Wadge Degrees

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Contents

Introduction

Descriptive set theory can be roughly described as the study of definable subsets of the Baire space ${}^{\omega}\omega$ which—as customary in this area—is identified with $\mathbb{R}$. The concept of "definable set" is a bit vague as it depends—essentially—on the amount of strong axioms of set theory that one is willing to accept: if we restrict ourselves to ZFC alone, then we should diet on Borel sets only, but if we are willing to assume large cardinals, we might consider projective sets, or sets in $\mathrm{L}(\mathbb{R})$, or even sets in larger inner models for determinacy. In any case, descriptive set theorists restrict their scope to some nice collection of sets which is immune from all pathologies spawned by the Axiom of Choice, e.g.: sets which are non-Lebesgue measurable, or do not have the property of Baire, etc. An obvious requirement is that such collection of sets (like the Borel sets, the projective sets, etc.) should be closed under continuous pre-images, and this is where the Wadge hierarchy comes into play. For A, B subsets of $\mathbb{R}$, say that A is continuously reducible to B if A is the continuous pre-image of B, in symbols $A \leq_{\mathrm{W}} B$. Although the notion of continuous pre-image harks back to the dawn of general topology, and had been used in a variety of situations, it was William W. Wadge, a PhD student of John Addison in Berkeley in the late 60s–early 70s that studied first the structural properties of the relation $\leq_{\mathrm{W}}$ *per se*. Wadge's breakthrough relied on a reformulation of $\leq_{\mathrm{W}}$ in terms of games, and it was during that time that games and determinacy hypotheses were becoming a standard tool in descriptive set theory. In Wadge's own words [Wad83, p. 3]

> Yet nowhere (to our knowledge) is the relation $A = f^{-1}(B)$ for some continuous f ever explicitly defined and studied as a partial order, not even in exhaustive work such as Kuratowski (1958) or Sierpiński (1952). In the latter, Sierpiński discusses preimage in general, continuous image and homeomorphic image, but not (explicitly) continuous preimage, which is perhaps the most natural. One possible explanation is that the investigation of $\leq$ naturally involves infinite games, and it is only recently that game methods

> have been fully understood and appreciated.[1]

Using games, Wadge proved a simple, but fundamental result, now known as Wadge's Lemma, which has ushered a whole slew of new results in descriptive set theory: Assume AD. Then for all $A, B \subseteq \mathbb{R}$

$$A \leq_{\mathrm{W}} B \quad \vee \quad \mathbb{R} \setminus B \leq_{\mathrm{W}} A\,.$$

The equivalence classes of the induced equivalence relation are called Wadge degrees and the pre-order $\leq_{\mathrm{W}}$ induces a partial order $\leq$ on them. A degree is self-dual if it is the degree of a set which is Wadge reducible to its complement: for example the degree of a clopen set is self-dual, but the degree of an open, but non-clopen set is non-self-dual. Wadge showed that pairs of non-self-dual degrees and self-dual degrees alternate: immediately above a pair of non-self-dual degrees there is a self-dual degree and immediately above a self-dual degree there is a pair of non-self-dual degrees. Wadge's Lemma says that there cannot be three mutually incomparable degrees—and hence the antichains have length at most 2. In other words: if we coalesce each non-self-dual degree with its dual, then $\leq$ becomes a linear order. Wadge then embarked on a thorough analysis of the degrees of Borel sets, showing that the relation $\leq$ is well-founded. Therefore, to every Borel subset of $\mathbb{R}$ an ordinal (its Wadge rank) measuring the complexity of the set can be assigned: if $A \leq B$ then the Wadge rank of A is no greater than the Wadge rank of B. Wadge also characterized the Borel sets with limit Wadge rank: a set is self-dual if and only if its Wadge rank has countable cofinality. By 1972, Wadge had complete the herculean task of giving a complete analysis of the structure of $\leq$ restricted to Borel sets, including a computation of the Wadge ranks of the sets involved. All these results were obtained assuming Borel determinacy (before this was proved by Martin in 1974) and are contained in his dissertation [Wad83] which did not appear until 1983. In the meanwhile other mathematicians started working on Wadge degrees: Tony Martin, building on partial results of Leonard Monk, showed in 1973 that AD implies that $\leq$ is well-founded, and hence *every* set of reals can be assigned an ordinal measuring its complexity. (For this reason $\leq$ is often called the Wadge hierarchy.) The technique of the proof became known as the *Martin-Monk method* and has been applied time and again in the study of Wadge degrees. For example a couple of years later John Steel and Robert Van Wesep (at the time both students in Berkeley) used the Martin-Monk method to generalize Wadge's result to all sets (assuming AD) by showing

[1] Kuratowski (1958) and Sierpiński (1952) are, respectively, the monographs [Kur58] and [Sie52], and $\leq$ is our $\leq_{\mathrm{W}}$.

that a set of limit Wadge rank λ is self-dual if and only if λ is of countable cofinality. In the forthcoming years many set theorists developed and extended Wadge's results turning the theory of Wadge degrees into a sophisticated area of set theory. Unfortunately most of this material is not easily accessible (for example Wadge's thesis was never published and can only be retrieved at Wadge's own web page `http://i.csc.uvic.ca/home/hei/WadgePhD.pdf`), and in any case there is no general introduction to the subject.

Which bring us to the reason for this manuscript. I wanted to collect and organize some of the material that I've been studying/working on, so these notes represent more of a personal journey than a treatise. It is certainly not meant (nor was ever meant) to be a scholarly work. My apologies to all set theorists who have worked in this area and whose work hasn't been properly presented (or hasn't been presented at all): this does not mean I consider certain results/topics unimportant or less important than the ones presented here. Again, this is just a very personal account of the subject and I make absolutely no claim of completeness.

Finally, it is not at all clear whether this manuscript will ever materialize into a real book, so probably you should not refer to it in published journals, but otherwise you can use and distribute it freely.

Prerequisites and Notation

We assume acquaintance with basic descriptive set theory—all undefined notions can be found in [Kec95] or [Mos80]. Since we will work under various determinacy assumptions, our base theory will be the Zermelo-Frænkel set theory ZF. We will state explicitly any (weak) form of the **Axiom of Choice**, AC, used in the arguments—among such choice principles, the most prominent are the **Axiom of Countable Choices over the Reals**:

$(\mathsf{AC}_\omega(\mathbb{R}))$

If $\emptyset \neq A_n \subseteq \mathbb{R}$, then there is an $f : \omega \to \mathbb{R}$ such that $f(n) \in A_n$, for all n,

and the **Axiom of Dependent Choices over the Reals**:

$(\mathsf{DC}(\mathbb{R}))$

If $R \subseteq \mathbb{R} \times \mathbb{R}$ is such that $\forall x \in \mathbb{R}\, \exists y \in \mathbb{R}\ (x, y) \in R$, and if $x_0 \in \mathbb{R}$, then there is a sequence $\langle x_n \mid n \in \omega\rangle$ such that $(x_n, x_{n+1}) \in R$, for all n.

These two choice principles can be generalized by replacing $\mathbb{R}$ with a nonempty set X in the definitions above—these revised statements are denoted by $\mathsf{AC}_\omega(X)$ and $\mathsf{DC}(X)$, respectively. AC_ω and DC are the global versions of these principles:

(AC_ω) $\qquad \forall X\,(X \neq \emptyset \Rightarrow \mathsf{AC}_\omega(X))$

(DC) $\qquad \forall X\,(X \neq \emptyset \Rightarrow \mathsf{DC}(X))\,.$

Since we will refrain from using the full **Axiom of Choice**, AC, care must be exerted when dealing with the notion of cardinality:

$$|X| \leq |Y|$$

means that there is an injection of X into Y, and

$$|X| = |Y|$$

means that $|X| \leq |Y| \wedge |Y| \leq |X|$, which by the Shröder–Bernstein theorem is equivalent to the assertion that there is a bijection between X and Y. "Finite" means "in bijection with a natural number", so that "X is infinite" does not necessarily imply that "$\omega \leq |X|$", unless, of course, the axiom of countable choices AC_ω is assumed.

Exercise 0.1. Show that:

1. If X is the surjective image of Y, then $\mathsf{AC}_\omega(Y) \Rightarrow \mathsf{AC}_\omega(X)$ and $\mathsf{DC}(Y) \Rightarrow \mathsf{DC}(X)$.
2. $\mathsf{AC}_\omega(\mathbb{R})$ implies that the countable union of countable sets of reals is countable, and that ω_1 is regular.

(These facts will often be used without specific acknowledgment).

Our terminology is quite standard, but for the reader's convenience we list some of the notation that will be used throughout this book. The **identity function on** X is the map

$$\mathrm{id}_X : X \to X$$

such that $\mathrm{id}_X(x) = x$ for every $x \in X$. The symbols "$\subseteq$" and "$\subset$" stand for "inclusion" and "proper inclusion", respectively, while

$$\mathscr{P}(X) = \{Y \mid Y \subseteq X\}$$

is the power-set of X. If A is a subset of the domain of f,

$$f(A) = \{f(a) \mid a \in A\}$$

is the pointwise image of A. This should not be source of confusion since we will hardly ever deal with transitive sets, and in those rare cases we shall use the notation $f``A$. If z is a finite or ω-sequence then $\mathrm{lh}(z)$ is the length of z, i.e., its domain $\mathrm{dom}(z)$. For s a finite sequence and z a finite or an ω-sequence,

$$s^\frown z$$

is the sequence obtained by concatenating s and z, that is $\mathrm{lh}(s^\frown z) = \mathrm{lh}(s) + \mathrm{lh}(z)$ and for $i < \mathrm{lh}(s) + \mathrm{lh}(z)$

$$(s^\frown z)(i) = \begin{cases} s(i) & \text{if } i \in \mathrm{dom}(s), \\ z(i - \mathrm{lh}(s)) & \text{otherwise.} \end{cases}$$

When z is of length 1, i.e. $z = \langle x \rangle$, we will often write $s^\frown x$ instead of $s^\frown \langle x \rangle$, if there is no possible source of confusion; similarly, $x^\frown s$ denotes $\langle x \rangle^\frown s$. For any x the symbol $\vec{x}$ denotes the ω-sequence

$$\langle x, x, x, \dots \rangle .$$

$\mathbb{N} = \{0, 1, 2, \dots\}$, the set of all natural numbers, is construed as ω, the first transfinite ordinal. If $x, y \in {}^\omega X$, then $x \oplus y$ is the element $z \in {}^\omega X$ defined by

$$\begin{aligned} z(2n) &= x(n) \\ z(2n+1) &= y(n) . \end{aligned} \tag{0.1}$$

Conversely, given a $z \in {}^\omega X$, then $(z)_{\mathbf{I}}$ and $(z)_{\mathbf{II}}$ are the unique elements of ${}^\omega X$ such that

$$(z)_{\mathbf{I}} \oplus (z)_{\mathbf{II}} = z . \tag{0.2}$$

These notions extend to finite sequences as well, but, obviously, when writing $s \oplus t$ we implicitly assume that $\mathrm{lh}(t) \leq \mathrm{lh}(s) \leq \mathrm{lh}(t) + 1$.

Int and Cl are, respectively, the interior and closure operators in a topological space. A subset X of a topological space is

nowhere dense if $\mathrm{Int}(\mathrm{Cl}(X)) = \emptyset$,

meager if $X = \bigcup_n X_n$ with X_n nowhere dense.

MGR, the collection of all meager subsets of the given topological space, is closed under subsets and (assuming AC_ω) under countable unions.

A topological space is **Polish** if it is separable and completely metrizable, i.e., it admits a complete metric compatible with its topology, and a **Polish metric space** is a complete metric space (X, d) such that the topology is separable. A topological space is **extremely disconnected** or **0-dimensional** if it has a basis of clopen sets. We will be mostly concerned with the **Baire space** ${}^\omega\omega$ and with the **Cantor space** ${}^\omega 2$, both of which are Polish and 0-dimensional.

Chapter I

Elementary Results on the Wadge Hierarchy

1 Prologue

If $\mathcal{X}$ and $\mathcal{Y}$ are topological spaces, and $A \subseteq \mathcal{X}$ and $B \subseteq \mathcal{Y}$, we say that A **is continuously reducible to** B, in symbols

$$A \leq_{\mathrm{W}}^{\mathcal{X},\mathcal{Y}} B\,, \tag{1}$$

just in case $A = f^{-1}(B)$ for some continuous $f\colon \mathcal{X} \to \mathcal{Y}$. The "W" in the sub-script is after William W. Wadge, and the formula above also reads: A **is Wadge reducible to** B **between the spaces** $\mathcal{X}$ **and** $\mathcal{Y}$. Any continuous function witnessing $A \leq_{\mathrm{W}}^{\mathcal{X},\mathcal{Y}} B$ is called a **(continuous) reduction of** A **to** B: checking whether "$x \in A$" reduces to the problem of verifying whether "$f(x) \in B$". When $\mathcal{X} = \mathcal{Y}$, (1) is written as

$$A \leq_{\mathrm{W}}^{\mathcal{X}} B\,. \tag{2}$$

The relation $\leq_{\mathrm{W}}^{\mathcal{X}}$ is a pre-order (i.e., a reflexive, transitive relation) on $\mathscr{P}(\mathcal{X})$ and the intuition behind the definition of (2) is that

the set A is no more complex than the set B.

Complexity here refers to some intuitive notion of topological intricacy: we require that no set can be strictly simpler than its complement. Thus if $A \leq_{\mathrm{W}}^{\mathcal{X}} B$ holds, any upper bound for the complexity of B yields an upper bound for the complexity of A and — conversely — a lower bound for the complexity of A yields a lower bound for the complexity of B. For example, if B is Borel (or $\mathbf{F}_\sigma$, closed, etc.), then also A must be Borel (resp. $\mathbf{F}_\sigma$, closed, etc), and if A is *not* Borel (or $\mathbf{F}_\sigma$, closed, etc.) then same can be said of B. It is quite natural to ask whether the converse property holds, that is: suppose A is no more complex than B, must A be the continuous pre-image of B? The answer, in general, is negative, as the following example shows.

Example 1.1. Let $\mathcal{X} = \mathbb{R} = (-\infty, +\infty)$ be the real line, let $A = \mathbb{Q}$, let $C \subset [0;1]$ be the usual 1/3-Cantor set, and let $B \subseteq C$. We claim that there is no continuous $f \colon \mathbb{R} \to \mathbb{R}$ reducing A to B: since B can be taken to be not in $\mathbf{G}_\delta$, and since $A \in \mathbf{F}_\sigma$, then $\mathbb{R} \setminus B \not\leq^{\mathbb{R}}_{\mathrm{W}} A$, and this will witness the failure of Wadge's Lemma for A and B. To-wards a contradiction, suppose $A = f^{-1}(B)$ for some continuous f. Then

$$\mathbb{R} = \mathrm{Cl}(\mathbb{Q}) = \mathrm{Cl}\big(f^{-1}(B)\big) \subseteq f^{-1}\big(\mathrm{Cl}(B)\big) \subseteq f^{-1}(C)\,,$$

where Cl denotes the closure. Thus $\mathrm{ran}(f) \subseteq C$. But $\mathrm{ran}(f)$ is connected, and since the connected components of C are its points, it follows that f is constant, and hence it cannot be a reduction of A to B.

Since the same argument applies to any Polish space $\mathcal{X}$ containing a non-empty connected open set, we shall focus on totally disconnected (i.e., 0-dimensional) spaces. Since all 0-dimensional Polish spaces are — up to homeomorphism — closed subsets of ${}^{\omega}\omega$ [Kec95, Theorem 7.8], it is natural to restrict our attention to the Baire space. When $\mathcal{X} = \mathbb{R} = {}^{\omega}\omega$, the superscript is dropped and we write

$$A \leq_{\mathrm{W}} B\,.$$

In this chapter we will show that the Axiom of Determinacy — introduced in section 2.C below — yields stringent structural properties for the pre-order $\leq_{\mathrm{W}}$, the most notable ones being that it is well-founded (Theorem 6.1) and that it has antichains of length at most 2 (Lemma 4.5).

2 Preliminaries

2.A Trees.

A **tree** T **on a set** X is a subset of ${}^{<\omega}X$ closed under initial segments, that is $t \in T \wedge s \subseteq t \Rightarrow s \in T$. The elements of T are called **nodes**. Two nodes s and t of ${}^{<\omega}X$ are said to be **incompatible**, in symbols $s \perp t$, if

$$\exists i \in \mathrm{dom}(s) \cap \mathrm{dom}(t)\, (s(i) \neq t(i))\,.$$

$\mathbf{tn}(T)$ is the set of all **terminal nodes** of T, that is, all $t \in T$ such that there is no $s \in T$ with $s \supset t$. The **boundary** of T is the set

$$\partial T = \{s \in {}^{<\omega}X \mid s \notin T \;\&\; s \restriction \mathrm{lh}(s) - 1 \in T\}\,.$$

A **branch of** T is a function $b\colon \omega \to X$ such that $b \restriction n \in T$, for all $n \in \omega$. The **body of the tree** T is the set of all branches of T

$$[T] = \{b \in {}^{\omega}X \mid \forall n\,(b \restriction n \in T)\}\,.$$

A tree T is

pruned if it has no terminal nodes, i.e., $\mathbf{tn}(T) = \emptyset$,

perfect if every node has two incompatible extensions,

$$\forall t \in T\, \exists s, u \in T\, (t \subset s, u \wedge s \perp u)\,,$$

finite branching if every node has a finite number of immediate successors, i.e., for all $t \in T$

$$|\{u \in T \mid t \subset u \wedge \mathrm{lh}(t) + 1 = \mathrm{lh}(u)\}| < \omega\,.$$

If either $A \subseteq {}^{<\omega}X$ or $A \subseteq {}^{\omega}X$, and $s \in {}^{<\omega}X$ let

$$s^\frown A = \{s^\frown x \mid x \in A\}$$

and let

$$A_{\lfloor s\rfloor} = \{x \mid s^\frown x \in A\}\,. \tag{3}$$

The set $A_{\lfloor s\rfloor}$ is called the **localization of** A **at** s. It is immediate to check that if $A \subseteq {}^{\omega}X$ then $s^\frown A, A_{\lfloor s\rfloor} \subseteq {}^{\omega}X$, and that if T is a tree on X then $T_{\lfloor s\rfloor}$ is also trees on X. When $s = \langle x\rangle$ and there is no danger of confusion, we will write $x^\frown A$ and $A_{\lfloor x\rfloor}$ instead of $\langle x\rangle^\frown A$ and $A_{\lfloor \langle x\rangle\rfloor}$. We will be mostly concerned with the **Baire space** ${}^{\omega}\omega$ and with the **Cantor space** ${}^{\omega}2$, both of which are Polish and 0-dimensional. In fact the Baire and Cantor spaces are particular cases of the following construction: given X a discrete space (i.e., every subset is open), endow the set ${}^{\omega}X$ with the product topology, i.e., the topology generated by the sets

$$\boldsymbol{N}_s = \boldsymbol{N}({}^{\omega}X; s) = \{f \in {}^{\omega}X \mid s \subset f\} \tag{4}$$

with $s \in {}^{<\omega}X$. A complete compatible metric is given by

$$d(f, g) = \begin{cases} 2^{-n} & \text{if } f \restriction n = g \restriction n \text{ and } f(n) \neq g(n), \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

The space ${}^{\omega}X$ is separable just in case $|X| \leq \omega$. Since

$$^{\omega}X \setminus \boldsymbol{N}_s = \bigcup \{\boldsymbol{N}_t \mid t \perp s\}\,,$$

it follows that each $\boldsymbol{N}_s$ is clopen and therefore ${}^{\omega}X$ is 0-dimensional. If T is a tree on X then $[T]$ is a closed subset of ${}^{\omega}X$ and, conversely, every closed subset of ${}^{\omega}X$ is of the form $[T]$ with T pruned. The function

$$\mathbb{h}_X \colon {}^{\omega}X \times {}^{\omega}X \to {}^{\omega}X \tag{6}$$

defined by

$$\mathbb{h}_X(f,g)(n) = \begin{cases} f(k) & \text{if } n = 2k, \\ g(k) & \text{if } n = 2k+1, \end{cases} \tag{7}$$

is a homeomorphism, and we will call it the **canonical homeomorphism** of ${}^{\omega}X \times {}^{\omega}X$ with ${}^{\omega}X$. When X is clear from the context, we shall write $\mathbb{h}$ instead of $\mathbb{h}_X$. Note that $X \subseteq Y$ implies that $\mathbb{h}_X = \mathbb{h}_Y \restriction {}^{\omega}X \times {}^{\omega}X$; in particular, the restriction to the Cantor space of the canonical homeomorphism of $\mathbb{R} \times \mathbb{R}$ and $\mathbb{R}$ yields the canonical homeomorphism of ${}^{\omega}2 \times {}^{\omega}2$ and ${}^{\omega}2$.

2.B Lipschitz and continuous functions.

A function $\varphi\colon S \to T$, where S and T are non-empty pruned trees, is said to be

monotone if $s \subseteq s' \Rightarrow \varphi(s) \subseteq \varphi(s')$,

Lipschitz if it is monotone and $\mathrm{lh}(s) = \mathrm{lh}(\varphi(s))$,

continuous if it is monotone and $\lim_{n\to\infty} \mathrm{lh}(\varphi(x \restriction n)) = \infty$, for all $x \in [S]$,

tame if it is monotone and such that for all n and $s^\frown n \in S$,

$$\varphi(s) \subset \varphi(s^\frown n) \Rightarrow \mathrm{lh}(\varphi(s^\frown n)) = \mathrm{lh}(\varphi(s)) + 1\,.$$

If φ is monotone, then

$$\mathcal{D}_\varphi = \{x \in [S] \mid \lim_{n\to\infty} \mathrm{lh}(\varphi(x \restriction n)) = \infty\}$$

is a $\boldsymbol{\Pi}^0_2$ subset of ${}^{\omega}X$ and hence of $[S]$, and every $\boldsymbol{\Pi}^0_2$ subset of ${}^{\omega}X$ is of this form. The function

$$f_\varphi \colon \mathcal{D}_\varphi \to [T]\,, \qquad x \mapsto \textstyle\bigcup_n \varphi(x \restriction n)$$

is continuous and for every continuous $f\colon G \to [T]$, with G a $\mathbf{\Pi}^0_2$ subset of $[S]$, there is a monotone, tame $\varphi\colon S \to T$ such that $\mathcal{D}_\varphi = G$ and $f_\varphi = f$ (see [Kec95, Proposition 2.6]). Clearly if $\varphi\colon S \to T$ is Lipschitz then it is also continuous, and, for all $x, y \in [S]$,

$$x \restriction n = y \restriction n \Rightarrow f_\varphi(x) \restriction n = f_\varphi(y) \restriction n$$

or equivalently

$$d^{[T]}(f_\varphi(x), f_\varphi(y)) \leq d^{[S]}(x, y)$$

where $d^{[S]}$ and $d^{[T]}$ are the two complete metrics induced on $[S]$ and $[T]$ given by (5). Thus $f_\varphi\colon [S] \to [T]$ is a Lipschitz function (in the usual sense of Real Analysis) with constant ≤ 1. Conversely, every Lipschitz function $f\colon [S] \to [T]$ with constant ≤ 1 is of the form f_φ with $\varphi\colon S \to T$ Lipschitz. A function $f\colon [S] \to [T]$ which is Lipschitz with constant $\leq 1/2$, i.e., such that

$$x \restriction n = y \restriction n \Rightarrow f(x) \restriction n + 1 = f(y) \restriction n + 1$$

it is called a **contraction** and it is induced by a monotone $\varphi\colon S \to T$ such that $\mathrm{lh}(\varphi(s)) = \mathrm{lh}(s) + 1$. In this book, we make the following convention: For a function $f\colon \mathbb{R} \to \mathbb{R}$, "Lipschitz" means "Lipschitz with constant ≤ 1," that is $f = f_\varphi$ for some Lipschitz φ as above. Similarly "f is a contraction" means "$f = f_\varphi$ for some contraction φ."

Since a continuous (Lipschitz) function $\mathbb{R} \to \mathbb{R}$ is completely determined by a continuous (res. Lipschitz) ${}^{<\omega}\omega \to {}^{<\omega}\omega$, and since these objects can be coded as subsets of ω, a map $x \mapsto f_x$ can be defined so that

$$\{f_x \mid x \in \mathbb{R}\}.$$

is the collection of all continuous (res. Lipschitz) functions $f\colon \mathbb{R} \to \mathbb{R}$. We call such a map a **parametrization** or **coding** of the continuous (res. Lipschitz) functions. We now fix once and for all a parametrization of all Lipschitz functions: For any $x \in \mathbb{R}$ let $\varphi_x\colon {}^{<\omega}\omega \to {}^{<\omega}\omega$ be defined by

(8)
$$\varphi_x(s) = t \Leftrightarrow \mathrm{lh}(s) = \mathrm{lh}(t) \wedge \forall n \, \forall i < \mathrm{lh}(s) \, (\boldsymbol{s}_n = s \restriction i + 1 \Rightarrow t(i) = x(n)),$$

where

$$\langle \boldsymbol{s}_n \mid n \in \omega \rangle \tag{9}$$

is a standard enumeration without repetitions of ${}^{<\omega}\omega \setminus \{\emptyset\}$ such that $\boldsymbol{s}_n \subset \boldsymbol{s}_m \Rightarrow n < m$. This is well-defined since $s \restriction i + 1 = \boldsymbol{s}_n$ for a unique n, and clearly each φ_x is Lipschitz. If $x, x' \in {}^{\omega}\omega$ and $x(n) \neq x'(n)$, then let i be such that $i+1 = \mathrm{lh}(\boldsymbol{s}_n)$: then $\varphi_x(\boldsymbol{s}_n) \neq \varphi_{x'}(\boldsymbol{s}_n)$, since these two sequences differ on their last element

$$\big(\varphi_x(\boldsymbol{s}_n)\big)(i) = x(n) \neq x'(n) = \big(\varphi_{x'}(\boldsymbol{s}_n)\big)(i)\,.$$

In other words: $x \neq x' \Rightarrow \varphi_x \neq \varphi_{x'}$. Suppose now $\varphi\colon {}^{<\omega}\omega \to {}^{<\omega}\omega$ is Lipschitz: if $x(n)$ is the last element of $\varphi(\boldsymbol{s}_n)$, then $\varphi = \varphi_x$. Therefore

$$\mathbb{R} \to \{\varphi \mid \varphi\colon {}^{<\omega}\omega \to {}^{<\omega}\omega \text{ is Lipschitz}\}\,, \qquad x \mapsto \varphi_x$$

is a bijection. Letting $\boldsymbol{\ell}_x\colon \mathbb{R} \to \mathbb{R}$ be the Lipschitz functions induced by φ_x, it is not hard to see that the evaluation map $(x,y) \mapsto \boldsymbol{\ell}_x(y)$ is continuous. Therefore we have shown that

Lemma 2.1. *There is a bijection*

$$\mathbb{R} \to \{f \in {}^{\mathbb{R}}\mathbb{R} \mid f \textit{ is Lipschitz}\}, \qquad x \mapsto \boldsymbol{\ell}_x$$

such that

$$\mathbb{R}^2 \to \mathbb{R}, \qquad (x,y) \mapsto \boldsymbol{\ell}_x(y)$$

is continuous.

Exercise 2.2. There is no parametrization $x \mapsto f_x$ of continuous functions such that the evaluation map $\mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto f_x(y)$, is continuous.

2.C Games.

Given a set $A \subseteq {}^{\omega}X$, a game $G(A)$ on X is defined where two player **I** and **II** take turns in picking points from X

I	x_0		x_1		$\dots$
II		y_0		y_1	

Player **I** wins just in case the sequence $\langle x_0, y_0, x_1, y_1, \dots\rangle$ is in A. In other words, **I** and **II** cooperatively construct an element z of ${}^{\omega}X$: **I**'s goal is to ensure $z \in A$, while **II**'s goal is to ensure $z \notin A$. We will say that $x = \langle x_0, x_1, \dots\rangle$ and $y = \langle y_0, y_1, \dots\rangle$ are the **plays** of **I** and **II**, respectively, while $x \oplus y$ is the **complete play**. A run of a game is organized in **rounds** or **innings** with **I** and **II** playing x_n and y_n in the n-th round. A **position** or **partial play** is simply a finite sequence $p \in {}^{<\omega}X$, and if it is of even length, we say it is the position after the n-th round or, equivalently, before the $n+1$-st round, where $2n = \mathrm{lh}(p)$. The sequences $(p)_{\mathbf{I}}$ and $(p)_{\mathbf{II}}$ are called **I**'s and **II**'s positions associated with p.

A **strategy for I** is any function $\sigma\colon {}^{<\omega}X \to X$, and σ is said to be **winning** if

$$\sigma * y = \langle \sigma(\emptyset), y_0, \sigma(\langle y_0\rangle), y_1, \sigma(\langle y_0, y_1\rangle), y_2, \sigma(\langle y_0, y_1, y_2\rangle)\dots\rangle \in A$$

for any $y \in {}^{\omega}X$. Conversely, a **strategy for II** is any function $\tau\colon {}^{<\omega}X \setminus \{\emptyset\} \to X$, and τ is said to be **winning** if

$$x * \tau = \langle x_0, \tau(\langle x_0\rangle), x_1, \tau(\langle x_0, x_1\rangle), x_2, \tau(\langle x_0, x_1, x_2\rangle)\dots\rangle \notin A$$

for any $x \in {}^{\omega}X$. A play $z \in {}^{\omega}X$ is said to be **according to** σ, strategy for **I** (or τ, strategy for **II**) iff $\exists y \in {}^{\omega}X\,(z = \sigma * y)$, (res. $\exists x \in {}^{\omega}X\,(z = x * \tau)$).

