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ASYMPTOTIC EXPANSIONS FOR HÖRMANDER SYMBOL CLASSES IN THE CALCULUS OF PSEUDO-DIFFERENTIAL OPERATORS

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Abstract. We establish formulas for asymptotic expansions for $S(m,g)$, the Hörmander class parameterized by the metric g and weight function m, defined on the phase space. By choosing m and g in appropriate ways, we cover some classical results on expansions for symbol classes of the form $S^{\tau}_{\rho,\delta}$, and by choosing m and g in other ways we obtain asymptotic expansions for (generalized) SG classes.

0. Introduction

An important property in the theory of pseudo-differential operators concerns asymptotic expansions. For the classical Hörmander symbols $S^{\tau}_{1,0}$, several of such expansions can be related to Proposition 18.1.3 in [7], which is equivalent to the following:

Let τ_j, $j=1,2,\dots$, tend to $-\infty$ as j tends to ∞ and $a_j \in S^{\tau_j}_{1,0}$, Then there is an element $a \in S^{\tau_1}_{1,0}$ such that

$$a - \sum_{k<j} a_k \in S^{\tau_j}_{1,0}, \tag{1}$$

for every k. The element a is uniquely determined modulo $S^{-\infty}_{1,0}$, and can be chosen such that $\operatorname{supp} a \subseteq \cup_{j\ge 1} \operatorname{supp} a_j$.

Here the uniqueness assertion means that if a is as above and (1) holds after a has been replaced by $b \in S^{\tau_1}_{1,0}$, then $a-b$ belongs to $S^{-\infty}_{1,0}$. See also Proposition 23.1 in [8] for similar results for the Shubin classes.

The previous result can be considered as a result on existence, since it ensures that the element a with convenient asserted properties exists. An other useful type of results related to the previous one can be considered as imposing types. For example, in Proposition 18.1.4 in [7] it is assumed that a here above exists, with certain relaxed assumptions. It is then proved that a possess similar properties as in the previous result. More precisely, Proposition 18.1.4 in [7] is equivalent to the following proposition. (See also [8, Prop. 23.2] for corresponding result

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for the Shubin classes.) Here and in what follows we write $A \lesssim B$ when $A \le cB$ for a suitable constant $c > 0$.

Proposition 0. *Let $P_j, \tau_j \in \mathbf{R}$, $j = 1, 2, \dots$, be such that*

$$P_j = \sup_{k \ge j} \tau_k \quad \text{and} \quad \lim_{j \to \infty} \tau_j = -\infty,$$

and let $a \in C^\infty(\mathbf{R}^{2d})$ be such that for every α and β, there are constants $C_{\alpha,\beta}$ and $\mu = \mu(\alpha, \beta)$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \le C_{\alpha,\beta} \langle \xi \rangle^\mu .$$

Also let $a_j \in S^{\tau_j}_{1,0}(\mathbf{R}^{2d})$, $j \ge 1$, be a sequence. If

$$|a - \sum_{k<j} a_k| \lesssim \langle \xi \rangle^{P_j}$$

holds for every j, then $a \in S^{P_1}_{1,0}(\mathbf{R}^{2d})$.

By the uniqueness property for asymptotic expansions it follows that a in Proposition 0 is the asymptotic expansion (modulo $S^{-\infty}_{1,0}$) of the sequence a_j.

In this paper we extend such type of results to symbol classes of the form $S(m, g)$, introduced by Hörmander in [7]. (Cf. Theorems 7, 8 and 9.) Here m is an appropriate weight, and g is an appropriate Riemannian metric on the phase space $W \simeq \mathbf{R}^{2d}$. By choosing m and g in appropriate ways we cover the classical results presented here above (cf. Propositions 18.1.3 and 18.1.4 in [7], and the related results, Propositions 23.1 and 23.2 in [8]).

The conditions on our metric g is that it should be slowly varying and its Planck's constant h_g should be bounded. We do not require that it should be σ-temperate, a property which is strongly needed in order for the symbolic calculus should in $S(m, g)$ classes work properly. Therefore our results can be applied also in the absence of needed prerequisites in the calculus. For example, we permit symbol classes which behaves like $S^{\tau_1}_{1,0}$ in some directions and like $S^{\tau_2}_{1,1}$ in other directions.

The key steps in our proofs are similar to those in the proof of Propositions 18.1.3 and 18.1.4 in [7] and corresponding results in [8]. On the other hand, in order to manage the general situation when dealing with symbols of the form $S(m, g)$, we need some additional arguments.

Finally, in the last section (Section 3) we apply our results on some important types of symbol classes. Especially we consider SG classes. In fact, the original motivation to consider questions on asymptotic expansions on such general level, was to find a common platform for results on asymptotic expansions for the SG classes, and the symbol classes in [7,8]. Note here that the classical results in [7,8] do not cover the case of SG symbols.

1. Preliminaries

In this section we recall the definition and some basic facts for the involved symbol classes. (See Sections 18.4–18.6 in [7] and Section 2 in [9].)

