Fundamentals of Extragalactic Celestial Mechanics *

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SUMMARY – There is a deep-rooted conviction that celestial mechanics has exhausted its role with its successes at the planetary level, etc. and that it is not possible to extend it to the study of the Universe considered on a large scale. This paper will summarise some of the results the Author has obtained in his research (made in cooperation with D.Galletto) that lead instead to such an extension. The assumptions these results are based on are only the following: 1) physical space is ordinary three-dimensional Euclidean space; 2) the principles of Newtonian mechanics hold true; 3) the Universe, at least on a large scale, is homogeneous; 4) typical galaxies are receding from our galaxy with a purely radial motion. By assuming the incoherent matter scheme (the scheme assumed in cosmology when pressure is negligible) as the Universe scheme, from such assumptions follow: the law which describes the expansion of the Universe, i.e. Hubble's law; the cosmological principle; the explicit equation of motion of any typical galaxy; the constancy (=1) of the density parameter (a result that holds true only for the case in which space is Euclidean and that follows from assumption 2 and from the cosmological principle); the existence of a constant that astronomical observation allows us to identify as the gravitational constant; the equivalence between the explicit equation of motion of any typical galaxy and

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the two-body problem equation; the result that the force exerted between any two typical galaxies is expressed by Newton's law of gravitation; the inconsistency of the gravitational paradox; the equations of extragalactic celestial mechanics which, when considered within a mathematical framework, do not differ from those obtained when operating within the framework of the general theory of relativity in the case of the homogeneous and isotropic relativistic model of the Universe with zero spatial curvature (Einstein-de Sitter model).

By interpreting appropriately the above-mentioned equations and taking into account the property of the speed of light being independent of its source, we obtain the well known results of cosmography concerning red-shift, horizons, etc., as well as the metric of the space-time manifold. Furthermore we obtain that there exists one and only one tensor form for the equations of extragalactic celestial mechanics compatible with this metric. Such form is given by Einstein's field equations of the general theory of relativity.

If we resort to the assumptions that the cosmological principle holds true, the principles of classical mechanics are valid and the speed of light is independent of its source, what has been summarised above, suitably adapted, allows us to construct, alongside the Einstein-de Sitter model, the homogeneous and isotropic relativistic models of the Universe with curvature other than zero (Friedmann's models without pressure). In this case too it is possible to find that there exists one and only one tensor form for the corresponding dynamical equations compatible with the metric of the corresponding space-time manifolds, which is the Robertson-Walker metric. This form is again given by Einstein's field equations. Therefore, it is in this sense, by building the Einstein-de Sitter model and the Friedmann models in this way, that cosmology, developed in this way, provides a great confirmation of Einstein's theory of gravitation. At the same time, if we bear in mind the weaknesses of the assumptions at the base of what we have just summarised, we can perceive the great role that classical mechanics can play in the study of the Universe.

The considerations and deductions made here are totally different from Milne and McCrea's "Newtonian cosmology", which, as it is formulated, performs a rather poor and open to criticism role in the dynamical description of the Universe.

1. In the introduction to his great treatise "Les Méthodes Nouvelles de la Mécanique Céleste", Henri Poincaré wrote that: "Le veritable but de la Mécanique Celeste n'est pas de calculer les éphémerides, [...], mais de reconnaître si la loi de Newton est suffisante pour expliquer tous les phénomenes" [1].

In a relatively recent treatise on celestial mechanics, we find written in the introduction to the chapter on binary stars[2]: "The preceding chapter may well have given one the impression that gravity plays a poorly understood role beyond the boundaries of the solar system. Such a supposition is false, which is why Newton's law of gravitation is usually referred as the *universal* law of gravitation and G is known as the *universal* gravitational constant. For within the realm of the *binary stars* [...], one can calculate orbits and predict positions in the same manner as for spacecraft and comets. It is the complete success of this enterprise for the many hundreds of binary stars, distributed at random over the celestial sphere, that allows us to stretch

$$\boldsymbol{F}_{12} = -\frac{Gm_1 m_2 \left(\boldsymbol{r}_1 - \boldsymbol{r}_2\right)}{\left|\boldsymbol{r}_1 - \boldsymbol{r}_2\right|^3}$$

beyond the boundaries of the solar system. Hence the adjective 'universal' ".

[1] "The real aim of celestial mechanics is not to calculate ephemerides, […], but to see if Newton's law is sufficient to explain all phenomena". Clearly Poincaré is referring to mechanical phenomena.
[2] See [14].

However, despite the fact that this law permits us to study the dynamics of binary stars, to estimate the period of revolution of the Sun about the nucleus of our galaxy, to calculate the mass of our galaxy, etc., all these results have a typically "local" significance and dimension, especially if compared to the dimensions of the Universe known today, and hence they are not such as to justify the term "universal" attributed to this law. On this matter it should be pointed out that all these results certainly cannot be considered as the confirmation of the validity of Newton's law of gravitation in that this law is the starting point for their deduction. Therefore they cannot be interpreted as an answer to the problem implicitly posed by Poincaré in his statement above. Meaningful is what A.P.French wrote in this regard in his excellent book *Newtonian Mechanics*, when he mentions the determination of our galaxy's mass[3]: "This is not really a figure that can be independently checked. It is a kind of ultimate tribute to our belief in the universality of the gravitational law that it is confidently used to draw conclusions like those above concerning masses of galactic systems".

Celestial mechanics has developed from the time of Newton to today, essentially within the framework of our planetary system[4], which is crossed by light in little more than ten hours, a time that is insignificant when compared to the dimensions of the part of the Universe that today can be reached by astronomical instruments, namely several billion light years.

It is also through such observation that cosmologists have become convinced that celestial mechanics, despite the considerable results that have been obtained within a planetary framework, does not have, or rather cannot have, a valid role in a large-scale study of the Universe. This conviction has become in-

[3] [2], p. 299.

[4] In this field Newton's law of gravitation follows as a necessary consequence of Kepler's laws and of the law of action and reaction.