Suppose τ is a strategy for **II** for a game on X and define $\tilde{\tau}(s)$ by induction on $\mathrm{lh}(s)$

$$\begin{aligned}\tilde{\tau}(\emptyset) &= \emptyset\\ \tilde{\tau}(s^\frown\langle x\rangle) &= \tilde{\tau}(s)^\frown\langle\tau(s^\frown\langle x\rangle)\rangle\,.\end{aligned}$$

Then $\tilde{\tau}\colon {}^{<\omega}X \to {}^{<\omega}X$ is Lipschitz, and, conversely, if $\varphi\colon {}^{<\omega}X \to {}^{<\omega}X$ is Lipschitz then

$$\begin{aligned}\hat{\varphi}(s) &= \varphi(s)(\mathrm{lh}(s) - 1)\\ &= \text{the last element of } \varphi(s)\end{aligned}$$

is a strategy for **II**. Since $\tau \mapsto \tilde{\tau}$ and $\varphi \mapsto \hat{\varphi}$ are inverse operations, the distinction between Lipschitz functions and strategies for **II** will often be blurred. Similarly, strategies for **I** can be identified with contractions. Let σ and τ be strategies for **I** and **II**, respectively, for games on X:

$$\sigma * \tau$$

is the element of ${}^{\omega}X$ obtained by pitting σ against τ, i.e., is the unique play according to σ and to τ. Since the play $\sigma * \tau$ lies either inside or outside a given set A, it follows that in any game *at most* one player can have a winning strategy. Our definition requires that a strategy acts against the sequence of moves provided by the opponent, but at times it is more convenient to think of a strategy as a recipe suggesting what to play next at any given position where we are to make a move. Therefore we could define strategies for **I** and **II** as maps

$$\bar{\sigma}\colon \bigcup_n {}^{2n}X \to X \quad \text{and} \quad \bar{\tau}\colon \bigcup_n {}^{2n+1}X \to X\,.$$

A $z \in {}^{\omega}X$ is according to $\bar{\sigma}$ iff $\forall n\,(z_{2n} = \bar{\sigma}(z \restriction 2n))$, and $\bar{\sigma}$ is winning for **I** in $G(A)$ just in case every play according to $\bar{\sigma}$ is in A. (The definition of $\bar{\tau}$ being winning for **II** is analogous.)

Exercise 2.3. Show that the two definitions of strategies are equivalent. In particular, show that σ (or τ) is a winning in $G(A)$ iff the same is true of $\bar{\sigma}$ (res. $\bar{\tau}$).

There is another, apparently more general notion of game on a set X. Let T be a non-empty pruned tree on X and let $A \subseteq [T]$: the game $G(T;A)$ is just like $G(A)$ except that both players must make moves so that the partial play is always in T, the first player to break this rule losing outright. A **legal position** is just a node of T, and a move that extends a legal position to another legal position is called a **legal move**. The definition of strategy must be modified accordingly so that they yield only legal moves. Note that all games of the form $G(A)$ can be considered as games in this new sense, $G({}^{<\omega}X; A)$.

Exercise 2.4. Show that any game $G(T;A)$ is equivalent to a game $G(\tilde{A})$, for some $\tilde{A} \subseteq {}^{<\omega}X$, in the sense that either player that has a winning strategy in $G(T;A)$ has also a winning strategy in $G(\tilde{A})$, and conversely.

A game is **determined** if one (and only one) of the two players admits a winning strategy. The **Axiom of Determinacy for games on** X is the statement

(AD_X) For every $A \subseteq {}^{\omega}X$ the game on X with pay-off A is determined.

Exercise 2.5. Show that $|X| \leq |Y|$ implies $\mathsf{AD}_Y \Rightarrow \mathsf{AD}_X$.

If X is empty or a singleton, then AD_X is trivial. When $X = \omega$ this is the usual Axiom of Determinacy, AD, and by a result of Mycielski, $\mathsf{AD}_2 \Rightarrow \mathsf{AD}$. Therefore if $2 \leq \alpha < \omega_1$ then by Exercise 2.5

$$\mathsf{AD}_\alpha \Leftrightarrow \mathsf{AD}\,.$$

The Axiom of Real Determinacy $\mathsf{AD}_{\mathbb{R}}$ is (much) stronger than AD. By deep results of Woodin, assuming large cardinals, both AD and $\mathsf{AD}_{\mathbb{R}}$ are consistent with $\mathsf{ZF}+\mathsf{DC}(\mathbb{R})$ — see [Kan03, pp. 464–468]. On the other hand, both AD_{ω_1} and $\mathsf{AD}_{\mathscr{P}(\mathbb{R})}$ are inconsistent with ZF — see Exercise 10.2.

AD contradicts the Axiom of Choice since, for example, it implies many regularity properties for sets of reals which are known to be inconsistent with AC (see section 2.D). Yet AD is consistent with weak forms of choice.

Lemma 2.6. $\mathsf{AD} \Rightarrow \mathsf{AC}_\omega(\mathbb{R})$.

Proof. Given $\emptyset \neq A_n \subseteq \mathbb{R}$ consider the game on ω where **I** plays n (and then his moves are irrelevant) and **II**, in order to win, must construct a real in A_n. Then **I** cannot have a winning strategy, and any strategy for **II** yields a choice function for the A_ns. □

On the other hand it is open whether AD implies $\mathsf{DC}(\mathbb{R})$ — see also the Open Problem 6.4.

Open problem 2.7. *Does* $\mathsf{AD} \Rightarrow \mathsf{DC}(\mathbb{R})$*?*

Kechris has shown [Kec84] that the answer is affirmative under the additional assumption that $\mathrm{V} = \mathrm{L}(\mathbb{R})$, showing thus that the consistency of $\mathsf{AD} + \mathsf{DC}$ from that of AD, i.e.,

$$\mathsf{Con}(\mathsf{ZF} + \mathsf{AD}) \Rightarrow \mathsf{Con}(\mathsf{ZF} + \mathsf{AD} + \mathsf{DC})\,.$$

AD does not imply neither DC nor AC_ω by results of [Sol78] and Woodin (unpublished). By unpublished results of Woodin's the theory $\mathsf{ZF} + \mathsf{AD} + \neg\mathsf{DC}(\mathbb{R})$ proves the consistency of $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$.

Although AD_X is inconsistent for any X containing ω_1, the same need not to be true for the weaker statement $\mathsf{Det}_X(\Gamma)$ requiring only the determinacy of games with payoff sets in some given pointclass $\Gamma \subseteq \mathscr{P}({}^\omega X)$,

($\mathsf{Det}_X(\Gamma)$) For every $A \in \Gamma$ the game on X with pay-off A is determined.

The pointclass Γ is usually some collection of sets defined in topological terms, e.g., the Borel pointclasses $\boldsymbol{\Sigma}^0_\alpha$, $\boldsymbol{\Pi}^0_\alpha$, and $\boldsymbol{\Delta}^0_\alpha$, or the projective pointclasses $\boldsymbol{\Sigma}^1_n$, $\boldsymbol{\Pi}^1_n$,and $\boldsymbol{\Delta}^1_n$. Gale and Stewart [**?**] showed that AC and $\forall X\, \mathsf{Det}_X(\boldsymbol{\Pi}^0_1)$ are equivalent over the base theory ZF. Thus ZFC proves the determinacy of all games with closed pay-off, on any space ${}^\omega X$. This was extended by Martin [Mar75, **?**] to $\boldsymbol{\Delta}^1_1$ — for uncountable X the pointclass $\boldsymbol{\Delta}^1_1$ is strictly larger than the collection of Borel sets. Martin's result is optimal (in ZFC) since the determinacy of games on ω with $\boldsymbol{\Pi}^1_1$ payoff implies the existence of $0^\sharp$ [**?**].

2.D Regularity properties.

We will now look at three regularity properties for sets of reals: perfect subsets, the property of Baire, and Lebesgue measurability.

The **perfect subset property** is the statement that every set of real numbers is either countable or else it contains a non-empty, perfect closed set. Since any non-empty perfect closed set contains a homeomorphic copy of the Cantor space, the perfect subset property can be stated as

(PSP) $\quad \forall X \subseteq \mathbb{R}\,(|X| \leq \omega \ \vee\ \exists f\colon {}^\omega 2 \to X,\ f \text{ continuous and injective})\,.$

PSP contradicts choice: if $\mathbb{R}$ is well-orderable then there is a **Bernstein set**, that is a set $B \subseteq \mathbb{R}$ such that $B \cap P \neq \emptyset$ and $P \setminus B \neq \emptyset$ for every non-empty perfect closed set P, [Kec95, Example 8.24]. Clearly the existence of Bernstein sets contradicts PSP.

Lemma 2.8. PSP *implies there is no ω_1-sequence of distinct real numbers.*

Proof. Towards a contradiction suppose $f\colon \omega_1 \rightarrowtail \mathbb{R}$ is injective. Then $\operatorname{ran}(f)$, being uncountable, must contain a copy C of the Cantor space. Then C, being a subset of $\operatorname{ran}(f)$, is well-orderable, and so is $\mathbb{R}$, since it is in bijection with C: a contradiction. □

Since closed sets are determined by pruned trees on ω, and since trees can be identified via their characteristic functions with a real, it follows that PSP implies there is no ω_1-sequence of closed or open sets.

A set $A \subseteq \mathbb{R}$ is said to have the **property of Baire** if there is an open set U such that $A \triangle U \in \mathrm{MGR}$. We abbreviate the expression "all sets have the property of Baire" with BP. Just as with the perfect subset property, BP implies that $\mathbb{R}$ cannot be well-ordered, and hence it contradicts AC.

The **Lebesgue measure** on ${}^{\omega}2$ is the completion μ of the measure defined on the basic open sets by $\mu(\boldsymbol{N}({}^{\omega}2; s)) = 2^{-\operatorname{lh}(s)}$, and it is often called the *coin-tossing measure*. The statement "all subsets of ${}^{\omega}2$ are Lebesgue measurable" is abbreviated by LM, and by the isomorphism theorem for measures [Kec95, Theorem 17.41] it is equivalent to the statement that every subset of $\mathbb{R}$ is Lebesgue measurable. (Here $\mathbb{R}$ is the usual Euclidean line with the traditional Lebesgue measure studied in analysis.) By a well known theorem of Vitali, if $\mathbb{R}$ is well-orderable, then LM fails. On the other hand, a weak form of the Axiom of Choice, $\mathsf{AC}_\omega(\mathbb{R})$, is needed for the construction of the Lebesgue measure on $\mathbb{R}$ (or for that matter: any non-discrete measure on an uncountable Polish space): the crucial issue is the verification that the outer measure $\mathscr{P}(\mathbb{R}) \to (0; \infty)$

$$A \mapsto \inf\{\sum_{n=0}^{\infty} \ell(I_n) \mid \langle I_n \mid n \in \omega\rangle \text{ is a sequence of half-open intervals such that } A \subseteq \bigcup_n I_n\}$$

where $\ell(I)$ is the length of the interval I, is indeed σ-subadditive — see [Fre00, Proposition 114D].

If D is non-principal ultrafilter on ω then, identifying each set of natural numbers with its characteristic function, D can be construed as a subset of ${}^{\omega}2$: under this identification, D is an example of a non-measurable set without the property of Baire [Kec95, Exercise 8.50].

The three regularity properties are consistent with and independent of ZF. We summarize here a few known facts

- assuming $\mathbb{R}$ is well orderable we have that $\neg$PSP, $\neg$BP, $\neg$LM;

- by a celebrated result of Solovay [Sol70], assuming the existence of an inaccessible cardinal it is consistent that

$$\mathrm{L}(\mathbb{R}) \models \mathsf{DC} + \mathsf{PSP} + \mathsf{BP} + \mathsf{LM}$$

- both PSP and LM imply the consistency of an inaccessible cardinal;
- Shelah in [She84] showed that $\mathsf{Con}(\mathsf{ZF}) \Rightarrow \mathsf{Con}(\mathsf{ZF} + \mathsf{DC} + \mathsf{BP})$, and therefore there is a model satisfying $\mathsf{DC} + \mathsf{BP} + \neg\mathsf{PSP} + \neg\mathsf{LM}$;
- Di Prisco and Todorcevic [DPT98] constructed a model of $\mathsf{ZF} + \mathsf{DC} + \mathsf{PSP}$ containing a non-principal ultrafilter on ω, and therefore in this model BP and LM fail.

Definition 2.9. A set $F \subseteq {}^{\omega}2$ is a **flip set** if the truth of the statement "$z \in F$" flips between true and false every time the value of z in one coordinate is changed, that is

$$\forall z, w \in {}^{\omega}2 \, (\exists! k \; z(k) \neq w(k) \Rightarrow (z \in F \Leftrightarrow w \notin F)) \, .$$

FS is the statement: there is a flip set.

Thus if F is a flip set and $|\{k \mid z(k) \neq w(k)\}| = n$ then

$$\begin{aligned} n \text{ even } &\Rightarrow (z \in F \Leftrightarrow w \in F) \, , \\ n \text{ odd } &\Rightarrow (z \in F \Leftrightarrow w \notin F) \, . \end{aligned}$$

Proposition 2.10. *The existence of a non-principal ultrafilter on ω implies* FS.

Proof. For any $z \in {}^{\omega}2$ let $\tilde{z} \in {}^{\omega}2$ be defined by

$$\begin{aligned} \tilde{z}(0) &= z(0) \\ \tilde{z}(n+1) &= \tilde{z}(n) + z(n+1) \end{aligned}$$

where $+$ denotes addition modulo 2. Let $N = \omega \times \{0, 1\}$ and let

$$A_z = \{(n, \tilde{z}(n)) \in N \mid n \in \omega\} \, .$$

If n_0 is the unique natural number such that $z(n_0) \neq w(n_0)$, then

$$\forall m \geq n_0 \, ((m, i) \in A_z \Leftrightarrow (m, i+1) \in A_w) \, ,$$

that is: A_z and A_w partition $N \setminus n_0 \times 2$. Since N is in bijection with ω we may assume that there is a non-principal ultrafilter D on N, and by the equation above

$$A_z \in D \Leftrightarrow A_w \notin D$$

whenever z and w differ in exactly one coordinate. Therefore

$$F = \{z \in {}^{\omega}2 \mid A_z \in D\}$$

is a flip-set. □

Therefore if $\mathbb{R}$ is well-orderable, then FS holds.

Proposition 2.11. *No flip set has the property of Baire.*

Proof. Towards a contradiction, suppose F is a flip set with the property of Baire. Since ${}^{\omega}2 \setminus F$ is also a flip set , by the Baire category theorem F and ${}^{\omega}2 \setminus F$ cannot be both meager or both co-meager. Therefore — possibly replacing F with its complement — we may assume that F is non-meager. Let U be open in ${}^{\omega}2$ and such that $F \triangle U$ is meager. Since $F \notin \mathrm{MGR}$ then $U \neq \emptyset$, so pick n and $s \in {}^{n}2$ such that $\boldsymbol{N}({}^{\omega}2; s) \subseteq U$. Let $f_n \colon {}^{\omega}2 \to {}^{\omega}2$ be the Lipschitz homeomorphism

$$f_n(z)(i) = \begin{cases} 1 - z(i) & \text{if } i = n, \\ z(i) & \text{otherwise.} \end{cases} \tag{10}$$

Then $f_n(\boldsymbol{N}_s) = \boldsymbol{N}_s$ and F is co-meager in $\boldsymbol{N}_s$, hence ${}^{\omega}2 \setminus F = f_n(F)$ is co-meager in $\boldsymbol{N}_s$, contradicting the Baire category theorem. □

Proposition 2.12 ($\mathsf{AC}_{\omega}(\mathbb{R})$)**.** *No flip set is Lebesgue measurable.*

Proof. Towards a contradiction, suppose F is a Lebesgue measurable flip set. Let s be any finite sequence and let $n = \mathrm{lh}(s)$. Since $\neg F = f_n(F)$, where f_n is as in (10), and since each f_n preserves the measure, then $\mu(F \cap \boldsymbol{N}_s) = \mu(\boldsymbol{N}_s \setminus F) = 1/2\mu(\boldsymbol{N}_s)$. In particular F is non-null, hence $\mu(\boldsymbol{N}_s \cap F) = \mu(\boldsymbol{N}_s)$ for some s: a contradiction. □

The Axiom of Determinacy implies all these regularity properties: PSP, BP, and LM (see [Kan03, pp.373–377]). Moreover they are all provable in ZFC if restricted to the realm of analytic sets.

3 Reducibilities

In order to study the theory of Wadge reducibility, it is often necessary to study other forms of reducibilities (like Lipschitz reducibility, or Borel reducibility). Let $\mathcal{X}$ be a topological space and let $\mathcal{F} \subseteq {}^{\mathcal{X}}\mathcal{X}$ be a family of functions closed under composition and containing the identity and all constant functions $\mathbf{c}_x$, i.e.,

$$\begin{aligned} &f, g \in \mathcal{F} \Rightarrow f \circ g \in \mathcal{F}\,, \\ &\mathrm{id}_{\mathbb{R}} \in \mathcal{F}\,, \\ &\mathbf{c}_x \in \mathcal{F} \qquad\qquad (x \in \mathcal{X})\,, \end{aligned} \tag{11}$$

where $\mathbf{c}_x$ is the constant function $x \in \mathcal{X}$. For $A, B \subseteq \mathcal{X}$, say that A is **$\mathcal{F}$-reducible** to B, in symbols

$$A \leq^{\mathcal{X}}_{\mathcal{F}} B\,,$$

just in case

$$\exists f \in \mathcal{F}\, \forall x \in \mathcal{X}\, (x \in A \Leftrightarrow f(x) \in B)$$

for some $f \in \mathcal{F}$. By our hypothesis $\leq^{\mathcal{X}}_{\mathcal{F}}$ is reflexive and transitive. Set

$$A \equiv_{\mathcal{F}} B \quad \Leftrightarrow \quad A \leq_{\mathcal{F}} B \;\wedge\; B \leq_{\mathcal{F}} A\,,$$

and

$$A <_{\mathcal{F}} B \quad \Leftrightarrow \quad A \leq_{\mathcal{F}} B \;\wedge\; B \not\leq_{\mathcal{F}} A\,.$$

An $\mathcal{F}$-degree is an equivalence class of $\equiv_{\mathcal{F}}$, and

$$[A]_{\mathcal{F}} = \{B \mid B \equiv_{\mathcal{F}} A\}$$

is the $\mathcal{F}$-degree of A. Notice that

$$A \leq_{\mathcal{F}} B \quad \Leftrightarrow \quad \neg A \leq_{\mathcal{F}} \neg B\,,$$

where for $A \subseteq \mathcal{X}$

$$\neg A = \mathcal{X} \setminus A$$

is the complement of A in $\mathcal{X}$. A set A is **$\mathcal{F}$-self-dual** iff $A \leq_{\mathcal{F}} \neg A$ iff $A \equiv_{\mathcal{F}} \neg A$, otherwise it is **$\mathcal{F}$-non-self-dual**. Since self-duality is invariant under $\equiv_{\mathcal{F}}$, it can be applied to $\mathcal{F}$-degrees as well. The **dual** of $[A]_{\mathcal{F}}$ is $[\neg A]_{\mathcal{F}}$, and a pair of distinct degrees of the form $\{[A]_{\mathcal{F}}, [\neg A]_{\mathcal{F}}\}$ is a **non-self-dual pair**. The pre-order $\leq_{\mathcal{F}}$ induces a partial order $\leq$ on the $\mathcal{F}$-degrees:

$$[A]_{\mathcal{F}} \leq [B]_{\mathcal{F}} \quad \Leftrightarrow \quad A \leq_{\mathcal{F}} B\,.$$

Similarly define

$$[A]_{\mathcal{F}} < [B]_{\mathcal{F}} \quad \Leftrightarrow \quad [A]_{\mathcal{F}} \leq [B]_{\mathcal{F}} \ \wedge \ [A]_{\mathcal{F}} \neq [B]_{\mathcal{F}} \quad \Leftrightarrow \quad A <_{\mathcal{F}} B\,.$$

If $\mathcal{F} \subseteq \mathcal{G}$ are sets of functions like in (11), then the pre-order $\leq_{\mathcal{G}}$ is coarser than $\leq_{\mathcal{F}}$, hence

$$A \leq_{\mathcal{F}} B \Rightarrow A \leq_{\mathcal{G}} B \tag{12}$$

$$A \text{ is } \mathcal{F}\text{-self-dual } \Rightarrow A \text{ is } \mathcal{G}\text{-self-dual} \tag{13}$$

$$[A]_{\mathcal{F}} \subseteq [A]_{\mathcal{G}}\,. \tag{14}$$

Notice that $[\mathcal{X}]_{\mathcal{F}} = \{\mathcal{X}\}$ and $[\emptyset]_{\mathcal{F}} = \{\emptyset\}$ form a non-self-dual pair, and since $\mathcal{F}$ has all constant functions they are $<$-least among $\mathcal{F}$-degrees. We say that $[A]_{\mathcal{F}}$ is a **successor degree** if there is a $[B]_{\mathcal{F}} < [A]_{\mathcal{F}}$ such that for no $C \subseteq \mathbb{R}$ we have $[B]_{\mathcal{F}} < [C]_{\mathcal{F}} < [A]_{\mathcal{F}}$. (In this case the degree $[B]_{\mathcal{F}}$ is an **immediate predecessor** of $[A]_{\mathcal{F}}$.) If an $\mathcal{F}$-degree is not a successor and it is neither $[\mathcal{X}]_{\mathcal{F}}$ nor $[\emptyset]_{\mathcal{F}}$, then we say it is a **limit degree**. A limit degree is said to be of **countable cofinality** if it is the least upper bound of an increasing sequence $[A_0]_{\mathcal{F}} < [A_1]_{\mathcal{F}} < \dots$ of $\mathcal{F}$-degrees; otherwise it is said to be of **uncountable cofinality**.

If $\mathcal{F}$ is the collection of all continuous functions, the "$\mathcal{F}$" in all subscripts is replaced by "W," while if $\mathcal{F}$ is the collection of all Lipschitz functions, we use "L." If $\mathcal{F}$ is the set of all functions from $\mathcal{X}$ to $\mathcal{X}$, then the structure of the $\mathcal{F}$-degrees is trivial, since there are only three degrees: $[\mathcal{X}]_{\mathcal{F}}$, $[\emptyset]_{\mathcal{F}}$, and $\mathscr{P}(\mathcal{X}) \setminus \{\mathcal{X}, \emptyset\}$. On the other hand, AD implies that for any reasonable $\mathcal{F} \subseteq {}^{\mathbb{R}}\mathbb{R}$ there is no largest $\mathcal{F}$-degree — Lemma 4.7 below.

4 Lipschitz and Wadge games

The **Lipschitz game for** A **and** B

$$G_{\mathrm{L}}(A, B)$$

with $A, B \subseteq {}^{\omega}X$, is the game in which **I** and **II** play elements of X a_i and b_i, respectively,

$$\begin{array}{lccccc} \textbf{I} & a_0 & & a_1 & & \\ & & & & & \dots \\ \textbf{II} & & b_0 & & b_1 & \end{array}$$

and player **II** wins just in case

$$a = \langle a_i \mid i \in \omega \rangle \in A \quad \Leftrightarrow \quad b = \langle b_i \mid i \in \omega \rangle \in B\,.$$

In the literature the game G_{L} is often called *Wadge game* rather than Lipschitz game, but we will use that name for another game to be introduced in section 7.

Exercise 4.1. Show that $G_{\mathrm{L}}(A,B)$ is indeed a game on X in the sense of section 22.C.

The notation $G_{\mathrm{L}}(A,B)$ is a tad ambiguous: if $A,B \subseteq {}^{\omega}X$ and $X \subseteq Y$, then we may consider the game $G_{\mathrm{L}}(A,B)$ as a game on X or as a game on Y, and these can be quite different: for example, if $A,B \subseteq {}^{\omega}2 \subseteq {}^{\omega}\omega$ with $A \neq {}^{\omega}2$ closed and $B = {}^{\omega}2$, then **II** has a winning strategy in $G_{\mathrm{L}}(A,B)$ if this is construed as a game on ω, but not if it is construed as a game on 2. Therefore whenever there is danger of confusion we will adopt the following convention: if $A,B \subseteq [T]$ where T is a pruned tree, then

$$G^T_{\mathrm{L}}(A,B)$$

is the game described above with the moves restricted to T.

Fix a space ${}^{\omega}X$. A winning strategy for **II** in $G_{\mathrm{L}}(A,B)$ is — essentially — a Lipschitz function $f\colon {}^{\omega}X \to {}^{\omega}X$ such that $A = f^{-1}(B)$, hence

$$\textbf{II} \text{ wins } G_{\mathrm{L}}(A,B) \quad \Leftrightarrow \quad A \leq_{\mathrm{L}} B \tag{15}$$

Conversely, a strategy for **I** yields a contraction thus, in particular a Lipschitz function $f\colon {}^{\omega}X \to {}^{\omega}X$ such that $\neg B = f^{-1}(A)$. Therefore

$$\textbf{I} \text{ wins } G_{\mathrm{L}}(A,B) \quad \Rightarrow \quad \neg B \leq_{\mathrm{L}} A\,. \tag{16}$$

In this case we cannot revert the implication since a Lipschitz function witnessing $\neg B \leq_{\mathrm{L}} A$ isn't necessarily a contraction.

Exercise 4.2. Let $A,B \subseteq {}^{\omega}X$.

(a) Let τ be a winning strategy for **II** in $G_{\mathrm{L}}(A,B)$ and suppose p and q are positions for **I** and **II**, respectively, according to τ, that is $q = (p * \tau)_{\mathbf{II}}$. Then $A_{\lfloor p\rfloor} \leq_{\mathrm{L}} B_{\lfloor q\rfloor}$ and $\tilde{\tau}(s) = \tau(p^\frown s)$ is winning for **II** on $G_{\mathrm{L}}(A_{\lfloor p\rfloor}, B_{\lfloor q\rfloor})$.

(b) If σ is a winning strategy for **I** in $G_{\mathrm{L}}(A,B)$, then $\tau = \sigma \setminus \{(\emptyset, \sigma(\emptyset))\}$ is a winning strategy for **II** in $G_{\mathrm{L}}(B, \neg A_{\lfloor\sigma(\emptyset)\rfloor})$. Conversely, for any $x \in X$, if τ is a winning strategy for **II** in $G_{\mathrm{L}}(B, \neg A_{\lfloor x\rfloor})$, then $\sigma = \tau \cup \{(\emptyset, x)\}$ is a winning strategy for **I** in $G_{\mathrm{L}}(A,B)$.

The **Wadge game for** A **and** B,

$$G_{\mathrm{W}}(A,B)\,,$$

is the variant of the Lipschitz game where **II** is allowed to pass (i.e., not to play) at any round, but he must play infinitely often otherwise he loses. Formally, this is a game on $X \cup \{\mathsf{p}\}$, where p (=the passing move) is an object that does not belong to X. An example of a run of $G_{\mathrm{W}}(A, B)$ is

I	a_0	a_1	$\dots$	a_n	a_{n+1}	a_{n+2}	$\dots$	a_m	
									$\dots\dots$
II		p	$\mathsf{p}\dots\mathsf{p}$	b_0	p	$\mathsf{p}\dots\mathsf{p}$	b_1		

As in the case of G_{L}, **II** wins just in case

$$a = \langle a_i \mid i \in \omega\rangle \in A \Leftrightarrow b = \langle b_i \mid i \in \omega\rangle \in B.$$

A strategy for either player for G_{W} is called a **Wadge strategy**, and by decree, a strategy for **II** in *must* play infinitely often on any given input. Although $G_{\mathrm{W}}(A, B)$ is a game on $X \cup \{\mathsf{p}\}$, it is more convenient to think of it as a game on X: all moves which are not p are called **active** moves, and the **position** q **of II at round** n is the sequence of *active* moves of **II** up to round n, disregarding all p's (thus $\mathrm{lh}(q) < n$ is possible). The sequence of length n recording *all* of **II** 's moves (including passing) is called the **true position** of **II** at round n, and has, obviously, length n.

Exercise 4.3. Show that if $\omega \leq |X|$, then the Wadge game for the space ${}^{\omega}X$ can be formulated as a game on X.

Just like the case of Lipschitz games, $G^{T}_{\mathrm{W}}(A, B)$ is the Wadge game for $A, B \subseteq [T]$, where the moves of the players are restricted to T, a pruned tree on X.