Let $N \in \mathbf{N}$, $W \simeq \mathbf{R}^{2d}$ be the phase space of dimension $2d$, $a \in C^N(W)$, g be an arbitrary Riemannian metric on W, and let $m > 0$ be a measurable function on W. For each $k = 0, \dots, N$, let

$$|a|_k^g(X) = \sup |a^{(k)}(X; Y_1, \dots, Y_k)|, \tag{2}$$

where the supremum is taken over all $Y_1, \dots, Y_k \in W$ such that $g_X(Y_j) \le 1$ for $j = 1, \dots, k$. Also set

$$\|a\|_{m,N}^g \equiv \sum_{k=0}^{N} \sup_{X \in W} \Big(|a|_k^g(X)/m(X) \Big), \tag{3}$$

let $S_N(m,g)$ be the set of all $a \in C^N(W)$ such that $\|a\|_{m,N}^g < \infty$, and let

$$S(m,g) \equiv \bigcap_{N \ge 0} S_N(m,g).$$

Next we recall some properties for the metric g on W (cf. [6, 7, 9]). It follows from Section 18.6 in [7] that for each $X \in W$, there are symplectic coordinates $Z = \sum_{j=1}^d (z_j e_j + \zeta_j \varepsilon_j)$ which diagonalize g_X, i. e. g_X takes the form

$$g_X(Z) = \sum_{j=1}^{d} \lambda_j(X)(z_j^2 + \zeta_j^2), \tag{4}$$

where

$$\lambda_1(X) \ge \lambda_2(X) \ge \cdots \ge \lambda_d(X) > 0, \tag{5}$$

only depend on g_X and are independent of the choice of symplectic coordinates which diagonalize g_X. Here $e_1, \dots, e_d, \varepsilon_1, \dots, \varepsilon_d$ is a symplectic basis for W.

The *dual metric* g^σ and *Planck's function* h_g with respect to g and the symplectic form σ, are defined by

$$g_X^\sigma(Z) \equiv \sup_{Y \ne 0} \frac{\sigma(Y,Z)^2}{g_X(Y)} \quad \text{and} \quad h_g(X) = \sup_{Z \ne 0} \Big(\frac{g_X(Z)}{g_X^\sigma(Z)} \Big)^{1/2},$$

respectively. It follows that if (4) and (5) are fulfilled, then $h_g(X) = \lambda_1(X)$ and

$$g_X^\sigma(Z) = \sum_{j=1}^{d} \lambda_j(X)^{-1}(z_j^2 + \zeta_j^2). \tag{4$'$}$$

In most of the applications we have that $h_g(X) \le 1$ everywhere, i. e. the *uncertainly principle* holds.

The metric g is called *symplectic* if $g_X = g_X^\sigma$ for every $X \in W$. It follows that g is symplectic if and only if $\lambda_1(X) = \cdots = \lambda_d(X) = 1$ in (4).

There are several investigations which have been done for metrics which occur in the symbolic calculi. (see e. g. [4, 9]). For exaple, in [9] it is proved that for every $t \in \mathbf{R}$ there is a symplectically invariant defined Riemannian metric $g^{(t)}$ which takes the form

$$g_X^{(t)}(Z) = \sum_{j=1}^{d} \lambda_j(X)^t (z_j^2 + \zeta_j^2), \tag{4$''$}$$

when (4) holds. We note that $g^{(t_1)} = g$ and $g^{(t_2)} = g^\sigma$, when $t_1 = 1$ and $t_2 = -1$, and that the dual metric for $g^{(t)}$ is given by $g^{(-t)}$. Furthermore, $g^{(0)}$ agrees with the symplectic metric g^0, given by

$$g_X^0(Z) = \sum_{j=1}^{d} (z_j^2 + \zeta_j^2),$$

when (4) holds.

The Riemannian metric g on W is called *slowly varying* if there are positive constants c and C such that

$$g_X(Y - X) \le c \quad \Longrightarrow \quad C^{-1} g_Y \le g_X \le C g_Y, \tag{6}$$

and the positive function m on W is called *g-continuous* if there are constants c and C such that

$$g_X(Y - X) \le c \quad \Longrightarrow \quad C^{-1} m(Y) \le m(X) \le C m(Y). \tag{7}$$

We observe that if g is slowly varying, $N \ge 0$ is an integer and m is g-continuous, then $S_N(m, g)$ is a Banach space when the topology is defined by the norm (3). Moreover, $S(m, g)$ is a Frechét space under the topology defined by the norms (3) for all $N \ge 0$.

Let g and G be Riemannian metrics on W. Then G is called *(σ,g)-temperate*, if there is an integer $N \ge 0$ such that

$$\begin{aligned} &G_Y(Z) \lesssim G_X(Z)(1 + g_Y^\sigma(X - Y))^N, \\ &G_Y(Z) \lesssim G_X(Z)(1 + g_X^\sigma(X - Y))^N, \quad \text{for all } X, Y, Z \in W. \end{aligned} \tag{8}$$

The metric g is called σ-temperate, if g is (σ, g)-temperate.

Let g be a Riemannian metric on W. The function m on W is called (σ, g)-temperate if m is positive everywhere and there is a constant N such that

$$\begin{aligned} &m(X) \lesssim m(Y)(1 + g_X^\sigma(X - Y))^N, \\ &m(X) \lesssim m(Y)(1 + g_Y^\sigma(X - Y))^N, \quad \text{for all } X, Y, Z \in W. \end{aligned} \tag{9}$$

If g is σ-temperate, then only one of the conditions in (8) and in (9) are needed.

