A CALCULUS OF FOURIER INTEGRAL OPERATORS WITH INHOMOGENEOUS PHASE FUNCTIONS ON $\mathbf{R}^d$

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Abstract. We construct a calculus for generalized SG Fourier integral operators, extending known results to a broader class of symbols of SG type. In particular, we do not require that the phase functions are homogeneous. An essential ingredient in the proofs is a general criterion for asymptotic expansions within the Weyl-Hörmander calculus. We also prove the $L^2(\mathbf{R}^d)$-boundedness of the generalized SG Fourier integral operators having regular phase functions and amplitudes uniformly bounded on $\mathbf{R}^{2d}$.

Contents

0. Introduction

The aim of this paper is to extend the calculus of Fourier integral operators based on the so-called *SG* symbol classes, originally studied by S. Coriasco [9], to the more general setting of generalized SG symbols introduced in [11] by S. Coriasco, K. Johansson and J. Toft.

Explicitly, for every $m, \mu \in \mathbf{R}$, the standard class $\mathrm{SG}^{m,\mu}(\mathbf{R}^d)$ of SG symbols, are functions $a(x,\xi) \in C^\infty(\mathbf{R}^d \times \mathbf{R}^d)$ with the property that, for any multiindices $\alpha \in \mathbf{N}^d$ and $\beta \in \mathbf{N}^d$, there exist constants $C_{\alpha\beta} > 0$ such that

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \le C_{\alpha\beta} \langle x \rangle^{m-|\alpha|} \langle \xi \rangle^{\mu-|\beta|}, \qquad x, \xi \in \mathbf{R}^d \times \mathbf{R}^d. \tag{0.1}$$

hold. Here $\langle x \rangle = (1+|x|^2)^{1/2}$ when $x \in \mathbf{R}^d$ and $\mathbf{N}$ is the set of natural numbers. These classes together with corresponding classes of pseudo-differential operators $\mathrm{Op}(\mathrm{SG}^{m,\mu})$, were first introduced in the '70s by

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H.O. Cordes [8] and C. Parenti [24]. See also R. Melrose [23]. They form a graded algebra, i. e.,

$$\mathrm{Op}(\mathrm{SG}^{m_1,\mu_1}) \circ \mathrm{Op}(\mathrm{SG}^{m_2,\mu_2}) \subseteq \mathrm{Op}(\mathrm{SG}^{m_1+m_2,\mu_1+\mu_2}),$$

whose residual elements are operators with symbols in

$$\mathrm{SG}^{-\infty,-\infty}(\mathbf{R}^d) = \bigcap_{(m,\mu)\in\mathbf{R}^2} \mathrm{SG}^{m,\mu}(\mathbf{R}^d) = \mathscr{S}(\mathbf{R}^{2d}),$$

that is, those having kernel in $\mathscr{S}(\mathbf{R}^{2d})$, continuously mapping $\mathscr{S}'(\mathbf{R}^d)$ to $\mathscr{S}(\mathbf{R}^d)$.

Operators in $\mathrm{Op}(\mathrm{SG}^{m,\mu})$ are continuous on $\mathscr{S}(\mathbf{R}^d)$, and extend uniquely to continuous operators on $\mathscr{S}'(\mathbf{R}^d)$ and from $H^{s,\sigma}(\mathbf{R}^d)$ to $H^{s-m,\sigma-\mu}(\mathbf{R}^d)$. Here $H^{t,\tau}(\mathbf{R}^d)$, $t,\tau\in\mathbf{R}$, denotes the weighted Sobolev space

$$H^{t,\tau}(\mathbf{R}^d) = \{u\in\mathscr{S}'(\mathbf{R}^d)\colon \|u\|_{t,\tau} = \|\langle\cdot\rangle^t\langle D\rangle^\tau u\|_{L^2} < \infty\},$$

An operator $A=\mathrm{Op}(a)$, is called *elliptic* (or $\mathrm{SG}^{m,\mu}$-*elliptic*) if $a\in \mathrm{SG}^{m,\mu}(\mathbf{R}^d)$ and there exists $R\geq 0$ such that

$$C\langle x\rangle^m\langle\xi\rangle^\mu \leq |a(x,\xi)|, \qquad |x|+|\xi|\geq R,$$

for some constant $C>0$.

An elliptic SG operator $A\in\mathrm{Op}(\mathrm{SG}^{m,\mu})$ admits a parametrix $P\in \mathrm{Op}(\mathrm{SG}^{-m,-\mu})$ such that

$$PA = I + K_1, \quad AP = I + K_2,$$

for suitable K_1, K_2, smoothing operators with symbols in $\mathrm{SG}^{-\infty,-\infty}(\mathbf{R}^d)$, and it turns out to be a Fredholm operator on the scale of functional spaces $H^{t,\tau}(\mathbf{R}^d)$, $t,\tau\in\mathbf{R}$.

In 1987, E. Schrohe [26] introduced a class of non-compact manifolds, the so-called SG manifolds, on which it is possible to transfer from $\mathbf{R}^d$ the whole SG calculus. These are manifolds which admit a finite atlas whose changes of coordinates behave like symbols of order $(0,1)$ (see [26] for details and additional technical hypotheses). An especially interesting example of SG manifolds are the manifolds with cylindrical ends, where also the concept of classical SG operator makes sense, see, e. g. [18, 23].

The calculus of corresponding classes of Fourier integral operators, in the forms

$$f\mapsto (\mathrm{Op}_\varphi(a)f)(x) = (2\pi)^{-d}\int_{\mathbf{R}^d} e^{i\varphi(x,\xi)}a(x,\xi)\widehat{f}(\xi)\,d\xi,$$

and

$$f\mapsto (\mathrm{Op}_\varphi(a)^*f)(x) = (2\pi)^{-d}\iint_{\mathbf{R}^{2d}} e^{i(\langle x,\xi\rangle-\varphi(y,\xi))}\overline{a(y,\xi)}f(y)\,dyd\xi,$$

$f\in\mathscr{S}(\mathbf{R}^d)$, started in [9]. Here the operators $\mathrm{Op}_\varphi(a)$ and $\mathrm{Op}^*_\varphi(a) = \mathrm{Op}_\varphi(a)^*$ are called Fourier integral operators of type I and type II, respectively, with amplitude a and phase function φ. Note that the type II operator $\operatorname{Op}_\varphi(a)^*$ is the formal L^2-adjoint of the type I operator $\operatorname{Op}_\varphi(a)$.

We assume that the phase function φ belongs $\mathrm{SG}^{1,1}(\mathbf{R}^d)$ and satisfy

$$\langle \varphi'_x(x,\xi)\rangle \asymp \langle \xi\rangle \quad \text{and} \quad \langle \varphi'_\xi(x,\xi)\rangle \asymp \langle x\rangle,$$

if nothing else is stated. Here and in what follows, $A \asymp B$ means that $A \lesssim B$ and $B \lesssim A$, where $A \lesssim B$ means that $A \le c\cdot B$, for a suitable constant $c>0$. In many cases, especially when studying the mapping properties of such operators, φ should also fulfill the usual (global) non-degeneracy condition

$$|\det(\varphi''_{x\xi}(x,\xi))| \ge c, \qquad x,\xi\in\mathbf{R}^d,$$

for some constant $c>0$. The calculus developed in [9] has been first applied to the analysis of the well-posedness, in the scale of weighted spaces $H^{t,\tau}(\mathbf{R}^d)$, of certain hyperbolic Cauchy problems. These involved linear operators whose coefficients have, at most, polynomial growth at infinity, and was studied in [10, 14].

The analysis of such Fourier integral operators subsequently developed into an interesting field of active research, with extensions in many different directions. For example, an approach involving more general phase functions compared to [9, 10] can be found in [1] by Andrews. In [5] Cappiello and Rodino deduce results involving Gelfand-Shilov spaces, and in [6, 7], boundedness on $\mathscr{F}L^p(\mathbf{R}^d)_{\mathrm{comp}}$ and the modulation spaces are obtained. The $L^p(\mathbf{R}^d)$-continuity of the above operators is studied in [15], extending to the global $\mathbf{R}^d$ situation a celebrated result by Seeger, Sogge and Stein in [27], valid on compact manifolds.

More general SG symbol classes, denoted by $\mathrm{SG}^{(\omega)}_{r,\rho}(\mathbf{R}^d)$, $r,\rho\ge 0$, $r+\rho>0$, have been introduced in the aforementioned paper [11]. In place of the estimates (0.1), $a\in \mathrm{SG}^{(\omega)}_{r,\rho}(\mathbf{R}^d)$ satisfies

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \le C_{\alpha\beta}\,\omega(x,\xi)\langle x\rangle^{-r|\alpha|}\langle \xi\rangle^{-\rho|\beta|} \tag{0.2}$$

for suitable weight ω and constants $C_{\alpha\beta}>0$, see Subsections 1.1 and 1.2 below. For the corresponding pseudo-differential operators, continuity results and the propagation of singularities, in terms of global wave-front sets are established in [11]. (See also [13, 23] for related results.) These generalized SG symbol classes are well suited when investigating singularities in the context of modulation and Fourier-Lebesgue spaces. (See [19–21] for details on these functional spaces.)

In Section 2 we extend the calculus developed in [9] to include operators $\operatorname{Op}_\varphi(a)$ and $\operatorname{Op}^*_\varphi(a)$ with phase functions in $\mathrm{SG}^{1,1}_{1,1}(\mathbf{R}^d)$ and amplitudes in the generalized SG classes (0.2). More precisely, in the first part of Section 2 we prove that for every $a,b\in \mathrm{SG}^{(\omega_0)}$ and $p\in \mathrm{SG}^{(\omega)}$

we have

$$\begin{aligned}
\operatorname{Op}(p) \circ \operatorname{Op}_\varphi(a) &= \operatorname{Op}_\varphi(c_1) \mod \operatorname{Op}(\mathcal{B}_0),\\
\operatorname{Op}(p) \circ \operatorname{Op}^*_\varphi(b) &= \operatorname{Op}^*_\varphi(c_2) \mod \operatorname{Op}(\mathcal{B}_0),\\
\operatorname{Op}_\varphi(a) \circ \operatorname{Op}(p) &= \operatorname{Op}_\varphi(c_3) \mod \operatorname{Op}(\mathcal{B}_0)\\
\operatorname{Op}^*_\varphi(b) \circ \operatorname{Op}(p) &= \operatorname{Op}^*_\varphi(c_4) \mod \operatorname{Op}(\mathcal{B}_0),
\end{aligned}$$

for some $c_j \in \mathrm{SG}^{(\omega_{0,j})}$, $j = 1, \dots, 4$, and suitable weights $\omega_{0,j}$. Here $\operatorname{Op}(\mathcal{B}_0)$ is a set of appropriate smoothing operators, depending on the symbols and the phase function. Furthermore, if $a \in \mathrm{SG}^{(\omega_1)}$ and $b \in \mathrm{SG}^{(\omega_2)}$, then it is also proved that $\operatorname{Op}^*_\varphi(b) \circ \operatorname{Op}_\varphi(a)$ and $\operatorname{Op}_\varphi(a) \circ \operatorname{Op}^*_\varphi(b)$ are equal to pseudo-differential operators $\operatorname{Op}(c_5)$ and $\operatorname{Op}(c_6)$, respectively, for some $c_5, c_6 \in \mathrm{SG}^{(\omega_{0,j})}$, $j = 5, 6$. We also present asymptotic formulae for c_j, $j = 1, \dots, 6$, in terms of a and b, or of a, b and p, modulo smoothing terms, with symbol which in most cases belong to $\mathrm{SG}^{-\infty,-\infty} = \mathscr{S}$.

The results shown in this paper are an essential part of the study of the propagation of singularities, in the context of general modulation spaces, from the data to the solutions of the Cauchy problems considered in [10, 13, 14]. Another application of the calculus developed here has been the proof of boundedness results between suitable couples of weighted modulation spaces for the class of Fourier integral operators studied here. Both these applications are examined in [12].

The paper is organized as follows. In Section 1 we recall the needed definitions and some basic results concerning the generalized SG symbol classes. In Section 2 we give the definition of the generalized SG Fourier integral operators, and prove our main results, i. e., the composition theorems between generalized pseudo-differential and generalized Fourier integral operators of SG type, as well as between the Fourier integral operators. The parametrices for the elliptic elements are also studied, together with an adapted version of the Egorov theorem. In Section 3 we discuss the global $L^2(\mathbf{R}^d)$-boundedness of the generalized SG Fourier integral operators under the hypotheses that the phase function is regular, see Subsection 2.1 below, and the amplitude is uniformly bounded on $\mathbf{R}^{2d}$.

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1. Preliminaries

We begin by fixing the notation and recalling some basic concepts which will be needed below. In Subsections 1.1-1.2 we mainly summarizes part of the contents of Sections 2 in [11]. In Subsection 1.3 we state a few lemmas which will be useful in the subsequent Section 2. Some of these, compared with their original formulation in the SG context, appeared in [9], are here given in a more general form, adapted to the definitions given in Subsection 1.2.

1.1. **Weight functions.** Let ω and v be positive measurable functions on $\mathbf{R}^d$. Then ω is called v-moderate if

$$\omega(x+y) \lesssim \omega(x)v(y) \tag{1.1}$$

If v in (1.1) can be chosen as a polynomial, then ω is called a function or weight of *polynomial type*. We let $\mathscr{P}(\mathbf{R}^d)$ be the set of all polynomial type functions on $\mathbf{R}^d$. If $\omega(x,\xi) \in \mathscr{P}(\mathbf{R}^{2d})$ is constant with respect to the x-variable or the ξ-variable, then we sometimes write $\omega(\xi)$, respectively $\omega(x)$, instead of $\omega(x,\xi)$, and consider ω as an element in $\mathscr{P}(\mathbf{R}^{2d})$ or in $\mathscr{P}(\mathbf{R}^d)$ depending on the situation. We say that v is submultiplicative if (1.1) holds for $\omega = v$. For convenience we assume that all submultiplicative weights are even, and v and v_j always stand for submultiplicative weights, if nothing else is stated.

Without loss of generality we may assume that every $\omega \in \mathscr{P}(\mathbf{R}^d)$ is smooth and satisfies the ellipticity condition $\partial^\alpha \omega/\omega \in L^\infty$. In fact, by Lemma 1.2 in [28] it follows that for each $\omega \in \mathscr{P}(\mathbf{R}^d)$, there is a smooth and elliptic $\omega_0 \in \mathscr{P}(\mathbf{R}^d)$ which is equivalent to ω in the sense

$$\omega \asymp \omega_0.$$

The weights involved in the sequel have to satisfy additional conditions. More precisely let $r, \rho \geq 0$. Then $\mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ is the set of all $\omega(x,\xi)$ in $\mathscr{P}(\mathbf{R}^{2d}) \bigcap C^\infty(\mathbf{R}^{2d})$ such that

$$\langle x\rangle^{r|\alpha|}\langle \xi\rangle^{\rho|\beta|}\frac{\partial_x^\alpha \partial_\xi^\beta \omega(x,\xi)}{\omega(x,\xi)} \in L^\infty(\mathbf{R}^{2d}), \tag{1.2}$$

for every multi-indices α and β. Any weight $\omega \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ is then called SG moderate on $\mathbf{R}^{2d}$, of order r and ρ. Note that there are elements in $\mathscr{P}(\mathbf{R}^{2d})$ which have no equivalent elements in $\mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$. On the other hand, if $s, t \in \mathbf{R}$ and $r, \rho \in [0,1]$, then $\mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ contains all weights of the form

$$\vartheta_{m,\mu}(x,\xi) \equiv \langle x\rangle^m \langle \xi\rangle^\mu, \tag{1.3}$$

which are one of the most common type of weights.

It will also be useful to consider SG moderate weights in one or three sets of variables. Let $\omega \in \mathscr{P}(\mathbf{R}^{3d}) \bigcap C^\infty(\mathbf{R}^{3d})$, and let $r_1, r_2, \rho \geq 0$.

Then ω is called SG moderate on $\mathbf{R}^{3d}$, of order r_1, r_2 and ρ, if it fulfills

$$\langle x_1\rangle^{r_1|\alpha_1|}\langle x_2\rangle^{r_2|\alpha_2|}\langle \xi\rangle^{\rho|\beta|}\frac{\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_\xi^\beta \omega(x_1,x_2,\xi)}{\omega(x_1,x_2,\xi)} \in L^\infty(\mathbf{R}^{3d}).$$

The set of all SG moderate weights on $\mathbf{R}^{3d}$ of order r_1, r_2 and ρ is denoted by $\mathscr{P}_{r_1,r_2,\rho}(\mathbf{R}^{3d})$. Finally, we denote by $\mathscr{P}_r(\mathbf{R}^d)$ the set of all SG moderate weights of order $r\ge 0$ on $\mathbf{R}^d$, which are defined in a similar fashion.

1.2. **Pseudo-differential operators and generalized** SG **symbol classes.** Let $a\in\mathscr{S}(\mathbf{R}^{2d})$, and $t\in\mathbf{R}$ be fixed. Then the pseudo-differential operator $\operatorname{Op}_t(a)$ is the linear and continuous operator on $\mathscr{S}(\mathbf{R}^d)$ defined by the formula

$$(\operatorname{Op}_t(a)f)(x) = (2\pi)^{-d}\iint e^{i\langle x-y,\xi\rangle}a((1-t)x+ty,\xi)f(y)\,dyd\xi \tag{1.4}$$

(cf. Chapter XVIII in [22]). For general $a\in\mathscr{S}'(\mathbf{R}^{2d})$, the pseudo-differential operator $\operatorname{Op}_t(a)$ is defined as the continuous operator from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$ with distribution kernel

$$K_{t,a}(x,y) = (2\pi)^{-d/2}(\mathscr{F}_2^{-1}a)((1-t)x+ty,x-y). \tag{1.5}$$

Here and in what follows, $\mathscr{F}f=\widehat f$ is the Fourier transform of $f\in\mathscr{S}'(\mathbf{R}^d)$ which takes the form

$$(\mathscr{F}f)(\xi)=\widehat f(\xi) = (2\pi)^{-d/2}\int_{\mathbf{R}^d} f(x)e^{-i\langle x,\xi\rangle}\,dx$$

when $f\in\mathscr{S}(\mathbf{R}^d)$, and $\mathscr{F}_2F$ is the partial Fourier transform of $(x,y)\mapsto F(x,y)$ with respect to the y-variable.

If $t=0$, then $\operatorname{Op}_t(a)$ is the Kohn-Nirenberg representation $\operatorname{Op}(a)=a(x,D)$, and if $t=1/2$, then $\operatorname{Op}_t(a)$ is the Weyl quantization.

In most of our situations, a belongs to a generalized SG symbol class, which we shall consider now. Let $m,\mu,r,\rho\in\mathbf{R}$ be fixed. Then the SG class $\mathrm{SG}^{m,\mu}_{r,\rho}(\mathbf{R}^{2d})$ is the set of all $a\in C^\infty(\mathbf{R}^{2d})$ such that

$$|D_x^\alpha D_\xi^\beta a(x,\xi)|\lesssim \langle x\rangle^{m-r|\alpha|}\langle\xi\rangle^{\mu-\rho|\beta|},$$

for all multi-indices α and β. Usually we assume that $r,\rho\ge 0$ and $\rho+r>0$.

