs and excision functions, are bisingular operators:

Proposition 1.2.8. $\mathrm{S}^0(\Omega_1 \times \Omega_2) \subset \mathrm{S}^{0,0}(\Omega_1, \Omega_2)$.

Proof. Let $a \in \mathrm{S}^0(\Omega_1 \times \Omega_2)$. Then for all pairs of multi-indices $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$ we have

$$|D^{\alpha_1}_{\xi_1} D^{\alpha_2}_{\xi_2} D^{\beta_1}_{x_1} D^{\beta_2}_{x_2} a(x_1, x_2, \xi_1, \xi_2)| = |D^\alpha_\xi D^\beta_x a(x, \xi)| \prec \langle \xi \rangle^{-|\alpha|}$$
$$\prec \langle \xi \rangle^{-|\alpha_1|} \langle \xi \rangle^{-|\alpha_2|} \prec \langle \xi_1 \rangle^{-|\alpha_1|} \langle \xi_2 \rangle^{-|\alpha_2|},$$

that is, $a \in \mathrm{S}^{0,0}(\Omega_1, \Omega_2)$. □

With this in mind, it is possible to study some model cases of operators of the form $A \otimes B$, and compare the notion of bi-ellipticity with the standard notion of ellipticity for pseudodifferential operators. In the following Table 1.1 we mean by ΨDO the classical pseudodifferential operators on $\Omega_1 \times \Omega_2$, and by ΨDO-order and ΨDO-ell. their order and ellipticity, respectively.

Operator	ΨDO-order	ΨDO-ell.	Bi-order	Bi-ell.
$I \otimes I$	0	√	$(0,0)$	√
$-\Delta_1 \otimes I + I \otimes (-\Delta_2)$	2	√	$(2,2)$	×
$-\Delta_1 \otimes (-\Delta_2)$	4	×	$(2,2)$	×
$(-\Delta_1 + I) \otimes (-\Delta_2)$	4	×	$(2,2)$	×
$(-\Delta_1 + I) \otimes (-\Delta_2 + I)$	4	×	$(2,2)$	√
$(-\Delta_1 + I)^{-1} \otimes (-\Delta_2 + I)^{-1}$	not a ΨDO		$(-2,-2)$	√

Table 1.1: Some model cases of bisingular operators of tensor product type

As in the case of the standard pseudodifferential calculus, it is possible to define a scale of adapted Sobolev spaces and prove some continuity results.

Definition 1.2.9. The Sobolev space of exponent (s_1, s_2), $s_1, s_2 \in \mathbb{R}$, is the space

$$\mathrm{H}^{s_1,s_2}(\Omega_1\times\Omega_2) = \mathrm{H}^{s_1,s_2} := \{u \in \mathcal{D}'(\Omega_1\times\Omega_2) : \mathrm{Op}(\langle\xi_1\rangle^{s_1}\langle\xi_2\rangle^{s_2})u \in L^2(\Omega_1\times\Omega_2)\}.$$

If $u \in \mathrm{H}^{s_1,s_2}$ then

$$\|u\|_{s_1,s_2} := \|\mathrm{Op}(\langle\xi_1\rangle^{s_1}\langle\xi_2\rangle^{s_2})u\|_2.$$

Similarly to weighted SG Sobolev spaces, the following result holds true, see [Cor95],

Proposition 1.2.10. If $s_1 \geq s_1'$, $s_2 \geq s_2'$, we have

$$\mathrm{H}^{s_1,s_2}(\Omega_1 \times \Omega_2) \subset \mathrm{H}^{s_1',s_2'}(\Omega_1 \times \Omega_2)$$

and the embedding $\mathrm{H}^{s_1,s_2}(\Omega_1 \times \Omega_2) \hookrightarrow \mathrm{H}^{s_1',s_2'}(\Omega_1 \times \Omega_2)$ is continuous. If $s_1 > s_1'$, $s_2 > s_2'$, the embedding is compact.

Proposition 1.2.11. A bisingular operator $A \in \mathrm{L}^{m_1,m_2}(\Omega_1, \Omega_2)$ extends to a continuous operator

$$A : \mathrm{H}^{s_1,s_2}(\Omega_1 \times \Omega_2) \to \mathrm{H}^{s_1-m_1,s_2-m_2}(\Omega_1 \times \Omega_2),$$

for every $s_1, s_2 \in \mathbb{R}$.

Theorem 1.2.12. Let $A \in \mathrm{L}^{m_1,m_2}_{\mathrm{cl}}(\Omega_1, \Omega_2)$ be bi-elliptic. Then, there exists $B \in \mathrm{L}^{-m_1,-m_2}_{\mathrm{cl}}(\Omega_1, \Omega_2)$ such that

$$AB = I + K_1, \qquad BA = I + K_2,$$

where I is the identity map and K_1, K_2 are smoothing operators. Moreover, the principal bi-symbol of B is $(\sigma^1(A)^{-1}, \sigma^2(A)^{-1})$.

From now on we will assume, for simplicity, that bi-symbols of bisingular operators have compact support with respect to the x_1, x_2 variables.

Theorem 1.2.13. Let $a \in \mathrm{S}^{m_1,m_2}(\Omega_1, \Omega_2), b \in \mathrm{S}^{p_1,p_2}(\Omega_1, \Omega_2)$. Then $AB \in \mathrm{L}^{m_1+p_1,m_2+p_2}(\Omega_1, \Omega_2)$, and its bi-symbol $c(x_1, x_2, \xi_1, \xi_2)$ has the asymptotic expansion

$$c \sim \sum_{j=0}^{\infty} c_{m_1+p_1-j,m_2+p_2-j}$$

where

$$\begin{aligned} c_{m_1+p_1-j,m_2+p_2-j} = {} & c^1_{m_1+p_1-j-1,m_2+p_2-j} + c^2_{m_1+p_1-j,m_2+p_2-j-1} \\ & + c^{12}_{m_1+p_1-j,m_2+p_2-j} \end{aligned}$$

and

$$c^1{}_{m_1+p_1-j-1,m_2+p_2-j}$$
$$= \sum_{|\alpha_2|=j} \frac{1}{\alpha_2!} \left\{ \partial_{\xi_2}^{\alpha_2} a \circ_1 D_{x_2}^{\alpha_2} b - \sum_{|\alpha_1|\le j} \frac{1}{\alpha_1!} \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} a \, D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} b \right\}$$
$$c^2{}_{m_1+p_1-j,m_2+p_2-j-1}$$
$$= \sum_{|\alpha_1|=j} \frac{1}{\alpha_1!} \left\{ \partial_{\xi_1}^{\alpha_1} a \circ_2 D_{x_1}^{\alpha_1} b - \sum_{|\alpha_2|\le j} \frac{1}{\alpha_2!} \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} a \, D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} b \right\}$$
$$c^{12}{}_{m_1+p_1-j,m_2+p_2-j}$$
$$= \sum_{|\alpha_1|=|\alpha_2|=j} \frac{1}{\alpha_1!\alpha_2!} \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} a \, D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} b$$

Corollary 1.2.14. Let $a \in \mathrm{S}^{m_1,m_2}(\Omega_1,\Omega_2), b \in \mathrm{S}^{p_1,p_2}(\Omega_1,\Omega_2)$. Then the commutator $[A,B] := AB - BA$ belongs to $\mathrm{L}^{m_1+p_1-1,m_2+p_2} + L^{m_1+p_1,m_2+p_2-1}$.

Proof. By Theorem 1.2.13, we find, as top order order terms ($j = 0$),

$$\begin{aligned} c^1_{m_1+p_1-1,m_2+p_2} &= a \circ_1 b - b \circ_1 a \\ c^2_{m_1+p_1,m_2+p_2-1} &= a \circ_2 b - b \circ_2 a \\ c^{12}_{m_1+p_1,m_2+p_2} &= 0. \end{aligned}$$

Then, expanding c^1 and c^2 according to the definition of $\circ_j$, $j = 1, 2$, we get

$$c^j = i\{a,b\}_j + \text{terms of order } (m_j + p_j - 2),$$

where with $\{a,b\}_j$ we denote the Poisson bracket of a and b in the j-argument. Therefore, the leading terms (up to order (m_1+p_1-2, m_2+p_2-2)) of the expansion of c can be written as

$$c = 0 + i(\{a,b\}_1 + \{a,b\}_2) \in \mathrm{S}^{m_1+p_1-1,m_2+p_2} + \mathrm{S}^{m_1+p_1,m_2+p_2-1}.$$

□

Remark 1.2.15. This behaviour under commutators is indeed something peculiar about bisingular calculus. For standard pseudodifferential calculus, $A \in \mathrm{L}^m(\Omega), B \in \mathrm{L}^p(\Omega)$ implies $[A,B] \in \mathrm{L}^{m+p-1}(\Omega)$, $\Omega \subset \mathbb{R}^n$ open.

Example 1.2.16. For a better understanding of this phenomenon, consider the model case of a tensor product where $A = A_1 \otimes A_2 \in \mathrm{L}^{m_1,m_2}$, $B = B_1 \otimes B_2 \in \mathrm{L}^{p_1,p_2}$. Then

$$\begin{aligned} [A,B] = [A_1 \otimes A_2, B_1 \otimes B_2] &= A_1B_1 \otimes A_2B_2 - B_1A_1 \otimes B_2A_2 \\ &= \underbrace{[A_1,B_1] \otimes A_2B_2}_{\mathrm{L}^{m_1+p_1-1,m_2+p_2}} + \underbrace{B_1A_1 \otimes [A_2,B_2]}_{\mathrm{L}^{m_1+p_1,m_2+p_2-1}}. \end{aligned}$$

Remark 1.2.17. Here, we deal with bisingular operators whose symbols follow Hörmander-type estimates (see e.g. [Hör85b]). In particular, in Chapter 2 we only study operators on closed manifolds, given explicitly in local coordinates. However a global version of bisingular calculus was defined by U. Battisti, T. Gramchev, S. Pilipović and L. Rodino in [BGPR13]. We will study this class more precisely in Chapter 4. We also notice that 'product-type' operators calculi, similar to bisingular calculus, were introduced by V. S. Pilidi [Pil73], R. V. Dudučava [Dud79a], [Dud79b], and, more recently, by R. Melrose and F. Rochon [MR06]. Moreover, multisingular calculi were considered by V. S. Pilidi [Pil71] and L. Rodino [Rod80].

Chapter 2

Wave front sets

In this chapter we will give a summary of different kinds of wave front sets, and define the bi-wave front set. Throughout this chapter, all the pseudodifferential operators are assumed properly supported.

2.1 The classical wave front set

Let $\Omega \subset \mathbb{R}^n$ be open and $u \in \mathcal{D}'(\Omega)$. The *singular support* of u is the set

$$\operatorname{sing\,supp}(u) = \Omega \setminus \{x \in \Omega : \exists \varphi \in \mathcal{C}_0^\infty(\Omega),\, \varphi(x) \neq 0,\, \varphi u \in \mathcal{C}^\infty(\Omega)\}.$$

From its definition, it is immediate that $\operatorname{sing\,supp}(u) \subset \Omega$ is closed. Now let $u \in \mathcal{D}'(\Omega)$ and $\varphi \in \mathcal{C}_0^\infty(\Omega)$. From the Paley-Wiener-Schwartz Theorem, we know that

$$|\widehat{\varphi u}(\xi)| \lesssim \langle \xi \rangle^N, \quad \xi \in \mathbb{R}^n,$$

for some $N \geq 0$ (depending on φ). Moreover, $\varphi u \in \mathcal{C}_0^\infty(\Omega)$ if and only if

$$|\widehat{\varphi u}(\xi)| \lesssim \langle \xi \rangle^{-N}, \quad \xi \in \mathbb{R}^n, \quad \forall N \geq 0. \tag{2.1}$$

So, the rapid decay of the Fourier trasform of u localized at $x_0 \in \Omega$ is equivalent to the smoothness of u in a neighborhood of x_0. The idea behind the *wave front set* is to refine the singular support, in the sense of finding the directions in the space of frequencies where rapid decay is lacking, i.e. directions which cause a singularity to appear in the singular support.

Definition 2.1.1. Let $u \in \mathcal{D}'(\Omega)$. The distribution u is microlocally $\mathcal{C}^\infty$ near $(x_0, \xi_0) \in \mathrm{T}^*\Omega \setminus 0$ if, there exists a cut-off $\varphi \in \mathcal{C}_0^\infty(\Omega)$ with $\varphi \equiv 1$ in an open set containing x_0 such that

$$|\widehat{\varphi u}(\xi)| \leq C_N \langle \xi \rangle^{-N} \quad \forall \xi \in \Gamma,\ |\xi| > R,$$

$\forall\, N \in \mathbb{N}$, where $\Gamma \subset \mathbb{R}^n \setminus 0$ is a conic open set containing ξ_0 and $C_N,\ R > 0$. The classical wave front set of $u \in \mathcal{D}'(\Omega)$, that we denote by $\mathrm{WF}_{\mathrm{cl}}(u)$, is the complement of the set of points where u is microlocally $\mathcal{C}^\infty$.

It is immediate to observe that $\mathrm{WF}_{\mathrm{cl}}(u)$ is closed is $\Omega \times (\mathbb{R}^n \setminus 0)$ and conic, in the sense that

$$(x, \xi) \in \mathrm{WF}_{\mathrm{cl}}(u) \Rightarrow (x, \lambda\xi) \in \mathrm{WF}_{\mathrm{cl}}(u)$$

for all $\lambda > 0$. Moreover, it is also easy to check that if $u, v \in \mathcal{D}'(\Omega), f \in \mathcal{C}^\infty(\Omega)$, we have

$$\begin{aligned} \mathrm{WF}_{\mathrm{cl}}(u+v) &\subseteq \mathrm{WF}_{\mathrm{cl}}(u) \cup \mathrm{WF}_{\mathrm{cl}}(v), \\ \mathrm{WF}_{\mathrm{cl}}(fu) &\subseteq \mathrm{WF}_{\mathrm{cl}}(u). \end{aligned}$$

Definition 2.1.2. Let $\xi_0 \in \mathbb{R}^n \setminus 0$. A *conic neighbourhood* of ξ_0 is an open neighbourhood $V \subset \mathbb{R}^n \times \mathbb{R}^n$ that contains ξ_0 and is conic with respect to the second variable, i.e. contains all the points of the form $(x, \lambda\xi_0)$, for all $\lambda > 0$.

Lemma 2.1.3. Let $\Gamma, \Gamma' \subset \mathbb{R}^n \setminus 0$ open conic sets such that $\overline{\Gamma' \cap \mathbb{S}^n} \subset \Gamma$, and let $f \in \mathscr{S}(\mathbb{R}^n)$. Then for all $g \in \mathscr{S}(\mathbb{R}^n)$ we have $f * g \in \mathscr{S}(\Gamma')$, i.e.

$$\sup_{\xi \in \Gamma'} \langle \xi \rangle^N |(f * g)(\xi)| \prec 1 \quad \text{for all } N \geq 0.$$

Proof. Let $N \geq 0$. Then

$$\begin{aligned} &|(f*g)(\xi)| \\ &\leq \int_{\langle\eta\rangle \leq \langle\xi\rangle^{1/2}} |f(\xi-\eta)|\,|g(\eta)|\,d\eta + \int_{\langle\eta\rangle > \langle\xi\rangle^{1/2}} |f(\xi-\eta)|\,|g(\eta)|\,d\eta \\ &\prec \int_{\langle\eta\rangle \leq \langle\xi\rangle^{1/2}} \langle \xi-\eta\rangle^{-N}\,|g(\eta)|\,d\eta + \int_{\langle\eta\rangle > \langle\xi\rangle^{1/2}} \langle \xi-\eta\rangle^{k}\,|g(\eta)|\,d\eta \\ &\prec \langle\xi\rangle^{-N} \int_{\mathbb{R}^n} \langle\eta\rangle^N\,|g(\eta)|\,d\eta + \langle\xi\rangle^k \int_{\langle\eta\rangle > \langle\xi\rangle^{1/2}} \langle\eta\rangle^{-2(N+k)} \langle\eta\rangle^{2N+3k}\,|g(\eta)|\,d\eta \\ &\prec \langle\xi\rangle^{-N} + \langle\xi\rangle^{k-N-k} \int_{\mathbb{R}^n} \langle\eta\rangle^{2N+3k}\,|g(\eta)|\,d\eta \\ &\prec \langle\xi\rangle^{-N}, \end{aligned}$$

where we used that if $\xi \in \Gamma'$ and $\langle\eta\rangle \leq \langle\xi\rangle^{1/2}$, then $(\xi - \eta) \in \Gamma$ for sufficiently large $|\xi|$. □

Proposition 2.1.4. If $\pi : \Omega \times (\mathbb{R}^n \setminus 0) \mapsto \Omega$ is the natural projection, then

$$\pi\big(\mathrm{WF}_{\mathrm{cl}}(u)\big) = \operatorname{sing\,supp}(u).$$

Proof. If $x_0 \notin \operatorname{sing\,supp}(u)$, choose $\varphi \in \mathcal{C}_0^\infty(\Omega)$ such that $\varphi \equiv 1$ in a neighbourhood of x_0, $\varphi \equiv 0$ in a neighbourhood of $\operatorname{sing\,supp}(u)$. Then, $\varphi u \in \mathcal{C}_0^\infty(\Omega)$, from which $\varphi u \in \mathscr{S}(\mathbb{R}^n)$ i.e. $x_0 \notin \pi\big(\mathrm{WF}_{\mathrm{cl}}(u)\big)$.
Conversely, let $x_0 \notin \pi\big(\mathrm{WF}_{\mathrm{cl}}(u)\big)$. Then, for any $\xi_0 \in \mathbb{R}^n \setminus 0$ there exists

a function $\varphi_{\xi_0}(x) \in \mathcal{C}_0^\infty(\Omega)$ and a conic neighbourhood Γ_{ξ_0} of ξ_0 such that $\varphi_{\xi_0} \equiv 1$ near x_0 and $\widehat{\varphi_{\xi_0} u}(\xi)$ decreases rapidly in Γ_{ξ_0}. Due to the compactness of the n-dimensional sphere, let $\Gamma_{\xi_1}, \dots, \Gamma_{\xi_N}$ be a covering of $\mathbb{R}^n \setminus 0$, and $\varphi_{\xi_j}, j = 1, \dots, N$, the corresponding functions. Setting

$$\varphi := \prod_{j=1}^{N} \varphi_{\xi_j},$$

we get

$$\widehat{\varphi u} = (2\pi)^{-n}\, \widehat{\varphi_{\xi_k} u} * \widehat{\Phi_k}$$

where

$$\Phi_k := \prod_{j \neq k} \varphi_{\xi_j},$$

for all $1 \leq k \leq N$. Choosing appropriate conic $\Gamma'_{\xi_j} \subset \Gamma_{\xi_j}$ such that $\overline{\Gamma'_{\xi_j} \cap \mathbb{S}^n} \subset \Gamma_{\xi_j}$, $j = 1, \dots, N$, we see from Lemma 2.1.3 that $\widehat{\varphi u}$ decreases rapidly everywhere so that $\varphi u \in \mathcal{C}_0^\infty(\Omega)$, i.e. $u \in \mathcal{C}_0^\infty$ in a neighbourhood of x_0, that is, $x_0 \notin \operatorname{sing\,supp}(u)$. □

Proposition 2.1.5. Let $u \in \mathcal{D}'(\Omega)$ and $(x_0, \xi_0) \notin \mathrm{WF}_{\mathrm{cl}}(u)$. Then there exists $A = \mathrm{Op}(a) \in \mathrm{L}^0_{\mathrm{cl}}(\Omega)$ such that $a \equiv 1$ modulo $\mathrm{S}^{-\infty}$ in a conic neighbourhood of (x_0, ξ_0), and $Au \in \mathcal{C}_0^\infty(\Omega)$.

Proof. Since $(x_0, \xi_0) \notin \mathrm{WF}_{\mathrm{cl}}(u)$, there exists $\varphi \in \mathcal{C}_0^\infty(\Omega)$, $\varphi(x_0) \equiv 1$ in a neighbourhood of x_0, such that $\widehat{\varphi u}$ decreases rapidly on a conic neighbourhood of ξ_0. Let $\chi(\xi)$ be supported in this neighbourhood, with $\chi(t\xi) = \chi(\xi)$ for $t \geq 1$, $|\xi| \geq 1$, and $\chi \in \mathcal{C}^\infty(\Omega)$ with $\chi(\xi) \equiv 1$ in some smaller conic neighbourhood of ξ_0. Then $\chi\widehat{\varphi u}(\xi)$ is rapidly decreasing everywhere, so that $\chi(D)(\varphi(x)u(x)) \in \mathcal{C}^\infty$. Then, if $h \in \mathcal{C}_0^\infty(\Omega)$, $h(x)\chi(D)(\varphi(x)u(x)) \in \mathcal{C}_0^\infty(\Omega)$. Choosing h such that $h \equiv 1$ in a neighbourhood of x_0 we get that the operator $A = \mathrm{Op}(a) := h(x)\chi(D)\varphi(x)$ satisfies all the required conditions, since

$$a(x,\xi) \sim h(x) \sum_{\alpha \geq 0} D_x^\alpha \varphi(x) \partial_\xi^\alpha \chi(\xi) \in \mathrm{S}^0_{\mathrm{cl}}(\Omega).$$

□

Lemma 2.1.6. Let $v \in \mathcal{E}'(\mathbb{R}^n), \chi \in \mathrm{S}^m$. Then, if $\operatorname{dist}(x, \operatorname{supp} v) \geq 1$, we have, for all $N \in \mathbb{N}$, $|x| \geq 1$,

$$|D^\alpha \chi(D) v(x)| \leq C_{\alpha,N} |x|^{-N}.$$

Proposition 2.1.7. Let $u \in \mathcal{E}'(\Omega)$, $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^n \setminus 0)$, and an operator $A \in \mathrm{L}^m_{\mathrm{cl}}(\Omega)$ with principal symbol $\sigma(A)$. Then, if $\sigma(A)(x_0, \xi_0) \neq 0$ and $Au \in \mathcal{C}^\infty(\Omega)$, we have $(x_0, \xi_0) \notin \mathrm{WF}_{\mathrm{cl}}(u)$.

Proof. Since $\sigma(A)(x_0, \xi_0) \neq 0$, by the standard parametrix construction in a conic neighbourhood of (x_0, ξ_0) (see, e.g., [Shu01]), we obtain an operator $B \in \mathrm{L}^{-m}_{\mathrm{cl}}(\Omega)$ such that $BA = \mathrm{Op}(c)$ and $c \equiv 1$ modulo $\mathrm{S}^{-\infty}$ in this neighbourhood. Since also $BAu \in \mathcal{C}^\infty(\Omega)$, up to replacing A by BA we may assume, without loss of generality, $A \equiv I \bmod \mathrm{S}^{-\infty}$ in the same neighbourhood of (x_0, ξ_0).
Let $\chi \equiv 1$ in a neighbourhood of ξ_0, $\chi \in \mathcal{C}^\infty(\mathbb{R}^n)$, and $\chi(\xi)$ homogeneous of degree 0 in ξ for $|\xi| \geq 1$. Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $\varphi \equiv 1$ in a neighbourhood of x_0, and the supports of φ and χ be chosen so that

$$\varphi(x)\chi(\xi)a(x,\xi) = \varphi(x)\chi(\xi) \qquad \bmod \mathrm{S}^{-\infty}.$$

From this we obtain

$$\chi(D)\varphi A - \chi(D)\varphi \in \mathrm{L}^{-\infty}, \tag{2.2}$$

and by $\chi(D)\varphi A u \in \mathcal{C}^\infty(\Omega)$, it follows from (2.2) that

$$\chi(D)\varphi u \in \mathcal{C}^\infty(\mathbb{R}^n). \tag{2.3}$$

Using (2.3) and Lemma 2.1.6 we obtain that

$$\chi(D)\varphi u \in \mathscr{S}(\mathbb{R}^n), \tag{2.4}$$

from which it follows that $\chi(\xi)\widehat{\varphi u}(\xi) \in \mathscr{S}(\mathbb{R}^n)$ and, in particular, that $\widehat{\varphi u}$ decreases rapidly in a conic neighbourhood of ξ_0, i.e. $(x_0, \xi_0) \notin \mathrm{WF}_{\mathrm{cl}}(u)$, as claimed. □

Definition 2.1.8. Let $A \in \mathrm{L}^m_{\mathrm{cl}}(\Omega)$. We say that A is non-characteristic at $(x_0, \xi_0) \in \mathrm{T}^*\Omega \setminus 0$ if $\sigma(A)(x_0, \xi_0) \neq 0$, where $\sigma(A)$ is the principal symbol of A.

Remark 2.1.9. The previous Definition 2.1.8 makes sense from the 1-1 correspondence between a classical pseudodifferential operator and its principal symbol. See [KG82] or [Cor95] for details.

Remark 2.1.10. One can alternatively define $\mathrm{WF}_{\mathrm{cl}}(u)$ as the complement of the set of points (x_0, ξ_0) such that there exists a pseudodifferential operator $A \in \mathrm{L}^0_{\mathrm{cl}}(\Omega)$, non-characteristic at (x_0, ξ_0), such that $Au \in \mathcal{C}^\infty(\Omega)$. Proposition 2.1.5 and Proposition 2.1.7 show that this definition and Definition 2.1.1 are equivalent.

We now recall the concept of classical characteristic set

Definition 2.1.11. Let $A \in \mathrm{L}^m_{\mathrm{cl}}(\Omega)$, $m \in \mathbb{R}$. The classical characteristic set of A is

$$\mathrm{Char}_{\mathrm{cl}}(A) = \{(x,\xi) \in \Omega \times (\mathbb{R}^n \setminus 0) : \sigma(A)(x,\xi) = 0\}.$$

We have

Theorem 2.1.12. Let $u \in \mathcal{D}'(\Omega)$. Then

$$\mathrm{WF}_{\mathrm{cl}}(u) = \bigcap_{\substack{A \in \mathrm{L}^0_{\mathrm{cl}}(\Omega) \\ Au \in \mathcal{C}^\infty}} \mathrm{Char}_{\mathrm{cl}}(A).$$

Proof. Let $(x_0, \xi_0) \notin \mathrm{WF}_{\mathrm{cl}}(u)$. Then, from Proposition 2.1.5, we can find $A \in \mathrm{L}^0_{\mathrm{cl}}(\Omega)$, $Au \in \mathcal{C}^\infty$, such that $\sigma(A)(x_0, \xi_0) = 1$ in a conic neighbourhood of (x_0, ξ_0), i.e. $(x_0, \xi_0) \notin \mathrm{Char}_{\mathrm{cl}}(A)$.
Conversely, let $A \in \mathrm{L}^0_{\mathrm{cl}}(\Omega)$, $Au \in \mathcal{C}^\infty$ and $(x_0, \xi_0) \notin \mathrm{Char}_{\mathrm{cl}}(A)$. By replacing u with χu, for an appropriate cut-off $\chi \in \mathcal{C}^\infty_0(\Omega)$, we can assume, without loss of generality, that $u \in \mathcal{E}'(\Omega)$. Then, from Proposition 2.1.7, we get $(x_0, \xi_0) \notin \mathrm{WF}_{\mathrm{cl}}(u)$. □

It is now interesting to compare, for given operators A, the sets $\mathrm{WF}_{\mathrm{cl}}(Au)$ and $\mathrm{WF}_{\mathrm{cl}}(u)$. An operator is *microlocal*, if $\mathrm{WF}_{\mathrm{cl}}(Au) \subset \mathrm{WF}_{\mathrm{cl}}(u)$. We have

Theorem 2.1.13 (Microlocality of pseudodifferential operators). Let $u \in \mathcal{D}'(\Omega)$ and $A \in \mathrm{L}^m_{\mathrm{cl}}(\Omega)$. Then,

$$\mathrm{WF}_{\mathrm{cl}}(Au) \subseteq \mathrm{WF}_{\mathrm{cl}}(u).$$

Proof. Let $(x_0, \xi_0) \notin \mathrm{WF}_{\mathrm{cl}}(u)$. Then we can find an operator $B = \mathrm{Op}(b) \in \mathrm{L}^0_{\mathrm{cl}}(\Omega)$ such that $Bu \in \mathcal{C}^\infty(\Omega)$ and $b \equiv 1$ modulo $\mathrm{S}^{-\infty}$ in a conic neighbourhood $U \times \Gamma$ of (x_0, ξ_0). By standard construction, cfr. [Shu01], we can find $B_1 = \mathrm{Op}(b_1) \in \mathrm{L}^0_{\mathrm{cl}}(\Omega)$ such that $B + B_1$ is elliptic, and

$$\mathrm{supp}(b_1 - r) \cap (U' \times \Gamma') \subset (\Omega \times \mathbb{S}^n), \quad r \in \mathrm{S}^{-\infty}(\Omega),$$

with $x_0 \in U' \subseteq U$ open, $\xi_0 \in \Gamma' \subseteq \Gamma$ open and conic. Let $P \in \mathrm{L}^0_{\mathrm{cl}}(\Omega)$ a parametrix of $B + B_1$, that is

$$I - P(B + B_1) = R \in \mathrm{L}^{-\infty}(\Omega).$$

Then

$$Au = APB_1u + APBu + ARu, \tag{2.5}$$

and $ARu \in \mathcal{C}^\infty(\Omega)$ because R is smoothing, while $APBu \in \mathcal{C}^\infty(\Omega)$ because $Bu \in \mathcal{C}^\infty(\Omega)$. Finally, take $\chi \in \mathcal{C}^\infty_0(\Omega)$, $\mathrm{supp}\,\chi \subset U'$, $\chi(x_0) \neq 0$, and $\psi \in \mathcal{C}^\infty(\mathbb{R}^n)$, $\mathrm{supp}\,\psi \subset \Gamma'$, $\psi(\xi) = 1$ for $\xi \in \Gamma'' \subset \Gamma'$ and $|\xi| \geq |\xi_0| > 0$, Γ'' open and conic. Now, setting $A_1 = \mathrm{Op}(a_1)$ with $a_1(x, \xi) := \chi(x)\psi(\xi)$, we have $A_1 \in \mathrm{L}^0_{\mathrm{cl}}(\Omega)$ and $\sigma(A_1)(x_0, \xi_0) \neq 0$. The support properties of a_1 and b_1 imply that $A_1APB_1 \in \mathrm{L}^{-\infty}(\Omega)$. Thus, from (2.5), we get that $A_1Au \in \mathcal{C}^\infty(\Omega)$, and from Proposition 2.1.7 it follows that $(x_0, \xi_0) \notin \mathrm{WF}_{\mathrm{cl}}(Au)$. □

As a direct consequence, we get

Corollary 2.1.14 (Pseudolocality of pseudodifferential operators). Let $u \in \mathcal{D}'(\Omega)$ and $A \in \mathrm{L}^m_{\mathrm{cl}}(\Omega)$. Then

$$\operatorname{sing\,supp}(Au) \subseteq \operatorname{sing\,supp}(u).$$

Proof. By Theorem 2.1.13, it follows, trivially,

$$\operatorname{sing\,supp}(Au) = \pi\big(\mathrm{WF}_{\mathrm{cl}}(Au)\big) \subseteq \pi\big(\mathrm{WF}_{\mathrm{cl}}(u)\big) = \operatorname{sing\,supp}(u).$$

□

The next result gives "an estimate" of $\mathrm{WF}_{\mathrm{cl}}(u)$ in terms of $\mathrm{WF}_{\mathrm{cl}}(Au)$ and $\mathrm{Char}_{\mathrm{cl}}(A)$. That is, if u is a solution of $Au = v \in \mathcal{D}'(\Omega)$, its wave front set depends on the characteristic set of A and wave front set of the datum v.

Theorem 2.1.15 (Microellipticity of pseudodifferential operators). Let $u \in \mathcal{D}'(\Omega)$ and $A \in \mathrm{L}^m_{\mathrm{cl}}(\Omega)$. Then

$$\mathrm{WF}_{\mathrm{cl}}(u) \subseteq \mathrm{WF}_{\mathrm{cl}}(Au) \cup \mathrm{Char}_{\mathrm{cl}}(A).$$

Proof. Let $(x_0, \xi_0) \notin \mathrm{WF}_{\mathrm{cl}}(Au) \cup \mathrm{Char}_{\mathrm{cl}}(A)$. Then, by Proposition 2.1.5, we can find an operator $P = \mathrm{Op}(p) \in \mathrm{L}^0_{\mathrm{cl}}(\Omega)$ such that $PAu \in \mathcal{C}^\infty(\Omega)$ and $p \equiv 1$ modulo $\mathrm{S}^{-\infty}$ in a conic neighbourhood of (x_0, ξ_0). But then, from Proposition 2.1.7 it follows that $(x_0, \xi_0) \notin \mathrm{WF}_{\mathrm{cl}}(u)$, since $\sigma(PA)(x_0, \xi_0) = \sigma(P)(x_0, \xi_0) \cdot \sigma(A)(x_0, \xi_0) \neq 0$. □

Corollary 2.1.16. Let $u \in \mathcal{D}'(\Omega)$, $A \in \mathrm{L}^m_{\mathrm{cl}}(\Omega)$, with A elliptic. Then,

$$\mathrm{WF}_{\mathrm{cl}}(Au) = \mathrm{WF}_{\mathrm{cl}}(u).$$

Proof. We already know that $\mathrm{WF}_{\mathrm{cl}}(Au) \subseteq \mathrm{WF}_{\mathrm{cl}}(u)$. Conversely, since A in elliptic, there exists $B \in \mathrm{L}^{-m}_{\mathrm{cl}}(\Omega)$ such that $BA - I = K \in \mathrm{L}^{-\infty}(\Omega)$. Thus,

$$\begin{aligned}\mathrm{WF}_{\mathrm{cl}}(u) = \mathrm{WF}_{\mathrm{cl}}((BA - K)u) &\subseteq \mathrm{WF}_{\mathrm{cl}}(BAu) \cup \mathrm{WF}_{\mathrm{cl}}(Ku)\\ &\subseteq \mathrm{WF}_{\mathrm{cl}}(Au) \cup \emptyset = \mathrm{WF}_{\mathrm{cl}}(Au).\end{aligned}$$

□

We conclude the section with some examples.