The following restatement of Proposition 1.2 in [6] shows that the functions λ_j posses appropriate symplectic invariance properties and appropriate continuity properties related to the metric g. We omit the proof since it can be found in [6]. Here we set

$$\Lambda_g(X) = \lambda_1(X) \cdots \lambda_d(X), \tag{10}$$

when g_X is given by (4).

Proposition 1. *Assume that g is a Riemannian metric on W, and that $X \in W$ is fixed. Also assume that the symplectic coordinates are chosen such that* (4) *holds. Then the following is true:*

(1) *λ_j for $1 \le j \le d$ and Λ_g are symplectically invariantly defined;*

(2) *if in addition g is slowly varying, then λ_j for $1 \le j \le d$ and Λ_g are g-continuous;*

(3) *if in addition g is σ-temperate, then λ_j for $1 \le j \le d$ and Λ_g are (σ, g)-temperate.*

The following definition is motivated by the general theory of Weyl calculus. (See e. g. [6,9], and Section 18.4–18.6 in [7].)

Definition 2. Assume that g is a Riemannian metric on W. Then g is called

(i) *feasible* if g is slowly varying and $h_g \le 1$ everywhere;

(ii) *strongly feasible* if g is feasible and σ-temperate.

If g is feasible and m is g-continuous, then $S(h_g^r m, g)$ decreases with respect to r. For conveniency we set

$$S(h_g^\infty m, g) \equiv \bigcap_{r \ge 0} S(h_g^r m, g),$$

in this situation.

Note that feasible and strongly feasible metrics are not standard terminology. In the literature it is common to use the term 'Hörmander metric' or 'admissible metric' instead of 'strongly feasible' for metrics which satisfy (ii) in Definition 2. (See [1–5].) An important reason for us to follow [6,9] concerning this terminology is that we permit metrics which are not admissible in the sense of [1–5], and that we prefer similar names for metrics which satisfy (i) or (ii) in Definition 2.

It is obvious that $g^{(t_1)} \le g^{(t_2)}$ when $t_1 \le t_2$ and $h_g \le 1$. In particular, $g \le g^{(t)} \le g^\sigma$ when $-1 \le t \le 1$ and $h_g \le 1$. In the following proposition we list some important properties for strongly feasible metrics. The proof is omitted since the result can be found in [9].

Proposition 3. *Let g be a strongly feasible metric on W, G be a Riemannian metric on W, and let $t_1, t_2 \in [-1, 1]$ be such that $t_2 > -1$. If G is (σ, g)-temperate, then $G^{(t_1)}$ is $(\sigma, g^{(t_2)})$-temperate.*

In particular, $g^{(t_1)}$ is $(\sigma, g^{(t_2)})$-temperate, and if $t \in [0,1]$, then $g^{(t)}$ is strongly feasible.

Remark 4. Assume that g is slowly varying on W and let c be the same constant as in (6). Then it follows from Theorem 1.4.10 in [7] that there is a constant $\varepsilon_0 > 0$, an integer $N_0 \ge 0$ and a sequence $\{X_j\}_{j\in\mathbf{N}}$ in W such that the following is true:

(1) there is a positive number ε such that $g_{X_j}(X_j - X_k) \ge \varepsilon_0$ for every $j,k \in \mathbf{N}$ such that $j \neq k$;

(2) $W = \bigcup_{j\in\mathbf{N}} U_j$, where U_j is the g_{X_j}-ball $\{\, X\, ;\, g_{X_j}(X - X_j) < c\,\}$;

(3) the intersection of more than N_0 balls U_j is empty.

Remark 5. It follows from Section 1.4 and Section 18.4 in [7] that if g is a slowly varying metric on W, and (1)–(3) in Remark 4 holds, then there is a sequence $\{\varphi_j\}_{j\in\mathbf{N}}$ in $C_0^\infty(W)$ such that the following is true:

(1) $0 \le \varphi_j \in C_0^\infty(U_j)$ for every $j \in \mathbf{N}$;

(2) $\sup_{j\in\mathbf{N}} \|\varphi_j\|_{1,N}^{g_{X_j}} < \infty$ for every integer $N \ge 0$ (i. e. $\{\varphi_j\}_{j\in\mathbf{N}}$ is a bounded sequence in $S(1,g)$);

(3) $\sum_{j\in\mathbf{N}} \varphi_j = 1$ on W.

1.1. **An important family of symbol classes.** A broad family of symbol classes concerns the following extended family of SG symbol classes. Let $t, \tau, r_l, \rho_l \in \mathbf{R}$ for $l = 1,2$. Then the (generalized) SG class $\mathrm{SG}^{t,\tau}_{(r_l,\rho_l)}(\mathbf{R}^{2d})$, $l = 1,2$, consists of all $a \in C^\infty(\mathbf{R}^{2d})$ such that for every multi-indices α, β, there is a constant $C_{\alpha,\beta}$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta} \langle x\rangle^{t - r_1|\alpha| + r_2|\beta|} \langle \xi\rangle^{\tau + \rho_1|\alpha| - \rho_2|\beta|}.$$

If m and g are given by $m(x,\xi) = \langle x\rangle^t \langle\xi\rangle^\tau$ and

$$g_{(x,\xi)}(y,\eta) = \langle x\rangle^{-2r_1}\langle\xi\rangle^{2\rho_1}|y|^2 + \langle x\rangle^{2r_2}\langle\xi\rangle^{-2\rho_2}|\eta|^2, \tag{11}$$

respectively, then it follows that $S(m,g) = \mathrm{SG}^{t,\tau}_{(r_j,\rho_j)}(\mathbf{R}^{2d})$. If

$$0 \le r_2 \le r_1 \le 1 \quad \text{and} \quad 0 \le \rho_1 \le \rho_2 \le 1$$

then g is feasible and m is g-continuous in this case. If in addition $r_2, \rho_1 < 1$, then g is strongly feasible. (Cf. [7].)