More generally, assume that $\omega\in\mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$. Then $\mathrm{SG}^{(\omega)}_{r,\rho}(\mathbf{R}^{2d})$ consists of all $a\in C^\infty(\mathbf{R}^{2d})$ such that

$$|D_x^\alpha D_\xi^\beta a(x,\xi)|\lesssim \omega(x,\xi)\langle x\rangle^{-r|\alpha|}\langle\xi\rangle^{-\rho|\beta|},\qquad x,\xi\in\mathbf{R}^d, \tag{1.6}$$

for all multi-indices α and β. We note that

$$\mathrm{SG}^{(\omega)}_{r,\rho}(\mathbf{R}^{2d}) = S(\omega,g_{r,\rho}), \tag{1.7}$$

when $g=g_{r,\rho}$ is the Riemannian metric on $\mathbf{R}^{2d}$, defined by the formula

$$\big(g_{r,\rho}\big)_{(y,\eta)}(x,\xi) = \langle y\rangle^{-2r}|x|^2+\langle\eta\rangle^{-2\rho}|\xi|^2 \tag{1.8}$$

(cf. Section 18.4–18.6 in [22]). Furthermore, $\mathrm{SG}^{(\omega)}_{r,\rho} = \mathrm{SG}^{m,\mu}_{r,\rho}$ when ω coincides with the weight $\vartheta_{m,\mu}$ defined in (1.3).

For conveniency we set

$$\mathrm{SG}^{(\omega\vartheta_{-\infty,0})}_{\rho}(\mathbf{R}^{2d}) = \mathrm{SG}^{(\omega\vartheta_{-\infty,0})}_{r,\rho}(\mathbf{R}^{2d}) \equiv \bigcap_{N\geq 0} \mathrm{SG}^{(\omega\vartheta_{-N,0})}_{r,\rho}(\mathbf{R}^{2d}),$$

$$\mathrm{SG}^{(\omega\vartheta_{0,-\infty})}_{r}(\mathbf{R}^{2d}) = \mathrm{SG}^{(\omega\vartheta_{0,-\infty})}_{r,\rho}(\mathbf{R}^{2d}) \equiv \bigcap_{N\geq 0} \mathrm{SG}^{(\omega\vartheta_{0,-N})}_{r,\rho}(\mathbf{R}^{2d}),$$

and

$$\mathrm{SG}^{(\omega\vartheta_{-\infty,-\infty})}(\mathbf{R}^{2d}) = \mathrm{SG}^{(\omega\vartheta_{-\infty,-\infty})}_{r,\rho}(\mathbf{R}^{2d}) \equiv \bigcap_{N\geq 0} \mathrm{SG}^{(\omega\vartheta_{-N,-N})}_{r,\rho}(\mathbf{R}^{2d}).$$

We observe that $\mathrm{SG}^{(\omega\vartheta_{-\infty,0})}_{r,\rho}(\mathbf{R}^{2d})$ is independent of r, $\mathrm{SG}^{(\omega\vartheta_{0,-\infty})}_{r,\rho}(\mathbf{R}^{2d})$ is independent of ρ, and that $\mathrm{SG}^{(\omega\vartheta_{-\infty,-\infty})}_{r,\rho}(\mathbf{R}^{2d})$ is independent of both r and ρ. Furthermore, for any $x_0,\xi_0\in\mathbf{R}^d$ we have

$$\mathrm{SG}^{(\omega\vartheta_{-\infty,0})}_{\rho}(\mathbf{R}^{2d}) = \mathrm{SG}^{(\omega_0\vartheta_{-\infty,0})}_{\rho}(\mathbf{R}^{2d}), \quad \text{when} \quad \omega_0(\xi) = \omega(x_0,\xi),$$

$$\mathrm{SG}^{(\omega\vartheta_{0,-\infty})}_{r}(\mathbf{R}^{2d}) = \mathrm{SG}^{(\omega_0\vartheta_{0,-\infty})}_{r}(\mathbf{R}^{2d}), \quad \text{when} \quad \omega_0(x) = \omega(x,\xi_0),$$

and

$$\mathrm{SG}^{(\omega\vartheta_{-\infty,-\infty})}(\mathbf{R}^{2d}) = \mathscr{S}(\mathbf{R}^{2d}).$$

The following result shows that the concept of asymptotic expansion extends to the classes $\mathrm{SG}^{(\omega)}_{r,\rho}(\mathbf{R}^{2d})$. We refer to [17, Theorem 8] for the proof.

Proposition 1.1. *Let $r,\rho\geq 0$ satisfy $r+\rho>0$, and let $\{s_j\}_{j\geq 0}$ and $\{\sigma_j\}_{j\geq 0}$ be sequences of non-positive numbers such that $\lim_{j\to\infty} s_j = -\infty$ when $r>0$ and $s_j=0$ otherwise, and $\lim_{j\to\infty}\sigma_j=-\infty$ when $\rho>0$ and $\sigma_j=0$ otherwise. Also let $a_j\in\mathrm{SG}^{(\omega_j)}_{r,\rho}(\mathbf{R}^{2d})$, $j=0,1,\dots$, where $\omega_j=\omega\cdot\vartheta_{s_j,\sigma_j}$. Then there is a symbol $a\in\mathrm{SG}^{(\omega)}_{r,\rho}(\mathbf{R}^{2d})$ such that*

$$a - \sum_{j=0}^{N} a_j \in \mathrm{SG}^{(\omega_{N+1})}_{r,\rho}(\mathbf{R}^{2d}). \tag{1.9}$$

The symbol a is uniquely determined modulo a remainder h, where

$$\begin{aligned} &h\in\mathrm{SG}^{\omega\vartheta_{-\infty,0})}_{\rho}(\mathbf{R}^{2d}) && \text{when} \quad r>0,\\ &h\in\mathrm{SG}^{(\omega\vartheta_{0,-\infty})}_{r}(\mathbf{R}^{2d}) && \text{when} \quad \rho>0,\\ &h\in\mathscr{S}(\mathbf{R}^{2d}) && \text{when} \quad r>0,\rho>0. \end{aligned} \tag{1.10}$$

Definition 1.2. The notation $a \sim \sum a_j$ is used when a and a_j fulfill the hypothesis in Proposition 1.1. Furthermore, the formal sum

$$\sum_{j \geq 0} a_j$$

is called (*generalized* SG) *asymptotic expansion*.

It is a well-known fact that SG operators give rise to linear continuous mappings from $\mathscr{S}(\mathbf{R}^d)$ to itself, extendable as linear continuous mappings from $\mathscr{S}'(\mathbf{R}^d)$ to itself. They also act continuously between general weighted modulation spaces, see [11].

1.3. **Composition and further properties of** SG **classes of symbols, amplitudes, and functions.** We define families of *smooth functions with* SG *behaviour*, depending on one, two or three sets of real variables (cfr. also [16]). We then introduce pseudo-differential operators defined by means of SG amplitudes. Subsequently, we recall sufficient conditions for maps of $\mathbf{R}^d$ into itself to keep the invariance of the SG classes.

In analogy of SG amplitudes defined on $\mathbf{R}^{2d}$, we consider corresponding classes of amplitudes defined on $\mathbf{R}^{3d}$. More precisely, for any $m_1, m_2, \mu, r_1, r_2, \rho \in \mathbf{R}$, let $\mathrm{SG}^{m_1,m_2,\mu}_{r_1,r_2,\rho}(\mathbf{R}^{3n})$ be the set of all $a \in C^\infty\left(\mathbf{R}^{3d}\right)$ such that

$$|\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\beta}_{\xi} a(x_1,x_2,\xi)| \lesssim \langle x_1\rangle^{m_1-r_1|\alpha_1|}\langle x_2\rangle^{m_2-r_2|\alpha_2|}\langle \xi\rangle^{\mu-\rho|\beta|}, \qquad (1.11)$$

for every multi-indices $\alpha_1, \alpha_2, \beta$. We usually assume $r_1, r_2, \rho \geq 0$ and $r_1+r_2+\rho > 0$. More generally, let $\omega \in \mathscr{P}_{r_1,r_2,\rho}(\mathbf{R}^{3d})$. Then $\mathrm{SG}^{(\omega)}_{r_1,r_2,\rho}(\mathbf{R}^{3d})$ is the set of all $a \in C^\infty\left(\mathbf{R}^{3d}\right)$ which satisfy

$$|\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\beta}_{\xi} a(x,y,\xi)| \lesssim \omega(x_1,x_2,\xi)\langle x_1\rangle^{-r_1|\alpha_1|}\langle x_2\rangle^{-r_2|\alpha_2|}\langle \xi\rangle^{-\rho|\beta|}, \quad (1.11)'$$

for every multi-indices $\alpha_1, \alpha_2, \beta$. The set $\mathrm{SG}^{(\omega)}_{r_1,r_2,\rho}(\mathbf{R}^{3n})$ is equipped with the usual Fréchet topology based upon the seminorms implicit in $(1.11)'$.

As above,

$$\mathrm{SG}^{(\omega)}_{r_1,r_2,\rho} = \mathrm{SG}^{m_1,m_2,\mu}_{r_1,r_2,\rho} \quad \text{when} \quad \omega(x_1,x_2,\xi) = \langle x_1\rangle^{m_1}\langle x_2\rangle^{m_2}\langle\xi\rangle^{\mu}.$$

Definition 1.3. Let $r_1, r_2, \rho \geq 0$, $r_1 + r_2 + \rho > 0$, and let $a \in \mathrm{SG}^{(\omega)}_{r_1,r_2,\rho}(\mathbf{R}^{3d})$, where $\omega \in \mathscr{P}_{r_1,r_2,\rho}(\mathbf{R}^{3d})$. Then, the pseudo-differential operator $\mathrm{Op}(a)$ is the linear and continuous operator from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$ with distribution kernel

$$K_a(x,y) = (2\pi)^{-d/2}(\mathscr{F}_3^{-1}a)(x,y,x-y).$$

For $f \in \mathscr{S}(\mathbf{R}^d)$, we have

$$(\mathrm{Op}(a)f)(x) = (2\pi)^{-d}\iint e^{i\langle x-y,\xi\rangle}a(x,y,\xi)f(y)\,dyd\xi.$$

The operators introduced in Definition 1.3 have properties analogous to the usual SG operator families described in [8]. They coincide with the operators defined in the previous subsection, where corresponding symbols are obtained by means of asymptotic expansions, modulo remainders of the type given in (1.2). For the sake of brevity, we here omit the details. Evidently, when neither the amplitude functions a, nor the corresponding weight ω, depend on x_2, we obtain the definition of SG symbols and pseudo-differential operators, given in the previous subsection.

Next we consider SG *functions*, also called *functions with* SG *behavior*. That is, amplitudes which depend only on one set of variables in $\mathbf{R}^d$. We denote them by $\mathrm{SG}^{(\omega)}_r(\mathbf{R}^d)$ and $\mathrm{SG}^m_r(\mathbf{R}^d)$, $r > 0$, respectively, for a general weight $\omega \in \mathscr{P}_r(\mathbf{R}^d)$ and for $\omega(x) = \langle x\rangle^m$. Furthermore, if $\phi\colon \mathbf{R}^{d_1} \to \mathbf{R}^{d_2}$, and each component ϕ_j, $j = 1, \dots, d_2$, of ϕ belongs to $\mathrm{SG}^{(\omega)}_r(\mathbf{R}^{d_1})$, we will occasionally write $\phi \in \mathrm{SG}^{(\omega)}_r(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$. We use similar notation also for other vector-valued SG symbols and amplitudes.

In the sequel we will need to consider compositions of SG amplitudes with functions with SG behavior. In particular, the latter will often be SG maps (or diffeomorphisms) with SG^0-parameter dependence, generated by phase functions (introduced in [9]), see Definitions 1.4 and 1.4, and Subsection 2.1 below. For the convenience of the reader, we first recall, in a form slightly more general than the one adopted in [9], the definition SG diffeomorphisms with SG^0-parameter dependence.

Definition 1.4. Let $\Omega_j \subseteq \mathbf{R}^{d_j}$ be open, $\Omega = \Omega_1 \times \cdots \times \Omega_k$ and let $\phi \in C^\infty(\mathbf{R}^d \times \Omega; \mathbf{R}^d)$. Then ϕ is called an SG map (with SG^0-parameter dependence) when the following conditions hold:

(1) $\langle \phi(x,\eta)\rangle \asymp \langle x\rangle$, uniformly with respect to $\eta \in \Omega$;

(2) for all $\alpha \in \mathbf{Z}^d_+$, $\beta = (\beta_1, \dots, \beta_k)$, $\beta_j \in \mathbf{Z}^{d_j}_+$, $j = 1, \dots, k$, and any $(x,\eta) \in \mathbf{R}^d \times \Omega$,

$$|\partial^\alpha_x \partial^{\beta_1}_{\eta_1} \cdots \partial^{\beta_k}_{\eta_k} \phi(x,\eta)| \lesssim \langle x\rangle^{1-|\alpha|}\langle \eta_1\rangle^{-|\beta_1|} \cdots \langle \eta_k\rangle^{-|\beta_k|},$$

where $\eta = (\eta_1, \dots, \eta_k)$ and $\eta_j \in \Omega_j$ for every j.

Definition 1.5. Let $\phi \in C^\infty(\mathbf{R}^d \times \Omega; \mathbf{R}^d)$ be an SG map. Then ϕ is called an SG diffeomorphism (with SG^0-parameter dependence) when there is a constant $\varepsilon > 0$ such that

$$|\det \phi'_x(x,\eta)| \ge \varepsilon, \tag{1.12}$$

uniformly with respect to $\eta \in \Omega$.

Remark 1.6. Condition (1) in Definition 1.4 and (1.12), together with abstract results (see, e.g., [3], page 221) and the inverse function theorem, imply that, for any $\eta \in \Omega$, an SG diffeomorphism $\phi(\,\cdot\,,\eta)$ is a smooth, global bijection from $\mathbf{R}^d$ to itself with smooth inverse $\psi(\,\cdot\,,\eta) = \phi^{-1}(\,\cdot\,,\eta)$. It can be proved that also the inverse mapping $\psi(y,\eta) =$

$\phi^{-1}(y,\eta)$ fulfills Conditions (1) and (2) in Definition 1.4, as well as (1.12), see [9].

Definition 1.7. Let $r,\rho \geq 0$, $r+\rho > 0$, $\omega \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$, and let $\phi, \phi_1, \phi_2 \in C^\infty(\mathbf{R}^d \times \mathbf{R}^{d_0}; \mathbf{R}^d)$ be SG mappings.

(1) ω is called $(\phi, 1)$-*invariant* when

$$\omega(\phi(x,\eta_1+\eta_2),\xi) \lesssim \omega(\phi(x,\eta_1),\xi),$$

for any $x,\xi \in \mathbf{R}^d$, $\eta_1,\eta_2 \in \mathbf{R}^{d_0}$, uniformly with respect to $\eta_2 \in \mathbf{R}^{d_0}$. The set of all $(\phi,1)$-invariant weights in $\mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ is denoted by $\mathscr{P}^{\phi,1}_{r,\rho}(\mathbf{R}^{2d})$;

(2) ω is called $(\phi, 2)$-*invariant* when

$$\omega(x,\phi(\xi,\eta_1+\eta_2)) \lesssim \omega(x,\phi(\xi,\eta_1)),$$

for any $x,\xi \in \mathbf{R}^d$, $\eta_1,\eta_2 \in \mathbf{R}^{d_0}$, uniformly with respect to $\eta_2 \in \mathbf{R}^{d_0}$. The set of all $(\phi,2)$-invariant weights in $\mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ is denoted by $\mathscr{P}^{\phi,2}_{r,\rho}(\mathbf{R}^{2d})$;

(3) ω is called (ϕ_1,ϕ_2)-*invariant* if ω is both $(\phi_1,1)$-invariant and $(\phi_2,2)$-invariant. The set of all (ϕ_1,ϕ_2)-invariant weights in $\mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ is denoted by $\mathscr{P}^{(\phi_1,\phi_2)}_{r,\rho}(\mathbf{R}^{2d})$

We now show that, under mild additional conditions, the families of weights introduced in Subsection 1.1 are indeed "invariant" under composition with SG maps with SG^0-parameter dependence. That is, the compositions introduced in Definition 1.7 are still weight functions in the sense of Subsection 1.1, belonging to suitable sets $\mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$.

Lemma 1.8. *Let $r,\rho \in [0,1]$, $r+\rho > 0$, $\omega \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$, and let $\phi\colon \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}^d$ be an* SG *map as in Definition 1.4. The following statements hold true.*

(1) *Assume $\omega \in \mathscr{P}^{\phi,1}_{1,\rho}(\mathbf{R}^{2d})$, and set $\omega_1(x,\xi) := \omega(\phi(x,\xi),\xi)$. Then, $\omega_1 \in \mathscr{P}_{1,\rho}(\mathbf{R}^{2d})$.*

(2) *Assume $\omega \in \mathscr{P}^{\phi,2}_{r,1}(\mathbf{R}^{2d})$, and set $\omega_2(x,\xi) := \omega(x,\phi(\xi,x))$. Then, $\omega_2 \in \mathscr{P}_{r,1}(\mathbf{R}^{2d})$.*

Proof. We prove only the first statement, since the proof of the second one follows by a completely similar argument, exchanging the role of x and ξ.