Since a strategy τ for **II** always produces a sequence with infinitely many entries different from p, the map

(17)
$$\hat{\tau}\colon {}^{<\omega}X \to {}^{<\omega}X, \qquad s \mapsto \text{the sequence of active moves of } \tau \text{ on input } s$$

is continuous. Formally:

$$\hat{\tau}(s) = \begin{cases} \emptyset & \text{if } s = \emptyset, \\ \hat{\tau}(s \restriction \mathrm{lh}(s) - 1)^\frown \tau(s) & \text{if } \tau(s) \neq \mathsf{p}, \\ \hat{\tau}(s \restriction \mathrm{lh}(s) - 1) & \text{otherwise,} \end{cases}$$

Conversely every continuous $\varphi\colon {}^{<\omega}X \to {}^{<\omega}X$ yields a strategy τ for **II** in the Wadge game on X such that $\hat{\tau} = \varphi$ — just take

$$\tau(s) = \begin{cases} \emptyset & \text{if } s = \emptyset, \\ \varphi(s)(\mathrm{lh}(s) - 1) & \text{if } s \neq \emptyset \text{ and } \varphi(s) \neq \varphi(s \restriction \mathrm{lh}(s) - 1), \\ \mathsf{p} & \text{otherwise.} \end{cases}$$

For this reason we will often blur the distinction between continuous functions and strategies for **II** in Wadge games. On the other hand a strategy for **I** in a Wadge game yields a whole family of contractions, since player **II** can reveal his moves as slowly as he wants. Summarizing:

$$\textbf{II} \text{ wins } G_{\mathrm{W}}(A,B) \quad \Leftrightarrow \quad A \leq_{\mathrm{W}} B \tag{18}$$

$$\textbf{I} \text{ wins } G_{\mathrm{W}}(A,B) \quad \Rightarrow \quad \neg B \leq_{\mathrm{L}} A\,. \tag{19}$$

The following is the analogue of Exercise 4.2(a) for G_{W}.

Exercise 4.4. Let τ be a winning strategy for **II** in $G_{\mathrm{W}}(A,B)$ and suppose p and q are positions for **I** and **II**, respectively, according to τ, that is $q = (p * \tau)_{\mathbf{II}}$. Then $A_{\lfloor p \rfloor} \leq_{\mathrm{W}} B_{\lfloor q \rfloor}$ and $\tilde{\tau}(s) = \tau(p^\frown s)$ is winning for **II** on $G_{\mathrm{W}}(A_{\lfloor p \rfloor}, B_{\lfloor q \rfloor})$.

The statement that all Lipschitz games on the space ${}^{\omega}X$ are determined:

$$(\mathsf{AD}^{\mathrm{L}}_X) \qquad \forall A, B \subseteq {}^{\omega}X \quad G_{\mathrm{L}}^{<\omega X}(A,B) \text{ is determined,}$$

is called **Lipschitz determinacy for the space** ${}^{\omega}X$, or simply Lipschitz determinacy (AD^{L}), when $X = \omega$. Similarly

$$(\mathsf{AD}^{\mathrm{W}}_X) \qquad \forall A, B \subseteq {}^{\omega}X \quad G_{\mathrm{W}}(A,B) \text{ is determined,}$$

is called **Wadge determinacy**. Clearly AD_X implies $\mathsf{AD}^{\mathrm{L}}_X$ and $\mathsf{AD}^{\mathrm{W}}_X$, hence Lipschitz determinacy is a consequence of the Axiom of Determinacy. Just like AD_X, the principles $\mathsf{AD}^{\mathrm{L}}_X$ and $\mathsf{AD}^{\mathrm{W}}_X$ are inconsistent if $\omega_1 \leq |X|$ — see Chapter II, Proposition 11.5. For this reason we will concentrate from now on to the case $X = \omega$. Whether AD follows from AD^{L} is an open problem — see section **??**.

Wadge observed the following simple, but crucial fact:

Lemma 4.5 (Wadge's Lemma)**.** *Assume* AD*, or even just* AD^{L}*. Then for every* $A, B \subseteq \mathbb{R}$

$$A \leq_{\mathrm{L}} B \quad \vee \quad \neg B \leq_{\mathrm{L}} A\,. \tag{20}$$

Clearly (20) implies

$$A \leq_{\mathrm{W}} B \quad \vee \quad \neg B \leq_{\mathrm{W}} A\,, \tag{21}$$

which is also a consequence of AD^{W}. Wadge's Lemma implies that if $A \nleq_{\mathrm{L}} B$ and $B \nleq_{\mathrm{L}} A$, then $\neg B \leq_{\mathrm{L}} A$ and $\neg A \leq_{\mathrm{L}} B$ and hence $A \equiv_{\mathrm{L}} \neg B$. In other words: if A and B are $\leq_{\mathrm{L}}$-incomparable, then $\{[A]_{\mathrm{L}}, [B]_{\mathrm{L}}\}$ is a non-self-dual pair, and hence if C is a set not equivalent to either A or B (i.e., $C \notin [A]_{\mathrm{L}} \cup [B]_{\mathrm{L}}$) then either

$$C <_{\mathrm{L}} A \quad \wedge \quad C <_{\mathrm{L}} B$$

or else

$$A <_{\mathrm{L}} C \quad \wedge \quad B <_{\mathrm{L}} C\,.$$

Thus the ordering induced on the Lipschitz degrees is *almost* a linear-order (if each degree is identified with its dual, then it is indeed linear) and for this reason (20) — the thesis of Wadge's Lemma — and (21) are called the **Semi-Linear Ordering Principle for Lipschitz maps** and **Semi-Linear Ordering Principle for continuous maps**, respectively, and are denoted by

$$\mathsf{SLO}^{\mathrm{L}} \qquad \text{and} \qquad \mathsf{SLO}^{\mathrm{W}}\,.$$

(We will show in Chapter IV that $\mathsf{SLO}^{\mathrm{W}} \Leftrightarrow \mathsf{SLO}^{\mathrm{L}} \Leftrightarrow \mathsf{AD}^{\mathrm{L}}$.) More generally, if $\mathcal{F}$ is a family of functions on a space $\mathcal{X}$ as in (11), the notion of Semi-Linear Ordering Principle for $\mathcal{F}$-maps is

$$(\mathsf{SLO}^{\mathcal{F}}(\mathcal{X})) \qquad \forall A, B \subseteq \mathcal{X} \left(A \leq^{\mathcal{X}}_{\mathcal{F}} B \ \vee\ \neg B \leq^{\mathcal{X}}_{\mathcal{F}} A \right),$$

with the understanding that $\mathsf{SLO}^{\mathcal{F}}$ means $\mathsf{SLO}^{\mathcal{F}}(\mathbb{R})$. If $\mathcal{F} \subseteq \mathcal{G}$ then, by (12),

$$\mathsf{SLO}^{\mathcal{F}}(\mathcal{X}) \Rightarrow \mathsf{SLO}^{\mathcal{G}}(\mathcal{X})\,. \tag{22}$$

In particular, if $\mathcal{F}$ contains all Lipschitz functions and $\mathcal{X} = \mathbb{R}$, then $\mathsf{SLO}^{\mathcal{F}}$ is a consequence of AD^{L}, and hence of AD. Notice that if A is $\mathcal{F}$ self-dual, then $\mathsf{SLO}^{\mathcal{F}}$ implies that $[A]_{\mathcal{F}}$ is a node of the ordering of the $\mathcal{F}$-degrees, that is it is comparable with all other $\mathcal{F}$-degrees

$$[B]_{\mathcal{F}} \neq [A]_{\mathcal{F}} \Rightarrow ([B]_{\mathcal{F}} < [A]_{\mathcal{F}} \ \vee\ [A]_{\mathcal{F}} < [B]_{\mathcal{F}})\,. \tag{23}$$

In order to study the Semi-Linear Ordering principle of different $\mathcal{X}$'s, it is convenient to introduce the following terminology: the **$\mathcal{F}$-structure** of $\mathcal{X}$ is the model-theoretic structure

$$\langle \mathscr{P}(\mathcal{X}), \neg^{\mathcal{X}}, \leq^{\mathcal{X}}_{\mathcal{F}} \rangle$$

where $\neg^{\mathcal{X}}$ is a 1-ary operation of taking complements, $\neg^{\mathcal{X}} A = \mathcal{X} \setminus A$. When $\mathcal{F}$ is the set of all continuous (Lipschitz) functions, we will speak of Wadge structure (Lipschitz structure). Suppose $\mathcal{X}$ is a **retract** of $\mathcal{Y}$, that is, $\mathcal{X}$ is a closed subset of $\mathcal{Y}$ and there is a continuous surjection $\pi \colon \mathcal{Y} \twoheadrightarrow \mathcal{X}$ which is the identity on $\mathcal{X}$. (The map π is called a **retraction** of $\mathcal{Y}$ onto $\mathcal{X}$.) We claim that the map $\Phi\colon \mathscr{P}(\mathcal{X}) \to \mathscr{P}(\mathcal{Y})$, $\Phi(A) = \pi^{-1}(A)$ is an embedding of the Wadge structure of $\mathcal{X}$ into the Wadge structure of $\mathcal{Y}$: it is easy to check that it is an injective map that preserves complementation, so it is enough to show that

$$A \leq_{\mathrm{W}}^{\mathcal{X}} B \Leftrightarrow \Phi(A) \leq_{\mathrm{W}}^{\mathcal{Y}} \Phi(B)\,.$$

If $f\colon \mathcal{X} \to \mathcal{X}$ is continuous and $A = f^{-1}(B)$, then $f \circ \pi\colon \mathcal{Y} \to \mathcal{X} \subseteq \mathcal{Y}$ witnesses $\Phi(A) \leq_{\mathrm{W}}^{\mathcal{Y}} \Phi(B)$; conversely, if $g\colon \mathcal{Y} \to \mathcal{Y}$ is continuous and $\Phi(A) = g^{-1}\big(\Phi(B)\big)$, then $\pi \circ g \restriction \mathcal{X}\colon \mathcal{X} \to \mathcal{X}$ witnesses $A \leq_{\mathrm{W}}^{\mathcal{X}} B$. Therefore we have shown

Proposition 4.6. *If $\pi\colon \mathcal{Y} \twoheadrightarrow \mathcal{X}$ is a retraction, then the map $A \mapsto \pi^{-1}(A)$ is an embedding of the Wadge structure of $\mathcal{X}$ into the Wadge structure of $\mathcal{Y}$. In fact*

$$\mathsf{SLO}^{\mathrm{W}}(\mathcal{Y}) \Rightarrow \mathsf{SLO}^{\mathrm{W}}(\mathcal{X})\,.$$

(The last statement is immediate since $\mathsf{SLO}^{\mathrm{W}}$ is a $\forall$-formula in the language for Wadge structures, and therefore down-ward absolute.) If $S \subseteq T$ are pruned trees, then by [Kec95, Proposition 2.8] (see also (6)) there is a Lipschitz retraction of $[T]$ onto $[S]$, hence $\mathsf{SLO}^{\mathrm{L}}$ implies $\mathsf{SLO}^{\mathrm{L}}({}^{\omega}2)$ and, more generally, $\mathsf{SLO}^{\mathrm{L}}([T])$ for any pruned tree T on a countable set.

If $\mathcal{F} = {}^{\mathbb{R}}\mathbb{R}$ then $\mathsf{SLO}^{\mathcal{F}}$ holds for trivial reasons, but if $\mathcal{F}$ is a reasonable collection of functions, then $\mathsf{SLO}^{\mathcal{F}}$ becomes highly non-trivial: for example, if we restrict ourselves to continuous functions, then the semi-linear ordering principle implies the prefect set property — see Corollary 11.4. The next result shows that if $\mathcal{F}$ is not too large, then there is a uniform way to construct from a set $A \subseteq \mathbb{R}$, a new set which is $<_{\mathcal{F}}$ larger than A and $\neg A$. Thus, in particular, there is no largest $\mathcal{F}$-degree.

Lemma 4.7. *Let $\mathcal{F} \subseteq {}^{\mathbb{R}}\mathbb{R}$ be as in* (11) *and suppose $\mathcal{F}$ is the surjective image of $\mathbb{R}$ and that $\mathsf{SLO}^{\mathcal{F}}$ holds. Then there is a map $J = J_{\mathcal{F}}\colon \mathscr{P}(\mathbb{R}) \to \mathscr{P}(\mathbb{R})$ such that*

$$\forall A \subseteq \mathbb{R}\, (A <_{\mathcal{F}} J(A) \ \wedge\ \neg A <_{\mathcal{F}} J(A))\,.$$

Proof. If $x \mapsto f_x$ is a surjection $\mathbb{R} \twoheadrightarrow \mathcal{F}$, let

$$J(A) = \{0^\frown x \mid f_x(0^\frown x) \notin A\} \cup \{1^\frown x \mid f_x(1^\frown x) \in A\}\,,$$

where $\{f_x \mid x \in \mathbb{R}\} = \mathcal{F}$. By diagonalization $J(A) \leq_{\mathcal{F}} A$ or $J(A) \leq_{\mathcal{F}} \neg A$ are impossible, hence by $\mathsf{SLO}^{\mathcal{F}}$, $A, \neg A <_{\mathcal{F}} J(A)$. □

Remarks 4.8. If we replace $\mathbb{R}$ with ${}^{\omega}X$, then everything so far goes through with minor or no modifications at all, but, of course, we must replace AD with AD_X in Lemma 4.5, and must assume that X has at least two elements in Lemma 4.7. This last assumption is not an issue since if X is empty or a singleton, then ${}^{\omega}X$ is utterly uninteresting. Let us take a closer look at the spaces ${}^{\omega}X$.

(a) If $|X| = n$, then ${}^{\omega}X$ is homeomorphic to the Cantor space, ${}^{\omega}2$, and AD_X is equivalent to AD.

(b) If $\omega \leq |X| < \omega_1$, then ${}^{\omega}X$ is homeomorphic to the Baire space, ${}^{\omega}\omega = \mathbb{R}$, and, again, AD_X is equivalent to AD.

(c) If $|X| = |\mathbb{R}|$, then ${}^{\omega}X$ is non-separable, yet $\mathsf{AD}_{\mathbb{R}}$ yields a theory of its Lipschitz/Wadge degrees completely analogous to the theory of Lipschitz/Wadge degrees for the Baire space.

(d) If $\omega_1 \leq |X|$, then AD_X is inconsistent. Yet it still might be the case that the analogue of Wadge's Lemma might hold for spaces like ${}^{\omega}\omega_1$. In other words, one could ask if (it is consistent that) for any $A, B \subseteq {}^{\omega}\omega_1$ it is the case that either $A \leq_{\mathrm{L}}^{{}^{\omega}\omega_1} B$ or ${}^{\omega}\omega_1 \setminus B \leq_{\mathrm{L}}^{{}^{\omega}\omega_1} A$. We shall show in Chapter II, Proposition 11.5 that this is not the case.

5 Lipschitz degrees

In this section we start our systematic study of Lipschitz degrees.

Lemma 5.1. *Suppose $A \leq_{\mathrm{L}} B$ and $s, t \in {}^{<\omega}\omega$ are sequences of the same length. Then $s^\frown A \leq_{\mathrm{L}} t^\frown B$.*

Proof. **II** wins $G_{\mathrm{L}}(s^\frown A, t^\frown B)$ as follows. As long as **I** enumerates s then **II** enumerates t, with the understanding that if **I** deviates from s at some round $n < \mathrm{lh}(s)$, then **II** deviates from t at the same round n and then plays whatever he likes, say, 0. If after the $\mathrm{lh}(s)$-round the two positions are s and t, then **II** applies his winning strategy in $G_{\mathrm{L}}(A, B)$ from now on. □

The preceding proof — like most proofs involving determinacy — is cast in mundane terms and uses freely many intuitions and suggestions borrowed from games that people play in everyday life. Some readers may feel a bit uneasy at first with this game-theoretic jargon, and may want to translate our proofs into a more classical lexicon. As a template, the proof above can be reworded as follows: A formal — but much less intelligible — rewording would be:

Let τ be a winning strategy for **II** in $G_{\mathrm{L}}(A, B)$. Then

$$\hat{\tau}(u) = \begin{cases} t(\mathrm{lh}(u) - 1) & \text{if } u \subset s, \\ t(\mathrm{lh}(u) - 1) + 1 & \text{if } u \restriction \mathrm{lh}(u) - 1 \subset s, \text{ but } u \not\subseteq s, \\ 0 & \text{if } \exists k < \mathrm{lh}(s) \, (u(k) \neq s(k) \ \wedge \ k + 1 < \mathrm{lh}(u)), \\ \tau(v) & \text{if } u = s^\frown v. \end{cases}$$

is a winning strategy for **II** in $G_{\mathrm{L}}(s^\frown A, t^\frown B)$.

We feel that this translation is much less intelligible than the original proof since it obscures the simple ideas behind it, and therefore in this book the game-theoretic jargon will always be used.

Exercise 5.2. Show that

(i) $A \leq_{\mathrm{L}} s^\frown A$,

(ii) if $\mathrm{lh}(s) \leq \mathrm{lh}(t)$ and $A \leq_{\mathrm{L}} B$ then $s^\frown A \leq_{\mathrm{L}} t^\frown B$,

(iii) $A_{\lfloor s \rfloor} \leq_{\mathrm{L}} A$,

(iv) if $A \leq_{\mathrm{L}} \neg A$ then $s^\frown A \leq_{\mathrm{L}} \neg(s^\frown A)$ for all s.

Thus $A \leq_{\mathrm{L}} 0^\frown A$, and if A is Lipschitz self-dual, the inequality is strict.

Lemma 5.3. *Suppose $A \leq_{\mathrm{L}} \neg A$. Then $A <_{\mathrm{L}} 0^\frown A$. Moreover, assuming AD^{L}, $[0^\frown A]_{\mathrm{L}}$ is the immediate successor of $[A]_{\mathrm{L}}$: if $B <_{\mathrm{L}} 0^\frown A$ then $B \leq_{\mathrm{L}} A$.*

Proof. Let τ be a winning strategy for **II** in $G_{\mathrm{L}}(A, \neg A)$,

$$\begin{array}{lccccc} \mathbf{I} & x_0 & & x_1 & & \\ & & \searrow^{\tau} & & \searrow^{\tau} & \dots \\ \mathbf{II} & & y_0 & & y_1 & \end{array}$$

Then **I** wins $G_{\mathrm{L}}(0^\frown A, A)$ by playing 0 and then following τ,

$$\begin{array}{lcccccc} \mathbf{I} & 0 & & y_0 & & y_1 & \\ & & \nearrow_{\tau} & & \nearrow_{\tau} & & \dots \\ \mathbf{II} & & x_0 & & x_1 & & \end{array}$$

Therefore **II** does not have a winning strategy in $G_{\mathrm{L}}(0^\frown A, A)$, hence $0^\frown A \not\leq_{\mathrm{L}} A$, and therefore $A <_{\mathrm{L}} 0^\frown A$.

Suppose now $B <_{\mathrm{L}} 0^\frown A$. Since $0^\frown A$ is Lipschitz self-dual, this is equivalent by (23) to $0^\frown A \not\leq_{\mathrm{L}} \neg B$. By AD^{L} let σ be a winning strategy for **I** in $G_{\mathrm{L}}(0^\frown A, \neg B)$. Then **II** wins $G_{\mathrm{L}}(B, A)$ by pretending that a 0 was played before the play started, and then following σ. In other words: **II**'s winning strategy is $\sigma \restriction ({}^{<\omega}\omega \setminus \{\emptyset\})$. $\square$

This should be contrasted with the continuous case: if $x \in \neg A$ then the function $\varphi\colon {}^{<\omega}\omega \to {}^{<\omega}\omega$

$$\varphi(s) = \begin{cases} \emptyset & \text{if } s = \emptyset, \\ t & \text{if } s = 0^\frown t, \\ x \restriction \mathrm{lh}(s) & \text{otherwise,} \end{cases}$$

is continuous and witnesses $0^\frown A \leq_{\mathrm{W}} A$. Therefore

$$\forall s \in {}^{<\omega}\omega \left(A \neq \mathbb{R} \Rightarrow s^\frown A \leq_{\mathrm{W}} A \right). \tag{24}$$

Proposition 5.4. *If $A \leq_{\mathrm{L}} \neg A$ then $\forall n \left(A_{\lfloor n \rfloor} <_{\mathrm{L}} A \right)$. Conversely, assuming $\mathsf{AC}_\omega(\mathbb{R}) + \mathsf{AD}^{\mathrm{L}}$, if $\forall n \left(A_{\lfloor n \rfloor} <_{\mathrm{L}} A \right)$, then $A \leq_{\mathrm{L}} \neg A$.*

Proof. Suppose $A \leq_{\mathrm{L}} \neg A$ and $A_{\lfloor n \rfloor} \equiv_{\mathrm{L}} A$, for some $n \in \omega$. Then $A_{\lfloor n \rfloor}$ is Lipschitz self-dual hence $A_{\lfloor n \rfloor} <_{\mathrm{L}} n^\frown A_{\lfloor n \rfloor} \leq_{\mathrm{L}} A$: a contradiction.

Vice versa, suppose $\forall n \left(A_{\lfloor n \rfloor} <_{\mathrm{L}} A \right)$. Since $A \not\leq_{\mathrm{L}} A_{\lfloor n \rfloor}$, choose σ_n a winning strategy for **I** in $G_{\mathrm{L}}(A, A_{\lfloor n \rfloor})$. Then **II** wins $G_{\mathrm{L}}(A, \neg A)$ as follows: if **I** plays n at round 0 then **II** follows σ_n from now on

I	n		a_0		a_1			$\dots$
II		$b_0 = \sigma_n(\emptyset)$		$b_1 = \sigma_n(\langle a_0 \rangle)$		$b_2 = \sigma_n(\langle a_0, a_1 \rangle)$		

so that

$$n^\frown a \in A \Leftrightarrow a \in A_{\lfloor n \rfloor} \Leftrightarrow b \in \neg A\,,$$

hence A is Lipschitz self-dual. $\square$

We now introduce an infinitary operation on sets: for $A_n \subseteq \mathbb{R}$ let

$$\textstyle\bigoplus_n A_n = \bigcup_n n^\frown A_n \tag{25}$$

and for $A, B \subseteq \mathbb{R}$ let

$$A \oplus B = \textstyle\bigoplus_n A_n \tag{26}$$

where $A_{2n} = A$ and $A_{2n+1} = B$.

Exercise 5.5. (i) Show that $A, B \leq_{\mathrm{L}} A \oplus B$ and $A_i \leq_{\mathrm{L}} \bigoplus_n A_n$, for all $i \in \omega$.

(ii) Show that $A \leq_{\mathrm{L}} A' \;\wedge\; B \leq_{\mathrm{L}} B' \Rightarrow A \oplus B \leq_{\mathrm{L}} A' \oplus B'$. Assuming $\mathsf{AC}_\omega(\mathbb{R})$, show that $A_n \leq_{\mathrm{L}} A'_n \Rightarrow \bigoplus_n A_n \leq_{\mathrm{L}} \bigoplus_n A'_n$.

(iii) Show that $\neg (A \oplus B) \equiv_{\mathrm{L}} \neg A \oplus \neg B$ and $\neg \bigoplus_n A_n = \bigoplus_n \neg A_n$.

(iv) Show that $A \oplus B \equiv_{\mathrm{L}} B \oplus A$ and that $\bigoplus_n A_n \equiv_{\mathrm{L}} \bigoplus_n A_{j(n)}$ for any surjection $j\colon \omega \twoheadrightarrow \omega$.

Thus the operations $\oplus$ and $\bigoplus$ can be extended to degrees:

Definition 5.6. Let $\mathcal{F} \subseteq {}^{\mathbb{R}}\mathbb{R}$ be a family of functions satisfying (11) and containing all Lipschitz functions.[1] Define

$$\begin{aligned} [A]_{\mathrm{L}} \oplus [B]_{\mathrm{L}} &= [A \oplus B]_{\mathrm{L}} \\ \textstyle\bigoplus_n [A_n]_{\mathrm{L}} &= [\textstyle\bigoplus_n A_n]_{\mathrm{L}} \,. \end{aligned}$$

It is easy to check that these operations are commutative.

If I is a countable set, then let

$$\textstyle\bigoplus_{i\in I} [A_i]_{\mathrm{L}} = \textstyle\bigoplus_n [A_{j(n)}]_{\mathrm{L}}$$

for some/any surjection $j \colon \omega \twoheadrightarrow I$.

Suppose $\varphi_n \colon {}^{<\omega}\omega \to {}^{<\omega}\omega$ is continuous and witnesses that $A_n \leq_{\mathrm{W}} A$. Then

$$\varphi(s) = \begin{cases} \emptyset & \text{if } s = \emptyset, \\ \varphi_n(t) & \text{if } s = n^\frown t, \end{cases}$$

is continuous and witnesses $\bigoplus_n A_n \leq_{\mathrm{W}} A$. Therefore $\mathsf{AC}_\omega(\mathbb{R})$ implies that

$$\forall n\, (A_n \leq_{\mathrm{W}} A) \Rightarrow \textstyle\bigoplus_n A_n \leq_{\mathrm{W}} A \,. \tag{27}$$

Lemma 5.7. *$A \oplus \neg A$ is Lipschitz self-dual. Moreover if AD^{L} holds, then $[A]_{\mathrm{L}} \oplus [\neg A]_{\mathrm{L}}$ is the least degree above $[A]_{\mathrm{L}}$ and $[\neg A]_{\mathrm{L}}$.*

Proof. Consider the strategy τ for **II** that in the first round plays

- 0, if **I** has played 1,
- 1, if **I** has played 0,
- $k + 2$, if **I** has played $k + 2$,

and then copies **I**'s moves from now on. Then τ is winning in $G_{\mathrm{L}}(A \oplus \neg A, \neg(A \oplus \neg A))$.

Suppose now A is Lipschitz non-self-dual. Then $A, \neg A <_{\mathrm{L}} A \oplus \neg A$ hence it is enough to show that if $B <_{\mathrm{L}} A \oplus \neg A$ then

$$B \leq_{\mathrm{L}} A \quad \vee \quad B \leq_{\mathrm{L}} \neg A \,. \tag{28}$$

[1]The $\mathcal{F}$ that we will consider in the next few sections are the set of al Lipschitz functions and the set of all continuous functions.

By assumption **II** does not win $G_{\mathrm{L}}(A \oplus \neg A, B)$ hence **I** has a winning strategy σ. If $B = \mathbb{R}$ then (28) follows at once, so we may as well assume otherwise. If in the first round σ plays an integer $\neq 0, 1$, then **I** will not produce a real in $A \oplus \neg A$, hence **II** can win by enumerating an arbitrary real not in B. Therefore $\sigma(\emptyset) = 0$ or $\sigma(\emptyset) = 1$. For the sake of definiteness assume the former. Then σ yields a winning strategy τ for **II** in $G_{\mathrm{L}}(B, \neg A)$ by letting $\tau(s) = \sigma(0^\frown s)$. □

Lemma 5.8. *Assume* $\mathsf{SLO}^{\mathrm{L}}$ *and* $\mathsf{AC}_\omega(\mathbb{R})$ *and suppose* $A_n \subseteq \mathbb{R}$ *are such that*

$$\forall n\, \exists m > n\, (A_n <_{\mathrm{L}} A_m)\,. \tag{29}$$

Then $\bigoplus_n A_n$ *is Lipschitz self-dual and* $A_i <_{\mathrm{L}} \bigoplus_n A_n$ *for each* i*. If moreover we assume* AD^{L} *then* $[\bigoplus_n A_n]_{\mathrm{L}}$ *is the least-upper bound of the* $[A_i]_{\mathrm{L}}$*'s: if* $A_n <_{\mathrm{L}} B$ *for all* n*, then* $\bigoplus_n A_n \leq_{\mathrm{L}} B$*.*

Proof. Let $f\colon \omega \to \omega$

$$f(n) = \text{ the least } m > n \text{ such that } A_n <_{\mathrm{L}} A_m\,.$$

By $\mathsf{SLO}^{\mathrm{L}}$ $A_n <_{\mathrm{L}} \neg A_{f(n)}$ holds, so, by $\mathsf{AC}_\omega(\mathbb{R})$, let us fix winning strategies τ_n for **II** in $G_{\mathrm{L}}(A_n, \neg A_{f(n)})$. Then **II** wins $G_{\mathrm{L}}(\bigoplus_n A_n, \neg \bigoplus_n A_n)$ as follows: in the first round, if **I** plays n then **II** responds $f(n)$; after this first round **II** follows τ_n.

Towards a contradiction, suppose $A_n \leq_{\mathrm{L}} B$, for all n, and that $\bigoplus_n A_n \not\leq_{\mathrm{L}} B$. Then player **I** has a winning strategy σ in $G_{\mathrm{L}}(\bigoplus_n A_n, B)$. Then $B \leq_{\mathrm{L}} \neg A_n$ for some n, and since $\mathsf{SLO}^{\mathrm{L}}$ implies that $\neg A_n <_{\mathrm{L}} A_{f(n)}$, a contradiction is reached. □

The assumption (11) can be relaxed a bit, but cannot be completely removed: assuming $A_n = A$ for all n, if A is non-self-dual, then $\bigoplus_n A_n$ is non-self-dual and if A is self-dual, then $\bigoplus_n A_n \equiv_{\mathrm{L}} 0^\frown A$ is self-dual, but it is not Lipschitz equivalent to A, the least upper bound of the A_n's.