The dual metric and Planck's constant are given by

$$g^\sigma_{(x,\xi)}(y,\eta) = \langle x\rangle^{-2r_2}\langle\xi\rangle^{2\rho_2}|y|^2 + \langle x\rangle^{2r_1}\langle\xi\rangle^{-2\rho_1}|\eta|^2,$$

$$h_g(x,\xi) = \langle x\rangle^{-(r_1-r_2)}\langle\xi\rangle^{-(\rho_2-\rho_1)},$$

$$g^0_{(x,\xi)}(y,\eta) = \langle x\rangle^{-(r_2+r_1)}\langle\xi\rangle^{\rho_1+\rho_2}|y|^2 + \langle x\rangle^{r_1+r_2}\langle\xi\rangle^{-(\rho_1+\rho_2)}|\eta|^2.$$

For future references we note that

$$S(m, h_g^{-N} g) = \mathrm{SG}^{t,\tau}_{(r_l - N, \rho_l - N)}(\mathbf{R}^{2d}), \quad l = 1,2.$$

In particular it follows that if $S(m,g)$ is a symbol class of the form $\mathrm{SG}^{t,\tau}_{(r_l,\rho_l)}$, then the same is true for $S(m,g^\sigma)$, $S(m,g^0)$ and $S(m,h_g^{-N}g)$.

We have the following two important special cases of the symbol classes here above.

(1) If $r_2=\rho_1=0$, $r_1=r$ and $\rho_2=\rho$, then $\mathrm{SG}^{t,\tau}_{(r_l,\rho_l)}(\mathbf{R}^{2d})$ agrees with the classical SG class $\mathrm{SG}^{t,\tau}_{r,\rho}(\mathbf{R}^{2d})$. In particular, in contrast to the extended family of SG symbol classes, the classical SG classes are in general not stable under replacements of the metric g in $S(m,g)$ here above, by g^σ, g^0 or $h_g^{-N}g$.

(2) If $t=r_1=r_2=0$, $\rho_1=\delta$ and $\rho_2=\rho$, here above, then $S(m,g)$ agrees with the Hörmander class $S^\tau_{\rho,\delta}(\mathbf{R}^{2d})$, which consists of all $a\in C^\infty(\mathbf{R}^{2d})$ such that for every multi-indices α,β, there is a constant $C_{\alpha,\beta}$ such that

$$|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)|\le C_{\alpha,\beta}\langle\xi\rangle^{\tau-\rho|\beta|+\delta|\alpha|}.$$

More generally, we have the following extended family of SG classes. Here a weight m on $\mathbf{R}^{2d}$ is called an SG weight on $\mathbf{R}^{2d}$ of order (r_l,ρ_l), $l=1,2$, if $0<m\in\mathrm{SG}^{t_0,\tau_0}_{(r_l,\rho_l)}(\mathbf{R}^{2d})$ for some $t_0,\tau_0\in\mathbf{R}$, and

$$m(x+y,\xi+\eta)\lesssim m(x,\xi)\langle(y,\eta)\rangle^N, \tag{12}$$

for some constant N.

Definition 6. Let $t_0,\tau_0,r_l,\rho_l\in\mathbf{R}$ and let m be an SG weight on $\mathbf{R}^{2d}$ of order (r_l,ρ_l), $l=1,2$. Then $\mathrm{SG}^{(m)}_{(r_l,\rho_l)}(\mathbf{R}^{2d})$, $j=1,2$, is the set of all $a\in C^\infty(\mathbf{R}^{2d})$ such that for every pairs of multi-indices α and β, it holds

$$|\partial_\xi^\alpha\partial_x^\beta a(x,\xi)|\le C_{\alpha,\beta}m(x,\xi)\langle x\rangle^{-r_1|\alpha|+r_2|\beta|}\langle\xi\rangle^{\rho_1|\alpha|-\rho_2|\beta|},$$

where the constants $C_{\alpha,\beta}$ only depends on α and β.

Let g be given by (11). Then it follows by straight-forward computations that $S(m,g)=\mathrm{SG}^{(m)}_{(r_l,\rho_l)}(\mathbf{R}^{2d})$. Furthermore, we have

$$S(m,g^\sigma)=\mathrm{SG}^{(m)}_{(p_j,\pi_j)}(\mathbf{R}^{2d}),\quad\text{when}\quad (p_1,p_2,\pi_1,\pi_2)=(r_2,r_1,\rho_2,\rho_1),$$

$$S(m,g^0)=\mathrm{SG}^{(m)}_{(p_j,\pi_j)}(\mathbf{R}^{2d}),\quad\text{when}\quad (p_1,p_2,\pi_1,\pi_2)=(r_1+r_2,r_1+r_2,\rho_1+\rho_2,\rho_1+\rho_2)/2,$$

and

$$S(m,h_g^{-2N}g)=\mathrm{SG}^{(m)}_{(p_j,\pi_j)}(\mathbf{R}^{2d}),\quad\text{when}\quad (p_1,p_2,\pi_1,\pi_2)=(r_1-N,r_2-N,\rho_1-N,\rho_2-N).$$

2. Asymptotic expansions

In this section we establish asymptotic expansion results for elements in the symbol class $S(m,g)$.