Example 2.1.17. Let us compute the wave front set of δ_{x_0}, $x_0 \in \mathbb{R}^n$. Taking ϕ as in Definition 2.1.1 around $x' \in \mathbb{R}^n$, we have $\widehat{\phi\delta_{x_0}} = \widehat{\phi(x_0)} \equiv 0$ if $x' \neq x_0$. Conversely, for $x' = x_0$, we have

$$\begin{aligned}\langle \widehat{\phi\delta_{x_0}}, \varphi\rangle = \langle \delta_{x_0}, \phi\hat{\varphi}\rangle &= \phi(x_0)\hat{\varphi}(x_0)\\ &= \hat{\varphi}(x_0) = \int e^{-ix_0\cdot\xi}\varphi(\xi)\,d\xi = \langle e^{-ix_0(\cdot)}, \varphi\rangle.\end{aligned}$$

Since $e^{-ix_0\cdot\xi}$ is not rapidly decreasing in any conical subset of $\mathbb{R}^n_\xi$, we conclude that

$$\mathrm{WF}_{\mathrm{cl}}(\delta_{x_0}) = \{x_0\} \times (\mathbb{R}^n \setminus 0).$$

Example 2.1.18. Let $A \in \mathrm{L}^m_{\mathrm{cl}}(\Omega)$, and $K_A \in \mathcal{D}'(\Omega \times \Omega)$ its distributional kernel. Then

$$\mathrm{WF}_{\mathrm{cl}}(K_A) \subseteq \{(x, x, \xi, -\xi) : (x, \xi) \in \Omega \times (\mathbb{R}^n \setminus 0)\}.$$

The proof of this fact can be found, for instance, in [Trè80].

2.2 The bi-wave front set

In this Section we establish a notion of wave front set for bisingular operators in terms of the bisingular calculus, and analyze its properties. In fact, it turns out to be quite similar to the notion studied in [CM03] for the *SG*-calculus, see also [Mel94, Mel95]. This is not surprising, as there is a strong similarity in the formulas arising in both calculi. In this Section all bisingular operators are assumed to be classical, and Ω_1, Ω_2 will denote two closed manifolds.

2.2.1 Classical microlocal analysis of bisingular operators

A natural way to start with the analysis of the microlocal properties of bisingular operators is to estimate their distributional kernel. From Theorem 1.1.4, we obtain that the Schwartz kernel of a bisingular operator with symbol a is defined by the oscillatory integral

$$K_A(x_1, x_2, y_1, y_2) = \int_{\mathbb{R}^{n_1+n_2}} e^{i(x_1-y_1, x_2-y_2)\cdot(\xi_1,\xi_2)} a(x_1, x_2, \xi_1, \xi_2) đ\xi_1 đ\xi_2.$$

The Schwartz kernel Theorem states the following smoothing property:

Proposition 2.2.1. A linear map $\mathcal{D}'(\Omega_1 \times \Omega_2) \to \mathcal{D}'(\Omega_1 \times \Omega_2)$ is actually a mapping $\mathcal{D}'(\Omega_1 \times \Omega_2) \to \mathcal{E}(\Omega_1 \times \Omega_2)$, i.e. *smoothing*, if and only if its distributional kernel is in $\mathcal{E}((\Omega_1 \times \Omega_2) \times (\Omega_1 \times \Omega_2))$.

Therefore, pseudodifferential operators with symbols in $S^{-\infty}$ and bisingular operators with symbol in $S^{-\infty,-\infty}$ are smoothing. As the prototype of a bisingular operator is the tensor product of two pseudodifferential operators, it makes sense also to define what is meant by an operator that is smoothing in one set of variables only.

Definition 2.2.2. We define $\mathcal{C}^\infty(\Omega_1, \mathcal{D}'(\Omega_2))$ as the set of all $u \in \mathcal{D}'(\Omega_1 \times \Omega_2)$ such that for each $f \in \mathcal{D}(\Omega_2)$, the distribution $\mathcal{D}'(\Omega_1) \ni u(g \otimes \cdot) : g \mapsto u(g \otimes f)$ is actually a smooth function. Correspondingly, we can define $\mathcal{C}^\infty(\Omega_2, \mathcal{D}'(\Omega_1))$.

We can now list the mapping properties of bisingular operators on these spaces, following the ideas in [Trè67].

Lemma 2.2.3.
- Bisingular operators map the spaces $\mathcal{C}^\infty(\Omega_1, \mathcal{D}'(\Omega_2))$ and $\mathcal{C}^\infty(\Omega_2, \mathcal{D}'(\Omega_1))$ into themselves[1].

[1] We recall that Ω_1 and Ω_2 are closed manifolds, hence $\mathcal{D}'(\Omega_i) = \mathcal{E}'(\Omega_i)$, $i = 1, 2$.

- Let $a \in S^{-\infty,m}$. Then, the bisingular operator $\mathrm{Op}(a)$ maps $\mathcal{D}'(\Omega_1 \times \Omega_2)$ to $\mathcal{C}^\infty(\Omega_1, \mathcal{D}'(\Omega_2))$ and $\mathcal{C}^\infty(\Omega_2, \mathcal{D}'(\Omega_1))$ to $\mathcal{C}^\infty(\Omega_1 \times \Omega_2)$.
- Let $a \in S^{m,-\infty}$. Then, the bisingular operator $\mathrm{Op}(a)$ maps $\mathcal{D}'(\Omega_1 \times \Omega_2)$ to $\mathcal{C}^\infty(\Omega_2, \mathcal{D}'(\Omega_1))$ and $\mathcal{C}^\infty(\Omega_1, \mathcal{D}'(\Omega_2))$ to $\mathcal{C}^\infty(\Omega_1 \times \Omega_2)$.

The following Lemma (see e.g. [GS94]) indicates how the singularities of a distribution transform under the action of a linear operator in terms of the singularities of its kernel.

Lemma 2.2.4. Let $K \in \mathcal{D}'(\Omega_1 \times \Omega_2)$, and denote by the same letter the corresponding operator $K : \mathcal{C}_0^\infty(\Omega_2) \to \mathcal{D}'(\Omega_1)$. Set

$$\mathrm{WF}'(K) := \{(x_1, x_2, \xi_1, -\xi_2) \in T^*(\Omega_1 \times \Omega_2) \setminus 0; (x_1, x_2, \xi_1, \xi_2) \in \mathrm{WF}_{\mathrm{cl}}(K)\},$$
$$\mathrm{WF}'_{\Omega_1}(K) := \{(x_1, \xi_1) \in T^*\Omega_1 \setminus 0;\ \exists y \in \Omega_2 \text{ with } (x_1, y, \xi_1, 0) \in \mathrm{WF}'(K)\},$$
$$\mathrm{WF}'_{\Omega_2}(K) := \{(x_2, \xi_2) \in T^*\Omega_2 \setminus 0;\ \exists x \in \Omega_1 \text{ with } (x, x_2, 0, \xi_2) \in \mathrm{WF}'(K)\}.$$

Then, if $u \in \mathcal{E}'(\Omega_2)$ and $\mathrm{WF}_{\mathrm{cl}}(u) \cap \mathrm{WF}'_{\Omega_2}(K) = \emptyset$, we have

$$\mathrm{WF}_{\mathrm{cl}}(Ku) \subset \big(\mathrm{WF}'(K) \circ (\mathrm{WF}_{\mathrm{cl}}(u))\big) \cup \mathrm{WF}'_{\Omega_1}(K),$$

where we considered $\mathrm{WF}'(K)$ as a relation.

Remark 2.2.5. From the previous Lemma we can obtain an alternative proof of the microlocality of pseudodifferential operators. In fact, if K is the kernel of a pseudodifferential operator A on $\Omega = \Omega_1 = \Omega_2$, then

$$\mathrm{WF}'_{\Omega_1}(K) = \emptyset = \mathrm{WF}'_{\Omega_2}(K),$$

and, from Example 2.1.18,

$$\mathrm{WF}'(K) \subseteq \{(x, x, \xi, -\xi) : (x, \xi) \in \Omega \times (\mathbb{R}^n \setminus 0)\}.$$

Hence, from Lemma 2.2.4 we get $\mathrm{WF}_{\mathrm{cl}}(Au) \subseteq \mathrm{WF}_{\mathrm{cl}}(u)$.

We now study the microlocal properties of bisingular operators.

Example 2.2.6. Consider $\Omega_1 = \Omega_2 = \mathbb{R}$. We further choose positive $\phi,\ \psi \in \mathcal{C}_0^\infty(\mathbb{R})$. Now, define the pseudodifferential operator T_ϕ on $\mathcal{C}_0^\infty(\mathbb{R})$ by

$$T_\phi(f) := \phi * f.$$

Then, the operator $A := T_\phi \otimes I$ is a tensor product of two pseudodifferential operators, and thus a bisingular operator on $\mathbb{R}^2$. Now, consider the distribution $u = \psi \otimes \delta$, which has the following wave front set:

$$\mathrm{WF}_{\mathrm{cl}}(u) = \{(x_1, 0, 0, \xi_2) |\, x_1 \in \mathrm{supp}(\psi),\ \xi_2 \in \mathbb{R} \setminus 0\}.$$

Then, it is easy to see that

$$\mathrm{WF}_{\mathrm{cl}}(Au) = \big\{(x_1, 0, 0, \xi_2) |\, x_1 \in \big(\mathrm{supp}(\psi) + \mathrm{supp}(\phi)\big),\ \xi_2 \in \mathbb{R} \setminus 0\big\},$$

thus $\mathrm{WF}_{\mathrm{cl}}(Au) \supset \mathrm{WF}_{\mathrm{cl}}(u)$. The example can be similarly given, using local coordinates, on a product of two closed manifolds.

The previous example shows that, in general, bisingular operators do not have the microlocal property. Let us start considering the model case of a tensor product of two pseudodifferential operators. For that, we use the following well-known estimate for the wave front set of a tensor product of distributions (cf. e.g. [Hör83]):

Lemma 2.2.7. Let $u \in \mathcal{D}'(\Omega_1)$, $v \in \mathcal{D}'(\Omega_2)$. Then,

$$\begin{aligned}\mathrm{WF}_{\mathrm{cl}}(u \otimes v) \subseteq & \big(\mathrm{WF}_{\mathrm{cl}}(u) \times \mathrm{WF}_{\mathrm{cl}}(v)\big) \cup \big((\mathrm{supp}(u) \times \{0\}) \times \mathrm{WF}_{\mathrm{cl}}(v)\big) \\ & \cup \big(\mathrm{WF}_{\mathrm{cl}}(u) \times (\mathrm{supp}(v) \times \{0\})\big).\end{aligned}$$

Example 2.2.8. Let $A \in \mathrm{L}^{m_1}(\Omega_1)$, $B \in \mathrm{L}^{m_2}(\Omega_2)$. Then, the kernel of the bisingular operator $A \otimes B \in \mathrm{L}^{m_1,m_2}(\Omega_1, \Omega_2)$ is given by $K := K_A \otimes K_B$. From Lemma 2.2.7 we have

$$\begin{aligned}\mathrm{WF}_{\mathrm{cl}}(K_A \otimes K_B) \subset & \big(\mathrm{WF}_{\mathrm{cl}}(K_A) \times \mathrm{WF}_{\mathrm{cl}}(K_B)\big) \\ & \cup \big((\Omega_1 \times \Omega_1 \times \{0\}) \times \mathrm{WF}_{\mathrm{cl}}(K_B)\big) \\ & \cup \big(\mathrm{WF}_{\mathrm{cl}}(K_A) \times (\Omega_2 \times \Omega_2 \times \{0\})\big).\end{aligned}$$

We know, from Example 2.1.18, that

$$\mathrm{WF}_{\mathrm{cl}}(K_A) \subseteq \{(x, x, \xi, -\xi) : (x, \xi) \in \Omega_1 \times (\mathbb{R}^{n_1} \setminus 0)\}.$$

If follows

$$\begin{aligned}\mathrm{WF}_{\mathrm{cl}}(K_A \otimes K_B) \subset & \{(x_1, x_2, x_1, x_2, \xi_1, \xi_2, -\xi_1, -\xi_2)\} \\ & \cup \{(x_1, x_2, y_1, x_2, 0, \xi_2, 0, -\xi_2)\} \\ & \cup \{(x_1, x_2, x_1, y_2, \xi_1, 0, -\xi_1, 0)\},\end{aligned}$$

that is

$$\begin{aligned}\mathrm{WF}_{\mathrm{cl}}(Ku) \subset & \{(x_1, x_2, \xi_1, \xi_2) \in \mathrm{WF}_{\mathrm{cl}}(u)\} \\ & \cup \{(x_1, x_2, 0, \xi_2); (y_1, x_2, 0, \xi_2) \in \mathrm{WF}_{\mathrm{cl}}(u)\} \\ & \cup \{(x_1, x_2, \xi_1, 0); (x_1, y_2, \xi_1, 0) \in \mathrm{WF}_{\mathrm{cl}}(u)\} = \mathrm{WF}'_{\mathrm{cl}}(K) \circ (\mathrm{WF}_{\mathrm{cl}}(u)),\end{aligned}$$

which is precisely the relation of Lemma 2.2.4, given the fact that in our case $\mathrm{WF}'_{\Omega_1}(K) = \emptyset$.

The approach of the previous Example 2.2.8 seems promising, but it strongly relies on the computation of the distributional kernel and its singularities in the sense of the classical Hörmander's wave front set. It is far more desirable to have a notion of singularities in terms of the actual bisingular calculus. This will be provided in the next subsection.

2.2.2 Microlocal properties of bisingular operators

While being the description that naturally arises when analysing the kernels of bisingular operators, the notion of classical wave front is not defined in terms of the bisingular calculus, but indeed with respect to the pseudodifferential one. It is the appropriate to provide a definition of wave front set that takes into account with the peculiar aspects of bisingular calculus.

Definition 2.2.9. Let $u \in \mathcal{D}'(\Omega_1 \times \Omega_2)$. We define $\mathrm{WF}_{\mathrm{bi}}(u) \subset \Omega_1 \times \Omega_2 \times (\mathbb{R}^{n_1+n_2} \setminus 0)$ as

$$\mathrm{WF}_{\mathrm{bi}}(u) = \mathrm{WF}^1_{\mathrm{bi}}(u) \cup \mathrm{WF}^2_{\mathrm{bi}}(u) \cup \mathrm{WF}^{12}_{\mathrm{bi}}(u),$$

where

- $(x_1, x_2, \xi_1, 0)$ is not in $\mathrm{WF}^1_{\mathrm{bi}}(u)$ if there exists $A \in \mathrm{L}^0_{\mathrm{cl}}(\Omega_1)$, non-characteristic at (x_1, ξ_1), such that
$$(A \otimes I)u \in \mathcal{C}^\infty(\Omega_1, \mathcal{D}'(\Omega_2)); \tag{2.6}$$
- $(x_1, x_2, 0, \xi_2)$ is not in $\mathrm{WF}^2_{\mathrm{bi}}(u)$ if there exists $A \in \mathrm{L}^0_{\mathrm{cl}}(\Omega_2)$, non-characteristic at (x_2, ξ_2), such that
$$(I \otimes A)u \in \mathcal{C}^\infty(\Omega_2, \mathcal{D}'(\Omega_1)); \tag{2.7}$$
- (x_1, x_2, ξ_1, ξ_2), $|\xi_1||\xi_2| \neq 0$, is not in $\mathrm{WF}^{12}_{\mathrm{bi}}(u)$ if there exist $A_i \in \mathrm{L}^0_{\mathrm{cl}}(\Omega_i)$, non-characteristic at (x_i, ξ_i), $i = 1, 2$, such that
$$(A_1 \otimes A_2)u \in \mathcal{C}^\infty(\Omega_1 \times \Omega_2) \tag{2.8}$$
$$(A_1 \otimes I)u \in \mathcal{C}^\infty(\Omega_1, \mathcal{D}'(\Omega_2)), \tag{2.9}$$
$$(I \otimes A_2)u \in \mathcal{C}^\infty(\Omega_2, \mathcal{D}'(\Omega_1)). \tag{2.10}$$

Remark 2.2.10. Notice that conditions (2.9) and (2.10) do not follow from (2.8). Take, for instance, $u \in \mathcal{D}'(\mathbb{R} \times \mathbb{R})$, $u = \delta(x-1)\delta(y+1)$ and ψ smooth such that $\psi \equiv 1$ for $x > 1/2$ and $\psi \equiv 0$ for $x \leq 0$. Then, $(\psi(x) \otimes \psi(y))u = 0$, as $(1, -1) \notin \operatorname{supp}(\psi \otimes \psi)$. However, for $g \in \mathcal{D}(\mathbb{R})$ with $g(-1) \neq 0$ we have $(\psi(x) \otimes I)u(\cdot \otimes g) = \delta(x-1)g(-1)$, which is not smooth.

We have the following inclusion result:

Lemma 2.2.11. If a point $(x_1^0, x_2^0, \xi_1^0, \xi_2^0)$, $|\xi_1^0||\xi_2^0| \neq 0$, is not in $\mathrm{WF}^{12}_{\mathrm{bi}}(u)$, then it is not in $\mathrm{WF}_{\mathrm{cl}}(u)$.

Proof. The proof is a variant of [Hör91], Proposition 2.8. By definition, there exists $A := A_1 \otimes A_2 \in \mathrm{L}^{0,0}(\Omega_1 \times \Omega_2)$, with $\sigma^{12}(A)(x_1^0, x_2^0, \xi_1^0, \xi_2^0) \neq 0$, such that $Au \in \mathcal{C}^\infty(\Omega_1 \times \Omega_2)$. Now take a ($\Psi$DO symbol) $\psi \in \mathrm{L}^0_{\mathrm{cl}}(\Omega_1 \times \Omega_2)$ such that

a) on the support of ψ we have $\langle\xi_1\rangle \lesssim \langle\xi_2\rangle \lesssim \langle\xi_1\rangle$;

b) ψ is non-characteristic at $(x_1^0, x_2^0, \xi_1^0, \xi_2^0)$.

Then, the (bi-singular) operator product

$$B := \psi(x_1, x_2, D_1, D_2) \circ A(x_1, x_2, D_1, D_2)$$

yields a pseudodifferential operator of order 0, plus a smoothing remainder, by virtue of the above inequality on the support of ψ and the symbol expansion in Theorem 1.2.13. It has the following properties:

a) its principal symbol is $\psi \cdot \sigma^{12}(A)$, and thus it is non-characteristic (in the sense of ΨDOs) at $(x_1^0, x_2^0, \xi_1^0, \xi_2^0)$ and of order zero;

b) $Bu = \psi(x_1, x_2, D_1, D_2)A(x_1, x_2, D_1, D_2)u \in \mathcal{C}^\infty$.

This proves the claim. □

Remark 2.2.12. Lemma 2.2.11 asserts that in the conic region where both covariables are non-vanishing we can pass from bisingular to pseudodifferential calculus by multiplying by a ΨDO. This has the consequence that the two operator classes have similar microlocal properties (with respect to the classical wave front set) in that region.

Definition 2.2.13. Let $x_0 \in \mathbb{S}^n$, and choose a positive function $\phi \in \mathcal{C}^\infty(\mathbb{S}^n)$ with $\|\phi\|_\infty = 1$ and $\phi \equiv 1$ in a neighbourhood of x_0. Denote by $\tilde{\phi}$ the homogeneous extension of ϕ to $\mathbb{R}^n \setminus 0$, and consider an excision function $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. The function

$$\psi(x) := \chi(x)\tilde{\phi}\left(\frac{x}{|x|}\right) \in \mathcal{C}^\infty(\mathbb{R}^n),$$

is called *conic localizer* around x_0. For further details on this construction, see, e.g., [Mel04].

The following Lemma gives a similar interpretation of the remaining components, illustrating the loss of localization of singularities already encountered in the previous section.

Lemma 2.2.14. Let $u \in \mathcal{E}'(\Omega_1 \times \Omega_2)$, $(x_1^0, \xi_1^0) \in \Omega_1 \times (\mathbb{R}^{n_1} \setminus 0)$. If for all $y \in \Omega_2$ we have $(x_1^0, y, \xi_1^0, 0) \notin \mathrm{WF}_{\mathrm{cl}}(u)$, then, for all $y \in \Omega_2$ we have $(x_1^0, y, \xi_1^0, 0) \notin \mathrm{WF}^1_{\mathrm{bi}}(u)$. Similarly, if for all $x \in \Omega_1$ we have $(x, x_2^0, 0, \xi_2^0) \notin \mathrm{WF}_{\mathrm{cl}}(u)$, then, for all $x \in \Omega_1$ we have $(x, x_2^0, 0, \xi_2^0) \notin \mathrm{WF}^2_{\mathrm{bi}}(u)$.

Proof. We prove only the first claim, since the proof of the second one is similar. Take $(x_1^0, y, \xi_1^0, 0) \notin \mathrm{WF}_{\mathrm{cl}}(u)$. Then, there exists a cut-off $\phi \in$

$\mathcal{C}_0^\infty(\Omega_1 \times \Omega_2)$ and a conic localizer $\psi_y \in \mathcal{C}^\infty(\mathbb{R}^{n_1+n_2})$, non-vanishing in a (conic) neighbourhood (x_1^0, y) and $(\xi_1^0, 0)$ respectively, such that

$$\psi_y(\xi_1, \xi_2)\mathcal{F}_{(x_1,x_2)\mapsto(\xi_1,\xi_2)}\{\phi(x_1, x_2)u\} \in \mathscr{S}(\mathbb{R}^{n_1+n_2}).$$

As this holds true for any y, due to the compactness of the support of u, there exists a cut-off $\phi_1 \in \mathcal{C}_0^\infty(\Omega_1)$ such that, for some conic localizer $\psi \in \mathcal{C}^\infty(\mathbb{R}^{n_1+n_2})$,

$$\psi(\xi_1, \xi_2)\mathcal{F}_{(x_1,x_2)\mapsto(\xi_1,\xi_2)}\{\phi(x_1)u\} \in \mathscr{S}(\mathbb{R}^{n_1+n_2}).$$

We can then find a conic localizer $\psi_1 \in \mathcal{C}^\infty(\mathbb{R}^{n_1})$, non-vanishing around ξ_1^0, such that $\psi(\xi_1)\mathcal{F}_{(x_1,x_2)\mapsto(\xi_1,\xi_2)}\{\phi(x_1)u\} \in \mathcal{C}^\infty(\mathbb{R}^{n_1+n_2})$ is rapidly decaying with respect to the first variable ξ_1 for fixed ξ_2, and polynomially bounded everywhere, by the Paley-Wiener-Schwartz Theorem. Define $A \in \mathrm{L}^0_{\mathrm{cl}}(\Omega_1)$ to be the operator

$$Au(y_1) = \mathcal{F}^{-1}_{\xi_1\mapsto y_1}(\psi(\xi_1)\mathcal{F}_{x_1\mapsto\xi_1}\{\phi(x_1)u\}.$$

By the assumptions on ϕ_1 and ψ_1, A is non-characteristic in the sense of pseudodifferential operators at (x_1^0, ξ_1^0). Taking $f \in \mathcal{D}(\Omega_2)$, we find that $[(A\otimes I)u](f)(x_1) = \mathcal{F}^{-1}_{\xi_1\mapsto x_1}\langle\psi(\xi_1)\mathcal{F}_{(y_1,y_2)\mapsto(\xi_1,\xi_2)}\{\phi(y_1)u\}, \widehat{f}\rangle$ is a smooth function, which means $(A \otimes I)u \in \mathcal{C}^\infty(\Omega_1, \mathcal{D}'(\Omega_2))$, as claimed. □

Proposition 2.2.15. Let $u \in \mathcal{E}'(\Omega_1 \times \Omega_2)$. Then

$$\mathrm{WF}_{\mathrm{bi}}(u) = \emptyset \Leftrightarrow u \in \mathcal{C}^\infty(\Omega_1 \times \Omega_2).$$

Proof. Assume $\mathrm{WF}_{\mathrm{bi}}(u) = \emptyset$. Then, in view of of Lemma 2.2.11, we have $\mathrm{WF}_{\mathrm{cl}}(u) \cap \{(x_1, x_2, \xi_1, \xi_2)| : |\xi_1||\xi_2| \neq 0\} = \emptyset$. Thus, $\widehat{u}$ is rapidly decaying on any ray $\mathbb{R} \cdot (\xi_1, \xi_2)$ where $|\xi_1||\xi_2| \neq 0$.
Since $\mathrm{WF}^1_{\mathrm{bi}}(u) = \emptyset$, for each (x_1, x_2, ξ_1, ξ_2) we can find $A \in \mathrm{L}^0_{\mathrm{cl}}(\Omega_1)$ non-characteristic at (x_1, ξ_1) such that, by Lemma 2.2.3, for any

$$B \in \mathrm{L}^{m,-\infty}(\Omega_1 \times \Omega_2),$$

we have

$$B(I \otimes A)u \in \mathcal{C}^\infty(\Omega_1 \times \Omega_2).$$

By compactness and a parametrix construction, see e.g. [Shu01], we can thus conclude that, for all

$$B \in \mathrm{L}^{m,-\infty}(\Omega_1 \times \Omega_2),$$

we get

$$Bu \in \mathcal{C}^\infty(\Omega_1 \times \Omega_2).$$

Now, choose $\phi \in \mathcal{C}_0^\infty(\Omega_1\times\Omega_2)$ with $\phi \equiv 1$ on a neighbourhood of the support of u, and define $b(x_1, x_2, \xi_1, \xi_2) = \phi(x_1, x_2)f(\xi_2)$, with $f \in \mathscr{S}(\mathbb{R}^{n_2})$. Then

$$\mathscr{S}(\mathbb{R}^{n_1+n_2}) \ni \mathcal{F}(Bu) = \mathcal{F}\big(b(x_1, x_2, D_1, D_2)u\big) = (1 \otimes f)\widehat{u}.$$

As f was arbitrary and rapidly decaying, this means that $\widehat{u}$ must already be rapidly decaying in the first variable. A repetition of the argument for the second variable proves the assertion. □

By the same arguments used in the proof of the previous Proposition 2.2.15 we get:

Proposition 2.2.16. Let $u \in \mathcal{E}'(\Omega_1 \times \Omega_2)$. Then,

$$\mathrm{WF}^1_{\mathrm{bi}}(u) = \emptyset \Leftrightarrow u \in \mathcal{C}^\infty(\Omega_1, \mathcal{D}'(\Omega_2));$$
$$\mathrm{WF}^2_{\mathrm{bi}}(u) = \emptyset \Leftrightarrow u \in \mathcal{C}^\infty(\Omega_2, \mathcal{D}'(\Omega_1)).$$

Remark 2.2.17. Notice that $u \in \mathcal{C}^\infty(\Omega_1, \mathcal{D}'(\Omega_2)) \bigcap \mathcal{C}^\infty(\Omega_2, \mathcal{D}'(\Omega_1))$ does not imply that $u \in \mathcal{C}^\infty(\Omega_1 \times \Omega_2)$. Following [Trè67], a counterexample is, for instance, $\delta(x_1 - x_2)$. The additional regularity needed such that $u \in \mathcal{C}^\infty(\Omega_1 \times \Omega_2)$ is therefore, by Proposition 2.2.15, encoded in $\mathrm{WF}^{12}_{\mathrm{bi}}(u)$.

The bisingular wave front set has the following properties:

Proposition 2.2.18. Let $u,\ v \in \mathcal{D}'(\Omega_1 \times \Omega_2)$, $f \in \mathcal{C}^\infty(\Omega_1 \times \Omega_2)$.

- $\mathrm{WF}_{\mathrm{bi}}(u)$ is a closed set and conic, jointly, with respect to both covariables.
- Let $\Omega_1 = \Omega_2 = \Omega$. Define, for $u \in \mathcal{C}^\infty(\Omega \times \Omega)$, $Au(x_1, x_2) = u(x_2, x_1)$. Then, we can define the pull-back A^* by duality as an endomorphism on $\mathcal{D}'(\Omega \times \Omega)$. We have $(x_1, x_2, \xi_1, \xi_2) \in \mathrm{WF}_{\mathrm{bi}}(Au) \Leftrightarrow (x_2, x_1, \xi_2, \xi_1) \in \mathrm{WF}_{\mathrm{bi}}(u)$.
- $\mathrm{WF}_{\mathrm{bi}}(u + v) \subseteq \mathrm{WF}_{\mathrm{bi}}(u) \cup \mathrm{WF}_{\mathrm{bi}}(v)$; $\mathrm{WF}_{\mathrm{bi}}(fu) \subseteq \mathrm{WF}_{\mathrm{bi}}(u)$.

Remark 2.2.19. These properties are quite similar to the corresponding ones of the SG-wave front set, examined in [CM03]. This is not very surprising, as the bisingular calculus is formally very similar, in its definition, to the SG-calculus, through which the SG-wave front set is introduced. We will explore this connection in Section 2.3.

Lemma 2.2.20. Let $C \in \mathrm{L}^{m_1, m_2}(\Omega_1 \times \Omega_2)$, $u \in \mathcal{D}'(\Omega_1 \times \Omega_2)$. Then,

$$\mathrm{WF}^1_{\mathrm{bi}}(Cu) \subset \mathrm{WF}^1_{\mathrm{bi}}(u).$$

Proof. Let $(x_1, x_2, \xi_1, 0) \notin \mathrm{WF}^1_{\mathrm{bi}}(u)$. By definition, there exists $A \in \mathrm{L}^0_{\mathrm{cl}}(\Omega_1)$, non-characteristic at (x_1, ξ_1), such that $(A \otimes I)u \in \mathcal{C}^\infty(\Omega_1, \mathcal{D}'(\Omega_2)$. In particular, by Lemma 2.2.3 we have for all $B \in \mathrm{L}^{m, -\infty}(\Omega_1 \times \Omega_2)$, that $B(A \otimes I)u$ is smooth. By the standard pseudodifferential calculus, we can find $A' \in \mathrm{L}^0(\Omega_1)$ such that $A + A'$ is elliptic in the sense of pseudodifferential operators and such that the symbol of A' vanishes on a conic neighbourhood Γ of (x_1, ξ_1). We thus have a parametrix $P \in \mathrm{L}^0(\Omega_1)$ such that

$P(A+A') = I - R$ with $R \in \mathrm{L}^{-\infty}(\Omega_1)$.
Take $H \in \mathrm{L}^0(\Omega_1)$ such that H is non-characteristic at (x_1, ξ_1) and such that the symbol of H vanishes outside a proper subcone of Γ. Then we have:

$$\begin{aligned}(H \otimes I)Cu &= (H \otimes I)C\big((P(A+A') + R) \otimes I\big)u \\ &= (H \otimes I)C(P \otimes I)(A \otimes I)u + (H \otimes I)C(PA' \otimes I)u + \\ &+ (H \otimes I)C(R \otimes I)u \in \mathcal{C}^\infty(\Omega_1, \mathcal{D}'(\Omega_2)).\end{aligned}$$

The first summand is in $\mathcal{C}^\infty(\Omega_1, \mathcal{D}'(\Omega_2))$, due to the definition of A. The second, in view of the symbol expansion given in Theorem 1.2.13, using the support properties of the symbols of H and A', gives an operator in $L^{-\infty,0}$. Finally, the third one is in $\mathcal{C}^\infty(\Omega_1, \mathcal{D}'(\Omega_2))$, as $R \in \mathrm{L}^{-\infty,0}$ is already a smoothing operator in the first variable. This proves the claim. $\square$

Lemma 2.2.21. Let $C \in \mathrm{L}^{m_1,m_2}(\Omega_1 \times \Omega_2)$, $u \in \mathcal{D}'(\Omega_1 \times \Omega_2)$. Then we have $\mathrm{WF}^{12}_{\mathrm{bi}}(Cu) \subset \mathrm{WF}^{12}_{\mathrm{bi}}(u)$.