It is obvious that $\omega_1 \in C^\infty(\mathbf{R}^d \times \mathbf{R}^d)$. The estimates (1.2) follows by Fàa di Bruno's formula (cf. [9]). Explicitly, for $|\alpha+\beta|>0$,

$$\langle x\rangle^{|\alpha|}\langle\xi\rangle^{\rho|\beta|}\partial_x^\alpha\partial_\xi^\beta\omega_1(x,\xi) = \langle x\rangle^{|\alpha|}\langle\xi\rangle^{\rho|\beta|}\partial_x^\alpha\partial_\xi^\beta(\omega(\phi(x,\xi),\xi))$$

belongs to the span of

$$\left\{ \langle x\rangle^{|\alpha|}\langle\xi\rangle^{\rho|\beta|}(\partial_x^{\gamma_0}\partial_\xi^{\delta_0}\omega)(\phi(x,\xi),\xi)\cdot\prod_{1\leq j\leq|\gamma_0|}\partial_x^{\gamma_j}\partial_\xi^{\delta_j}\phi(x,\xi) \,:\, \sum_{j\geq 1}\gamma_j=\alpha, \sum_{j\geq 0}\delta_j=\beta \right\}.$$

Denoting by $f_{\alpha\beta\gamma\delta}$, $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_{|\gamma_0|})$, $\delta = (\delta_1, \dots, \delta_{|\gamma_0|})$, the terms in braces above, in view of the hypotheses we have

$$
\begin{aligned}
&|f_{\alpha\beta\gamma\delta}(x,\xi)| \\
&\lesssim \langle x\rangle^{|\alpha|}\langle\xi\rangle^{\rho|\beta|}\cdot\omega(\phi(x,\xi),\xi)\langle\phi(x,\xi)\rangle^{-|\gamma_0|}\langle\xi\rangle^{-\rho|\delta_0|}\cdot\prod_{1\le j\le|\gamma_0|}\langle x\rangle^{1-|\gamma_j|}\langle\xi\rangle^{-|\delta_j|} \\
&\lesssim \omega(\phi(x,\xi),\xi)\cdot\langle x\rangle^{|\alpha|}\langle\xi\rangle^{\rho|\beta|}\cdot\langle x\rangle^{-|\gamma_0|}\langle\xi\rangle^{-\rho|\delta_0|}\cdot\langle x\rangle^{|\gamma_0|}\langle x\rangle^{-\sum_{j\ge1}|\gamma_j|}\langle\xi\rangle^{-\sum_{j\ge1}|\delta_j|} \\
&\qquad\qquad= \omega_1(x,\xi)\cdot\langle\xi\rangle^{\rho|\beta|}\cdot\langle\xi\rangle^{-\rho|\beta|} = \omega_1(x,\xi),
\end{aligned}
$$

which implies (1.2) with $r=1$, $\rho\in[0,1]$, $|\alpha+\beta|>0$. The estimate for $\alpha=\beta=0$ is trivial. Then, (1.2) holds true for ω_1 with $r=1$, $\rho\in[0,1]$, as claimed. It remains to prove (1.1). To this aim, observe that, by the moderateness of ω, using the properties of ϕ we find, for some polynomial v,

$$
\begin{aligned}
\omega_1(x+y,\xi+\eta) &= \omega(\phi(x+y,\xi+\eta),\xi+\eta) \\
&= \omega\left(\phi(x,\xi+\eta) + \overbrace{\int_0^1 \phi'_x(x+ty,\xi+\eta)\cdot y\,dt}^{=z}, \xi+\eta\right) \\
&\qquad\lesssim \omega(\phi(x,\xi+\eta),\xi)\,v(z,\eta).
\end{aligned}
$$

Since $|\phi'_x(x+ty,\xi+\eta)|\lesssim 1$ for any $x,y,\xi,\eta\in\mathbf{R}^d$, $t\in[0,1]$, so that $|z|\lesssim|y|$, we conclude, in view of the $(\phi,1)$-invariance of ω, that

$$
\begin{aligned}
\omega_1(x+y,\xi+\eta) &\lesssim \omega(\phi(x,\xi+\eta),\xi)\cdot\tilde v(y,\eta) \\
&\lesssim \omega(\phi(x,\xi),\xi)\,\tilde v(y,\eta) \lesssim \omega_1(x,\xi)\,\tilde v(y,\eta),
\end{aligned}
$$

for some other suitable polynomial $\tilde v$ and any $x,y,\xi,\eta\in\mathbf{R}^d$. The proof is complete. ☐

Remark 1.9. It is obvious that, when dealing with Fourier integral operators, the requirements for ϕ and ω in Lemma 1.8 need to be satisfied only on the support of the involved amplitude. By Lemma 1.8, it also follows that if $a\in \mathrm{SG}^{(\omega)}_{1,1}(\mathbf{R}^{2d})$ and $\phi=(\phi_1,\phi_2)$, where $\phi_1\in\mathrm{SG}^{1,0}_{1,1}(\mathbf{R}^{2d})$ and $\phi_2\in\mathrm{SG}^{0,1}_{1,1}(\mathbf{R}^{2d})$ are SG maps with SG^0 parameter dependence, then $a\circ\phi\in\mathrm{SG}^{(\omega_0)}_{1,1}(\mathbf{R}^{2d})$ when $\omega_0:=\omega\circ\phi$, provided ω is (ϕ_1,ϕ_2)-invariant. Similar results hold for SG amplitudes and weights defined on $\mathbf{R}^{3d}$.

Remark 1.10. By the definitions it follows that any weight $\omega=\vartheta_{s,\sigma}$, $s,\sigma\in\mathbf{R}$, is $(\phi,1)$-, $(\phi,2)$-, and (ϕ_1,ϕ_2)-invariant with respect to any SG diffeomorphism with SG^0 parameter dependence ϕ, (ϕ_1,ϕ_2).

We conclude the section by recalling the definition, taken from [9], of the sets of SG compatible cutoff and 0-excision functions, which we will use in the sequel. By a standard construction, it is easy to prove that the sets $\Xi^\Delta(k)$ and $\Xi(R)$ introduced in Definition 1.11 below are non-empty, for any $k, R > 0$.

Definition 1.11. The sets $\Xi^\Delta(k)$, $k > 0$, of the SG compatible cut-off functions along the diagonal of $\mathbf{R}^d \times \mathbf{R}^d$, consist of all $\chi = \chi(x, y) \in \mathrm{SG}^{0,0}_{1,1}(\mathbf{R}^{2d})$ such that

$$\begin{aligned} |y - x| \leq k\langle x\rangle/2 \quad &\Longrightarrow \quad \chi(x, y) = 1, \\ |y - x| > k\langle x\rangle \quad &\Longrightarrow \quad \chi(x, y) = 0. \end{aligned} \tag{1.13}$$

If not otherwise stated, we always assume $k \in (0, 1)$.
$\Xi(R)$ with $R > 0$ will instead denote the sets of all SG compatible 0-excision functions, namely, the set of all $\varsigma = \varsigma(x, \xi) \in \mathrm{SG}^{0,0}_{1,1}(\mathbf{R}^{2d})$ such that

$$\begin{aligned} |x| + |\xi| \geq R \quad &\Longrightarrow \quad \varsigma(x, \xi) = 1, \\ |x| + |\xi| \leq R/2 \quad &\Longrightarrow \quad \varsigma(x, \xi) = 0. \end{aligned} \tag{1.14}$$

2. Symbolic calculus for generalized FIOs of SG type

We here introduce the class of Fourier integral operators we are interested in, generalizing those studied in [9]. In particular, we show how a symbolic calculus can be developed for them. We examine their compositions with the generalized SG pseudo-differential operators introduced in [11], and the compositions between Type I and Type II operators. A key tool in the proofs of the composition results below are the results on asymptotic expansions in the Weyl-Hörmander calculus obtained in [17].

2.1. Phase functions of SG type.

We recall the definition of the class of admissible phase functions in the SG context, as it was given in [9]. We then observe that the subclass of *regular phase functions* generates (parameter-dependent) mappings of $\mathbf{R}^d$ onto itself, which turn out to be SG maps with SG^0 parameter-dependence. Finally, we define some *regularizing operators*, which are used to prove the properties of the SG Fourier integral operators introduced in the next subsection.

Definition 2.1. A real-valued function $\varphi \in \mathrm{SG}^{1,1}_{1,1}(\mathbf{R}^{2d})$ is called a *simple phase function* (or *simple phase*), if

$$\langle \varphi'_\xi(x, \xi)\rangle \asymp \langle x\rangle \text{ and } \langle \varphi'_x(x, \xi)\rangle \asymp \langle \xi\rangle, \tag{2.1}$$

are fulfilled, uniformly with respect to ξ and x, repectively. The set of all simple phase functions is denoted by $\mathfrak{F}$. Moreover, the simple phase function φ is called *regular*, if $\left|\det(\varphi''_{x\xi}(x, \xi))\right| \geq c$ for some $c > 0$ and all $x, \xi \in \mathbf{R}^d$. The set of all regular phases is denoted by $\mathfrak{F}^r$.

We observe that a regular phase function φ defines two globally invertible mappings, namely $\xi \mapsto \varphi'_x(x,\xi)$ and $x \mapsto \varphi'_\xi(x,\xi)$, see the analysis in [9]. Then, the following result holds true for the mappings ϕ_1 and ϕ_2 generated by the first derivatives of the admissible regular phase functions.

Proposition 2.2. *Let $\varphi \in \mathfrak{F}$. Then, for any $x_0, \xi_0 \in \mathbf{R}^d$, $\phi_1 \colon \mathbf{R}^d \to \mathbf{R}^d \colon x \mapsto \varphi'_\xi(x,\xi_0)$ and $\phi_2 \colon \mathbf{R}^d \to \mathbf{R}^d \colon \xi \mapsto \varphi'_x(x_0,\xi)$ are* SG *maps (with* SG^0 *parameter dependence), from $\mathbf{R}^d$ to itself. If $\varphi \in \mathfrak{F}^r$, ϕ_1 and ϕ_2 give rise to* SG *diffeomorphism with* SG^0 *parameter dependence.*

For any $\varphi \in \mathfrak{F}$, the operators $\Theta_{1,\varphi}$ and $\Theta_{2,\varphi}$ are defined by

$$(\Theta_{1,\varphi} f)(x,\xi) \equiv f(\varphi'_\xi(x,\xi),\xi) \quad \text{and} \quad (\Theta_{2,\varphi} f)(x,\xi) \equiv f(x,\varphi'_x(x,\xi)),$$

when $f \in C^1(\mathbf{R}^{2d})$, and remark that the modified weights

$$(\Theta_{1,\varphi}\omega)(x,\xi) = \omega(\varphi'_\xi(x,\xi),\xi) \quad \text{and} \quad (\Theta_{2,\varphi}\omega)(x,\xi) = \omega(x,\varphi'_x(x,\xi)), \tag{2.2}$$

will appear frequently in the sequel. In the following lemma we show that these weights belong to the same classes of weights as ω, provided they additionally fulfill

$$\Theta_{1,\varphi}\omega \asymp \Theta_{2,\varphi}\omega \tag{2.3}$$

when φ is the involved phase function. That is, (2.3) is a sufficient condition to obtain $(\phi_1,1)$- and/or $(\phi_2,2)$-invariance of ω in the sense of Definition 1.7, depending on the values of the parameters $r,\rho \ge 0$.

Lemma 2.3. *Let φ be a simple phase on $\mathbf{R}^{2d}$, $r,\rho \in [0,1]$ be such that $r = 1$ or $\rho = 1$, and let $\Theta_{j,\varphi}\omega$, $j = 1,2$, be as in* (2.2)*, where $\omega \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ satisfies* (2.3)*. Then*

$$\Theta_{j,\varphi}\omega \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d}), \quad j = 1,2.$$

Proof. Evidently, the estimates (1.2) for $\Theta_{1,\varphi}\omega$ and $\Theta_{2,\varphi}\omega$ follow from Lemma 1.8. We need to show that $\Theta_{1,\varphi}\omega$ and $\Theta_{2,\varphi}\omega$ are moderate.

By Taylor expansion, and the fact that ω is moderate, there are numbers $\theta = \theta(x,y) \in [0,1]$ and $N_1 \ge 0$ such that

$$\begin{aligned}(\Theta_{1,\varphi}\omega)(x+y,\xi) &= \omega(\varphi'_\xi(x+y,\xi),\xi) = \omega(\varphi'_\xi(x,\xi)+\langle \varphi''_{x,\xi}(x+\theta y,\xi),y\rangle,\xi) \\ &\lesssim \omega(\varphi'_\xi(x,\xi),\xi)\langle\langle \varphi''_{x,\xi}(x+\theta y,\xi),y\rangle\rangle^{N_1} \lesssim \omega(\varphi'_\xi(x,\xi),\xi)\langle y\rangle^{N_1}.\end{aligned}$$

This gives

$$(\Theta_{1,\varphi}\omega)(x+y,\xi) \lesssim (\Theta_{1,\varphi}\omega)(x,\xi)\langle y\rangle^{N_1}.$$

In the same way we get

$$(\Theta_{2,\varphi}\omega)(x,\xi+\eta) \lesssim (\Theta_{2,\varphi}\omega)(x,\xi)\langle \eta\rangle^{N_2},$$

for some $N_2 \ge 0$. From these estimates we obtain

$$
\begin{aligned}
(\Theta_{2,\varphi}\omega)(x+y,\xi+\eta) &\lesssim (\Theta_{2,\varphi}\omega)(x+y,\xi)\langle \eta\rangle^{N_2} \\
&\asymp (\Theta_{1,\varphi}\omega)(x+y,\xi)\langle \eta\rangle^{N_2} \lesssim (\Theta_{1,\varphi}\omega)(x,\xi)\langle y\rangle^{N_1}\langle \eta\rangle^{N_2} \\
&\asymp (\Theta_{2,\varphi}\omega)(x,\xi)\langle y\rangle^{N_1}\langle \eta\rangle^{N_2}.
\end{aligned}
$$

Hence $\Theta_{2,\varphi}\omega$, and thereby $\Theta_{1,\varphi}\omega$, are v-moderate, when $v(x,\xi) = \langle x\rangle^{N_1}\langle \xi\rangle^{N_2}$. □

In the following lemma we recall some facts on mapping properties for the operators R_1 and $\mathscr{D}$, given in [9]. For $\varphi \in \mathfrak{F}$, are defined by the formulas

$$
R_1 = \frac{1-\Delta_\xi}{\langle \varphi'_\xi(x,\xi)\rangle^2 - i\Delta_\xi \varphi(x,\xi)}, \tag{2.4}
$$

and

$$
(\mathscr{D}a)(x,\xi) = \frac{a(x,\xi)}{\langle \varphi'_\xi(x,\xi)\rangle^2 - i\Delta_\xi \varphi(x,\xi)}. \tag{2.5}
$$

Here and in what follows we let

$$
{}^t a(x,\xi) = a(\xi,x) \quad \text{and} \quad (a^*)(x,\xi) = \overline{a(\xi,x)},
$$

when $a(x,\xi)$ is a function.

Lemma 2.4. *Let $\varphi \in \mathfrak{F}$ and let R_1 and $\mathscr{D}$ be defined by* (2.4) *and* (2.5)*. Then the following is true:*

(1) $R_1 e^{i\varphi} = e^{i\varphi}$*;*

(2) $R_1 = \mathscr{D}(1-\Delta_\xi)$*:*

(3) *for any positive integer l,*

$$
({}^t R_1)^l = \underbrace{(1-\Delta_\xi)\mathscr{D}\cdots(1-\Delta_\xi)\mathscr{D}}_{l \text{ times}} = \mathscr{D}^l + Q_l(\mathscr{D},\Delta_\xi), \tag{2.6}
$$

where $Q_l(\mathscr{D},\Delta_\xi)$ is a suitable differential operator depending on $l, \mathscr{D}, \Delta_\xi$, whose terms contains exactly l factors equal to $\mathscr{D}$ and at least one equal to Δ_ξ.

(4) *If $\omega \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$, where $r,\rho \in [0,1]$ are such that $r+\rho > 0$, then the mappings*

$$
\mathscr{D}^l \, : \, \mathrm{SG}^{(\omega)}_{r,\rho}(\mathbf{R}^{2d}) \to \mathrm{SG}^{(\omega\cdot\vartheta_{-2l,0})}_{r,\rho}(\mathbf{R}^{2d}),
$$

$$
Q_l(\mathscr{D},\Delta_\xi) \, : \, \mathrm{SG}^{(\omega)}_{r,\rho}(\mathbf{R}^{2d}) \to \mathrm{SG}^{(\omega\cdot\vartheta_{-2l,-2})}_{r,\rho}(\mathbf{R}^{2d})
$$

are continuous.

The next two lemma follows by induction (see e. g. [9]). The details are left for the reader.

Lemma 2.5. *Let $\varphi \in \mathrm{SG}^{1,1}_{1,1}(\mathbf{R}^{2d})$, and let α and β be multi-indices. Then $\partial^\alpha_x \partial^\beta_\xi e^{i\varphi(x,\xi)} = b_{\alpha,\beta}(x,\xi)e^{i\varphi(x,\xi)}$, for some $b_{\alpha,\beta} \in \mathrm{SG}^{|\beta|,|\alpha|}_{1,1}(\mathbf{R}^{2d})$.*

2.2. **Generalised Fourier integral operators of** SG **type.** In analogy with the definition of generalized SG pseudo-differential operators, recalled in Subsection 1.1, we define the class of Fourier integral operators we are interested in terms of their distributional kernels. These belong to a class of tempered oscillatory integrals, studied in [16]. Thereafter we prove that they posses convenient mapping properties.

Definition 2.6. Let $\omega \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$ satisfy (2.3), $r, \rho \geq 0$, $r+\rho > 0$, $\varphi \in \mathfrak{F}$, $a, b \in \mathrm{SG}^{(\omega)}_{r,\rho}(\mathbf{R}^{2d})$.

(1) The generalized Fourier integral operator $A = \mathrm{Op}_\varphi(a)$ of SG *type I* (SG *FIOs of type I*) with phase φ and amplitude a is the linear continuous operator from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$ with distribution kernel $K_A \in \mathscr{S}'(\mathbf{R}^{2d})$ given by
$$K_A(x,y) = (2\pi)^{-d/2}(\mathscr{F}_2(e^{i\varphi}a))(x,y);$$

(2) The generalized Fourier integral operator $B = \mathrm{Op}^*_\varphi(b)$ of SG *type II* (SG *FIOs of type II*) with phase φ and amplitude b is the linear continuous operator from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$ with distribution kernel $K_B \in \mathscr{S}'(\mathbf{R}^{2d})$ given by
$$K_B(x,y) = (2\pi)^{-d/2}(\mathscr{F}_2^{-1}(e^{-i\varphi}\overline{b}))(y,x).$$

Evidently, if $u \in \mathscr{S}(\mathbf{R}^d)$, and A and B are the operators in Definition 2.6, then
$$Au(x) = \mathrm{Op}_\varphi(a)u(x) = (2\pi)^{-d/2}\int e^{i\varphi(x,\xi)}\, a(x,\xi)\,(\mathscr{F}u)(\xi)\,d\xi, \tag{2.7}$$
and
$$\begin{aligned} Bu(x) &= \mathrm{Op}^*_\varphi(b)u(x) \\ &= (2\pi)^{-d}\iint e^{i(\langle x,\xi\rangle - \varphi(y,\xi))}\,\overline{b(y,\xi)}\,u(y)\,dyd\xi. \end{aligned} \tag{2.8}$$

Remark 2.7. In the sequel the formal (L^2-)adjoint of an operator Q is denoted by Q^*. By straightforward computations it follows that the SG type I and SG type II operators are formal adjoints to each others, provided the amplitudes and phase functions are the same. That is, if b and φ are the same as in Definition 2.6, then $\mathrm{Op}^*_\varphi(b) = \mathrm{Op}_\varphi(b)^*$.

Obviously, for any $\omega \in \mathscr{P}_{r,\rho}(\mathbf{R}^{2d})$, ${}^t\omega = \omega^*$ is also an admissible weight which belongs to $\mathscr{P}_{\rho,r}(\mathbf{R}^{2d})$. Similarly, for arbitrary $\varphi \in \mathfrak{F}$ and $a \in \mathrm{SG}^{(\omega)}_{r,\rho}(\mathbf{R}^{2d})$, we have ${}^t\varphi = \varphi^* \in \mathfrak{F}$ and ${}^ta, a^* \in \mathrm{SG}^{(\omega^*)}_{\rho,r}(\mathbf{R}^{2d})$. Furthermore, by Definition 2.6 we get
$$\begin{gathered} \mathrm{Op}^*_\varphi(b) = \mathscr{F}^{-1}\circ \mathrm{Op}_{-\varphi^*}(b^*)\circ \mathscr{F}^{-1} \\ \Longleftrightarrow \\ \mathrm{Op}_\varphi(a) = \mathscr{F}\circ \mathrm{Op}^*_{-\varphi^*}(a^*)\circ \mathscr{F}. \end{gathered} \tag{2.9}$$

The following result shows that type I and type II operators are linear and continuous from $\mathscr{S}(\mathbf{R}^d)$ to itself, and extendable to linear and continuous operators from $\mathscr{S}'(\mathbf{R}^d)$ to itself.