Let g be a feasible metric, and let m be g-continuous. Then the following proposition shows that $S(m,g)$ fulfills convenient asymptotic expansion properties.

Theorem 7. *Let g be feasible and let m be g-continuous. If $0 \le r_j \in \mathbf{R}$, $j \ge 1$, is strictly increasing and satisfies*

$$\lim_{j\to\infty} r_j = \infty,$$

and $a_j \in S(h_g^{r_j} m, g)$, then there is an element $a \in S(h_g^{r_1} m, g)$ such that

$$a - \sum_{k<j} a_k \in S(h_g^{r_j} m, g), \quad \textit{for every} \quad j \ge 1. \tag{13}$$

The element a is uniquely determined modulo $S(h_g^{\infty} m, g)$, and can be chosen such that $\operatorname{supp} a \subseteq \cup_{j\ge 1} \operatorname{supp} a_j$.

Proof. Let U_j and φ_j be the same as in Remarks 4 and 5. For any integer $k \ge 1$, let J_k be the set of all $l \ge 1$ such that

$$|a_k\varphi_l|_n^g(X) \le 2^{-k} h_g^{r_{k-1}}(X)m(X), \qquad n \le k,\ X \in W, \tag{14}$$

and let $\psi_k = \sum_{i\in J_k} \varphi_i$. We also let $b_1 = a_1$, and define inductively

$$b_{k+1} = b_k + \psi_{k+1}a_{k+1}, \qquad k \ge 1.$$

We claim that

(1) $b \equiv \lim_{k\to\infty} b_k$ exists and defines an element in $S(m,g)$. Furthermore, $\lim_{k\to\infty} \|b-b_k\|^g_{N,h_g^r m} = 0$, for any $r>0$ and $N \ge 0$;

(2) $b - \sum_{j<k} a_j \in S(h_g^{r_k} m, g)$.

In fact, by the definitions and Weierstrass theorem it follows that b exists in $S(m,g)$, and that for N fixed, then $\lim_{k\to\infty} \|b-b_k\|^g_{N,h_g^r m} = 0$, for every fixed $r>0$. This gives (1) in the claim.

Next we prove (2). We have

$$b - \sum_{j<k} a_j = u_1 + u_2,$$

where

$$u_1 = \sum_{j=2}^{k-1} (\psi_j - 1)a_j \qquad \text{and} \qquad u_2 = \sum_{j=k}^{\infty} \psi_j a_j.$$

The result follows if we prove that $u_1 \in S(h_g^r m, g)$ and $u_2 \in S(h_g^{r_k} m, g)$ for every $r>0$.

Let $K_j = \complement J_j = \mathbf{Z}_+ \setminus J_j$. We have

$$u_1 = -\sum_{j=2}^{k-1} v_j,$$

where

$$v_j = \sum_{i \in K_j} \varphi_i a_j,$$

and we shall investigate the terms v_j separately. First let $c_0 > 0$ be fixed and let $K_{1,j}$ be the set of all $i \in K_j$ such that $h_g(X) \ge c_0$ when $X \in U_i$. Also let $K_{2,j} = K_j \setminus K_{1,j}$, and let $\Omega_{k,j} = \cup_{i \in K_{k,j}} U_i$. Then $v_j = v_{1,j} + v_{2,j}$, where

$$v_{k,j} = \sum_{i \in K_{k,j}} \varphi_i a_j.$$

Since $h_g \ge c_0$ on $\Omega_{1,j}$, it follows that if $r \ge 0$ and v is smooth on W with support in $\Omega_{1,j}$, then

$$v \in S(h_g^r m, g) \quad \Longleftrightarrow \quad v \in S(m, g).$$

In particular, $v_{1,j} \in S(h_g^{r_k} m, g)$.

Next we consider $v_{2,j}$. From the fact that (14) is violated for some $n \in [0, j]$, $a_j \in S(h_g^{r_j} m, g)$, $r_{j-1} < r_j$ and $h_g(X) < C c_0$ when $X \in \cup_{i \in K_{2,j}} U_i$, it follows that $K_{2,j}$ is a finite set, provided c_0 was chosen small enough. Here C is the same as in (6). This implies that

$$v_{2,j} \in C_0^\infty \subseteq S(h_g^{r_k} m, g).$$

Consequently, $v_{2,j}$, and thereby u_1 belong to $S(h_g^\infty m, g)$.

It remains to consider u_2. We have $u_2 = b - b_{k-1}$ and $\psi_j a_j \in S(h_g^{r_j} m, g) \subseteq S(h_g^{r_k} m, g)$ when $j \ge k$. Since $\|b - b_{k-1}\|_{N, h_g^r m} \to 0$ when $k \to \infty$, for every fixed $r > 0$, it follows that $u_2 \in S(h_g^{r_k} m, g)$. This gives (13) with $a = b$. Furthermore, due to the construction we also have $\operatorname{supp} a \subseteq \cup_{j \ge 1} \operatorname{supp} a_j$.