Exercise 5.9. (i) Show that if $A_n <_{\mathrm{L}} A_{n+1}$, for all n, and $f\colon \omega \to \omega$ is strictly increasing, then $\bigoplus_n A_n \equiv_{\mathrm{L}} \bigoplus_n A_{f(n)}$.

(ii) Suppose $\forall n\, \exists m\, (A_n \leq_{\mathrm{L}} B_m)$ and $\forall n\, \exists m\, (B_n \leq_{\mathrm{L}} A_m)$. Then $\bigoplus_n A_n \equiv_{\mathrm{L}} \bigoplus_n B_n$.

Proposition 5.10. *Assume* $\mathsf{AD}^{\mathrm{L}} + \mathsf{AC}_\omega(\mathbb{R})$*. A limit Lipschitz degree is self-dual if and only if it is of countable cofinality.*

Proof. If the A_n's witness that $[A]_L$ is of countable cofinality, then Lemma 5.8 implies that $\bigoplus_n A_n \equiv_L A$ is self-dual.

Suppose $[A]_L$ is self-dual, so that by Lemma 5.4 $A_{\lfloor n \rfloor} <_L A$ for all n. Let $B_0 = A_{\lfloor 0 \rfloor}$ and

$$B_{n+1} = \begin{cases} A_{\lfloor n+1 \rfloor} & \text{if } B_n <_L A_{\lfloor n+1 \rfloor}, \\ 0^\frown (B_n \oplus \neg B_n) & \text{otherwise.} \end{cases}$$

By induction $A_{\lfloor n \rfloor} \leq_L B_n <_L A$ and $B_n <_L B_{n+1}$, hence $\bigoplus_n B_n$ is Lipschitz self-dual and

$$A = \bigoplus_n A_{\lfloor n \rfloor} \leq_L \bigoplus_n B_n \,.$$

But $\bigoplus_n B_n$ is the least upper bound of the $B_n \leq_L A$ and thus $\bigoplus_n B_n \leq_L A$. Therefore $\bigoplus_n B_n \equiv_L A$, and the $[B_n]_L$'s witness that $[A]_L$ is of countable cofinality. □

Lemma 5.3 shows that if $[A]_L$ is self-dual, then $[0^\frown A]_L$ is its immediate successor and it is self-dual. Therefore we can define the successor operator $\mathcal{L}$ on self-dual Lipschitz degrees

$$\mathcal{L}([A]_L) = [0^\frown A]_L \,. \tag{30}$$

This operation can be iterated through the countable ordinals:

$$\begin{aligned} \mathcal{L}^0([A]_L) &= [A]_L \,, \\ \mathcal{L}^{\alpha+1}([A]_L) &= \mathcal{L}(\mathcal{L}^\alpha([A]_L)) \,, \\ \mathcal{L}^\lambda([A]_L) &= \textstyle\bigoplus_n [A_n]_L \,, && \text{if } \lambda \text{ is limit,} \end{aligned}$$

where $A_n \in \mathcal{L}^{\alpha_n}([A]_L)$, and the α_ns are increasing and cofinal in λ. Thus above a self-dual Lipschitz degree there is an ω_1-sequence of self-dual Lipschitz degrees.

Remark 5.11. (a) If in the definition of $\mathcal{L}^\lambda([A]_L)$ we choose another increasing cofinal sequence α'_n and sets $A'_n \in \mathcal{L}^{\alpha'_n}([A]_L)$, then $\bigoplus_n A_n \equiv_L \bigoplus_n A'_n$ by Exercise 5.9, so the operation $\mathcal{L}^\lambda$ is well-defined.

(b) In order to choose the $A_n \in \mathcal{L}^{\alpha_n}([A]_L)$ we only need to appeal to $\mathsf{AC}_\omega(\mathbb{R})$. To see this it is enough to verify by induction on $\alpha < \omega_1$ that

$$B \in \mathcal{L}^\alpha([A]_L) \Rightarrow B \leq_W A \,. \tag{31}$$

In fact if (31) holds at λ and $\{f_x \mid x \in \mathbb{R}\}$ is an enumeration of all continuous functions, then

$$X_n = \{x \in \mathbb{R} \mid f_x^{-1}(A) \in \mathcal{L}^{\alpha_n}([A]_L)\}$$

is non-empty, so by $\mathsf{AC}_\omega(\mathbb{R})$ we can choose $x_n \in X_n$ and hence $A_n = f_{x_n}^{-1}(A)$. To show (31) notice that the successor case follows from (24), while the limit case follows from (27).

(c) Had we defined $\mathcal{L}$ on *sets* rather than degrees (by putting, say, $\mathcal{L}(A) = 0^\frown A$) we would have run into troubles when trying to iterate the operation in the transfinite, since there is no uniform, definable way to pick an increasing cofinal sequence converging to a limit ordinal. In fact, PSP (which is a consequence of AD, and in fact of $\mathsf{SLO}^{\mathrm{W}}$) implies that there is no ω_1-sequence of distinct clopen sets — see Lemma 2.8. (The argument above shows that there is an ω_1-sequence of distinct Lipschitz degrees of clopen sets.)

(d) If we look at the Lipschitz degrees in the Cantor space — rather than in the Baire space — then Lemmas 5.3 and 5.7 imply that $[A \oplus \neg A]_{\mathrm{L}}$ is self-dual and the immediate successor of $\{[A]_{\mathrm{L}}, [\neg A]_{\mathrm{L}}\}$ and that above a self-dual degree there are ω self-dual degrees. Proposition 5.4 and its proof hold also for ${}^{\omega}2$ (if in the statement we understand n to vary in $\{0, 1\}$), and this implies that every limit degree is non-self-dual: if $[A]_{\mathrm{L}}$ is self-dual, then $A = A_{\lfloor 0 \rfloor} \oplus A_{\lfloor 1 \rfloor}$ is the least upper bound of $A_{\lfloor 0 \rfloor}$, $A_{\lfloor 1 \rfloor}$, hence $[A]_{\mathrm{L}}$ cannot be limit.

6 The well-foundedness of the Lipschitz hierarchy

Wadge's Lemma implies that if every Lipschitz degree is combined with its dual then $\leq_{\mathrm{L}}$ is a linear order. In fact under this identification it is a *well-order.*

Theorem 6.1 (Martin–Monk)**.** *Assume* AD*. There is no sequence* $\langle A_n \mid n < \omega\rangle$ *of sets of reals such that* $\forall n \in \omega\, (A_{n+1} <_{\mathrm{L}} A_n)$ *or* $\forall n \in \omega\, (A_{n+1} <_{\mathrm{W}} A_n)$*.*

Corollary 6.2. *Assume* $\mathsf{AD} + \mathsf{DC}(\mathbb{R})$*. Then* $<_{\mathrm{L}}$ *and* $<_{\mathrm{W}}$ *are well-founded on* $\mathscr{P}(\mathbb{R})$*.*

Proof. Since $\leq_{\mathrm{L}}$ refines $\leq_{\mathrm{W}}$ it is enough to prove the result for the Lipschitz case. Towards a contradiction suppose $\emptyset \neq \mathcal{A} \subseteq \mathscr{P}(\mathbb{R})$ does not have an $<_{\mathrm{L}}$ minimal element, and let $A \in \mathcal{A}$. Fix an enumeration $\{f_x \mid x \in \mathbb{R}\}$ of all Lipschitz functions and let

$$T = \{\langle x_0, x_1, \ldots, x_n\rangle \in {}^{<\omega}\mathbb{R} \mid \forall i < n \left(f_{x_{i+1}}(A) <_{\mathrm{L}} f_{x_i}(A)\right)\}.$$

T is the tree on $\mathbb{R}$ searching for an infinite $<_{\mathrm{L}}$-descending chain $\langle f_{x_i}(A) \mid i \in \omega\rangle$ in $\mathcal{A}$. By the Theorem such a sequence does not exist, hence T cannot have an infinite branch. But by the assumption on $\mathcal{A}$, the tree T is pruned, hence by $\mathsf{DC}(\mathbb{R})$ it must have an infinite branch: a contradiction. □

In view of this result we can give the following

Definition 6.3 ($\mathsf{AD} + \mathsf{DC}(\mathbb{R})$)**.** The **Lipschitz rank** of $A \subseteq \mathbb{R}$ is the rank of A in the well-founded relation $<_{\mathrm{L}}$ and it is denoted by

$$\|A\|_{\mathrm{L}} \,.$$

Similarly

$$\|A\|_{\mathrm{W}}$$

denotes the **Wadge rank** of A in the $<_{\mathrm{W}}$ pre-order.

For technical reasons, Lipschitz and Wadge ranks take values in the non-zero ordinals, that is

$$\|\emptyset\|_{\mathrm{L}} = \|\mathbb{R}\|_{\mathrm{L}} = 1 = \|\emptyset\|_{\mathrm{W}} = \|\mathbb{R}\|_{\mathrm{W}} \,.$$

Therefore $\mathsf{AD}+\mathsf{DC}(\mathbb{R})$ yields a fine calibration of sets of reals that enables us to prove results about $\mathscr{P}(\mathbb{R})$ by induction on the Wadge rank. This is most remarkable since AD forbids any kind of well ordering of $\mathbb{R}$.

It is a well-known open problem in descriptive set theory whether $\mathsf{AD} \Rightarrow \mathsf{DC}(\mathbb{R})$. The following also seems to be open:

Open problem 6.4. *Is* $\mathsf{DC}(\mathbb{R})$ *needed for Corollary 6.2?*

Proof of Theorem 6.1. Towards a contradiction suppose $A_{n+1} <_{\mathrm{L}} A_n$ for all n. Then $A_n \not\leq_{\mathrm{L}} A_{n+1}$ and $A_n \not\leq_{\mathrm{L}} \neg A_{n+1}$ so that by AD^{L} and $\mathsf{AC}_\omega(\mathbb{R})$ there are winning strategies σ^0_n and σ^1_n for **I** in $G_{\mathrm{L}}(A_n, A_{n+1})$ and in $G_{\mathrm{L}}(A_n, \neg A_{n+1})$, respectively. For $z \in {}^\omega 2$ let

$$\mathcal{G}^z_n = \begin{cases} G_{\mathrm{L}}(A_n, A_{n+1}) & \text{if } z(n) = 0\,, \\ G_{\mathrm{L}}(A_n, \neg A_{n+1}) & \text{if } z(n) = 1\,. \end{cases}$$

The games $\{\mathcal{G}^z_n \mid n \in \omega\}$ will be played simultaneously with **I** using $\sigma^{z(n)}_n$ in $\mathcal{G}^z_n$ and **II** copying **I**'s moves from the next game, $\mathcal{G}^z_{n+1}$. To see how this is done consider the diagram of Figure 1 where $\xrightarrow{\sigma_n}$ is the strategy $\sigma^{z(n)}_n$ and $\Longrightarrow$ is the copying strategy. **I** starts by filling-in the first column, then **II** copies and fills-in the second column. Now **I** can use his strategies to fill-in the third column. And so on. Let $a^z_n = \langle a^z_n(i) \mid i \in \omega\rangle$ be the real played by **I** is $\mathcal{G}^z_n$. Then a^z_{n+1} is also the real played by **I** in $\mathcal{G}^z_n$. So we have the following

$$\begin{aligned} &z(n) = 0 \Rightarrow \left(a^z_n \in A_n \Leftrightarrow a^z_{n+1} \notin A_{n+1}\right) \\ &z(n) = 1 \Rightarrow \left(a^z_n \in A_n \Leftrightarrow a^z_{n+1} \in A_{n+1}\right). \end{aligned} \tag{32}$$

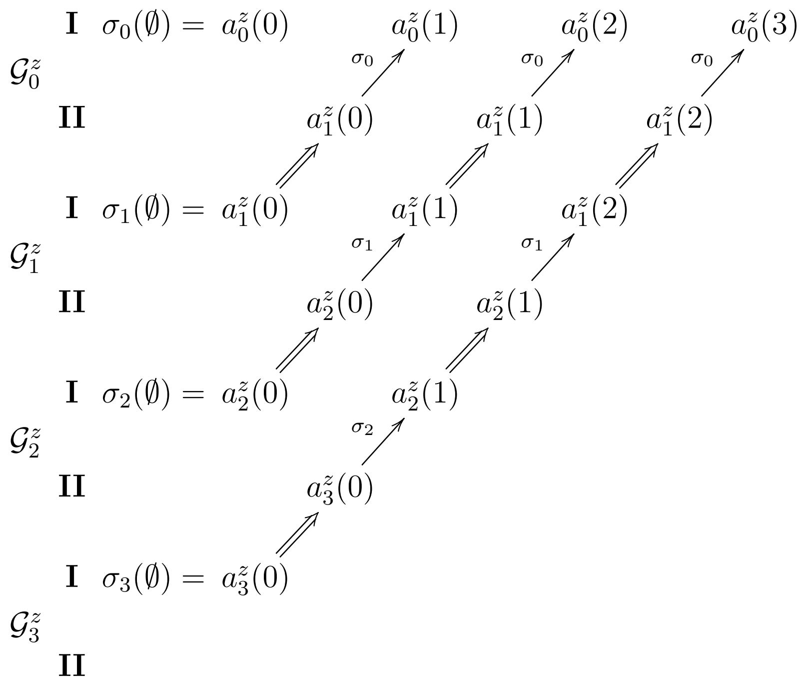


Figure 1: A Martin-Monk diagram

Notice that if z and w are eventually equal, that is for some m_0

$$\forall m \geq m_0 \, (z(m) = w(m)) \, ,$$

then

$$\forall m \geq m_0 \, (a_m^z = a_m^w) \, .$$

Claim 6.2.1. *Suppose $z, w \in {}^{\omega}2$ are such that $\exists ! k \in \omega \, (z(k) \neq w(k))$. Then*

$$a_0^z \in A_0 \Leftrightarrow a_0^w \notin A_0 \, .$$

Proof. As z and w agree after k, then $a_{k+1}^z = a_{k+1}^w$. By (32) above, $a_k^z \in A_k \Leftrightarrow a_k^w \notin A_k$. As $z \restriction k = w \restriction k$ and again by repeated applications of (32), $a_0^z \in A_0 \Leftrightarrow a_0^w \notin A_0$. □

Therefore $F = \{z \in {}^{\omega}2 \mid a_0^z \in A_0\}$ is a flip set, contradicting BP and LM, and hence AD. □

Notice that the existence of a flip set implies the ill-foundedness of $<_{\mathrm{L}}$ in the Cantor space — see Exercise 10.3.

Remarks 6.5. (a) The strategies σ_n^i in the proof of Theorem 6.1 are (or better: induce) contractions f_n, and having them play against each other amounts to construct a some sort of projective limit

$$\dots \xrightarrow{f_2} \mathbb{R} \xrightarrow{f_1} \mathbb{R} \xrightarrow{f_0} \mathbb{R}$$

and the real in the first row of the diagram of Figure 1 is the unique element of $\bigcap_n B_n$, where

$$B_n = f_0 \circ \cdots \circ f_n(\mathbb{R}) . \tag{33}$$

(b) By looking at Figure 1 it is clear that if $z \restriction n = w \restriction n$ then the diagrams relative to z and to w agree in the upper triangular part, that is

$$i + j \leq n \Rightarrow a_i^z(j) = a_i^w(j) ,$$

so that the map $z \mapsto a_0^z$ is Lipschitz and $F \leq_{\mathrm{L}} A_0$. Therefore, in order to reach the desired contradiction, we only need the property of Baire to hold of sets $\leq_{\mathrm{L}} A_0$.

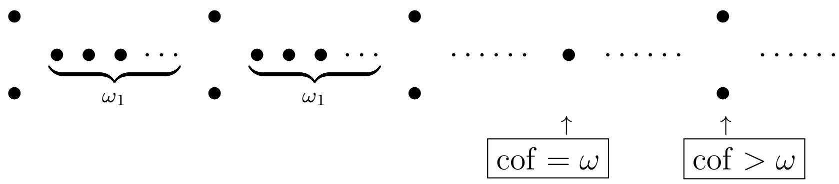
(c) Determinacy was used in a "local" way in the proofs of Wadge's Lemma and the Martin-Monk Theorem. For example, in order to prove Wadge's Lemma for sets in some pointclass $\boldsymbol{\Gamma}$ it is enough to assume the determinacy of all sets contained in the smallest Boolean algebra containing $\boldsymbol{\Gamma}$. So, for example, the well-foundedness of the Wadge hierarchy restricted to Borel sets can be proved in ZFC. In fact it can be proved in second order arithmetic [LSR88], while Borel determinacy is not provable in Zermelo's set theory with choice, ZC, that is ZFC without replacement [Fri71]. On the other hand by results of Harrington and Martin the determinacy of $\boldsymbol{\Pi}_1^1$ sets implies the determinacy of boolean combination of $\boldsymbol{\Pi}_1^1$ sets, hence it is enough to prove $\mathsf{SLO}^{\mathrm{L}}$ and the well-foundedness of $<_{\mathrm{L}}$ for $\boldsymbol{\Pi}_1^1$ and $\boldsymbol{\Sigma}_1^1$ sets.

(d) The statements and proofs of Theorem 6.1 and Corollary 6.2 were given for $\mathbb{R}$, but work for any space $[T]$, with T a pruned tree on ω.

Corollary 6.6. *Assume* $\mathsf{AD} + \mathsf{DC}(\mathbb{R})$*. The order* $\leq$ *on the Lipschitz degrees in any space of the form* $[T]$ *with* T *a pruned tree on* ω *is well-founded and the anti-chains have size at most* 2*.*

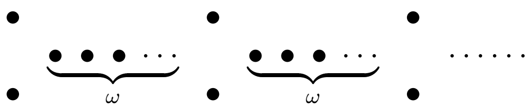
(a) *If the space is* $\mathbb{R}$*, there are blocks of length* ω_1 *of consecutive self-dual degrees and non-self-dual pairs at levels of uncountable cofinality. There-*

fore the Lipschitz hierarchy looks like this:



with non-self-dual pairs occurring exactly at levels $\omega_1 \cdot \alpha$*, with* α *successor or limit of cofinality* $> \omega$*.*

(b) *If the space is* ${}^{\omega}k$*, there are blocks of length* ω *of consecutive self-dual degrees, and non-self-dual pairs occur at all limit levels.*



7 Wadge games

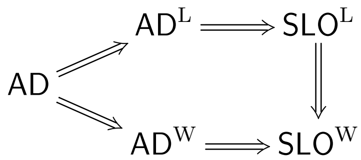
The analogue of Lemma 4.5 for G_{W} is also called Wadge's Lemma:

Lemma 7.1. *Assume* AD*, or even just* AD^{W}*. Then for every* $A, B \subseteq \mathbb{R}$

$$(\mathsf{SLO}^{\mathrm{W}}) \qquad A \leq_{\mathrm{W}} B \quad \vee \quad \neg B \leq_{\mathrm{W}} A\,.$$

Thus AD^{W} and AD^{L} imply $\mathsf{SLO}^{\mathrm{W}}$ and $\mathsf{SLO}^{\mathrm{L}}$, respectively, and are both consequences of AD. Keeping in mind (22) we have that

(34)



In Chapter IV we shall prove that, assuming $\neg\mathsf{FS} + \mathsf{DC}(\mathbb{R})$,

$$\mathsf{SLO}^{\mathrm{W}} \Leftrightarrow \mathsf{SLO}^{\mathrm{L}} \Leftrightarrow \mathsf{AD}^{\mathrm{W}} \Leftrightarrow \mathsf{AD}^{\mathrm{L}}\,.$$

For the time being let us point out that many of the preceding proofs did go through under weaker assumptions. For example:

Theorem 7.2. *Assume* $\mathsf{AC}_{\omega}(\mathbb{R}) + \neg\mathsf{FS} + \mathsf{AD}^{\mathrm{L}}$*. Then there is no sequence* $\langle A_n \mid n \in \omega \rangle$ *of sets of reals such that* $A_{n+1} <_{\mathrm{L}} A_n$*.*

and

Corollary 7.3. *Assume* $\mathsf{DC}(\mathbb{R}) + \neg\mathsf{FS} + \mathsf{AD}^{\mathrm{L}}$. *Then* $<_{\mathrm{L}}$ *is well-founded on* $\mathscr{P}(\mathbb{R})$.

The following seems to be open:

Open problem 7.4. *Does* $\mathsf{SLO}^{\mathrm{W}}$ *imply* $\neg\mathsf{FS}$*?*

Many of the results in the forthcoming pages are stated under rather technical assumptions like the ones in 7.2 and 7.3, since these assumptions will be of importance in the subsequent Chapters, but on first reading, the reader should probably replace them with AD.

8 The Wadge hierarchy

As was observed in part (b) of Remark 5.11, each ω_1-block of consecutive self-dual Lipschitz degrees is included in the same self-dual Wadge degree. The next result — whose proof is deferred to the next section — implies that this Wadge degree contains no other Lipschitz degrees.

Theorem 8.1 (Steel, Van Wesep)**.** *Assume* $\mathsf{AD}^{\mathrm{L}} + \neg\mathsf{FS}$. *For all* $A \subseteq \mathbb{R}$

$$A \leq_{\mathrm{W}} \neg A \Rightarrow A \leq_{\mathrm{L}} \neg A\,.$$

Corollary 8.2. *Assume* $\mathsf{AD}^{\mathrm{L}} + \mathsf{AC}_{\omega}(\mathbb{R}) + \neg\mathsf{FS}$.

(a) $[A]_{\mathrm{L}}$ *is self-dual iff* $[A]_{\mathrm{W}}$ *is self-dual, and in this case* $[A]_{\mathrm{W}} = [A]_{\mathrm{L}}$ *and*

$$\{B \subseteq \mathbb{R} \mid B \leq_{\mathrm{L}} A\} = \{B \subseteq \mathbb{R} \mid B \leq_{\mathrm{W}} A\}\,.$$

(b) *Every self-dual degree* $[A]_{\mathrm{W}}$ *is the union of an* ω_1*-block of consecutive Lipschitz degrees and*

$$\{B \subseteq \mathbb{R} \mid B \leq_{\mathrm{L}} A\} \subset \{B \subseteq \mathbb{R} \mid B \leq_{\mathrm{W}} A\}\,.$$

(c) *Every self-dual Wadge degree has a non-self-dual pair of immediate successors.*

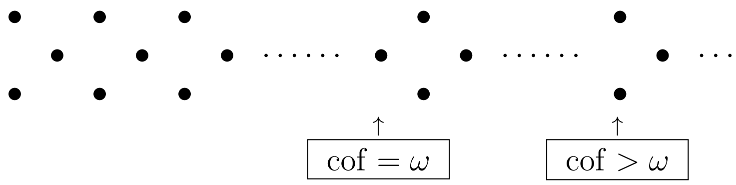
(d) *Every non-self-dual pair of Wadge degrees has a self-dual degree as immediate successor.*

(e) *Assume also* $\mathsf{DC}(\mathbb{R})$.

- *If we work with the Baire space ${}^{\omega}\omega$, then at limit levels of countable cofinality there is a self-dual Wadge degree; at levels of uncountable cofinality there is a non-self-dual pair.*
- *If we work with the Cantor space ${}^{\omega}2$, then at all limit levels there is a non-self-dual pair of degrees.*

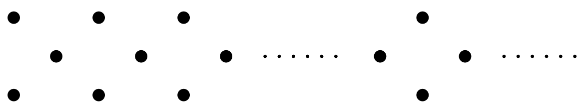
Therefore the Wadge degrees on ${}^{\omega}\omega$ look like this:

(35)



and on ${}^{\omega}2$ the Wadge degrees look like this:

(36)



8.A The length of the Wadge hierarchy

The two diagrams (35) and (36) suggest the following question: what is the length of the Wadge hierarchy? A binary relation $\trianglelefteq$ is a **pre-well-order** (pwo for short) on X if it is reflexive, transitive, well-founded and connected, i.e.,

$$\forall x, y \in X\,(x \trianglelefteq y \;\vee\; y \trianglelefteq x)\,.$$

Any pwo yields a **norm** on X, i.e., an ordinal valued function $|\cdot|_{\trianglelefteq} \colon X \to \mathrm{Ord}$:

$$|x|_{\trianglelefteq} = \text{the rank of } x \text{ in the well-founded relation } \trianglelefteq\,.$$

Conversely any norm $f \colon X \to \mathrm{Ord}$ gives rise to a pwo on X. A norm is regular if its range is an ordinal — all canonical norms associated to pre-well-orders are regular. The length $\mathrm{lh}(\trianglelefteq)$ of a pwo $\trianglelefteq$ is the range of its canonical norm.

Definition 8.3. The ordinal Θ is defined by

$$\begin{aligned}\Theta &= \sup\{\mathrm{lh}(\trianglelefteq) \mid \trianglelefteq \text{ is a pwo on } \mathbb{R}\}\\ &= \sup\{\alpha \mid \exists f \colon \mathbb{R} \to \alpha \text{ is surjective}\}\,.\end{aligned}$$

Exercise 8.4. Show in ZF that Θ is a cardinal and $\Theta \geq \omega_2$.

Assuming AD, Θ is a fixed point of the $\aleph$-functions, i.e., $\aleph_\Theta = \Theta$, but it is not the least such fixed point; in fact it is larger than the least fixed point of the enumerating function of all fixed points of the $\aleph$-function, and so on (see [Kan03, pp. 396–399]).

Theorem 8.5. *Assume $\mathsf{AD} + \mathsf{DC}(\mathbb{R})$. Then Θ is the length of the Wadge hierarchy:*

$$\Theta = \sup\{\|A\|_{\mathrm{W}} \mid A \subseteq \mathbb{R}\}\,.$$

Proof. Let $f \colon \mathbb{R} \to \alpha$ be a surjection. Define $\langle A_\nu \mid \nu < \alpha\rangle$ by

$$A_\nu = J\left(\{x \oplus y \in \mathbb{R} \mid f(x) < \nu \wedge y \in A_{f(x)}\}\right)$$

where $J = J_{\mathcal{F}}$ is as in Lemma 4.7 and $\mathcal{F}$ is the collection of all continuous functions. Then $\nu < \xi \Rightarrow A_\nu <_{\mathrm{W}} A_\xi$, hence

$$\Theta \leq \sup\{\|A\|_{\mathrm{W}} \mid A \subseteq \mathbb{R}\}\,.$$

For the other direction, fix $\{g_x \mid x \in \mathbb{R}\}$ a parametrization of all continuous functions: if $\|A\|_{\mathrm{W}} = \alpha$ then $f \colon \mathbb{R} \to \alpha + 1$

$$f(x) = \left\|g_x^{-1}(A)\right\|_{\mathrm{W}}$$

is a surjection. □

Since Θ is a cardinal bigger than ω_1 then $\omega_1 \cdot \Theta = \Theta$, and Θ is also the length of the Lipschitz hierarchy.

8.B Boldface pointclasses

The theory of Wadge degrees can be seen as the general theory of boldface pointclasses. A **pointclass** in $\mathcal{X}$ is simply a non-empty $\Gamma \subset \mathscr{P}(\mathcal{X})$. If $\mathcal{X}$ is a topological space a **boldface pointclass** is a pointclass closed under continuous pre-images; boldface pointclasses are denoted by capital boldface Greek letters, like $\boldsymbol{\Gamma}$, $\boldsymbol{\Lambda}$, etc. If $\mathcal{X}$ is not mentioned, it is understood to be $\mathbb{R}$. Assuming $\mathsf{SLO}^{\mathrm{W}}$, every boldface pointclass $\boldsymbol{\Gamma}$ (on $\mathbb{R}$ or ${}^{\omega}2$) is down-ward closed under $\leq_{\mathrm{W}}$ and it is of the form

$$\boldsymbol{\Gamma} = \{A \subseteq \mathbb{R} \mid A <_{\mathrm{W}} C\} \tag{37}$$

or

$$\boldsymbol{\Gamma} = \{A \subseteq \mathbb{R} \mid A \leq_{\mathrm{W}} C\}\,. \tag{38}$$

In the latter case $\boldsymbol{\Gamma}$ is said to be **principal and generated** by C, and the set C (and hence any other $C' \in [C]_{\mathrm{W}}$) is said to be **$\boldsymbol{\Gamma}$-complete**, or **complete for $\boldsymbol{\Gamma}$**. Not every pointclass is principal for example $\boldsymbol{\Delta}^1_1$, the collection of Borel sets, is not. The **dual** of a pointclass $\boldsymbol{\Gamma}$ is the pointclass

$$\breve{\boldsymbol{\Gamma}} = \boldsymbol{\Gamma}^{\smile} = \{\neg A \mid A \in \boldsymbol{\Gamma}\} .$$

A pointclass $\boldsymbol{\Gamma}$ is **self-dual** if $\breve{\boldsymbol{\Gamma}} = \boldsymbol{\Gamma}$, that is, if it is closed under complements; otherwise it is called **non-self-dual**. If $\boldsymbol{\Gamma}$ is principal and generated by A, then

$$\boldsymbol{\Gamma} \text{ is self-dual} \Leftrightarrow A \text{ is self-dual.}$$

Exercise 8.6. Show that $\mathsf{SLO}^{\mathrm{W}}$ implies Wadge's Lemma for boldface pointclasses, i.e. for any $\boldsymbol{\Gamma}$ and $\boldsymbol{\Lambda}$,

$$\boldsymbol{\Gamma} \subseteq \boldsymbol{\Lambda} \quad \vee \quad \breve{\boldsymbol{\Lambda}} \subseteq \boldsymbol{\Gamma} .$$

Show that the non-self-dual pointclasses are exactly those which are principal and generated by a non-self-dual set.