Proof. Let $(x_1, x_2, \xi_1, \xi_2) \notin \mathrm{WF}^{12}_{\mathrm{bi}}(u)$. Then, by definition, we know that there exist $A_i \in \mathrm{L}^0_{\mathrm{cl}}(\Omega_i)$, non-characteristic at (x_i, ξ_i), $i = 1, 2$, such that

$$\begin{aligned}(A_1 \otimes A_2)u &\in \mathcal{C}^\infty(\Omega_1 \times \Omega_2), \\ (A_1 \otimes I)u &\in \mathcal{C}^\infty(\Omega_1, \mathcal{D}'(\Omega_2)), \\ (I \otimes A_2)u &\in \mathcal{C}^\infty(\Omega_2, \mathcal{D}'(\Omega_1)).\end{aligned}$$

By the standard pseudodifferential calculus we can thus find $A_i' \in \mathrm{L}^0(\Omega_i)$, such that $A_i + A_i'$ is elliptic in the sense of pseudodifferential operators and such that the symbol of A_i' vanishes on a conic neighborhood Γ_i of (x_i, ξ_i), $i = 1, 2$. We then have two parametrices $P_i \in \mathrm{L}^0(\Omega_i)$, $i = 1, 2$, such that

$$(P_1 \otimes P_2)\big((A_1 + A_1') \otimes (A_2 + A_2')\big) = I \otimes I - R_1 \otimes I - I \otimes R_2 - \underbrace{R_1 \otimes R_2}_{:=R},$$

with $R_i \in \mathrm{L}^{-\infty}(\Omega_i)$, $i = 1, 2$. Now, choose $H_i \in \mathrm{L}^0(\Omega_i)$ such that H_i is non-characteristic at (x_i, ξ_i) and such that the symbol of H_i vanishes outside a proper subcone of Γ_i, $i = 1, 2$. Recall that, by the standard pseudodifferential calculus, if two operators have symbols with disjoint supports, their product is a smoothing operator. Then, using Lemma 2.2.3,

$$
\begin{aligned}
&(H_1 \otimes H_2)Cu = \\
&(H_1 \otimes H_2)C\Big((P_1 \otimes P_2)\big((A_1 + A_1') \otimes (A_2 + A_2')\big) + R_1 \otimes I + I \otimes R_2 + R\Big)u \\
&= \underbrace{(H_1 \otimes H_2)C(P_1 \otimes P_2)(A_1 \otimes A_2)u}_{\in \mathcal{C}^\infty \text{by eq (2.8)}} + \underbrace{(H_1 \otimes H_2)C(P_1 \otimes P_2)(A_1' \otimes A_2)u}_{\in \mathcal{C}^\infty \text{by (2.10) and by the support of } H_1, A_1'} \\
&+ \underbrace{(H_1 \otimes H_2)C(P_1 \otimes P_2)(A_1 \otimes A_2')u}_{\in \mathcal{C}^\infty \text{by (2.9) and by the support of } H_2, A_2'} + \underbrace{(H_1 \otimes H_2)C(P_1 \otimes P_2)(A_1' \otimes A_2')u}_{\in \mathcal{C}^\infty \text{by the support of } H_1, H_2, A_1', A_2'} \\
&+ (H_1 \otimes H_2)C(R_1 \otimes I)u + (H_1 \otimes H_2)C(I \otimes R_2)u + \underbrace{(H_1 \otimes H_2)CRu}_{\in \mathcal{C}^\infty \text{because } R \in \mathrm{L}^{-\infty,-\infty}} \\
&= (H_1 \otimes H_2)C(R_1 \otimes I)u + (H_1 \otimes H_2)C(I \otimes R_2)u \quad \mathrm{mod}\, \mathcal{C}^\infty(\Omega_1 \times \Omega_2).
\end{aligned}
$$

Now, without loss of generality, we proceed with the calculations only for the term $(H_1 \otimes H_2)C(R_1 \otimes I)u$. We have

$$
\begin{aligned}
(H_1 \otimes H_2)C(R_1 \otimes I)u &= (H_1 \otimes H_2)C(R_1 \otimes (P_2(A_2 + A_2') + R_2))u \\
&= \underbrace{(H_1 \otimes H_2)C(R_1 \otimes P_2)(I \otimes A_2)u}_{\in \mathcal{C}^\infty \text{by (2.10)}} \\
&+ \underbrace{(H_1 \otimes H_2)C(R_1 \otimes P_2)(I \otimes A_2')u}_{\in \mathcal{C}^\infty \text{by the support of } H_2, A_2'} \\
&+ \underbrace{(H_1 \otimes H_2)C(R_1 \otimes I)(I \otimes R_2)u}_{\in \mathcal{C}^\infty \text{because } R \in \mathrm{L}^{-\infty,-\infty}} \in \mathcal{C}^\infty(\Omega_1 \times \Omega_2),
\end{aligned}
$$

therefore $(H_1 \otimes H_2)Cu \in \mathcal{C}^\infty(\Omega_1 \times \Omega_2)$. With similar computations, one can check that

$$
\begin{aligned}
&(H_1 \otimes I)Cu \in \mathcal{C}^\infty(\Omega_1, \mathcal{D}'(\Omega_2)), \\
&(I \otimes H_2)Cu \in \mathcal{C}^\infty(\Omega_2, \mathcal{D}'(\Omega_1)),
\end{aligned}
$$

and this proves the claim. □

The previous Lemmas leads to the next Proposition 2.2.22, dealing with the microlocality of bisingular operators. It means that this definition of wave front set is indeed suitable for the calculus of bisingular operators.

Proposition 2.2.22 (**Microlocality of bisingular operators**)**.** Let $C \in \mathrm{L}^{m_1,m_2}(\Omega_1 \times \Omega_2)$, $u \in \mathcal{D}'(\Omega_1 \times \Omega_2)$. Then we have $\mathrm{WF}_{\mathrm{bi}}(Cu) \subseteq \mathrm{WF}_{\mathrm{bi}}(u)$.

Proof. The claim follows directly from Lemma 2.2.20 and Lemma 2.2.21, recalling Definition 2.2.9. □

2.2.3 Microelliptic properties of bisingular operators

From the previous Proposition 2.2.22, we can conclude that bielliptic operators preserve the bi-wave front set:

Corollary 2.2.23. Let $A \in \mathrm{L}^{m_1,m_2}(\Omega_1,\Omega_2)$ be bi-elliptic. Then $\mathrm{WF}_{\mathrm{bi}}(Au) = \mathrm{WF}_{\mathrm{bi}}(u)$.

Proof. One inclusion follows directly form Proposition 2.2.22. The other follows arguing as in Corollary 2.1.16. □

Next, we study the microellipticity properties of bisingular operators. To this aim, we need a suitable definition of a characteristic set. As in Definition 1.2.5, Ω_1 and Ω_2 are closed manifolds.

Definition 2.2.24. Let $B \in \mathrm{L}^{m_1,m_2}(\Omega_1,\Omega_2)$ and $v_0 = (x_0,\xi_0) \in \Omega_1 \times (\mathbb{R}^{n_1} \setminus 0)$. We say that B is not 1-characteristic on $V := \pi_2^{-1}(v_0) := \{(x_0, y, \xi_0, 0) : y \in \Omega_2\}$ if

1. for all $v \in V$ there exists an open conic neighbourhood Θ of v such that $\sigma^{12}(B) \neq 0$ on $\Theta \setminus (\mathbb{R}^+ v)$;
2. $\sigma^1(B) \in \mathrm{L}^{m_2}_{\mathrm{cl}}(\Omega_2)$ is invertible with inverse in $\mathrm{L}^{-m_2}_{\mathrm{cl}}(\Omega_2)$ in an open conic neighbourhood Γ of v_0.

Let $\mathrm{Char}^1_{\mathrm{bi}}(B)$ be the set of all V such that B is 1-characteristic at V. We define $\mathrm{Char}^2_{\mathrm{bi}}(B)$ similarly for $W := \pi_2^{-1}(w_0)$, $w_0 \in \Omega_2 \times (\mathbb{R}^{n_2} \setminus 0)$, by exchanging the roles of $\sigma^1(B)$ and $\sigma^2(B)$. Finally, we define $\mathrm{Char}^{12}_{\mathrm{bi}}(B)$ as the set of points $z = (x_1, x_2, \xi_1, \xi_2)$, $|\xi_1||\xi_2| \neq 0$, where $\sigma^{12}(B)(z) = 0$. Set

$$\mathrm{Char}_{\mathrm{bi}}(B) := \mathrm{Char}^1_{\mathrm{bi}}(B) \cup \mathrm{Char}^2_{\mathrm{bi}}(B) \cup \mathrm{Char}^{12}_{\mathrm{bi}}(B).$$

Remark 2.2.25. B is bi-elliptic if and only if $\mathrm{Char}_{\mathrm{bi}}(B) = \emptyset$.

Remark 2.2.26. As consequence of Remark 2.2.25, an equivalent definition of $\mathrm{WF}_{\mathrm{bi}}(u)$ is the following one:

$$\mathrm{WF}^1_{\mathrm{bi}}(u) = \bigcap_{\substack{A \in \mathrm{L}^0_{\mathrm{cl}}(\Omega_1) \\ (A\otimes I)u \in \mathcal{C}^\infty(\Omega_1, \mathcal{D}'(\Omega_2))}} \underbrace{\mathrm{Char}^1_{\mathrm{bi}}(A \otimes I),}_{\mathrm{Char}_{\mathrm{cl}}(A)\times(\Omega_2\times(\mathbb{R}^{n_2}\setminus 0))}$$

$$\mathrm{WF}^2_{\mathrm{bi}}(u) = \bigcap_{\substack{A \in \mathrm{L}^0_{\mathrm{cl}}(\Omega_2) \\ (I\otimes A)u \in \mathcal{C}^\infty(\Omega_2, \mathcal{D}'(\Omega_1))}} \underbrace{\mathrm{Char}^2_{\mathrm{bi}}(I \otimes A),}_{(\Omega_1\times(\mathbb{R}^{n_1}\setminus 0))\times\mathrm{Char}_{\mathrm{cl}}(A)}$$

$$\mathrm{WF}^{12}_{\mathrm{bi}}(u) = \bigcap_{\substack{A_i \in \mathrm{L}^0_{\mathrm{cl}}(\Omega_1) \\ (A_1\otimes A_2)u \in \mathcal{C}^\infty \\ (A_1\otimes I)u \in \mathcal{C}^\infty(\Omega_1, \mathcal{D}'(\Omega_2)) \\ (I\otimes A_2)u \in \mathcal{C}^\infty(\Omega_2, \mathcal{D}'(\Omega_1))}} \mathrm{Char}^{12}_{\mathrm{bi}}(A_1 \otimes A_2).$$

Operator A	$\mathrm{Char}^1_{\mathrm{bi}}(A)$	$\mathrm{Char}^2_{\mathrm{bi}}(A)$	$\mathrm{Char}^{12}_{\mathrm{bi}}(A)$
$I \otimes I$	$\emptyset$	$\emptyset$	$\emptyset$
$-\Delta_1 \otimes I + I \otimes (-\Delta_2)$	$\Omega \times \mathbb{R}^{n_1}_* \times \{0\}$	$\Omega \times \{0\} \times \mathbb{R}^{n_2}_*$	$\Omega \times \mathbb{R}^{n_{12}}_*$
$-\Delta_1 \otimes (-\Delta_2)$	$\Omega \times \mathbb{R}^{n_1}_* \times \{0\}$	$\Omega \times \{0\} \times \mathbb{R}^{n_2}_*$	$\emptyset$
$(-\Delta_1 + I) \otimes (-\Delta_2)$	$\Omega \times \mathbb{R}^{n_1}_* \times \{0\}$	$\emptyset$	$\emptyset$
$(-\Delta_1 + I) \otimes (-\Delta_2 + I)$	$\emptyset$	$\emptyset$	$\emptyset$
$(-\Delta_1 + I)^{-1} \otimes (-\Delta_2 + I)^{-1}$	$\emptyset$	$\emptyset$	$\emptyset$

Table 2.1: $\mathrm{Char}_{\mathrm{bi}}(A)$ for model cases of bisingular operators

With the definition of $\mathrm{Char}_{\mathrm{bi}}(u)$ we can review the model cases of Table 1.1 into the next Table 2.1, setting $\mathbb{R}^n_* = \mathbb{R}^n \setminus 0$, $\Omega = \Omega_1 \times \Omega_2$ and $\mathbb{R}^{n_{12}}_* = \mathbb{R}^{n_1}_* \times \mathbb{R}^{n_2}_*$.

Lemma 2.2.27. Let $C \in \mathrm{L}^{m_1,m_2}(\Omega_1, \Omega_2)$ be such that

$$\mathrm{Char}^1_{\mathrm{bi}}(C) \cap (\Gamma \times \Omega_2 \times \{0\}) = \emptyset.$$

Let $a \in \mathrm{S}^0(\Omega_1)$ have support in a closed cone Γ and be non-characteristic (in the sense of ΨDO) in Γ^0. Then, there exists $H \in \mathrm{L}^{-m_1,-m_2}(\Omega_1, \Omega_2)$ such that

$$HC = A \otimes I - R,$$

where $R \in \mathrm{L}^{-\infty,0}$.

Proof. The requirements on the support of a mean precisely that C is elliptic with respect to a in the sense of [Cor95], Theorem 2.3.3. Therefore, by the classical calculus of pseudodifferential operators, we can find a symbol $b \in \mathrm{S}^{-m_1,-m_2}$ such that for all fixed (x_2, ξ_2) the operator $B(x_2, \xi_2) = b(x_1, x_2, D_1, \xi_2)$ is a local parametrix with respect to a namely,

$$B(x_2, \xi_2)\, \sigma^2(C)(x_2, \xi_2) = R(x_2, \xi_2) + (A \otimes 1)(x_2, \xi_2),$$

where $R(x_2, D_2) \in \mathrm{L}^{-\infty,0}(\Omega_1, \Omega_2)$ and $B(x_2, D_2) \in \mathrm{L}^{-m_1,-m_2}(\Omega_1, \Omega_2)$. Define H as the operator with symbol

$$h = \psi_1(x_1, \xi_1) a_{m_1;\cdot} b_{m_1;\cdot} + \psi_2(x_2, \xi_2) a_{\cdot;m_2} c^{-1}_{\cdot;m_2} - \psi_1(x_1, \xi_1)\psi_2(x_2, \xi_2) a_{m_1;m_2} c^{-1}_{m_1;m_2}.$$

Using the calculus and Theorem 1.2.13, it is straightforward to check that H matches the requirements. □

Lemma 2.2.28. Let $C \in \mathrm{L}^{m_1,m_2}(\Omega_1 \times \Omega_2)$, $u \in \mathcal{D}'(\Omega_1 \times \Omega_2)$. Then we have

$$\mathrm{WF}^1_{\mathrm{bi}}(u) \subseteq \mathrm{Char}^1_{\mathrm{bi}}(C) \cup \mathrm{WF}^1_{\mathrm{bi}}(Cu).$$

Proof. Let $(x_1, x_2, \xi_1, 0) \notin \mathrm{Char}^1_{\mathrm{bi}}(C) \cup \mathrm{WF}^1_{\mathrm{bi}}(Cu)$. Then, there exists $A \in \mathrm{L}^0(\Omega_1)$ such that $(A \otimes I)HCu \in \mathcal{C}^\infty(\Omega_1, \mathcal{D}'(\Omega_2))$, with H as in Lemma 2.2.27, due to microlocality of H (see Proposition 2.2.22). Then we find

$$\begin{aligned}(A^2 \otimes I)u &= (A \otimes I)(R - HC)u \\ &= \underbrace{(A \otimes I)R}_{\in \mathrm{L}^{-\infty,0}} u - \underbrace{(A \otimes I)HCu}_{\in \mathcal{C}^\infty(\Omega_1, \mathcal{D}'(\Omega_2)) \text{ by assumption on } A} \in \mathcal{C}^\infty.\end{aligned}$$

Using Lemma 2.2.3 on the first summand, this proves the claim. □

Using the previous Lemma 2.2.28, we obtain the microellipticity of bisingular operators with respect to the components $\mathrm{WF}^1_{\mathrm{bi}}(u)$ and $\mathrm{WF}^2_{\mathrm{bi}}(u)$ of $\mathrm{WF}_{\mathrm{bi}}(u)$.

Proposition 2.2.29. Let $C \in \mathrm{L}^{m_1,m_2}(\Omega_1, \Omega_2)$, $u \in \mathcal{D}'(\Omega_1 \times \Omega_2)$. Then

$$\mathrm{WF}^i_{\mathrm{bi}}(u) \subseteq \mathrm{Char}^i_{\mathrm{bi}}(C) \cup \mathrm{WF}^i_{\mathrm{bi}}(Cu), \quad i = 1, 2.$$

It is remarkable that we do not obtain full microellipticity, i.e. with respect to the $\mathrm{WF}^{12}_{\mathrm{bi}}$-component. This can be seen through the following example.

Example 2.2.30. Consider $C = -\Delta \otimes (-\Delta)$, $u = \delta \otimes 1 + 1 \otimes \delta$. Then $\mathrm{Char}^{12}_{\mathrm{bi}}(C) = \emptyset$ and $Cu = 0 \in \mathcal{C}^\infty(\Omega_1 \times \Omega_2)$, i.e. $\mathrm{WF}^{12}_{\mathrm{bi}}(Cu) = \emptyset$. Take any A_1 non-characteristic at $(0, \xi_1)$, $\xi_1 \neq 0$. Then $(A_1 \otimes I)u = (A_1\delta) \otimes 1$, which is never in $\mathcal{C}^\infty(\Omega_1, \mathcal{D}'(\Omega_2))$. This means that $\mathrm{WF}^{12}_{\mathrm{bi}}(u)$ is non-empty, because the requirements (2.9) fail to hold. By a similar argument, one can show that also the requirements (2.10) are not satisfied.

Remark 2.2.31. The problems encountered in Example 2.2.30 could be circumvented by imposing stronger invertibility conditions in the definition of $\mathrm{Char}^{12}_{\mathrm{bi}}(A)$. This would, however, break the characterization of the wave front set in Remark 2.2.26, and lead to a loss of local information. Moreover, it would yield no additional interesting cases, compared to those already covered by Corollary 2.2.23.

With our definition we have instead the following Lemma, which can be regarded as a microellipticity result for the 12-component of Definition 2.2.9, for operators given by a tensor product.

Lemma 2.2.32. Let $C_i \in \mathrm{L}^{m_i}_{\mathrm{cl}}(\Omega_i)$, $i = 1, 2$, $u \in \mathcal{D}'(\Omega_1 \times \Omega_2)$. If

$$(x_1, x_2, \xi_1, \xi_2) \notin \mathrm{Char}^{12}_{\mathrm{bi}}(C_1 \otimes C_2) \cup \mathrm{WF}^{12}_{\mathrm{bi}}((C_1 \otimes C_2)u),$$

there exist operators $H_i \in \mathrm{L}^0(\Omega_i)$, non-characteristic at (x_i, ξ_i), $i = 1, 2$, such that $(H_1 \otimes H_2)u \in \mathcal{C}^\infty(\Omega_1 \times \Omega_2)$.

Proof. By the standard pseudodifferential calculus we can choose $B_i \in \mathrm{L}^{-m_i}_{\mathrm{cl}}(\Omega_i)$ non-characteristic at (x_i, ξ_i), $i = 1, 2$. Then the product $B_iC_i \in \mathrm{L}^0_{\mathrm{cl}}(\Omega_i)$ is non-characteristic at (x_i, ξ_i), $i = 1, 2$. Proposition 2.2.22 and the definition of the bi-wave front set guarantee us the existence of $A_i \in \mathrm{L}^0(\Omega_i)$, non-characteristic at (x_i, ξ_i),$i = 1, 2$, such that $(A_1 \otimes A_2)(B_1C_1 \otimes B_2C_2)u \in \mathcal{C}^\infty(\Omega_1 \times \Omega_2)$. Therefore $H_i := A_iB_iC_i$, $i = 1, 2$, fulfill the claim. □

2.3 The SG wave front set

In this section we will compare the bisingular calculus with the so-called SG calculus, introduced on $\mathbb{R}^n$ by H. O. Cordes [Cor95] and C. Parenti [Par72], see also R. Melrose [Mel95], Y. Egorov and B.-W. Schulze [ES97], and E. Schrohe [Sch87]. SG calculus is a global calculus obtained from the classical calculus by treating the variables and covariables equivalently, that is, by imposing on the symbols similar estimates as in bisingular calculus. These a priori formal similarities lead to interesting similarities in the calculus and in the analysis of singularities. However, the two calculi also differ in important aspects, as we will point out throughout the section.

Definition 2.3.1. A function $a(x,\xi) \in \mathcal{C}^\infty(\mathbb{R}^{2n})$ is called a SG symbol belonging to $\mathrm{SG}^{\mu,m}(\mathbb{R}^n) := \mathrm{SG}^{\mu,m}$ if, for every $\alpha, \beta \in \mathbb{Z}_+^n$, there exists a constant $C_{\alpha,\beta} > 0$ such that

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta} \langle\xi\rangle^{\mu-|\beta|} \langle x\rangle^{m-|\alpha|}$$

for every $x, \xi \in \mathbb{R}^n$. A SG pseudodifferential operator is an operator of the form

$$Au(x) := \int e^{ix\cdot\xi}\, a(x,\xi)\hat{u}(\xi)\, đ\xi,\ u \in \mathscr{S}(\mathbb{R}^n),$$

and the class of operators with symbols in $\mathrm{SG}^{\mu,m}$ is denoted by $\mathrm{LG}^{\mu,m}$.

A symbol $a \in \mathrm{SG}^{\mu,m}$ is called SG classical if it admits a homogeneous expansion with respect to ξ, for $|\xi| >> 1$, a homogeneous expansion in x, for $|x| >> 1$, and the two expansions satisfy certain compatibility conditions. We refer here to [Cor95, ES97] for a precise definition of classical SG symbols. We limit our attention (in this context) to classical operators, i.e. such that their symbols are SG classical. As usual, one proceeds to develop a calculus for these operators. As every classical SG operator A is also a classical pseudodifferential operator in the Hörmander calculus, it admits a principal symbol $\sigma^\psi(A)$, homogeneous in the covariable. In addition, by exchanging the roles of the variables and covariables, one obtains a symbol $\sigma^e(A)$, homogeneous in the variable x. The two functions satisfy a compatibility condition, i.e. that there is a third, bihomogeneous principal symbol component $\sigma^{\psi e}(A)$, the leading term of the corresponding expansions of the ψ and e-symbols. The *principal homogeneous symbol* of the operator is then the couple $(\sigma^\psi(A), \sigma^e(A))$ which gives rise to the principal symbol

$$\mathrm{Sym}_p(x,\xi) = \phi_\psi(\xi)\sigma^\psi(A) + \phi_e(x)\sigma^e(A) - \phi_\psi(x)\phi_e(\xi)\sigma^{\psi e}(A),$$

where ϕ_*, $* = \phi, e, \phi e$, are 0-excision functions.
So far, this is very similar to the bisingular calculus, but the expansion formula for the symbol of a product is in fact a lot simpler. The operator

compositions arising there are not present, and the composition formula is just the one corresponding to the c^{12}-term in Theorem 1.2.13. This leads to a definition of ellipticity close to our notion of 12-ellipticity, as no such thing as full invertibility of the symbols as operators is needed in the parametrix construction.

Definition 2.3.2. A symbol $a \in \mathrm{SG}^{\mu,m}$ is SG-elliptic if there exist constants $R, C_1, C_2 > 0$ such that

$$C_1 \langle \xi \rangle^{\mu} \langle x \rangle^{m} \leq |a(x,\xi)| \leq C_2 \langle \xi \rangle^{\mu} \langle x \rangle^{m}$$

when $|x| + |\xi| \geq R$.

With this notion of ellipticity, we have the Fredholm property, i.e. an SG-elliptic operator admits a parametrix in the calculus. Another important aspect to notice is that in the context of the SG calculus we have the following property of the commutator of two operators.

Proposition 2.3.3. Let $p \in \mathrm{SG}^{m,\mu}(\mathbb{R}^n), q \in \mathrm{SG}^{r,\nu}(\mathbb{R}^n)$. Then the commutator $[P,Q] := PQ - QP$ belongs to $\mathrm{LG}^{m+r-1,\mu+\nu-1}(\mathbb{R}^n)$.

We stress here this relevant difference in comparison with the bisingular setting, see Corollary 1.2.14 and Example 1.2.16. The notion of SG calculus can be used to introduce a concept of global analysis of singularities. In the sequel, we refer to [Cor95], [CM03], see also [CJT13a], [CJT13b]. In [Mel94] a geometric scattering version of the SG wave front set is studied. First, we introduce SG characteristic sets and the SG wave front set.

Definition 2.3.4. Let $A \in \mathrm{LG}^{\mu,m}$. Define the SG characteristic set of A as

$$\mathrm{Char}_{\mathrm{SG}}(A) = \mathrm{Char}^{\psi}_{\mathrm{SG}}(A) \cup \mathrm{Char}^{e}_{\mathrm{SG}}(A) \cup \mathrm{Char}^{\psi e}_{\mathrm{SG}}(A),$$

where

$$\begin{aligned}
\mathrm{Char}^{\psi}_{\mathrm{SG}}(A) &= \{(x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0) : \sigma^{\psi}(A)(x,\xi) = 0\},\\
\mathrm{Char}^{e}_{\mathrm{SG}}(A) &= \{(x,\xi) \in (\mathbb{R}^n \setminus 0) \times \mathbb{R}^n : \sigma^{e}(A)(x,\xi) = 0\},\\
\mathrm{Char}^{\psi e}_{\mathrm{SG}}(A) &= \{(x,\xi) \in (\mathbb{R}^n \setminus 0) \times (\mathbb{R}^n \setminus 0) : \sigma^{\psi e}(A)(x,\xi) = 0\}.
\end{aligned}$$

Definition 2.3.5. Let $u \in \mathscr{S}'(\mathbb{R}^n)$. Define the SG wave front set of u as

$$\mathrm{WF}_{\mathrm{SG}}(u) = \mathrm{WF}^{\psi}_{\mathrm{SG}}(u) \cup \mathrm{WF}^{e}_{\mathrm{SG}}(u) \cup \mathrm{WF}^{\psi e}_{\mathrm{SG}}(u),$$

where

$$\mathrm{WF}^{\psi}_{\mathrm{SG}}(u) = \bigcap_{\substack{A\in \mathrm{LG}^{0,0}(\mathbb{R}^n)\\ Au\in \mathscr{S}(\mathbb{R}^n)}} \mathrm{Char}^{\psi}_{\mathrm{SG}}(A),$$
$$\mathrm{WF}^{e}_{\mathrm{SG}}(u) = \bigcap_{\substack{A\in \mathrm{LG}^{0,0}(\mathbb{R}^n)\\ Au\in \mathscr{S}(\mathbb{R}^n)}} \mathrm{Char}^{e}_{\mathrm{SG}}(A),$$
$$\mathrm{WF}^{\psi e}_{\mathrm{SG}}(u) = \bigcap_{\substack{A\in \mathrm{LG}^{0,0}(\mathbb{R}^n)\\ Au\in \mathscr{S}(\mathbb{R}^n)}} \mathrm{Char}^{\psi e}_{\mathrm{SG}}(A).$$

This notion has the following properties:

Proposition 2.3.6. Let $u, v \in \mathscr{S}'(\mathbb{R}^n)$, $f \in \mathscr{S}(\mathbb{R}^n)$. Then:

1. $\mathrm{WF}_{\mathrm{SG}}(u)$ is a closed set and $\mathrm{WF}^{\psi}_{\mathrm{SG}}(u)$ is conical with respect to the covariable ξ, $\mathrm{WF}^{e}_{\mathrm{SG}}(u)$ is conical with respect to the variable x and $\mathrm{WF}_{\mathrm{SG}}(u)$ is conical with respect to both of them, independently;

2. $(x, \xi) \in \mathrm{WF}_{\mathrm{SG}}(u) \Leftrightarrow (\xi, -x) \in \mathrm{WF}_{\mathrm{SG}}(\mathcal{F}u)$;

3. $\mathrm{WF}_{\mathrm{SG}}(u + v) \subseteq \mathrm{WF}_{\mathrm{SG}}(u) \cup \mathrm{WF}_{\mathrm{SG}}(v)$; $\mathrm{WF}_{\mathrm{SG}}(fu) \subseteq \mathrm{WF}_{\mathrm{SG}}(u)$;

4. $\mathrm{WF}_{\mathrm{SG}}(u) = \emptyset \Leftrightarrow u \in \mathscr{S}(\mathbb{R}^n)$.

It is immediate to notice the similarities, but also apparent differences, with the bisingular notion:

- Fourier transformation (i.e. exchange of variables and covariables) corresponds to the exchange of variables x_1 and x_2 in Proposition 2.2.18;

- The conical properties of the individual components of the wave front sets correspond to the homogeneity properties of the corresponding principal symbol part;

- The global (that is, $\mathscr{S}$-)regularity is of course due to the fact that *SG*-calculus imposes bounds on the variables also.

Moreover, *SG* operators satisfy *SG* microlocality and microellipticity, namely

Proposition 2.3.7. Let $u \in \mathscr{S}'(\mathbb{R}^n)$ and $C \in \mathrm{LG}^{m,\mu}$. Then we have the inclusion

$$\mathrm{WF}_{\mathrm{SG}}(Cu) \subseteq \mathrm{WF}_{\mathrm{SG}}(u) \subseteq \mathrm{Char}_{\mathrm{SG}}(C) \cup \mathrm{WF}_{\mathrm{SG}}(Cu).$$

This again stems from an individual double inclusion with respect to each wave front set component. A capital difference lies, however, in the structure of the wave front set. For the components of the SG wave front set one finds

$$\mathrm{WF}_{\mathrm{SG}}(u) \subseteq \big(\mathbb{R}^n \times (\mathbb{R}^n \setminus 0)\big) \cup \big((\mathbb{R}^n \setminus 0) \times \mathbb{R}^n\big) \cup \big((\mathbb{R}^n \setminus 0) \times (\mathbb{R}^n \setminus 0)\big).$$

Here, the $\big(\mathbb{R}^n \times (\mathbb{R}^n \setminus 0)\big) \ni (x, \xi)$ component corresponds exactly to singularities at finite arguments x with propagation direction ξ, the $\big(\mathbb{R}^n \times (\mathbb{R}^n \setminus 0)\big)$ component yields the same interpretation in the Fourier transformed space (growth singularities of u become differential singularities of $\hat{u}$) and the $\big((\mathbb{R}^n \setminus 0) \times (\mathbb{R}^n \setminus 0)\big)$ component corresponds to high oscillations or lack of decay present at infinite arguments.
In the bisingular case, new phenomena are present. The 12-component has the classical interpretation, in the sense that it includes all the 'classical' singularities (see Lemma 2.2.11), but the other components lose some amount of localization. This is reflected in the structure of the (1- and 2-components) of the bi-wave front set. In fact,

$$\begin{aligned}\mathrm{WF}_{\mathrm{bi}}(u) \subset (\Omega_1 \times \Omega_2) \times (\mathbb{R}^{n_1+n_2} \setminus 0) = (\Omega_1 \times \Omega_2) &\times \big((\mathbb{R}^{n_1} \setminus 0) \times \{0\}\big) \\ \cup\, (\Omega_1 \times \Omega_2) &\times \big(\{0\} \times (\mathbb{R}^{n_2} \setminus 0)\big) \\ \cup\, (\Omega_1 \times \Omega_2) &\times \big((\mathbb{R}^{n_1} \setminus 0) \times (\mathbb{R}^{n_2} \setminus 0)\big),\end{aligned}$$

where if, for instance, for some x_1 we have that $(x_1, x_2, 0, \xi_2)$ belongs to $\mathrm{WF}_{\mathrm{bi}}(u)$, all $(y, x_2, 0, \xi_2)$ belong to $\mathrm{WF}_{\mathrm{bi}}(u)$ as well. This is due to the fact that bi-ellipticity involves true invertibility, i.e. a non-local requirement.
Another difference arises as follows. The 1- and 2-component can be understood as the boundary faces of the 12-component, whereas in the SG case the ψe-component is interpreted as the corner of the wave front space where the e- and ψ-component meet, i.e. the roles as boundaries are interchanged, see Figure 2.1.
The above observation imply the next result.

Example 2.3.8. Consider the one dimensional case. Following here Example 2.7 in [CM03], there exists a temperate distribution $u(x) = e^{ix^2/2}$, $x \in \mathbb{R}$, such that $\mathrm{WF}^{\psi}_{\mathrm{SG}}(u) = \emptyset = \mathrm{WF}^{e}_{\mathrm{SG}}(u)$ and $\mathrm{WF}^{\psi e}_{\mathrm{SG}}(u) = (\mathbb{R} \setminus 0) \times (\mathbb{R} \setminus 0)$.
However, there cannot exist a distribution $v \in \mathcal{E}'(\Omega_1 \times \Omega_2)$, $\Omega_1, \Omega_2 \subset \mathbb{R}$, such that $\mathrm{WF}^{1}_{\mathrm{bi}}(v) = \emptyset = \mathrm{WF}^{2}_{\mathrm{bi}}(v)$ and $\mathrm{WF}^{12}_{\mathrm{bi}}(v) = \Omega_1 \times \Omega_2 \times (\mathbb{R} \setminus 0) \times (\mathbb{R} \setminus 0)$. This is because $\mathrm{WF}^{12}_{\mathrm{bi}}(v) = \Omega_1 \times \Omega_2 \times (\mathbb{R} \setminus 0) \times (\mathbb{R} \setminus 0)$ is an open set (and not a clopen set), and $\mathrm{WF}_{\mathrm{bi}}(v)$ has to be closed.
Similarly $\tilde{v} = \delta \otimes 1$ satisfies $\mathrm{WF}^{1}_{\mathrm{bi}}(\tilde{v}) = (\{0\} \times \Omega_2) \times ((\mathbb{R} \setminus 0) \times \{0\})$, $\mathrm{WF}^{12}_{\mathrm{bi}}(\tilde{v}) = \emptyset = \mathrm{WF}^{2}_{\mathrm{bi}}(\tilde{v})$, but there cannot exist a distribution $\tilde{u}$ such that $\mathrm{WF}^{e}_{\mathrm{SG}}(\tilde{u}) = \mathbb{R} \times (\mathbb{R} \setminus 0)$ with $\mathrm{WF}^{\psi e}_{\mathrm{SG}}(\tilde{u}) = \emptyset$, again due to closedness .

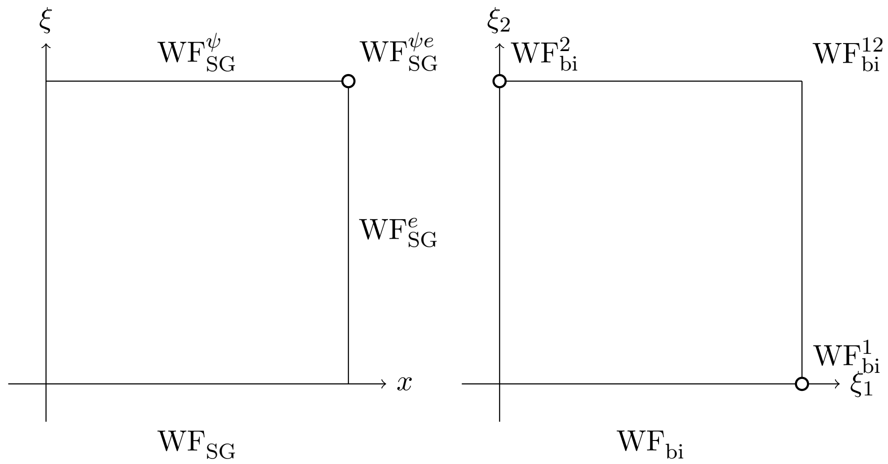


Figure 2.1: A schematic comparison of the components of $\mathrm{WF}_{\mathrm{SG}}$ and $\mathrm{WF}_{\mathrm{bi}}$

Chapter 3

Spectral asymptotics

In this chapter we will prove an estimate for the spectral counting function of the tensor product of two pseudodifferential operators. Then, we will show that the estimate is sharp. In the whole chapter, $\sigma(A)$ denotes the spectrum of the operator A.

3.1 Spectrum and functional calculus

We begin recalling some elements of the functional calculus for elliptic pseudodifferential operators.

3.1.1 Complex powers of elliptic operators

Let A be a classical elliptic pseudodifferential operators of order $m > 0$, over an n dimensional closed manifold M, and let $z \in \mathbb{C}$. In this subsection we will give meaning to the formula

$$A_z = \frac{i}{2\pi} \int_\Gamma \lambda^z (A - \lambda I)^{-1} \, d\lambda, \tag{3.1}$$

where Γ is a special path in the complex plane and the integral is to be understood in the Dunford-Schwartz sense, see e.g. [DS58].