Attempts to start a study of the Universe within the framework of Newtonian mechanics were made at the end of the last century. It was believed that the Universe were static and infinite in space and time, basing the study on Newton's law of gravitation which was assumed to be valid also at great distances. Such fruitless attempts gave rise to the so-called *gravitational paradox*, a point we shall return to in Sec. 7, as well as to proposals for modifying Newton's law of gravitation which are completely unacceptable manipulations of this law. See [6].

creasingly deep-rooted in the course of this century, both as a consequence of the considerable successes that have been achieved in the development of cosmology by the general theory of relativity and for the fact that scholars of celestial mechanics have always limited their studies to traditional sectors, without ever trying to extend the field of application of celestial mechanics to the large-scale study of the Universe[5], the knowledge of which has seen enormous progress through astronomical observation in the last few decades. The result has been a sort of fracture between celestial mechanics and astronomy, as the close ties that in the past had always existed between the two disciplines began to fade.

Actually this conviction may be considered at least debatable, if we make a careful analysis of the large-scale structure of the Universe. Indeed it is sufficient to reflect on the fact that the Universe in its present phase, the "matter-dominated era" (which if we consider the "big-bang" theory, which now is almost universally accepted, has lasted at least twelve, thirteen billion years), is made of countless free bodies (the galaxies) that astronomical observation, quite naturally, indicates to be in motion. Such motion was defined by Hubble in 1929 in his famous empirical law: galaxies recede from our galaxy at a velocity that is proportional to their distance. Therefore it would be surprising that classical mechanics, at least for lesser distances (namely those where the receding velocity of galaxies is less than the speed of light), were not able to provide a dynamical description of the Universe. If this were so, classical mechanics in its relationship with the Universe, would be a discipline that, understood as celestial mechanics, has exhausted all its resources in its successes at the planetary level. Therefore, this discipline would be completely obsolete as a result of the enor-

[5] Milne and McCrea are exceptions. In the 1930s they tried to construct a "*Newtonian cosmology*" (see [13], [12]), which, however, offers certain highly unsatisfactory aspects such as the implicit assumption made that Newton's law of gravitation is valid for the whole Universe. We shall return to this construction in Sec. 11.

mous limitation it would have from the comparison between the dimensions of our planetary system and those of the visible Universe.

Therefore, in the light of what has been said now, the problem arises of seeing to what extent it is possible, by operating within the realm of classical mechanics, namely Newtonian mechanics, to provide a large-scale dynamical description of the Universe, by applying, of course, a suitable model for this and, conforming to the aim of celestial mechanics according to Poincaré, without resorting to Newton's law of gravitation. In brief, and more precisely, the problem should be posed in these terms: *remaining within the framework of the principles of Newtonian mechanics, to see to what extent, with the Universe obviously represented by a suitable model, it is possible to provide a large-scale dynamical description, as well as a kinematical one, of the Universe and at the same time to see to what extent Newton's law of gravitation is verified.*

2. The aim of this paper in particular is to give an answer to the problem formulated above.

Since the dimension of galaxies is negligible when compared to the dimensions of that part of the Universe that is today accessible using astronomical instruments, galaxies can be represented by particles, the fundamental particles the Universe is made of. We shall assume that such particles form a continuous medium, the so-called *cosmological fluid* which is used in cosmology. The fluid will be indicated by $\mathcal{U}$. Taking into account, in agreement with the aim of this lecture, that we will consider, as already said, the matter-dominated era, the fluid $\mathcal{U}$ will be supposed without internal stresses, namely without pressure ("incoherent matter scheme" or "dust scheme").

Having adopted this model for the Universe, the assumptions on which this lecture is based are solely the following:

1. physical space is ordinary three-dimensional Euclidean space;

2. the principles of Newtonian mechanics hold true;
3. the Universe, at least on a large-scale, is homogeneous and extended to all space, and hence the fluid $\mathcal{U}$ will be considered homogeneous and extended to all space[6].
4. with respect to our galaxy the Universe expands *radially*, in the sense that every typical[7] galaxy recedes from our galaxy with a purely radial motion[8]; this means that there exists a particle O of the fluid $\mathcal{U}$ (our galaxy) with respect to which the particles of $\mathcal{U}$ recede with a purely radial motion.

Assumptions 1 and 2 are in common with classical celestial mechanics. Assumption 2 includes, in particular, the principle of superposition of simultaneous forces, as well as the principle of conservation of mass. Assumptions 3 and 4 are suggested by astronomical observation. As far as assumption 4 is concerned, it should be pointed out explicitly that, in holding that the Universe, namely the fluid $\mathcal{U}$, expands radially, we do not intend to assume that the velocity of any typical galaxy (i.e. of any particle of $\mathcal{U}$) is only a function of its distance from our galaxy (i.e. from the particle O).

Furthermore, it is important to underline that assumption 4 is far less restrictive than that of isotropy which, together with assumption 3 on homogeneity, is made in the studies, within the framework of the general theory of relativity, of the usual models of the Universe, whereby isotropy is even assumed to exist at every point of the Universe (homogeneous and isotropic models), in line with the so-called *cosmological principle*, which we shall return to in Sec. 3.

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[6]The fact that the Universe is homogeneous only on a large scale would mean, strictly speaking, the assumption, not of galaxies, but of clusters (and even clusters of clusters) of galaxies as particles composing the fluid $\mathcal{U}$. The assumption of galaxies that has been made here is used for the sake of simplicity and brevity of exposition.

[7]Typical in the sense that local motions cannot be detected or are negligible, which are important only for galaxies that are very near our galaxy, namely belonging to the Local Group.

[8] That is in the sense that, the centre of mass of our galaxy being indicated by G, every typical galaxy P during the expansion process has its velocity directed constantly from G to P. Hence its trajectory is on a half-line from centre of mass G, which, with the adopted scheme, is identified with the particle O of $\mathcal{U}$.

Finally it should be pointed out explicitly that, in formulating assumptions 1, 2, 3 and 4, *no assumption has been made for the forces acting between galaxies, i.e. between the particles of* $\mathcal{U}$.