Give an example of self-dual pointclass which is principal, and one which is non-principal.

For $A \subseteq X \times Y$, $\bar{x} \in X$ and $\bar{y} \in Y$,

$$A_{\{\bar{x}\}} = \{y \mid (\bar{x}, y) \in A\}$$
$$A^{\{\bar{y}\}} = \{x \mid (x, \bar{y}) \in A\} ,$$

are, respectively the vertical and horizontal sections of A through $\bar{x}$ and $\bar{y}$. A set $U \subseteq \mathbb{R} \times \mathbb{R}$ is **universal for $\boldsymbol{\Gamma}$** if

$$\boldsymbol{\Gamma} = \{U_{\{x\}} \mid x \in \mathbb{R}\}$$

and $\mathbb{h}(U) \in \boldsymbol{\Gamma}$, where $\mathbb{h} \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the canonical homeomorphism of (7).

Proposition 8.7. *Assume $\mathsf{SLO}^{\mathrm{L}}$. Every non-self-dual $\boldsymbol{\Gamma}$ admits a universal set U, and $\mathbb{h}(U) \in \boldsymbol{\Gamma}$ is complete.*

Proof. Let $\boldsymbol{\Gamma}$ be non-self-dual and generated by A, and let

$$U = \{(x, y) \mid \boldsymbol{\ell}_x(y) \in A\} .$$

Then $\mathbb{h}(U) \in \boldsymbol{\Gamma}$, and since every Lipschitz function is continuous

$$\{B \mid B \leq_{\mathrm{L}} A\} \subseteq \boldsymbol{\Gamma} .$$

Conversely, if $B \in \boldsymbol{\Gamma}$, and, towards a contradiction, $B \not\leq_{\mathrm{L}} A$, then $A \leq_{\mathrm{L}} \neg B$: since $B \leq_{\mathrm{W}} A$, then by transitivity $B \leq_{\mathrm{W}} \neg B$ and hence $A \equiv_{\mathrm{W}} B$ would be Wadge self-dual, contrarily to our assumption. Therefore U is universal for $\boldsymbol{\Gamma}$. $\square$

8.C An overview of the Wadge hierarchy

A clopen partition of a topological space $\mathcal{X}$ is a family $\{D_i \mid i < N\}$ such that

(i) $1 \leq N \leq \omega$

(ii) each D_i is clopen and non-empty,

(iii) $i < j < N \Rightarrow D_i \cap D_j = \emptyset$,

(iv) $\bigcup_{i<N} D_i = \mathcal{X}$.

The partition is said to be finite iff $N < \omega$. In the Baire space any non-empty clopen set D can be written as $D = \bigcup_{n\in\omega} D_n$ with each D_n clopen, and $n \neq m \Rightarrow D_n \cap D_m = \emptyset$, so any clopen partition can be refined to an infinite clopen partition.

Exercise 8.8. Let $\mathcal{X}$ be a topological space. Show that:

(i) If $\{D_i \mid i < N\}$ is a clopen partition of $\mathcal{X}$ and $f_i \colon D_i \to \mathcal{X}$ is continuous, then $\bigcup_{i<N} f_i \colon \mathcal{X} \to \mathcal{X}$ is continuous.

(ii) If $D \subseteq \mathcal{X}$ is clopen, $f \colon D \to \mathcal{X}$ is continuous, $B \subset \mathcal{X}$, and $A = f^{-1}(B)$, then $A \leq^{\mathcal{X}}_{\mathrm{W}} B$.

(iii) If $\{D_i \mid i < N\}$ is a clopen partition of $\mathcal{X}$, then

$$A \leq^{\mathcal{X}}_{\mathrm{W}} B \Leftrightarrow \forall i < N \left(A \cap D_i \leq^{\mathcal{X}}_{\mathrm{W}} B \right).$$

Proposition 8.9. *Let $A, B_n \subseteq \mathbb{R}$.*

(a) *$A \leq_{\mathrm{W}} \bigoplus_{n\in\omega} B_n$ iff there is a clopen partition $\{D_n \mid n < \omega\}$ of $\mathbb{R}$ such that*

$$\forall n \left(A \cap D_n \leq^{D_n,\mathbb{R}}_{\mathrm{W}} B_n \right). \tag{39}$$

(b) *$A \leq_{\mathrm{W}} B_0 \oplus B_1$ iff there is a clopen partition $\{D_0, D_1\}$ of $\mathbb{R}$ such that $A \cap D_n \leq_{\mathrm{W}} B_n$, for all $n = 0, 1$.*

Remark 8.10. The definition of $\leq^{D_n,\mathcal{X}}_{\mathrm{W}}$ is given in (1), and when $B_n \neq \mathbb{R}$, (39) is equivalent to $A \cap D_n \leq_{\mathrm{W}} B_n$ by Exercise 8.8(ii).

Proof of 8.9. If f is continuous, then $D_n = f^{-1}(\boldsymbol{N}_{\langle n\rangle})$ is clopen and $\{D_n \mid n \in \omega\}$ is partition. (For (b) use $D_0 = \bigcup_n f^{-1}(\boldsymbol{N}_{\langle 2n\rangle})$ and $D_1 = \bigcup_n f^{-1}(\boldsymbol{N}_{\langle 2n+1\rangle})$.) Then

$$g_n \stackrel{\text{def}}{=} f \upharpoonright D_n \colon D_n \to \mathbb{R}$$

is continuous and such that $g_n^{-1}(B_n) = A \cap D_n$, so we are done by part (iii) of Exercise 8.8. □

The very same proof yields

Proposition 8.11. *If $A, B_0, B_1 \subseteq {}^{\omega}2$, then $A \leq_{\mathrm{W}}^{{}^{\omega}2} B_0 \oplus B_1$ iff there is a clopen partition $\{D_0, D_1\}$ of ${}^{\omega}2$ such that $A \cap D_n \leq_{\mathrm{W}}^{D_n, {}^{\omega}2} B_n$, for $n = 0, 1$.*

If T is a well-founded tree on ω, then

$$\forall x \in \mathbb{R}\, \exists! s \in \partial T\, (s \subset x)$$

so that $\mathbb{R} = \bigcup_{s \in \partial T} \boldsymbol{N}_s$, and since

$$\forall s, t \in \partial T\, (s \neq t \Rightarrow \boldsymbol{N}_s \cap \boldsymbol{N}_t = \emptyset)\,,$$

it follows that $\{\boldsymbol{N}_s \mid s \in \partial T\}$ is a clopen partition of $\mathbb{R}$. Conversely, if $\{D_n \mid n < N\}$ is a clopen partition of $\mathbb{R}$, then

$$T = \{t \in {}^{<\omega}\omega \mid \forall n\, (\boldsymbol{N}_t \nsubseteq D_n)\}$$

is a well-founded tree on ω. Since every real belongs to a unique D_n, then $\{\boldsymbol{N}_s \mid s \in \partial T\}$ is a clopen partition refining $\{D_n \mid n < N\}$. We shall refer to the $\{\boldsymbol{N}_s \mid s \in \partial T\}$ thus obtained as the **canonical clopen partition refining** $\{D_n \mid n < N\}$.

Suppose T is a tree on ω, and suppose $\bar{A} = \langle A_s \mid s \in \partial T\rangle$ is a sequence of sets of reals. Then let

$$\boldsymbol{S}(T; \bar{A}) = \bigcup_{s \in \partial T} s^\frown A_s\,. \tag{40}$$

There are two cases when this definition is problematic:

- if $T = \emptyset$, then $\partial T = \{\emptyset\}$, so $\boldsymbol{S}(T; \langle A_\emptyset\rangle) = A_\emptyset$,
- and if $T = {}^{<\omega}\omega$ then $\partial T = \emptyset$, so $\bar{A} = \emptyset$ and $\boldsymbol{S}(T; \bar{A}) = \emptyset$.

Proposition 8.9 can be strengthened to

Proposition 8.12. *Let $B_n \subseteq \mathbb{R}$. The pointclass generated by $\bigoplus_n B_n$ is*

$$\{\boldsymbol{S}(T;\bar{A}) \mid T \text{ is well-founded, and } \forall s \in \partial T\, \exists n\, (A_s \leq_{\mathrm{W}} B_n)\}\,.$$

Similarly, then pointclass generated by $B_0 \oplus B_1$ is

$$\{\boldsymbol{S}(T;\bar{A}) \mid T \text{ is well-founded, and } \forall s \in \partial T\, \exists i \in 2\, (A_s \leq_{\mathrm{W}} B_i)\}\,.$$

Proof. For the sake of definiteness, let us focus on the pointclass generated by $\bigoplus_n B_n$, the other case being left as an exercise. By Proposition 8.9, its elements are sets A such that $A \cap D_n \leq_{\mathrm{W}}^{D_n,\mathbb{R}} B_n$, for some carefully chosen clopen partition $\{D_n \mid n < N\}$: if $\{\boldsymbol{N}_s \mid s \in \partial T\}$ is the canonical refinement of $\{D_n \mid n < N\}$, and if $\boldsymbol{N}_s \subseteq D_n$, then $A = \boldsymbol{S}(T;\langle A_{\lfloor s \rfloor} \mid s \in \partial T)$ and $s^\frown A_{\lfloor s \rfloor} \leq_{\mathrm{W}}^{\boldsymbol{N}_s,\mathbb{R}} B_n$. Since $A_{\lfloor s \rfloor} \leq_{\mathrm{W}}^{\mathbb{R},\boldsymbol{N}_s} s^\frown A_{\lfloor s \rfloor}$, the result follows. □

We are now going to look at the first few Wadge degrees or — equivalently — at the first few boldface pointclasses. By the general remark after (14) on page 14, at the bottom of the Wadge hierarchy there is the non-self-dual pair $[\emptyset]_{\mathrm{W}} = \{\emptyset\}$ and $[\mathbb{R}]_{\mathrm{W}} = \{\mathbb{R}\}$. (For the sake of definiteness, these facts are stated for $\mathbb{R}$ only, but they also hold for ${}^{\omega}2$.) Notice that these are also boldface pointclasses, albeit trivial ones. The self-dual degree immediately above it is $[\emptyset \oplus \mathbb{R}]_{\mathrm{W}}$, and the pointclass it generates is $\boldsymbol{\Delta}^0_1$, the collection of all clopen sets. Thus

$$[\emptyset \oplus \mathbb{R}]_{\mathrm{W}} = \boldsymbol{\Delta}^0_1 \setminus \{\emptyset, \mathbb{R}\}\,.$$

The two non-self-dual pointclasses immediately above are $\boldsymbol{\Sigma}^0_1$ and $\boldsymbol{\Pi}^0_1$, the collection of open and closed sets. To see this notice that if $B <_{\mathrm{W}} A$ and A is open, then B is open and by $\mathsf{SLO}^{\mathrm{W}}$, $\neg B <_{\mathrm{W}} A$ and hence $B <_{\mathrm{W}} \neg A$, which implies B is closed. Therefore B is clopen. Notice that the appeal to $\mathsf{SLO}^{\mathrm{W}}$ is not necessary since the games involved are Borel and therefore determined in ZFC.

The next ω_1 levels of the Wadge hierarchy are occupied by the Hausdorff difference pointclasses over $\boldsymbol{\Pi}^0_1$ sets (see Exercise **??**) so that the sets of countable Wadge rank are the $\boldsymbol{\Delta}^0_2$ sets. In other words, Hausdorff's analysis of $\boldsymbol{\Delta}^0_2$ sets is exhaustive: all non-self-dual boldface pointclasses between $\boldsymbol{\Pi}^0_1$ and $\boldsymbol{\Delta}^0_2$ are the α-$\boldsymbol{\Pi}^0_1$ and their dual. On the other hand between $\boldsymbol{\Sigma}^0_2$ and $\boldsymbol{\Pi}^0_2$ (which live at level ω_1) and $\boldsymbol{\Sigma}^0_3$ and $\boldsymbol{\Pi}^0_3$ there are many more non-self-dual pointclasses than the α-$\boldsymbol{\Pi}^0_2$ and their dual. In fact the Wadge rank of a complete $\boldsymbol{\Sigma}^0_3$ (or $\boldsymbol{\Pi}^0_3$) set is not $\omega_1 + \omega_1$, but $\omega_1^{\omega_1}$, see section **??** for more on this.

9 Proof of the Steel-Van Wesep Theorem

The proof of Theorem 8.1 uses a variant of the technique in the proof of Theorem 6.1, usually called the Martin–Monk method. This technique figures prominently in the theory of Wadge degrees, so we start with some remarks on how to formulate it in general terms.

9.A The Martin-Monk method.

Given a sequence σ_n of strategies for **I** in a Lipschitz game, we can pit them against each other, pretending that the opponent of the player using σ_n is the player using σ_{n+1}. Therefore the diagram of Figure 1 can be conveyed in a more compact form

(41)

$$\begin{array}{c|lllll|}
\hline
\sigma_0 & a^0_0=\sigma_0(\emptyset) & a^0_1=\sigma_0(\langle a^1_0\rangle) & a^0_2=\sigma_0(\langle a^1_0,a^1_1\rangle) & a^0_3=\sigma_0(\langle a^1_0,a^1_1,a^1_2\rangle) & \cdots\cdots \\
\hline
\sigma_1 & a^1_0=\sigma_1(\emptyset) & a^1_1=\sigma_1(\langle a^2_0\rangle) & a^1_2=\sigma_1(\langle a^2_0,a^2_1\rangle) & a^1_3=\sigma_1(\langle a^2_0,a^2_1,a^2_2\rangle) & \cdots\cdots \\
\hline
\sigma_2 & a^2_0=\sigma_2(\emptyset) & a^2_1=\sigma_2(\langle a^3_0\rangle) & a^2_2=\sigma_2(\langle a^3_0,a^3_1\rangle) & a^2_3=\sigma_2(\langle a^3_0,a^3_1,a^3_2\rangle) & \cdots\cdots \\
\hline
\end{array}$$
$$\vdots$$

The player using σ_n operates on row n and contrasts the moves of his opponent on row $n+1$

$$\text{(42)}\qquad \begin{array}{cc|lllll|l}
\hline
\mathbf{I} & \sigma_n & a^n_0 & & a^n_1 & & \cdots\cdots & \text{row } n \\
\hline
\mathbf{II} & & & a^{n+1}_0 & & a^{n+1}_1 & \cdots\cdots & \text{row } n+1 \\
\hline
\end{array}$$

while the player on row $n+1$ is using σ_{n+1} to contrast the moves of the next player on row $n+2$, and so on. Thus Remark 6.5(b) says that knowing $\sigma_0,\ldots,\sigma_n$ (or even just a finite part of them) allows us to fill-in the upper-left triangular part of the Martin-Monk diagram with entries a^i_j, with $i+j\le n$,

$$\text{(43)}\qquad \begin{array}{lllll}
a^0_0 & a^0_1 & \cdots & a^0_{n-1} & a^0_n \\
a^1_0 & a^1_1 & \cdots & a^1_{n-1} & \\
\vdots & & & & \\
a^{n-1}_0 & a^{n-1}_1 & & & \\
a^n_0 & & & &
\end{array}$$

(Formally, the finite region filled-in by $\sigma_0,\ldots,\sigma_n$ is an $f\colon D\to\omega$ with $D\subset\omega\times\omega$ finite, and with the obvious requirement that the values $f(k,0), f(k,1),\ldots$ are the output $a^k_0, a^k_1,\ldots$ on row k computed by σ_k on the input given by row $k+1$.)

Suppose now we want to repeat the construction when some of the σ_n are strategies for **II**. To this end let's convene that the player using σ_n always makes his moves are on row n while his opponent moves on row $n+1$. Thus if σ_n is a strategy for **I** we have (42), if σ_n is a strategy for **II** we have

(44)
$$\begin{array}{ll|cccc|l} \mathbf{II} & \sigma_n & & a_0^n & a_1^n & \cdots\cdots & \text{row } n \\ \hline \mathbf{I} & & a_0^{n+1} & a_1^{n+1} & & \cdots\cdots & \text{row } n+1 \end{array}$$

Therefore if row $n+1$ has the first k entries filled-in, then row n has the first $k+1$ or k entries filled in, depending whether σ_n is a strategy for **I** or **II**. The presence of strategies for **II** slows-down the filling-in procedure, and the diagram cannot be completely filled-in unless σ_n is a strategy for **I** for infinitely many n. A more topological way to say this is that, if A_n are as in equation (33) then $\mathrm{diam}(A_{n+1}) < \mathrm{diam}(A_n)$ if and only if f_{n+1} is a contraction, i.e., σ_{n+1} is a strategy for **I**. The region filled in by a finite number of strategies $\sigma_0, \ldots, \sigma_n$ is, in general, smaller than the triangular region of (43). For example, if among $\sigma_0, \ldots, \sigma_5$ only σ_0, σ_2, and σ_3 are strategies for **II** , then the region of the Martin-Monk diagram filled-in is

(45)
$$\begin{array}{l|ccc} \sigma_0 & a_0^0 & a_1^0 & a_2^0 \\ \sigma_1 & a_0^1 & a_1^1 & a_2^1 \\ \sigma_2 & a_0^2 & a_1^2 & \\ \sigma_3 & a_0^3 & a_1^3 & \\ \sigma_4 & a_0^4 & a_1^4 & \\ \sigma_5 & a_0^5 & & \end{array}$$

Finally, and this is the most interesting case, suppose each σ_n is either a strategy for **I** or a Wadge strategy for **II** (and this second case includes the possibility that σ_n is actually a strategy in a Lipschitz game). In this case it is not true anymore in the region filled in by $\sigma_0, \ldots, \sigma_n$ that rows with smaller indexes are at least as long as rows with larger indexes. If, for example, in the example of (45) above σ_3 passes in the first two innings, the region filled in by $\sigma_0, \ldots, \sigma_5$ is

(46)
$$\begin{array}{l|cc} \sigma_0 & a_0^0 & \\ \sigma_1 & a_0^1 & \\ \sigma_2 & & \\ \sigma_3 & & \\ \sigma_4 & a_0^4 & a_1^4 \\ \sigma_5 & a_0^5 & \end{array}$$

Suppose $\langle \sigma_n \mid n \in \omega \rangle$ is a sequence of Wadge strategies for **II** and Lipschitz strategies for **I**, and let $f_n \colon D_n \to \omega$ be the finite region of the Martin-Monk diagram filled-in by $\sigma_0, \ldots, \sigma_n$. The following is a tedious, but straightforward verification.

Exercise 9.1. Show that $n < m$ implies $D_n \subseteq D_m$ and $f_m \upharpoonright D_n = f_n$.

We will say that $\langle \sigma_n \mid n \in \omega \rangle$ admits a **global play** iff $\bigcup_n D_n = \omega \times \omega$, and $\bigcup_n f_n$ is the fill-in of the Martin-Monk diagram. It is not hard to see that the existence of infinitely many n's such that σ_n is a strategy for **I** does not guarantee that there is a global play, i.e., that the diagram can be completely filled-in, since a Wadge strategy for **II** can take a lot of naps and can drive the filling-in procedure to a grinding halt. The cure for this is to guarantee that the rows on which a Wadge strategy for **II** is used are sparse enough compared to the rows where a Lipschitz strategy for **I** is used. We will see an example of such a situation in the next proof.

9.B Proof of Theorem 8.1.

Suppose, towards a contradiction, that σ_0 is a winning strategy for **II** in $G_{\mathrm{W}}(A, \neg A)$ and σ_1 be **II**'s copying strategy, i.e., $\sigma_1(s^\frown n) = n$. Let also σ_2 be a winning strategy for **I** in $G_{\mathrm{L}}(A, \neg A)$. We shall define an increasing sequence of natural numbers

$$0 = M_0 < M_1 < M_2 < \dots$$

and let

$$\mathcal{G}_n = \begin{cases} G_{\mathrm{W}}(A, \neg A) & \text{if } n \in \{M_i \mid i \in \omega\}, \\ G_{\mathrm{L}}(A, \neg A) & \text{otherwise.} \end{cases}$$

When $n = M_i$, player **II** uses either σ_0 or σ_1 in the game $\mathcal{G}_n = G_{\mathrm{W}}(A, \neg A)$ on row n: the choice will depend on the value of $z(i)$, with $z \in {}^{\omega}2$. On all remaining rows, player **I** uses σ_2. If the M_i's are sparse enough, the diagram can be filled-in, no matter what z is used, and an argument as in the proof of Theorem 6.1 will yield the desired contradiction. Since $M_0 = 0$, player **II** can use either σ_0 or σ_1 on the 0-th row — for the sake of definiteness, let's assume that the former holds. Choose n' large enough so that if σ_2 is used on rows $1, 2, \dots, n'$ and hence n' values of row 1 are determined, then σ_0 makes a move in row 0 (remember that σ_0 may pass.) Let $M_1 = n' + 1$. Since σ_1 is the copying strategy, we can conclude that no matter what **I** uses on row M_0, then at least the first value of this row is determined. Choose now $n'' > M_1$ large enough such that if on rows $M_1 + 1 < M_1 + 2 < \dots < n''$ player **I** uses σ_2, then at least the first two entries on row M_0 will be filled-in, no matter what strategies **II** uses on rows M_0 and M_1. Let $M_2 = n'' + 1$, and so on (see picture **??**).

Here is the formal construction. Suppose $M_0, \dots, M_k$ have been defined, and let $s \in {}^{M_k+1}3$ be such that

(47) $$\forall n \leq M_k \, (s(n) \in \{0, 1\} \Leftrightarrow n \in \{M_i \mid i \leq k\}).$$

Claim 9.1.1. *There is an* m *such that letting* $t = s^\frown 0^{(m)}$*, the* $\sigma_{t(0)}, \dots, \sigma_{t(M_k+m)}$ *fill-in (at least) the first* $k+1$ *entries in the* 0*-th row, i.e., if* $f\colon D \to \omega$ *is the region filled-in by the* $\sigma_{t(0)}, \dots, \sigma_{t(M_k+m)}$*, then* $n \leq M_k \Rightarrow (0, n) \in D$*.*

Proof. Consider the element $w = s^\frown \vec{0} \in {}^\omega 3$. Then $\langle \sigma_{w(n)} \mid n \in \omega \rangle$ admits a global play, since all strategies from some point on are for **I**. Let x be the real in the $M_k + 1$-st row. Since each $\sigma_{w(0)}, \dots, \sigma_{w(M_k)}$ are (or better: yield) continuous functions, there is an m such that $x \restriction m$ is enough to compute the value of the real in the 0-th row up to $k+1$. This m will do. □

Let m_s be the least m as in the Claim and let

$$M_{k+1} = M_k + \sup\{m_s \mid s \in {}^{M_k+1}3 \wedge s \text{ satisfies } (47)\}\,.$$

This concludes the definition of $\langle M_k \mid k \in \omega \rangle$. Finally for any $z \in {}^\omega 2$ let $w_z \in {}^\omega 3$ be obtained by coping the values of z on the set $\{M_k \mid k \in \omega\}$ and set the value 2 in the other spots, that is

$$w_z(n) = \begin{cases} z(i) + 1 & \text{if } n = M_i, \\ 2 & \text{otherwise.} \end{cases}$$

The sequence of strategies $\langle \sigma_{w_z(n)} \mid n \in \omega \rangle$ admits a global play $f^z\colon \omega \times \omega \to \omega$, for each $z \in {}^\omega 2$, since the first M_k strategies are enough to determine the first $k+1$ entries of row M_0. Let $h\colon {}^\omega 2 \to {}^\omega \omega$

$$\begin{aligned} h(z) &= \langle f^z(0, m) \mid m \in \omega \rangle \\ &= \text{the real in the 0-th row of the diagram.} \end{aligned}$$

Arguing as is the proof of Theorem 6.1, the set $\{z \in {}^\omega 2 \mid h(z) \in A\}$ is a flip set, contradicting $\neg\mathsf{FS}$. This concludes the proof of Theorem 8.1. □

In view of Corollary 8.2 above we will say that a set A is (non-)self-dual iff the Wadge degree $[A]_{\mathrm{W}}$, or equivalently the Lipschitz degree $[A]_{\mathrm{L}}$, is (non-)self-dual. Notice that Theorem 8.1 is non-trivial even for Borel sets. For example it implies that if $A \in \boldsymbol{\Sigma}^0_\alpha \setminus \boldsymbol{\Pi}^0_\alpha$ and $B = f^{-1}(A)$ for some continuous f then $B = g^{-1}(A)$ for some Lipschitz g.

Exercise 9.2. Assume $\neg\mathsf{FS}$. Show that if $A \leq_{\mathrm{W}} \neg A$ then there is no contraction f such that $A = f^{-1}(A)$.

10 Other ambient spaces

In the previous sections we have seen how AD (or some of its consequences) plus $\mathsf{DC}(\mathbb{R})$ yields a complete description of the Lipschitz and Wadge hierarchies on $\mathbb{R}$ and on the Cantor spaces ${}^{\omega}k$. Assuming the stronger $\mathsf{AD}_{\mathbb{R}}$, the same analysis can be carried out for ${}^{\omega}\mathbb{R}$, where $\mathbb{R}$ is given the discrete topology: in this case the Lipschitz and Wadge hierarchies look like their analogues on $\mathbb{R}$. Suppose now we work in the space $[T]$, where T is a perfect pruned tree on ω. These spaces are not (in general) homogeneous, in the sense that

$$[T_{\lfloor s \rfloor}] = [T] \qquad \text{and} \qquad s^\frown [T] = \boldsymbol{N}([T]\,; s)$$

need not to hold.

Additional exercises

Exercise 10.1. Let T be a pruned, perfect, finite branching tree. Construct a $\varphi \colon T \to {}^{<\omega}2$ such that $f_\varphi \colon [T] \to {}^{\omega}2$ is a homeomorphism.

Show that ${}^{\omega}k$ and ${}^{\omega}h$ are not Lipschitz isometric when $2 \leq h < k < \omega$.

Exercise 10.2. Consider the game G on ω_1 where **I** plays an ordinal $\alpha \geq \omega$ in his first move (and then his other moves are irrelevant) and **II** plays natural numbers n_i, with the condition that **II** wins iff $\langle n_i \mid i \in \omega \rangle$ codes a well-order of length α. Show that the determinacy of G implies the failure of the PSP. Conclude that AD_{ω_1} is inconsistent.

Show that $\mathsf{AD}_{\mathscr{P}(\mathbb{R})}$ is inconsistent.

Exercise 10.3. Show that a flip set F is Lipschitz self-dual, and that $F_{\lfloor s \rfloor}$ is a flip set, for all $s \in {}^{<\omega}2$. Conclude that the existence of a flip set implies that $<_{\mathrm{L}}$ is ill-founded.

Notes and references

The basic results of this Chapter are from Wadge's Ph.D. thesis [Wad83]. Our account follows closely that of [VW78], where Theorems 6.1 and 8.1 are presented: Martin never published Theorem 6.1, but other accounts of this result are in [Mos80, Exercise 7D.14s] and [Kec95, Theorem 21.15], while Theorem 8.1 was proved independently by Steel and Van Wesep and recorded in their dissertations [Ste77], [VW77]. Lemma 4.7 and Theorem 8.5 are from [Sol78]. Example 1.1 is taken from [Woo99, Remark 9.26, page 624]. Proposition 2.10 is due to Camillo Costantini, while the notion of Wadge structure and Proposition 4.6 are due to Alberto Marcone.

Chapter II

$\mathsf{SLO}^{\mathrm{W}}$ and Wadge degrees

In this Chapter we start a systematic investigation of the consequences of $\mathsf{SLO}^{\mathrm{W}}$. It turns out that many results proved in Chapter I under various assumptions like AD, AD^{L}, AD^{W}, and $\mathsf{SLO}^{\mathrm{L}}$ are already consequences of $\mathsf{SLO}^{\mathrm{W}}$ (see diagram (34)). For example the axiom of countable choices over the reals and the perfect subset property follow from $\mathsf{SLO}^{\mathrm{W}}$, and hence from AD^{L}, $\mathsf{SLO}^{\mathrm{L}}$ and AD^{W}. In fact $\mathsf{SLO}^{\mathrm{W}}$ is strong enough to recover the basic structure (diagram (35)) of the Wadge hierarchy. The detailed analysis of the Wadge hierarchy will be carried out under the assumption of $\mathsf{SLO}^{\mathrm{W}}$ or some "local" version of it—the reason for this technical twists will be clear in Chapter IV.

11 Countable choices over $\mathbb{R}$ and perfect subsets

The axiom of countable choices over the reals, $\mathsf{AC}_\omega(\mathbb{R})$, is used all the time in descriptive set theory (and many other parts of mathematics). It is equivalent to a seemingly weaker statement: given any family of non-empty sets of reals A_n, it is possible to choose an element from A_n, for infinitely many ns.

Lemma 11.1. *The following are equivalent:*

(a) $\mathsf{AC}_\omega(\mathbb{R})$,

(b) *For any sequence* $\langle A_n \mid n \in \omega \rangle$ *of non-empty sets of reals, there are* $\langle a_n \mid n \in \omega \rangle$ *and* $k_0 < k_1 < \dots$ *such that* $\forall n\,(a_n \in A_{k_n})$.