The uniqueness follows from the uniqueness of the next result. $\square$

We also have the following extension of the previous result.

Theorem 8. *Let g be feasible and m be g-continuous. If $r_j, R_j \in \mathbf{R}$, $j \ge 1$, satisfy*

$$\lim_{j \to \infty} r_j = \infty \quad \text{and} \quad R_j = \min_{k \ge j} r_k,$$

and $a_j \in S(h_g^{r_j} m, g)$, then there is an element $a \in S(h_g^{R_1} m, g)$ such that

$$a - \sum_{k<j} a_k \in S(h_g^{R_j} m, g), \quad \text{for every} \quad j \ge 1. \tag{15}$$

The element a is uniquely determined modulo $S(h_g^\infty m, g)$, and can be chosen such that $\operatorname{supp} a \subseteq \cup_{j \ge 1} \operatorname{supp} a_j$.

Let m, g, r_j and R_j be the same as in Theorem 8. Then we write

$$a \sim \sum a_j \tag{15$'$}$$

(with respect to the weight m and the metric g), when (15) holds.

Proof. Let n be the largest number such that $r_n < 0$. By replacing a with

$$a - \sum_{j \le n} a_j,$$

it follows that we may assume that $R_1 \ge 0$. Since $r_j \ge R_j$, it follows that $S(h_g^{r_j} m, g) \subseteq S(h_g^{R_j} m, g)$. Hence it is no restriction to assume that $r_j = R_j$, which in particular implies that r_j increases with j. Finally, by letting

$$b_k = \sum_{r_j = r_k} a_j,$$

and considering the sequence $\{b_k\}$ instead of $\{a_j\}$, we reduce ourself to the case that $r_j \ge 0$ are strictly increasing. The expansion (15) is now an immediate consequence of Theorem 7.

If $b \in S(h_g^r m, g)$ satisfies $b \sim \sum a_j$, then it follows from (15) that $a - b \in S(h_g^\infty m, g)$. This gives the asserted uniqueness, and the result follows. □

We have now the following proposition.

Theorem 9. *Let g be a feasible metric on W, and let m and m_j, $j \ge 0$, be g-continuous weights such that*

$$m_j \le C h_g^{-s_j} m,$$

for some real numbers s_j, and let r_j, R_j be the same as in Theorem 8. Also let $a \in C^\infty(\mathbf{R}^{2d})$ be such that

$$\|a\|^g_{m_j, j} < \infty$$

for every $j \ge 0$, and let $a_j \in S(h_g^{r_j} m, g)$, $j \ge 1$, be a sequence. If

$$|a - \sum_{k<j} a_k| \lesssim h_g^{R_j} m$$

holds for every $j \ge 1$, then $a \in S(h_g^{R_1} m, g)$, and (15)$'$ *holds.*

Proof. We shall use the same framework as in the proof of Proposition 18.1.4 in [7]. We may assume that $s_j \le R_j$.

By Theorem 8, there is an element $b \in S(h_g^{R_1} m, g)$ such that $b \sim \sum a_k$. Let $u = a - b$. Then it follows from the assumptions that

$$\|u\|^g_{m_1,1} < \infty, \|u\|^g_{m_2,2} < \infty \quad \text{and} \quad |u| \lesssim h_g^{2N+2s_2} m, \quad \text{for every } N \ge 0.$$

The result follows if we prove that $u \in S(h_g^\infty m, g)$.

Let c be chosen such that (6) holds, $N \geq 0$ and $\varepsilon \in (0,1)$, and let $X, Y \in W$ be fixed such that $g_X(Y) < c$. By Taylor's formula we have

$$|u(X+\varepsilon Y) - u(X) - \varepsilon(\partial_Y u)(X)| \leq 2^{-1}\varepsilon^2 |(\partial^2_{Y,Y} u)(X+\theta Y)|,$$

for some $\theta \in [0,\varepsilon]$. This gives

$$\begin{aligned} |(\partial_Y u)(X)| &\leq \varepsilon^{-1}(|u(X+Y)| + |u(X)|) + 2^{-1}\varepsilon |(\partial^2_{Y,Y} u)(X+\theta Y)|. \\ &\leq C_1\Big(\varepsilon^{-1}\big(h_g^{2N+2s_2}(X+Y)m(X+Y) + h_g^{2N+2s_2}(X)m(X)\big) \\ &\qquad + \varepsilon h_g^{-2s_2}(X+\theta Y)m(X+\theta Y)\Big) \\ &\leq C_2\left(\varepsilon^{-1} h_g^{2N+2s_2}(X) + \varepsilon h_g^{-2s_2}(X)\right) m(X), \end{aligned}$$

for some constants C_1 and C_2 which only depend on m_2, N, the constants in (6) and (7), and $\|a\|^g_{m_2,2}$. In the last step we have used the fact that m is g-continuous, $g_X(Y) < c$ and $g_X(\theta Y) < c$.

By taking the supremum of the left-hand side over all possible Y and choosing $\varepsilon = h_g^{N+2s_2}(X)$, we obtain

$$\sqrt{c}|u|^g_1(X) \leq C_3 h_g(X)^N m(X),$$

which gives $\|u\|^g_{h_g^N m,1} < \infty$.