3. Let t be the time and μ the density of the fluid $\mathcal{U}$ (i.e. the density of the Universe considered on a large scale). From the homogeneity of $\mathcal{U}$ it follows: $\mu=\mu(t)$.

Let P be any particle (i.e. any typical galaxy) and $\boldsymbol{r}$ the vector OP. If we introduce the function

$$h(t)=-\frac{1}{3}\frac{\dot{\mu}(t)}{\mu(t)} \quad , \tag{1}$$

by resorting to the equation of continuity, which translates into a mathematical form the physical principle of conservation of mass, as well as by the direct application of this principle, we arrive at the following expression for the velocity of P with respect to the particle O, namely with respect to our galaxy[9]:

$$\frac{d\boldsymbol{r}}{dt}=h(t)\boldsymbol{r} \, . \tag{2}$$

This result expresses that the velocity of the particle P of $\mathcal{U}$, namely of the galaxy P, with respect to our galaxy (which, taking into account assumption 4, is directed as the vector $\boldsymbol{r} = OP$), depends only on the distance of P from our galaxy, as well as on time t. More precisely, this result expresses that the velocity

[9] The deduction of equation (1) obtained from the assumptions that have been made will be presented in a forthcoming work.

of the galaxy *P* is proportional to its distance *r* from our galaxy *O.* This is what is expressed by Hubble's law (as mentioned in Sec. 1) which was obtained by Hubble from astronomical observation:

I. Whatever the typical galaxy P is, it recedes from our galaxy O at a velocity that is proportional to its distance from O.

The factor of proportionality *h(t)* is expressed by (1) and in cosmology is incorrectly called *Hubble's constant*. In this lecture it will be called *Hubble's parameter*.

The result now obtained, expressed by (2), seems at first sight to put our galaxy *O* in a Ptolemaic situation, that is as if it were placed at the centre of the Universe. But actually such a conclusion, as can be easily ascertained, is only apparent.

In fact, let us observe that the particles of $\mathcal{U}$ (i.e. typical galaxies) in their purely radial receding motion with respect to *O*, identify a frame of reference with its origin in *O*[10], a frame that will be indicated by $\mathfrak{R}_O$. Let us now consider any other particle *O'* of the fluid $\mathcal{U}$ (i.e. any other typical galaxy *O'*) and the frame of reference $\mathfrak{R}_{O'}$ with its origin in *O'* and in translational motion with respect to the frame of reference $\mathfrak{R}_O$. As can be easily ascertained for the velocity of *P* with respect to the frame of reference $\mathfrak{R}_{O'}$, we still obtain

$$\frac{d\boldsymbol{r}'}{dt} = h(t)\boldsymbol{r}' \,, \tag{2'}$$

where $\boldsymbol{r}' = O'P$. This result expresses that for this velocity, i.e. for the velocity of the galaxy *P* with respect to the frame of reference $\mathfrak{R}_{O'}$, Hubble's law still holds

[10] In this regard bear in mind the observations made in note 8.

with the same Hubble's parameter *h(t)*: galaxy *P* recedes from galaxy *O'* at a velocity that now takes the direction of the vector ***r'*** and that is still proportional to its distance *r'* from galaxy *O'*, with the factor of proportionality still expressed by Hubble's parameter *h(t)*.

We shall call *natural frames of reference* those frames of reference which, such as $\mathfrak{R}_{O'}$, have their origins in particles of the fluid $\mathcal{U}$ (i.e. in typical galaxies) and are in translational motion with respect to the frame of reference $\mathfrak{R}_O$. We can therefore state that:

II. All natural frames of reference are equivalent to one another from the kinematical point of view, in the sense that, whatever we consider a natural frame of reference, the motion of the particle P of the fluid $\mathcal{U}$ *(i.e. of the galaxy P) with respect to this frame is always described by the same law, that is by Hubble's law, expressed by (2) and (2'). The frame of reference* $\mathfrak{R}_O$ *is a natural frame of reference too and, from the kinematical point of view, it is indistinguishable from any other natural frame of reference and consequently does not have a privileged aspect.*

In other words:

III. The fluid $\mathcal{U}$*, i.e. the Universe, has the same kinematical behaviour with respect to any natural frame of reference. Whatever the natural frame of reference* $\mathfrak{R}_{O'}$ *is, such behaviour is described by Hubble's law (2').*

The latter result, a strict consequence of assumptions 1, 2, 3, and 4, expresses that:

III'. The fluid $\mathcal{U}$*, i.e. the Universe, observed from any point, has at any instant*

the same aspect, which varies from instant to instant.

This is just what the *cosmological principle* states. Therefore we can say that from assumptions 1, 2, 3, and 4, it follows that for $\mathcal{U}$ the cosmological principle holds true. Considered in the present context, it therefore becomes a theorem.

Finally from (2') we obtain the following expression for the acceleration of *P* with respect to any natural frame of reference $\mathfrak{R}_{O'}$:

$$\frac{d^2\boldsymbol{r}'}{dt^2} = \left(\dot{h}(t) + h^2(t)\right)\boldsymbol{r}' \,. \tag{3}$$

This expression provides a considerable indication of what must be the form, in such an arbitrary natural frame of reference $\mathfrak{R}_{O'}$, of the resultant of the forces acting on *P*.

The results so far obtained give the kinematical description of the Universe, founded on the assumptions made.

At this point we still have to study the dynamical description of the Universe and, in particular, we still have to give an answer to the problem of seeing to what extent Newton's law of gravitation holds true. Taking into account equation (3), this problem may be now reformulated in the following way: *as we know the law whereby the fluid $\mathcal{U}$ (i.e. the Universe) expands, expressed by Hubble's law (2'), a law which has as a consequence equation (3), we have to determine, if possible, the forces that regulate the process of expansion of $\mathcal{U}$, or, at least, their resultant acting on the particle (galaxy) P*.

We can even add that the problem of seeing to what extent Newton's law of gravitation holds true is thus taken back to an "inverse" problem in mechanics: *given the law whereby motion takes place (law (2')), go back to the forces that cause it, or, at least, to their resultant.*

An inverse problem of this type is, for example, the famous classical problem that consists of starting from Kepler's laws (laws whereby motion takes place) and deduce from these the force that regulates the motion of the planets.