Theorem 3.1.1. Let $A \in \mathrm{L}^m_{\mathrm{cl}}(M)$, $m > 0$, be elliptic[1], with principal symbol $a_m(x, \xi)$. Let Λ be a closed angle in the complex plane with vertex $0 \in \mathbb{C}$, and set

$$\Lambda_R := \{\lambda \in \Lambda : |\lambda| \geq R\},$$

$R \in \mathbb{R}$. If, for every $\xi \neq 0$, $a_m(x, \xi)$ does not take values in Λ, then

a) there exists $R > 0$ such that $\Lambda_R \cap \sigma(A) = \emptyset$, that is, for every $\lambda \in \Lambda_R$ the resolvent $R_\lambda = (A - \lambda I)^{-1}$ exists and is a bounded linear operator;

[1] We recall that A can be extended to a closed, unbounded operator with dense domain $A : \mathrm{H}^m(M) \subset L^2(M) \to L^2(M)$. For details, see [Shu01].

b) the following norm estimate holds true

$$\|(A-\lambda I)^{-1}\|_{s,s+l} \leq \frac{C_{s,l}}{|\lambda|^{1-\frac{l}{m}}}, \quad 0 \leq l \leq m, \quad \lambda \in \Lambda_R,$$

where $\|\cdot\|_{s,s+l}$ is the norm in the space of linear continuous operators from $\mathrm{H}^s(M)$ to $\mathrm{H}^{s+l}(M)$.

Due to the previous Theorem 3.1.1, we see that if $a_m(x,\xi)$ does not take values in Γ, we can find $R>0$ such that $(A-\lambda I)^{-1}$ exists for every $\lambda \in \Lambda_R$. Moreover, due to the norm estimate, the absolute convergence of the integral in (3.1) is granted by $\Re(z)<0$. Thus, we assume

1) $a_m(x,\xi)-\lambda \neq 0$ for $\xi \neq 0$ and $\lambda \in (-\infty,0]$;

2) $\sigma(A) \cap (-\infty,0] = \emptyset$.

In what follows, we consider the path $\Gamma := -\Gamma_1 - \Gamma_2 + \Gamma_3$, with

$$\begin{aligned}
\Gamma_1 &= re^{i\pi} \quad \rho \leq r < +\infty,\\
\Gamma_2 &= \rho e^{i\theta} \quad -\pi \leq \theta \leq \pi,\\
\Gamma_3 &= re^{-i\pi} \quad \rho \leq r < +\infty,
\end{aligned}$$

with ρ such that $\sigma(A)$ does not intersect the disk $|z| \leq 2\rho$ in the complex plane, see Figure 3.1.

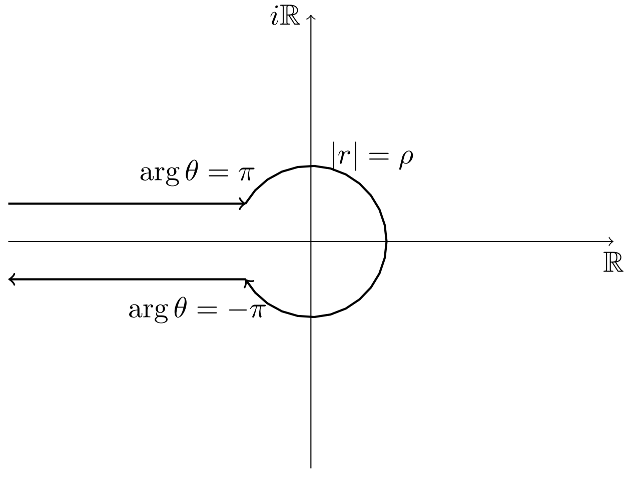


Figure 3.1: The path Γ.

Since $m>0$, the conditions above can always be achieved. In fact, by Theorem 3.1.1, $(A-\lambda I)^{-1}$ exists for every $|\lambda|>R$, then $\sigma(A) \neq \mathbb{C}$, thus

$\sigma(A)$ is discrete, see [Shu01]. Since $0 \notin \sigma(A)$, we can always find ρ with the required property. Moreover, we can always consider a sector

$$\Lambda' = \{z \in \mathbb{C} : \pi - \varepsilon \leq \arg(z) \leq \pi + \varepsilon\},$$

$\varepsilon > 0$, such that

1') $a_m(x, \xi) - \lambda \neq 0$ for $\xi \neq 0$ and $\lambda \in \Lambda'$,

2') $\sigma(A) \cap \Lambda' = \emptyset$.

From now on, we will always consider sectors Λ' with these properties. Then, again for the norm estimate in Theorem 3.1.1, the integral (3.1) converges in the operator norm of $\mathscr{L}(L^2(M))$ if $\Re(z) < 0$, and also A_z is a bounded operator on $L^2(M)$. Moreover, again due to this estimate, A_z converges in the operator norm of $\mathscr{L}(\mathrm{H}^s(M))$, for arbitrary $s \in \mathbb{R}$, and maps $\mathrm{H}^s(M)$ to itself. Hence, this implies that A_z maps $\mathcal{C}^\infty(M)$ as well as $\mathcal{D}'(M)$ to itself, since

$$\mathcal{C}^\infty(M) = \bigcap_s \mathrm{H}^s(M)$$

and

$$\mathcal{D}'(M) = \bigcup_s \mathrm{H}^s(M).$$

Proposition 3.1.2. a) For $\Re(z) < 0$ and $\Re(w) < 0$, A_z has the semigroup property

$$A_z A_w = A_{z+w}.$$

b) If $k \in \mathbb{Z}$ and $k > 0$ then

$$A_{-k} = (A^{-1})^k = \underbrace{A^{-1} \circ \cdots \circ A^{-1}}_{k \text{ times}}.$$

c) For arbitrary $s \in \mathbb{R}$ and $\Re(z) < 0$, A_z is an holomorphic operator with values in the algebra of bounded operators on the Hilbert space $\mathrm{H}^s(M)$.

Proof. a) Fist, we recall the formula

$$(A - \lambda I)^{-1}(A - \mu I)^{-1} = \frac{1}{\lambda - \mu}[(A - \lambda I)^{-1} - (A - \mu I)^{-1}], \qquad (3.2)$$

which can be proved by multiplying both sides of (3.2) by $(A - \lambda I)(A - \mu I)$. Then, we set $\Gamma' = -\Gamma'_1 - \Gamma'_2 + \Gamma'_3$, with

$$\Gamma'_1 = re^{i(\pi-\varepsilon)} \quad \frac{3}{2}\rho \leq r < +\infty$$
$$\Gamma_2 = \frac{3}{2}\rho e^{i\theta} \quad -\pi + \varepsilon \leq \theta \leq \pi - \varepsilon$$
$$\Gamma_3 = re^{-i(\pi-\varepsilon)} \quad \frac{3}{2}\rho \leq r < +\infty$$

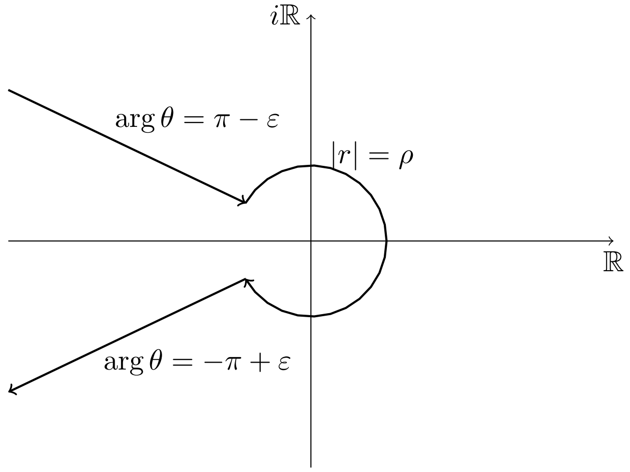


Figure 3.2: The path Γ'.

with $\varepsilon > 0$ as in the definition of Λ'. We obtain the path in Figure 3.2. Thus, we can write

$$A_z A_w = -\frac{1}{4\pi^2}\int_{\Gamma'}\int_{\Gamma}(A-\lambda I)^{-1}(A-\mu I)^{-1}\lambda^z\mu^w\,d\mu\,d\lambda.$$

Using (3.2) and the Cauchy formula (since the contour Γ is contained within Γ'), we get

$$\begin{aligned}
A_z A_w &= -\frac{1}{4\pi^2}\int_{\Gamma'}\int_{\Gamma}(A-\lambda I)^{-1}(A-\mu I)^{-1}\lambda^z\mu^w\,d\mu\,d\lambda\\
&= -\frac{1}{4\pi^2}\int_{\Gamma'}\int_{\Gamma}\frac{\lambda^z\mu^w}{\lambda-\mu}[(A-\lambda I)^{-1}-(A-\mu I)^{-1}]\,d\mu\,d\lambda\\
&= \frac{i}{2\pi}\int_{\Gamma'}(A-\lambda I)^{-1}\lambda^{z+w}\,d\lambda + \frac{1}{4\pi^2}\int_{\Gamma}\int_{\Gamma'}(A-\mu I)^{-1}\frac{\lambda^z\mu^w}{\lambda-\mu}\,d\lambda\,d\mu\\
&= A_{z+w} + 0 = A_{z+w},
\end{aligned}$$

which is the claim.

b) Consider the identity

$$(A-\mu^{-1}I)^{-1} = \mu A^{-1}(\mu I - A^{-1})^{-1}, \tag{3.3}$$

which can be proved taking the inverses of both sides of (3.3). Now, notice that for $z = -1, -2, \dots$, we have $(re^{i\pi})^z = (re^{-i\pi})^z$, and the integrals along the straight line parts of Γ cancel each others. Therefore,

$$A_{-k} = \frac{i}{2\pi}\int_{|\lambda|=\rho}\lambda^{-k}(A-\lambda I)^{-1}\,d\lambda, \tag{3.4}$$

moving on the path clockwise. Now, with the change of variable $\lambda = \frac{1}{\mu}$, using (3.3), we find

$$\begin{aligned} A_{-k} &= -\frac{i}{2\pi}\int_{|\mu|=\frac{1}{\rho}} \mu^k (A-\mu^{-1}I)^{-1}\mu^{-2}\,d\mu \\ &= -\frac{iA^{-1}}{2\pi}\int_{|\mu|=\frac{1}{\rho}} \mu^{k-1}(\mu I - A^{-1})^{-1}\,d\mu \\ &= A^{-1}(A^{-1})^{k-1} = (A^{-1})^k, \end{aligned}$$

since the spectrum of the bounded operator A^{-1} is contained in the ball of radius $\frac{1}{\rho}$, so that the Cauchy formula holds.

c) Differentiating (3.1) with respect to z gives

$$\frac{d}{dz}A_z = \frac{i}{2\pi}\int_\Gamma \lambda^z (\ln z)(A-\lambda I)^{-1}\,d\lambda, \tag{3.5}$$

converging in operator norm from $\mathrm{H}^s(M)$ to itself uniformly for $\Re(z) \leq -\varepsilon < 0$, due to Theorem 3.1.1. Thus, the operator A_z is holomorphic in z and the derivative $\dfrac{d}{dz}A_z$ is equal to the integral (3.5).

□

We can now define the complex powers of an elliptic pseudodifferential operator.

Definition 3.1.3. Let A be as is Proposition 3.1.2, $z \in \mathbb{C}$, $k \in \mathbb{Z}$ such that $\Re(z) < k$. Then

$$A^z := A^k A_{z-k}. \tag{3.6}$$

This operator can be considered either from $\mathcal{C}^\infty(M)$ or $\mathcal{D}'(M)$ to itself.

We need to show that the definition of A^z does not depend on the choice of k:

Proposition 3.1.4. a) The operator A^z is independent of the choice of $k \in \mathbb{Z}$, provided $\Re(z) < k$.

b) If $\Re(z) < 0$, then $A^z = A_z$.

c) The group property holds

$$A^z A^w = A^{z+w}, \quad z, w \in \mathbb{C}. \tag{3.7}$$

d) If $k \in \mathbb{Z}$, $k > 0$, $A^k = \underbrace{A \circ \cdots \circ A}_{k\,\text{times}}$. In particular, $A^1 = A$. If $k \in \mathbb{Z}$, $k < 0$, $A^k = \underbrace{A^{-1} \circ \cdots \circ A^{-1}}_{-k\,\text{times}}$, where A^{-1} is the inverse of A. Moreover, $A^0 = I$.

e) For arbitrary $k \in \mathbb{Z}$ and $s \in \mathbb{R}$, A^z is a holomorphic operator in the half plane $\Re(z) < k$ with values in the space $\mathscr{L}(\mathrm{H}^s(M), \mathrm{H}^{s-mk}(M))$.

Proof. a) Let $z \in \mathbb{C}$ and $l, k \in \mathbb{Z}$ such that $\Re(z) < l \leq k$. We need to show that

$$A^k A_{z-k} = A^l A_{z-l}.$$

Setting $k - l = p > 0$ and $z - k = w \Rightarrow \Re(w) < 0$, we get the equivalent expression

$$A_w = A^{-p} A_{w+p}.$$

To conclude that this holds true, we notice that, by Proposition 3.1.2(b), $A^{-p} = A_{-p}$ and then use Proposition 3.1.2(a).

b) It is enough to set $k = 0$ in (3.6).

c) By definition, we have

$$A^z A^w = A^p A_{z-p} A^k A_{w-k},$$

with $\Re(z) < p$, $\Re(w) < k$. Set $p = l - k$, and choose k, l such that $k < \Re(z + w) < l$. Then, using Proposition 3.1.2(a) , we get

$$\begin{aligned} A^z A^w = A^p A_{z-p} A^k A_{w-k} = A^{l-k} A_{z-l+k} A^k A_{w-k} &= A^l A^{-k} A_{z-l+k} A^k A_{w-k} \\ &= A^l A_{-k} A_{z-l+k} A^k A_{w-k} \\ &= A^l A_{z-l+k-k} A^k A_{w-k} \\ &= A^l A_{z-l+k} A_{-k} A^k A_{w-k} \\ &= A^l A_{z-l+k} A^{-k} A^k A_{w-k} \\ &= A^l A_{z-l+k} A_{w-k} \\ &= A^l A_{z+w-l} = A^{z+w}. \end{aligned}$$

d) If $k \in \mathbb{Z}$, $k > 0$, set $p = k+1$ and notice that $k-p = k-k-1 = -1 < 0$. We have

$$A^k = A^p A_{k-p} = A^{k+1} A_{-1} = A^{k+1} A^{-1} = \underbrace{A \circ \cdots \circ A}_{k\,\text{times}}.$$

If $k \in \mathbb{Z}$, $k < 0$, using Proposition 3.1.2(b), we obtain

$$A^k = A_k = \underbrace{A^{-1} \circ \cdots \circ A^{-1}}_{-k\,\text{times}}.$$

Finally, from (3.7),

$$A^0 = A^{1-1} = A^1 A^{-1} = I.$$

e) This follows directly from Proposition 3.1.2(a) and c, remembering that A^k, for k integer, maps $\mathrm{H}^s(M)$ continuously into $\mathrm{H}^{s-mk}(M)$, and that $A^{-1} \in \mathrm{L}^{-m}_{\mathrm{cl}}(M)$.

□

To conclude this subsection, we recall two important results, whose proof can be found, e.g., in [Shu01].

Theorem 3.1.5. Let A be self-adjoint. Let $f \in \mathcal{D}'(M)$, and let

$$f(x) = \sum_{j=1}^{+\infty} f_j \varphi_j(x)$$

be the Fourier expansion of f in the eigenfunctions of the operator A. Then

$$A^z f(x) = \sum_{j=1}^{+\infty} \lambda_j^z f_j \varphi_j(x).$$

In particular, φ_j are the eigenfuntions of the operator A^z with eigenvalues λ_j^z.

Remark 3.1.6. In the self-adjoint case, the principal symbol $a_m(x, \xi)$ is real-valued. In particular, the previous conditions translates into

$$\begin{aligned} &a_m(x,\xi) > 0, \quad \xi \neq 0, \\ &A \geq \delta I, \quad \delta > 0, \end{aligned}$$

i.e. $(Au, u) \geq \delta(u, u)$ for every $u \in \mathcal{C}^\infty(M)$.

Theorem 3.1.7. For any $z \in \mathbb{C}$, $A^z \in \mathrm{L}^{mz}_{\mathrm{cl}}(M)$[2]. Moreover, the principal symbol of A^z is $(a_m(x, \xi))^z$.

3.1.2 The spectral ζ-function

Let A be an elliptic operator satisfying the same hypoteses of Theorem 3.1.1, and consider the kernel K_{A^z} of A^z, $z \in \mathbb{C}$. We have

Theorem 3.1.8. Let Ω be a fixed arbitrary coordinate neighbourhood on M, and $K_{A^z}(x, y)$ be defined for $x, y \in \Omega$. Then

i) For $\Re(z) < -\dfrac{n}{m}$ the kernel $K_{A^z}(x, y)$ is holomorphic.

[2]For pseudodifferential operators with complex order see, for instance, [Shu01].

ii) $K_{A^z}(x,x)$ can be extended to a meromorphic function in the whole complex plane with at most simple poles

$$z_j = \frac{j-n}{m} \quad j = 0, 1, \ldots,$$

and residues

$$\mathrm{Res}(K_{A^z}, z = z_j) = -\frac{1}{m} \int_{\mathbb{S}^{n-1}} a_{-n}^{(z_j)}(x,\xi)\, đ\xi,$$

where $a_p^{(z)}$ is the p-homogeneous component in the expansion of A^z.

Remark 3.1.9. Notice that, with the notation of Theorem 3.1.8, $a_{-n}^{(z_j)} = a_{mz_j-j}^{(z_j)}$ since

$$mz_j - j = m\frac{j-n}{m} - j = -n.$$

Remark 3.1.10. The residue in the left-most pole $z_0 = -\frac{n}{m}$ is

$$\mathrm{Res}(K_{A^z}, z = z_0) = -\frac{1}{m} \int_{\mathbb{S}^{n-1}} \frac{1}{(a_m(x,\xi))^{\frac{n}{m}}}\, đ\xi.$$

In the important case $a_m(x,\xi) > 0$ for $\xi \neq 0$, the previous equations implies that $\mathrm{Res}(K_{A^z}, z = z_0) < 0$.

We can now define the spectral ζ-function:

Definition 3.1.11. Let A be as in Theorem 3.1.1. The function

$$\zeta(A,z) := \int_M K_{A^{-z}}(x,x)\, dx$$

is called the *spectral ζ-function* of A.

Due to Theorem 3.1.8, we have

Theorem 3.1.12. The function $\zeta(A,z)$ is holomorphic for $\Re(z) > \frac{n}{m}$. Moreover, it can be continued to a meromorphic function in the entire complex plane, with at most simple poles

$$z_j = \frac{n-j}{m} \quad j = 0, 1, \ldots,$$

and residues

$$\mathrm{Res}(\zeta(A,z), z = z_j) = -\frac{1}{m} \int_M \int_{\mathbb{S}^{n-1}} \frac{1}{a_n^{(z_j)}(x,\xi)}\, đ\xi\, dx.$$

Remark 3.1.13. Again, in the case $a_m(x,\xi) > 0$ for $\xi \neq 0$, the residue in the right-most pole $z_0 = \dfrac{n}{m}$ is

$$\operatorname{Res}(\zeta(A,z), z = z_0) = -\frac{1}{m}\int_M \int_{\mathbb{S}^{n-1}} \frac{1}{(a_m(x,\xi))^{\frac{n}{m}}}\, đ\xi\, dx.$$

In the self-adjoint case, we have

Theorem 3.1.14. Let A be a self-adjoint, classical, positive, elliptic operator on M, and let $\{\lambda_j\}$, $j \in \mathbb{N}$, be its eingenvalues. Then

$$\zeta(A,z) = \sum_{j=1}^{+\infty} \frac{1}{\lambda_j^z}, \quad \Re(z) > \frac{n}{m} \tag{3.8}$$

and the series converges absolutely. Moreover, this convergence is uniform for z in the half-plane $\Re(z) > \dfrac{n}{m} + \varepsilon$, for arbitrary $\varepsilon > 0$.

For more detail on the meromorphic properties of the spectral ζ-function see, e.g., [See67].

3.2 Asymptotics of N_A spectral counting function of A

In this section we will review some classical results on the asymptotics of the counting function of the operator A. As in Theorem 3.1.14, let A be a self-adjoint, classical positive elliptic operator on M, with eigenvalues $\{\lambda_j\}$, $j \in \mathbb{N}$.

Definition 3.2.1. The *(spectral) counting function* of A is

$$N_A(\tau) := \sum_{\lambda_j \in \sigma(A)\cap(0,\tau)} 1 = \sum_{\lambda_j < \tau} 1. \tag{3.9}$$

Hence, for fixed $\tau \in \mathbb{R}$, $N_A(\tau)$ represents the number of eigenvalues of A less than τ (counted with their multiplicity). Clearly, $N(\tau)$ is a non-decreasing function, and we can assume, without loss of generality, that the eigenvalues are arranged in nondecreasing order

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$$

Our aim will now be to estimate the asymptotics of $N(\tau)$ as $\tau \to +\infty$. Without loss of generality, we can assume $\lambda_1 > 1$[3].

[3] If not, we set $l = \lambda_1 - \varepsilon$, $\varepsilon > 0$ small enough, consider $\tilde{A} := \frac{1}{l}A$, and notice that

$$N_{\tilde{A}}(\tau) = \sum_{l^{-1}\lambda_i < \tau} 1 = \sum_{\lambda_i < \tau l} 1 = N_A(\tau l). \tag{3.10}$$

That is, $N_A(\tau)$ and $N_{\tilde{A}}(\tau)$ have the same asymptotics as τ goes to infinity.

Remark 3.2.2. The classical way to describe the asymptotic expansion of the counting function $N_P(\tau)$ as τ goes to infinity is the Weyl's law. In the standard settings, it is well known that that the leading term of the Weyl's law depends on the dimension of the space, on the order of the operator and on its principal symbol. See [Hör85a] and [Shu01] for classical theory in the case of closed manifolds and Shubin calculus respectively. Weyl's law can be described in many other pseudodifferential calculi, for example SG-operators [BC11, CM13], bisingular operators [Bat12], global bisingular operators [BGPR13], cusp manifolds [Mor08] and many others. Some versions of the Weyl's law will be described in the following sections.

3.2.1 A Tauberian Theorem

To obtain the first term in the asymptotic expansion, we will use the following Tauberian Theorem of Ikehara:

Theorem 3.2.3. Let $f(t)$ be a non-decreasing function equal to 0 for $t \leq 1$, and such that the Stieltjes integral

$$g(t) = \int_1^{+\infty} t^{-z} df(t)$$

converges for $\Re(z) > k$ for some $k > 0$. Assume that there exists $B \neq 0$ such that the function

$$g(z) - \frac{B}{z-k} \tag{3.11}$$

con be extended by continuity to the closed half plane $\Re(z) \geq k$. Then,

$$f(t) \sim -\frac{B}{k} t^k$$

as $t \to +\infty$.

From (3.8) and (3.9) we can write

$$\zeta(A, z) = \int_1^{+\infty} \tau^{-z} dN(\tau).$$

From Theorem 3.1.12 and Remark 3.1.13 we have that (3.11) is satisfied with $g(z) = \zeta(A, z)$, $k = \dfrac{n}{m}$, and

$$B = -\frac{1}{m} \int_M \int_{\mathbb{S}^{n-1}} \frac{1}{(a_m(x,\xi))^{\frac{n}{m}}} \, đ\xi \, dx.$$

Thus, using Theorem 3.2.3 on $f(\tau) = N(\tau)$, we get, for $\tau \to +\infty$,

$$\begin{aligned} N(\tau) &\sim \left(\frac{m}{n} \frac{1}{m} \int_M \int_{\mathbb{S}^{n-1}} \frac{1}{(a_m(x,\xi))^{\frac{n}{m}}} \, đ\xi \, dx \right) \tau^{\frac{n}{m}} \\ &= \left(\frac{1}{n} \frac{1}{(2\pi)^n} \int_{a_m(x,\xi') < 1} d\xi' \, dx \right) \tau^{\frac{n}{m}}, \end{aligned}$$

where in the last equality we used the change of variables $\xi' = \tau^{-\frac{1}{m}}\xi$. In conclusion, we have, for the first term,

$$N(\tau) \sim \frac{C}{n}\tau^{\frac{n}{m}},$$

for $\tau \to +\infty$, with

$$C = \frac{1}{(2\pi)^n}\int_{a_m(x,\xi)<1} d\xi\, dx. \tag{3.12}$$

3.2.2 Stationary phase approximation

In this subsection we will find an estimate of the remainder term in the asymptotic expansion of $N(\tau)$ as τ goes to infinity. To do so, we will use Fourier Integral Operators (FIOs) and the Stationary Phase Theorem. This method is one of the most powerful in order to determine the sharp remainder in Weyl's law, see [DG75, GS94, Hör68, Hör85a, Shu01]. First, we recall the important Stationary Phase Theorem.

Theorem 3.2.4. Let $X \subset \mathbb{R}^n$ be open. Let $\varphi \in \mathcal{C}^\infty(X)$ have a non-degenerate critical point $x_0 \in X$, and assume that $\varphi'(x) \neq 0$ for $x \neq x_0$. Then, there are differential operators $P_{2\nu}(D)$ of order less or equal to 2ν such that, for every compact $K \subset X$ and every $N \in \mathbb{N}$, there is a constant $C = C_{K,N} > 0$ such that, for every $u \in \mathcal{C}^\infty(X) \cap \mathcal{D}'(K)$ we have

$$\begin{aligned}\left| \int e^{i\lambda\varphi(x)}u(x)\,dx - \left(\sum_{\nu=0}^{N-1}(P_{2\nu}(D_x)u)(x_0)\lambda^{-\nu-\frac{n}{2}}\right)e^{i\lambda\varphi(x_0)}\right| \\ \leq C\lambda^{-N-\frac{n}{2}}\sum_{|\alpha|\leq 2N+n+1}\sup_{x\in K}|\partial_x^\alpha u(x)|,\end{aligned}$$

$\lambda \geq 1$. Moreover,

$$P_0 = \frac{(2\pi)^{\frac{n}{2}}e^{i\frac{\pi}{4}\operatorname{sgn}\varphi''(x_0)}}{|\det\varphi''(x_0)|^{\frac{1}{2}}}.$$

For a proof of Theorem 3.2.4 we refer to [Hör83, GS94], see also, e.g., [MS75] and [Sjö82] for extensions.

Now consider the operator $Q := A^{\frac{1}{m}}$. From Theorem 3.1.7, we have $Q \in \mathrm{L}^1_{\mathrm{cl}}(M)$, with principal symbol $q_1(x,\xi) = (a_m(x,\xi))^{\frac{1}{m}}$, eigenvalues $\mu_j := \lambda_j^{\frac{1}{m}}$ and eigenfunctions $\{\varphi_j\}$, $j \in \mathbb{N}$. In the remaining part of this subsection, we refer to [GS94].

A trace formula

Let $u \in L^2(M), t \in \mathbb{R}$, and set

$$U(t)u := \sum_{j=1}^{+\infty} e^{it\mu_j}(u,\varphi_j)\varphi_j,$$

where the series converges in the L^2 norm. For every t, $U(t)$ is a unitary operator, and we have

$$U(0) = I,$$
$$U(t+s) = U(t)U(s), \quad t, s \in \mathbb{R}.$$

Moreover, for every $k \in \mathbb{N}$, we have

$$U(t)u = \mathcal{C}^k(\mathbb{R}, \mathrm{H}^0(M)) \cap \mathcal{C}^{k-1}(\mathbb{R}, \mathrm{H}^1(M)) \cap \cdots \cap \mathcal{C}^0(\mathbb{R}, \mathrm{H}^{k-1}(M)), \quad (3.13)$$

for every $u \in \mathrm{H}^k(M)$. For $u \in \mathrm{H}^1(M)$, we also have

$$\begin{cases} D_t U(t)u - QU(t)u &= 0 \\ U(0)u &= u, \end{cases}$$

so that $v(t,x) := U(t)u(x)$ is a solution to the Cauchy problem

$$\begin{cases} (D_t - Q)v &= 0 \\ v|_{t=0} &= u. \end{cases}$$

Now, let $\chi \in \mathscr{S}(\mathbb{R})$, and consider the operator $S := \int \chi(t)U(t)\,dt$, defined by

$$Su = \left(\int \chi(t)U(t)\,dt \right) u := \int \chi(t)U(t)u\,dt,$$

for $u \in L^2(M)$. S is a bounded operator from $L^2(M)$ to itself, since its norm is bounded by $\|\chi\|_{L^1}$. If we apply S to finite linear combinations, and then use a density argument, we get

$$\begin{aligned} Su = \left(\int \chi(t)U(t)\,dt \right) u &= \sum_{j=1}^{+\infty} \int e^{it\mu_j} \chi(t)\,dt (u, \varphi_j)\varphi_j \\ &= \sum_{j=1}^{+\infty} \hat{\chi}(-\mu_j)(u, \varphi_j)\varphi_j. \end{aligned}$$

To show the convergence of the series in $\mathcal{C}^\infty(M \times M)$, we recall that, since $\chi \in \mathscr{S}(\mathbb{R})$, we have, for every[4] $N \geq 0$,

$$|\hat{\chi}(\mu)| \leq C_N \mu^{-N}, \quad \mu > 1.$$

Moreover, the Sobolev inequalities give

$$\|\varphi_j\|_{C^k} \leq C_k \|\varphi_j\|_{k+n+1} \leq C'_k \|Q^{k+n+1}\varphi_j\|_0 \leq C'_k \mu_j^{k+n+1},$$

hence $\|\varphi_j(x)\overline{\varphi_j(y)}\|_{C^k} \leq C'_k \mu_j^{k+2(n+1)}$. To conclude, we use the following Proposition:

[4]Remember that all the eigenvalues are greater the one.

Proposition 3.2.5. There exists a sufficiently large $N_0 > 0$ such that

$$\sum_{j=1}^{+\infty} \mu_j^{-N_0} < +\infty.$$

Proof. We have

$$\sum_j \frac{1}{\mu_j^{N_0}} = \zeta(P, N_0).$$

Then, by Theorem 3.1.14, it suffices to choose $N_0 > n$. □

Now let $K_\chi(x,y) \in \mathcal{C}^\infty(M \times M)$ be the kernel of $\int \chi(t)U(t)\,dt$:

$$\left(\int \chi(t)U(t)\,dt\,u\right)(x) = \int K_\chi(x,y)u(y)\,dy.$$

Thus, we have the following trace formula:

$$\sum_{j=1}^{+\infty} \hat{\chi}(-\mu_j) = \int K_\chi(x,x)\,dx.$$

Construction of the approximate solution of the Cauchy problem

Consider the Cauchy problem

$$\begin{cases} (D_t - Q)u &= 0 \\ u|_{t=0} &= v \end{cases}.$$

It is well known, see, e.g. [Hör85a] and [KG82], that we can construct an operator V with the following properties:

a) There exists $T > 0$ such that V is continuous as an operator

$$V : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty((-T,T) \times M),$$
$$V : \mathcal{D}'(M) \to \mathcal{C}^\infty((-T,T), \mathcal{D}'(M)).$$

b) For all $u \in \mathcal{D}'(M)$

$$\begin{aligned} (D_t - Q) \circ V :=& R_0 \in \mathrm{L}^{-\infty}((-T,T) \times M \times M) \\ Vu|_{t=0} =& u, \end{aligned} \tag{3.14}$$

where we denote by $\mathrm{L}^{-\infty}((-T,T) \times M \times M)$ the space of smoothing operators $\mathcal{D}'(M) \to \mathcal{C}^\infty((-T,T) \times M)$.

More precisely, V has the form

$$(V(t))u(x) = Vu(t,x) = \sum_{1}^{N} \chi_j(x)(V_j\psi_j(\cdot)u(\cdot))(t,x),$$

where $\chi_j, \psi_j \in \mathcal{C}_0^\infty(M)$, $\chi_j \equiv 1$ near $\operatorname{supp}\psi_j$, $\sum \psi_j = 1$, and in suitable local coordinates, depending on j,

$$V_j(t)u(x) = \int e^{i(\varphi_j(t,x,\xi)-y\cdot\xi)} a_j(t,x,\xi)u(y)\,dy\,đ\xi. \tag{3.15}$$

In (3.15) $\varphi = \varphi_j$ is the solution of the *eikonal equation*

$$\begin{cases} \partial_t\varphi(t,x,\xi) - q_1(x,\partial_x\varphi(t,x,\xi)) &= 0 \\ \varphi(0,x,\xi) &= x\cdot\xi \end{cases}, \tag{3.16}$$

and $a = a_j \in \mathrm{S}^0_{\mathrm{cl}}$ satisfies $a|_{t=0} = 1$.