Proof. (a)⇒(b) is trivial, so we may assume (b) towards proving (a). Given $\langle A_n \mid n \in \omega \rangle$ be as in the statement, let $B_0 = A_0$ and

$$B_{n+1} = \{x \oplus y \mid x \in A_{n+1} \wedge y \in B_n\}\,.$$

By induction on n, $\emptyset \neq B_n$, hence fix an increasing sequence of integers $k_0 < k_1 < \dots$ and elements $b_n \in B_{k_n}$. By composing the surjections

$$B_{n+1} \to A_{n+1}, \qquad z \mapsto (z)_{\mathbf{I}}$$

and

$$B_{n+1} \to B_n, \qquad z \mapsto (z)_{\mathbf{II}}$$

where $(z)_{\mathbf{I}}$ and $(z)_{\mathbf{II}}$ are as in (0.2), canonical surjections

$$\pi_{m,n} : B_m \to A_n \qquad (n \leq m)$$

are obtained. Define for $k_{m-1} < n \leq k_m$ (and $k_{-1} = -1$),

$$a_n = \pi_{k_m,n}(b_m) \in A_n .$$

□

For $A \subseteq \mathbb{R}$ let

(1) $$A^{\triangledown} = \bigcup_n 0^{(n)}{}^\frown 1^\frown A \qquad \text{and} \qquad A^{\circ} = \{\vec{0}\} \cup A^{\triangledown} .$$

The operations $A \mapsto A^{\triangledown}$ and $A \mapsto A^{\circ}$ will be studied extensively in section 15. For the time being it is enough to notice that $A \leq_{\mathrm{L}} A^{\triangledown}$ and $A \leq_{\mathrm{L}} A^{\circ}$ via the function $x \mapsto 1^\frown x$.

Theorem 11.2. $\mathsf{SLO}^{\mathrm{W}} \Rightarrow \mathsf{AC}_{\omega}(\mathbb{R})$.

Proof. Let $\langle A_n \mid n \in \omega \rangle$ be a sequence of non-empty sets of reals. We want to replace the A_n's with sets B_n that code the A_n and are $<_{\mathrm{W}}$-increasing: using the function J_{W} of Lemma 4.7, define inductively

$$\begin{aligned} B_0 &= \{x \oplus y \mid x \in A_0 \wedge y \in \mathbb{R}\}, \\ B_{n+1} &= \{x \oplus y \mid x \in A_{n+1} \wedge y \in J_{\mathrm{W}}\big(B_n^{\triangledown} \oplus B_n^{\circ}\big)\}. \end{aligned}$$

Since $B_n \leq_{\mathrm{W}} B_n^{\triangledown} \oplus B_n^{\circ}$ via $x \mapsto \langle 0, 1\rangle^\frown x$, since $B_n^{\triangledown} \oplus B_n^{\circ} <_{\mathrm{W}} J_{\mathrm{W}}\big(B_n^{\triangledown} \oplus B_n^{\circ}\big)$ by definition of J_{W}, and since $J_{\mathrm{W}}\big(B_n^{\triangledown} \oplus B_n^{\circ}\big) \leq_{\mathrm{W}} B_{n+1}$ via $y \mapsto \bar{x} \oplus y$, where $\bar{x}$ is some fixed element of A_{n+1}, it follows that

$$\emptyset, \mathbb{R} <_{\mathrm{W}} B_0 <_{\mathrm{W}} B_1 <_{\mathrm{W}} \dots .$$

Notice that for any fixed $\bar{n}$

(2) $$B_{\bar{n}}^{\triangledown}, B_{\bar{n}}^{\circ}, \neg B_{\bar{n}}^{\triangledown}, \neg B_{\bar{n}}^{\circ} <_{\mathrm{W}} B_{\bar{n}+1} .$$

Let

$$C = \bigcup_n n^\frown B_n^\circ$$

and

$$D = \bigcup_{h,k\in\omega} 0^{(h)\frown} k + 1^\frown B_k^\triangledown .$$

Claim 11.2.1. $D \nleq_{\mathrm{W}} \neg C$.

Proof of the Claim. Suppose otherwise, and let σ be a winning strategy for **II** in $G_{\mathrm{W}}(D, \neg C)$. Let **I** play 0's as long as **II** passes: since σ is winning, there is a first round h when **II** plays some natural number $\bar{n}$, that is $\sigma(0^{(h)}) = \bar{n}$. Let **I** answer $\bar{n} + 2$, so that after this round the true positions of the two players in $G_{\mathrm{W}}(D, \neg C)$ are

$$\begin{array}{lcccc} \mathbf{I} & \overbrace{0\cdots\cdots 0}^{h} & & \bar{n}+2 & \\ \mathbf{II} & \underbrace{\mathsf{p}\cdots\mathsf{p}}_{h-1} & \bar{n} & & i \end{array}$$

where $i \in \omega \cup \{\mathsf{p}\}$. Since σ is winning, **II** can use σ to win what's left of the game, namely $G_{\mathrm{W}}(D_{\lfloor 0^{(h)\frown}\bar{n}+2\rfloor}, (\neg C)_{\lfloor\langle\bar{n},i\rangle\rfloor})$. But

$$(\neg C)_{\lfloor\langle\bar{n},i\rangle\rfloor} = \begin{cases} \neg B_{\bar{n}}^\circ & \text{if } i \in \{\mathsf{p}, 0\}, \\ \neg B_{\bar{n}} & \text{if } i = 1, \\ \emptyset & \text{otherwise,} \end{cases}$$

so in any case, by (2)

$$(\neg C)_{\lfloor\langle\bar{n},i\rangle\rfloor} <_{\mathrm{W}} B_{\bar{n}+1} \leq_{\mathrm{W}} B_{\bar{n}+1}^\triangledown = D_{\lfloor 0^{(h)\frown}\bar{n}+2\rfloor} ,$$

a contradiction. □

By $\mathsf{SLO}^{\mathrm{W}}$ $C \leq_{\mathrm{W}} D$ holds so let τ be a winning strategy for **II** in $G_{\mathrm{W}}(C, D)$. Since $n^\frown\vec{0} \in C$ but $\vec{0} \notin D$, for every n there is a least m such that $\tau(n^\frown 0^{(m)})$ is a non-zero natural number $k_n + 1$ and let h be the number of 0's played so far by τ; that is τ responds with $0^{(h)\frown} k_n + 1$ to **I** playing $n^\frown 0^{(m)}$.

Claim 11.2.2. $n \leq k_n$.

Proof of the Claim. Suppose otherwise. Then τ would yield a continuous reduction of B_n° to $D_{\lfloor 0^{(h)\frown}k_n+1\rfloor} = B_{k_n}^\triangledown <_{\mathrm{W}} B_n \leq_{\mathrm{W}} B_n^\circ$: a contradiction. □

For every n let m and h be as above: since $0^{(h)} {}^\frown k_n + 1^\frown \vec{0} \notin D$ and $n^\frown \vec{0} \in C$, and since τ is winning, there are $i \in \omega$ and $g(n) \in \mathbb{R}$ be such that

$$0^{(h)} {}^\frown k_n + 1^\frown 0^{(i)} {}^\frown 1^\frown g(n) \in D$$

is the result of applying τ to $n^\frown \vec{0} \in C$. Then $\forall n \in \omega\, (g(n) \in B_{k_n})$ hence $a_n = g(n)_{\mathbf{I}} \in A_{k_n}$. Therefore we have constructed a sequence of integers k_n and reals $a_n \in A_{k_n}$. By Claim 11.2.2 $\{k_n \mid n \in \omega\}$ is infinite, so by passing to a subsequence we may assume that the k_n's are increasing, so we are done by Lemma 11.1. □

We now turn to the PSP. Let

$$P = \{x \in \mathbb{R} \mid \exists^\infty n\; x(n) = 0\} \tag{3}$$

be the set of all sequences containing infinitely many 0s. It is easy to check that $P \in \mathbf{\Pi}^0_2$.

Lemma 11.3. *For every $A \subset \mathbb{R}$, if $P \leq_{\mathrm{W}} A$ then A contains a perfect set.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and such that $f^{-1}(A) = P$. Let also $\varphi : {}^{<\omega}\omega \to {}^{<\omega}\omega$ be such that $f = f_\varphi$. We will construct a complete binary tree T such that $[T] \subseteq P$ and $f \restriction [T]$ is injective, and therefore $f\big([T]\big)$ will be a perfect subset of A. The tree will be

$$T = \{t \in {}^{<\omega}\omega \mid \exists w \in {}^{<\omega}2\, (t \subseteq s_w)\}\,,$$

where the map ${}^{<\omega}2 \to {}^{<\omega}\omega$, $w \mapsto s_w$ is such that

- $z \subset w \Rightarrow s_z \subset s_w$
- $z \perp w \Rightarrow s_z \perp s_w$.

Set $s_\emptyset = \emptyset$. Since $\vec{0} \in P$ and $\vec{1} \notin P$, then $f(\vec{0}) \in A$ and $f(\vec{1}) \notin A$, hence there are n, m such that $\varphi(0^{(n)}) \perp \varphi(1^{(m)})$. Let $s_0 = 0^{(n)}$ and $s_1 = 1^{(m)\frown}0$: since φ is monotone $\varphi(s_0) \perp \varphi(s_1)$. Arguing as before, $f(s_i{}^\frown\vec{0}) \neq f(s_i{}^\frown\vec{1})$, for $i = 0, 1$, so there are n', m' such that $\varphi(s_{i,0}) \perp \varphi(s_{i,0})$, where $s_{i,0} = s_i{}^\frown 0^{(n')}$ and $s_{i,0} = s_i{}^\frown 1^{(m')\frown} 0$. We can now repeat the argument for $s_{0,0}, s_{0,1}, s_{1,0}, s_{1,1}$, constructing all s_w for w of length 3, and so on. □

Corollary 11.4. SLO$^{\mathrm{W}}$ $\Rightarrow$ PSP. *In particular* SLO$^{\mathrm{W}}$ *implies that $\mathbb{R}$ cannot be well-ordered.*

Proof. We must show that if $A \subseteq \mathbb{R}$ is uncountable then it contains a perfect set. By $\mathsf{SLO}^{\mathrm{W}}$

$$P \leq_{\mathrm{W}} A \ \vee \ A \leq_{\mathrm{W}} \neg P\,.$$

If the former holds, then we are done by the Lemma. Otherwise $A \in \mathbf{\Sigma}^0_2$, and hence $A = \bigcup_n C_n$, where each C_n is closed. By the Cantor-Bendixson theorem (which is provable in ZF) each C_n is either countable, or it contains a perfect set: if some C_n contains a perfect set, then so does A; otherwise each C_n is countable, hence by Theorem 11.2 and the remark following it, A is countable. □

Let us consider the notion of continuous reducibility in the generalized Baire space ${}^{\omega}\omega_1$. In this space, the games G_{W} are games on ω_1 and we know that AD_{ω_1} is false in ZF (Exercise 10.2). Therefore this does not bode too well for the consistency of the statement: "all games $G_{\mathrm{W}}(A, B)$, with $A, B \subseteq {}^{\omega}\omega_1$, are determined." In order to avoid unpleasant surprises we might just retreat to $\mathsf{SLO}^{\mathrm{W}}({}^{\omega}\omega_1)$, the analogue of $\mathsf{SLO}^{\mathrm{W}}$ for ${}^{\omega}\omega_1$. Unfortunately, even this move won't save us:

Theorem 11.5. $\mathsf{SLO}^{\mathrm{W}}({}^{\omega}\omega_1)$ *is inconsistent with* ZF.

Proof. The proof is an elaboration of the ideas behind Theorem 11.2. Suppose $\mathsf{SLO}^{\mathrm{W}}({}^{\omega}\omega_1)$ holds, and let $\pi : {}^{\omega}\omega_1 \twoheadrightarrow {}^{\omega}\omega$ be the retraction

$$\pi(x)(n) = \begin{cases} x(n) & \text{if } x(n) < \omega, \\ 2 & \text{otherwise.} \end{cases}$$

(The rationale for the 2 in the formula above is explained in (4) below.) Then, by Proposition 4.6, $\mathsf{SLO}^{\mathrm{W}}$ follows and hence Theorem 11.2 holds. We will show that there exist an ω_1 sequence of reals, contradicting PSP and hence Corollary 11.4. In order to simplify the notation, let's agree that $\mathcal{G}$ is the Wadge game on the space ${}^{\omega}\omega_1$, i.e., $\mathcal{G} = G^T_{\mathrm{W}}$ where $T = {}^{<\omega}\omega_1$. Similarly $\leq^{{}^{\omega}\omega_1}_{\mathrm{W}}$ and $<^{{}^{\omega}\omega_1}_{\mathrm{W}}$ are abbreviated with $\preceq$ and $\prec$. The definition of $A^{\triangledown}$ and A° (1) work also in the case of $A \subseteq {}^{\omega}\omega_1$. If $A \subseteq {}^{\omega}2$, then $A^{\triangledown}, A^{\circ} \subseteq {}^{\omega}2$ and

$$\pi^{-1}(A^{\triangledown}) = A^{\triangledown} \quad \wedge \quad \pi^{-1}(A^{\circ}) = A^{\circ}\,. \tag{4}$$

(This is the point of the specific definition of π.) For any $x \in {}^{\omega}2$ let E_x be the binary relation on ω given by

$$n \, E_x \, m \Leftrightarrow x(\langle n, m\rangle) = 1$$

where $\langle \cdot, \cdot \rangle : \omega \times \omega \to \omega$ is a standard bijection, e.g., $\langle n, m\rangle = 2^n(2m + 1)$. As usual, WO_α is the set of all reals coding a well-order of type $\alpha \geq \omega$, i.e.,

$$\mathrm{WO}_\alpha = \{x \in {}^{\omega}2 \mid \langle \omega, E_x\rangle \cong \alpha\}\,.$$

By a theorem of J. Stern [Ste78]

$$W_\alpha \stackrel{\text{def}}{=} \mathrm{WO}_{\omega^\alpha} \in \mathbf{\Sigma}^0_{2\alpha+2} \setminus \mathbf{\Sigma}^0_{2\alpha+1}\,.$$

Since W°_α is a countable union of sets in $\mathbf{\Sigma}^0_{2\alpha+2}$, then $\mathbb{R} \setminus W^\circ_\alpha \in \mathbf{\Pi}^0_{2\alpha+2}$, and therefore

$$\text{(5)} \qquad \alpha < \beta \Rightarrow (W_\beta \not\leq_{\mathrm{W}} \mathbb{R} \setminus W^\circ_\alpha,\ \mathbb{R} \setminus W_\alpha,\ W^\triangledown_\alpha)\,.$$

Let

$$A = \bigcup_{1 \leq \alpha < \omega_1} \alpha^\frown W^\circ_\alpha \quad \text{and} \quad B = \bigcup_{\substack{h \in \omega \\ 1 \leq \alpha < \omega_1}} 0^{(h)\frown} \alpha^\frown W^\triangledown_\alpha\,.$$

We claim that

$$A \not\preceq B \quad \wedge \quad B \not\preceq {}^\omega\omega_1 \setminus A\,,$$

and hence $\mathsf{SLO}({}^\omega\omega_1)$ fails.

Lemma 11.6. *$B \not\preceq {}^\omega\omega_1 \setminus A$.*

Proof. Suppose otherwise, and let τ be a winning strategy for **II** in $\mathcal{G}(B, {}^\omega\omega_1 \setminus A)$. Let **I** play 0's as long as **II** passes: since τ is winning, there is a first round h when **II** plays some ordinal number $\bar\alpha$, that is $\tau(0^{(h)}) = \bar\alpha$. Let **I** answer $\bar\alpha + 1$, so that after this inning is over, the positions of the two players in are

$$\begin{array}{llll} \mathbf{I} & \overbrace{0 \cdots\cdots 0}^{h} & & \bar\alpha + 1 \\[2ex] \mathbf{II} & \underbrace{\mathsf{p} \cdots \mathsf{p}}_{h-1} & \bar\alpha & \qquad i \end{array}$$

where $i \in \omega_1 \cup \{\mathsf{p}\}$ and p denotes "passing". Since τ is winning, **II** can use τ to win what's left of the game. In other words, recalling the notion of localization of a set at a given sequence (3):

$$W^\triangledown_{\bar\alpha+1} = B_{\lfloor 0^{(h)\frown}\bar\alpha+1 \rfloor} \preceq ({}^\omega\omega_1 \setminus A)_{\lfloor \langle \bar\alpha, i\rangle \rfloor}\,,$$

and this last set is

$$({}^\omega\omega_1 \setminus A)_{\lfloor \langle \bar\alpha, i\rangle \rfloor} = \begin{cases} {}^\omega\omega_1 \setminus W^\circ_{\bar\alpha} & \text{if } i \in \{\mathsf{p}, 0\}, \\ {}^\omega\omega_1 \setminus W_{\bar\alpha} & \text{if } i = 1, \\ {}^\omega\omega_1 & \text{if } i < 1. \end{cases}$$

The third case is impossible, since the only X such that $X \preceq {}^\omega\omega_1$ is ${}^\omega\omega_1$ itself and $W^\triangledown_{\bar\alpha+1} \neq {}^\omega\omega_1$. Therefore

$$W^\triangledown_{\bar\alpha+1} \preceq {}^\omega\omega_1 \setminus W^\circ_{\bar\alpha} \quad \vee \quad W^\triangledown_{\bar\alpha+1} \preceq {}^\omega\omega_1 \setminus W_{\bar\alpha}\,.$$

Proposition 4.6 and (4) imply that

$$W^{\triangledown}_{\bar{\alpha}+1} \leq_{\mathrm{W}} \mathbb{R} \setminus W^{\circ}_{\bar{\alpha}} \quad \vee \quad W^{\triangledown}_{\bar{\alpha}+1} \leq_{\mathrm{W}} \mathbb{R} \setminus W_{\bar{\alpha}} ,$$

and since $W_{\bar{\alpha}+1} \leq_{\mathrm{W}} W^{\triangledown}_{\bar{\alpha}+1}$ either of these contradicts (5). □

Suppose now $A \preceq B$ and let τ be a winning strategy for **II** in $\mathcal{G}(A, B)$. If **I** plays $\alpha^\frown\vec{0} \in A$, then **II** cannot respond with $\vec{0}$ since $\vec{0} \notin B$, so for any $\alpha \geq 1$ let $m_\alpha \in \omega$ be least such that $\tau(\alpha^\frown 0^{(m_\alpha)}) = \gamma_\alpha > 0$, and let h_α be the number of 0s played so far; that is, after the m_α-th inning is completed, the sequences of actual moves[1] played by **I** and **II** are $\alpha^\frown 0^{(m_\alpha)}$ and $0^{(h_\alpha)\frown}\gamma_\alpha$, respectively. (Notice that $h_\alpha < m_\alpha$ is possible, since τ can pass.) If $\gamma_\alpha < \alpha$ then

$$\begin{aligned} W_\alpha &\preceq W^{\circ}_\alpha \\ &= A_{\lfloor \alpha^\frown 0^{(m_\alpha)} \rfloor} \\ &\preceq B_{\lfloor 0^{(h_\alpha)\frown}\gamma_\alpha \rfloor} \\ &= W^{\triangledown}_{\gamma_\alpha} \end{aligned}$$

and hence, by Proposition 4.6, $W_\alpha \leq_{\mathrm{W}} W^{\triangledown}_{\gamma_\alpha}$, contradicting (5).

Therefore $\alpha \leq \gamma_\alpha$, for all $1 \leq \alpha < \omega_1$. Since $\alpha^\frown\vec{0} \in A$ but $0^{(h_\alpha)\frown}\gamma_\alpha^\frown\vec{0} \notin B$, player **II** cannot go on playing 0s forever, and arguing as before, the first non-zero ordinal played must be a 1. Let $g(\alpha) \in W_{\gamma_\alpha}$ be such that

$$0^{(h_\alpha)\frown}\gamma_\alpha{}^\frown 0^{(i)\frown} 1^\frown g(\alpha)$$

is τ's answer to **I** playing $\alpha^\frown\vec{0}$. Let $C \subseteq \omega_1$ be unbounded and such that the map $\alpha \mapsto \gamma_\alpha$ is injective. Since the W_α's are disjoint, then the map $C \to \mathbb{R}$, $\alpha \mapsto g(\alpha)$, yields an ω_1 sequence of distinct reals, contradicting PSP.

This concludes the proof of Theorem 11.5. □

Some remarks about the proof. Recall that a set $A \subseteq {}^\omega X$ is $\boldsymbol{\Sigma}^1_1$ if it is the projection of a closed subset of ${}^\omega X \times \mathbb{R}$, and it is $\boldsymbol{\Delta}^1_1$ if it and its complement ${}^\omega X \setminus A$ are both in $\boldsymbol{\Sigma}^1_1$. Every Borel set of ${}^\omega X$ is $\boldsymbol{\Delta}^1_1$ and if X is countable the converse holds by a theorem of Suslin [Kec95, Theorem 14.11]. For X uncountable Suslin's theorem fails and $\boldsymbol{\Delta}^1_1$ is strictly larger than the collection of Borel sets, and an alternative definition of $\boldsymbol{\Delta}^1_1$ is the following: it is the smallest $\mathcal{D} \subseteq \mathscr{P}({}^\omega X)$ containing all open sets, closed under complements, countable unions, and **open-separated unions**, i.e., for any family $A_i \in \mathcal{D}$ $(i \in I)$ such that there are open *disjoint* sets $U_i \supseteq A_i$, then $\bigcup_{i\in I} A_i \in \mathcal{D}$. Working in ZFC, Martin showed in [Mar90] that all $\boldsymbol{\Delta}^1_1$

[1] That is, disregarding the p's.

games with pay-off in ${}^{\omega}X$ are determined and therefore under AC, no pair of $\mathbf{\Delta}^1_1$ sets can witness the failure of $\mathsf{SLO}^{\mathrm{W}}({}^{\omega}\omega_1)$. On the other hand, the determinacy of closed set (and hence of $\mathbf{\Delta}^1_1$ sets) of ${}^{\omega}X$, for any X, implies AC, so Martin's theorem does not apply in ZF. Our construction gave (in $\mathsf{ZF} + \mathsf{AC}_\omega(\mathbb{R})$) a pair of $\mathbf{\Delta}^1_1$ sets A and B such that $B \not\preceq {}^{\omega}\omega_1 \setminus A$ and such that $A \preceq B$ implies the existence of an ω_1 sequence of distinct reals: this contradicts PSP, but it is a consequence of AC, and in fact $\mathsf{ZFC} \vdash A \preceq B$.

12 $\mathsf{SLO}^{\mathrm{W}}$ and the well-foundedness of $<_{\mathrm{W}}$

As we already mentioned before on page 19, if $S \supseteq T$ are pruned trees on some set X, then there is a retraction of S onto T, i.e., there is a Lipschitz map $\varrho : S \twoheadrightarrow T$ which is the identity on T. Obviously there are many retractions of S onto T, so we will now define a specific one, called the **canonical retraction of S onto T**, when $S \supseteq T$ are non-empty pruned trees on ω

$$\boldsymbol{\varrho}_{S,T} : S \twoheadrightarrow T \tag{6}$$

by setting $\boldsymbol{\varrho}_{S,T}(s) = t$, if $s \in T$, and for $s^\frown n \notin T$

$$\boldsymbol{\varrho}_{S,T}(s^\frown x) = \begin{cases} \boldsymbol{\varrho}_{S,T}(s)^\frown n & \text{if } \boldsymbol{\varrho}_{S,T}(s)^\frown n \in T\,, \\ \boldsymbol{\varrho}_{S,T}(s)^\frown m & \text{if } \boldsymbol{\varrho}_{S,T}(s)^\frown n \notin T \text{ and } m \text{ is least} \\ & \qquad \text{such that } \boldsymbol{\varrho}_{S,T}(s)^\frown m \in T\,. \end{cases}$$

Notice that if

$$t \in T_{\lfloor \boldsymbol{\varrho}_{S,T}(s) \rfloor} \Rightarrow \boldsymbol{\varrho}_{S,T}(s^\frown t) = \boldsymbol{\varrho}_{S,T}(s)^\frown t\,. \tag{7}$$

The Lipschitz surjection induced by $\boldsymbol{\varrho}_{S,T}$ is denoted by

$$\boldsymbol{r}_{S,T} : [S] \twoheadrightarrow [T]\,, \tag{8}$$

and it is also called the canonical retraction of $[S]$ onto $[T]$. When $S = {}^{<\omega}\omega$ we shall drop it from the index and write $\boldsymbol{\varrho}_T : {}^{<\omega}\omega \twoheadrightarrow T$ and $\boldsymbol{r}_T : \mathbb{R} \twoheadrightarrow [T]$.

In order to state many results in this book at the right level of generality we need a "local form" of the semilinear-ordering principle: for any $A \subseteq \mathbb{R}$, $\mathsf{SLO}^{\mathrm{W}}_{\leq A}$ holds iff

$$\forall B, C \leq_{\mathrm{W}} A\, (B \leq_{\mathrm{W}} C \ \vee\ \neg C \leq_{\mathrm{W}} B)\,. \tag{$\mathsf{SLO}^{\mathrm{W}}_{\leq A}$}$$

Exercise 12.1. Show that:

(i) $\mathsf{SLO}^{\mathrm{W}}_{\leq A} \Leftrightarrow \mathsf{SLO}^{\mathrm{W}}_{\leq \neg A}$,

(ii) $\mathsf{SLO}^{\mathrm{W}}_{\leq A} \Rightarrow \forall B <_{\mathrm{W}} A\,(B <_{\mathrm{W}} \neg A)$,

(iii) If $B_{\lfloor n \rfloor} <_{\mathrm{W}} B \leq_{\mathrm{W}} A$ for all n, and $\mathsf{SLO}^{\mathrm{W}}_{\leq A}$ holds, then B is self-dual.

Lemma 12.2. (a) *If $A \leq_{\mathrm{W}} B$ then there is $A' \equiv_{\mathrm{W}} A$ such that $A' \leq_{\mathrm{L}} A$ and $A' \leq_{\mathrm{L}} B$.*

(b) *Assume $\mathsf{SLO}^{\mathrm{W}}_{\leq B}$ and $A <_{\mathrm{W}} B$. Then there is $A' \equiv_{\mathrm{W}} A$ such that $A' \leq_{\mathrm{L}} B$ and $A' \leq_{\mathrm{L}} \neg B$ via a contraction, i.e.,* **I** *has a winning strategy in $G_{\mathrm{L}}(\neg B, A')$ and $G_{\mathrm{L}}(B, A')$.*

Proof. (a) Let $\varphi : {}^{<\omega}\omega \to {}^{<\omega}\omega$ be a tame map whose induced function witnesses $A \leq_{\mathrm{W}} B$. Then

$$S = \{s \in {}^{<\omega}\omega \mid \forall t \subset s \ \ \mathrm{lh}(\varphi(t)) < \mathrm{lh}(\varphi(s))\}$$

is the set of all sequences where φ properly extends its previous values. The idea is to replace S with ${}^{<\omega}\omega$ via some enumeration of ${}^{<\omega}\omega$, so that the resulting map will be Lipschitz. Recall from (9) that $\langle \boldsymbol{s}_i \mid i \in \omega \rangle$ is some fixed enumeration without repetitions of ${}^{<\omega}\omega$ such that $\boldsymbol{s}_0 = \emptyset$. Let $\gamma : {}^{<\omega}\omega \to {}^{<\omega}\omega$ be the continuous map

$$\gamma(t) = \begin{cases} \boldsymbol{s}_{k_0+1}{}^\frown \cdots {}^\frown \boldsymbol{s}_{k_n+1} & \text{if } t = \langle k_0, \dots, k_n \rangle\,, \\ \emptyset & \text{if } t = \emptyset\,, \end{cases}$$

and let $g = f_\gamma : \mathbb{R} \to \mathbb{R}$ be its induced function. Let

$$\begin{aligned} C &= \{z \in \mathbb{R} \mid \forall n \ \ \mathrm{lh}(\varphi(\gamma(z \restriction n))) = n\} \\ &= \{z \in \mathbb{R} \mid \forall n \ \gamma(z \restriction n) \in S\}\,. \end{aligned}$$

Given a real $x \in \mathbb{R}$ we can construct, continuously in x, a real $h(x) = z \in C$ such that $x = \boldsymbol{s}_{z(0)+1}{}^\frown \boldsymbol{s}_{z(1)+1}{}^\frown \cdots$: if $z \restriction n$ has been defined, let s be of least length (which exists since φ is continuous) such that $\boldsymbol{s}_{z(0)+1}{}^\frown \cdots {}^\frown \boldsymbol{s}_{z(n-1)+1}{}^\frown s \subset x$ and $\boldsymbol{s}_{z(0)+1}{}^\frown \cdots {}^\frown \boldsymbol{s}_{z(n-1)+1}{}^\frown s \in S$, that is $\mathrm{lh}(\varphi(\boldsymbol{s}_{z(0)+1}{}^\frown \cdots {}^\frown \boldsymbol{s}_{z(n-1)+1}{}^\frown s)) = n + 1$, and set $z(n) =$ the unique k such that $\boldsymbol{s}_{k+1} = s$. C is closed since $C = [T]$ where

$$T = \{s \restriction n \mid s \in S \wedge n \in \omega\}$$

is a pruned tree. Clearly the function $g \restriction C : C \to \mathbb{R}$, $z \mapsto \boldsymbol{s}_{z(0)+1}{}^\frown \boldsymbol{s}_{z(1)+1}{}^\frown \cdots$ is Lipschitz. Let

$$A' = (g \circ \boldsymbol{r}_T)^{-1}(A)\,,$$

where $\boldsymbol{r}_T : \mathbb{R} \to [T] = C$ is the Lipschitz retraction of (8). Then $A' \leq_{\mathrm{L}} A$, and $A \leq_{\mathrm{W}} A'$ since $A = h^{-1}(A')$. Moreover the Lipschitz map $t \mapsto \varphi\big(\,\gamma(\boldsymbol{r}_T(t))\,\big)$ witnesses $A' \leq_{\mathrm{L}} B$.