4. From the kinematical point of view the starting frame of reference $\mathfrak{R}_O$ being indistinguishable from any other natural frame of reference, the arbitrarily chosen natural frame of reference $\mathfrak{R}_{O'}$ will from now on be indicated simply by $\mathfrak{R}_O$.

Recourse to the second principle of dynamics and to the principle of superposition of simultaneous forces (a principle which allows us to replace the forces acting on any particle *P* of the fluid $\mathcal{U}$, namely acting on any typical galaxy *P*, with their resultant) allows us, through considerations which for the sake of brevity are not possible to summarise here, to deduce from equation (3) (which follows from Hubble's law) the following equation for the motion of *P* with respect to the natural frame of reference $\mathfrak{R}_O$[11]:

$$\frac{d^2\mathbf{r}}{dt^2} = \kappa\mu(t)\mathbf{r}\,, \tag{4}$$

where κ is a constant different from 0 and for the moment indetermined. In deducing equation (4) the mass of *P* has been assumed to be the unit mass (i.e. recourse has been made to the so-called "specific forces").

For reasons which will become clear shortly, it is the case to introduce the constant *k* linked to κ by:

[11] For an outline of the deduction of equation (4) see [4], 3; [5], 3; [7], 2.2, 2.3; [10], 6. A detailed deduction of this equation highlighting the important implications from a physical point of view will be presented in a forthcoming paper.

$$k = -\frac{3}{4\pi}\kappa .$$

With such a position equation (4) assumes the form

(5) $$\frac{d^2\mathbf{r}}{dt^2} = -\frac{4}{3}\pi k\mu(t)\mathbf{r} .$$

Since the natural frame of reference $\Re_o$ was chosen in an arbitrary way, equation (5) keeps its form unchanged in every natural frame of reference, as can be easily verified.

Therefore we can state that:

IV. From assumptions 2, 3, and 4 follows the explicit form of the equation of motion of the particle P of the fluid $\mathcal{U}$ (i.e. of the galaxy P) with respect to every natural frame of reference.

V. All natural frames of reference are equivalent to one another both, as has been seen, from the kinematical point of view (equivalence expressed by Hubble's law (2)) and also from the dynamical point of view (equivalence expressed by equation (5), which has the same form in every natural frame of reference).

VI. With respect to every natural frame of reference the fluid $\mathcal{U}$, i.e. the Universe, has the same behaviour, both from the kinematical point of view, expressed by Hubble's law (2), and from the dynamical point of view, expressed by equation (5).

These results, apart from the value of the constant *k* which is still indeter-

mined, are the answer to the problem of providing a dynamical description of the Universe, based on the assumptions made. At the same time, equation (5), with its right side determined (apart from the value of the constant *k*), provides a first answer, albeit partial, to the problem that was formulated at the end of Sec. 3.

From equation (5) it follows, among other things, that the natural frames of reference are accelerated with respect to one another.

The natural frames of reference, as a limit case, become inertial frames of reference, once we consider zero the value of the density $\mu\,(t)$ in equation (5) (empty Universe).

5. So far in equation (5) the constant *k* has remained indetermined.

To determine it let us introduce the function

(6) $$\Omega(t) = \frac{8\pi}{3} k \frac{\mu(t)}{h^2(t)} \ ,$$

which depends, as well as on time t, on the indetermined constant *k*.

Long and complex mathematical considerations, which cannot even be summarised here, and which are based on assumption 2 (in particular once again on the principle of superposition of simultaneous forces) and on the cosmological principle[12], allow us to prove that if physical space is Euclidean (assumption 1) it necessarily follows

(7) $$\Omega(t) = cost. = 1.$$

[12] These will be the subject of a forthcoming paper. A brief outline of this deduction is implicitly presented in what will be briefly said at the end of Sec. 8. In this regard it is the case to point out that the deduction of (7) needs considerations that go beyond the context of assumptions 1, 2, 3, 4, because, even if it depends only on assumption 2, this deduction needs the intervention of a space (a spherical hypersurface) which is not Euclidean.

This result is obtained in cosmology (where the function $\Omega(t)$ is called the *density parameter*) by working within the corpus of the general theory of relativity. On the contrary, in this lecture it is obtained by resorting to the principles of Newtonian mechanics.

From (6) and (7) it follows that

$$k = \frac{3}{8\pi}\frac{h^2(t)}{\mu(t)}. \tag{8}$$

The most recent determinations of the present value of Hubble's parameter have confirmed for it a value very near, if not less than, 50 km s^{-1} Mpc^{-1} (1 Mpc = $3{,}26\cdot10^6$ light years). The present value of the density, which is known with much less accuracy, seems to be between 10^{-27} and 10^{-26} kg m^{-3}. With such values, equation (8) implies the following limitations for the value of the constant *k*:

$$3{,}13\cdot10^{-11} < k < 31{,}3\cdot10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}.$$

And even more so, maintaining the above-mentioned value for *h* and considering μ in the order of $5\cdot10^{-27}$ kg m^{-3} (a value that can be considered plausible considering that certainly there is a large quantity of dark matter in the Universe which is far higher than the quantity of visible matter), from (8) we obtain

$$k = 6{,}26\cdot10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}},$$

a value that is surprisingly near the value of the gravitational constant *G*:

$$G = 6{,}67\cdot10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}. \tag{9}$$

Hence we can say that the identification of the constant k with the gravitational constant G:

(10) $$k \equiv G,$$

is legitimated by astronomical observation, in that such observations provide indications that agree, albeit only roughly, with (10)[13]. Hence we can conclude that:

VII. Constant k, the existence of which follows from the assumptions made and from the principles of Newtonian mechanics, can be identified with the gravitational constant G [14].