Proposition 3.2.6. *$U - V := R$ belongs to $\mathrm{L}^{-\infty}((-T,T)\times M\times M)$.*

Proof. Consider $W(t) := U(-t)V(t)$, where $V(t)u(x) = (Vu)(t,x)$. For $u \in \mathrm{H}^1$, we have

$$\begin{aligned} D_tW(t)u &= -U(-t)QV(t)u + U(-t)QV(t)u + U(-t)R_0(t)u \\ &= U(-t)R_0(t)u. \end{aligned}$$

Using (3.13) and (3.14), we get that

$$U(-t)R_0(t) : \mathcal{D}'(M) \to \mathcal{C}^\infty((-T,T)\times M)$$

is smoothing, hence

$$U(-t)R_0(t) \in \mathrm{L}^{-\infty}((-T,T)\times M\times M).$$

Since $W(0) = I$, integrating from 0 to t we obtain that $W(t) \equiv I$ modulo $\mathrm{L}^{-\infty}((-T,T)\times M\times M)$. Hence, using again (3.13), $V(t) \equiv U(t)$ modulo $\mathrm{L}^{-\infty}((-T,T)\times M\times M)$. □

Asymptotics of $\sum_{j=1}^{+\infty} \hat{\chi}(\tau - \mu_j)$

Let R be as in Proposition 3.2.6, $R(t,x,y)$ be its kernel. Let $\chi \in \mathcal{C}_0^\infty((-T,T))$. For $u \in \mathcal{C}^\infty(M)$ we have

$$\begin{aligned} &\int \chi(t)U(t)u(x)\,dt \\ &= \sum_{j=1}^{N} \chi_j(x) \int \chi(t)e^{i(\varphi_j(t,x,\xi)-y\cdot\xi)} a_j(t,x,\xi)\psi_j(y)u(y)\,dy\,đ\xi\,dt \\ &\qquad + \int \chi(t)R(t,x,y)u(y)\,dy\,dt, \end{aligned}$$

where the local coordinates in each term depend on j. Thus, the kernel of $\int \chi(t)U(t)\,dt$ is

$$K_\chi(x,y) = \sum_{j=1}^{N} \chi_j(x) \int \chi(t) e^{i(\varphi_j(t,x,\xi)-y\cdot\xi)} a_j(t,x,\xi)\,dt\,đ\xi\,\psi_j(y) + \int \chi(t) R(t,x,y)\,dt.$$

In particular,

$$\int K_\chi(x,x)\,dx = \sum_{j=1}^{N} \int \chi(t) e^{i(\varphi_j(t,x,\xi)-x\cdot\xi)} a_j(t,x,\xi)\psi_j(x)\,dt\,dx\,đ\xi + \int \chi(t) R(t,x,x)\,dt\,dx. \tag{3.17}$$

Now, we replace $\chi(t)$ by $\chi(t)e^{-it\tau}$, $\tau \in \mathbb{R}$, and study the asymptotics as τ goes to infinity. The trace formula gives

$$\int K_{\chi e^{-i(\cdot)\tau}}(x,x)\,dx = \sum_{j=1}^{+\infty} \hat{\chi}(\tau - \mu_j),$$

so (3.17) implies, for $\tau \to +\infty$,

$$\sum_{j=1}^{+\infty} \hat{\chi}(\tau - \mu_j) = \sum_{j=1}^{N} \int \chi(t) e^{i(-\tau t + \varphi_j(t,x,\xi)-x\cdot\xi)} a_j(t,x,\xi)\psi_j(x)\,dt\,dx\,đ\xi + \mathcal{O}(|\tau|^{-\infty}). \tag{3.18}$$

Then, we shall study the asymptotics of each of the integrals. For that we fix j, and consider

$$I(x,\tau) = \int e^{i\Phi(t,x,\xi,\tau)} \chi(t) a(t,x,\xi)\,dt\,đ\xi,$$

where

$$\Phi(t,x,\xi,\tau) := -\tau t + \varphi(t,x,\xi) - x\cdot\xi = t q_1(x,\xi) + |\xi|\mathcal{O}(t^2) - \tau t.$$

Observe that, for $\tau \to -\infty$, we have

$$\partial_t \Phi(t,x,\xi,\tau) \succ \langle\xi\rangle + |\tau|,$$

which means, via integration by parts in t, that the term is $\mathcal{O}(|\tau|^{-\infty})$.
Consider next the case where $\tau \to +\infty$. Choose a cut-off function $H \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\operatorname{supp} H \subseteq [K^{-1}, K]$, $0 \le H \le 1$ and $H \equiv 1$ on $[K^{-1}, K]$,

for a suitable $K >> 1$. Again, for $\xi \in \operatorname{supp}\left(1 - H\left(\frac{(\cdot)}{\tau}\right)\right)$, we have $\partial_t \Phi(t, x, \xi, \tau) \succ \langle \xi \rangle + |\tau|$, thus

$$I(x,\tau) = \int e^{i\Phi(t,x,\xi,\tau)} \chi(t) H\left(\frac{\xi}{\tau}\right) a(t,x,\xi)\, dt\, d\!\!\bar{}\,\xi + \mathcal{O}(|\tau|^{-\infty}).$$

To study the remaining part, we switch to polar coordinates $\xi = \tau r \omega$, $|\omega| = 1$, $d\xi = \tau^n r^{n-1}\, dr\, d\omega$. Then,

$$I(x,\tau) = \frac{\tau^n}{(2\pi)^n} \int_{\mathbb{S}^{n-1}} d\omega \int e^{i\tau((\varphi(t,x,\omega) - x\cdot\omega)r - t)} \chi(t) H(r\omega) a(t,x,\tau r\omega) r^{n-1}\, dt\, dr. \tag{3.19}$$

We want to apply the Stationary Phase Theorem to the $dt\, dr$ integral. The phase function is

$$\Phi'(t,x,\omega,r) = (\varphi(t,x,\omega) - x\cdot\omega)r - t = trq_1(x,\omega) + r\mathcal{O}(t^2) - t,$$

due to (3.16). Then,

$$\partial_r \Phi' = tq_1(x,\omega) + \mathcal{O}(t^2)$$
$$\partial_t \Phi' = rq_1(x,\omega) - 1 + \mathcal{O}(t^2),$$

and we have an unique critical point given by

$$t = 0, \qquad r = \frac{1}{q_1(x,\omega)}.$$

The elements of the Hessian matrix at the critical point are given by

$$\Phi''_{t,r}\left(0,x,\omega,\frac{1}{q_1}\right) = \Phi''_{r,t}\left(0,x,\omega,\frac{1}{q_1}\right) = q_1(x,\omega), \quad \Phi''_{r,r}\left(0,x,\omega,\frac{1}{q_1}\right) = 0,$$

hence $|\det \Phi''| = (q_1(x,\omega))^2$, and the signature of Φ'' is 0. The Stationary Phase Theorem 3.2.4 then gives

$$\begin{aligned} I(x,\tau) &= \frac{\tau^{n-1}}{(2\pi)^{n-1}} \int_{\mathbb{S}^{n-1}} \chi(0) \left(\frac{1}{q_1(x,\omega)}\right)^{n-1} \frac{1}{q_1(x,\omega)} d\omega + \mathcal{O}(\tau^{n-2}) \\ &= \frac{\tau^{n-1}}{(2\pi)^{n-1}} \int_{\mathbb{S}^{n-1}} \frac{1}{(q_1(x,\omega))^n} d\omega + \mathcal{O}(\tau^{n-2}). \end{aligned}$$

Remark 3.2.7. In the case $n = 1$, (3.19) becomes

$$\begin{aligned} &I(x,\tau) \\ &= \frac{\tau}{2\pi} \int_{\mathbb{S}^0} d\omega \int e^{i\tau t((q_1(x,\omega)+\mathcal{O}(t))r-1)} \chi(t) H(r\omega) \left(1 + \mathcal{O}(t) + \frac{\mathcal{O}(t)}{\tau}\right) dt\, dr, \end{aligned}$$

and, computing the first two terms in the stationary phase expansion, we see that the coefficient of the τ^{-1} term is 0. Therefore, we get the improved estimate

$$I(x,\tau) = \int_{\mathbb{S}^{n-1}} \frac{1}{q_1(x,\omega)} d\omega + \mathcal{O}(\tau^{-2}).$$

To conclude, in suffices to integrate with respect to $\psi_j(x)\,dx$ and add the N integrals in (3.18), obtaining

Theorem 3.2.8. Let $\chi \in \mathcal{C}_0^\infty((-T,T))$, $\chi(0) = 1$. Then

$$\sum_{j=1}^{+\infty} \hat{\chi}(\tau - \mu_j) = \begin{cases} \dfrac{\tau^{n-1}}{(2\pi)^{n-1}} \displaystyle\iint_{\mathbb{S}^*M} \frac{1}{(q_1(x,\omega))^n} d\omega\, dx + \mathcal{O}(\tau^{n-2}) & \text{for} \quad \tau \to +\infty \\ \mathcal{O}(|\tau|^{-\infty}) & \text{for} \quad \tau \to -\infty. \end{cases} \tag{3.20}$$

Estimate of the counting function

First, notice that, in Theorem 3.2.8, we may choose χ with the additional property that $\hat{\chi} \geq 0$. To do so, set $\chi := \psi * \overline{\check{\psi}}$ for a suitable $\psi \in \mathcal{C}_0^\infty$, where $\check{\psi}(t) := \psi(-t)$. Now rewrite (3.20) as a Stieltjes integral, so that

$$\int \hat{\chi}(\tau - \sigma)\, dN(\sigma) = \begin{cases} \dfrac{\tau^{n-1}}{(2\pi)^{n-1}} \displaystyle\iint_{\mathbb{S}^*M} \frac{1}{(q_1(x,\omega))^n} d\omega\, dx + \mathcal{O}(\tau^{n-2}) & \text{for} \quad \tau \to +\infty \\ \mathcal{O}(|\tau|^{-\infty}) & \text{for} \quad \tau \to -\infty. \end{cases} \tag{3.21}$$

Setting

$$G(\tau) := \int_{-\infty}^{\tau} \hat{\chi}(\lambda)\, d\lambda,$$

and integrating (3.21), we get

$$\int G(\tau - \sigma)\, dN(\sigma) = \begin{cases} \dfrac{\tau^{n}}{n(2\pi)^{n-1}} \displaystyle\iint_{\mathbb{S}^*M} \frac{1}{(q_1(x,\omega))^n} d\omega\, dx + \mathcal{O}(\tau^{n-1}) & \text{for} \quad \tau \to +\infty \\ \mathcal{O}(|\tau|^{-\infty}) & \text{for} \quad \tau \to -\infty. \end{cases} \tag{3.22}$$

We observe that, in the case $n = 1$, we have used Remark 3.2.7, replacing $\mathcal{O}(\lambda^{-1})$ with $\mathcal{O}(\lambda^{-2})$ in (3.21). Then, indicating by $\Theta(t)$ the Heaviside function,

$$\begin{aligned}\int G(\tau-\sigma)\,dN(\sigma) = \sum_j G(\tau-\mu_j) &= \sum_j \int_{-\infty}^{\tau-\mu_j} \hat{\chi}(\lambda)\,d\lambda \\ &= \sum_j \int \Theta(\tau-\mu_j-\lambda)\hat{\chi}(\lambda)\,d\lambda \\ &= \int \sum_j \Theta(\tau-\mu_j-\lambda)\hat{\chi}(\lambda)\,d\lambda \\ &= \int N_Q(\tau-\lambda)\hat{\chi}(\lambda)\,d\lambda \\ &= 2\pi N_Q(\tau) + Y(\tau),\end{aligned}$$

where $N_Q(\tau)$ is the counting function,

$$Y(\tau) := \int (N_Q(\tau-\lambda) - N_Q(\tau))\hat{\chi}(\lambda)\,d\lambda,$$

and we used that $\int \hat{\chi}(\lambda)\,d\lambda = 2\pi$. From 3.20 (see [GS94] for the details), we get that

$$\#\left\{\sigma(Q)\cap[\tau,\tau+1]\right\} = N_Q(\tau+1) - N_Q(\tau) = \mathcal{O}(\tau^{n-1}).$$

Therefore, for $\tau > 1$,

$$|N_Q(\tau-\lambda) - N_Q(\tau)| \prec (1+|\lambda|)(\tau+|\lambda|)^{n-1} \prec (1+|\lambda|)^n \tau^{n-1}.$$

Since

$$\int (1+|\lambda|)^n \hat{\chi}(\lambda)\,d\lambda < +\infty,$$

we get $Y(\tau) = \mathcal{O}(\tau^{n-1})$, and

$$\int G(\tau-\sigma)\,dN(\sigma) = 2\pi N_Q(\tau) + \mathcal{O}(\tau^{n-1}). \tag{3.23}$$

Finally, (3.22) and (3.23) give

Proposition 3.2.9. We have

$$N_Q(\tau) = \frac{1}{n}\frac{\tau^n}{(2\pi)^n}\iint_{\mathbb{S}^*M}\frac{1}{(q_1(x,\omega))^n}d\omega\,dx + \mathcal{O}(\tau^{n-1}),$$

for $\tau \to +\infty$.

Switching back to the operator $A = Q^m$, since $\mu_j := \lambda_j^{\frac{1}{m}}$, we have

$$N_A(\tau) = \sum_{\lambda_j<\tau} 1 = \sum_{\lambda_j^{\frac{1}{m}}<\tau^{\frac{1}{m}}} 1 = \sum_{\mu_j<\tau^{\frac{1}{m}}} 1 = N_Q(\tau^{\frac{1}{m}}),$$

and, in conclusion,

Theorem 3.2.10. Let A be a self-adjoint, classical, positive, elliptic operator on M, with eigenvalues $\{\lambda_j\}$, $j \in \mathbb{N}$. Then,

$$\begin{aligned} N_A(\tau) &= \frac{1}{n} \frac{\tau^{\frac{n}{m}}}{(2\pi)^n} \iint_{\mathbb{S}^* M} \frac{1}{(a_m(x,\xi))^{\frac{n}{m}}} d\xi \, dx + \mathcal{O}(\tau^{\frac{n-1}{m}}) \\ &= \frac{C}{n} \tau^{\frac{n}{m}} + \mathcal{O}(\tau^{\frac{n-1}{m}}) \end{aligned}$$

for $\tau \to +\infty$, with C as in (3.12).

3.2.3 Sharpness of the result

In this subsection we show that the estimate of the remainder in Theorem 3.2.10 is sharp. To do so, we need the precise knowledge of the spectrum of some operator with the required properties on a closed manifold. We consider

$$A := -\Delta_{\mathbb{S}^n} \in L^2(\mathbb{S}^n),$$

the Laplace-Beltrami operator on the n-dimensional sphere $\mathbb{S}^n$. It is well known, see for example [Shu01] and [BEM71], that

Theorem 3.2.11. The eigenvalues of the operator $A = -\Delta_{\mathbb{S}^n}$ on $\mathbb{S}^n$ are

$$\lambda_k = k(k+n-1), \quad k \in \mathbb{N}.$$

Moreover, each λ_k has multiplicity

$$\sharp\,(\lambda_k) = N_k - N_{k-2},$$

where

$$N_k := \binom{n+k}{n}.$$

Now,

$$N_A(\lambda_k) = \sum_{\lambda_l < \lambda_k} \sharp\,(\lambda_l) = \sum_{l<k} (N_l - N_{l-2}) = N_k + N_{k-1}. \tag{3.24}$$

Moreover,

$$N_k = \binom{n+k}{n} = \frac{(n+k)!}{n!\,k!} = \frac{1}{n!}(k+n)(k+n-1)\dots(k+1),$$

thus N_k is a polynomial of degree n in k with leading coefficient $\dfrac{1}{n!}$. Then, from (3.24),

$$N_A(\lambda_k) \sim \frac{2}{n!} k^n \sim \frac{2}{n!} (k(k+n-1))^{\frac{n}{2}} = \frac{2}{n!} (\lambda_k)^{\frac{n}{2}}. \tag{3.25}$$

From (3.24) it also follows that

$$N_A(\lambda_k) - N_A(\lambda_{k-1}) = N_k - N_{k-2} = P_{n-1}(k),$$

where $P_{n-1}(k)$ is a polynomial of degree $n-1$ in k. Therefore

$$N_A(\lambda_k) - N_A(\lambda_{k-1}) \geq ck^{n-1} \geq c'(k(k+n-1))^{\frac{n-1}{2}} = c'(\lambda_k)^{\frac{n-1}{2}} \tag{3.26}$$

for some $c, c' > 0$. Combining (3.25) and (3.26) we get

$$N_A(\tau) = \frac{2}{n!}\tau^{\frac{n}{2}} + \mathcal{O}(\tau^{\frac{n-1}{2}}), \tag{3.27}$$

which shows that the estimate in Theorem 3.2.10 is sharp.

Remark 3.2.12. Notice that also the constant in the first term of (3.27) is correct. We have $a_2(x, \xi) = |\xi|^2$, and, by direct computation,

$$\iint_{\mathbb{S}^n \times \mathbb{S}^{n-1}} \frac{1}{|\xi|^{\frac{n}{2}}} d\xi \, dx = \frac{2(2\pi)^n}{(n-1)!}.$$

Thus,

$$\frac{C}{n} = \frac{1}{n}\frac{1}{(2\pi)^n}\iint_{\mathbb{S}^n \times \mathbb{S}^{n-1}} \frac{1}{|\xi|^{\frac{n}{2}}} d\xi \, dx = \frac{1}{n}\frac{1}{(2\pi)^n}\frac{2(2\pi)^n}{(n-1)!} = \frac{2}{n!}.$$

3.3 Asymptotics of $N_{A_1 \otimes A_2}$ spectral counting function of $A_1 \otimes A_2$

Let M_1, M_2 be two closed manifolds of dimension n_1, n_2 respectively. Let $A = A_1 \otimes A_2$, $A_i \in \mathrm{L}^{m_i}_{\mathrm{cl}}(M_i)$, $m_i > 0$, $i = 1, 2$, and

$$\sigma(A_1) = \{\lambda_i\}_{i \in \mathbb{N}}, \quad \sigma(A_2) = \{\mu_k\}_{k \in \mathbb{N}}.$$

Thus, by Proposition 1.1.13,

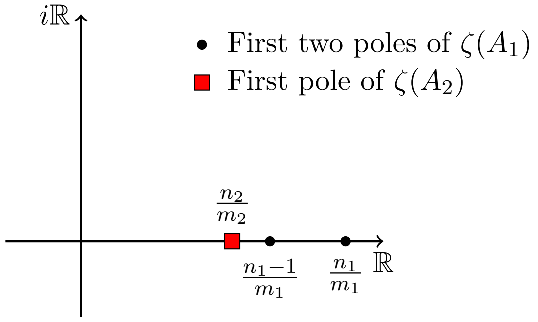
$$\sigma(A) = \{\lambda_i \cdot \mu_k : \lambda_i \in \sigma(A_1),\, \mu_k \in \sigma(A_2)\}.$$

In this section we will study the asymptotics of the counting function of $A = A_1 \otimes A_2$,

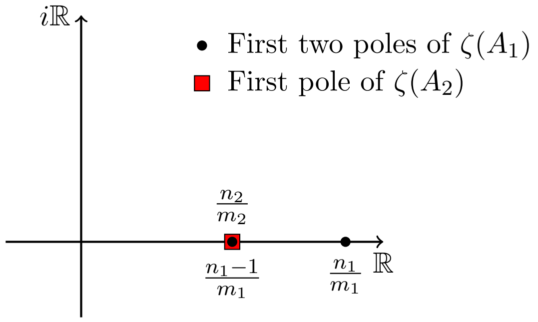
$$N_A(\tau) = \sum_{\lambda_i \, \mu_k < \tau} 1. \tag{3.28}$$

The asymptotic expansion of $N_{A_1 \otimes A_2}(\tau)$ is related with the position of the first poles of the spectral ζ-function associated with A_1 and A_2, as sketched in the following pictures.

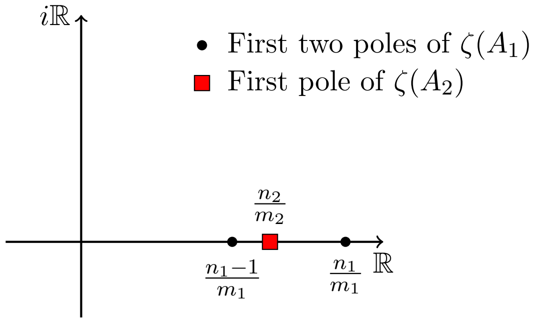
Case $\frac{n_2}{m_2} < \frac{n_1-1}{m_2}$



Case $\frac{n_2}{m_2} = \frac{n_1-1}{m_2}$



Case $\frac{n_2}{m_2} > \frac{n_1-1}{m_2}$



In [GPRV14] the authors analyze the same class of operators and find

$$N_A(\tau) = \frac{C_1}{n_1}\zeta\left(A_2, \frac{n_1}{m_1}\right)\tau^{\frac{n_1}{m_1}} + \mathcal{O}(\tau^\delta)$$

where $\min\left\{\frac{n_1-1}{m_1}, \frac{n_2}{m_2}\right\} < \delta < \frac{n_1}{m_1}$. We are able to refine the above estimate.

3.3.1 The main Theorem

In this subsection we will prove our main Theorem. The next Proposition 3.3.1 will be crucial in our proof of the Weyl law with sharp remainder for tensor products. It follows as consequence of well known properties of the spectra of positive self-adjoint operators.

Proposition 3.3.1. Let M be a closed manifold of dimension n, and $A \in \mathrm{L}^m_{\mathrm{cl}}(M)$, $m > 0$, be elliptic, positive and self-adjoint, with spectrum $\sigma(A) = \{\mu_k\}_{k\in\mathbb{N}}$. Define

$$F_A(\tau, c) = \sum_{\mu_k<\tau} \frac{1}{\mu_k^c} = \begin{cases} F_1(\tau) & \text{if} \quad c > \dfrac{n}{m}, \\ F_2(\tau) & \text{if} \quad c = \dfrac{n}{m}, \\ F_3(\tau) & \text{if} \quad c < \dfrac{n}{m}. \end{cases} \tag{3.29}$$

Then,

$$\limsup_{\tau\to+\infty} \frac{\zeta(A,c) - F_1(\tau)}{\tau^{\frac{n}{m}-c}} = \kappa_1, \ \limsup_{\tau\to+\infty} \frac{F_2(\tau)}{\log \tau} = \kappa_2, \ \limsup_{\tau\to+\infty} \frac{F_3(\tau)}{\tau^{\frac{n}{m}-c}} = \kappa_3,$$

for suitable positive constants κ_1, κ_2, κ_3. That is, for $\tau \to +\infty$,

$$\zeta(A,c) - F_1(\tau) = \mathcal{O}\left(\tau^{\frac{n}{m}-c}\right), \ F_2(\tau) = \mathcal{O}(\log \tau), \ F_3(\tau) = \mathcal{O}\left(\tau^{\frac{n}{m}-c}\right).$$

Proof. If $c > \frac{n}{m}$ it is immediate that the series $\sum_{k=0}^{\infty} \frac{1}{\mu_k^c}$ is convergent, in view of the holomorphic properties of the spectral ζ-function associated with A. To prove the asymptotic properties of $\zeta(A,c) - F_1(\tau)$, we switch to $B = A^{1/m}$, so that the order of B is one and $\sigma(B) = \mu_k^{1/m}$. We have

$$\begin{aligned} \zeta(A,c) - F_1(\tau) &= \sum_{\mu_k\geq\tau} \frac{1}{\mu_k^c} = \sum_{\mu_k^{1/m}\geq\tau^{1/m}} \frac{1}{\left(\mu_k^{1/m}\right)^{cm}} \\ &= \int_{\tau^{1/m}}^{+\infty} \frac{1}{\mu^{cm}} dN_B(\mu). \end{aligned} \tag{3.30}$$

Since B is of order one, it is well known that

$$N_B(\lambda+1) - N_B(\lambda) \leq \sharp\{\sigma(B) \cap [\lambda, \lambda+1]\} = \mathcal{O}(\lambda^{n-1}), \lambda \to +\infty \tag{3.31}$$

(see, e.g., [GS94, § 12]). Using (3.31) and the properties of Stieltjes integral, we obtain, for $\tau \to +\infty$,

$$\begin{aligned}
\zeta(A,c) - F_1(\tau) &= \int_{\tau^{1/m}}^{+\infty} \frac{1}{\mu^{c\,m}} dN_B(\mu) \\
&\leq \sum_{j=[\tau^{1/m}]-1}^{\infty} \sup_{\mu\in[j,j+1]} \left(\frac{1}{\mu^{c\,m}}\right) (N_B(j+1) - N_B(j)) \\
&\leq \kappa \sum_{j=[\tau^{1/m}]-1}^{\infty} \frac{1}{j^{c\,m-n+1}} \\
&\leq \kappa \int_{[\tau^{1/m}]-1}^{+\infty} \frac{1}{(t-1)^{c\,m-n+1}} dt \\
&= \kappa \frac{1}{c\,m-n} [\tau^{1/m} - 2]^{n-m\,c} \in \mathcal{O}\left(\tau^{\frac{n}{m}-c}\right).
\end{aligned}$$

where $[a]$ denotes the minimum integer such that $[a] \geq a$.

To prove the results for F_2 and F_3 we can assume, without loss of generality, that $\mu_0 = \tilde{\mu}_0 = 1$. Using again the properties of the Stieltjes integral, we write

$$F_A(\tau,c) = \int_1^{\tau^{1/m}} \frac{1}{\mu^{c\,m}} dN_B(\mu) \leq \sum_{j=1}^{[\tau^{1/m}]} \sup_{\mu\in[j,j+1]} \left(\frac{1}{\mu^{c\,m}}\right) (N_B(j+1) - N_B(j)).$$

Let us initially suppose that $c > 0$, so that $\frac{1}{x^c}$ is a decreasing function on $[1,+\infty)$. In view of (3.31), we have

$$\begin{aligned}
\int_1^{\tau^{1/m}} \frac{1}{\mu^{c\,m}} dN_B(\mu) &\leq \sum_{j=1}^{[\tau^{1/m}]} \frac{1}{j^{c\,m}} \mathcal{O}(j^{n-1}) \leq \widetilde{\kappa} \sum_{j=1}^{[\tau^{1/m}]} \frac{1}{j^{c\,m-n+1}} \\
&\leq \widetilde{\kappa} \left(\int_1^{[\tau^{1/m}]} t^{n-c\,m-1} dt + 1 \right). \qquad (3.32)
\end{aligned}$$

By integration, we find

$$F_A(\tau,c) = \int_1^{\tau^{1/m}} \frac{1}{\mu^{c\,m}} dN_B(\mu) \leq \begin{cases} \dfrac{\widetilde{\kappa}_1}{n-c\,m} \tau^{\frac{n}{m}-c} & \text{if } \ 0 < c < \dfrac{n}{m}, \\ \dfrac{\widetilde{\kappa}_2}{m} \log\tau & \text{if } \ c = \dfrac{n}{m}, \end{cases}$$

as claimed. Finally, if $c \leq 0$, then $\frac{1}{\mu^c}$ is a non-decreasing function and also in this case, similarly to (3.32), we obtain

$$F_A(\tau,c) \leq \kappa \int_1^{[\tau^{1/m}]} (x+1)^{n-c\,m-1}\, dx \leq \frac{\widetilde{\kappa}_3}{n-c\,m} \tau^{\frac{n}{m}-c}.$$

The proof is complete. □

Now, we consider the tensor product of 2 operators $A = A_1 \otimes A_2$, $A_i \in \mathrm{L}^{m_i}_{\mathrm{cl}}(M_i)$, $m_i > 0$, $i = 1, 2$. For simplicity, we start with the case $m_1 = 1$ and $n_1 > \frac{n_2}{m_2}$. Without loss of generality, we can assume[5] $\lambda_j > 1$ and $\mu_k > 1$ for all j, k. Let us now summarize the hypotheses on the factors A_1, A_2.

Assumptions 3.3.2.

$$\begin{aligned}
&M_1, M_2 \text{ smooth closed manifolds of dimensions } n_1, n_2, \text{ respectively;}\\
&A = A_1 \otimes A_2,\ A_1 \in \mathrm{L}^1_{\mathrm{cl}}(M_1), A_2 \in \mathrm{L}^{m_2}_{\mathrm{cl}}(M_2), \quad m_2 > 0, n_1 > \frac{n_2}{m_2};\\
&A_1, A_2 \text{ positive, self-adjoint, elliptic;}\\
&\sigma(A_1) = \{\lambda_j\}_{j\in\mathbb{N}},\ \sigma(A_2) = \{\mu_k\}_{k\in\mathbb{N}}, \quad \lambda_j > 1, \mu_k > 1, \text{ for all } j, k.
\end{aligned}$$

Since $\lambda_j, \mu_k > 1$ for all j, k, using (3.10), we have[6]

$$\begin{aligned}
N_A(\tau) &= \sum_{\lambda_j\cdot\mu_k<\tau} 1 = \sum_{\mu_k<\tau}\left(\sum_{\lambda_j\cdot\mu_k<\tau} 1\right) =\\
&= \sum_{\mu_k<\tau} N_{\mu_k A_1}(\tau) = \sum_{\mu_k<\tau} N_{A_1}\left(\frac{\tau}{\mu_k}\right). \qquad (3.33)
\end{aligned}$$

Proposition 3.3.3. Let A, A_1 and A_2 be as in Assumptions 3.3.2. Then,

$$N_A(\tau) = \sum_{\mu_k<\tau}\left(\frac{C_1}{n_1}\left(\frac{\tau}{\mu_k}\right)^{n_1} + \frac{1}{\mu_k^{n_1-1}} r_k(\tau)\right),$$

with

$$C_1 = \frac{1}{(2\pi)^{n_1}} \iint_{\mathbb{S}^* M_1} \frac{d\theta_1 dx_1}{[a_{m_1}(x_1,\theta_1)]^{\frac{n_1}{m_1}}}, \qquad (3.34)$$

and $r_k(\tau)$ is $\mathcal{O}\left(\tau^{n_1-1}\right)$, uniformly with respect to μ_k. That is, there exists a positive constant C such that

$$|r_k(\tau)| \le C\tau^{n_1-1}, \quad \text{for all } k \in \mathbb{N}. \qquad (3.35)$$

Proof. By (3.33) we have

$$N_A(\tau) = \sum_{\mu_k<\tau} N_{A_1}\left(\frac{\tau}{\mu_k}\right).$$

[5] In fact, if that condition were not true, we could consider the operator c^2A, with $c = (\min\{\lambda_j, \mu_k\} - \varepsilon)^{-1}$, $\varepsilon > 0$ small enough, and reason as in (3.10).

[6] Recall that $\lambda_j > 1$ for all j. In the first term of (3.33) we can reduce the summation to $\mu_k < \tau$ since, otherwise, we would have $\lambda_k \cdot \mu_k \ge \tau$ for all k, and the second summation would be zero.

Using Theorem 3.2.10, we can write

$$N_A(\tau) = \sum_{\mu_k<\tau} \left(\frac{C_1}{n_1} \frac{\tau^{n_1}}{\mu_k^{n_1}} + R_A\left(\frac{\tau}{\mu_k}\right)\right). \tag{3.36}$$

Theorem 3.2.10 implies that

$$|R_A(t)| \leq \kappa t^{n_1-1}, \quad t>1,$$

for a suitable constant κ. Since $\mu_k < \tau \Rightarrow \frac{\tau}{\mu_k} > 1$ in the summation (3.36), we can write

$$\left| R_A\left(\frac{\tau}{\mu_k}\right)\right| \leq C\left(\frac{\tau}{\mu_k}\right)^{n_1-1}.$$

Hence, setting

$$r_k(\tau) = \mu_k^{n_1-1} R_A\left(\frac{\tau}{\mu_k}\right),$$

we have the assertion. □

Remark 3.3.4. For an alternative proof of Proposition 3.3.3 involving FIO theory, see Appendix 3.27.

Lemma 3.3.5. Let A, A_1, A_2 be as in Assumptions 3.3.2, and assume $n_1 > \dfrac{n_2}{m_2}$. Then we have, for $\tau \to +\infty$,

$$N_A(\tau) = \begin{cases} \dfrac{C_1}{n_1}\zeta\left(A_2, n_1\right)\tau^{n_1} + \mathcal{O}(\tau^{n_1-1}) & \text{if } \dfrac{n_2}{m_2} < n_1 - 1, \\ \dfrac{C_1}{n_1}\zeta\left(A_2, n_1\right)\tau^{n_1} + \mathcal{O}\left(\tau^{n_1-1}\log\tau\right) & \text{if } \dfrac{n_2}{m_2} = n_1 - 1, \\ \dfrac{C_1}{n_1}\zeta\left(A_2, n_1\right)\tau^{n_1} + \mathcal{O}\left(\tau^{\frac{n_2}{m_2}}\right) & \text{if } \dfrac{n_2}{m_2} > n_1 - 1, \end{cases}$$

where C_1 is given by (3.34).

Proof. Using Proposition 3.3.3 we obtain

$$N_A(\tau) = \sum_{\mu_k<\tau} \left(\frac{C_1}{n_1}\left(\frac{\tau}{\mu_k}\right)^{n_1} + \frac{1}{{\mu_k}^{n_1-1}} r_k\left(\tau\right)\right),$$

where $r_k(\tau)$ is uniformly $\mathcal{O}\left(\tau^{n_1-1}\right)$ for $\tau \to +\infty$, in the sense of (3.35). We can then write

$$\begin{aligned}
&\left|N_A(\tau) - \frac{C_1}{n_1}\zeta(A_2,n_1)\tau^{n_1}\right| \\
&\quad = \left|\sum_{\mu_k<\tau}\left(\frac{C_1}{n_1}\frac{\tau^{n_1}}{\mu_k^{n_1}} + \frac{1}{\mu_k^{n_1-1}}r_k(\tau^{n_1-1})\right) - \frac{C_1}{n_1}\zeta(A_2,n_1)\tau^{n_1}\right| \\
&\quad \leq \frac{C_1}{n_1}\tau^{n_1}\left|F_{A_2}(\tau,n_1) - \zeta(A_2,n_1)\right| + \left|\sum_{\mu_k<\tau}\frac{1}{\mu_k^{n_1-1}}r_k(\tau^{n_1-1})\right| \\
&\quad \leq \frac{C_1}{n_1}\tau^{n_1}\left|F_{A_2}(\tau,n_1) - \zeta(A_2,n_1)\right| + C\tau^{n_1-1}F_{A_2}(\tau,n_1-1). \qquad (3.37)
\end{aligned}$$

Let us start with the case $n_1 - 1 > \frac{n_2}{m}$. Using (3.37), we find

$$\begin{aligned}
&\limsup_{\tau\to+\infty}\frac{\left|N_A(\tau) - \frac{C_1}{n_1}\zeta(A_2,n_1)\tau^{n_1}\right|}{\tau^{n_1-1}} \\
&\quad \leq \frac{C_1}{n_1}\limsup_{\tau\to+\infty}\tau\left|\zeta(A_2,n_1) - F_{A_2}(\tau,n_1)\right| + C\limsup_{\tau\to+\infty}F_{A_2}(\tau,n_1-1).
\end{aligned}$$

Since

$$n_1 > n_1 - 1 > \frac{n_2}{m_2} \Rightarrow \frac{n_2}{m_2} - n_1 < -1,$$

$\zeta(A_2,n_1) - F_1(\tau) = \mathcal{O}\left(\tau^{\frac{n_2}{m_2}-n_1}\right)$ for $\tau\to+\infty$, in view of Proposition 3.3.1. It follows that

$$\limsup_{\tau\to+\infty}\tau|\zeta(A_2,n_1) - F_1(\tau)| \leq \widetilde{C}\limsup_{\tau\to+\infty}\tau^{\frac{n_2}{m_2}-n_1+1} = 0,$$

which implies

$$\limsup_{\tau\to+\infty}\frac{\left|N_A(\tau) - \frac{C_1}{n_1}\zeta(A_2,n_1)\tau^{n_1}\right|}{\tau^{n_1-1}} \leq C\limsup_{\tau\to+\infty}F_{A_2}(\tau,n_1-1) = C\zeta(A_2,n_1-1).$$

Since $n_1 - 1 > \frac{n_2}{m_2}$, $\zeta(A_2,n_1-1)$ is finite, and we have the desired assertion.