By induction, using similar arguments after u has been replaced by $(\partial_{Y_1}\cdots\partial_{Y_k})u$, we get $\|u\|^g_{h_g^N m,k} < \infty$ for all $k \geq 0$ and $N \geq 0$. That is $u \in S(h_g^\infty m, g)$, and the proof is complete. □

Remark 10. Let $t \in (-1,1]$, and assume that λ_j in (4) satisfy $\lambda_1 \lesssim h_g^{-M}\lambda_d$, for some constant $M \geq 0$. These conditions are usually fulfilled, e. g. they are fulfilled for any of the symbol classes in Subsection 1.1. Since the metric $G \equiv g^{(t)}$ is g-continuous, it follows from Proposition 3 that Theorem 9 in this case remains the same after the condition $\|a\|^g_{m_j,j} < \infty$ has been replaced by the weaker condition $\|a\|^G_{m_j,j} < \infty$.

3. Applications to more specific types of symbol classes

In this section we apply the results in the previous section to symbol classes of the form $\mathrm{SG}^{(m)}_{(r_l,\rho_l)}$, $l = 1,2$, where

$$0 \leq r_2 \leq r_1 \leq 1 \quad \text{and} \quad 0 \leq \rho_1 \leq \rho_2 \leq 1. \tag{16}$$

(Cf. Subsection 1.1.)

The following results are immediate consequences of the listed properties in Subsection 1.1, and Theorems 8 and 9. Here and in what follows we let

$$m_{t,\tau}(x,\xi) = m(x,\xi)\langle x\rangle^t \langle \xi\rangle^\tau, \tag{17}$$

$$\mathrm{SG}^{(m_{t,-\infty})}_{(r_l,\rho_l)} \equiv \bigcap_{j=0}^{\infty} \mathrm{SG}^{(m_{t,-j})}_{(r_l,\rho_l)} \quad \text{and} \quad \mathrm{SG}^{(m_{-\infty,\tau})}_{(r_l,\rho_l)} \equiv \bigcap_{j=0}^{\infty} \mathrm{SG}^{(m_{-j,\tau})}_{(r_l,\rho_l)},$$

and observe that

$$\bigcap_{j_1,j_2=0}^{\infty} \mathrm{SG}_{(r_l,\rho_l)}^{(m_{-j_1,-j_2})} = \mathscr{S}.$$

Theorem 11. *Let $t_j, \tau_j, R_j, P_j, r_l, \rho_l \in \mathbf{R}$, $l = 1,2$, be such that* (16) *holds,*

$$R_j = \max_{k\ge j} t_k, \quad P_j = \max_{k\ge j} \tau_k, \quad \text{and} \quad \lim_{j\to\infty} t_j = \lim_{j\to\infty} \tau_j = -\infty,$$

$l = 1,2,\dots$. *Also let m be an* SG *weight on $\mathbf{R}^{2d}$ of order (r_l,ρ_l), $l = 1,2$ and let $m_{t,\tau}$ be given by* (17). *If $a_j \in \mathrm{SG}_{(r_l,\rho_l)}^{(m_{t_j,\tau_j})}(\mathbf{R}^{2d})$, then there is an element $a \in \mathrm{SG}_{(r_l,\rho_l)}^{(m_{R_1,P_1})}(\mathbf{R}^{2d})$ such that*

$$a - \sum_{k<j} a_k \in \mathrm{SG}_{(r_l,\rho_l)}^{(m_{R_j,P_j})}(\mathbf{R}^{2d}).$$

Furthermore,

(1) *if $r_2 < r_1$, then a is uniquely determined modulo $\mathrm{SG}_{(r_l,\rho_l)}^{(m_{-\infty,P_1})}(\mathbf{R}^{2d})$;*

(2) *if $\rho_1 < \rho_2$, then a is uniquely determined modulo $\mathrm{SG}_{(r_l,\rho_l)}^{(m_{R_1,-\infty})}(\mathbf{R}^{2d})$;*

(3) *if $r_2 < r_1$ and $\rho_1 < \rho_2$, then a is uniquely determined modulo $\mathscr{S}(\mathbf{R}^{2d})$.*

The element a and can be chosen such that $\operatorname{supp} a \subseteq \cup_{j\ge1} \operatorname{supp} a_j$.

Theorem 12. *Let $t_j, \tau_j, R_j, P_j, r_l, \rho_l \in \mathbf{R}$, $j = 1,2,\dots$, $l = 1,2$, and $m_{t,\tau}$ be the same as in Theorem 11. Also let $a \in C^\infty(\mathbf{R}^{2d})$ be such that for every α and β there are constants $\mu = \mu(\alpha,\beta)$ and $C_{\alpha,\beta}$ such that*

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \lesssim m(x,\xi)\langle x\rangle^{(r_1-r_2)\mu}\langle \xi\rangle^{(\rho_2-\rho_1)\mu},$$

and let $a_j \in \mathrm{SG}_{(r_l,\rho_l)}^{(m_{t_j,\tau_j})}(\mathbf{R}^{2d})$, $j \ge 1$, be a sequence. If

$$|a - \sum_{k<j} a_k| \lesssim m(x,\xi)\langle x\rangle^{-R_j}\langle \xi\rangle^{-P_j}$$

holds for every $j \ge 1$, then $a \in \mathrm{SG}_{(r_l,\rho_l)}^{(m_{R_1,P_1})}(\mathbf{R}^{2d})$ and $(15)'$ *holds.*

The following results are immediate consequences of the previous theorems. Here the second corollary is a slight extension of Proposition 0 in the introduction.