Therefore the equation (5) of motion of the particle P (i.e. of the typical galaxy P) with respect to the natural frame of reference $\mathfrak{R}_o$ becomes completely determined:

(11) $$\frac{d^2\mathbf{r}}{dt^2} = -\frac{4}{3}\pi G\mu(t)\mathbf{r}\ .$$

[13] At this point the objection could be raised that the result now obtained is not surprising and that it would seem to present, at least at first view, tautological aspects, in that in determining the present value of the density $\mu(t)$, that appears in equation (8), recourse is made to our galaxy's mass, which is determined (as already said in Sec. 1) by applying Newton's law of gravitation, considered valid at the galaxy level and with the value of the constant G expressed by (9).

Actually this is not true because one cannot say *a priori* that the right side of equation (8), albeit with the present value of $\mu(t)$ determined by the application of Newton's law of gravitation, gives G as the k value, because the above right side also depends on the present value of $h(t)$ which is deduced independently of this law.

[14] Such identification certainly does not mean that physical space is necessarily Euclidean, and namely that for the density parameter it must be $\Omega=1$, a property which, as can be proven with the considerations outlined at the beginning of this section, is characteristic of the Euclidean case. If anything, we can say (leaving the context of classical celestial mechanics where physical space is considered Euclidean) that, once it is held $k=G$, physical space, whether it is open or closed (i.e., more precisely, conforming to the cosmological principle, whether it has the characteristics of a pseudospherical hypersurface or those of a spherical hypersurface), locally differs little from Euclidean space and therefore, locally (i.e. with distances of not more than 1000 Mpc = $3{,}26\cdot10^9$ light years), with a good approximation it can be considered Euclidean.

It should be pointed out the importance of the fact that *the recourse to the principle of superposition of simultaneous forces not only leads to equation (5) of motion of P, but, above all, it entails the existence of a constant k that astronomical observation allows us to identify with the gravitational constant G*.

6. By introducing mass M of the material sphere $\mathcal{S}_r$ with its centre in the origin O of the natural frame of reference $\mathfrak{R}_o$ and radius $r=|\boldsymbol{r}|=|OP|$ (a mass that does not change with time), equation (11) takes the form:

$$\frac{d^2\boldsymbol{r}}{dt^2} = -G\frac{M}{r^3}\boldsymbol{r}\,, \tag{12}$$

expressing that:

VIII. To the effects of the motion of P, the particle (galaxy) chosen arbitrarily and supposed to have unit mass, everything happens as if the natural frame of reference $\mathfrak{R}_o$ with respect to which the motion is considered were inertial, the part of $\mathcal{U}$ (i.e. of the Universe) external to the material sphere $\mathcal{S}_r$ did not contribute to the motion of P, the mass M of this sphere were concentrated in its centre O and attracted P according to Newton's law of gravitation.

The problem of the study of the motion of P has thus been traced back to the celebrated two-body problem of classical celestial mechanics, considered in the case in which one of the two bodies is fixed.

This result has been obtained by resorting only to assumptions 3 and 4 and to the principles of classical mechanics, without resorting to any theory of gravitation.

The two-body problem that, formulated by resorting to Newton's law of gravitation and at the planetary level, is substantially at the base of classical celestial mechanics, therefore reappears on a large scale, i.e. at extragalactic level.

What we have seen now may be considered in itself as an answer to the second part of the problem posed at the end of Sec. 1, thus as it has been reformulated at the end of Sec. 3: *equation (12) expresses that the motion of the particle (galaxy) P, however far it is from the origin O of the frame of reference with respect to which this motion is considered, is governed by Newton's law of gravitation.*

From equation (12) we obtain, by integration:

$$\frac{1}{2}\left(\frac{d\boldsymbol{r}}{dt}\right)^2 - G\frac{M}{r} = \alpha\,, \tag{13}$$

an equation that expresses the conservation of energy for the particle (galaxy) *P*, assumed to have unit mass, and where α is the energy constant of *P*. Equation (7), however, implies

$$\alpha = 0\,. \tag{14}$$

In other words *the mechanical energy of every particle of* $\mathcal{U}$*, i.e. of every galaxy, is zero*.

Thus the equation expressing the conservation of energy, whatever the particle (galaxy) *P* is, is expressed by

$$\frac{1}{2}\left(\frac{d\boldsymbol{r}}{dt}\right)^2 = G\frac{M}{r}\,.$$

7. Despite equation (12), we have still to solve the problem to determine the forces that are exerted between the particles of the fluid $\mathcal{U}$, i.e. between galaxies.

On this aspect, considerations that for the sake of brevity cannot be summarised here[15] and which are always and only based on the assumptions made, lead to the following result[16]:

IX. However we consider two particles P and Q of the fluid $\mathcal{U}$, i.e. two typical galaxies P and Q, there is one and only one force that is exerted between P and Q and which governs the expansion process of $\mathcal{U}$, i.e. of the Universe. This force is expressed by Newton's law of universal gravitation.

This result provides a full answer to the second part of the problem formulated at the beginning of this paper, and namely: remaining within the framework of Newtonian mechanics, to see to what extent, in a large-scale description of the Universe, Newton's law of gravitation holds true. The answer, as we have seen, is the following:

X. With the assumptions made, Newton's law of gravitation holds true in all the Universe.

Furthermore, this result proves the inconsistency of the so-called *gravitational paradox* (mentioned in note 4), which asserts the total impossibility of applying the Newtonian theory of gravitation to a homogeneous fluid extended to all space, a paradox which is quoted in several treatises on the general theory of relativity and on cosmology, including recent ones, where it is considered to

[15] These considerations will be presented in detail in a forthcoming paper.
[16] An attempt to obtain this result starting from Hubble's law, but through arguments totally different from those summarised here, has been made by P.T.Landsberg in [11], but this attempt does not hold true because it is tautological. On this matter see [3].

hold true.

From the considerations now made it emerges that not only does the gravitational paradox not exist but that the motion of a homogeneous fluid without external stresses and extended to all space is necessarily governed by Newton's law of gravitation, whatever the motion of the fluid is[17]. Moreover this result invalidates the modifications to this law proposed in the past in order to overcome the above-mentioned paradox.