Theorem 2.8. *Let a, b and φ be the same as in Definition 2.6. Then $\operatorname{Op}_\varphi(a)$ and $\operatorname{Op}^*_\varphi(b)$ are linear and continuous operators on $\mathscr{S}(\mathbf{R}^d)$, and uniquely extendable to linear and continuous operators on $\mathscr{S}'(\mathbf{R}^d)$.*

Proof. First we consider the operator $\operatorname{Op}_\varphi(a)$. By differentiation under the integral sign, using Lemma 2.5 and the facts that differentiations and multiplications by polynomials maps SG classes into SG classes, it is enough to prove that

$$|Au(x)| \lesssim p(u), \quad u \in \mathscr{S}(\mathbf{R}^d),$$

for some seminorm p on $\mathscr{S}(\mathbf{R}^d)$. By a regularization argument, using the operator R_1 defined in (2.6), in view of Lemma 2.4 we find, for arbitrary l and $\mathfrak{D} = \langle \varphi'_\xi \rangle^2 - i\Delta_\xi \varphi$,

$$
\begin{aligned}
Au(x) &= (2\pi)^{-d} \int e^{i\varphi(x,\xi)} ({}^t R_1)^l [a(x,\xi)(\mathscr{F}u)(\xi)] d\xi = \\
&= (2\pi)^{-d} \int e^{i\varphi(x,\xi)} \left\{ \frac{a(x,\xi)}{(\mathfrak{D}(x,\xi))^l} (\mathscr{F}u)(\xi) + Q_l(\mathscr{D}, \Delta_\xi) \, [a(x,\xi)(\mathscr{F}u)(\xi)] \right\} d\xi \\
&= (2\pi)^{-d} \int e^{i\varphi(x,\xi)} \left[\frac{a(x,\xi)}{(\mathfrak{D}(x,\xi))^l} (\mathscr{F}u)(\xi) + \sum_{|\gamma| \le 2l} c_\gamma(x,\xi) D^\gamma (\mathscr{F}u)(\xi) \right] d\xi
\end{aligned}
$$

with coefficients $c_\gamma \in \mathrm{SG}^{(\omega\cdot\vartheta_{-2l,-2})}_{r,\rho}(\mathbf{R}^{2d})$ depending only on a and $\mathfrak{D}$, and $\dfrac{a(x,\xi)}{(\mathfrak{D}(x,\xi))^l} \in \mathrm{SG}^{(\omega\cdot\vartheta_{-2l,0})}_{r,\rho}(\mathbf{R}^{2d})$. Since ω is polynomially bounded and $u \in \mathscr{S}(\mathbf{R}^d)$, it follows that, for any l and a suitable $m \in \mathbf{R}$, there is a semi-norm p on $\mathscr{S}$ such that

$$|Au(x)| \lesssim \langle x \rangle^{m-2l} p(u) \int \langle \xi \rangle^{-d-1} d\xi \lesssim p(u),$$

as desired, choosing l and k large enough. The $\mathscr{S}$-continuity of the operators of type II follows by similar argument. The details are left for the reader.

Finally, the continuity and uniqueness on $\mathscr{S}'(\mathbf{R}^d)$ of the operators $\operatorname{Op}_\varphi(a)$ and $\operatorname{Op}^*_\varphi(b)$ now follows by duality, recalling Remark 2.7. □

2.3. **Compositions with pseudo-differential operators of SG type.** The composition theorems presented in this and the subsequent subsections are variants of those originally appeared in [9]. We include anyway some of their proofs, focusing on the role of the parameters in the classes of the involved amplitudes and symbols, as well as on the different notion of asymptotic expansions needed here, see [17]. The notation used in the statements of the composition theorems are those introduced in Subsections 1.2 and 2.1.

Theorem 2.9. *Let $r_j, \rho_j \in [0,1]$, $\varphi \in \mathfrak{F}$ and let $\omega_j \in \mathscr{P}_{r_j,\rho_j}(\mathbf{R}^{2d})$, $j = 0,1,2$, be such that*

$$\rho_2 = 1, \quad r_0 = \min\{r_1, r_2, 1\}, \quad \rho_0 = \min\{\rho_1, 1\}, \quad \omega_0 = \omega_1 \cdot (\Theta_{2,\varphi}\omega_2),$$

and $\omega_2 \in \mathscr{P}_{r,1}(\mathbf{R}^{2d})$ is $(\phi, 2)$-invariant with respect to $\phi\colon \xi \mapsto \varphi'_x(x,\xi)$. Also let $a \in \mathrm{SG}^{(\omega_1)}_{r_1,\rho_1}(\mathbf{R}^{2d})$, $p \in \mathrm{SG}^{(\omega_2)}_{r_2,1}(\mathbf{R}^{2d})$, and let

$$\psi(x,y,\xi) = \varphi(y,\xi) - \varphi(x,\xi) - \langle y - x, \varphi'_x(x,\xi)\rangle. \tag{2.10}$$

Then

$$\mathrm{Op}(p) \circ \mathrm{Op}_\varphi(a) = \mathrm{Op}_\varphi(c) \,\mathrm{Mod}\, \mathrm{Op}_\varphi(\mathrm{SG}_0^{(\omega\vartheta_{0,-\infty})}), \quad r_1 = 0,$$

$$\mathrm{Op}(p) \circ \mathrm{Op}_\varphi(a) = \mathrm{Op}_\varphi(c) \,\mathrm{Mod}\, \mathrm{Op}(\mathscr{S}), \qquad r_1 > 0,$$

where $c \in \mathrm{SG}^{(\omega_0)}_{r_0,\rho_0}(\mathbf{R}^{2d})$ admits the asymptotic expansion

$$c(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} (D_\xi^\alpha p)(x, \varphi'_x(x,\xi))\, D_y^\alpha \big[e^{i\psi(x,y,\xi)} a(y,\xi)\big]_{y=x}. \tag{2.11}$$

As usual, we split the proof of Theorem 2.9 into various intermediate steps. We first need an expression for the derivatives of the exponential functions appearing in (2.11). Again, Lemma 2.10 is a special case of the Fàa di Bruno formula, and can be proved by induction. For the proof of Lemma 2.11, see [9]. Then, in view of these two results, in Lemma 2.12 we can prove that the terms which appear in the right-hand side of (2.11) indeed give a generalized SG asymptotic expansion, in the sense described in Definition 1.2. and [17].

Lemma 2.10. *Let $\varphi \in C^\infty(\mathbf{R}^{2d})$, and let ψ be as in (2.10). If $\alpha \in \mathbf{N}^d$ satisfies $|\alpha| \ge 1$, then*

$$D_y^\alpha e^{i\psi} = \tau_\alpha e^{i\psi}$$

where

$$\tau_\alpha = \big(\varphi'_y - \varphi'_x\big)^\alpha + \sum_j c_j \big(\varphi'_y - \varphi'_x\big)^{\delta_j} \prod_{k=1}^{N_j} D_y^{\beta_{jk}} \varphi \tag{2.12}$$

for suitable constants $c_j \in \mathbf{R}$, and the summation in last sum should be taking over all multi-indices δ_j and β_{jk} such that

$$\delta_j + \sum_{k=1}^{N_j} \beta_{jk} = \alpha, \quad \text{and} \quad |\beta_{jk}| \ge 2. \tag{2.13}$$

In (2.12), $\varphi'_x = \varphi'_x(x,\xi)$, $\varphi'_y = \varphi'_y(y,\xi)$ and $\partial_y^\alpha \varphi = \partial_y^\alpha \varphi(y,\xi)$ is to be understood.

Note that, by (2.13), we have, in each term appearing in (2.12),

$$|\alpha| \ge \sum_{k=1}^{N_j} |\beta_{jk}| \ge 2N_j \Rightarrow N_j \le \frac{|\alpha|}{2}. \tag{2.14}$$

Lemma 2.11. *Let $\varphi \in \mathrm{SG}^{1,1}_{1,1}(\mathbf{R}^{2d})$, and let ψ be as in* (2.10)*. If $\alpha \in \mathbf{N}^d$ satisfies $|\alpha| \geq 1$, then*

$$\partial_y^\alpha e^{i\psi(x,y,\xi)}\big|_{y=x} \in \mathrm{SG}^{[-|\alpha|/2],[|\alpha|/2]}_{1,1}(\mathbf{R}^{2d})$$
$$\Rightarrow \partial_y^\alpha e^{i\psi(x,y,\xi)}\big|_{y=x} \lesssim \vartheta_{-|\alpha|/2,|\alpha|/2}(x,\xi).$$

Moreover, $|y-x| \leq \varepsilon_1\langle x\rangle, \varepsilon_1 \in (0,1)$, implies that each summand in the right-hand side of (2.12) *can be estimated by the product of a suitable power $|y-x|^{m_0}$ times a weight of the form $\langle x\rangle^m\langle\xi\rangle^\mu$, with $0 \leq m_0 \leq \mu \leq |\alpha|$, $m \leq -\frac{|\alpha|}{2}$.*

Lemma 2.12. *Let $\varphi \in \mathfrak{F}$, ψ be as in* (2.10)*, and let a, p, ω_j, r_j and ρ_j, $j = 0,1,2$, be as in Theorem 2.9. Then*

$$\sum_\alpha \frac{c_\alpha(x,\xi)}{\alpha!},$$
$$\text{with} \quad c_\alpha(x,\xi) = i^{|\alpha|}(D_\xi^\alpha p)(x,\varphi'_x(x,\xi))\, D_y^\alpha\big[e^{i\psi(x,y,\xi)}a(y,\xi)\big]_{y=x} \tag{2.15}$$

is a generalized SG *asymptotic expansion which defines an amplitude $c \in \mathrm{SG}^{(\omega_0)}_{r_0,\rho_0}(\mathbf{R}^{2d})$, modulo a remainder of the type described in* (1.10)*.*

Proof. Using Lemma 2.11, the hypothesis $a \in \mathrm{SG}^{(\omega_1)}_{r_1,\rho_1}$, and the properties of the symbolic calculus, we see that

$$\begin{aligned} D_y^\alpha\left[e^{i\psi(x,y,\xi)}a(y,\xi)\right]_{y=x} &= \sum_{0\leq\beta\leq\alpha}\binom{\alpha}{\beta} D_y^\beta e^{i\psi(x,y,\xi)} D_y^{\alpha-\beta}a(y,\xi)\Bigg|_{y=x} \\ &\in \sum_{0\leq\beta\leq\alpha} \mathrm{SG}^{(\vartheta_{-|\beta|/2,|\beta|/2})}_{1,1}\cdot \mathrm{SG}^{(\omega_0\cdot\vartheta_{-r_1(|\alpha|-|\beta|),0})}_{r_1,\rho_1} \\ &= \sum_{0\leq\beta\leq\alpha} \mathrm{SG}^{(\omega_1\cdot\vartheta_{-r_1|\alpha|+(r_1-1/2)|\beta|),|\beta|/2})}_{\min\{r_1,1\},\min\{\rho_1,1\}} \\ &\subseteq \mathrm{SG}^{(\omega_1\cdot\vartheta_{-\min\{r_1,1/2\}|\alpha|,|\alpha|/2})}_{\min\{r_1,1\},\min\{\rho_1,1\}}. \end{aligned}$$

Using $\varphi \in \mathfrak{F}$, in particular (2.1), and the results in Subsections 1.2 and 2.1, we also easily have:

$$(D_\xi^\alpha p)(x,\varphi'_x(x,\xi)) \in \mathrm{SG}^{(\Theta_{2,\varphi}\omega_2\cdot\vartheta_{0,-|\alpha|})}_{r_2,1}.$$

Summing up, we obtain, for any multi index α,

$$c_\alpha(x,\xi) \in \mathrm{SG}^{(\omega_2\cdot\vartheta_{-\min\{r_1,1/2\}|\alpha|,-|\alpha|/2})}_{\min\{r_1,r_2,1\},\min\{\rho_1,1\}},$$

which proves the lemma, by the hypotheses and the general properties of the symbolic calculus. $\square$

The next two lemmas are well-known, see, e.g., [8, 9], and can be proved by induction on l.

Lemma 2.13. *Let*

$$\Omega = \{\, (x,y,\eta) \in \mathbf{R}^{3d}\,;\ |x-y|>0\,\},$$

and let R_2 *be the operator on* Ω*, given by*

$$R_2 = \sum_{j=1}^{d} \frac{x_j - y_j}{|x-y|^2} D_{\eta_j}. \tag{2.16}$$

Then $R_2 e^{i\langle x-y,\eta\rangle} = e^{i\langle x-y,\eta\rangle}$ *when* $(x,y,\eta)\in\Omega$*, and for any positive integer* l*,*

$$({}^tR_2)^l = \sum_{|\theta|=l} c_\theta \frac{(x-y)^\theta}{|x-y|^{2l}} D_\eta^\theta,$$

for suitable coefficients c_θ*.*

Lemma 2.14. *Let* $\Omega\subseteq\mathbf{R}^d$ *be open,* $f\in C^\infty(\Omega)$ *be such that* $|f'_y(y)|\neq 0$*, and let*

$$R_3 = \frac{1}{|f'_y(y)|^2}\sum_{k=1}^{d} f'_{y_k}(y) D_{y_k}. \tag{2.17}$$

Then $R_3 e^{if} = e^{if}$*, and for any positive integer* l*,*

$$({}^tR_3)^l = \frac{1}{|f'_y(y)|^{4l}} \sum_{|\alpha|\le l} P_{l\alpha}(y) D_y^\alpha, \tag{2.18}$$

with

$$P_{l\alpha} = \sum c^{l\alpha}_{\gamma\delta_1\cdots\delta_l} (f'_y)^\gamma\, D_y^{\delta_1} f \cdots D_y^{\delta_l} f, \tag{2.19}$$

where the last sum should be taken over all γ *and* δ *such that*

$$|\gamma| = 2l \quad \text{and} \quad |\delta_j|\ge 1,\ \sum_{j=1}^{l} |\delta_j| + |\alpha| = 2l, \tag{2.20}$$

and $c^{l\alpha}_{\gamma\delta_1\cdots\delta_l}$ *are suitable constants.*

Lemma 2.15. *Let* $\varphi, a, p, r_j, \rho_j$ *be as in Theorem 2.9,* $\chi\in\Xi^\Delta(\varepsilon_1)$*, and let*

$$h(x,\xi) = (2\pi)^{-d}\iint e^{i(\varphi(y,\xi)-\varphi(x,\xi)-\langle y-x,\eta\rangle)}(1-\chi(x,y))a(y,\xi)p(x,\eta)\,dyd\eta.$$

Then $h\in\mathscr{S}(\mathbf{R}^{2d})$*.*

For the proof of Lemma 2.15 we recall that for every $\varepsilon>0$ it exists an $\varepsilon_0>0$ such that

$$|y-x|\ge\varepsilon_0\langle y\rangle \quad \text{when} \quad |y-x|\ge\varepsilon\langle x\rangle. \tag{2.21}$$

Hence,

$$(\langle x\rangle\langle y\rangle)^{\frac12} \le \langle x\rangle + \langle y\rangle \lesssim |y-x| \quad \text{when} \quad |y-x|\ge\varepsilon\langle x\rangle. \tag{2.22}$$

Proof. We make use of the operators

$$\widetilde{R}_1 = \frac{1-\Delta_y}{\langle\varphi'_y(y,\xi)\rangle^2 - i\Delta_y\varphi(y,\xi)},$$

which has properties similar to those of the operator R_1 defined in (2.4), and R_2, defined in (2.16). For any couple of positive integers l_1, l_2 we have

$$\begin{aligned} h(x,\xi) =& (2\pi)^{-d}\iint e^{i(\varphi(y,\xi)-\varphi(x,\xi)-\langle y-x,\eta\rangle)}(1-\chi(x,y))a(y,\xi)\left[({}^tR_2)^{l_2}p\right](x,\eta)\,dyd\eta \\ =& (2\pi)^{-d}\iint e^{i(\varphi(y,\xi)-\varphi(x,\xi)+\langle x,\eta\rangle)}({}^t\widetilde{R}_1)^{l_1}\left[e^{-i\langle y,\eta\rangle}q(x,y,\xi,\eta)\right]dyd\eta \end{aligned} \tag{2.23}$$

when

$$q(x,y,\xi,\eta) = (1-\chi(x,y))a(y,\xi)\left[({}^tR_2)^{l_2}p\right](x,\eta).$$

By Lemma 2.13, we get

$$\begin{aligned} &\partial_y^\alpha q(x,y,\xi,\eta) = \\ &\quad = \partial_y^\alpha\left[(1-\chi(x,y))a(y,\xi)\sum_{|\theta|=l_2} c_\theta\frac{(x-y)^\theta}{|x-y|^{2l_2}}(D_\eta^\theta p)(x,\eta)\right] \\ &\quad = \sum_{|\theta|=l_2}(D_\eta^\theta p)(x,\eta)\sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\frac{\alpha!}{\alpha_1!\alpha_2!\alpha_3!}(\delta_{|\alpha_1|,0}-(\partial_y^{\alpha_1}\chi)(x,y))\cdot \\ &\qquad \cdot(\partial_y^{\alpha_2}a)(y,\xi)\sum_{\beta_1+\beta_2=\alpha_3}\frac{\alpha_3!}{\beta_1!\beta_2!}c_{\theta\beta_1}(x-y)^{\theta-\beta_1}\frac{P_{\beta_2}(x-y)}{|x-y|^{2(r_2+|\beta_2|)}}, \end{aligned}$$

with P_{β_2} homogeneous polynomial of degree $|\beta_2|$, while $\delta_{|\alpha_1|,0} = 1$ for $\alpha_1 = 0$, $\delta_{|\alpha_1|,0} = 0$ otherwise. Then we obtain

$$\begin{aligned} &|\partial_y^\alpha q(x,y,\xi,\eta)| \\ &\quad \lesssim \sum_{|\theta|=l_2}\omega_2(x,\eta)\vartheta_{0,-|\theta|}(x,\eta)\sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\langle y\rangle^{-|\alpha_1|}\omega_1(y,\xi)\vartheta_{-r_1|\alpha_2|,0} \\ &\qquad\qquad \sum_{\beta_1+\beta_2=\alpha_3}|x-y|^{|\theta|-|\beta_1|+|\beta_2|-2l_2-2|\beta_2|} \\ &\qquad\qquad \lesssim \omega_1(x,\eta)\omega_2(y,\xi)\cdot\vartheta_{0,-l_2}(x,\eta)\cdot \\ &\qquad\qquad \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha}\vartheta_{-\min\{r_1,1\}(|\alpha_1|+|\alpha_2|),0}(y,\xi)|x-y|^{-l_2-|\alpha_3|}. \end{aligned}$$

In view of the fact that $|y-x| \geq \frac{\varepsilon_1}{2}\langle x\rangle$ on $\mathrm{supp}(q)$, from (2.21) and (2.22) we also obtain

$$|y-x| \geq \frac{\varepsilon_1}{2}\langle x\rangle \Rightarrow |y-x| \gtrsim \langle y\rangle \Rightarrow |y-x| \gtrsim \langle x\rangle + \langle y\rangle \geq (\langle x\rangle\langle y\rangle)^{\frac{1}{2}},$$

and we can conclude

$$
\begin{aligned}
&|\partial_y^\alpha q(x,y,\xi,\eta)| \\
&\quad \lesssim \omega_1(x,\eta)\omega_2(y,\xi)\cdot \vartheta_{-l_2/2,-l_2/2}(x,y)\cdot \langle\eta\rangle^{-l_2}\langle y\rangle^{-\min\{r_1,1/2\}|\alpha|}. \quad (2.24)
\end{aligned}
$$

Finally, since admissible weight functions are polynomially moderate, it follows by choosing l_2 large enough that that the order of q can be made arbitrary low with respect to x, y, η. Moreover, when derivatives with respect to y are involved, q behaves as an SG symbol.