Corollary 13. *Let $P_j, \tau_j, \rho, \delta \in \mathbf{R}$ be such that $0 \le \delta < \rho \le 1$,*

$$P_j = \min_{k\ge j} \tau_k \quad \text{and} \quad \lim_{j\to\infty} \tau_j = -\infty,$$

$j = 1,2,\dots$. *If $a_j \in S_{\rho,\delta}^{\tau_j}(\mathbf{R}^{2d})$, then there is an element $a \in S_{\rho,\delta}^{P_1}(\mathbf{R}^{2d})$ such that*

$$a - \sum_{k<j} a_k \in S_{\rho,\delta}^{P_j}(\mathbf{R}^{2d}).$$

The element a is uniquely determined modulo $S^{-\infty}_{\rho,\delta}(\mathbf{R}^{2d})$, and can be chosen such that $\operatorname{supp} a \subseteq \cup_{j\ge 1} \operatorname{supp} a_j$.

Corollary 14. *Let $P_j, \tau_j, \rho, \delta \in \mathbf{R}$ be the same as in Corollary 13. Also let $a \in C^\infty(\mathbf{R}^{2d})$ be such that for every multi-indices α and β, there is a constant $\mu = \mu(\alpha, \beta)$ such that*

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \lesssim \langle \xi \rangle^\mu,$$

and let $a_j \in S^{\tau_j}_{\rho,\delta}(\mathbf{R}^{2d})$, $j \ge 1$, be a sequence. If

$$|a - \sum_{k<j} a_k| \lesssim \langle \xi \rangle^{P_j}$$

holds for every $j \ge 1$, then $a \in S^{P_1}_{\rho,\delta}(\mathbf{R}^{2d})$ and $(15)'$ holds.

The corresponding results for the SG classes are the following.

Corollary 15. *Let $t_j, \tau_j, R_j, P_j, r, \rho \in \mathbf{R}$ be such that $r, \rho \ge 0$,*

$$R_j = \min_{k\ge j} t_k, \quad P_j = \min_{k \ge j} \tau_k, \quad \text{and} \quad \lim_{j\to\infty} t_j = \lim_{j\to\infty} \tau_j = -\infty,$$

$j = 1, 2, \dots$. Also let m be an SG *weight on $\mathbf{R}^{2d}$ of order (r, ρ), and let $m_{t,\tau}$ be given by (17). If $a_j \in \mathrm{SG}^{(m_{t_j,\tau_j})}_{\rho_1,\rho_2}(\mathbf{R}^{2d})$, then there is an element $a \in \mathrm{SG}^{(m_{R_1,P_1})}_{r,\rho}(\mathbf{R}^{2d})$ such that*

$$a - \sum_{k<j} a_k \in \mathrm{SG}^{(m_{R_j,P_j})}_{r,\rho}(\mathbf{R}^{2d}).$$

If $r, \rho > 0$, then the element a is uniquely determined modulo $\mathscr{S}(\mathbf{R}^{2d})$, and can be chosen such that $\operatorname{supp} a \subseteq \cup_{j\ge 1} \operatorname{supp} a_j$.

Corollary 16. *Let $t_j, \tau_j, R_j, P_j \in \mathbf{R}$ and $m_{t,\tau}$ be the same as in Corollary 15, $r, \rho > 0$. Also let $a \in C^\infty(\mathbf{R}^{2d})$ be such that for every multi-indices α and β, there is a constant $\mu = \mu(\alpha, \beta)$ such that*

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \lesssim m(x,\xi) \langle x \rangle^\mu \langle \xi \rangle^\mu,$$

and let $a_j \in \mathrm{SG}^{(m_{t_j,\tau_j})}_{r,\rho}(\mathbf{R}^{2d})$, $j \ge 1$, be a sequence. If

$$|a - \sum_{k<j} a_k| \lesssim m(x,\xi) \langle x \rangle^{-R_j} \langle \xi \rangle^{-P_j}$$

holds for every $j \ge 1$, then $a \in \mathrm{SG}^{(m_{R_1,P_1})}_{r,\rho}(\mathbf{R}^{2d})$ and $(15)'$ holds.

The function m in the previous corollary satisfies

$$m(x,\xi) \lesssim \langle x \rangle^{\mu_0} \langle \xi \rangle^{\mu_0},$$

for some μ_0. Hence the result does not change if the conditions on the derivatives of a are replaced by

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \lesssim \langle x \rangle^\mu \langle \xi \rangle^\mu.$$

Finally we remark that if the metric g is chosen as

$$g_{(x,\xi)}(y,\eta) = \frac{|y|^2 + |\eta|^2}{1 + |x|^2 + |\xi|^2},$$

and the weight functions are given by

$$m(x,\xi) = (1 + |x|^2 + |\xi|^2)^{\tau/2} \quad \text{and} \quad m_j(x,\xi) = (1 + |x|^2 + |\xi|^2)^{\tau_j/2},$$

then Theorems 8 and 9 give Propositions 23.1 and 23.2 in [8].

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