The result we have now seen (namely that, by operating within the realm of Newtonian mechanics, the entire process of expansion of the Universe is governed by Newton's law of gravitation) fully justifies the use of the adjective "universal" attributed to this law and to the constant *G*. A justification that goes well beyond the indeed very limited one referring to binary stars which has been reported at the beginning of this paper.

Furthermore it should be added on this point that, as it has been already observed in Sec. 1, whilst in the study of binary stars this law is assumed as a starting point, in this lecture this law is deduced from the assumptions made, proving at the same time that the large-scale motion of the Universe is indeed governed by this law.

8. Let us fix the instant t_o once and for all and let us introduce the function

$$R(t) = \exp\left(\int_{t_0}^{t_1} h(t)\, dt \right). \tag{15}$$

From equation (2), considered with respect to any natural frame of reference $\mathfrak{R}_O$, we obtain by integration

[17] Result IX has indeed an intrinsic character and therefore holds true whatever the motion of the fluid is.

(16) $$\boldsymbol{r}(t) = R(t)\boldsymbol{r}_0,$$

where $\boldsymbol{r}_0$ is the vector $\boldsymbol{r} = OP$ considered at the instant t_0: $\boldsymbol{r}_0 = \boldsymbol{r}(t_0)$.

Taking into account equation (14) and equation (16) for the vector $\boldsymbol{r}$, equations (11) e (13) become

(17) $$\frac{\ddot{R}(t)}{R(t)} = -\frac{4}{3}\pi\, G\, \mu(t)\ ,$$

(18) $$\left(\frac{\dot{R}(t)}{R(t)}\right)^2 = \frac{8}{3}\pi\, G\, \mu(t)\ .$$

Bearing in mind (16), equations (17) and (18) are the equations which lie at the bases of *extragalactic celestial mechanics*. They are of almost immediate integration[18].

XI. When considered in a mathematical framework, equations (17) and (18) are the same that would be obtained instead of operating in the context of classical mechanics by operating within the framework of the general theory of relativity in the case of the homogeneous and isotropic relativistic model of the Universe with zero spatial curvature.

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[18] From (18) we obtain, bearing in mind equations (15) and (1):

(a) $\mu(t) = \dfrac{1}{6\pi G t^2}$, (b) $h(t) = \dfrac{2}{3t}$, (c) $R(t) = bt^{2/3}$,

where it is assumed as the initial instant, i.e. as instant 0, the instant in which expansion starts, in correspondence to which density is considered infinitely great.

From (a) and (b) we obtain the following expressions

$$t = \frac{2}{3h(t)} \quad , \quad t = \frac{1}{\sqrt{6\pi G\mu^2(t)}}\ ,$$

which allow us to calculate the age of the Universe as soon as the present value of h or μ is known. E.g. for h = 50 km s^{-1} Mpc^{-1}, or for $\mu = 5\cdot10^{-27}$ kg m^{-3}, we obtain a value for the age of the Universe in the order of 13 billion years, a value that is perhaps a little below the true one.

However, their physical meaning is different, according to whether we operate within the framework of classical mechanics or that of the general theory of relativity.

By operating within the framework of classical mechanics, as has been done in this lecture, equations (17) and (18), through equation (16), provide the description of the motion of the particles P of the fluid $\mathcal{U}$, i.e. of typical galaxies P of the Universe, in the space where the fluid $\mathcal{U}$, i.e. the Universe, is contained and expands.

On the other hand, by operating within the framework of the general theory of relativity, still through equation (16) such equations provide the description of an expansion process of space, that drags with it the particles of $\mathcal{U}$, i.e. the galaxies of the Universe, linked to the expanding space[19].

However the latter interpretation given to the above-mentioned equations may be obtained without resorting to the general theory of relativity. This can be done by considering physical space, that has been considered Euclidean, as a "singular" case of the homogeneous and isotropic expanding model of the Universe with positive curvature (hence having the geometrical characteristics of an expanding three-dimensional spherical hypersurface) when one tries to have the curvature equal to zero. Such a model can be built, like the one built in the present paper, by operating within the framework of classical mechanics and starting only from the principles of classical mechanics and the cosmological principle[20] (already mentioned at the end of Sec. 2 and in Sec. 3), and includes as a "singular" case that of expanding Euclidean space that drags with it the particles of the cosmological fluid, i.e. the galaxies, and with the expansion process de-

[19] More precisely (cfr. note 6) the clusters (and even the clusters of clusters). Within these there is no expansion for space because of the strong gravitational field that they generate.

[20] Such a construction, with all its properties, has been performed in [8],II. The equations obtained by this construction differ from those obtained by operating within the framework of the general theory of relativity only for the fact that in this construction the value of the energy constant is indetermined. This value becomes zero when the curvature becomes zero, which in turn implies equation (7).

scribed by equations (16), (17) and (18)[21].

9. Ignoring for the moment the interpretation seen now, the considerations and the results outlined briefly in the previous sections form the foundations of extragalactic celestial mechanics, understood in the classic sense of the term, namely, as has been implicitly said in Sec. 2 by introducing assumptions 1 and 2 (i.e. physical space is Euclidean and the principles of classical mechanics hold true), in the sense of ordinary celestial mechanics. Of course this entails, at least at first sight, limitations on its applicability, due to the limit of validity of Hubble's law (2), a limit represented by the insuperability of the speed of light. Bearing in mind (2) as well as the said limit, Hubble's law, and all the equations we have seen, would have a significance only for distances less than $c/h(t)$, namely the distance that in cosmology is called *Hubble's radius*.

But since the equations that are at the basis of extragalactic celestial mechanics (which, as follows immediately from (18) and (16), include Hubble's law) are, as has been mentioned in Sec. 8, formally the same as those obtained by operating as indicated at the end of that section, at this point it is sufficient to resort to the corresponding interpretation given there to be able to conclude that, with such an interpretation, no limit is any longer imposed on the velocity of expansion and consequently on Hubble's law. Therefore, as occurs in the general theory of relativity, in this new context Hubble's law maintains its form unchanged, however great the distances may be and hence the velocities.