We now estimate the integrand of (2.23). As shown in Lemma 2.4, we have

$$
\begin{aligned}
&({}^t\widetilde{R}_1)^{l_1}\left[e^{-i\langle y,\eta\rangle}q(x,y,\xi,\eta)\right] = \\
&= e^{-i\langle y,\eta\rangle}\frac{q(x,y,\xi,\eta)}{(\langle\varphi'_\xi(y,\xi)\rangle^2 - i\Delta_\xi\varphi(y,\xi))^{l_1}} + Q(\mathscr{D},\Delta_y)\left[e^{-i\langle y,\eta\rangle}q(x,y,\xi,\eta)\right],
\end{aligned}
$$

as in (2.6). Due to the presence of the exponential in the argument of $Q(\mathscr{D},\Delta_y)$, in the second term there are powers of η of height not greater than $2l_1$. Owing to (2.24) we finally find

$$
\begin{aligned}
|x^\alpha\xi^\beta h(x,\xi)| \;\lesssim\; & \langle x\rangle^{-\frac{l_2}{2}+|\alpha|}\langle\xi\rangle^{-2l_1+|\beta|} \\
& \int \omega(y,\xi)\,\langle y\rangle^{-\frac{l_2}{2}}dy\int \omega_0(x,\eta)\,\langle\eta\rangle^{-\frac{l_2}{2}+2l_1}d\eta \lesssim 1,
\end{aligned}
$$

for all multi-indices α,β, provided that l_1 and l_2 are large enough, since ω_1 and ω_2 are polynomially bounded. Here l_1 is chosen first, and thereafter l_2 is fixed accordingly. Differentiating h_2 and multiplying it by powers of x and ξ would give a linear combination of expressions similar to (2.23), with different ω_1, ω_2 and parameters for the involved symbols, which are then similarly estimated by constants. The proof is complete. □

Proof of Theorem 2.9. Let

$$
c(x,\xi) = (2\pi)^{-d}\iint e^{i(\varphi(y,\xi)-\varphi(x,\xi)-\langle y-x,\eta\rangle)}a(y,\xi)p(x,\eta)\,dyd\eta
$$

By explicitly writing $\mathrm{Op}(p)\circ\mathrm{Op}_\varphi(a)u(x)$ with $u\in\mathscr{S}$, we obtain

$$
\begin{aligned}
&\mathrm{Op}(p)\circ\mathrm{Op}_\varphi(a)u(x) = \\
&= (2\pi)^{-3d/2}\int e^{i\langle x,\xi\rangle}p(x,\xi)\int e^{-i\langle y,\xi\rangle}\int e^{i\varphi(y,\eta)}a(y,\eta)\widehat{u}(\eta)\,d\eta dyd\xi \\
&= (2\pi)^{-d/2}\int e^{i\varphi(x,\eta)}c(x,\eta)\widehat{u}(\eta)\,d\eta \\
&= (2\pi)^{-d/2}\int e^{i\varphi(x,\xi)}c(x,\xi)\widehat{u}(\xi)\,d\xi.
\end{aligned}
$$

We have to show that $c \in \mathrm{SG}^{(\omega_0)}_{r_0,\rho_0}$. Choosing $\chi \in \Xi^\Delta(\varepsilon_1)$, with $\varepsilon_1 \in (0,1)$ fixed below (after equation (2.41)), we write $c = c_0 + h$, where

$$c_0(x,\xi) = (2\pi)^{-d} \iint e^{i(\varphi(y,\xi)-\varphi(x,\xi)-\langle y-x,\eta\rangle)} \chi(x,y) a(y,\xi) p(x,\eta)\, dy d\eta$$

and

$$h(x,\xi) = (2\pi)^{-d} \iint e^{i(\varphi(y,\xi)-\varphi(x,\xi)-\langle y-x,\eta\rangle)} (1-\chi(x,y)) a(y,\xi) p(x,\eta)\, dy d\eta.$$

By Lemma 2.15 we get $h \in \mathscr{S}$. We shall prove that $c_0 \in \mathrm{SG}^{(\omega_0)}_{r_0,\rho_0}$, and admits the asymptotic expansion in Lemma 2.12.

In fact, let $\eta = \varphi'_x(x,\xi) + \theta$. Then

$$\begin{aligned} p(x,\eta) &= \sum_{|\alpha|<M} \frac{i^{|\alpha|}\theta^\alpha}{\alpha!} (D^\alpha_\xi p)(x, \varphi'_x(x,\xi)) + \sum_{|\alpha|=M} \frac{i^{|\alpha|}\theta^\alpha}{\alpha!} r_\alpha(x,\xi,\theta) \\ r_\alpha(x,\xi,\theta) &= M \int_0^1 (1-t)^{M-1} (D^\alpha_\xi p)(x, \varphi'_x(x,\xi) + t\theta)\, dt, \end{aligned}$$

by Taylor's formula. Also let

$$\begin{aligned} H_\alpha(x,\xi,\theta) &= \theta^\alpha \mathscr{F}\left(e^{i\psi(x,\,\cdot\,,\xi)}\chi(x,\,\cdot\,)a(\,\cdot\,,\xi)\right)(\theta) \\ &= \mathscr{F}\left(D^\alpha(e^{i\psi(x,\,\cdot\,,\xi)}\chi(x,\,\cdot\,)a(\,\cdot\,,\xi))\right)(\theta). \end{aligned}$$

Then

$$c_0(x,\xi) = c_{0,1}(x,x,\xi) + c_{0,2}(x,x,\xi),$$

where

$$\begin{aligned} c_{0,1}(x,y,\xi) &= \sum_{|\alpha|<M} \frac{(i^{|\alpha|} D^\alpha_\xi p)(x, \varphi'_x(x,\xi))}{\alpha!} (\mathscr{F}^{-1} H_\alpha(x,\xi,\,\cdot\,))(y) \\ c_{0,2}(x,y,\xi) &= \sum_{|\alpha|=M} \frac{i^{|\alpha|}}{\alpha!} (\mathscr{F}^{-1}(r_\alpha(x,\xi,\,\cdot\,) H_\alpha(x,\xi,\,\cdot\,))(y) \end{aligned}$$

Now, since every derivative of χ vanishes in a neighbourhood of the diagonal of $\mathbf{R}^d \times \mathbf{R}^d$, and $\chi(x,x) = 1$, we get

$$\begin{aligned} c_{0,1}(x,x,\xi) &= \sum_{|\alpha|<M} \frac{c_\alpha(x,\xi)}{\alpha!} \\ c_{0,2}(x,x,\xi) &= \sum_{|\alpha|=M} \frac{c_{0,\alpha}(x,\xi)}{\alpha!}, \end{aligned}$$

where c_α is the same as in (2.15), and

$$c_{0,\alpha}(x,\xi) = (2\pi)^{-d/2} \int e^{i\langle x,\theta\rangle} r_\alpha(x,\xi,\theta) H_\alpha(x,\xi,\theta) d\theta.$$

By the properties of the generalized SG asymptotic expansions, we only have to estimate $c_{0,\alpha}$, $|\alpha| = M$ to complete the proof (cf. [17]).

Let $\chi_{0,\xi} = \chi_0(\langle\xi\rangle^{-1}\cdot)$, where $\chi_0 \in C_0^\infty(\mathbf{R}^d)$ is identically equal to 1 in the ball $B_{\varepsilon_2/2}(0)$ and supported in the ball $B_{\varepsilon_2}(0)$, where $\varepsilon_2 \in (0,1)$ will be fixed later (after equation(2.28)). Then,

$$\operatorname{supp}\chi_{0,\xi} \subset B_{\varepsilon_2\langle\xi\rangle}(0).$$

Next we split $c_{0,\alpha}$ into the sum of the two integrals

$$\begin{aligned} c_{1,\alpha}(x,\xi) =& (2\pi)^{-d/2}\int e^{i\langle x,\theta\rangle} r_\alpha(x,\xi,\theta)\chi_{0,\xi}(\theta)H_\alpha(x,\xi,\theta)d\theta;\\ c_{2,\alpha}(x,\xi) =& (2\pi)^{-d/2}\int e^{i\langle x,\theta\rangle} r_\alpha(x,\xi,\theta)\big(1-\chi_{0,\xi}(\theta)\big)H_\alpha(x,\xi,\theta)d\theta; \end{aligned}$$

We claim that for some integer $N_0 \ge 0$, depending on ω_2 only, it holds

$$|c_{1,\alpha}(x,\xi)| \lesssim \omega_1(x,\xi)(\Theta_{2,\varphi}\omega_2)(x,\xi)\langle x\rangle^{-\min(r_1,1/2)|\alpha|}\langle\xi\rangle^{N_0-|\alpha|/2}, \quad (2.25)$$

and that for every integers N_1 and N_2 it holds

$$|c_{2,\alpha}(x,\xi)| \lesssim \langle x\rangle^{-N_1}\langle\xi\rangle^{-N_2}. \quad (2.26)$$

In order to prove (2.25) we set

$$f_\alpha(x,\xi,y) = \mathscr{F}^{-1}\left(r_\alpha(x,\xi,\,\cdot\,)\chi_{0,\xi}\right)(y)$$

and use Parseval's formula to rewrite $c_{1,\alpha}$ into

$$c_{1,\alpha}(x,\xi) = (2\pi)^{-d/2}\int f_\alpha(x,\xi,x-y)D_y^\alpha\left(e^{i\psi(x,y,\xi)}\chi(x,y)a(y,\xi)\right)\,dy. \quad (2.27)$$

By our choice of χ_0 and $\varphi \in \mathfrak{F}$ it follows that for any multiindex β, on the support of the integrand of $c_{1,\alpha}$,

$$\begin{aligned} &|D_\theta^\beta r_\alpha(x,\xi,\theta)|\\ &\lesssim \int_0^1 \omega_2(x,\varphi_x'(x,\xi)+t\theta)\langle\varphi_x'(x,\xi)+t\theta\rangle^{-(|\alpha|+|\beta|)}\cdot(1-t)^{M-1}\,t^{|\beta|}\,dt\\ &\qquad\qquad \lesssim (\Theta_{2,\varphi}\omega_2)(x,\xi)\,\langle\xi\rangle^{N_0-(|\alpha|+|\beta|)}, \quad (2.28) \end{aligned}$$

for a suitable $N_0 \in \mathbf{Z}_+$. In fact, ω is polynomially moderate, while the presence of χ_0 in the integrand of $c_{1,\alpha}$ and $t \in [0,1]$ imply

$$|\theta| \le \varepsilon_2\langle\xi\rangle, \quad |t\theta| \le \varepsilon_2\langle\xi\rangle \quad \text{and} \quad \langle\varphi_x'(x,\xi)+t\theta\rangle \asymp \langle\xi\rangle.$$

We have also, for any multi-indices α,β,

$$\begin{aligned} \left|y^\beta f_\alpha(x,\xi,y)\right| &= \left|\mathscr{F}^{-1}\left(D_\theta^\beta\left(r_\alpha(x,\xi,\,\cdot\,)\chi_{0,\xi}\right)\right)\right|\\ &\lesssim |B_{\varepsilon_2\langle\xi\rangle}(0)|\cdot\sup_{\theta\in B_{\varepsilon_2\langle\xi\rangle}(0)}\left|D_\theta^\beta\left(r_\alpha(x,\xi,\theta)\chi_{0,\xi}\left(\frac{\theta}{\langle\xi\rangle}\right)\right)\right|, \quad (2.29) \end{aligned}$$

where $|B_{\varepsilon_2\langle\xi\rangle}(0)|$ is the volume of $B_{\varepsilon_2\langle\xi\rangle}(0)$. In view of (2.28),

$$\begin{aligned}\left|D_\theta^\beta\left(r_\alpha(x,\xi,\theta)\chi_{0,\xi}(\theta)\right)\right| &\lesssim \sum_{\gamma\le\beta}|D_\theta^\gamma r_\alpha(x,\xi,\theta)|\left|D_\theta^{\beta-\gamma}\chi_{0,\xi}\right| \\ &\lesssim \sum_{\gamma\le\beta}(\Theta_{2,\varphi}\omega_2)(x,\xi)\langle\xi\rangle^{N_0-(|\alpha|+|\gamma|)}\langle\xi\rangle^{(|\gamma|-|\beta|)} \\ &\lesssim (\Theta_{2,\varphi}\omega_2)(x,\xi)\langle\xi\rangle^{N_0-(|\alpha|+|\beta|)}, \quad (2.30)\end{aligned}$$

Since $|B_{\varepsilon_2\langle\xi\rangle}(0)|\lesssim\langle\xi\rangle^d$, uniformly with respect to $\varepsilon_2\in(0,1)$, (2.28), (2.29), and (2.30) imply, for any multi-indices α,β and integer N,

$$\left|y^\beta f_\alpha(x,\xi,y)\right|\lesssim(\Theta_{2,\varphi}\omega_2)(x,\xi)\,\langle\xi\rangle^{d+N_0-|\alpha|-|\beta|}$$

giving that

$$\left||y|^N\langle\xi\rangle^N f_\alpha(x,\xi,y)\right|\lesssim(\Theta_{2,\varphi}\omega_2)(x,\xi)\,\langle\xi\rangle^{d+N_0-|\alpha|}.$$

This in turn gives

$$|f_\alpha(x,\xi,y)|\lesssim\omega_{2,\varphi}(x,\xi)\,\langle\xi\rangle^{d+N_0-|\alpha|}\,(1+|y|\langle\xi\rangle)^{-N}.$$

for any multi-index α and integer N.

By letting $N=N_1+d+1$ with N_1 arbitrary integer, the previous estimates and (2.27) give

$$\begin{aligned}&|c_{1,\alpha}(x,\xi)| \\ &\quad\lesssim(\Theta_{2,\varphi}\omega_2)(x,\xi)K_\alpha(x,\xi)\langle\xi\rangle^{d+N_0-|\alpha|}\cdot\int(1+|y-x|\langle\xi\rangle)^{-(d+1)}\,dy\end{aligned}$$

where

$$K_\alpha(x,\xi):=\sup_y\left(\left|(\mathscr{F}_3^{-1}H_\alpha)(x,\xi,y)\right|(1+|y-x|\langle\xi\rangle)^{-N_1}\right). \qquad (2.31)$$

That is,

$$|c_{1,\alpha}(x,\xi)|\lesssim(\Theta_{2,\varphi}\omega_2)(x,\xi)K_\alpha(x,\xi)\langle\xi\rangle^{d+N_0-d-|\alpha|}. \qquad (2.32)$$

In order to estimate $K_\alpha(x,\xi)$, we notice that

$$\begin{aligned}&D_y^\alpha\left(e^{i\psi(x,y,\xi)}\chi(x,y)a(y,\xi)\right)= \\ &\qquad=\sum_{\beta+\gamma+\delta=\alpha}\frac{\alpha!}{\beta!\gamma!\delta!}\tau_\beta(x,y,\xi)e^{i\psi(x,y,\xi)}\,D_y^\gamma\chi(x,y)\,D_y^\delta a(y,\xi),\end{aligned}$$

where τ_β are the same as in Lemma 2.10. Furthermore, by the support properties of χ, Lemma 2.11 shows that

$$\langle x\rangle\asymp\langle y\rangle,\quad\text{and}\quad\omega_1(y,\xi)\lesssim\omega_1(x,\xi)\langle y-x\rangle^{M_1}$$

in the support of $(\mathscr{F}_3^{-1}H_\alpha)(x,\xi,y)$, for some constant $M_1 \ge 0$. Hence, if $s=\min(r_1,1/2)$, we get

$$
\begin{aligned}
&|D_y^\alpha \left(e^{i\psi(x,y,\xi)}\chi(x,y)a(y,\xi)\right)| \\
&\qquad \le \sum_{\beta+\gamma+\delta=\alpha} \frac{\alpha!}{\beta!\gamma!\delta!} |\tau_\beta(x,y,\xi)\ D_y^\gamma \chi(x,y)\ D_y^\delta a(y,\xi)| \\
&\qquad \lesssim \sum_{\beta+\gamma+\delta=\alpha} |\tau_\beta(x,y,\xi)|\ \langle y\rangle^{-|\gamma|}\ \omega_1(y,\xi)\ \langle y\rangle^{-r_1|\delta|} \\
&\lesssim \omega_1(x,\xi)\langle y-x\rangle^{M_1} \sum_{\beta+\gamma+\delta=\alpha} \sum_j |y-x|^{|\delta_j|}\langle x\rangle^{N_j-|\beta|}\langle \xi\rangle^{N_j+|\theta_j|}\langle y\rangle^{-r_1|\gamma+\delta|} \\
&\lesssim \omega_1(x,\xi)\langle y-x\rangle^{M_1} \sum_{\beta+\gamma+\delta=\alpha} \sum_j |y-x|^{|\delta_j|}\langle x\rangle^{-|\beta|/2}\langle \xi\rangle^{N_j+|\delta_j|}\langle x\rangle^{-r_1|\gamma+\delta|} \\
&\quad \lesssim \omega_1(x,\xi)\langle y-x\rangle^{M_2} \sum_{\beta+\gamma+\delta=\alpha} \sum_j (|y-x|\langle \xi\rangle)^{|\delta_j|}\langle \xi\rangle^{N_j}\langle x\rangle^{-s|\beta+\gamma+\delta|} \\
&\qquad\qquad \lesssim \omega_1(x,\xi)\langle x\rangle^{-s|\alpha|}\big\langle |x-y|\langle \xi\rangle\big\rangle^{M_3} \sum_{\beta+\gamma+\delta=\alpha} \sum_j \langle \xi\rangle^{N_j},
\end{aligned}
$$

for some constants M_2 and M_3. Note that all terms in the last sum, are never larger than $\langle \xi\rangle^{|\beta|/2} \lesssim \langle \xi\rangle^{|\alpha|/2}$ in view of (2.14). Moreover, (2.13) implies

$$
\begin{aligned}
N_j \le N_j + \frac{1}{2}|\delta_j| &= \frac{1}{2}(2N_j+|\delta_j|) \\
&\le \frac{1}{2}\left(|\delta_j| + \sum_{k=1}^{N_j} |\beta_{jk}| \right) = \frac{|\beta|}{2} \le \frac{|\alpha|}{2}.
\end{aligned}
$$

We conclude that, for N_1 large enough, we get

$$
\begin{aligned}
&|c_{1,\alpha}(x,\xi)| \\
&\quad \lesssim \omega_1(x,\xi)(\Theta_{2,\varphi}\omega_2)(x,\xi)\langle x\rangle^{-s|\alpha|}\langle \xi\rangle^{N_0-|\alpha|/2} \sup_{y\in \mathbf{R}^d} \big\langle |x-y|\langle \xi\rangle\big\rangle^{M_3-N_1} \\
&\qquad \le \omega_1(x,\xi)(\Theta_{2,\varphi}\omega_2)(x,\xi)\langle x\rangle^{-r_1|\alpha|/2}\langle \xi\rangle^{N_0-|\alpha|/2},
\end{aligned}
$$

and (2.25) follows.