(b) If $\forall k \left(A \not\leq_{\mathrm{W}} B_{\lfloor k \rfloor}\right)$ then, by $\mathsf{SLO}^{\mathrm{W}}_{\leq B}$, $\forall n \left(B_{\lfloor k \rfloor} \leq_{\mathrm{W}} \neg A\right)$ hence $B \leq_{\mathrm{W}} \neg A <_{\mathrm{W}} \neg B$: a contradiction. Similarly $\forall m \left(A \not\leq_{\mathrm{W}} \neg B_{\lfloor m \rfloor}\right)$ does not hold, so fix $\bar{k}$ and $\bar{m}$ such that $A \leq_{\mathrm{W}} B_{\lfloor \bar{k} \rfloor}, \neg B_{\lfloor \bar{m} \rfloor}$. Choose $\varphi, \psi : {}^{<\omega}\omega \to {}^{<\omega}\omega$ continuous, tame and such that $f_\varphi^{-1}(B_{\lfloor \bar{k} \rfloor}) = A$ and $f_\psi^{-1}(B_{\lfloor \bar{m} \rfloor}) = A$. Let γ and $g : \mathbb{R} \to \mathbb{R}$ be as above, and let

$$C = \left\{z \in \mathbb{R} \mid \forall n \ \min\left\{\mathrm{lh}\big(\varphi(\gamma(z \restriction n))\big), \mathrm{lh}\big(\psi(\gamma(z \restriction n))\big)\right\} = n\right\}$$

Arguing as in part (a), C is closed, so $C = [T]$ for some pruned tree T, and let $\pi = \boldsymbol{r}_T : \mathbb{R} \to [T]$, $h : \mathbb{R} \to C$, and let $A' = (g \circ \boldsymbol{r}_T)^{-1}(A)$ be as above. Then $A' \leq_{\mathrm{L}} A$ via $g \circ \boldsymbol{r}_T$ and $A \leq_{\mathrm{W}} A'$ via h, and $t \mapsto \varphi\big(\,\gamma(\boldsymbol{r}_T(t))\,\big)$ and $t \mapsto \psi\big(\,\gamma(\boldsymbol{r}_T(t))\,\big)$ witness $A' \leq_{\mathrm{L}} B_{\lfloor \bar{n} \rfloor}$ and $A' \leq_{\mathrm{L}} \neg B_{\lfloor \bar{m} \rfloor}$, respectively. Therefore $A' \leq_{\mathrm{L}} B, \neg B$ via contractions. □

We can now re-prove Martin's Theorem 7.2 for $<_{\mathrm{W}}$:

Theorem 12.3. *Assume* $\neg\mathsf{FS} + \mathsf{DC}(\mathbb{R}) + \mathsf{SLO}^{\mathrm{W}}$. *There is no sequence* $\langle A_n \mid n \in \omega\rangle$ *such that* $A_{n+1} <_{\mathrm{W}} A_n$ *for all* $n \in \omega$.

Proof. Using Lemma 12.2(b), by induction on n choose $A'_n \in [A_n]_{\mathrm{W}}$ such that **I** has winning strategies σ^0_n in $G_{\mathrm{L}}(A'_n, A'_{n+1})$ and σ^1_n in $G_{\mathrm{L}}(\neg A'_n, A'_{n+1})$. (The σ^i_n are reals and the A'_n are of the form $\boldsymbol{\ell}^{-1}_{x_n}(A_0)$, so can be coded by reals; therefore choosing A'_n, σ^0_n, and σ^1_n requires only $\mathsf{DC}(\mathbb{R})$.) We can now repeat *verbatim* the argument of the proof in Theorem 6.1 and reach a contradiction. □

In fact the proof shows that

Theorem 12.4. *Assume* $\mathsf{DC}(\mathbb{R}) + \mathsf{SLO}^{\mathrm{W}}_{\leq A}$. *Then there is no sequence* $\langle A_n \mid n \in \omega\rangle$ *such that* $A_{n+1} <_{\mathrm{W}} A_n <_{\mathrm{W}} A$ *for all* $n \in \omega$.

Exercise 12.5. Suppose $A_0 <_{\mathrm{W}} A_1 <_{\mathrm{W}} A_2 <_{\mathrm{W}} \ldots$ and assume that $\mathsf{SLO}^{\mathrm{W}}_{\leq A_n}$ holds, for all n. Show that if $\bigoplus_n A_n$ is self-dual and $[\bigoplus_n A_n]_{\mathrm{W}}$ is the least upper bound of the A_n's.

13 The Wadge tree of a set

Wadge introduced the following definition:

(9) $$\boldsymbol{T}(A) = \{s \in {}^{<\omega}\omega \mid A_{\lfloor s \rfloor} \equiv_{\mathrm{W}} A\}\,.$$

Exercise 5.2((iii)) implies that $\boldsymbol{T}(A)$ is a tree, and since $A_{\lfloor \emptyset \rfloor} = A$, it is a non-empty tree.

Exercise 13.1. Show that $\boldsymbol{T}(\neg A) = \neg \boldsymbol{T}(A)$ and that for $t \in \boldsymbol{T}(A)$

$$\boldsymbol{T}(A_{\lfloor t \rfloor}) = \boldsymbol{T}(A)_{\lfloor t \rfloor}\,.$$

If s is a terminal node of $\boldsymbol{T}(A)$, then $A_{\lfloor s^\frown n \rfloor} <_{\mathrm{W}} A \equiv_{\mathrm{W}} A_{\lfloor s \rfloor}$ for all n, which by $\mathsf{SLO}^{\mathrm{W}}_{\leq A}$ implies that $\neg A_{\lfloor s^\frown n \rfloor} <_{\mathrm{W}} A_{\lfloor s \rfloor}$ for all n, and hence $\bigoplus_n \neg A_{\lfloor s^\frown n \rfloor} \leq_{\mathrm{W}} A_{\lfloor s \rfloor} = \bigoplus_n A_{\lfloor s^\frown n \rfloor}$ is self-dual. Therefore we have shown that

Proposition 13.2. *Assume* $\mathsf{SLO}^{\mathrm{W}}_{\leq A} + \mathsf{AC}_\omega(\mathbb{R})$. *If $\boldsymbol{T}(A)$ is not pruned then A is self-dual. In particular, if $\boldsymbol{T}(A)$ is well-founded, then A is self-dual.*

Conversely:

Theorem 13.3. *Assume* $\mathsf{AC}_\omega(\mathbb{R}) + \neg\mathsf{FS}$. *If A is Wadge-self-dual, then $\boldsymbol{T}(A)$ is well-founded.*

Corollary 13.4. *Assume* $\mathsf{SLO}^{\mathrm{W}}_{\leq A} + \mathsf{AC}_\omega(\mathbb{R}) + \neg\mathsf{FS}$. *Then*

A *is self-dual* $\Leftrightarrow \boldsymbol{T}(A)$ *is well-founded* $\Leftrightarrow \boldsymbol{T}(A)$ *has a terminal node,*
A *is non-self-dual* $\Leftrightarrow \boldsymbol{T}(A)$ *is ill-founded* $\Leftrightarrow \boldsymbol{T}(A)$ *is pruned.*

Proof of 13.3. Towards a contradiction, let $x \in [\boldsymbol{T}(A)]$ and $A \leq_{\mathrm{W}} \neg A$. Clearly for every $n \in \omega$, **II** wins $G_{\mathrm{W}}(A_{\lfloor x \restriction n \rfloor}, \neg A_{\lfloor x \restriction n \rfloor})$ via some strategy τ_n, and **I** wins $G_{\mathrm{L}}(A_{\lfloor x \restriction n \rfloor}, \neg A_{\lfloor x \restriction n+1 \rfloor})$ via the strategy σ_n that plays $x(n)$ in the first round and then copies **II**'s moves. Let τ be **II**'s copying strategy, i.e., $\tau(s) =$ the last element of s. We are now going to apply the Martin-Monk method. The basic idea is to pit τ against the τ_n's so that a flip set is constructed, contradicting our assumption $\neg\mathsf{FS}$. The problem is that τ and τ_n are strategies for **II**, while we need strategies for **I** in order to start the filling-in procedure of the Martin-Monk diagram. This is where the σ_n's come into the picture: an increasing sequence of natural numbers $0 = M_0 < M_1 < M_2 < \cdots$ is defined so that $M_k + 1 < M_{k+1}$ and on the M_k-th row **II** plays using either τ or τ_n for some n, and on the m-th row with $m \notin \{M_k \mid k \in \omega\}$ **I** plays using σ_n for an appropriate n. More precisely, letting $\langle N_j \mid j \in \omega \rangle$ be the enumeration of $\omega \setminus \{M_k \mid k \in \omega\}$

- on the M_k-th row **II** is playing $G_{\mathrm{W}}(\neg A_{\lfloor x \restriction i \rfloor}, A_{\lfloor x \restriction i \rfloor})$ using τ_i or τ, where i is such that $N_i = M_k + 1$;
- on the N_j-th row **I** is playing $G_{\mathrm{L}}(A_{\lfloor x \restriction j \rfloor}, \neg A_{\lfloor x \restriction j+1 \rfloor})$ using σ_j.

The choice of using τ versus τ_i on the M_k-th row, with $M_k + 1 = N_i$, will be given by $z(k)$ where $z \in {}^{\omega}2$: τ will be used when $z(k) = 0$, and τ_i will be used when $z(k) = 1$. For example, if the first few M_k's are $\langle 0, 3, 5, 8 \dots \rangle$ and $z = \langle 0, 1, 1, 0, \dots \rangle$ then the strategies in the Martin-Monk diagram are: $\langle \tau, \sigma_0, \sigma_1, \tau_2, \sigma_2, \tau_3, \sigma_3, \sigma_4, \tau, \sigma_5, \dots \rangle$. Since τ is the copying strategy, for any $s \in {}^{<\omega}\omega$, τ's response to s is a sequence of the same length as s, that is $\mathrm{lh}((s * \tau)_{\mathbf{II}}) = \mathrm{lh}(s)$. On the hand, the τ_n's might pass, so it can happen that τ_n's response to s is a sequence of shorter length, $\mathrm{lh}((s * \tau_n)_{\mathbf{II}}) < \mathrm{lh}(s)$. Therefore the M_k's are to be taken sparse enough so that the filling-in procedure does not come to a grinding halt. Arguing as in the proof of Theorem 8.1, suppose $\langle M_k \mid k \leq n \rangle$ has been defined and for every $s \in {}^{n+1}2$ the finite diagram with $n + 1$ rows where τ is used on the M_k-th row iff $s(k) = 0$, has the first n entries of the 0-th row are filled-in. Fix any $s \in {}^{n+1}2$. Suppose the $M_n + 1$-st row is filled-in with $\langle x(m + i) \mid m \in \omega \rangle$, where i is such that $N_i = M_n + 1$. Then the 0-th row is completely determined, so by continuity of the strategies there is a $j = j(s) > 0$ such that $\langle x(i), x(i + 1), \dots, x(i + j - 1) \rangle$ is enough to fill-in the first $n + 1$ entries of the 0-th row. Therefore if the σ's are used in the rows $M_n + 1, \dots, M_n + j$, then the first $n + 1$ entries of the 0-th row are filled-in. Let

$$M_{n+1} = \sup\{M_n + j(s) + 1 \mid s \in {}^{n+1}2\} .$$

By construction of the M_k's, for any $z \in {}^{\omega}2$ the Martin-Monk diagram relative to z can be filled-in. Then $\{z \in {}^{\omega}2 \mid$ the real in the 0-th row of the Martin-Monk diagram relative to z is in $A\}$ is a flip set, contradicting $\neg\mathsf{FS}$. □

Lemma 13.5. *Assume* $\neg\mathsf{FS} + \mathsf{AC}_{\omega}(\mathbb{R})$ *and that* $A \equiv_{\mathrm{W}} B$.

(a) *If* $f : \mathbb{R} \to \mathbb{R}$ *witnesses* $A \leq_{\mathrm{W}} B$ *then* $f\big([\boldsymbol{T}(A)]\big) \subseteq [\boldsymbol{T}(B)]$.

(b) $A \cap [\boldsymbol{T}(A)] \equiv_{\mathrm{W}} B \cap [\boldsymbol{T}(B)]$

Proof. (a) follows from Exercise 4.4.

(b) If A and B are self-dual or if $A = B = \mathbb{R}$, then result follows for trivial reasons, so we may assume that A and B are non-self-dual and different from $\mathbb{R}$. Let τ be **II**'s winning strategy in $G_{\mathrm{W}}(A, B)$. The **II** wins $G_{\mathrm{W}}(A \cap [\boldsymbol{T}(A)], B \cap [\boldsymbol{T}(B)])$ as follows:

> As long as **II** plays inside $\boldsymbol{T}(A)$ then **II** follows τ, so that if **I** ends up with a real in $[\boldsymbol{T}(A)]$, then $x \in A \Leftrightarrow (x * \tau)_{\mathbf{II}} \in B \cap [\boldsymbol{T}(B)]$. If at some round the positions of **I** and **II** are $p \in \boldsymbol{T}(A)$ and $q \in \boldsymbol{T}(B)$, respectively, and if **I** plays n such that $p^\frown n \notin \boldsymbol{T}(A)$,

> then **II** starts enumerating a real outside $B_{\lfloor q \rfloor}$, which exists since $B_{\lfloor q \rfloor} \equiv_{\mathrm{W}} B \neq \mathbb{R}$.

□

Proposition 13.6. *Assume* $\mathsf{AC}_\omega(\mathbb{R}) + \neg\mathsf{FS} + \mathsf{SLO}^{\mathrm{W}}_{\leq A}$. *Then* $A \cap [\boldsymbol{T}(A)] \leq_{\mathrm{L}} A$.

Proof. If A is self-dual then the result holds trivially, so we may assume that A is non-self-dual and hence $\boldsymbol{T}(A)$ is pruned. We may also assume that $A \neq \mathbb{R}$. We claim that if $t^\frown n \in \partial\boldsymbol{T}(A)$, then $\boldsymbol{N}_t \setminus A \neq \emptyset$. In fact if $\boldsymbol{N}_t \subseteq A$ then $\mathbb{R} = A_{\lfloor t \rfloor} \equiv_{\mathrm{W}} A$ so that $A = \mathbb{R}$, contrarily to our assumption. Then **II** wins $G_{\mathrm{L}}(A \cap [\boldsymbol{T}(A)], A)$ as follows:

> **II** copies **I**'s moves as long as they yield positions inside $\boldsymbol{T}(A)$, so that if **I**'s play is in $[\boldsymbol{T}(A)]$ then **II** wins this game. Suppose at some round **I** plays n so that his resulting position is $t^\frown n \in \partial\boldsymbol{T}(A)$: then **II** follows from now on a real in $\boldsymbol{N}_t \setminus A$.

□

In general, $A \cap [\boldsymbol{T}(A)]$ is much simpler than A —see Lemma 15.2(c). If $A \cap [\boldsymbol{T}(A)]$ is of the same degree as A, then the operation $\boldsymbol{T}$ applied to A or to $A \cap [\boldsymbol{T}(A)]$ yields the same tree.

Proposition 13.7. *Assume* $\neg\mathsf{FS} + \mathsf{AC}_\omega(\mathbb{R})$. *Then*

$$A \equiv_{\mathrm{W}} A \cap [\boldsymbol{T}(A)] \Rightarrow \boldsymbol{T}(A) = \boldsymbol{T}\big(A \cap [\boldsymbol{T}(A)]\big)\,.$$

Proof. Suppose $A \equiv_{\mathrm{W}} A \cap [\boldsymbol{T}(A)]$. If $A = \mathbb{R}, \emptyset$ then the result follows at once, so we may assume $A \neq \mathbb{R}, \emptyset$. If $t \in \boldsymbol{T}(A)$ then

$$\begin{aligned}\big(A \cap [\boldsymbol{T}(A)]\big)_{\lfloor t \rfloor} &= A_{\lfloor t \rfloor} \cap [\big(\boldsymbol{T}(A)\big)_{\lfloor t \rfloor}] \\ &= A_{\lfloor t \rfloor} \cap [\boldsymbol{T}(A_{\lfloor t \rfloor})] \qquad \text{by Exercise 13.1} \\ &\equiv_{\mathrm{W}} A \cap [\boldsymbol{T}(A)]\end{aligned}$$

the third line following from $A_{\lfloor t \rfloor} \equiv_{\mathrm{W}} A$ and Lemma 13.5(b). Therefore $t \in \boldsymbol{T}\big(A \cap [\boldsymbol{T}(A)]\big)$.

Conversely, suppose $t \notin \boldsymbol{T}(A)$: then $\emptyset = (A \cap [\boldsymbol{T}(A)])_{\lfloor t \rfloor} \leq_{\mathrm{W}} A \cap [\boldsymbol{T}(A)]$, and by case assumption the inequality is strict. Hence $t \notin \boldsymbol{T}(A \cap [\boldsymbol{T}(A)])$. □

A set $A \subseteq \mathbb{R}$ is said to be **homogeneous** just in case $\boldsymbol{T}(A) = {}^{<\omega}\omega$. For example $\mathbb{R}$ and $\emptyset$ are homogeneous, and so is the set P of (3). On the other hand no open or closed set other than $\mathbb{R}$ and $\emptyset$ is homogeneous, nor is any self-dual set. Since $\boldsymbol{T}(\neg A) = \boldsymbol{T}(A)$ then homogeneity is preserved by taking complements; on the other hand it is not preserved by Wadge equivalence: for example, if A is homogeneous, then $A \equiv_{\mathrm{W}} 0^\frown A$ but $0^\frown A$ is not homogeneous. Van Wesep characterized in [VW77] all degrees which contain a homogeneous set—see Chapter ?? For the time being let us prove the following result.

Lemma 13.8. *If A is non-self-dual and $A \equiv_{\mathrm{W}} A \cap \boldsymbol{T}(A)$, then there is $B \equiv_{\mathrm{L}} A \cap [\boldsymbol{T}(A)]$ such that B is homogeneous.*

Proof. If A is closed the result is trivial, so we may assume otherwise. Let

$$\rho = \boldsymbol{\varrho}_{\boldsymbol{T}(A)} : {}^{<\omega}\omega \twoheadrightarrow \boldsymbol{T}(A)$$

be the canonical surjection of ${}^{<\omega}\omega$ onto $\boldsymbol{T}(A)$ and let $r : \mathbb{R} \twoheadrightarrow [\boldsymbol{T}(A)]$ be the induced Lipschitz retraction—see (6),(8). By (7) $s \in \boldsymbol{T}(A)_{\lfloor \rho(t) \rfloor} \Rightarrow \rho(t^\frown s) = \rho(t)^\frown s$. Let $B = r^{-1}(A \cap [\boldsymbol{T}(A)])$.

Claim 13.8.1. $\forall t \in {}^{<\omega}\omega \left(A_{\lfloor \rho(t) \rfloor} \cap [\boldsymbol{T}(A)]_{\lfloor \rho(t) \rfloor} \leq_{\mathrm{L}} B_{\lfloor t \rfloor} \right)$.

Proof of Claim. Fix a $t \in {}^{<\omega}\omega$. Player **II** wins $G_{\mathrm{W}}(A_{\lfloor \rho(t) \rfloor} \cap [\boldsymbol{T}(A)_{\lfloor \rho(t) \rfloor}], B_{\lfloor t \rfloor})$ as follows:

> As long as **I** plays inside $\boldsymbol{T}(A)_{\lfloor \rho(t) \rfloor}$, then **II** copies **I**'s moves: if **I**'s complete play is $a \in [\boldsymbol{T}(A)_{\lfloor \rho(t) \rfloor}]$ then **II**' complete play will also be a and
>
> $$a \in A_{\lfloor \rho(t) \rfloor} \Leftrightarrow a = r^{-1}(a) \in B\,,$$
>
> since $r \restriction A_{\lfloor \rho(t) \rfloor} \cap [\boldsymbol{T}(A)_{\lfloor \rho(t) \rfloor}]$ is the identity.
>
> Suppose that at some round **I** plays an n such that $s^\frown n \in \partial\big(\boldsymbol{T}(A)_{\lfloor \rho(t) \rfloor}\big)$, that is, $\rho(t)^\frown s^\frown n \in \partial \boldsymbol{T}(A)$. Since **I**'s play will be for sure outside $A_{\lfloor \rho(t) \rfloor} \cap [\boldsymbol{T}(A)_{\lfloor \rho(t) \rfloor}]$ we must guarantee that **II**'s play will be outside $B_{\lfloor t \rfloor}$ Since $\rho(t)^\frown s \in \boldsymbol{T}(A)$, then $A \equiv_{\mathrm{W}} A_{\lfloor \rho(t)^\frown s \rfloor}$ and hence
>
> $$A \cap [\boldsymbol{T}(A)] \equiv_{\mathrm{W}} A_{\lfloor \rho(t)^\frown s \rfloor} \cap [\boldsymbol{T}(A_{\lfloor \rho(t)^\frown s \rfloor})]$$
>
> by Lemma 13.5(b), and since $\boldsymbol{T}(A_{\lfloor \rho(t)^\frown s \rfloor}) = \boldsymbol{T}(A)_{\lfloor \rho(t)^\frown s \rfloor}$ by Exercise 13.1 and $A \cap [\boldsymbol{T}(A)] \equiv_{\mathrm{W}} A$ is not closed by case assumption, then $[\boldsymbol{T}(A)]_{\lfloor \rho(t)^\frown s \rfloor} \setminus A_{\lfloor \rho(t)^\frown s \rfloor} \neq \emptyset$. Then **II** can enumerate a real $\bar{a} \in [\boldsymbol{T}(A)]_{\lfloor \rho(t)^\frown s \rfloor} \setminus A_{\lfloor \rho(t)^\frown s \rfloor}$ from now on, so that his final play will be $s^\frown \bar{a}$. Since $s^\frown \bar{a} \in [\boldsymbol{T}(A)]_{\lfloor \rho(t) \rfloor}$, then $r(s^\frown \bar{a}) = s^\frown \bar{a}$ and therefore $s^\frown \bar{a} \in B_{\lfloor t \rfloor}$ would imply $s^\frown \bar{a} \in A_{\lfloor \rho(t) \rfloor}$, contrarily to the choice of $\bar{a}$. Therefore $s^\frown \bar{a} \notin B_{\lfloor t \rfloor}$.

□

By taking $t = \emptyset$ in the Claim, we have that $A \cap [\boldsymbol{T}(A)] \leq_{\mathrm{L}} B$ and since r witnesses that $B \leq_{\mathrm{W}} A \cap [\boldsymbol{T}(A)]$,

$$B \equiv_{\mathrm{L}} A \cap [\boldsymbol{T}(A)] .$$

By Proposition 13.7 $\boldsymbol{T}(A \cap [\boldsymbol{T}(A)]) = \boldsymbol{T}(A)$, and since $\rho(t) \in \boldsymbol{T}(A)$ for every t, then

$$\forall t \in {}^{<\omega}\omega\big((A \cap [\boldsymbol{T}(A)])_{\lfloor \rho(t) \rfloor} \equiv_{\mathrm{W}} A \cap [\boldsymbol{T}(A)] \equiv_{\mathrm{W}} A\big) .$$

Therefore by the Claim $\forall t \in {}^{<\omega}\omega\big(B_{\lfloor t \rfloor} \equiv_{\mathrm{W}} A\big)$, that is $\boldsymbol{T}(B) = {}^{<\omega}\omega$, which is what we had to prove. □

14 $\mathsf{SLO}^{\mathrm{W}}$ and the Wadge hierarchy

Proposition 14.1. (a) *Assume* $\mathsf{SLO}^{\mathrm{W}}_{\leq A} + \mathsf{AC}_\omega(\mathbb{R})$. *If there is* $\{D_n \mid n < N\}$ *a clopen partition of* $\mathbb{R}$ *such that* $\forall n < N\,(A \cap D_n <_{\mathrm{W}} A)$, *then* $A \leq_{\mathrm{W}} \neg A$.

(b) *Assume* $\mathsf{AC}_\omega(\mathbb{R}) + \neg\mathsf{FS}$. *If* A *is not clopen and* $A \leq_{\mathrm{W}} \neg A$ *then there is* $\{D_n \mid n < \omega\}$ *a clopen partition of* $\mathbb{R}$ *such that* $\forall n < \omega\,(A \cap D_n <_{\mathrm{W}} A)$.

Proof. (a) $A \cap D_n \leq_{\mathrm{W}} \neg A$ by $\mathsf{SLO}^{\mathrm{W}}_{\leq A}$, so we can choose continuous reductions $f_n : \mathbb{R} \to \mathbb{R}$ such that $f_n^{-1}(\neg A) = A \cap D_n$. Then $\bigcup_n (f_n \restriction D_n) : \mathbb{R} \to \mathbb{R}$ is continuous and witnesses $A \leq_{\mathrm{W}} \neg A$.

(b) By Theorem 13.3 $\boldsymbol{T}(A)$ is well-founded so

$$A = \bigcup_{t \in \partial \boldsymbol{T}(A)} t^\frown A_{\lfloor t \rfloor} .$$

Let $\{D_n \mid n < \omega\}$ be an enumeration without repetitions of $\{\boldsymbol{N}_t \mid t \in \partial \boldsymbol{T}(A)\}$ so that it is a clopen partition of $\mathbb{R}$. Then each $A \cap D_n$ is of the form $t^\frown A_{\lfloor t \rfloor}$ with $t \in \partial \boldsymbol{T}(A)$: if $A_{\lfloor t \rfloor} \neq \mathbb{R}$ then $t^\frown A_{\lfloor t \rfloor} \equiv_{\mathrm{W}} A_{\lfloor t \rfloor} <_{\mathrm{W}} A$, and if $A_{\lfloor t \rfloor} = \mathbb{R}$ then $t^\frown A_{\lfloor t \rfloor}$ is clopen, hence $<_{\mathrm{W}} A$. Therefore $\forall n < \omega\,(A \cap D_n <_{\mathrm{W}} A)$. □

Exercise 14.2. Assume $\mathsf{SLO}^{\mathrm{W}}_{\leq A} + \mathsf{AC}_\omega(\mathbb{R}) + \neg\mathsf{FS}$. Then $[A]_{\mathrm{W}}$ is self-dual iff the clopen partition of $\mathbb{R}$ can be taken to be finite—in fact of size 2.

Lemma 14.3. *Assume* $\neg\mathsf{FS} + \mathsf{DC}(\mathbb{R})$. *If* $A <_{\mathrm{W}} B$ *are self-dual and* $\mathsf{SLO}^{\mathrm{W}}_{\leq B}$ *holds, then there is a non-self-dual* C *such that* $A <_{\mathrm{W}} C <_{\mathrm{W}} B$.

Proof. It is enough to show that there is *some* C such that $A <_{\mathrm{W}} C <_{\mathrm{W}} B$, since if every such C were self-dual, an infinite descending chain would result

$$A <_{\mathrm{W}} \ldots <_{\mathrm{W}} C_2 <_{\mathrm{W}} C_1 <_{\mathrm{W}} C_0 <_{\mathrm{W}} B\,,$$

contrarily to Theorem 12.3. Towards a contradiction, suppose there is no C such that $A <_{\mathrm{W}} C <_{\mathrm{W}} B$. If $t \in \mathbf{tn}(\boldsymbol{T}(B))$, then for every $n \in \omega$

$$B_{\lfloor t^\frown n \rfloor} <_{\mathrm{W}} B_{\lfloor t \rfloor} \equiv_{\mathrm{W}} B$$

and by $\mathsf{SLO}^{\mathrm{W}}_{\leq B}$ and $A \equiv_{\mathrm{W}} \neg A$,

$$B_{\lfloor t^\frown n \rfloor} \leq_{\mathrm{W}} A\,. \tag{10}$$

By Theorem 13.3 $\boldsymbol{T}(B)$ is well-founded. □

The following is the analogue of Lemma 5.8 for the Wadge hierarchy.

Exercise 14.4. Assume $\mathsf{SLO}^{\mathrm{W}}_{\leq A}$ and $\mathsf{AC}_\omega(\mathbb{R})$ and suppose $A_n <_{\mathrm{W}} A$ are such that

$$\forall n\, \exists m > n\, (A_n <_{\mathrm{W}} A_m)\,. \tag{11}$$

Then $[\bigoplus_n A_n]_{\mathrm{W}}$ is self-dual and it is the least-upper bound of the $[A_i]_{\mathrm{W}}$'s.

We are now able to prove Corollary 8.2 under weaker hypotheses.