In the case $n_1 - 1 = \frac{n_2}{m_2}$, from (3.37) we analogously get

$$\begin{aligned}
&\limsup_{\tau\to+\infty}\frac{\left|N_A(\tau) - \frac{C_1}{n_1}\zeta(A_2,n_1)\tau^{n_1}\right|}{\tau^{n_1-1}\,\log\tau} \\
&\quad \leq \frac{C_1}{n_1}\limsup_{\tau\to+\infty}\frac{\tau}{\log\tau}\left|\zeta(A_2,n_1) - F_{A_2}(\tau,n_1)\right| + C\limsup_{\tau\to+\infty}\frac{1}{\log\tau}F_{A_2}\!\left(\tau,\frac{n_2}{m_2}\right).
\end{aligned}$$

Since $n_1 > n_1 - 1 = \frac{n_2}{m_2}$, in view of Proposition 3.3.1 we find

$$\zeta(A_2,n_1) - F_1(\tau) = \mathcal{O}\left(\tau^{-1}\right), \quad F_{A_2}\left(\tau,\frac{n_2}{m_2}\right) = F_2(\tau) = \mathcal{O}\left(\log\tau\right),$$

so that

$$\limsup_{\tau\to+\infty} \frac{\left|N_A(\tau) - \frac{C_1}{n_1}\zeta(A_2, n_1)\tau^{n_1}\right|}{\tau^{n_1-1}\log\tau} \le \widetilde{C},$$

as claimed.

Finally, in the case $n_1 - 1 < \frac{n_2}{m_2}$, (3.37) gives

$$\begin{aligned}\limsup_{\tau\to+\infty} &\frac{\left|N_A(\tau) - \frac{C_1}{n_1}\zeta(A_2, n_1)\tau^{n_1}\right|}{\tau^{\frac{n_2}{m_2}}} \\ &\le \frac{C_1}{n_1}\limsup_{\tau\to+\infty} \tau^{n_1-\frac{n_2}{m_2}} \left|\zeta(A_2, n_1) - F_{A_2}(\tau, n_1)\right| + \\ &\quad C \limsup_{\tau\to+\infty} \tau^{n_1-1-\frac{n_2}{m_2}} F_{A_2}(\tau, n_1-1).\end{aligned}$$

Since $n_1 > \frac{n_2}{m_2} > n_1 - 1$, Proposition 3.3.1 implies

$$\zeta(A_2, n_1) - F_1(\tau) = \mathcal{O}\left(\tau^{\frac{n_2}{m_2}-n_1}\right), \quad F_{A_2}(\tau, n_1-1) = F_3(\tau) = \mathcal{O}\left(\tau^{\frac{n_2}{m}-n_1+1}\right).$$

Therefore,

$$\limsup_{\tau\to+\infty} \frac{\left|N_A(\tau) - \frac{C_1}{n_1}\zeta(A_2, n_1)\tau^{n_1}\right|}{\tau^{\frac{n_2}{m_2}}} < +\infty.$$

The proof is complete. □

We can now prove our main result.

Theorem 3.3.6. Let M_1, M_2 be two closed manifolds of dimension n_1, n_2, respectively. Let $A = A_1 \otimes A_2$, where $A_j \in \mathrm{L}^{m_j}_{\mathrm{cl}}(M_j)$, $m_j > 0$, $j = 1, 2$, are positive, self-adjoint, invertible operators, with $\dfrac{n_1}{m_1} > \dfrac{n_2}{m_2}$. Then, for $\tau \to +\infty$,

$$N_A(\tau) = \begin{cases} \dfrac{C_1}{n_1}\zeta\left(A_2, \dfrac{n_1}{m_1}\right)\tau^{\frac{n_1}{m_1}} + \mathcal{O}\left(\tau^{\frac{n_1-1}{m_1}}\right) & \text{if } \dfrac{n_2}{m_2} < \dfrac{n_1-1}{m_1}, \\ \dfrac{C_1}{n_1}\zeta\left(A_2, \dfrac{n_1}{m_1}\right)\tau^{\frac{n_1}{m_1}} + \mathcal{O}\left(\tau^{\frac{n_1-1}{m_1}}\log\tau\right) & \text{if } \dfrac{n_2}{m_2} = \dfrac{n_1-1}{m_1}, \\ \dfrac{C_1}{n_1}\zeta\left(A_2, \dfrac{n_1}{m_1}\right)\tau^{\frac{n_1}{m_1}} + \mathcal{O}\left(\tau^{\frac{n_2}{m_2}}\right) & \text{if } \dfrac{n_2}{m_2} > \dfrac{n_1-1}{m_1}, \end{cases}$$

where C_1 is given by (3.34).

Proof. Without loss of generality, we can assume $m_1 = 1$, possibly considering an appropriate power of A. Moreover, again without loss of the generality, we can assume that all the eigenvalues are strictly larger than one, so that the Assumptions 3.3.2 are fulfilled. Then, the claim follows from Lemma 3.3.5. □

Remark 3.3.7. In this chapter we always worked in the case of usual Hörmander pseudodifferential operators on closed manifolds, $A_i \in \mathrm{L}^{m_i}(M_i)$, $i = 1, 2$. However, it should be noticed that the case of global Shubin operators, $A_i \in \mathrm{G}^{m_i}(\mathbb{R}^{n_i})$, $i = 1, 2$ can be treated in a completely similar fashion.

3.3.2 Sharpness of the result

In this subsection we show that the estimates of Theorem 3.3.6 are sharp. Again, we consider $-\Delta_{\mathbb{S}^n}$, the usual Laplace operator on the n-dimensional sphere $\mathbb{S}^n$, and recall Theorem 3.2.11. In particular, for the 1-dimensional sphere $-\Delta_{\mathbb{S}^1}$ we have

$$\begin{aligned} \lambda_k &= k^2 \\ \sharp\left(k^2\right) &= \binom{1+k}{1} - \binom{k-1}{1} = 2; \end{aligned}$$

for the 2-dimensional sphere $-\Delta_{\mathbb{S}^2}$ we have

$$\begin{aligned} \lambda_k &= k(k+1) = k^2 + k \\ \sharp\left(k^2+k\right) &= \binom{2+k}{2} - \binom{k}{2} = \frac{(k+2)(k+1)}{2} - \frac{k(k-1)}{2} = 2k+1. \end{aligned}$$

Now, we set

$$A_1 = (-\Delta_{\mathbb{S}^2} + 2) - 2\left(-\Delta_{\mathbb{S}^2} + \frac{1}{4}\right)^{\frac{1}{2}} \in \mathrm{L}^2_{\mathrm{cl}}(\mathbb{S}^2),\ A_2 = -\Delta_{\mathbb{S}^1} + 1 \in \mathrm{L}^2_{\mathrm{cl}}(\mathbb{S}^1),$$

where A_1 is considered as an unbounded operator on $L^2(\mathbb{S}^2)$, and A_2 is considered as an unbounded operator on $L^2(\mathbb{S}^1)$. By the functional calculus of operators,

$$\sigma(A_1) = \left\{k^2 - k + 1 \mid k \in \mathbb{N},\ \sharp\left(k^2 - k + 1\right) = (2k+1)\right\}, \tag{3.38}$$
$$\sigma(A_2) = \left\{n^2 + 1 \mid n \in \mathbb{N},\ \sharp\left(n^2 + 1\right) = 2\right\}, \tag{3.39}$$

since the eigenfunction of A_1 and $-\Delta_{\mathbb{S}^2}$ are the same. Notice that all the eigenvalues of A_1 are larger then 1, therefore

$$N_{A_1}(\tau) = 0, \quad \tau \leq 1. \tag{3.40}$$

Knowing precisely the eigenvalues of A_1 together with their multiplicities, we can write, for $\tau > 1$,

$$\begin{aligned} N_{A_1}(\tau) &= \sum_{k^2-k+1<\tau} \sharp\left(k^2 - k + 1\right) \\ &= \sum_{k^2-k+1<\tau} (2k+1) = \sum_{k=0}^{\bar{k}} (2k+1) \end{aligned}$$

where

$$\bar{k}^2 - \bar{k} + 1 < \tau \leq \left(\bar{k}+1\right)^2 - \left(\bar{k}+1\right) + 1 = \bar{k}^2 + \bar{k} + 1, \tau > 1.$$

That is,

$$N_{A_1}(\tau) = \sum_{k=0}^{\bar{k}} (2k+1) = \sum_{k^2+k \leq \bar{k}^2+\bar{k}} \sharp\left(k^2+k\right) = N_{-\Delta_{\mathbb{S}^2}}\left(\bar{k}^2 + \bar{k} + \frac{1}{2}\right), \tag{3.41}$$

provided that

$$\bar{k}^2 - \bar{k} + 1 < \tau \leq \left(\bar{k}+1\right)^2 - \left(\bar{k}+1\right) + 1 = \bar{k}^2 + \bar{k} + 1, \tau > 1.$$

Using (3.24), we have, for each $\bar{k} \in \mathbb{N}$,

$$N_{-\Delta_{\mathbb{S}^2}}\left(\bar{k}^2 + \bar{k} + \frac{1}{2}\right) = \bar{k}^2 + 2\bar{k} + 1.$$

So, in view of (3.41), supposing $\tau > 1$, we find

$$\begin{aligned} N_{A_1}(\tau) &= \bar{k}^2 + 2\bar{k} + 1, \\ &\qquad \bar{k}^2 - \bar{k} + 1 < \tau \leq \left(\bar{k}+1\right)^2 - \left(\bar{k}+1\right) + 1 = \bar{k}^2 + \bar{k} + 1. \end{aligned}$$

The asymptotic expansion in Theorem 3.2.10 implies that

$$N_{A_1}(\tau) = \tau + R(\tau), \quad R = \mathcal{O}\left(\tau^{\frac{1}{2}}\right).$$

We can then obtain a bound for $R(\tau)$:

$$\begin{aligned} R(\tau) &= N_{A_1}(\tau) - \tau \\ &= \bar{k}^2 + 2\bar{k} + 1 - \tau, \quad \bar{k}^2 - \bar{k} + 1 < \tau \leq \bar{k}^2 + \bar{k} + 1. \end{aligned}$$

Therefore, for $\tau > 16$,

$$R(\tau) \geq \bar{k}^2 + 2\bar{k} + 1 - \bar{k}^2 - \bar{k} - 1 = \bar{k} > \frac{3\sqrt{\tau}}{4},$$

which implies, in particular, that the remainder is positive for $\tau > 16$. We also have

$$R(\tau) < \bar{k}^2 + 2\bar{k} + 1 - \bar{k}^2 + \bar{k} - 1 = 2\bar{k} < 4\sqrt{\tau},$$

and we can conclude that

$$\frac{3\sqrt{\tau}}{4} \leq R(\tau) \leq 4\sqrt{\tau}, \quad \tau > 16. \tag{3.42}$$

Summing up, we proved that

$$N_{A_1}(\tau) = \tau + R(\tau), \tag{3.43}$$

$$N_{A_2}(\tau) = 2\,\tau^{1/2} + \mathcal{O}(1), \tag{3.44}$$

where the $R(\tau)$ in (3.43) satisfies (3.42). Notice that both A_1 and A_2 are elliptic, invertible and positive, so it is possible to consider powers of these operators of arbitrary exponent. Now, we examine separately the three different situations that can arise.

Case $\frac{n_1}{m_1} > \frac{n_2}{m_2}$ and $\frac{n_1-1}{m_1} > \frac{n_2}{m_2}$

Let us consider the operator

$$B = A_1 \otimes A_2^2.$$

Clearly $\frac{n_1}{m_1} = \frac{2}{2} = 1 > \frac{n_2}{m_2} = \frac{1}{4}$ and $\frac{n_1-1}{m_1} = \frac{1}{2} > \frac{n_2}{m_2} = \frac{1}{4}$, so we are in the first case of Theorem 3.3.6, which states that

$$N_B(\tau) = \zeta(A_2^2, 1)\tau + \mathcal{O}\left(\tau^{1/2}\right). \tag{3.45}$$

By equations (3.38) and (3.39) we obtain

$$\sigma(B) = \{\left(k^2-k+1\right)\left(n^2+1\right)^2 \mid k, n \in \mathbb{N}, \\ \sharp\left((k^2-k+1)(n^2+1)^2\right) = 2(2k+1)\}.$$

Therefore,

$$\begin{aligned} N_B(\tau) &= \sum_{(k^2-k+1)(n^2+1)^2<\tau}^{n\in\mathbb{N},\, k\in\mathbb{N}} \sharp\left(\left(k^2-k+1\right)\left(n^2+1\right)^2\right) \\ &= \sum_{(k^2-k+1)(n^2+1)^2<\tau}^{n\in\mathbb{N},\, k\in\mathbb{N}} 2\,(2k+1) \\ &= 2 \sum_{(k^2-k+1)<\frac{\tau}{(n^2+1)^2}}^{n\in\mathbb{N},\, k\in\mathbb{N}} \sharp\left(k^2-k+1\right) \\ &= 2 \sum_{(n^2+1)^2<\tau}^{n\in\mathbb{N}} N_{A_1}\left(\frac{\tau}{\left(n^2+1\right)^2}\right) \qquad (3.46) \\ &= 2\left(\sum_{(n^2+1)^2<\tau}^{n\in\mathbb{N}} \frac{\tau}{(n^2+1)^2} + R\left(\frac{\tau}{(n^2+1)^2}\right)\right). \qquad (3.47) \end{aligned}$$

Notice that in (3.46) we have made use of (3.40) to reduce the summation. Let us now show that the estimate (3.45) is indeed sharp, that is

$$\limsup_{\tau\to+\infty} \frac{\left|N_B(\tau) - \zeta\left(A_2^2, 1\right)\tau\right|}{\tau^{1/2}} > 0,$$

by direct computation. In view of (3.47), we can write

$$\begin{aligned}
&\limsup_{\tau\to+\infty} \frac{\left|N_B(\tau) - \zeta\left(A_2^2, 1\right)\tau\right|}{\tau^{1/2}} \\
&= \limsup_{\tau\to+\infty} \frac{\left|2\sum_{(n^2+1)^2<\tau}\left(\frac{\tau}{(n^2+1)^2} + R\left(\frac{\tau}{(n^2+1)^2}\right)\right) - \zeta\left(A_2^2, 1\right)\tau\right|}{\tau^{1/2}} \\
&= \limsup_{\tau\to+\infty} \frac{\left|2\sum_{(n^2+1)^2<\tau}\frac{\tau}{(n^2+1)^2} - \zeta\left(A_2^2, 1\right)\tau + 2\sum_{(n^2+1)^2<\tau} R\left(\frac{\tau}{(n^2+1)^2}\right)\right|}{\tau^{1/2}}.
\end{aligned} \tag{3.48}$$

We notice that

$$\begin{aligned}
&\limsup_{\tau\to+\infty} \frac{\left|2\sum_{(n^2+1)^2<\tau}\frac{\tau}{(n^2+1)^2} - \zeta\left(A_2^2, 1\right)\tau\right|}{\tau^{1/2}} \\
&\qquad\qquad = \limsup_{\tau\to+\infty} \tau^{1/2}\left(F_{A_2^2}(\tau,1) - \zeta\left(A_2^2, 1\right)\right),
\end{aligned}$$

where we have used the notation introduced in Proposition 3.3.1. By Proposition 3.3.1, $F_{A_2^2}(\tau,1) - \zeta\left(A_2^2, 1\right) = \mathcal{O}\left(\tau^{-\frac{3}{4}}\right)$, therefore[7],

$$\limsup_{\tau\to+\infty} \frac{\left|2\sum_{(n^2+1)^2<\tau}\frac{\tau}{(n^2+1)^2} - \zeta\left(A_2^2, 1\right)\tau\right|}{\tau^{1/2}} = 0.$$

Since, for all τ,

$$\sum_{(n^2+1)^2<\tau} 2\frac{\tau}{(n^2+1)^2} - \zeta\left(A_2^2, 1\right)\tau \leq 0,$$

[7] Actually, here one could prove directly that $F_1(\tau) - \zeta\left(A_2^2, 1\right)$ is asymptotic to $\tau^{-\frac{3}{4}}$.

(3.48) becomes

$$\begin{aligned}
&\limsup_{\tau\to+\infty} \frac{\left|N_B(\tau) - \zeta\left(A_2^2, 1\right)\tau\right|}{\tau^{1/2}} \\
&\geq -\limsup_{\tau\to+\infty} \frac{\zeta\left(A_2^2, 1\right) - 2\sum_{(n^2+1)^2<\tau} \frac{\tau}{(n^2+1)^2}}{\tau^{1/2}} + 2\limsup_{\tau\to+\infty} \sum_{(n^2+1)^2<\tau} \frac{\left|R\left(\frac{\tau}{(n^2+1)^2}\right)\right|}{\tau^{1/2}} \\
&\geq \frac{3}{2}\limsup_{\tau\to+\infty} \sum_{(n^2+1)^2<\tau} \frac{\tau^{1/2}}{(n^2+1)\tau^{1/2}} \\
&= \frac{3}{2}\zeta\left(A_2^2, \frac{1}{2}\right).
\end{aligned}$$

Here, we have used the estimates (3.42) and that the quantities $\frac{n_1}{m_1} = 1$ and $\frac{n_1-1}{n_2} = \frac{1}{2}$ are larger than $\frac{n_2}{m_2} = \frac{1}{4}$. The latter implies that $\zeta\left(A_2^2, \frac{1}{2}\right)$ is a finite, positive quantity[8], in view of the holomorphic properties of the spectral ζ-function of elliptic positive pseudodifferential operators on closed manifolds, see [See67]. This proves the desired result.

Case $\frac{n_1}{m_1} > \frac{n_2}{m_2}$ and $\frac{n_1-1}{m_1} = \frac{n_2}{m_2}$

We consider the operator

$$C = A_1 \otimes A_2.$$

Clearly $\frac{n_1}{m_1} = \frac{2}{2} = 1 > \frac{n_2}{m_2} = \frac{1}{2}$ and $\frac{n_1-1}{m_1} = \frac{1}{2} = \frac{n_2}{m_2}$ so that we are in second case of Theorem 3.3.6, which now states that

$$N_C(\tau) = \zeta(A_2, 1)\tau + \mathcal{O}\left(\tau^{1/2}\log\tau\right).$$

Using (3.38) and (3.39) we obtain explicitly the spectrum of C, namely

$$\sigma(C) = \left\{\left(k^2 - k + 1\right)\left(n^2 + 1\right) \mid \sharp\left(\left(k^2 - k + 1\right)\left(n^2 + 1\right)\right) = 2(2k+1)\right\}.$$

[8]The convergence of the involved series is straightforward.

Therefore, using (3.40),

$$\begin{aligned}
N_C(\tau) &= \sum_{(k^2-k+1)(n^2+1)<\tau}^{n\in\mathbb{N},k\in\mathbb{N}} 2\,(2k+1) \\
&= 2 \sum_{(k^2-k+1)<\frac{\tau}{n^2+1}}^{n\in\mathbb{N},k\in\mathbb{N}} \sharp\left(k^2+k+1\right) \\
&= 2 \sum_{(n^2+1)<\tau}^{n\in\mathbb{N}} N_{A_1}\left(\frac{\tau}{n^2+1}\right) \\
&= 2 \sum_{n^2+1<\tau}^{n\in\mathbb{N}} \left(\frac{\tau}{n^2+1} + R\left(\frac{\tau}{n^2+1}\right)\right). \qquad (3.49)
\end{aligned}$$

Let us check directly that

$$\limsup_{\tau\to+\infty} \frac{|N_C(\tau) - \zeta(A_2,1)\tau|}{\tau^{1/2}\log\tau} > 0. \qquad (3.50)$$

Using (3.49) and (3.42) we can write

$$\begin{aligned}
&\limsup_{\tau\to+\infty} \frac{|N_C(\tau) - \zeta(A_2,1)\tau|}{\tau^{1/2}\log\tau} \\
&\quad = \limsup_{\tau\to+\infty} \frac{\left|2\sum_{n^2+1<\tau}\left(\frac{\tau}{n^2+1} + R\left(\frac{\tau}{n^2+1}\right)\right) - \zeta(A_2,1)\tau\right|}{\tau^{1/2}\log\tau} \\
&\quad \geq -\limsup_{\tau\to+\infty} \frac{\tau^{1/2}\left(\zeta(A_2,1) - 2\sum_{n^2+1<\tau}\frac{1}{n^2+1}\right)}{\log\tau} \\
&\qquad + \limsup_{\tau\to+\infty} \frac{3}{4}\tau^{1/2}\frac{2\sum_{n^2+1<\tau}\frac{1}{(n^2+1)^{1/2}}}{\tau^{1/2}\log\tau} \\
&\quad \geq -\limsup_{\tau\to+\infty} \tau^{\frac{1}{2}}\frac{2\sum_{n^2+1\geq\tau}\frac{1}{n^2+1}}{\log\tau} + \limsup_{\tau\to+\infty}\frac{3}{2}\frac{\sum_{n^2+1<\tau}\frac{1}{(n^2+1)^{1/2}}}{\log\tau} \qquad (3.51)
\end{aligned}$$

Finally, using the results of Proposition 3.3.1 (or directly, by integral inequalities), we obtain that

$$\limsup_{\tau\to+\infty} \tau^{\frac{1}{2}}\frac{2\sum_{n^2+1\geq\tau}\frac{1}{n^2+1}}{\log\tau} = \lim_{\tau\to+\infty} \tau^{\frac{1}{2}}\frac{2\sum_{n^2+1\geq\tau}\frac{1}{n^2+1}}{\log\tau} = 0.$$

Moreover,

$$\limsup_{\tau\to+\infty} \frac{3}{2}\frac{\sum_{n^2+1<\tau}\frac{1}{n^2+1}}{\log\tau} = \frac{3}{4},$$

so that, by means of (3.51), the desired result is proven also in this second case.

Case $\frac{n_1}{m_1} > \frac{n_2}{m_2}$ and $\frac{n_1-1}{m_1} < \frac{n_2}{m_2}$

In this situation we consider the operator

$$D = A_1 \otimes A_2^{\frac{3}{4}}.$$

Clearly, $\frac{n_1}{m_1} = \frac{2}{2} = 1 > \frac{n_2}{m_2} = \frac{2}{3}$ and $\frac{n_1-1}{m_1} = \frac{1}{2} < \frac{n_2}{m_2} = \frac{2}{3}$, so we are in the third case of Theorem 3.3.6, which implies that

$$N_D(\tau) = \zeta\left(A_2^{\frac{3}{4}}, 1\right)\tau + \mathcal{O}\left(\tau^{\frac{2}{3}}\right). \tag{3.52}$$

It is immediate to observe that

$$\sigma(D) = \{\left(k^2+k+1\right)\left(n^2+1\right)^{3/4} \mid \\ \sharp\left((k^2+k+1)(n^2+1)^{3/4}\right) = 2\,(2k+1)\}. \tag{3.53}$$

Therefore, using again (3.40), we obtain

$$\begin{aligned} N_D(\tau) &= \sum_{(k^2-k+1)(n^2+1)^{3/4}<\tau}^{n\in\mathbb{N},k\in\mathbb{N}} 2\,(2k+1) \\ &= 2\sum_{(k^2-k+1)<\frac{\tau}{(n^2+1)^{3/4}}}^{n\in\mathbb{N},k\in\mathbb{N}} \sharp\left(k^2-k+1\right) \\ &= 2\sum_{(n^2+1)^{3/4}<\tau}^{n\in\mathbb{N}} N_{A_1}\left(\frac{\tau}{(n^2+1)^{3/4}}\right) \\ &= 2\sum_{(n^2+1)^{3/4}<\tau}^{n\in\mathbb{N}} \left(\frac{\tau}{(n^2+1)^{3/4}} + R\left(\frac{\tau}{(n^2+1)^{3/4}}\right)\right). \end{aligned} \tag{3.54}$$

Let us now compute directly

$$\limsup_{\tau\to+\infty} \frac{\left|N_D(\tau) - \zeta\left(A_2^{3/4}, 1\right)\tau\right|}{\tau^{2/3}}.$$

By (3.54), we find

$$\limsup_{\tau\to+\infty} \frac{\left|N_D(\tau) - \zeta(A_2^{3/4},1)\tau\right|}{\tau^{2/3}}$$

$$= \limsup_{\tau\to+\infty} \frac{\left|2\sum_{(n^2+1)^{3/4}<\tau}\left(\frac{\tau}{(n^2+1)^{3/4}} + R\left(\frac{\tau}{(n^2+1)^{\frac{3}{4}}}\right)\right) - \zeta\left(A_2^{3/4},1\right)\tau\right|}{\tau^{2/3}}$$

$$= \limsup_{\tau\to+\infty} \tau^{-2/3}\cdot\left|2\sum_{(n^2+1)^{3/4}<\tau}\frac{\tau}{(n^2+1)^{3/4}} - \zeta\left(A_2^{3/4},1\right)\tau + \right.$$

$$\left. +2\sum_{(n^2+1)^{3/4}<\tau} R\left(\frac{\tau}{(n^2+1)^{\frac{3}{4}}}\right)\right|.$$

We also notice that

$$\lim_{\tau\to+\infty}\frac{\left|2\sum_{(n^2+1)^{3/4}<\tau}\frac{\tau}{(n^2+1)^{3/4}} - \zeta\left(A_2^{3/4},1\right)\tau\right|}{\tau^{2/3}}$$

$$= \lim_{\tau\to+\infty}\frac{\zeta\left(A_2^{3/4},1\right)\tau - 2\sum_{(n^2+1)^{3/4}<\tau}\frac{\tau}{(n^2+1)^{3/4}}}{\tau^{2/3}}$$

$$= \lim_{\tau\to+\infty} 2\,\tau^{1/3}\sum_{(n^2+1)^{3/4}\geq\tau}\frac{1}{(n^2+1)^{3/4}},$$

and that

$$\sum_{(n+1)^{3/2}\geq\tau}\frac{1}{(n+1)^{3/2}} \leq \sum_{(n^2+1)^{3/4}\geq\tau}\frac{1}{(n^2+1)^{3/4}} \leq \sum_{n^{3/2}\geq\tau}\frac{1}{n^{3/2}}.$$

Using the standard integral criteria of series convergence, one can easily check that

$$\lim_{\tau\to+\infty}\tau^{1/3}\sum_{(n+1)^{3/2}\geq\tau}\frac{1}{(n+1)^{3/2}} = \lim_{\tau\to+\infty}\tau^{1/3}\sum_{n^{3/2}\geq\tau}\frac{1}{n^{3/2}} = 2.$$

Hence

$$\lim_{\tau\to+\infty} 2\,\tau^{1/3}\sum_{(n^2+1)^{3/4}\geq\tau}\frac{1}{(n^2+1)^{3/4}} = 4. \tag{3.55}$$

By a similar argument, we also have that

$$\lim_{\tau\to+\infty}\tau^{-1/6}\sum_{(n^2+1)^{3/4}<\tau}\frac{1}{(n^2+1)^{3/8}} = 4. \tag{3.56}$$

In view of (3.42), (3.55) and (3.56) we finally obtain

$$
\begin{aligned}
\limsup_{\tau\to+\infty} & \frac{\left|N_D(\tau)-\zeta(A_2^{3/4},1)\tau\right|}{\tau^{2/3}} \\
&\geq \limsup_{\tau\to+\infty}\frac{N_D(\tau)-\zeta(A_2^{3/4},1)\tau}{\tau^{2/3}} \\
&= -\lim_{\tau\to+\infty} 2\,\tau^{1/3}\sum_{(n^2+1)^{3/4}\geq\tau}\frac{1}{(n^2+1)^{3/4}} \\
&\quad + \limsup_{\tau\to+\infty} 2\frac{\sum_{(n^2+1)^{3/4}<\tau} R\left(\frac{\tau}{(n^2+1)^{3/4}}\right)}{\tau^{2/3}} \\
&\geq -4+\frac{3}{2}\limsup_{\tau\to+\infty}\tau^{-1/6}\sum_{(n^2+1)^{3/4}<\tau}\frac{1}{(n^2+1)^{3/8}} \\
&\geq -4+6=2>0. \qquad (3.57)
\end{aligned}
$$

Equation (3.57) proves the desired result also in this last case.

3.3.3 Appendix. An alternative proof of Proposition 3.3.3

In this appendix we will give an alternative proof of Proposition 3.3.3, which stresses the connection with FIO theory. First, we recall the following Tauberian Theorem, whose proof can be found in [Hel84].

Theorem 3.3.8. Let $\rho\in\mathscr{S}(\mathbb{R})$ such that $\rho\geq 0$, $\rho(0)>0$, $\hat{\rho}$ is even, compactly supported, and such that $\hat{\rho}(0)=1$. If

$$\int\rho(\lambda-\mu)\,dN(\mu)=C_1\lambda^{n-1}+\mathcal{O}(\lambda^{n-2}),\qquad \lambda\to+\infty, \qquad (3.58)$$

then

$$N(\lambda)=\frac{C_1}{n}\lambda^n+\mathcal{O}(\lambda^{n-1}),\qquad \lambda\to+\infty.$$

Remark 3.3.9. A key point in the proof of Theorem 3.3.8 is the decomposition

$$\int_{-\infty}^{\tau}\int_{-\infty}^{+\infty}\rho(\lambda-\mu)dN(\mu)d\lambda = N(\tau)+\mathcal{A}(\tau)+\mathcal{B}_2(\tau)+\mathcal{B}_3(\tau)+\mathcal{C}(\tau),$$

with

$$\begin{aligned}
\mathcal{A}(\tau) &= \int_{-\infty}^{\tau} \int_{\mu>\tau+K} \rho(\lambda-\mu)dN(\mu)d\lambda;\\
\mathcal{B}_2(\tau) &= -\int_{\tau}^{+\infty} \int_{\mu<\tau-K} \rho(\lambda-\mu)dN(\mu)d\lambda;\\
\mathcal{B}_3(\tau) &= -\int_{\tau-K\le\mu<\tau} dN(\mu);\\
\mathcal{C}(\tau) &= \int_{-\infty}^{\tau} \int_{|\mu-\tau|\le K} \rho(\lambda-\mu)dN(\mu)d\lambda;
\end{aligned}$$

for a fixed $K > 0$. Moreover, all the terms $\mathcal{A}(\cdot), \mathcal{B}_2(\cdot), \mathcal{B}_3(\cdot)$ and $\mathcal{C}(\cdot)$ are $\mathcal{O}(\tau^{n-1})$ for $\tau > 1$. For further details see [Hel84, §10].

Lemma 3.3.10. Let ρ be as in (3.58). Then, for $n_1 > 1$,

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \frac{1}{\mu_k}\rho\left(\frac{\lambda-\sigma}{\mu_k}\right) dN_{\mu_k A_1}(\sigma)\\
&\qquad = \begin{cases} \frac{1}{\mu_k}\left(C_1\left(\frac{\lambda}{\mu_k}\right)^{n_1-1} + R_{n_1-2}\left(\frac{\lambda}{\mu_k}\right)\right) & \text{for } \frac{\lambda}{\mu_k} > 1\\ \frac{1}{\mu_k} R_{-\infty}\left(\frac{\lambda}{\mu_k}\right) & \text{for } \frac{\lambda}{\mu_k} \to -\infty, \end{cases}
\end{aligned}$$

where

$$C_1 = \frac{1}{(2\pi)^{n_1}} \int_{a_1(z_1)\le 1} dz_1,$$

$R_{n_1-2}(\tau) = \mathcal{O}\left(\tau^{n_1-2}\right)$ and $R_{-\infty}(\tau) = \mathcal{O}(\tau^{-\infty})$. If $n_1 = 1$, then the term $R_{n_1-2}(\tau)$ actually is $\mathcal{O}\left(\tau^{-2}\right)$.