At this point nothing prevents us, once chosen (in a completely arbitrary way) the particle O of $\mathscr{U}$, i.e. the typical galaxy O, from considering, alongside the Euclidean space which is expanding without limits, an *ideal non-expanding* Euclidean space, with respect to which O is fixed and all the other particles of $\mathscr{U}$,

[21] Such considerations will be adequately developed in forthcoming papers.

i.e. the galaxies, are receding from *O*. In such ideal Euclidean space both Hubble's law and the equations of extragalactic celestial mechanics reacquire their original meaning. Their deduction remains the same as has been briefly summarised in this lecture and their validity with respect to the above ideal space now holds true without any limit. It is with respect to this space, introduced in this way, that extragalactic celestial mechanics, understood in the classic sense, should be considered and where it is fully valid without any limit for the velocities[22].

10. Let us now interpret equation (16) as representing an expansion process for space, instead of the expansion process of $\mathscr{U}$. At this point, with a suitable redefinition of time, that is by introducing *cosmic time* (which for the particles of the fluid $\mathscr{U}$, i.e. for the galaxies, does not differ substantially from the universal time of Newtonian mechanics), if we take into account the property of the speed of light being independent of its source, we can deduce, in the present case (i.e. that physical space is Euclidean and expanding) the several well known results of cosmography concerning the propagation of light, such as the relation between red-shift and expansion of the Universe, the problem of horizons, etc. The results obtained are the same as those obtained by operating within the framework of the general theory of relativity[23].

If we consider time as a fourth coordinate added to the three spatial coordinates, we obtain in this way a differential manifold (the s*pace-time* manifold) which can be made Riemannian by introducing into it the metric that follows

[22] Obviously, because of the arbitrary way in which *O* has been chosen, such ideal spaces with respect to which extragalactic celestial mechanics is fully valid are infinite, and consist of the non-expanding Euclidean spaces fixed to the natural frames of reference, each of which is accelerated with respect to the others.
[23] The results mentioned here will be presented in detail in forthcoming papers.

from the condition that the above property of light is verified[24] and furthermore that its sections *t=const*. are, as must be the case, the original Euclidean space (which now, as already said, is regarded as expanding). This metric is the following:

$$ds^2 = R^2(t)\sum_1^3{}_i \left(dy^i\right)^2 - c^2dt^2 , \tag{19}$$

where the spatial coordinates y^i (*co-moving coordinates*) are, as has been mentioned in note 24, rectangular Cartesian coordinates of the Euclidean space regarded as expanding and considered at the instant t_0 , an instant that has been fixed once and for all.

Equations (17) and (18) express that there is a link between the geometry of the space-time manifold characterised by metric (19) and the matter of the Universe, represented by the density $\mu(t)$. At this point we can formulate the following question: in the light of metric (19), what can be the tensor form of equations (17) and (18) that, as we have said before, are the same as those that would be obtained by operating within the framework of the general theory of relativity?

The answer to this question is one and only one and is given by:

XII. There exists one and only one possible tensor form for the above-mentioned equations that is compatible with metric (19). This tensor form is given by the

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[24] Taking into account (16), this condition is expressed by

$$R^2(t)\sum_1^3{}_i \left(dy^i\right)^2 = c^2dt^2 \quad ,$$

where y^i are rectangular Cartesian coordinates of the Euclidean space regarded as expanding and considered at the instant t_0. This equation becomes, in the case of a photon that propagates according to the direction *OP*:

$$d\bar{r} = \frac{c}{R(t)}dt \quad , \tag{a}$$

where $\bar{r}$ is the radial coordinate, etc.

From (a) easily follow the well known results of cosmography mentioned above.

gravitational equations of the general theory of relativity, i.e. of the Einstein theory of gravitation[25].

With the assumption made at the beginning of this section and with such results we can say:

XIII. The model of the Universe that in this lecture has been constructed by starting from assumptions 1, 2, 3 and 4 does not differ from the homogeneous and isotropic relativistic model of the Universe with zero spatial curvature.

This model, already mentioned in result XI, is the celebrated *Einstein-de Sitter model of the Universe*, introduced by Einstein and de Sitter in 1932[26].

It is this model which, for its simplicity, is quite often used by astronomers. It is often called the *standard model* of the Universe.

11. The content of this lecture is totally different, both for how it has been developed and for assumptions 1, 2, 3 and 4 which are its starting point, from the so-called "*Newtonian cosmology*", developed in 1934 by Milne and McCrea[27], the bases of which have in a certain way contributed to keep alive and widespread the conviction that the gravitational paradox, mentioned in Sec. 7, is well founded.

In fact Milne and McCrea assume as the starting points for their theory, in addition to assumptions 1, 2 and 3, that the frame of reference $\mathfrak{R}_O$ they consider is inertial, that with respect to it every particle P of the cosmological fluid $\mathcal{U}$ moves radially at a velocity that depends only on distance $r=|OP|$, that the part

[25] See [5], 12,13,14,15 and, in particular, [9].
[26] See [1].
[27] See [13], [12].

of $\mathcal{U}$ external to the material sphere $\mathcal{S}_r$ with centre O and radius r gives no contribution to the motion of P [28], that the forces that are exerted between the particles of the fluid $\mathcal{U}$ are expressed by Newton's law of gravitation.

These assumptions (which are extremely strong if compared to assumptions 1, 2, 3 and 4 made here) become theorems when examined in the light of what has been developed in this lecture.

Moreover, it should be said that in Newtonian cosmology, as it has so far been considered, it has not been proved at all that the energy constant α is zero. Therefore in Newtonian cosmology scholars took into account the case $\alpha \neq 0$ [29], with results that, bearing in mind what we have seen in Sec. 6, are not compatible with the assumption that physical space is Euclidean.

Therefore we can conclude, obviously taking into account what has been observed above, that the considerations developed in this lecture provides (within the framework of classical mechanics) Newtonian cosmology with that validity that until today has been denied to it.