Next we show that (2.26) holds. Let

$$
\begin{aligned}
f(x,y,\xi,\theta) &= \langle y,\theta\rangle - \psi(x,y,\xi) \\
&= \langle y,\theta\rangle - (\varphi(y,\xi)-\varphi(x,\xi)-\langle y-x,\varphi'_x(x,\xi)\rangle), \quad (2.33)
\end{aligned}
$$

which implies

$$
f'_y(x,y,\xi,\theta) = \theta - (\varphi'_y(y,\xi)-\varphi'_x(x,\xi)),
$$

giving that

$$
\langle f'_y(x,y,\xi,\theta)\rangle \lesssim \langle \theta\rangle + \langle \xi\rangle.
$$

Let

$$R_4 = \frac{1-\Delta_\theta}{\langle x\rangle^2}$$

Then ${}^tR_4 = R_4$ and $R_4 e^{i\langle x,\theta\rangle} = e^{i\langle x,\theta\rangle}$. By induction we get

$$\begin{aligned} c_{2,\alpha}(x,\xi) &= (2\pi)^{-d/2}\int e^{i\langle x,\theta\rangle} R_4^l\big(r_\alpha(x,\xi,\,\cdot\,)(1-\chi_{0,\xi})\cdot H_\alpha(x,\xi,\,\cdot\,)\big)(\theta)d\theta \\ &= \sum_j \int e^{i\langle x,\theta\rangle} r_{j,\alpha}(x,\xi,\theta)\chi_{j,\xi}(\theta) D_\theta^{\beta_j} H(x,\xi,\theta)\,d\theta, \quad (2.34)\end{aligned}$$

for every integer $l\ge 0$, where $\chi_{j,\xi}\equiv \chi_j(\,\cdot\,/\langle\xi\rangle)$, and χ_j and $r_{j,\alpha}$ are smooth functions which satisfy

$$\chi_j \in L^\infty\cap C^\infty,\quad \operatorname{supp}\chi_j \subseteq \mathbf{R}^d\setminus B_{\varepsilon_2}(0) \tag{2.35}$$

and

$$|r_{j,\alpha}(x,\xi,\theta)| \lesssim \omega_{2,\varphi}(x,\xi)\langle\theta\rangle^N \vartheta_{-2l_1,-|\alpha|}(x,\xi). \tag{2.36}$$

Here $|\beta_j|\le 2l$ and the induction is done over $l\ge 0$.

We need to estimate the integrals in the sum (2.34). It is then convenient to set

$$g^j_{\beta,\gamma,\delta}(x,y,\xi) \equiv \tau_\beta(x,y,\xi)\;\partial_y^\gamma\chi(x,y)\;y^{\beta_j}\partial_y^\delta a(y,\xi). \tag{2.37}$$

and

$$J^j_{\beta,\gamma,\delta}(x,\xi,\theta) \equiv (2\pi)^{-d/2}\int e^{-if(x,y,\xi,\theta)} g^j_{\beta,\gamma,\delta}(x,y,\xi)\,dy. \tag{2.38}$$

In fact, by expanding the Fourier transform in (2.33), and using the same notation as in Lemma 2.10, we have that $c_{2,\alpha}$ is a (finite) linear combination of

$$\begin{aligned} &c_{2,j,\alpha}(x,\xi) \\ &\qquad = \sum \frac{\alpha!}{\beta!\gamma!\delta!}\int e^{i\langle x,\theta\rangle} r_{j,\alpha}(x,\xi,\theta)\chi_{j,\xi}(\theta) J^j_{\beta,\gamma,\delta}(x,\xi,\theta)\,d\theta, \quad (2.39)\end{aligned}$$

where the sum is taken over all multi-indices β,γ,δ such that $\beta+\gamma+\delta=\alpha$.

In order to estimate $c_{2,j,\alpha}$ we first consider $J^j_{\beta,\gamma,\delta}$ and the factor $g^j_{\beta,\gamma,\delta}$ in its integrand. By the relations

$$\tau_\beta \in \mathrm{SG}^{0,0,|\beta|}_{1,1,1}\subseteq \mathrm{SG}^{0,0,|\alpha|}_{1,1,1},\quad \chi\in \mathrm{SG}^{0,0,0}_{1,1,1},\quad a\in \mathrm{SG}^{(\omega_0)}_{r_0,\rho_0},$$

and $|\beta_j|\le 2l$, it follows that

$$\partial_y^\gamma\chi \in \mathrm{SG}^{0,-|\gamma|,0}_{1,1,1}\subseteq \mathrm{SG}^{0,0,0}_{1,1,1}\quad\text{and}\quad y^{\beta_j}a(y,\xi)\in \mathrm{SG}^{(\omega_0\cdot\vartheta_{2l_1,0})}_{\min\{r_0,1\},\rho_0}.$$

This in turn gives

$$\begin{gathered} g^j_{\beta\gamma\delta}\in \mathrm{SG}^{(\omega_3)}_{1,\min\{r_0,1\},\min\{\rho_0,1\}}, \\ \text{where}\quad \omega_3(x,y,\xi) = \omega_0(y,\xi)\,\vartheta_{2l_1,|\alpha|}(y,\xi). \end{gathered} \tag{2.40}$$

In order to estimate $|J^j_{\beta,\gamma,\delta}|$ we consider the operator R_3 in (2.17), which is admissible, since

$$
\begin{aligned}
|f'_y(x,y,\xi,\theta)| &= |\theta - (\varphi'_y(y,\xi) - \varphi'_x(x,\xi))| \\
&\ge |\theta| - |\varphi'_y(y,\xi) - \varphi'_x(x,\xi)| \gtrsim \langle \theta \rangle + \langle \xi \rangle \asymp \langle (\xi,\theta) \rangle \gtrsim (\langle \xi \rangle \langle \theta \rangle)^{\frac{1}{2}}, \\
&\qquad \text{when} \quad (x,y) \in \operatorname{supp} \chi,\ \theta \in \operatorname{supp} \chi_{j,\xi}, \qquad (2.41)
\end{aligned}
$$

provided $\varepsilon_1 \in (0,1)$ in the definition of χ is chosen small enough.

In fact, if $\theta \in \operatorname{supp} \chi_{j,\xi}$, then $|\theta| \ge \varepsilon_2 \langle \xi \rangle /2$. Moreover, if $(x,y) \in \chi$, then $|y-x| \le \varepsilon_1 \langle x \rangle$, which gives

$$
\begin{aligned}
&\varphi'_{x_j}(y,\xi) - \varphi'_{x_j}(x,\xi) = \\
&\qquad = \sum_{k=1}^d \int_0^1 \varphi''_{x_j x_k}(x + t(y-x), \xi)(y_k - x_k)\, dt \\
&\qquad \lesssim \varepsilon_1 \langle x \rangle \langle \xi \rangle \int_0^1 \langle x + t(y-x) \rangle^{-1} dt \lesssim \varepsilon_1 \langle \xi \rangle \langle x \rangle \langle x \rangle^{-1} = \varepsilon_1 \langle \xi \rangle,
\end{aligned}
$$

and (2.41) follows by straight-forward applications of these estimates.

We note that ${}^tR_3$ acts only on $g^j_{\beta\gamma\delta}$, leaving $e^{i\langle x,\theta\rangle}$, $r_{j,\alpha}$ and $\chi_{j,\xi}$ unchanged. By applying (2.18), (2.19), (2.39) and (2.40) we get, for any integer l_0,

$$
\begin{aligned}
&|J^j_{\beta,\gamma,\delta}(x,\xi,\theta)| \\
&\qquad = \left| \int e^{-if(x,y,\xi,\theta)} ({}^tR_3)^{l_0} g^j_{\beta,\gamma,\delta}(x,y,\xi,\theta)\, dy \right| \\
&\qquad \le \int \frac{1}{|f'_y(x,y,\xi,\theta)|^{4l_0}} \sum_{|\kappa| \le l_0} |P_{\kappa,l_0} \partial^\kappa_y g^j_{\beta\gamma\delta}(x,y,\xi,\theta)|\, dy. \qquad (2.42)
\end{aligned}
$$

In the support of the latter integrands we have $|x-y| \le \varepsilon_1 \langle x \rangle$, which gives

$$
\begin{gathered}
\langle x \rangle \asymp \langle y \rangle, \quad |x-y| \lesssim \langle y \rangle, \quad v(x-y) \lesssim \langle y \rangle^{m_0}, \\
v(y) \lesssim \langle y \rangle^{m_0} \quad \text{and} \quad |y| \lesssim \langle x \rangle,
\end{gathered} \qquad (2.43)
$$

for a suitable $m_0 \in \mathbf{Z}_+$, which only depends on ω_1. Here $v \in \mathscr{P}(\mathbf{R}^d)$ is chosen such that $\omega_1(x+y,\xi) \lesssim \omega_1(x,\xi) v(y)$. Hence it suffices to evaluate the integrals in (2.42) over the set

$$
\Omega = \{\, y \in \mathbf{R}^d\, ;\, |y| \le C_2 \langle x \rangle | \text{ and } C_1 \langle x \rangle \le \langle y \rangle \le C_2 \langle x \rangle \,\},
$$

provided $C_1 > 0$ is small enough and $C_2 > 0$ is large enough.

By brute-force computations, (2.41), (2.42), (2.43) and (2.42) we get

$$
\begin{aligned}
&|J^j_{\beta,\gamma,\delta}(x,\xi,\theta)|\\
&\lesssim \langle \theta\rangle^N\langle(\xi,\theta)\rangle^{-4l_0}\sum_{|\kappa|\le l_0}\int_\Omega \omega_1(y,\xi)\vartheta_{2l-|\kappa|,|\alpha|}(y,\xi)(\langle\theta\rangle+\langle\xi\rangle)^{3l_0}\langle y\rangle^{|\kappa|-l_0}\,dy\\
&\qquad\lesssim \langle \theta\rangle^N\langle(\xi,\theta)\rangle^{-4l_0+3l_0}\omega_1(x,\xi)\vartheta_{m_0+2l-l_0,|\alpha|}(x,\xi)\sum_{|\kappa|\le l_0}\int_{|y|\le C\langle x\rangle}dy\\
&\qquad\qquad\lesssim \langle\xi\rangle^{-l_0/2}\langle\theta\rangle^{N-l_0/2}\omega_1(x,\xi)\vartheta_{d+m_0+2l-l_0,|\alpha|}(x,\xi)
\end{aligned}\tag{2.44}
$$

Inserting this into (2.39), we get

$$
\begin{aligned}
|c_{2,j,\alpha}(x,\xi)| &\lesssim (\Theta_{2,\varphi}\omega_2)(x,\xi)\vartheta_{-2l,-|\alpha|}(x,\xi)\sum\int|J^j_{\beta,\gamma,\delta}(x,\xi,\theta)|\,d\theta\\
&\lesssim \omega_1(x,\xi)(\Theta_{2,\varphi}\omega_2)(x,\xi)\langle x\rangle^{d+m_0-l_0}\sum\int\langle\xi\rangle^{-l_0/2}\langle\theta\rangle^{N-l_0/2}\,d\theta\\
&\lesssim \omega_1(x,\xi)(\Theta_{2,\varphi}\omega_2)(x,\xi)\langle x\rangle^{-2l}\langle\xi\rangle^{-l_0/2},
\end{aligned}\tag{2.45}
$$

provided

$$
l_0>\max\{2N,2l+d+m_0\}.
$$

Here the sums should be taken over all j and β, γ and δ such that $\beta+\gamma+\delta=\alpha$.

Since l and l_0 can be chosen arbitrarily large, and

$$
\omega_{2,\varphi}(x,\xi)\,\omega_0(x,\xi)\lesssim\langle x\rangle^m\langle\xi\rangle^\mu,
$$

for suitable $m,\mu\ge 0$, it follows that (2.26) is true for every integers N_1 and N_2. In particular it follows that the hypothesis in [17, Corollary 16] is fulfilled with $a=h$ and a_j being a suitable linear combination of c_α. This gives the result. □

The next three theorems can be proved by modifying the arguments given in [9], similarly to the above proof of Theorem 2.9. The relations between Type I and Type II operators, and the formulae for the formal-adjoints of the involved operators, explained in Remark 2.7, are useful in the corresponding arguments.

Theorem 2.16. *Let $r_j,\rho_j\in[0,1]$, $\varphi\in\mathfrak{F}$ and let $\omega_j\in\mathscr{P}_{r_j,\rho_j}(\mathbf{R}^{2d})$, $j=0,1,2$, be such that*

$$
r_2=1,\quad r_0=\min\{r_1,1\},\quad \rho_0=\min\{\rho_1,\rho_2,1\},\quad \omega_0=\omega_1\cdot(\Theta_{1,\varphi}\omega_2),
$$

and $\omega_2\in\mathscr{P}_{r,1}(\mathbf{R}^{2d})$ is $(\phi,1)$-invariant with respect to $\phi\colon x\mapsto\varphi'_\xi(x,\xi)$. Also let $a\in\mathrm{SG}^{(\omega_1)}_{r_1,\rho_1}(\mathbf{R}^{2d})$ and $p\in\mathrm{SG}^{(\omega_2)}_{1,\rho_2}(\mathbf{R}^{2d})$. Then

$$
\begin{aligned}
&\mathrm{Op}_\varphi(a)\circ\mathrm{Op}(p)=\mathrm{Op}_\varphi(c)\,\mathrm{Mod}\,\mathrm{Op}_\varphi(\mathrm{SG}_0^{(\omega\vartheta_{-\infty,0})}),\quad &\rho_1=0,\\
&\mathrm{Op}_\varphi(a)\circ\mathrm{Op}(p)=\mathrm{Op}_\varphi(c)\,\mathrm{Mod}\,\mathrm{Op}(\mathscr{S}),\quad &\rho_1>0,
\end{aligned}
$$

where the transpose ${}^t c$ *of* $c \in \mathrm{SG}^{(\omega_0)}_{r_0,\rho_0}(\mathbf{R}^{2d})$ *admits the asymptotic expansion* (2.11)*, after* p *and* a *have been replaced by* ${}^t p$ *and* ${}^t a$*, respectively.*

Theorem 2.17. *Let* $r_j, \rho_j \in [0,1]$*,* $\varphi \in \mathfrak{F}$ *and let* $\omega_j \in \mathscr{P}_{r_j,\rho_j}(\mathbf{R}^{2d})$*,* $j = 0,1,2$*, be such that*

$$\rho_2 = 1, \quad r_0 = \min\{r_1, r_2, 1\}, \quad \rho_0 = \min\{\rho_1, 1\}, \quad \omega_0 = \omega_1 \cdot (\Theta_{2,\varphi}\omega_2),$$

and $\omega_2 \in \mathscr{P}_{r,1}(\mathbf{R}^{2d})$ *is* $(\phi, 2)$*-invariant with respect to* $\phi\colon \xi \mapsto \varphi'_x(x,\xi)$*. Also let* $b \in \mathrm{SG}^{(\omega_1)}_{r_1,\rho_1}(\mathbf{R}^{2d})$*,* $p \in \mathrm{SG}^{(\omega_2)}_{r_2,1}(\mathbf{R}^{2d})$*,* ψ *be the same as in* (2.10)*, and let* $q \in \mathrm{SG}^{(\omega_2)}_{r_2,1}(\mathbf{R}^{2d})$ *be such that*

$$q(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D_x^\alpha D_\xi^\alpha \overline{p(x,\xi)}. \tag{2.46}$$

Then

$$\begin{aligned}
&\mathrm{Op}^*_\varphi(b) \circ \mathrm{Op}(p) = \mathrm{Op}_\varphi(c) \,\mathrm{Mod}\, \mathrm{Op}^*_\varphi(\mathrm{SG}_0^{(\omega\vartheta_{0,-\infty})}), && r_1 = 0,\\
&\mathrm{Op}^*_\varphi(b) \circ \mathrm{Op}(p) = \mathrm{Op}_\varphi(c) \,\mathrm{Mod}\, \mathrm{Op}(\mathscr{S}), && r_1 > 0,
\end{aligned}$$

where $c \in \mathrm{SG}^{(\omega_0)}_{r_0,\rho_0}(\mathbf{R}^{2d})$ *admits the asymptotic expansion*

$$c(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} (D_\xi^\alpha q)(x, \varphi'_x(x,\xi)) D_y^\alpha \big[e^{i\psi(x,y,\xi)} b(y,\xi) \big]_{y=x}. \tag{2.47}$$

Theorem 2.18. *Let* $r_j, \rho_j \in [0,1]$*,* $\varphi \in \mathfrak{F}$ *and let* $\omega_j \in \mathscr{P}_{r_j,\rho_j}(\mathbf{R}^{2d})$*,* $j = 0,1,2$*, be such that*

$$r_2 = 1, \quad r_0 = \min\{r_1, 1\}, \quad \rho_0 = \min\{\rho_1, \rho_2, 1\}, \quad \omega_0 = \omega_1 \cdot (\Theta_{1,\varphi}\omega_2),$$

and $\omega_2 \in \mathscr{P}_{r,1}(\mathbf{R}^{2d})$ *is* $(\phi, 1)$*-invariant with respect to* $\phi\colon x \mapsto \varphi'_\xi(x,\xi)$*. Also let* $a \in \mathrm{SG}^{(\omega_1)}_{r_1,\rho_1}(\mathbf{R}^{2d})$ *and* $p \in \mathrm{SG}^{(\omega_2)}_{1,\rho_2}(\mathbf{R}^{2d})$*. Then*

$$\begin{aligned}
&\mathrm{Op}(p) \circ \mathrm{Op}^*_\varphi(b) = \mathrm{Op}_\varphi(c) \,\mathrm{Mod}\, \mathrm{Op}^*_\varphi(\mathrm{SG}_0^{(\omega\vartheta_{-\infty,0})}), && \rho_1 = 0,\\
&\mathrm{Op}(p) \circ \mathrm{Op}^*_\varphi(b) = \mathrm{Op}_\varphi(c) \,\mathrm{Mod}\, \mathrm{Op}(\mathscr{S}), && \rho_1 > 0,
\end{aligned}$$

where the transpose ${}^t c$ *of* $c \in \mathrm{SG}^{(\omega_0)}_{r_0,\rho_0}(\mathbf{R}^{2d})$ *admits the asymptotic expansion* (2.47)*, after* q *and* b *have been replaced by* ${}^t q$ *and* ${}^t b$*, respectively.*

2.4. **Composition between** SG **FIOs of type I and type II.** The subsequent Theorems 2.19 and 2.20 deal with the composition of a type I operator with a type II operator, and show that such compositions are pseudo-differential operators with symbols in natural classes. We give the argument only for Theorem 2.19, since the proof of Theorem 2.20 follows, with similar modifications, from the one given in [9] for the corresponding composition result.