Proof. We have

$$\int_{-\infty}^{+\infty} \frac{1}{\mu_k}\rho\left(\frac{\lambda-\sigma}{\mu_k}\right) dN_{\mu_k A_1}(\sigma) = \frac{1}{\mu_k}\int_{-\infty}^{+\infty} \rho\left(\frac{\lambda}{\mu_k}-\sigma'\right) dN_{A_1}(\sigma').$$

Now let us consider

$$\int_{-\infty}^{+\infty} \rho\left(\frac{\lambda}{\mu_k}-\sigma'\right) dN_{A_1}(\sigma').$$

Following the ideas in Subsection 3.2.2, we have

$$\begin{aligned}\int_{-\infty}^{+\infty} \rho\left(\frac{\lambda}{\mu_k}-\sigma'\right) dN_{A_1}(\sigma') &= \sum_{\sigma_i'\in\sigma(A_1)} \rho\left(\frac{\lambda}{\mu_k}-\sigma_i\right)\\ &= \sum_{k=1}^{N}\int \psi_j(t)e^{i\left(-\frac{\lambda}{\mu_k}t+\varphi_j(t,x_1,\xi_1)-x_1\xi_1\right)}\\ &\qquad\qquad \hat\rho(t)a_j(t,x_1,\xi_1)\,dx_1\,dt\,d\xi_1\\ &\quad+\mathcal{O}\left(\left|\frac{\lambda}{\mu_k}\right|^{-\infty}\right),\end{aligned}$$

where $\{\psi_j\}$ is a partition of unity over M_1. Then, we need to study, for each k, an integral of the form

$$I\left(x_1,\frac{\lambda}{\mu_k}\right)=\int e^{i\left(-\frac{\lambda}{\mu_k}t+\varphi(t,x_1,\xi_1)-x_1\xi_1\right)}\hat\rho(t)a(t,x_1,\xi_1)\,dt\,d\xi_1,$$

where a is of order zero and φ is the solution of the eikonal equation related to A_1. We set

$$\Phi\left(t,x_1,\xi_1,\frac{\lambda}{\mu_k}\right):=-\frac{\lambda}{\mu_k}t+\varphi(t,x_1,\xi_1)-x_1\xi_1$$

and observe that, for $\frac{\lambda}{\mu_k}\to-\infty$ we have

$$\partial_t\Phi\left(t,x_1,\xi_1,\frac{\lambda}{\mu_k}\right)\succ\langle\xi_1\rangle+\left|\frac{\lambda}{\mu_k}\right|,$$

which means, via integration by parts with respect to t, that the term is $\mathcal{O}\left(\left|\frac{\lambda}{\mu_k}\right|^{-\infty}\right)$. Now we choose a cut-off function $H\in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp} H\subseteq[C^{-1},C]$, $0\le H\le 1$ and $H\equiv 1$ on $[C^{-1},C]$, for a suitable $C>>1$. Again, for $\xi_1\in\operatorname{supp}\left(1-H\left(\frac{(\cdot)}{\lambda/\mu_k}\right)\right)$, we have $\partial_t\Phi\left(t,x_1,\xi_1,\frac{\lambda}{\mu_k}\right)\succ\langle\xi_1\rangle+\left|\frac{\lambda}{\mu_k}\right|$, thus

$$I\left(x_1,\frac{\lambda}{\mu_k}\right)=\int e^{i\Phi}H\left(\frac{\xi_1}{\lambda/\mu_k}\right)\hat\rho(t)a(t,x_1,\xi_1)\,dt\,d\xi_1+\mathcal{O}\left(\left|\frac{\lambda}{\mu_k}\right|^{-\infty}\right).$$

To study the remaining part, we switch to polar coordinates $\xi_1=\frac{\lambda}{\mu_k}r_1\omega_1$, $r_1\in[0,+\infty)$, $\omega_1\in\mathbb{S}^{n_1-1}$, and use the Taylor approximation of Φ at $t=0$. We get

$$\begin{aligned}I\left(x_1,\frac{\lambda}{\mu_k}\right)=\left(\frac{\lambda}{\mu_k}\right)^{n_1}\int_{\mathbb{S}^{n_1-1}}\int e^{i\frac{\lambda}{\mu_k}(-t+tr_1a_1(x_1,\omega_1)+r_1\mathcal{O}(t^2))}&\\ \hat\rho(t)H(r_1\omega_1)a\left(t,x_1,\frac{\lambda}{\mu_k}r_1\omega_1\right)r_1^{n_1-1}\,dr_1\,dt\,d\omega_1.&\end{aligned}$$

To conclude, we apply the Stationary Phase Theorem to the $dr_1\, dt$ integral around the unique critical point $t = 0, r_1 = \frac{1}{a_1(x_1,\omega_1)}$, observing that, since $r \sim 1$, $D_r^2 a\left(t, x_1, \frac{\lambda}{\mu_k} r_1\omega_1\right)$ is in the span of

$$(\partial_\xi^\alpha a)\left(t, x_1, \frac{\lambda}{\mu_k} r_1\omega_1\right)\left(\frac{\lambda}{\mu_k}\right)^2 \prec \left\langle \frac{\lambda}{\mu_k} r_1\omega_1 \right\rangle^{-2}\left(\frac{\lambda}{\mu_k}\right)^2 \prec 1,$$

$|\alpha| = 2$. We obtain

$$I\left(x_1, \frac{\lambda}{\mu_k}\right) = \left(\frac{\lambda}{\mu_k}\right)^{n_1-1} \hat{\rho}(0)\frac{1}{(2\pi)^{n_1}} \int_{\mathbb{S}^{n_1-1}} \frac{1}{(a_1(x_1,\omega_1))^{n_1}}\, d\omega_1 + \mathcal{O}\left(\frac{\lambda}{\mu_k}\right)^{n_1-2},$$

which is our claim. If $n_1 = 1$ we conclude as in [GS94]. □

We can now prove the following Proposition.

Proposition 3.3.11. Let ρ be as in (3.58). Then

$$\begin{aligned} &\sum_{\mu_k<\tau} \int_{-\infty}^{\tau}\int_{-\infty}^{+\infty} \frac{1}{\mu_k}\rho\left(\frac{\lambda-\sigma}{\mu_k}\right) dN_{\mu_k A_1}(\sigma) d\lambda \\ &= N_A(\tau) + \sum_{\mu_k<\tau}\left(\mathcal{A}\left(\frac{\tau}{\mu_k}\right) + \mathcal{B}_2\left(\frac{\tau}{\mu_k}\right) + \mathcal{B}_3\left(\frac{\tau}{\mu_k}\right) + \mathcal{C}\left(\frac{\tau}{\mu_k}\right)\right), \end{aligned} \tag{3.59}$$

and $\mathcal{A}(\tau)$, $\mathcal{B}_2(\tau)$, $\mathcal{B}_3(\tau)$ and $\mathcal{C}(\tau)$ are are as in Remark 3.3.9, that is,

$$\mathcal{A}_j(\tau), \mathcal{B}_{2,j}(\tau), \mathcal{B}_{3,j}(\tau), \mathcal{C}_j(\tau) = \mathcal{O}\left(\left(\frac{\tau}{\mu_k}\right)^{n_1-1}\right),$$

uniformly with respect to μ_k.

Proof. First, we have, as a consequence of (3.59) and (3.33) ,

$$\begin{aligned} &\sum_{\mu_k<\tau} \int_{-\infty}^{\tau}\int_{-\infty}^{+\infty} \frac{1}{\mu_k}\rho\left(\frac{\lambda-\sigma}{\mu_k}\right) dN_{\mu_k A_1}(\sigma) d\lambda \\ &= \sum_{\mu_k<\tau} (N_{\mu_k A_1}(\tau) + \mathcal{A}_j(\tau) + \mathcal{B}_{2,j}(\tau) + \mathcal{B}_{3,j}(\tau) + \mathcal{C}_j(\tau)) && (3.60) \\ &= N_A(\tau) + \sum_{\mu_k<\tau} (\mathcal{A}_j(\tau) + \mathcal{B}_{2,j}(\tau) + \mathcal{B}_{3,j}(\tau) + \mathcal{C}_j(\tau)) && (3.61) \\ &= N_A(\tau) + \sum_{\mu_k<\tau}\left(\mathcal{A}\left(\frac{\tau}{\mu_k}\right) + \mathcal{B}_2\left(\frac{\tau}{\mu_k}\right) + \mathcal{B}_3\left(\frac{\tau}{\mu_k}\right) + \mathcal{C}\left(\frac{\tau}{\mu_k}\right)\right). \end{aligned}$$

Now, fix $K > 0$, and divide the leftover term in (3.61) as

$$\mathcal{A}_j(\tau) = \int_{-\infty}^{\tau} \int_{\sigma > \tau + \mu_k K} \frac{1}{\mu_k} \rho \left(\frac{\lambda - \sigma}{\mu_k} \right) dN_{\mu_k A_1}(\sigma) d\lambda; \tag{3.62}$$

$$\mathcal{B}_j(\tau) = \int_{-\infty}^{\tau} \int_{\sigma < \tau - \mu_k K} \frac{1}{\mu_k} \rho \left(\frac{\lambda - \sigma}{\mu_k} \right) dN_{\mu_k A_1}(\sigma) d\lambda; \tag{3.63}$$

$$\mathcal{C}_j(\tau) = \int_{-\infty}^{\tau} \int_{|\sigma - \tau| \leq \mu_k K} \frac{1}{\mu_k} \rho \left(\frac{\lambda - \sigma}{\mu_k} \right) dN_{\mu_k A_1}(\sigma) d\lambda. \tag{3.64}$$

It is clear that

$$\int_{-\infty}^{\tau} \int_{-\infty}^{+\infty} \frac{1}{\mu_k} \rho \left(\frac{\lambda - \sigma}{\mu_k} \right) dN_{\mu_k A_1}(\sigma) d\lambda = \mathcal{A}_j(\tau) + \mathcal{B}_j(\tau) + \mathcal{C}_j(\tau).$$

We begin to analyze $\mathcal{B}_j(\tau)$. The integrand function is uniformly bounded and integrable in the domain of integration, in view of Lemma 3.3.10. So, changing the order of integration, we can write

$$\begin{aligned}
\mathcal{B}_j(\tau) &= \int_{\sigma < \tau - \mu_k K} dN_{\mu_k A_1}(\sigma) - \int_{\sigma < \tau - \mu_k K} \int_{\tau}^{+\infty} \frac{1}{\mu_k} \rho \left(\frac{\lambda - \sigma}{\mu_k} \right) dN_{\mu_k A_1}(\sigma) d\lambda \\
&= \mathcal{B}_{1,j}(\tau) + \mathcal{B}_{2,j}(\tau).
\end{aligned}$$

In the above equation we have used the hypothesis

$$\int_{-\infty}^{+\infty} \frac{1}{\mu_k} \rho \left(\frac{\lambda - \sigma}{\mu_k} \right) d\lambda = \int_{-\infty}^{+\infty} \rho(\lambda) d\lambda = \widehat{\rho}(0) = 1.$$

We can decompose $\mathcal{B}_{1,j}(\tau)$

$$\begin{aligned}
\mathcal{B}_{1,j}(\tau) &= N_{\mu_k A_1}(\tau) - \int_{\tau - \mu_k K \leq \sigma < \tau} dN_{\mu_k A_1}(\sigma) \\
&= N_{\mu_k A_1}(\tau) + \mathcal{B}_{3,j}(\tau).
\end{aligned}$$

The above equation clarifies the presence of $N_A(\tau)$ in the first term of (3.59). We can write

$$\begin{aligned}
\mathcal{A}_j(\tau) &= \int_{\sigma > \tau + \mu_k K} \int_{-\infty}^{\tau} \frac{1}{\mu_k} \rho \left(\frac{\lambda - \sigma}{\mu_k} \right) dN_{\mu_k A_1}(\sigma) d\lambda \\
&= \int_{\mu_k \sigma' > \tau + \mu_k K} \int_{-\infty}^{\tau} \frac{1}{\mu_k} \rho \left(\frac{\lambda}{\mu_k} - \sigma' \right) dN_{A_1}(\sigma') d\lambda \qquad (3.65) \\
&= \int_{\mu_k \sigma' > \tau + \mu_k K} \int_{-\infty}^{\frac{\tau}{\mu_k}} \rho \left(\lambda' - \sigma' \right) dN_{A_1}(\sigma') d\lambda' \qquad (3.66) \\
&= \int_{\sigma' > \frac{\tau}{\mu_k} + K} \int_{-\infty}^{\frac{\tau}{\mu_k}} \rho \left(\lambda' - \sigma' \right) dN_{A_1}(\sigma') d\lambda' = \mathcal{A} \left(\frac{\tau}{\mu_k} \right), \qquad (3.67)
\end{aligned}$$

where, in (3.65), we have used the fact that, for all $S \subseteq \mathbb{R}$,

$$\int_{\theta\in S} f(\theta) dN_{\mu_k A_1}(\theta) = \sum_{\theta\in S\cap\sigma(\mu_k A_1)} f(\theta)$$
$$= \sum_{\mu_k\theta'\in S\wedge\theta'\in\sigma(A_1)} f(\mu_k\,\theta') = \int_{\mu_k\theta'\in S} f(\mu_k\theta') dN_{A_1}(\theta'),$$

while in (3.66) we have used the change of variable $\frac{\lambda}{\mu_k} = \lambda'$. Again from Remark 3.3.9, we have

$$\mathcal{A}(\eta) = \mathcal{O}\left(\eta^{n_1-1}\right), \quad \eta > 1,$$

and in our case $\eta = \frac{\tau}{\mu_k}$, since the sum in (3.61) is over $\mu_k < \tau \Leftrightarrow \frac{\tau}{\mu_k} > 1$. Hence

$$\mathcal{A}_j(\tau) = \mathcal{A}\left(\frac{\tau}{\mu_k}\right) = \mathcal{O}\left(\left(\frac{\tau}{\mu_k}\right)^{n_1-1}\right).$$

We can repeat the same argument for the other terms:

$$\mathcal{B}_{j,2}(\tau) = \int_{\sigma<\tau-\mu_k K}\int_{\tau}^{\infty}\frac{1}{\mu_k}\rho\left(\frac{\lambda-\sigma}{\mu_k}\right) dN_{\mu_k A_1}(\sigma)d\lambda$$
$$= \int_{\mu_k\sigma'<\tau-\mu_k K}\int_{\tau}^{\infty}\frac{1}{\mu_k}\rho\left(\frac{\lambda}{\mu_k}-\sigma'\right) dN_{A_1}(\sigma')d\lambda$$
$$= \int_{\mu_k\sigma'<\tau-\mu_k K}\int_{\frac{\tau}{\mu_k}}^{\infty}\rho\left(\lambda'-\sigma'\right) dN_{A_1}(\sigma')d\lambda'$$
$$= \int_{\sigma'<\frac{\tau}{\mu_k}-K}\int_{\frac{\tau}{\mu_k}}^{\infty}\rho\left(\lambda'-\sigma'\right) dN_{A_1}(\sigma')d\lambda' = \mathcal{B}_2\left(\frac{\tau}{\mu_k}\right);$$

$$\mathcal{B}_{j,3}(\tau) = \int_{\tau-\mu_k K\leq\sigma<\tau} dN_{\mu_k A_1}(\sigma)$$
$$= \int_{\tau-\mu_k K\leq\mu_k\sigma'<\tau} dN_{A_1}(\sigma')$$
$$= \int_{\frac{\tau}{\mu_k}-K\leq\sigma'<\frac{\tau}{\mu_k}} dN_{A_1}(\sigma') = \mathcal{B}_{j,3}\left(\frac{\tau}{\mu_k}\right);$$

$$\begin{aligned}
\mathcal{C}_j(\tau) &= \int_{|\sigma-\tau|\le\mu_k K}\int_{-\infty}^{\tau}\frac{1}{\mu_k}\rho\left(\frac{\lambda-\sigma}{\mu_k}\right)dN_{\mu_k A_1}(\sigma)d\lambda\\
&= \int_{|\mu_k\sigma'-\tau|\le\mu_k K}\int_{-\infty}^{\tau}\frac{1}{\mu_k}\rho\left(\frac{\lambda}{\mu_k}-\sigma'\right)dN_{A_1}(\sigma')d\lambda\\
&= \int_{|\mu_k\sigma'-\tau|\le\mu_k K}\int_{-\infty}^{\frac{\tau}{\mu_k}}\rho\left(\lambda'-\sigma'\right)dN_{A_1}(\sigma')d\lambda'\\
&= \int_{\left|\sigma'-\frac{\tau}{\mu_k}\right|\le K}\int_{-\infty}^{\frac{\tau}{\mu_k}}\rho\left(\lambda'-\sigma'\right)dN_{A_1}(\sigma')d\lambda' = \mathcal{C}\left(\frac{\tau}{\mu_k}\right).
\end{aligned}$$

This proves the claim in view of the properties of $\mathcal{A}(\cdot)$, $\mathcal{B}_2(\cdot)$, $\mathcal{B}_3(\cdot)$ and $\mathcal{C}(\cdot)$ and the fact that $\frac{\tau}{\mu_k} > 1$. □

In view of Proposition 3.3.11 and Lemma 3.3.10, we have:

Proposition 3.3.12. Let A, A_1 and A_2 be as in Assumptions 3.3.2. Then

$$N_A(\tau) = \sum_{\mu_k<\tau}\left(\frac{C_1}{n_1}\left(\frac{\tau}{\mu_k}\right)^{n_1} + \frac{1}{\mu_k^{n_1}}r_j(\tau)\right),$$

where C_1 is given by (3.34), and $r_j(\tau)$ is $\mathcal{O}\left(\tau^{n-1}\right)$ uniformly with respect to μ_k, that is there exists a constant C such that

$$r_j(\tau) \le C\tau^{n_1-1}, \quad \text{for all } j\in\mathbb{N}. \tag{3.68}$$

Proof. By Proposition 3.3.11, we have

$$\begin{aligned}
N_A(\tau) = &\sum_{\mu_k<\tau}\int_{-\infty}^{\tau}\int_{-\infty}^{+\infty}\frac{1}{\mu_k}\rho\left(\frac{\lambda-\sigma}{\mu_k}\right)dN_{\mu_k A_1}d\lambda\\
&-\sum_{\mu_k<\tau}\left(\mathcal{A}\left(\frac{\tau}{\mu_k}\right)+\mathcal{B}_2\left(\frac{\tau}{\mu_k}\right)+\mathcal{B}_3\left(\frac{\tau}{\mu_k}\right)+\mathcal{C}\left(\frac{\tau}{\mu_k}\right)\right).
\end{aligned}$$

Lemma 3.3.10 implies that

$$\begin{aligned}
&\int_{\mu_k}^{\tau}\int_{-\infty}^{+\infty}\frac{1}{\mu_k}\rho\left(\frac{\lambda-\sigma}{\mu_k}\right)dN_{\mu_k A_1}(\sigma)d\lambda\\
&\qquad= \int_{\mu_k}^{\tau}\frac{1}{\mu_k}\left(C_1\left(\frac{\lambda}{\mu_k}\right)^{n_1-1}+\mathcal{O}\left(\frac{\lambda}{\mu_k}\right)^{n_1-2}\right)d\lambda\\
&\qquad= \frac{C_1}{n_1}\left(\frac{\tau}{\mu_k}\right)^{n_1}+\mathcal{O}\left(\left(\frac{\tau}{\mu_k}\right)^{n_1-1}\right),
\end{aligned}$$

since $\frac{\lambda}{\mu_k} > 1$. Moreover, we know that

$$g(\lambda) = \int_{-\infty}^{+\infty}\rho(\lambda-\sigma)\,dN_{A_1}(\sigma) = \mathcal{O}(|\lambda|^{-\infty})$$

for $\lambda \to -\infty$, and that g is bounded on every compact sets, due to the Schwartz decay of ρ and of the polynomial growth of the counting function. Thus,

$$\begin{aligned}
&\int_{-\infty}^{\mu_k}\int_{-\infty}^{+\infty} \frac{1}{\mu_k}\rho\left(\frac{\lambda-\sigma}{\mu_k}\right) dN_{\mu_k A_1}(\sigma)d\lambda \\
&\qquad = \int_{-\infty}^{\mu_k}\int_{-\infty}^{+\infty} \frac{1}{\mu_k}\rho\left(\frac{\lambda}{\mu_k}-\sigma'\right) dN_{A_1}(\sigma')d\lambda \\
&\qquad = \int_{-\infty}^{1}\int_{-\infty}^{+\infty} \rho(\lambda'-\sigma')\, dN_{A_1}(\sigma')\, d\lambda' \\
&\qquad = \int_{-\infty}^{1} (1+|\lambda'|)^{-3}(1+|\lambda'|)^{3}\int_{-\infty}^{+\infty} \rho(\lambda'-\sigma')\, dN_{A_1}(\sigma')\, d\lambda' \\
&\qquad \le \int_{-\infty}^{1} (1+|\lambda'|)^{-3} C'\, d\lambda' \le C
\end{aligned}$$

for some constant $C > 0$ independent from μ_k. Hence, we get the claim from the properties of $\mathcal{A}(\cdot)$, $\mathcal{B}_2(\cdot)$, $\mathcal{B}_3(\cdot)$ and $\mathcal{C}(\cdot)$. □

Chapter 4

Fredholm properties

In this chapter we will show the equivalence of bi-ellipticity and Fredholm property for a class of global bisingular operator.

4.1 Fredholm operators

First, let us recall some general properties of Fredholm operators. For more details, see, e.g., [Shu01].

Definition 4.1.1. Let E_1, E_2 be Banach spaces. Denote by $\mathcal{K}(E_1, E_2) \subset \mathcal{L}(E_1, E_2)$ the subset of all compact linear operators $E_1 \to E_2$.

Definition 4.1.2. Let E_1, E_2 be Banach spaces, and $A \in \mathcal{L}(E_1, E_2)$. A is called a Fredholm operator if $\dim \operatorname{Ker} A < +\infty$ and $\dim \operatorname{Coker} A < +\infty$, where we recall that

$$\operatorname{Coker} A := E_2 / \operatorname{Im} A.$$

The next two results give relations between the class of Fredholm operators and of compact operators.

Proposition 4.1.3. Let E be a Banach space, and $R \in \mathcal{K}(E, E)$. Then $I + R$ is a Fredholm operator from E to itself.

Proof. Since

$$I|_{\operatorname{Ker}(I+R)} = -R|_{\operatorname{Ker}(I+R)} \in \mathcal{K}(E, E),$$

the closed unit ball in $\operatorname{Ker}(I + R)$ is a compact set, because, in $\operatorname{Ker}(I + R)$,

$$\overline{B_1(0)} = \overline{I(B_1(0))} = \overline{-R(B_1(0))},$$

and the image of a bounded set under the action of a compact operator has compact closure. Thus, $\dim \operatorname{Ker}(I + R) < +\infty$. Moreover, R^* is also compact, which means that $\dim \operatorname{Ker}(I + R^*) < +\infty$. To prove that $I + R$

is a Fredholm operator we only need to show the closedness of $\mathrm{Im}(I+R)$, since then

$$\dim \mathrm{Coker}(I+R) = \dim \mathrm{Ker}(I+R^*) < +\infty.$$

Let $\{x_n\} \subset E$, and $y_n := (I+R)x_n \to y \in E$, as $n \to +\infty$. We need to verify the existence of an $x \in E$ such as $(I+R)x = y$.
Let L be any closed subspace complementary to $\mathrm{Ker}(I+R)$ in E. For every $x_n \in E$, we have a unique decomposition $x_n = v_n + v_n'$, where $v_n \in \mathrm{Ker}(I+R)$ and $v_n' \in L$. Thus, adding to $\{x_n\}$ vectors from $\mathrm{Ker}(I+R)$, which obviously does not change y_n, we may assume that $x_n \in L$ for all n.
Let us show that the sequence $\{x_n\}$ is bounded. Indeed, if this is not the case, taking a subsequence of $\{x_n\}$, we may assume that $\|x_n\| \to +\infty$, as $n \to +\infty$. But then, setting

$$x_n' := x_n/\|x_n\|, \quad y_n' := (I+R)x_n' = y_n/\|x_n\|,$$

we have that $y_n' \to 0$, as $n \to +\infty$, since $\{y_n\}$ is bounded. By construction, $\|x_n'\| = 1$ for all n, so that we have, up to a subsequence, the existence of

$$\lim_{n\to+\infty} Rx_n' = x'.$$

But then

$$\lim_{n\to+\infty} x_n' = \lim_{n\to+\infty} (-Rx_n') = -x',$$

and clearly $\|x'\| = 1$, $x' \in L$. Moreover

$$(I+R)x' = \lim_{n\to+\infty} (I+R)x_n' = -x' + x' = 0,$$

and this contradicts the choice of L. Thus, the sequence $\{x_n\}$ is bounded, and we have, up to a subsequence, that

$$\lim_{n\to+\infty} Rx_n$$

exists, and, consequently, so does

$$\lim_{n\to+\infty} x_n = y - \lim_{n\to+\infty} Rx_n.$$

Denoting $x := \lim_{n\to+\infty} x_n$, we have that $(I+R)x = y$, proving the closedness of $\mathrm{Im}(I+R)$. □

Theorem 4.1.4. Let $A \in \mathcal{L}(E_1, E_2)$ and let $B_1, B_2 \in \mathcal{L}(E_2, E_1)$ be such that

$$B_1 A = I + R_1, \qquad AB_2 = I + R_2,$$

where $R_j \in \mathcal{K}(E_j, E_j)$, $j = 1, 2$. Then, A is a Fredholm operator from E_1 to E_2.

Proof. By Proposition 4.1.3 one has that $I + R_j$, is a Fredholm operator from E_j to itself, $j = 1, 2$. Then

$$\operatorname{Ker} A \subset \operatorname{Ker}(B_1 A) = \operatorname{Ker}(I + R_1),$$

thus $\dim \operatorname{Ker} A \leq \dim \operatorname{Ker}(I + R_1) < +\infty$. Moreover, we have that

$$\operatorname{Im} A \supset \operatorname{Im}(AB_2) = \operatorname{Im}(I + R_2).$$

Given $x \in E_2$, let us denote by $x + \operatorname{Im} A$ the lateral class of x in $\operatorname{Coker} A = E_2/\operatorname{Im} A$. Since $\operatorname{Im}(I + R_2) \subset \operatorname{Im} A$, one has a homomorphism $f : E_2/\operatorname{Im}(I + R_2) \to E_2/\operatorname{Im} A$ given by $x + \operatorname{Im}(I + R_2) \mapsto x + \operatorname{Im} A$. Notice that f is well defined: taking $x + v_1, x + v_2 \in x + \operatorname{Im}(I + R_2)$, $v_1, v_2 \in \operatorname{Im}(I + R_2)$, we have that $f(x + v_1), f(x + v_2) \in x + \operatorname{Im} A$, since

$$\begin{aligned} f(x + v_1) - f(x + v_2) &= x + v_1 - (x + v_2) \\ &= v_1 - v_2 \in \operatorname{Im}(I + R_2) \subset \operatorname{Im} A \Rightarrow v_1 - v_2 \in \operatorname{Im} A. \end{aligned}$$

Next, let us show that f is surjective. Given $y + \operatorname{Im} A$, we have to find $x + \operatorname{Im}(I + R_2)$ such that $f(x + \operatorname{Im}(I + R_2)) = y + \operatorname{Im} A$. Choose an element $v \in y + \operatorname{Im} A$ and set $x + \operatorname{Im}(I + R_2) = v + \operatorname{Im}(I + R_2)$. By definition of f,

$$f(x + \operatorname{Im}(I + R_2)) = v + \operatorname{Im} A = y + \operatorname{Im} A,$$

as desired. Of course, choosing $v, v' \in y + \operatorname{Im} A$, $v' \neq v$, one obtains, in general, different classes $v' + \operatorname{Im}(I + R_2), v + \operatorname{Im}(I + R_2)$, in $\operatorname{Coker}(I + R_2)$. However, again by definition of f,

$$f(v' + \operatorname{Im}(I + R_2)) = v' + \operatorname{Im} A = v + \operatorname{Im} A = y + \operatorname{Im} A.$$

Finally, by assumption, $\operatorname{Coker}(I+R_2) = E_2/\operatorname{Im}(I+R_2)$ is finite dimensional, and, by the surjectivity of $f : \operatorname{Coker}(I + R_2) \to \operatorname{Coker}(A)$, this holds also for $\operatorname{Coker} A = E_2/\operatorname{Im} A$. □

4.2 Global bisingular operators

In this section we present a class of global bisingular operators whose bi-symbol are adapted to Shubin calculus, see [Shu01]. This calculus was recently studied by U. Battisti, T. Gramchev, S. Pilipović and L. Rodino (for additional details and the proofs, see [BGPR13]).

Definition 4.2.1. $\Gamma^{m_1,m_2}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$ is the set of all functions in $\mathcal{C}^\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ such that, for all multi-indices α_i, β_i, $i = 1, 2$, there exists a constant $C_{\alpha_1,\alpha_2,\beta_1,\beta_2} > 0$ such that

$$\begin{aligned} &|D^{\alpha_1}_{\xi_1} D^{\alpha_2}_{\xi_2} D^{\beta_1}_{x_1} D^{\beta_2}_{x_2} a(x_1, x_2, \xi_1, \xi_2)| \\ &\qquad \leq C_{\alpha_1,\alpha_2,\beta_1,\beta_2} \langle (x_1, \xi_1) \rangle^{m_1 - |\alpha_1| - |\beta_1|} \langle (x_1, \xi_2) \rangle^{m_2 - |\alpha_2| - |\beta_2|}, \end{aligned}$$

for all $x_i, \xi_i \in \mathbb{R}^{n_i}$. An element of $\Gamma^{m_1,m_2}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$ is called a (global) bi-symbol.

Definition 4.2.2. A linear operator $A : \mathcal{C}_0^\infty(\mathbb{R}^{n_1+n_2}) \to \mathcal{C}^\infty(\mathbb{R}^{n_1+n_2})$ is called a (global) bisingular operator if it can be written in the form

$$\begin{aligned} A(u)(x_1, x_2) &= (\mathrm{Op}(a)[u])(x_1, x_2) \\ &= \frac{1}{(2\pi)^{n_1+n_2}} \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} e^{i(x_1\cdot\xi_1 + x_2\cdot\xi_2)}\, a(x_1, x_2, \xi_1, \xi_2)\hat{u}(\xi_1, \xi_2)\, d\xi_1 d\xi_2, \end{aligned}$$

where $a \in \Gamma^{m_1,m_2}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$. $\mathrm{G}^{m_1,m_2}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$ denotes the set of all bisingular operators with bi-symbol in $\Gamma^{m_1,m_2}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$. Moreover, we set

$$\begin{aligned} \Gamma^{\infty,\infty}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2}) &:= \bigcup_{m_1,m_2} \Gamma^{m_1,m_2}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2}), \\ \Gamma^{-\infty,-\infty}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2}) &:= \bigcap_{m_1,m_2} \Gamma^{m_1,m_2}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2}), \end{aligned}$$

and we denote by $\mathrm{G}^{\infty,\infty}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$, $\mathrm{G}^{-\infty,-\infty}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$, respectively, the corresponding class of operators. The operators in $\mathrm{G}^{-\infty,-\infty}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$ are called *smoothing operators*.

For the operators in $\Gamma^{\infty,\infty}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$ we have $\mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ and $\mathscr{S}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ continuity, and

$$\Gamma^{-\infty,-\infty}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2}) = \mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}).$$

We associate with every $A \in \Gamma^{m_1,m_2}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$ the two maps

$$\begin{aligned} A^1 &: \mathbb{R}^{n_1} \times \mathbb{R}^{n_1} \to \mathrm{G}^{m_2}(\mathbb{R}^{n_2}) : (x_1, \xi_1) \mapsto a(x_1, x_2, \xi_1, D_2), \\ A^2 &: \mathbb{R}^{n_2} \times \mathbb{R}^{n_2} \to \mathrm{G}^{m_1}(\mathbb{R}^{n_1}) : (x_2, \xi_2) \mapsto a(x_1, x_2, D_1, \xi_2). \end{aligned} \tag{4.1}$$

Remark 4.2.3. Also the global version of bisingular calculus in Definition 4.2.2 takes the form of an operator-valued calculus, cfr. Chapter 1.

Theorem 4.2.4. Let $A \in \Gamma^{m_1,m_2}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$ and $B \in \Gamma^{p_1,p_2}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$. Then, $AB \in \Gamma^{m_1+p_1,m_2+p_2}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$.

Definition 4.2.5. Let $a \in \Gamma^{m_1,m_2}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$. Then a has a homogeneous principal bi-symbol if

i) there exists $a_{m_1;\cdot} \in \Gamma^{m_1,m_2}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$ such that

$$\begin{aligned} a_{m_1;\cdot}(tx_1, x_2, t\xi_1, \xi_2) &= t^{m_1} a_{m_1;\cdot}(x_1, x_2, \xi_1, \xi_2), \\ a - \psi_1(x_1, \xi_1) a_{m_1;\cdot} &\in \Gamma^{m_1-1,m_2}, \end{aligned}$$

for all x_2, ξ_2, for all x_1, ξ_1 such that $|x_1|+|\xi_1| > 1$, and all $t > 0$, where ψ_1 is an 0-excision function. Moreover, $a_{m_1;\cdot}(x_1, x_2, \xi_1, D_2) \in \mathrm{G}^{m_2}_{\mathrm{cl}}(\mathbb{R}^{n_2})$, so, being a classical global symbol on $\mathbb{R}^{n_2}$, it admits an asymptotic expansion with respect to (x_2, ξ_2).

ii) there exists $a_{\cdot;m_2} \in \Gamma^{m_1,m_2}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$ such that

$$\begin{aligned} a_{\cdot;m_2}(x_1, tx_2, \xi_1, t\xi_2) &= t^{m_2} a_{\cdot;m_2}(x_1, x_2, \xi_1, \xi_2), \\ a - \psi_2(x_2, \xi_2) a_{\cdot;m_2} &\in \Gamma^{m_1,m_2-1}, \end{aligned}$$

for all x_1, ξ_1, for all x_2, ξ_2 such that $|x_2|+|\xi_2| > 1$, and all $t > 0$, where ψ_2 is an 0-excision function. Moreover, $a_{\cdot;m_2}(x_1, x_2, D_1, \xi_2) \in \mathrm{G}^{m_1}_{\mathrm{cl}}(\mathbb{R}^{n_1})$, so, being a classical global symbol on $\mathbb{R}^{n_1}$, it admits an asymptotic expansion with respect to (x_1, ξ_1).

iii) the symbols $a_{m_1;\cdot}$ and $a_{\cdot;m_2}$ have the same leading term, so there exists $a_{m_1;m_2}$ such that

$$\begin{aligned} a_{m_1;\cdot} - \psi_2(x_2, \xi_2) a_{m_1;m_2} &\in \Gamma^{m_1,m_2-1}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2}), \\ a_{\cdot;m_2} - \psi_1(x_1, \xi_1) a_{m_1;m_2} &\in \Gamma^{m_1-1,m_2}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2}), \end{aligned}$$

and

$$a - \psi_1 a_{m_1;\cdot} - \psi_2 a_{\cdot;m_2} + \psi_1 \psi_2 a_{m_1;m_2} \in \Gamma^{m_1-1,m_2-1}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2}).$$

The bi-symbols which admit a full bi-homogeneous expansion in (x_1, ξ_1) and (x_2, ξ_2) are called *classical bi-symbols*, their class is denoted by $\Gamma^{m_1,m_2}_{\mathrm{cl}}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$, and the corresponding operator class by $\mathrm{G}^{m_1,m_2}_{\mathrm{cl}}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$.
Given $A \in \mathrm{G}^{m_1,m_2}_{\mathrm{cl}}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$, we can define maps $\sigma^1(A)$, $\sigma^2(A)$, $\sigma^{12}(A)$ as follows:

$$\begin{aligned} \sigma^1(A) : & T^*\mathbb{R}^{n_1} \setminus 0 \to \mathrm{G}^{m_2}_{\mathrm{cl}}(\mathbb{R}^{n_2}) \\ & (x_1, \xi_1) \mapsto a_{m_1;\cdot}(x_1, x_2, \xi_1, D_2), \\ \sigma^2(A) : & T^*\mathbb{R}^{n_2} \setminus 0 \to \mathrm{G}^{m_1}_{\mathrm{cl}}(\mathbb{R}^{n_1}) \\ & (x_2, \xi_2) \mapsto a_{\cdot;m_2}(x_1, x_2, D_1, \xi_2), \\ \sigma^{12}(A) : & (T^*\mathbb{R}^{n_1} \setminus 0) \times (T^*\mathbb{R}^{n_2} \setminus 0) \to \mathbb{C} \\ & (x_1, x_2, \xi_1, \xi_2) \mapsto a_{m_1;m_2}(x_1, x_2, \xi_1, \xi_2). \end{aligned}$$

In this way, denoting by $\sigma_P(x, \xi)$ the principal symbol of an operator P, we have

$$\begin{aligned} \sigma_{\sigma^1(A)(x_1,\xi_1)}(x_2, \xi_2) &= \sigma_{\sigma^2(A)(x_2,\xi_2)}(x_1, \xi_1) \\ &= \sigma^{12}(A)(x_1, x_2, \xi_1, \xi_2) = a_{m_1;m_2}(x_1, x_2, \xi_1, \xi_2). \end{aligned} \tag{4.2}$$

We call the couple $(\sigma^1(A), \sigma^2(A))$ the *principal bi-symbol* of A.