12. The considerations developed in this paper, with suitable adaptations, can be extended to the case in which physical space, instead of being Euclidean, has the geometrical features of a spherical hypersurface (as already mentioned at the end of Sec. 8) or of a pseudospherical hypersurface. As has already been said at the end of Sec. 8 for the hyperspherical case, the corresponding models of the Universe can be built starting only from the principles of classical mechanics and from the cosmological principle and then taking account of the property of the speed of light being independent of its source[30].

[28] As far as this assumption is concerned, it is a general firm belief that it can be proved working only within the corpus of the general theory of relativity. See, for instance, [15], p. 475.
[29] See [12].
[30] See [8] for the case in which the curvature is positive. A general study, including all possible cases, namely the cases in which curvature is positive, negative or zero, will be presented in forthcoming papers.

Thus, in this way, in addition to the Einstein-de Sitter model, we obtain the celebrated homogeneous and isotropic relativistic cosmological models with non-zero spatial curvature, considered (conforming to how the Universe is today) in the absence of pressure, models that usually are called *Friedmann's models of the Universe*. For such models too, we can prove that there exists one and only one tensor form for the dynamical equations obtained for them that is compatible with the metric obtained for the corresponding space-time manifolds (which turns out to be the well known Robertson-Walker metric), a form that is again provided by Einstein's equations of the general theory of relativity.

Extragalactic celestial mechanics developed as outlined in this lecture, at this point becomes as a chapter of cosmology – especially if we bear in mind the property (that has been mentioned several times) of the speed of light being independent of its source. More precisely, the chapter regarding the "matter-dominated era", with all the indetermination and uncertainties to be found in cosmology, due to the inaccuracy, at times still very great, in determining the various astronomical parameters (the present values of $h(t)$, $\mu(t)$, distances, the so-called deceleration parameter, the density parameter $\Omega(t)$, etc.). These uncertainties as yet do not even allow us to say what is the geometrical structure of physical space, let alone whether it is Euclidean, as we have assumed in this lecture, following what is done in classical celestial mechanics. By making this assumption, we naturally ignore the case of phenomena on a "very small" scale, such as anomalies in the behaviour of the perihelion of the planets, particularly Mercury, anomalies that can be only explained within the framework of the general theory of relativity, namely of Einstein's theory of gravitation.

However, on the other hand, the large-scale study of the Universe too, performed within the framework of classical mechanics, leads, as we have seen or said, to this theory. But it must be pointed out that in the theory exposed in this lecture Einstein's gravitational equations are the arrival point and not the starting point. Thus it is in this sense, with the Einstein-de Sitter model and the

Friedmann models built as indicated above, that cosmology, developed in this way, provides a great confirmation of the validity of Einstein's theory of gravitation.

REFERENCES

[1] EINSTEIN A., de SITTER W., *On the Relation between the Expansion and the Mean Density of the Universe*, Proc. Nat. Acad. Sci., **18**, pp. 213-214 (1932).

[2] FRENCH A.P., *Newtonian Mechanics*, Norton & Company, New York, 1971.

[3] GALLETTO D., BARBERIS B., *Further Remarks on a Paper by P.T.Landsberg*, Atti Accad. Sci. Torino, **112**, pp. 103-107 (1978).

[4] GALLETTO D., *Sui fondamenti della meccanica classica, della teoria newtoniana della gravitazione e della cosmologia newtoniana*, in: «Atti del 3° Convegno Nazionale di Relatività Generale e Fisica della Gravitazione, Torino, 18-21 settembre 1978», Accademia delle Scienze, Torino, 1981; pp. 111-157.

[5] GALLETTO D., *Des principes de la mécanique classique aux théories de la gravitation de Newton et Einstein*, in: «Atti del Convegno Internazionale "Aspetti matematici della teoria della relatività", Roma, 5-6 giugno 1980», Accademia Nazionale dei Lincei, Roma, 1983; pp. 59-83.

[6] GALLETTO D., BARBERIS B., *Considerazioni sul paradosso newtoniano*, in: «Atti del Convegno celebrativo dell'80° anniversario della nascita di Renato Calapso, Messina-Taormina, 1-4 aprile 1981», Vieschi, Roma, 1983; pp. 153-164.

[7] GALLETTO D., BARBERIS B., *Extragalactic Celestial Mechanics: the Equation of Evolution*, in: «Bhatnagar K.B. (ed.), "Space Dynamics and Celestial Mechanics", Proceedings of "The International Workshop on Space Dynamics and Celestial Mechanics", Delhi, November 14-16, 1985», Reidel, Dordrecht, 1986; pp. 203-219.

[8] GALLETTO D., BARBERIS B., *Fondamenti di meccanica celeste extragalattica e costruzione dei modelli relativistici d'universo omogenei e isotropi*, in: «AA.VV., "Formalismo matematico e realtà fisica", Atti del Convegno di studi in ricordo di Antonio Pignedoli, Cesena, 19-20 marzo 1993», Accademia Nazionale di Scienze Lettere e Arti, Modena, 1996; pp. 25-80.

[9] GALLETTO D., BARBERIS B., *Remarks on the Propagation of Light in the Universe*, in: AA.VV., «Proceedings of the International Meeting "New Trends in Mathematical Physics" in Honour of the Salvatore Rionero 70th Birthday», Napoli, January, 24-25, 2003», World Scientific, Singapore, 2004; pp. 54-67.

[10] GALLETTO D., BARBERIS B., *Meccanica celeste extragalattica: cinematica*, Istituto Lombardo (Rend. Sc.), A **138**, pp. 69-84 (2004).

[11] LANDSBERG P.T., *A Deduction of the Inverse Square Law from Newtonian Cosmology*, Nature phys. Sci., **244**, pp. 66-67 (1973).

[12] McCREA W.H., MILNE E.A., *Newtonian Universes and the Curvature of Space*, Quart. J. Math. (Oxford Ser.), **5**, pp. 73-80 (1934).

[13] MILNE E.A., *A Newtonian Expanding Universe*, Quart. J. Math. (Oxford Ser.), **5**, pp. 64-72 (1934).

[14] TAFF L.C., *Celestial Mechanics*, John Wiley & Sons, New York, 1985.

[15] WEINBERG S., *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, John Wiley & Sons, New York, 1972.