The main difference, with respect to the arguments in [9] for the analogous composition results, is that we again make use, in both cases, of the generalized asymptotic expansions introduced in Definition 1.2. This allows to overcome the additional difficulty, not arising there, that the amplitudes appearing in the computations below involve weights which are still polynomially bounded, but which do not satisfy, in general, the moderateness condition (1.1). On the other hand, all the terms appearing in the associated asymptotic expansions belong to SG classes with weights of the form $\widetilde{\omega}_{2,\varphi} \cdot \vartheta_{-k,-k}$, where $\widetilde{\omega} = \omega_1 \cdot \omega_2$. In view of the results in [17], this allows to conclude as desired, since the remainders are of the forms given in Proposition 1.1.

In order to formulate our next result, it is convenient to let S_φ with $\varphi \in \mathfrak{F}$, be the operator, defined by the formulas

$$
\begin{gathered}
(S_\varphi f)(x,y,\xi) = f(x,y,\Phi(x,y,\xi)) \cdot \left|\det \Phi'_\xi(x,y,\xi)\right| \\
\text{where} \quad \int_0^1 \varphi'_x(y+t(x-y),\Phi(x,y,\xi))\,dt = \xi.
\end{gathered}
\tag{2.48}
$$

That is, for every fixed $x,y \in \mathbf{R}^d$, $\xi \mapsto \Phi(x,y,\xi)$ is the inverse of the map

$$
\xi \mapsto \int_0^1 \varphi'_x(y+t(x-y),\xi)\,dt. \tag{2.49}
$$

Notice that, as proved in [9], the map (2.49) is indeed invertible for (x,y) belonging to the support of the elements of $\Xi^\Delta(\varepsilon)$, provided ε is chosen suitably small, and it turns out to be, in that case, a SG diffeomorphism with SG^0 parameter dependence.

Theorem 2.19. *Let $r_j \in [0,1]$, $\varphi \in \mathfrak{F}$ and let $\omega_j \in \mathscr{P}_{r_j,1}(\mathbf{R}^{2d})$, $j = 0,1,2$, be such that ω_1 and ω_2 are $(\phi,2)$-invariant with respect to $\phi\colon \xi \mapsto (\varphi'_x)^{-1}(x,\xi)$,*

$$
r_0 = \min\{r_1,r_2,1\} \quad \text{and} \quad \omega_0(x,\xi) = \omega_1(x,\phi(x,\xi))\omega_2(x,\phi(x,\xi)),
$$

Also let $a \in \mathrm{SG}^{(\omega_1)}_{r_1,1}(\mathbf{R}^{2d})$ and $b \in \mathrm{SG}^{(\omega_2)}_{r_2,1}(\mathbf{R}^{2d})$. Then

$$
\mathrm{Op}_\varphi(a) \circ \mathrm{Op}^*_\varphi(b) = \mathrm{Op}(c),
$$

for some $c \in \mathrm{SG}^{(\omega_0)}_{r_0,1}(\mathbf{R}^{2d})$. Furthermore, if $\varepsilon \in (0,1)$, $\chi \in \Xi^\Delta(\varepsilon)$, $c_0(x,y,\xi) = a(x,\xi)b(y,\xi)\chi(x,y)$ and S_φ is given by (2.48), then c admits the asymptotic expansion

$$
c(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}(D_y^\alpha D_\xi^\alpha(S_\varphi c_0))(x,y,\xi)\big|_{y=x}. \tag{2.50}
$$

For the proof Fourier integral operators involving amplitudes $q(x,y,\xi)$ and phase functions $\psi(x,y,\xi)$, defined on $\mathbf{R}^d \times \mathbf{R}^d \times \mathbf{R}^d \simeq \mathbf{R}^{3d}$ appear in natural ways. The corresponding operator $\mathrm{Op}_\psi(x,y)$ is defined as the operator with distribution kernel

$$
(x,y) \mapsto (2\pi)^{-d} \int e^{i\psi(x,y,\xi)} q(x,y,\xi)\,d\xi,
$$

provided the integral is well-defined as an oscillatory integral. As usual we write

$$\mathrm{Op}_\psi(q)u(x) = (2\pi)^{-d} \iint e^{i\psi(x,y,\xi)} q(x,y,\xi)u(y)\, dyd\xi$$

Proof. Let $\omega \in \mathscr{P}_{r_1,r_2,1}(\mathbf{R}^{3d})$ be given by $\omega(x,y,\xi) = \omega_1(x,\xi)\omega_2(y,\xi)$, let $u \in \mathscr{S}(\mathbf{R}^d)$, and write

$$\begin{aligned}
&\mathrm{Op}_\varphi(a) \circ \mathrm{Op}^*_\varphi(b)u(x) \\
&\qquad = (2\pi)^{-d} \int e^{i\varphi(x,\xi)}\, a(x,\xi) \left[\int e^{-i\varphi(y,\xi)}\, \overline{b(y,\xi)}\, u(y)\, dy \right] d\xi \\
&\qquad\qquad = (2\pi)^{-d} \int e^{i\psi(x,y,\xi)}\, q(x,y,\xi)\, u(y)\, dyd\xi = \mathrm{Op}_\psi(c_0)u(x)
\end{aligned}$$

where $\psi(x,y,\xi) = \varphi(x,\xi) - \varphi(y,\xi)$ and $c_0(x,y,\xi) = a(x,\xi) \cdot \overline{b(y,\xi)} \in \mathrm{SG}^{(\omega)}_{r_1,r_2,1}$.

Let $\chi \in \Xi^\Delta(\varepsilon)$, $\varepsilon \in (0,1)$, and let c_1 and c_2 be given by $c_1(x,y,\xi) = \chi(x,y)q(x,y,\xi)$ and $c_2(x,y,\xi) = (1-\chi(x,y))q(x,y,\xi)$. Then $c_1, c_2 \in \mathrm{SG}^{(\omega)}_{r,s,1}$, and

$$\mathrm{Op}_\varphi(a) \circ \mathrm{Op}^*_\varphi(b) = \mathrm{Op}_\psi(c_1) + \mathrm{Op}_\psi(c_2).$$

We shall prove that $\mathrm{Op}_\psi(c_2)$ is a smoothing operator and that $\mathrm{Op}_\psi(c_1)$ be written as an SG pseudo-differential operator described in the statement, provided $\varepsilon \in (0,1)$ is chosen suitably small.

In order to prove that $\mathrm{Op}_\psi(c_2)$ is a smoothing operator, we first notice that $|x-y| \ge \frac{\varepsilon}{2}\langle x\rangle$ on $\operatorname{supp} c_2$. We shall apply the operator

$$R_3 = \frac{1}{|\psi'_\xi(x,y,\xi)|^2} \sum_{k=1}^n \psi'_{\xi_j}(x,y,\xi)\, D_{\xi_j},$$

analogous to the operator in (2.17), in the integral formula of $\mathrm{Op}_\psi(c_2)u(x)$.

In fact, let $v = \varphi'_\xi(x,\xi)$ and $w = \varphi'_\xi(y,\xi)$. By Proposition 2.2 and the fact that $\varphi \in \mathrm{SG}^{1,1}_{1,1}$, we get

$$|x-y| \le M|\varphi'_\xi(x,\xi) - \varphi'_\xi(y,\xi)| = M|f'_\xi(x,y,\xi)|,$$

for a suitable constant $M > 0$, which implies

$$|\psi'_\xi(x,y,\xi)| \gtrsim |x-y| \gtrsim \langle x\rangle + \langle y\rangle$$

on $\operatorname{supp} c_2$. Then, using $R_3 e^{if} = e^{if}$, (2.18)–(2.20) and that $\varphi \in \mathrm{SG}^{1,1}_{1,1}$, for any integer l,

$$\mathrm{Op}_\psi(c_2)u(x) = (2\pi)^{-d} \int e^{i\psi(x,y,\xi)}\, (({}^tR_3)^l c_1)(x,y,\xi)\, u(y)\, dyd\xi$$

and

$$(({}^tR_3)^l c_1)(x,y,\xi) = \frac{1}{|\psi'_\xi(x,y,\xi)|^{4l}} \sum_{|\alpha|\le l} P_{l\alpha}\partial_\xi^\alpha c_2(x,y,\xi)$$

$$\lesssim \frac{\sum_{|\alpha|\le l} (\langle x\rangle + \langle y\rangle)^{3l} \langle \xi\rangle^{|\alpha|-l} \omega_1(x,\xi)\,\omega_2(y,\xi)\langle\xi\rangle^{-|\alpha|}}{(\langle x\rangle + \langle y\rangle)^{4l}}$$

$$\lesssim \frac{\omega_1(x,y,\xi)}{(\langle x\rangle + \langle y\rangle)^{l}}\langle\xi\rangle^{-l}$$

Hence

$$\mathrm{Op}_\psi(c_2)u(x) = \int K(x,y)\,u(y)\,dy,$$

where

$$K(x,y) = (2\pi)^{-d}\int e^{if(x,y,\xi)}\,({}^tR_3)^l c_2(x,y,\xi)\,d\xi.$$

where l can be chosen arbitrarily large. Since ω is polynomially bounded, and $\langle x\rangle + \langle y\rangle \ge (\langle x\rangle\langle y\rangle)^{\frac{1}{2}}$, it follows $K(x,y) \lesssim (\langle x\rangle\langle y\rangle)^{-N}$ for any integer N. The derivatives of $D_x^\alpha D_y^\beta K_1(x,y)$ can be estimated in the same way, since any such expression is of the same form as K, for a suitable replacement of c_2 and ω.

This proves that $K \in \mathscr{S}(\mathbf{R}^{2d})$, and hence $\mathrm{Op}_\psi(c_2)$ is a smoothing operator.

Next we prove that $\mathrm{Op}_\psi(c_1)$ is a generalized SG pseudo-differential operator. On $\operatorname{supp} c_1$ we have $|x-y| \le \varepsilon\langle x\rangle$ which gives $\langle x\rangle \asymp \langle y\rangle$. Let

$$\Psi(x,y,\xi) = \int_0^1 \varphi'_x(y+t(x-y),\xi)\,dt.$$

In [9] it has been proved that

$$\phi\colon \mathbf{R}^d\times\mathbf{R}^{2d} : (\xi,(x,y)) \mapsto \phi(\xi,x,y) = \Psi(x,y,\xi)$$

is, on the support of c_1, an SG diffeomorphism with SG^0 parameter dependence.

We now prove that

$$\Psi(x,y,\xi) \asymp \langle\xi\rangle.$$

on $\operatorname{supp} c_1$. In fact, the upper bound is immediate, and we also have

$$\begin{aligned}|w(x,y,\xi)| &\le |x-y|\cdot \sup_{t\in[0,1]} \|H(y+t(x-y),\xi)\| \lesssim \varepsilon\langle x\rangle\langle y\rangle^{-1}\langle\xi\rangle\\ &\lesssim \varepsilon\langle\xi\rangle \lesssim \varepsilon\langle\varphi'_x(y,\xi)\rangle\\ \langle\Psi(x,y,\xi)\rangle &= \langle\varphi'_x(y,\xi)+w(x,y,\xi)\rangle \asymp \langle\varphi'_x(y,\xi)\rangle \asymp \langle\xi\rangle.\end{aligned}$$

For a suitable choice of $\varepsilon \in (0,1)$, $\Psi(x,y;\xi)$ satisfies all the requirements of Definition 1.5. Furthermore, $\Psi(x,y,\xi)$ is an SG diffeomorphism with SG^0 parameter dependence on $\operatorname{supp} c_1$. From these properties we have

$$\mathrm{Op}_\psi(c_1)u(x) = (2\pi)^{-d} \iint e^{i\langle x-y, \Psi(x,y,\eta)\rangle}\, c_1(x,y,\eta)\, u(y)\, d\eta dy.$$

By making the substitution

$$\xi = \Psi(x,y,\eta), \quad \text{or equivalently,} \quad \eta = \Psi^{-1}(x,y,\xi), \tag{2.51}$$

we get

$$\mathrm{Op}_\psi(c_1)u(x) = (2\pi)^{-d} \int e^{i\langle x-y,\xi\rangle}\, (S_\varphi c_1)(x,y,\xi) u(y)\, dy d\xi,$$

where

$$(S_\varphi c_1)(x,y,\xi) = c_1(x,y,(\Psi^{-1}(x,y,\xi)) \cdot \left|\det\left((\Psi^{-1})'_\xi(x,y,\xi)\right)\right|. \tag{2.52}$$

Here the inverse of Ψ in (2.51) should be taken only with respect to the η variable.

Consequently, $\mathrm{Op}_\psi(c_1)$ is a pseudo-differential operator with amplitude (2.52). We claim that this can be written as a pseudo-differential operator with a symbol in $\mathrm{SG}^{(\omega_0)}_{r_0,1}(\mathbf{R}^{2d})$, obeying the expansion (2.50).

In fact, ω is polynomially moderate, which implies that

$$\omega(x,y,(\Psi^{-1})'_\xi(x,y,\xi)) \lesssim \langle x\rangle^{m_1}\langle y\rangle^{m_2}\langle \xi\rangle^{\mu},$$

for some m_1, m_2, μ. By a straight-forward applications of [17, Theorem 12] and [22, Lemma 18.2.1] it follows that $\mathrm{Op}(S_\varphi c_1) = \mathrm{Op}(c_{1,0})$ for some $c_{1,0} \in \mathrm{SG}^{(\omega_0)}_{r_0,1}(\mathbf{R}^{2d})$ which satisfies the asymptotic expansion

$$c(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}\, (D_y^\alpha D_\xi^\alpha (S_\varphi c_0))(x,y,\xi)\big|_{y=x}.$$

Here we note that the α-derivatives of the first factor in (2.52) with respect to y and ξ, evaluated for $y=x$, contain only the derivatives of the SG diffeomorphism with SG^0 parameter dependence $\phi\colon (\xi,x) \mapsto (\varphi'_x)^{-1}(x,\xi)$ and the derivatives of a and b evaluated at the image $(x,(\varphi'_x)^{-1}(x,\xi))$. In view of the properties of the SG diffeomorphism ϕ and the $(\phi,1)$-invariance of ω_1 and ω_2, Lemma 1.8 implies that $(x,\xi) \mapsto \omega(x,x,\phi(x,\xi))$ is again a polynomially moderate weight, and the order of the terms decreases with respect to ξ in desired ways. The proof is complete. $\square$

For the next result it is convenient to modify the operator S_φ in (2.48) such that it fulfills the formulas

$$\begin{gathered}(S_\varphi f)(x,\xi,\eta) = f(\Phi(x,y,\xi),\xi,\eta) \cdot |\det \Phi'_x(x,\xi,\eta)| \\ \text{where} \quad \int_0^1 \varphi'_\xi(\Phi(x,\xi,\eta),\eta + t(\xi-\eta),)\, dt = x.\end{gathered} \tag{2.53}$$

Theorem 2.20. *Let $\rho_j \in [0,1]$, $\varphi \in \mathfrak{F}^r$ and let $\omega_j \in \mathscr{P}_{1,\rho_j}(\mathbf{R}^{2d})$, $j = 0,1,2$, be such that ω_1 and ω_2 are $(\phi,1)$-invariant with respect to $\phi\colon x \mapsto (\varphi'_\xi)^{-1}(x,\xi)$,*

$$\rho_0 = \min\{\rho_1, \rho_2, 1\} \quad \textit{and} \quad \omega_0(x,\xi) = \omega_1(\phi(x,\xi),\xi)\omega_2(\phi(x,\xi),\xi),$$

Also let $a \in \mathrm{SG}^{(\omega_1)}_{1,\rho_1}(\mathbf{R}^{2d})$ and $b \in \mathrm{SG}^{(\omega_2)}_{1,\rho_2}(\mathbf{R}^{2d})$. Then

$$\mathrm{Op}^*_\varphi(b) \circ \mathrm{Op}_\varphi(a) = \mathrm{Op}(c),$$

for some $c \in \mathrm{SG}^{(\omega_0)}_{1,\rho_0}(\mathbf{R}^{2d})$. Furthermore, if $\varepsilon \in (0,1)$, $\chi \in \Xi^\Delta(\varepsilon)$, $c_0(x,\xi,\eta) = a(x,\xi)b(x,\eta)\chi(\xi,\eta)$ and S_φ is given by (2.53)*, then c admits the asymptotic expansion*

$$c(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}(D^\alpha_x D^\alpha_\eta (S_\varphi c_0))(x,\xi,\eta)\big|_{\eta=\xi}. \tag{2.54}$$

2.5. **Elliptic FIOs of generalized** SG **type and parametrices. Egorov Theorem.** The results about the parametrices of the subclass of generalized (SG) elliptic Fourier integral operators are achieved in the usual way, by means of the composition theorems in Subsections 2.3 and 2.4. The same holds for the versions of the Egorov's theorem adapted to the present situation. The additional conditions, compared with the statements in [9], concern the invariance of the weights, so that the hypotheses of the composition theorems above are fulfilled. Here we omit the proofs.

Definition 2.21. A type I or a type II SG FIO, $\mathrm{Op}_\varphi(a)$ or $\mathrm{Op}^*_\varphi(b)$, respectively, is said (SG) elliptic if $\varphi \in \mathfrak{F}^r$ and the amplitude a, respectively b, is (SG) elliptic.