We can now introduce the notion of bi-ellipticity. As in the case of the bisingular calculus on closed manifolds, we restrict ourselves to classical operators.

Definition 4.2.6. Let $A \in \mathrm{G}^{m_1,m_2}_{\mathrm{cl}}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$. We say that A is bi-elliptic if

i) $\sigma^{12}(A)(v_1, v_2) \neq 0$ for all $(T^*\mathbb{R}^{n_1} \setminus 0) \times (T^*\mathbb{R}^{n_2} \setminus 0)$;

ii) $\sigma^1(A)(v_1)$ is invertible as an operator in $\mathrm{G}^{m_2}_{\mathrm{cl}}(\mathbb{R}^{n_2})$ for all $v_1 \in T^*\mathbb{R}^{n_1} \setminus 0$,

iii) $\sigma^2(A)(v_2)$ is invertible as an operator in $\mathrm{G}^{m_1}_{\mathrm{cl}}(\mathbb{R}^{n_1})$ for all $v_2 \in T^*\mathbb{R}^{n_2} \setminus 0$.

Also for this global calculus, it is possible to define adapted Sobolev spaces and prove some continuity results.

Definition 4.2.7. The Sobolev space of exponent (s_1, s_2), $s_1, s_2 \in \mathbb{R}$, is the space

$$\begin{aligned} &\mathrm{Q}^{s_1,s_2} = \mathrm{Q}^{s_1,s_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \\ &:= \{u \in \mathscr{S}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) : \mathrm{Op}(\langle (x_1, \xi_1)\rangle^{s_1} \langle (x_2, \xi_2)\rangle^{s_2})u \in L^2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})\}. \end{aligned}$$

In particular,

$$\mathrm{Q}^{0,0}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) = L^2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}).$$

For $u \in \mathrm{Q}^{s_1,s_2}$ set

$$\|u\|_{s_1,s_2} := \|\mathrm{Op}(\langle (x_1, \xi_1)\rangle^{s_1} \langle (x_2, \xi_2)\rangle^{s_2})u\|_2.$$

Remark 4.2.8. We notice that, if $A \in \mathrm{G}^{m_1,m_2}_{\mathrm{cl}}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$ satisfies condition ii) and condition iii) of Definition 4.2.6, one can prove that both the operators $\sigma^1(A)(x_1, \xi_1) \in \mathrm{G}^{m_2}_{\mathrm{cl}}(\mathbb{R}^{n_2})$ and $\sigma^2(A)(x_2, \xi_2) \in \mathrm{G}^{m_1}_{\mathrm{cl}}(\mathbb{R}^{n_1})$ are injective Fredholm operators, therefore invertible operators also in the Q^s spaces[1], for all $s \in \mathbb{R}$. Therefore, in Definition 4.2.6, it is equivalent to require invertibility of the operators on the Schwartz spaces or on the Sobolev Q^s spaces.

Proposition 4.2.9. If $s_1 \geq s_1'$, $s_2 \geq s_2'$, we have

$$\mathrm{Q}^{s_1,s_2} \subset \mathrm{Q}^{s_1',s_2'}$$

and the embedding $\mathrm{Q}^{s_1,s_2} \hookrightarrow \mathrm{Q}^{s_1',s_2'}$ is continuous. If $s_1 > s_1'$, $s_2 > s_2'$, the embedding is compact.

Proposition 4.2.10. A bisingular operator $A \in \mathrm{G}^{m_1,m_2}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$ extends to a continuous operator

$$A : \mathrm{Q}^{s_1,s_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \to \mathrm{Q}^{s_1-m_1,s_2-m_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}),$$

for every $s_1, s_2 \in \mathbb{R}$.

[1]The Q^s spaces are the scale of Sobolev spaces adapted to the global Shubin calculus. For more details, see [Shu01].

We conclude this section with the analogue of Theorem 1.2.12:

Theorem 4.2.11. Let $A \in \mathrm{G}^{m_1,m_2}_{\mathrm{cl}}(\mathbb{R}^{n_1},\mathbb{R}^{n_2})$ be bi-elliptic. Then, there exists $B \in \mathrm{L}^{-m_1,-m_2}_{\mathrm{cl}}(\mathbb{R}^{n_1},\mathbb{R}^{n_2})$ such that

$$AB = I + K_1 \qquad BA = I + K_2,$$

where I is the identity map and K_1, K_2 are smoothing operators. Moreover, the principal bi-symbol of B is $(\sigma^1(A)^{-1}, \sigma^2(A)^{-1})$.

4.3 Fredholm property for global bi-elliptic operators

Let $A \in \mathrm{G}^{m_1,m_2}_{\mathrm{cl}}(\mathbb{R}^{n_1},\mathbb{R}^{n_2})$. In this section we will prove the main result of this chapter, namely, that A is bi-elliptic if and only if A is a Fredholm operator.

First, let us see that we can, without loss of generality, restrict ourselves to operators $\tilde{A} \in \mathrm{G}^{0,0}_{\mathrm{cl}}(\mathbb{R}^{n_1},\mathbb{R}^{n_2})$. To do so, consider the so-called order reduction operators, that is, operators

$$\Lambda_{m_1,m_2} \in \mathrm{G}^{m_1,m_2}_{\mathrm{cl}}(\mathbb{R}^{n_1},\mathbb{R}^{n_2}),$$

invertible, with inverse $(\Lambda_{m_1,m_2})^{-1} = \Lambda_{-m_1,-m_2}$. These operators are an isometry $\mathrm{Q}^{m_1,m_2} \to \mathrm{Q}^{0,0} = L^2$. For the construction of these operators in the Shubin calculus, see, e.g., [NR11]. Thus, from Proposition 4.2.10 and Theorem 4.2.4, the study of the operator

$$A : \mathrm{Q}^{s_1,s_2} \to \mathrm{Q}^{s_1-m_1,s_2-m_2}$$

is equivalent to the study of

$$\tilde{A} : \mathrm{Q}^{0,0} \to \mathrm{Q}^{0,0},$$

where

$$\tilde{A} := \Lambda_{s_1-m_1,s_2-m_2} A \Lambda_{-s_1,-s_2} \in \mathrm{G}^{0,0}_{\mathrm{cl}}(\mathbb{R}^{n_1},\mathbb{R}^{n_2}).$$

In fact, the following diagram is commutative.

$$\begin{array}{ccc}
\mathrm{Q}^{s_1,s_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}) & \xrightarrow{A} & \mathrm{Q}^{s_1-m_1,s_2-m_2}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}) \\
{\scriptstyle (\Lambda_{-s_1,-s_2})^{-1}=\Lambda_{s_1,s_2}}\downarrow & & \downarrow{\scriptstyle \Lambda_{s_1-m_1,s_2-m_2}} \\
\mathrm{Q}^{0,0}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}) & \xrightarrow{\tilde{A}} & \mathrm{Q}^{0,0}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})
\end{array}$$

4.3.1 Sufficiency of bi-elliptic

In this subsection we will prove that bi-ellipticity is a sufficient condition for a global bisingular operator to have the Fredholm property.

Proposition 4.3.1. Given $m_1, m_2 > 0$, $A \in \mathrm{G}_{\mathrm{cl}}^{-m_1,-m_2}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$ is a compact operator from H^{s_1,s_2} to itself, for every $s_1, s_2 \in \mathbb{R}$.

Proof. Since $m_1, m_2 > 0$, for every $(s_1, s_2) \in \mathbb{R}^2$ we obviously have $s_1 < s_1 + m_1$, $s_2 < s_2 + m_2$, and, by Proposition 4.2.9, this means that $\mathrm{Q}^{s_1+m_1,s_2+m_2}$ is compactly embedded into Q^{s_1,s_2}. Thus, by Proposition 4.2.4,

$$A : \mathrm{Q}^{s_1,s_2} \to \mathrm{Q}^{s_1-(-m_1),s_2-(-m_2)} = \mathrm{Q}^{s_1+m_2,s_2+m_2} \subset \mathrm{Q}^{s_1,s_2},$$

and the embedding is compact. □

Theorem 4.3.2. Let $A \in \mathrm{G}_{\mathrm{cl}}^{0,0}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$, and let A be bi-elliptic. Then, A is a Fredholm operator from $L^2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) = \mathrm{Q}^{0,0}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ to itself.

Proof. From Theorem 4.2.11, we know that there exists $B \in \mathrm{L}_{\mathrm{cl}}^{0,0}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$ such that

$$AB = I + K_1 \qquad BA = I + K_2,$$

where K_1, K_2 are smoothing operators, that is, operators in $\mathrm{L}^{-\infty,-\infty}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$. Thus, from Proposition 4.3.1, K_1 and K_2 are compact operators from $L^2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ to itself. In view of the abstract result from Section 4.1, this proves our claim. □

4.3.2 Necessity of bi-elliptic

In this subsection we prove that bi-ellipticity is a necessary condition for a global bisingular operator in order to have the Fredholm property. To this aim, we show a result in the more general class of operators with operator-valued symbols in a Hilbert space. Theories of operator-valued symbols were originally developed by Schulze, see, e.g., [Sch98]. The following definitions are variants of his original concepts, adapted to the presently used Shubin-type calculus. They are defined as follows:

Definition 4.3.3. Let E be a Hilbert space. $\Gamma^\mu(\mathbb{R}^n, \mathscr{L}(E))$ is the set of all $a \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n, \mathscr{L}(E))$ such that, for all multi-indices α, β there exists a constant $C_{\alpha,\beta} > 0$ such that

$$\|D_\xi^\alpha D_x^\beta a(x,\xi)\|_{\mathscr{L}(E)} \le C_{\alpha,\beta} \langle (x,\xi) \rangle^{\mu-|\alpha|-|\beta|},$$

for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$. An element of $\Gamma^\mu(\mathbb{R}^n, \mathscr{L}(E))$ is called an operator-valued symbol. The corresponding pseudodifferential operator $A : \mathcal{C}_0^\infty(\mathbb{R}^n, E) \to \mathcal{C}^\infty(\mathbb{R}^n, E)$ is defined as

$$Au(x) := (\mathrm{Op}(a)[u])(x) := \int e^{ix\xi} a(x,\xi) \hat{u}(\xi)\, đ\xi, \quad u \in \mathcal{C}_0^\infty(\mathbb{R}^n, E),$$

and the corresponding class by $\mathrm{G}^{\mu}(\mathbb{R}^n, \mathscr{L}(E))$.

For this class of "abstract" symbols all the standard features of pseudodifferential calculus work essentially as in the case of scalar symbols. In particular, it is possible to define classical symbols, that is, the elements of $\Gamma^{\mu}(\mathbb{R}^n, \mathscr{L}(E))$ which admit a homogeneous expansion in (x, ξ), and their principal symbols σ_A. For more details and properties of operator-valued symbols, we refer to [Sch98] and [Tay74]. From now on, we will assume that all operator-valued symbols are classical.

Remark 4.3.4. Given $A \in \mathrm{G}^{0,0}_{\mathrm{cl}}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$, we have

$$\mathrm{Op}(A^1) \in \mathrm{G}^0(\mathbb{R}^{n_1}, \mathscr{L}(L^2(\mathbb{R}^{n_2}))), \quad \mathrm{Op}(A^2) \in \mathrm{G}^0(\mathbb{R}^{n_2}, \mathscr{L}(L^2(\mathbb{R}^{n_1}))),$$

with A^1, A^2 as in (4.1). Moreover,

$$\sigma_{\mathrm{Op}(A^1)}(x_1, \xi_1) = \sigma^1(A)(x_1, \xi_1), \quad \sigma_{\mathrm{Op}(A^2)}(x_2, \xi_2) = \sigma^2(A)(x_2, \xi_2).$$

Definition 4.3.5. Let $b \in \Gamma^{\mu}(\mathbb{R}^n, \mathscr{L}(E))$, $\lambda > 1$, $0 < \tau < 1/2$ and $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n$, $|(x_0, \xi_0)| = 1$, be fixed. We set

$$b_\lambda(x, \xi) := b(\lambda x_0 + \lambda^{-\tau} x, \lambda \xi_0 + \lambda^{\tau} \xi). \tag{4.3}$$

Lemma 4.3.6. Let $b \in \Gamma^{\mu}(\mathbb{R}^n, \mathscr{L}(E))$, b_λ as in (4.3), $\mu \leq 0$, $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n$, $|(x_0, \xi_0)| = 1$. Set

$$\rho := \frac{\tau}{1 - \tau} \tag{4.4}$$

(notice that $0 < \rho < 1$, since $0 < \tau < 1/2$). Then, for any $\lambda \geq 1$, and any $\alpha, \beta \in \mathbb{Z}^n_+$, we have

$$\|D^{\alpha}_{\xi} D^{\beta}_{x} b_\lambda(x, \xi)\|_{\mathscr{L}(E)} \leq C_{\alpha,\beta} \lambda^{(1-\tau)\mu - |\beta|\tau} \langle (x, \xi) \rangle^{\rho|\alpha| - \mu}, \tag{4.5}$$

uniformly in $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$.

Proof. By direct computation, we get

$$\begin{aligned}
&\|D^{\alpha}_{\xi} D^{\beta}_{x} b(\lambda x_0 + \lambda^{-\tau} x, \lambda \xi_0 + \lambda^{\tau} \xi)\|_{\mathscr{L}(E)} \\
&\qquad \leq C_{\alpha,\beta} \lambda^{|\alpha|\tau - |\beta|\tau} \langle (\lambda x_0 + \lambda^{-\tau} x, \lambda \xi_0 + \lambda^{\tau} \xi) \rangle^{\mu - |\alpha| - |\beta|} \\
&\qquad \prec \lambda^{|\alpha|\tau - |\beta|\tau} \langle (\lambda x_0 + \lambda^{-\tau} x, \lambda \xi_0 + \lambda^{\tau} \xi) \rangle^{\mu - \rho|\alpha|} \\
&\qquad \prec \lambda^{|\alpha|\tau - |\beta|\tau} \langle (\lambda x_0, \lambda \xi_0) \rangle^{\mu - \rho|\alpha|} \langle (\lambda^{-\tau} x, \lambda^{\tau} \xi) \rangle^{\rho|\alpha| - \mu} \\
&\qquad \prec \lambda^{|\alpha|\tau - |\beta|\tau} \lambda^{\mu - \rho|\alpha|} \lambda^{\rho\tau|\alpha| - \tau\mu} \langle (x, \xi) \rangle^{\rho|\alpha| - \mu} \\
&\qquad \prec \lambda^{(1-\tau)\mu - |\beta|\tau} \lambda^{|\alpha|(\tau - \rho + \rho\tau)} \langle (x, \xi) \rangle^{\rho|\alpha| - \mu}
\end{aligned}$$

where we used Peetre's inequality (1.1.1) and that $\rho|\alpha| - \mu \geq 0$. From (4.4),

$$\tau - \rho + \rho\tau = \tau - \frac{\tau}{1 - \tau} + \frac{\tau^2}{1 - \tau} = 0.$$

The proof is complete. □

We can now prove the following Proposition:

Proposition 4.3.7. Let $(x_0, \xi_0) \in \mathbb{R}^n \times \mathbb{R}^n$, $|(x_0, \xi_0)| = 1$, and $0 < \tau < 1/2$ be fixed. Set

$$R_\lambda u(x) = \lambda^{\tau n/2} \, e^{i\lambda x \xi_0} \, u(\lambda^\tau (x - \lambda x_0)), \quad u \in \mathcal{C}_0^\infty(\mathbb{R}^n, E).$$

Then

i) R_λ is an invertible isometry $L^2(\mathbb{R}^n, E) \to L^2(\mathbb{R}^n, E)$;

ii) $R_\lambda u \to 0$ weakly in $L^2(\mathbb{R}^n, E)$ for $\lambda \to +\infty$;

iii) For any $A \in \mathrm{G}^0(\mathbb{R}^n, \mathscr{L}(E))$ and $u \in L^2(\mathbb{R}^n, E)$ we have

$$\|R_\lambda^{-1} A R_\lambda u - \sigma_A(x_0, \xi_0) u\|_{L^2(\mathbb{R}^n, E)} \to 0.$$

Proof. In this proof we follow the ideas in [RS82, §2.3.4].

i) Obviously, we have $\|R_\lambda u\|_{L^2(\mathbb{R}^n,E)} = \|u\|_{L^2(\mathbb{R}^n,E)}$ for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^n, E)$. We get the claim by density.

ii) Pick $u, v \in \mathcal{C}_0^\infty(\mathbb{R}^n, E)$, and notice that the inverse of R_λ in given by

$$R_\lambda^{-1} u(x) = \lambda^{-\tau n/2} e^{-i\lambda(\lambda^{-\tau} x + \lambda x_0)\xi_0} u(\lambda^{-\tau} x + \lambda x_0).$$

Hence,

$$\begin{aligned}
|(R_\lambda u, v)_{L^2(\mathbb{R}^n,E)}| &\le \lambda^{\tau n/2} \int |(u(\lambda^\tau (x - \lambda x_0)), v(x))_E| \, dx \\
&\le \lambda^{-\tau n/2} \int |(u(x), v(\lambda^{-\tau} x + \lambda x_0))_E| \, dx \\
&\le \lambda^{-\tau n/2} \|u\|_{L^1(\mathbb{R}^n,E)} \|v\|_{L^\infty(\mathbb{R}^n,E)} \to 0
\end{aligned}$$

for $\lambda \to +\infty$. Moreover, observe that

$$|(R_\lambda u, v)| \le \|R_\lambda u\|_{L^2(\mathbb{R}^n,E)} \, \|v\|_{L^2(\mathbb{R}^n,E)} = \|u\|_{L^2(\mathbb{R}^n,E)} \, \|v\|_{L^2(\mathbb{R}^n,E)}.$$

Fix $u, v \in L^2(\mathbb{R}^n, E)$, choose $\{u_j\}, \{v_j\} \subset \mathcal{C}_0^\infty(\mathbb{R}^n, E)$ such that $u_j \to u, v_j \to v$ in $L^2(\mathbb{R}^n, E)$ for $j \to +\infty$. Then,

$$\begin{aligned}
|(R_\lambda u, v)| &\le |(R_\lambda (u - u_j), v)| + |(R_\lambda u_j, v - v_j))| + |(R_\lambda u_j, v_j)| \\
&\le \|u - u_j\|_{L^2} \, \|v\|_{L^2} + \|u_j\|_{L^2} \, \|v - v_j\|_{L^2} + |(R_\lambda u_j, v_j)|.
\end{aligned}$$

Thus, for every $\varepsilon > 0$ we can choose $N > 0$ such that, for all $j > N$,

$$\|u - u_j\|_{L^2} \, \|v\|_{L^2} + \|u_j\|_{L^2} \, \|v - v_j\|_{L^2} \le \frac{\varepsilon}{2},$$

and $\lambda >> 1$ such that
$$|(R_\lambda u_j, v_j)| \leq \frac{\varepsilon}{2},$$
since $\|u_j\|, \|v_j\|$ are (uniformly) bounded with respect to j. This implies
$$(R_\lambda u, v)_{L^2(\mathbb{R}^n,E)} \to 0 \quad \text{for} \quad \lambda \to +\infty,$$
for all $v \in L^2(\mathbb{R}^n, E)$, that is, $R_\lambda u \to 0$ weakly in $L^2(\mathbb{R}^n, E)$.

iii) Let $A = \mathrm{Op}(a)$ with $a \in \Gamma^0(\mathbb{R}^n, \mathscr{L}(E))$, and choose an excision function $\chi(x, \xi)$ such that $a(x, \xi) = \chi(x, \xi)\sigma_A(x, \xi) + r(x, \xi)$, where $r \in \Gamma^{-1}(\mathbb{R}^n, \mathscr{L}(E))$. From
$$\widehat{R_\lambda u}(\xi) = \lambda^{-\tau n/2} e^{-i\lambda x_0(\xi - \lambda\xi_0)} \hat{u}((\xi - \lambda\xi_0)\lambda^{-\tau}),$$
$u \in \mathcal{C}_0^\infty(\mathbb{R}^n, E)$, we get, by direct computation,
$$\begin{aligned}
&R_\lambda^{-1}\mathrm{Op}(a)R_\lambda u(x)\\
&= \int e^{ix\xi}\big(\chi(\lambda x_0 + \lambda^{-\tau}x, \lambda^\tau\xi + \lambda\xi_0)\sigma_A(\lambda x_0 + \lambda^{-\tau}x, \lambda^\tau\xi + \lambda\xi_0)\\
&\qquad + r(\lambda x_0 + \lambda^{-\tau}x, \lambda^\tau\xi + \lambda\xi_0)\big)\hat{u}(\xi)\, đ\xi\\
&= \int e^{ix\xi}\big(\chi(\lambda x_0 + \lambda^{-\tau}x, \lambda^\tau\xi + \lambda\xi_0)\sigma_A(x_0 + \lambda^{-\tau-1}x, \xi_0 + \lambda^{\tau-1}\xi)\\
&\qquad + r(\lambda x_0 + \lambda^{-\tau}x, \lambda^\tau\xi + \lambda\xi_0)\big)\hat{u}(\xi)\, đ\xi\\
&= \int e^{ix\xi}\big(a'_\lambda(x,\xi) + r_\lambda(x,\xi)\big)\hat{u}(\xi)\, đ\xi\\
&= \big(\mathrm{Op}(a'_\lambda) + \mathrm{Op}(r_\lambda)\big)u(x),
\end{aligned}$$
with
$$a'_\lambda(x,\xi) = \chi(\lambda x_0 + \lambda^{-\tau}x, \lambda^\tau\xi + \lambda\xi_0)\sigma_A(x_0 + \lambda^{-\tau-1}x, \xi_0 + \lambda^{\tau-1}\xi),$$
and $r_\lambda \in \Gamma^{-1}(\mathbb{R}^n, \mathscr{L}(E))$ as in (4.3). First, we want to show that, for $\lambda \to +\infty$,
$$\mathrm{Op}(r_\lambda)u \to 0 \quad \text{in } L^2(\mathbb{R}^n, E). \tag{4.6}$$
Due to Lebesgue's Theorem on dominated convergence, this is true provided we verify

a) $\mathrm{Op}(r_\lambda)u(x) \to 0$ in E pointwise for $x \in \mathbb{R}^n$;
b) There exists $g \in L^1(\mathbb{R}^n)$ such that $\|\mathrm{Op}(r_\lambda)u(x)\|_E \prec g(x)$ for all $x \in \mathbb{R}^n$ and $\lambda \geq 1$.

Using (4.5) with $\mu = -1$, for fixed x we get
$$\begin{aligned}
\|r_\lambda(x,\xi)\hat{u}(\xi)\|_E &\leq \|r_\lambda(x,\xi)\|_{\mathscr{L}(E)}\|\hat{u}(\xi)\|_E\\
&\prec \lambda^{\tau-1}\langle(x,\xi)\rangle\|\hat{u}(\xi)\|_E \to 0
\end{aligned}$$

for $\lambda \to +\infty$ for every $\xi \in \mathbb{R}$, because $\tau - 1 < 0$. Since $u \in \mathcal{C}_0^\infty(\mathbb{R}^n, E)$, for any $N \in \mathbb{N}$ we have $\|\hat{u}(\xi)\|_E \prec \langle \xi \rangle^{-N}$, thus

$$\|r_\lambda(x,\xi)\hat{u}(\xi)\|_E \prec \lambda^{\tau-1}\langle (x,\xi)\rangle \|\hat{u}(\xi)\|_E \prec \langle \xi \rangle^{1-N},$$

which is in $L^1(\mathbb{R}^n_\xi, E)$ for $N > 1 + n$. This, by Lebesgue's Theorem on dominated convergence, implies a).
Now, for fixed $M \in \mathbb{N}$, consider $\langle x \rangle^{2M} \mathrm{Op}(r_\lambda)u(x)$. By an integration by parts argument, we get

$$\begin{aligned}\langle x \rangle^{2M} \mathrm{Op}(r_\lambda)u(x) &= \langle x \rangle^{2M} \int e^{ix\xi} r_\lambda(x,\xi)\hat{u}(\xi)\, đ\xi \\ &= \int e^{ix\xi}(1+\Delta_\xi)^M (r_\lambda(x,\xi)\hat{u}(\xi))\, đ\xi.\end{aligned}$$

Using again that $u \in \mathcal{C}_0^\infty(\mathbb{R}^n, E)$, (4.5) and Leibniz's product rule,

$$\begin{aligned}\|\langle x \rangle^{2M} \mathrm{Op}(r_\lambda)u(x)\|_E &\le \int \|(1+\Delta_\xi)^M (r_\lambda(x,\xi)\hat{u}(\xi))\|_E\, đ\xi \\ &\prec \int \lambda^{\tau-1}\langle (x,\xi)\rangle^{2M\rho+1} \|\hat{u}(\xi)\|_E\, đ\xi \\ &\prec \langle x \rangle^{2M\rho+1} \int \langle \xi \rangle^{2M\rho+1} \|\hat{u}(\xi)\|_E\, đ\xi \\ &\prec \langle x \rangle^{2M\rho+1} \int \langle \xi \rangle^{2M\rho+1-N}\, đ\xi < +\infty\end{aligned}$$

for any $N > 2M\rho + 1 + n$. Hence, for any $M \in \mathbb{N}$ with , $M > \dfrac{n+1}{2(1-\rho)}$,

$$\|\mathrm{Op}(r_\lambda)u(x)\|_E \prec \langle x \rangle^{2M(\rho-1)+1} := g(x) \in L^1(\mathbb{R}^n),$$

i.e. b) is verified. Thus, we have(4.6).
Now we consider

$$\begin{aligned}&\mathrm{Op}(a'_\lambda)u(x) \\ &= \int e^{ix\xi}\big(\chi(\lambda x_0 + \lambda^{-\tau}x, \lambda^\tau \xi + \lambda \xi_0)\sigma_A(x_0 + \lambda^{-\tau-1}x, \xi_0 + \lambda^{\tau-1}\xi)\big)\hat{u}(\xi)\, đ\xi\end{aligned}$$

For fixed x, we get

$$\big(\chi(\lambda x_0 + \lambda^{-\tau}x, \lambda^\tau \xi + \lambda \xi_0)\sigma_A(x_0 + \lambda^{-\tau-1}x, \xi_0 + \lambda^{\tau-1}\xi)\big)\hat{u}(\xi) \to \sigma_A(x_0,\xi_0)\hat{u}(\xi)$$

for $\lambda \to +\infty$ for all $\xi \in \mathbb{R}^n$. Then, for fixed $x \in \mathbb{R}^n$,

$$\|(a'_\lambda(x,\xi) - \sigma_A(x,\xi))\hat{u}(\xi)\|_E \prec \|\hat{u}(\xi)\|_E \sup_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n} \|\sigma(x,\xi)\|_{\mathcal{L}(E)} \prec \langle \xi \rangle^{-N'},$$

hence

$$\mathrm{Op}(a'_\lambda)u(x) \to \sigma_A(x_0,\xi_0) \int e^{ix\xi}\hat{u}(\xi)\, đ\xi = \sigma_A(x_0,\xi_0)u(x)$$

for $\lambda \to +\infty$. Again, integration by parts yields

$$\langle x\rangle^{2M}\mathrm{Op}(a'_\lambda)u(x) = \int e^{ix\xi}(1+\Delta_\xi)^M(a'_\lambda(x,\xi)\hat{u}(\xi))\, đ\xi$$

for arbitrary $M \in \mathbb{N}$, and

$$\|\mathrm{Op}(a'_\lambda)u(x)\|_E \prec \langle x\rangle^{2M(\rho-1)} \in L^1(\mathbb{R}^n) \text{ for } M > \frac{n}{2(1-\rho)}.$$

In the end, by Lebesgue's Theorem on dominated convergence, we get

$$\mathrm{Op}(a'_\lambda)u \to \sigma_A(x_0,\xi_0)u \quad \text{in } L^2(\mathbb{R}^n,E).$$

for $\lambda \to +\infty$, and this, together with (4.6), completes the proof.

□

Using the previous Proposition 4.3.7, we can prove the following result on the invertibility of the principal symbol for operators in $\mathrm{G}^0(\mathbb{R}^n,\mathscr{L}(E))$ with the Fredholm property.

Theorem 4.3.8. Let $A = \mathrm{Op}(a) \in \mathrm{G}^0(\mathbb{R}^n,\mathscr{L}(E))$ induce a Fredholm operator $L^2(\mathbb{R}^n,E) \to L^2(\mathbb{R}^n,E)$. Then, σ_A is invertible.

Proof. In this proof we set $\|\cdot\| := \|\cdot\|_{L^2(\mathbb{R}^n,E)}$. Let $B \in \mathcal{L}(L^2(\mathbb{R}^n,E))$ denote a Fredholm inverse of A, i.e.

$$BA = I + K,$$

where K is compact in $L^2(\mathbb{R}^n,E)$. Choose $u \in L^2(\mathbb{R}^n,E)$. Using the isometric operator R_λ, for fixed $(x_0,\xi_0) \in \mathbb{R}^n\times\mathbb{R}^n$, $|(x_0,\xi_0)| = 1$, as introduced in Proposition 4.3.7, we get

$$\begin{aligned}\|u\| = \|R_\lambda u\| &= \|(BA-K)R_\lambda u\| \\ &\le \|B\|_{\mathscr{L}(L^2(\mathbb{R}^n,E))}\,\|AR_\lambda u\| + \|KR_\lambda u\| \\ &\le \|B\|_{\mathscr{L}(L^2(\mathbb{R}^n,E))}\,\|R_\lambda^{-1}AR_\lambda u\| + \|KR_\lambda u\| \\ &\to \|B\|_{\mathscr{L}(L^2(\mathbb{R}^n,E))}\,\|\sigma_A(x_0,\xi_0)u\|\end{aligned}$$

for $\lambda \to +\infty$ in $L^2(\mathbb{R}^n,E)$, since $KR_\lambda u \to 0$, in view of the fact that K is compact, and compact operators map weakly convergent sequences into strongly convergent ones, cfr., e.g., [Bre86]. Choosing $u = \phi\otimes e$, with $\phi \in L^2(\mathbb{R}^n)$, $\|\phi\|_{L^2(\mathbb{R}^n)} = 1$, and $e \in E$, we get by the previous equation

$$\|\sigma_A(x_0,\xi_0)e\|_E \ge \frac{\|e\|_E}{\|B\|_{\mathscr{L}(L^2(\mathbb{R}^n,E))}},$$

i.e. σ_A is injective. Moreover, since A is Fredholm, also A^*, the adjoint of A, is Fredholm, i.e.

$$B'A^* = I + K'$$

for some $B' \in \mathcal{L}(L^2(\mathbb{R}^n, E))$ and K' compact. Analogously

$$\|\sigma_{A^*}(x_0, \xi_0)e\|_E \geq \frac{\|e\|_E}{\|B'\|_{\mathscr{L}(L^2(\mathbb{R}^n,E))}},$$

and since $(\sigma_A)^* = \sigma_{A^*}$, we get that σ_A is bijective, as claimed. □

Theorem 4.3.9. Let $A \in \mathrm{G}^{0,0}_{\mathrm{cl}}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$ induce a Fredholm operator from $L^2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ to itself. Then A is bi-elliptic.

Proof. In view of Remark 4.3.4, we can consider A as an operator belonging to $\mathrm{G}^0(\mathbb{R}^{n_1}, \mathscr{L}(E))$ with $E = L^2(\mathbb{R}^{n_2})$. Then, by Theorem 4.3.8, the operator valued principal symbol $\sigma_{\mathrm{Op}(A^1)}(x_1, \xi_1) = \sigma^1(A)(x_1, \xi_1)$ is everywhere invertible.
Analogously, by viewing A as an operator in $\mathrm{G}^0(\mathbb{R}^{n_2}, \mathscr{L}(E))$ with $E = L^2(\mathbb{R}^{n_1})$, also $\sigma^2(A)(x_2, \xi_2)$ is everywhere invertible.
Finally, fix $(x_1, x_2, \xi_1, \xi_2) \in (T^*\mathbb{R}^{n_1} \setminus 0) \times (T^*\mathbb{R}^{n_2} \setminus 0)$. Since $\sigma^1(A)(x_1, \xi_1) \in \mathrm{G}^{m_2}_{\mathrm{cl}}(\mathbb{R}^{n_2})$ is everywhere invertible, it is, in particular, non-characteristic at (x_2, ξ_2). Thus

$$\sigma^{12}(A)(x_1, x_2, \xi_1, \xi_2) = \sigma_{\sigma^1(A)(x_1,\xi_1)}(x_2, \xi_2) \neq 0, \tag{4.7}$$

where we used (4.2). This, according to Definition 4.2.6, proves our claim. □

Summing up, we have the following Theorem.

Theorem 4.3.10. Let $A \in \mathrm{G}^{0,0}_{\mathrm{cl}}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$. Then A is bi-elliptic if and only if it induces a Fredholm operator from $L^2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ to itself.

Proof. The claim follows from Theorem 4.3.2 and Theorem 4.3.9. □

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