Lemma 2.22. *Let a type I* SG *FIO $\mathrm{Op}_\varphi(a)$ be elliptic, with $a \in \mathrm{SG}^{(\omega)}_{1,1}(\mathbf{R}^{2d})$. Assume that ω is ϕ-invariant, $\phi = (\phi_1,\phi_2)$, where ϕ_2 and ϕ_1 are the* SG *diffeomorphisms appearing in Theorems 2.19 and 2.20, respectively. Then, the two pseudo-differential operators $\mathrm{Op}_\varphi(a) \circ \mathrm{Op}^*_\varphi(a)$ and $\mathrm{Op}^*_\varphi(a) \circ \mathrm{Op}_\varphi(a)$ are* SG *elliptic.*

Theorem 2.23. *Let $\varphi \in \mathfrak{F}^r$, $a \in \mathrm{SG}^{(\omega)}_{1,1}(\mathbf{R}^{2d})$, with a* SG *elliptic. Assume that ω is ϕ invariant, $\phi = (\phi_1,\phi_2)$, where ϕ_2 and ϕ_1 are the* SG *diffeomorphisms appearing in the Theorems 2.19 and 2.20, respectively. Then, the elliptic* SG *FIOs $\mathrm{Op}_\varphi(a)$ and $\mathrm{Op}^*_\varphi(a)$ admit a parametrix. These are elliptic* SG *FIOs of type II and type I, respectively.*

As usual, in the next two results we need the canonical transformation $\phi\colon (x,\xi) \mapsto (y,\eta)$ generated by the phase function φ, namely

$$\begin{cases} \xi = \varphi'_x(x,\eta) \\ y = \varphi'_\xi(x,\eta). \end{cases} \tag{2.55}$$

Theorem 2.24. *Let $A = \mathrm{Op}_\varphi(a)$ be an* SG *FIO of type I with $a \in \mathrm{SG}^{(\omega_0)}_{1,1}(\mathbf{R}^{2d})$ and $P = \mathrm{Op}(p)$ a pseudo-differential operator with $p \in \mathrm{SG}^{(\omega)}_{1,1}(\mathbf{R}^{2d})$. Assume that ω is ϕ-invariant, where ϕ is the canonical transformation* (2.55)*, associated with φ. Assume also that ω_0 is $(\tilde{\phi}, 2)$-invariant, where $\tilde{\phi}\colon \xi \mapsto (\varphi'_x)^{-1}(x,\xi)$. Then, setting $\eta = (\varphi'_x)^{-1}(x,\xi)$ we have*

$$\begin{aligned}\mathrm{Sym}\,(A \circ P \circ A^*)\,(x,\xi) = p(\varphi'_\xi(x,\eta),\eta)\,|a(x,\eta)|^2\,|\det \varphi''_{x\xi}(x,\eta)|^{-1}\\ \bmod\ \mathrm{SG}^{(\widetilde{\omega}\cdot\vartheta_{-1,-1})}_{1,1}(\mathbf{R}^{2d}),\end{aligned} \tag{2.56}$$

which is an element of $\mathrm{SG}^{(\widetilde{\omega})}_{1,1}(\mathbf{R}^{2d})$ with

$$\widetilde{\omega}(x,\xi) = \omega(\phi(x,\xi)) \cdot \omega_0(x,(\varphi'_x)^{-1}(x,\xi))^2.$$

Theorem 2.25. *Let $A = \mathrm{Op}_\varphi(a)$ be an elliptic* SG *FIO of type I with $a \in \mathrm{SG}^{(\omega_0)}_{1,1}(\mathbf{R}^{2d})$ and $P = \mathrm{Op}(p)$ a pseudo-differential operator with $p \in \mathrm{SG}^{(\omega)}_{1,1}(\mathbf{R}^{2d})$. Assume that ω is ϕ-invariant, where ϕ is the canonical transformation* (2.55)*, associated with φ. Then, we have*

$$\mathrm{Sym}\left(A \circ P \circ A^{-1}\right)(x,\xi) = p(\phi(x,\xi)) \mod \mathrm{SG}^{(\widetilde{\omega}\cdot\vartheta_{-1,-1})}_{1,1}(\mathbf{R}^{d}), \tag{2.57}$$

with $\widetilde{\omega}(x,\xi) = \omega(\phi(x,\xi))$.

3. $L^2(\mathbf{R}^d)$-continuity of regular generalized SG FIOs with uniformly bounded amplitude

In this section we deduce L^2-estimates for type I SG Fourier-integral operators. More precisely, we have the following.

Theorem 3.1. *Let $A = \mathrm{Op}_\varphi(a)$ be a type I* SG *Fourier integral operator with $\varphi \in \mathfrak{F}^r$ and $a \in \mathrm{SG}^{0,0}_{r,\rho}(\mathbf{R}^{2d})$, $r,\rho \geq 0$. Then, $A \in \mathcal{L}(L^2(\mathbf{R}^d))$.*

The proof of Theorem 3.1 is given as an adapted version of a general L^2-boundedness result by Asada and Fujiwara [2]. The argument below is an adapted version of the one originally given in [9] for the case $a \in \mathrm{SG}^{0,0}_{1,1}$ (see also, e.g., [4] and [25], and the references quoted therein). We illustrate below the full argument, since Theorem 3.1 is a first relevant mapping property for the class of generalized SG Fourier integral operators. Other mapping properties of this type, including a continuity result between suitable weighted modulation spaces, are given in [12].

We need some preparations for the proof and begin to recall the classical Schur's lemma.

Lemma 3.2. *If $K \in C(\mathbf{R}^d \times \mathbf{R}^d)$,*

$$\sup_y \int |K(x,y)|\,dx \leq M \quad \text{and} \quad \sup_x \int |K(x,y)|\,dy \leq M,$$

then the integral operator on $L^2(\mathbf{R}^d)$ with kernel K has norm less than or equal to M.

For the proof of Theorem 3.1 we also needs the following version of Cotlar's lemma.

Lemma 3.3. *Let $x \mapsto T_x$ be a measurable function from $\mathbf{R}^n$ to the set of linear and continuous operators on $L^2(\mathbf{R}^d)$, and let $h_j(x,y)$, $j = 1,2$, be positive functions on $\mathbf{R}^{2n}$ such that*

$$\|T_x T_y^*\| \le h_1(x,y)^2, \quad \|T_x^* T_y\| \le h_2(x,y)^2. \tag{3.1}$$

If h_1 and h_2 statisfy

$$\int h_1(x,y))\,dx \le M \quad \text{and} \quad \int h_2(x,y)\,dx \le M, \tag{3.2}$$

for some constant M, then

$$\Big\| \int (T_x f)\,dx \Big\|_{L^2} \le M\|f\|_{L^2}, \quad f \in L^2(\mathbf{R}^d).$$

Proof of Theorem 3.1. Let $g \in C^\infty(\mathbf{R})$ be decreasing and such that $g(t) = 1$ for $t < \frac{1}{2}$ and $g(t) = 0$ for $t > 1$, and set $\chi(x) = g(|x|)$, $x \in \mathbf{R}^d$, and

$$\psi_Z(x,\xi) = \frac{\chi(|x-z|)\chi(|\xi-\zeta|)}{\|\chi\|_{L^1}^2}, \qquad Z = (z,\zeta) \in \mathbf{R}^{2d}.$$

Then

$$\operatorname{supp}\psi_Z \subseteq U_Z \equiv \{\,(x,\xi) \in \mathbf{R}^{2d}\,;\, |x-z| \le 1,\ |\xi-\zeta| \le 1\,\}, \tag{3.3}$$

$$\max_{|\alpha+\beta|\le N} \sup_{x,\xi\in\mathbf{R}^d} \big|\partial_\xi^\alpha \partial_x^\beta \psi_Z(x,\xi)\big| \le C_N, \tag{3.4}$$

$$\int \psi_Z(x,\xi)\,dZ = 1,$$

where the constants C_N are independent of Z. For Z fixed, let

$$a_Z(x,\xi) = \psi_Z(x,\xi)a(x,\xi), \quad \text{and} \quad A_Z = \operatorname{Op}_\varphi(a_Z). \tag{3.5}$$

Now (3.3), (3.4) and (3.5) imply that A_Z is linear from $C_0^\infty(\mathbf{R}^d)$ to itself, and $\|A_Z f\|_{L^2} \le C\|f\|_{L^2}$, where the constant C is independent of Z. In fact, a_Z has compact support and (3.4) holds. Moreover,

$$\psi_Z \in C_0^\infty \subseteq \mathrm{SG}^{0,0}_{\min\{r,1\},\min\{\rho,1\}}$$

and

$$Af(x) = \lim_{N\to\infty} \int_{|Z|\le N} A_Z f(x)\,dZ,$$

where the limit exists pointwise for all $x \in \mathbf{R}^d$ and with respect to the strong topology of L^2.

The result follows if we prove that for all compact sets $K \subset \mathbf{R}^{2d}$

$$\Big\| \int_K A_Z f \, dZ \Big\|_{L^2} \le M \|f\|_{L^2}, \qquad f \in C_0^\infty(\mathbf{R}^d), \tag{3.6}$$

for some constant M independent of f and K. To this aim, we shall prove that A_Z obey the hypothesis in Lemma 3.3.

For this reason we consider the kernel $K_{Z_1,Z_2}(x,y)$ of $A_{Z_1}A_{Z_2}^*$, which can be written as

$$K_{Z_1,Z_2}(x,y) = (2\pi)^{-d/2} \int e^{i(\varphi(x,\xi)-\varphi(y,\xi))}\, q_{Z_1,Z_2}(x,y,\xi)\, d\xi, \tag{3.7}$$

with

$$q_{Z_1,Z_2}(x,y,\xi) = a_{Z_1}(x,\xi)\overline{a_{Z_2}(y,\xi)} \in \mathscr{S}(\mathbf{R}^{3d})$$

supported in

$$\{\, (x,y,\xi)\,;\, |x-z_1| \le 1,\ |y-z_2| \le 1,\ |\xi-\zeta_1| \le 1,\ |\xi-\zeta_2| \le 1 \,\}.$$

We shall prove that K_{Z_1,Z_2} satisfies the hypotheses of Lemma 3.2 for a suitable M.

Let T be the operator

$$T = H_\varphi \cdot (1-L),$$

where

$$L = i \sum_{j=1}^{n} \Big(\varphi'_{\xi_j}(x,\xi) - \varphi'_{\xi_j}(y,\xi) \Big) \partial_{\xi_j}$$

and

$$H_\varphi(x,y,\xi) = (1 + \big|\varphi'_\xi(x,\xi) - \varphi'_\xi(y,\xi)\big|^2)^{-1}.$$

Then

$$T e^{i(\varphi(x,\xi)-\varphi(y,\xi))} = e^{i(\varphi(x,\xi)-\varphi(y,\xi))},$$

and since

$$\big|\varphi'_\xi(x,\xi) - \varphi'_\xi(y,\xi)\big| \gtrsim |x-y|,$$

by the first part of the proof of Theorem 2.19, we get

$$H_\varphi(x,y,\xi) \lesssim \langle x-y \rangle^2.$$

Consequently, if $\mathscr{D}$ is the map $F \mapsto H_\varphi \cdot F$, then L and $\mathscr{D}$ are continuous on $\mathscr{S}(\mathbf{R}^{3d})$. Since an analogous formula to (2.6) holds for $({}^tT)^N$, by the hypotheses and the above observations we have, for arbitrary $N \in \mathbf{N}$ and suitable differential operators $V_N(\mathscr{D}, L)$, depending on $\mathscr{D}$, L and

N,

$$
\begin{aligned}
K_{Z_1,Z_2}(x,y) &= (2\pi)^{-d/2}\int T^N\, e^{i(\varphi(x,\xi)-\varphi(y,\xi))}\, q_{Z_1,Z_2}(x,y,\xi)\, d\xi \\
&= (2\pi)^{-d/2}\int e^{i(\varphi(x,\xi)-\varphi(y,\xi))}\, ({}^tT)^N\, q_{Z_1,Z_2}(x,y,\xi)\, d\xi \\
&= (2\pi)^{-d/2}\int e^{i(\varphi(x,\xi)-\varphi(y,\xi))}\, (\mathscr{D}^N + V_N(\mathscr{D},L))\, q_{Z_1,Z_2}(x,y,\xi)\, d\xi
\end{aligned}
$$

Since each term appearing in $V_N(\mathscr{D},L)$ contains exactly N operators with $\mathscr{D}$, by standard arguments we find

$$
K_{Z_1,Z_2}(x,y) \lesssim \tau\left(\frac{\zeta_1-\zeta_2}{2}\right)\tau(x-z_1)\,\tau(y-z_2)\,(1+|x-y|^2)^{-N}, \quad (3.8)
$$

where $\tau = \chi_{B_1(0)}$ is the characteristic function of the unit ball in $\mathbf{R}^d$. Then:

$$
\begin{aligned}
&\sup_y \int |K_{Z_1,Z_2}(x,y)|\, dx \\
&\quad\lesssim \tau\left(\frac{\zeta_1-\zeta_2}{2}\right) \sup_{y\in B_1(z_2)} \int_{B_1(0)} (1+|x+(z_1-y)|^2)^{-N}\, dx \\
&\quad\quad\lesssim \tau\left(\frac{\zeta_1-\zeta_2}{2}\right) \sup_{y\in B_1(z_2)} (1+|z_1-y|^2)^{-N} \\
&\quad\quad\quad\lesssim \tau\left(\frac{\zeta_1-\zeta_2}{2}\right) (1+|z_1-z_2|^2)^{-N}
\end{aligned}
$$

and analogously for $\sup_x \int |K_{Z_1,Z_2}(x,y)|\, dy$, owing to the symmetry in the estimate (3.8). So, all requirements of Lemma 3.2 are satisfied and summing up, we have:

$$
\begin{aligned}
|\zeta_1-\zeta_2| \geq 2 \quad &\Rightarrow \quad A_{Z_1}A^*_{Z_2} = 0 \\
|\zeta_1-\zeta_2| \leq 2 \quad &\Rightarrow \quad \|A_{Z_1}A^*_{Z_2}\| \lesssim (1+|z_1-z_2|^2)^{-N}.
\end{aligned}
$$

An analogous estimate can be obtained for $A^*_{Z_1}A_{Z_2}$, in view of the symmetry in the role of variables and covariables in SG phases and amplitudes. Then, also the requirements (3.1) and (3.2) of Lemma 3.3 are satisfied. This gives the result. □

References

[1] G.D. Andrews, *A Closed Class of* SG *Fourier Integral Operators with Applications.* PhD thesis, Imperial College, London (2004).

[2] K. Asada, D. Fujiwara, *On Some Oscillatory Transformation in* $L^2(\mathbf{R}^n)$. Japan J. Math., **4** (1978), 229–361.

[3] M. S. Berger, *Nonlinearity and Functional Analysis.* Academic Press (1977).

[4] P. Boggiatto, E. Buzano, L. Rodino, *Global Hypoellipticity and Spectral Theory.* Mathematical Research, 92, Akademie Verlag, Berlin (1996).

[5] M. Cappiello, L. Rodino, SG-*pseudodifferential operators and Gelfand-Shilov spaces*. Rocky Mountain J. Math., **36**, 4 (2006), 1117–1148.

[6] E. Cordero, F. Nicola and L. Rodino, *Boundedness of Fourier Integral Operators on $\mathcal{F}L^p$ spaces*. Trans. Amer. Math. Soc. **361** (2009), 6049–6071.

[7] E. Cordero, F. Nicola, L. Rodino, *On the Global Boundedness of Fourier Integral Operators*. Ann. Global Anal. Geom. **38** (2010), 373–398.

[8] H. O. Cordes, *The Technique of Pseudodifferential Operators.* Cambridge Univ. Press (1995).

[9] S. Coriasco, *Fourier Integral Operators in* SG *classes I. Composition Theorems and Action on* SG *Sobolev spaces.* Rend. Sem. Mat. Univ. Politec. Torino **57** (1999), 4, 249-302 (2002).

[10] S. Coriasco, *Fourier Integral Operators in* SG *Classes II: Application to* SG *Hyperbolic Cauchy Problems*. Ann. Univ. Ferrara, Sez. VII – Sc. Mat. **44** (1998), 81–122.

[11] S. Coriasco, K. Johansson, J. Toft, *Global wave-front sets of Banach, Fréchet and Modulation space types, and pseudo-differential operators*. J. Differential Equations **254** (2013), 3228–3258.

[12] S. Coriasco, K. Johansson, J. Toft, *Global wave-front properties for Fourier integral operators and hyperbolic problems*. J. Fourier Anal. Appl. (2015) DOI 10.1007/s00041-015-9422-1.

[13] S. Coriasco, L. Maniccia, *Wave front set at infinity and hyperbolic linear operators with multiple characteristics*. Ann. Global Anal. Geom. **24** (2003), 375–400.

[14] S. Coriasco, L. Rodino, *Cauchy problem for* SG*-hyperbolic equations with constant multiplicities*. Ric. di Matematica, **48** (Suppl.) (1999), 25–43.

[15] S. Coriasco, M. Ruzhansky, *Global L^p-continuity of Fourier Integral Operators*. Trans. Amer. Math. Soc. (2014), **366**, 5, 2575–2596.

[16] S. Coriasco, R. Schulz, *Global Wave Front Set of Tempered Oscillatory Integrals with Inhomogeneous Phase Functions*. J. Fourier Anal. Appl. **19**, 5 (2013), 1093-1121.

[17] S. Coriasco, J. Toft, *Asymptotic expansions for Hörmander symbol classes in the calculus of pseudo-differential operators*. J. Pseudo-Differ. Oper. Appl. **5**, 1 (2014), 27-41.

[18] Y. V. Egorov, B.-W. Schulze, *Pseudo-differential operators, singularities, applications*. Volume 93 of *Operator Theory: Advances and Applications.* Birkhäuser Verlag, Basel (1997).

[19] H. G. Feichtinger, *Modulation spaces on locally compact abelian groups. Technical report*, University of Vienna, Vienna, 1983; also in: M. Krishna, R. Radha, S. Thangavelu (Eds) Wavelets and their applications, Allied Publishers Private Limited, NewDehli Mumbai Kolkata Chennai Hagpur Ahmedabad Bangalore Hyderbad Lucknow (2003), pp. 99–140.

[20] H. G. Feichtinger, K. H. Gröchenig, *Banach spaces related to integrable group representations and their atomic decompositions, I.* J. Funct. Anal. **86** (1989), 307–340.

[21] H. G. Feichtinger, K. H. Gröchenig, *Modulation spaces: Looking back and ahead*. Sampl. Theory Signal Image Process. **5** (2006), 109–140.

[22] L. Hörmander, *The Analysis of Linear Partial Differential Operators*, vol I–IV. Springer-Verlag, Berlin Heidelberg NewYork Tokyo (1983, 1985).

[23] R. Melrose, *Geometric scattering theory.* Stanford Lectures. Cambridge University Press, Cambridge (1995).

[24] C. Parenti, Operatori pseudodifferenziali in $\mathbb{R}^n$ e applicazioni. *Ann. Mat. Pura Appl.* **93** (1972) 359–389.

[25] M. Ruzhansky, M. Sugimoto, *Global L^2 boundedness theorems for a class of Fourier integral operators.* Comm. Partial Differential Equations **31** (2006), 547–569.

[26] E. Schrohe, *Spaces of weighted symbols and weighted Sobolev spaces on manifolds* In: H. O. Cordes, B. Gramsch, and H. Widom (eds), Proceedings, Oberwolfach, **1256** Springer LMN, New York, (1986), pp. 360-377.

[27] A. Seeger, C.D. Sogge, E.M. Stein, *Regularity properties of Fourier integral operators.* Ann. of Math. **134** (1991), 231–251.

[28] J. Toft, *Continuity properties for modulation spaces with applications to pseudo-differential calculus, II.* Ann. Global Anal. Geom. **26** (2004), 73–106.

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