Theorem 14.5. *Assume* $\mathsf{SLO}^{\mathrm{W}} + \neg\mathsf{FS} + \mathsf{DC}(\mathbb{R})$. *Then*

(a) $<_{\mathrm{W}}$ *is well-founded,*

(b) *immediately above a self-dual Wadge degree there is a non-self-dual pair of Wadge degrees,*

(c) *immediately above a non-self-dual pair of Wadge degrees there is a self-dual Wadge degree,*

(d) *at limit levels of countable cofinality there is a single self-dual Wadge degree, and at all other limit levels there is a non-self-dual pair of Wadge degrees.*

Proof. (a) and (b) follow from Theorem 12.3 and Lemma 14.3, respectively.

If $A \nleq_{\mathrm{W}} \neg A$, then, arguing as in Lemma 5.7, $A \oplus \neg A$ is Wadge self-dual and above A and $\neg A$. If $A \leq_{\mathrm{W}} B$ and $\neg A \leq_{\mathrm{W}} B$ via reductions φ and ψ, then **II** wins $G_{\mathrm{W}}(A \oplus \neg A, B)$ by passing in the first round, and then following the appropriate reduction depending on wether **I** played an even or an odd number. Therefore $[A \oplus \neg A]_{\mathrm{W}}$ is the self-dual degree immediately above $[A]_{\mathrm{W}}$ and $[\neg A]_{\mathrm{W}}$. This settles (c).

Now for (d). If $A_0 <_{\mathrm{W}} A_1 <_{\mathrm{W}} \cdots$ witness $[A]_{\mathrm{W}}$ is of countable cofinality, then by Exercise 12.5, $\bigoplus_n A_n$ is Wadge self-dual and $\bigoplus_n A_n \equiv_{\mathrm{W}} A$. Conversely, suppose $[A]_{\mathrm{W}}$ is limit and self-dual, so that by Lemma 13.3 $\boldsymbol{T}(A)$ is well-founded. Then $\{\boldsymbol{N}_s \mid s \in \partial\boldsymbol{T}(A)\}$ is a clopen partition of $\mathbb{R}$. If B is such that

$$\forall s \in \partial\boldsymbol{T}(A) \left(A_{\lfloor s\rfloor} \leq_{\mathrm{W}} B\right),$$

then $A = \bigcup_{s\in\partial\boldsymbol{T}(A)} s^\frown A_{\lfloor s\rfloor} \leq_{\mathrm{W}} B$ by Exercise 8.8, and therefore $\|A\|_{\mathrm{W}} = \sup\{\|A_{\lfloor s\rfloor}\|_{\mathrm{W}} \mid s \in \partial\boldsymbol{T}(A)\}$, witnessing that $[A]_{\mathrm{W}}$ is of countable cofinality. □

Note that the same proof shows that the Wadge hierarchy has the expected behavior below A, if $\mathsf{SLO}^{\mathrm{W}}_{\leq A}$ rather than the full $\mathsf{SLO}^{\mathrm{W}}$ is assumed.

15 The next non-self-dual pair

Recall from (1) the definition of $A^\triangledown = \bigcup_n 0^{(n)\frown}1^\frown A$ and $A^\circ = A^\triangledown \cup \{\vec{0}\}$.

Exercise 15.1. Show that:

(i) $A \leq_{\mathrm{L}} A^\triangledown, A^\circ$;

(ii) $A \leq_{\mathrm{W}} B \Rightarrow A^\triangledown \leq_{\mathrm{W}} B^\triangledown \wedge A^\circ \leq_{\mathrm{W}} B^\circ$, and similarly for $\leq_{\mathrm{L}}$ instead of $\leq_{\mathrm{W}}$;

(iii) $(A^\triangledown)^\triangledown \equiv_{\mathrm{L}} A^\triangledown$ and $(A^\circ)^\circ \equiv_{\mathrm{L}} A^\circ$;

(iv) $\mathbb{R}^\triangledown$ is open and $\mathbb{R}^\circ$ is the closure of $\mathbb{R}^\triangledown$;

(v) $\emptyset^\circ = \{\vec{0}\}$ and $\emptyset^\triangledown = \emptyset$.

There is an annoying lack of symmetry between ((iv)) and ((v)) due to the specific definition of $A^\triangledown$ in (1): had we defined it as

$$\bigcup_{\substack{n\geq 0\\ m>0}} 0^{(n)\frown}m^\frown A$$

all of the previous result would still hold (with minor changes in the proofs) and ((iv)) would become

((iv))* $\mathbb{R}^\triangledown = \mathbb{R} \setminus \{\vec{0}\}$ and $\mathbb{R}^\circ = \mathbb{R}$.

The reason for the specific definition in (1) is that it makes sense even if A is a subset of the Cantor space.

In view of ((ii)) the $^\triangledown$ and $^\circ$ operations can be defined on Lipschitz and Wadge degrees:

$$[A]_{\mathrm{L}}^{\triangledown} = [A^{\triangledown}]_{\mathrm{L}} \quad \text{and} \quad [A]_{\mathrm{L}}^{\circ} = [A^{\circ}]_{\mathrm{L}} .$$
$$[A]_{\mathrm{W}}^{\triangledown} = [A^{\triangledown}]_{\mathrm{W}} \quad \text{and} \quad [A]_{\mathrm{W}}^{\circ} = [A^{\circ}]_{\mathrm{W}} .$$

Lemma 15.2. (a) *If A is self-dual, then $A^{\triangledown} \equiv_{\mathrm{W}} \neg A^{\circ}$.*

(b) *If $\neg\mathsf{FS}+\mathsf{AC}_{\omega}(\mathbb{R})$ holds then $A^{\triangledown}$ and A° are non-self-dual and $\{[A]_{\mathrm{W}}^{\triangledown}, [A]_{\mathrm{W}}^{\circ}\}$ is a non-self-dual pair above $[A]_{\mathrm{W}}$.*

(c) *If $\neg\mathsf{FS} + \mathsf{AC}_{\omega}(\mathbb{R})$ hold and $A \neq \emptyset$, then $[\boldsymbol{T}(A^{\triangledown})] \cap A^{\triangledown} <_{\mathrm{W}} A^{\triangledown}$.*

Proof. (a) Let τ be the winning strategy for **II** in $G_{\mathrm{W}}(A, \neg A)$. Then **II** wins $G_{\mathrm{W}}(A^{\triangledown}, \neg A^{\circ})$ as follows:

> As long as **I** plays 0 then **II** answers 0. If at some round **I** plays a non-zero integer n, then **II** plays n, and after that follows τ.

The proof that $A^{\circ} \leq_{\mathrm{W}} A^{\triangledown}$ is similar.

(b) Since $\boldsymbol{T}(A^{\triangledown}) = \boldsymbol{T}(A^{\circ}) = \{0^{(n)} \mid n \in \omega\}$ is ill-founded, Lemma 13.3 implies that $A^{\triangledown}$ and A° are non-self-dual. Therefore $\{[A^{\triangledown}]_{\mathrm{W}}, [A^{\circ}]_{\mathrm{W}}\}$ is a non-self-dual pair above $[A]_{\mathrm{W}}$.

(c) By part (b) $A^{\triangledown}$ is non-self-dual and $A^{\triangledown} \neq \emptyset$. By Proposition 13.6, $\emptyset = [\boldsymbol{T}(A^{\triangledown})] \cap A^{\triangledown} \leq_{\mathrm{L}} A^{\triangledown}$, and the result follows at once. □

Proposition 15.3. *Assume $\mathsf{SLO}^{\mathrm{W}}_{\leq B}$ with B non-self-dual, and let $A \equiv_{\mathrm{W}} \neg A <_{\mathrm{W}} B$.*

(a) *If $[\boldsymbol{T}(B)] \setminus B \neq \emptyset$ then $A^{\triangledown} \leq_{\mathrm{W}} B$.*

(b) *If $[\boldsymbol{T}(B)] \cap B \neq \emptyset$ then $A^{\circ} \leq_{\mathrm{W}} B$.*

Proof. Suppose $x \in [\boldsymbol{T}(B)] \setminus B$. Then **II** wins $G_{\mathrm{W}}(B, A^{\triangledown})$ as follows:

> **II** enumerates x as long as **I** plays 0s. Suppose n is the first round when **I** plays $k > 0$: up to this round **II**'s position is $x \restriction n \in \boldsymbol{T}(B)$ and
>
> (12) $$B \equiv_{\mathrm{W}} B_{\lfloor x \restriction n \rfloor}$$
>
> We distinguish two cases:

- $k = 1$. Then **II** passes and then by (12) follows a reduction of A to $B_{\lfloor x \restriction n \rfloor}$.
- $k > 1$. Then **II** from now on enumerates some $y \notin B_{\lfloor x \restriction n \rfloor}$ —such y exists by (12) and $B \neq \mathbb{R}$.

This proves (a). The proof of (b) is analogous and it is left to the reader. □

Corollary 15.4. *Suppose A is self-dual and assume $\mathsf{SLO}^{\mathrm{W}}_{\leq A^{\triangledown}}$ or, equivalently, $\mathsf{SLO}^{\mathrm{W}}_{\leq A^{\circ}}$. Then $\{[A^{\triangledown}]_{\mathrm{W}}, [A^{\circ}]_{\mathrm{W}}\}$ is the least non-self-dual pair above $[A]_{\mathrm{W}}$, and for any $B \equiv_{\mathrm{W}} A^{\triangledown}$ and $C \equiv_{\mathrm{W}} A^{\circ}$*

$$[\boldsymbol{T}(B)] \cap B = \emptyset \quad \wedge \quad [\boldsymbol{T}(C)] \subseteq C\,.$$

Proof. Suppose B is non-self-dual and, say, $A <_{\mathrm{W}} B \leq_{\mathrm{W}} A^{\triangledown}$, so that $\mathsf{SLO}^{\mathrm{W}}_{\leq B}$ holds. (The case of $A <_{\mathrm{W}} B \leq_{\mathrm{W}} A^{\circ}$ is analogous.) If there is an $x \in [\boldsymbol{T}(B)] \cap B$ then $A^{\circ} \leq_{\mathrm{W}} B$ and hence $A^{\circ} \leq_{\mathrm{W}} A^{\triangledown}$, contradicting Lemma 15.2, so $[\boldsymbol{T}(B)] \cap B = \emptyset$. Therefore $A^{\triangledown} \leq_{\mathrm{W}} B$, which implies $A^{\triangledown} \equiv_{\mathrm{W}} B$ and $A^{\circ} \equiv_{\mathrm{W}} \neg B$. □

The hypothesis $\mathsf{SLO}^{\mathrm{W}}_{\leq A^{\triangledown}}$ in the Corollary can be weakened to $\mathsf{SLO}^{\mathrm{W}}_{\leq A}$—see section 17. Conversely

Proposition 15.5. *Suppose A is non-self-dual and $\mathsf{SLO}^{\mathrm{W}}_{\leq A} + \mathsf{AC}_{\omega}(\mathbb{R})$ holds. Then either $[\boldsymbol{T}(A)] \subseteq A$ or $[\boldsymbol{T}(A)] \cap A = \emptyset$ imply $[A]_{\mathrm{W}}$ is a successor degree or $A \in \{\mathbb{R}, \emptyset\}$.*

Proof. For the sake of definiteness, suppose $[\boldsymbol{T}(A)] \subseteq A$ and, to avoid trivialities, suppose $A \notin \{\mathbb{R}, \emptyset\}$, so that $\boldsymbol{T}(A) \neq {}^{<\omega}\omega$. Let $\{B_n \mid n \in \omega\}$ be an enumeration of $\{A_{\lfloor s \rfloor}, \neg A_{\lfloor s \rfloor} \mid s \in \partial \boldsymbol{T}(A)\}$. The set $B = \bigoplus_n B_n$ is self-dual, as each B_n is the dual of some B_m. By $\mathsf{SLO}^{\mathrm{W}}_{\leq A}$ each $B_n <_{\mathrm{W}} A$ and therefore $B \leq_{\mathrm{W}} A$. As A is non-self-dual the inequality is strict, i.e., $B <_{\mathrm{W}} A$. We will show that $A \leq_{\mathrm{W}} B^{\circ}$ and therefore $A \equiv_{\mathrm{W}} B^{\circ}$, completing the proof. Player **II** wins $G_{\mathrm{W}}(A, B^{\circ})$ as follows:

> As long as **I** plays in $\boldsymbol{T}(A)$ then **II** plays 0. Suppose at some round **I** reaches a position $s \in \partial \boldsymbol{T}(A)$: then **II** plays n, where n is such that $B_n = A_{\lfloor s \rfloor}$ and copies **I**'s moves from now on.

This completes the proof. □

Exercise 15.6. Show that:

(a) If T is a tree on ω, $B_s \leq_{\mathrm{W}} A$ for all $s \in \partial T$, and $\mathsf{SLO}^{\mathrm{W}}_{\leq A} + \mathsf{AC}_{\omega}(\mathbb{R})$ holds, then

$$\boldsymbol{S}(T; \bar{B}) \leq_{\mathrm{W}} A^{\triangledown} \wedge \boldsymbol{S}(T; \bar{B}) \cup [T] \leq_{\mathrm{W}} A^{\circ}\,.$$

(b) If $A \leq_{\mathrm{W}} \neg A$ and $B \leq_{\mathrm{W}} A^{\triangledown}$, then $B = \boldsymbol{S}(T; \bar{B})$ for some sequence $\bar{B} = \langle B_s \mid s \in \partial T \rangle$ such that $B_s \leq_{\mathrm{W}} A$.

Similarly, if $A \leq_{\mathrm{W}} \neg A$ and $B \leq_{\mathrm{W}} A^{\circ}$, then $B = \boldsymbol{S}(T; \bar{B}) \cup [T]$ for some sequence $\bar{B} = \langle B_s \mid s \in \partial T \rangle$ such that $B_s \leq_{\mathrm{W}} A$.

Using this we can characterize the sets in non-self-dual successor degrees.

Proposition 15.7. *Assume* $\neg\mathsf{FS} + \mathsf{AC}_{\omega}(\mathbb{R})$ *and* $A \leq_{\mathrm{W}} \neg A$. *Then* $B \in [A^{\triangledown}]_{\mathrm{W}}$ *iff* $B = \boldsymbol{S}(T; \bar{B})$ *for some tree* T *on* ω *and some* $\bar{B} = \langle B_s \mid s \in \partial T \rangle$ *such that*

$$\exists \bar{x} \in [T]\, \forall A' <_{\mathrm{W}} A\, \forall n \in \omega\, \exists s \in \partial T\, (s \supset \bar{x} \restriction n \wedge A' \leq_{\mathrm{W}} B_s)\,. \tag{13}$$

Similarly, $B \in [A^{\circ}]_{\mathrm{W}}$ *iff* $B = \boldsymbol{S}(T; \bar{B}) \cup [T]$ *for some* T, $\bar{B}$ *satisfying* (13)

Proof. We consider only the case of $A^{\triangledown}$, the other one being similar. All sets $\boldsymbol{S}(T; \bar{B})$ as above are $\leq_{\mathrm{W}} A^{\triangledown}$ by the Exercise 15.6, so we must check that **II** wins $G_{\mathrm{W}}(A^{\triangledown}, \boldsymbol{S}(T; \bar{B}))$:

As long as **I** plays 0's, **II** enumerates $\bar{x}$. If at some point

□

Assume $\mathsf{SLO}^{\mathrm{W}} + \neg\mathsf{FS}$, and let $[A]_{\mathrm{W}}$ be self-dual. Then $[A]_{\mathrm{W}}^{\triangledown}$, $[A]_{\mathrm{W}}^{\circ}$ are the immediate successors of $[A]_{\mathrm{W}}$, and by Exercise 15.1

$$[A]_{\mathrm{W}}^{\triangledown} \leq [A]_{\mathrm{W}}^{\triangledown\circ}, [A]_{\mathrm{W}}^{\circ\triangledown} \quad \text{and} \quad [A]_{\mathrm{W}}^{\circ} \leq [A]_{\mathrm{W}}^{\triangledown\circ}, [A]_{\mathrm{W}}^{\circ\triangledown}$$

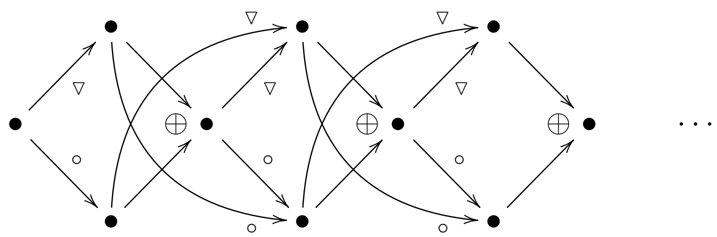
and since $[A]_{\mathrm{W}}^{\triangledown}$, $[A]_{\mathrm{W}}^{\circ}$ are dual to each other, the inequality is strict. By Lemma 15.2 $[A]_{\mathrm{W}}^{\triangledown\circ}$ and $[A]_{\mathrm{W}}^{\circ\triangledown}$ are non-self-dual so

$$[A]_{\mathrm{W}}^{\triangledown} \oplus [A]_{\mathrm{W}}^{\circ} < [A]_{\mathrm{W}}^{\triangledown\circ}, [A]_{\mathrm{W}}^{\circ\triangledown}\,.$$

By Exercise 15.1

$$\left([A]_{\mathrm{W}}^{\triangledown} \oplus [A]_{\mathrm{W}}^{\circ}\right)^{\circ} \leq [A]_{\mathrm{W}}^{\triangledown\circ\circ} = [A]_{\mathrm{W}}^{\triangledown\circ} \leq \left([A]_{\mathrm{W}}^{\triangledown} \oplus [A]_{\mathrm{W}}^{\circ}\right)^{\circ}$$

hence $\left([A]_{\mathrm{W}}^{\triangledown} \oplus [A]_{\mathrm{W}}^{\circ}\right)^{\circ} = [A]_{\mathrm{W}}^{\triangledown\circ}$ and, similarly $\left([A]_{\mathrm{W}}^{\triangledown} \oplus [A]_{\mathrm{W}}^{\circ}\right)^{\triangledown} = [A]_{\mathrm{W}}^{\circ\triangledown}$. Therefore $\{[A]_{\mathrm{W}}^{\circ\triangledown}, [A]_{\mathrm{W}}^{\triangledown\circ}\}$ is the least non-self-dual pair above $\{[A]_{\mathrm{W}}^{\triangledown}, [A]_{\mathrm{W}}^{\circ}\}$. The following diagram summarizes the situation:



The computations above are done under the assumption of $\mathsf{SLO}^{\mathrm{W}}$ for simplicity reasons, but by the results in the next section, only $\mathsf{SLO}^{\mathrm{W}}_{\leq A}$ is needed.

Proposition 15.8. *Suppose A is self-dual and $\mathsf{SLO}^{\mathrm{W}}_{\leq A^\circ}$ holds. If $B, C \in [A^\circ]_{\mathrm{W}}$ and $B \cap C = \emptyset$, then there is $D \leq_{\mathrm{W}} A$ such that $B \subseteq D$ and $D \cap C = \emptyset$.*

Proof. By Corollary 15.4 $[\boldsymbol{T}(B)] \subseteq B$ and $[\boldsymbol{T}(C)] \subseteq C$, and $[\boldsymbol{T}(B)] \cap [\boldsymbol{T}(C)] = \emptyset$ so that $T = \boldsymbol{T}(B) \cap \boldsymbol{T}(C)$ is well-founded. We will define

$$D = \bigcup_{t \in \partial T} t^\frown D_{\lfloor t \rfloor}$$

as follows. Let $t \in \partial T$: we consider two cases.

Case 1 $t \in \partial \boldsymbol{T}(B)$. Then $B_{\lfloor t \rfloor} <_{\mathrm{W}} A^\circ$ so $B_{\lfloor t \rfloor} \leq_{\mathrm{W}} A$. Let $D_{\lfloor t \rfloor} = B_{\lfloor t \rfloor}$ so that $D_{\lfloor t \rfloor} \cap C_{\lfloor t \rfloor} = \emptyset$.

Case 2 otherwise, that is, $t \in \boldsymbol{T}(C) \setminus \boldsymbol{T}(B)$. Then $C_{\lfloor t \rfloor} <_{\mathrm{W}} A^\circ$ so $C_{\lfloor t \rfloor} \leq_{\mathrm{W}} A$. Let $D_{\lfloor t \rfloor} = \neg C_{\lfloor t \rfloor}$ so that $D_{\lfloor t \rfloor} \supseteq B_{\lfloor t \rfloor}$.

It is easy to check that the set D is as required. □

16 Addition of degrees

For any $x \in {}^{<\omega}\omega \cup \mathbb{R}$, let $x + 1 = \langle x(n) + 1 \mid n \in \mathrm{lh}(x) \rangle$, and for $A \subseteq {}^{<\omega}\omega \cup \mathbb{R}$, let

$$A^+ = \{x + 1 \mid x \in A\} . \tag{14}$$

The **addition of two sets** A **and** B is defined as

$$A + B = \{s + 1^\frown \langle 0 \rangle^\frown x \mid x \in A\} \cup B^+ .$$

Notice that the definition of A^+, and hence of $A + B$ does not make sense in the Cantor space.

Exercise 16.1. Show that:

(i) $B^+ = \emptyset + B$;

(ii) $(A + B)_{\lfloor s \rfloor} = A + B_{\lfloor s+1 \rfloor}$;

(iii) $\neg (A + B) = \neg A + \neg B$;

(iv) $\bigoplus_n (A + B_n) \equiv_{\mathrm{L}} A + \bigoplus_n B_n$;

(v) $A \leq_{\mathrm{L}} A + B$;

(vi) $A \leq_{\mathrm{L}} A' \wedge B \leq_{\mathrm{L}} B' \Rightarrow A + B \leq_{\mathrm{L}} A' + B'$;

(vii) $(A + B) + C \equiv_{\mathrm{L}} A + (B + C)$;

and similarly for the $\leq_{\mathrm{W}}$ pre-order.

Lemma 16.2. *If* $A \subseteq \mathbb{R}$ *and* $\mathrm{Int}(A) = \emptyset$*, then* $A \equiv_{\mathrm{L}} A^+$*.*

Proof. The map $x \mapsto x^+$ witnesses that $A \leq_{\mathrm{L}} A^+$ so it is enough to check that **II** wins $G_{\mathrm{L}}(A^+, A)$. Here is a strategy:

> As long as **I** plays integers of the form $n + 1$ then **II** answers n. Suppose at some round **I** plays 0, and let p be **II**'s position before this round. Since $\mathrm{Int}(A) = \emptyset$, there is an $x \notin A_{\lfloor p \rfloor}$, and **II** can follow x from now on.

□

The addition operation can be defined on Wadge degrees by

$$[A]_{\mathrm{W}} + [B]_{\mathrm{W}} = [A + B]_{\mathrm{W}} .$$

This definition makes sense also for Lipschitz degrees or, more generally, for $\mathcal{F}$-degrees, where $\mathcal{F} \subseteq {}^{\mathbb{R}}\mathbb{R}$ is a family of functions containing all Lipschitz functions and satisfying (11), but we have no use for such generalizations. Although addition on degrees is associative, it is far from commutative; in fact, as we shall see, addition of sets pretty much resembles ordinal addition.

Exercise 16.3. Show that

(i) $\mathbb{R} + \mathbb{R} = \mathbb{R}$ and $\emptyset + \emptyset = \emptyset$.

(ii) Assume $\mathsf{AC}_\omega(\mathbb{R})$ and $A \neq \mathbb{R}$. Then $A + \emptyset \equiv_{\mathrm{L}} A^\triangledown$ and $A + \mathbb{R} \equiv_{\mathrm{L}} A^\circ$.

(iii) Suppose $A \leq_{\mathrm{W}} \neg A$. Assume $\mathsf{AC}_\omega(\mathbb{R})$ and $\mathsf{SLO}^{\mathrm{W}}_{\leq A+\mathbb{R}}$ or—equivalently—$\mathsf{SLO}^{\mathrm{W}}_{\leq A^\circ}$. Then $\forall B \, (A <_{\mathrm{W}} A + B)$.

The assumption in (iii) that A be self-dual is crucial, since there are non-self-dual B such that $B + B \equiv_{\mathrm{W}} B$—see (**??**)—while by the results of section 17 the hypothesis $\mathsf{SLO}^{\mathrm{W}}_{\leq A+\mathbb{R}}$ could be weakened to $\mathsf{SLO}^{\mathrm{W}}_{\leq A}$.

Suppose that A is self-dual and that τ is a winning strategy for **II** in $G_{\mathrm{W}}(A + B, A + B')$. We claim that if **II** follows τ then he is not the first player to play a 0. Suppose otherwise: then at some round **I** reaches a position $s + 1$ and **II** plays 0 reaching a position $(t + 1)^\frown\langle 0\rangle$. Then **II** must be able to win what's left of the game and hence

$$A + B_{\lfloor s \rfloor} = (A + B)_{\lfloor s+1 \rfloor} \leq_{\mathrm{W}} (A + B')_{\lfloor (t+1)^\frown\langle 0\rangle \rfloor} = A ,$$

contradicting Exercise 16.3(iii). Therefore τ yields a winning strategy $s \mapsto \tau(s+1)-1$ for **II** in $G_{\mathrm{W}}(B, B')$. Since the implication $B \leq_{\mathrm{W}} B' \Rightarrow A+B \leq_{\mathrm{W}} A+B'$ is immediate, we have shown that

Proposition 16.4. *Assume* $\mathsf{AC}_{\omega}(\mathbb{R}) + \mathsf{SLO}^{\mathrm{W}}_{\leq A^{\circ}}$ *and that* $A \leq_{\mathrm{W}} \neg A$. *Then*

$$B \leq_{\mathrm{W}} B' \Leftrightarrow A+B \leq_{\mathrm{W}} A+B' .$$

Thus set-addition is strictly increasing in the second variable; on the other hand, it need not be strictly increasing in the first variable, as the next result shows.

Lemma 16.5. *Suppose* B *is non-self-dual and* $B+B \equiv_{\mathrm{W}} B$. *Then*

$$\neg B + B \equiv_{\mathrm{W}} (B \oplus \neg B) + B .$$

Proof. For notational ease, let $A = B \oplus \neg B$. By Exercise 16.1 $\neg B \leq_{\mathrm{W}} A$ implies $\neg B + B \leq_{\mathrm{W}} A + B$, so it is enough to show that $A + B \leq_{\mathrm{W}} \neg B + B$. Again by Exercise 16.1 it is enough to prove that **II** has a winning strategy in $G_{\mathrm{W}}(A+B, \neg B + (B+B))$. Here is such a strategy:

> As long as **I** plays non-0 natural numbers, **II** copies and adds 1. If at some round **I** plays 0 for the first time, then **II** passes. In the next round, if **I** plays an odd integer (choosing the $\neg B$-side of A) then **II** plays 0 and then copies; if otherwise **I** plays an even integer (choosing the B-side of A) then **II** plays 1 and then copies and adds 1.

□

Lemma 16.6 (Wadge). *Suppose* $A <_{\mathrm{W}} X$ *with* A *self-dual, and assume* $\mathsf{SLO}^{\mathrm{W}}_{\leq X}$. *Then there is a* $Y \leq_{\mathrm{W}} X$ *such that* $X \equiv_{\mathrm{W}} A + Y$.

Proof. The tree

$$T = \{t \in {}^{<\omega}\omega \mid X_{\lfloor t \rfloor} \not\leq_{\mathrm{W}} A\}$$

is pruned, since $X_{\lfloor t^\frown \langle n \rangle \rfloor} \leq_{\mathrm{W}} A$ for all n implies $X_{\lfloor t \rfloor} \leq_{\mathrm{W}} A$. Let $\boldsymbol{\varrho}_T : {}^{<\omega}\omega \twoheadrightarrow T$ and $\boldsymbol{r}_T : \mathbb{R} \twoheadrightarrow [T]$ be as in (6) and (8). Then

$$Y = \boldsymbol{r}_T^{-1}(X \cap [T]) \leq_{\mathrm{W}} X .$$

It is enough to show that $A + Y \equiv_{\mathrm{W}} X$.

To show that $X \leq_{\mathrm{W}} A + Y$ consider the following strategy for **II** in $G_{\mathrm{W}}(X, A+Y)$:

> As long as **I** plays in T, then **II** copies and adds 1. If **I** ever reaches a $t \in \partial T$, then $X_{\lfloor t \rfloor} \leq_{\mathrm{W}} A$. Then **II** plays 0 and from now on follows a strategy witnessing $X_{\lfloor t \rfloor} \leq_{\mathrm{W}} A$.

To show that $A + Y \leq_{\mathrm{W}} X$ consider the following strategy for **II** in $G_{\mathrm{W}}(A+Y, X)$

> As long as **I** plays non-zero integers, then **II** subtracts 1 and uses $\boldsymbol{\varrho}_T$. Suppose at some round **I** plays first a 0, and let s and t be **I**'s and **II**'s positions reached before this round: then $t \in T$, hence $X_{\lfloor t \rfloor} \not\leq_{\mathrm{W}} A = (A+Y)_{\lfloor s^\frown 0 \rfloor}$. Since $X_{\lfloor t \rfloor} \leq_{\mathrm{W}} X$ and $A \leq_{\mathrm{W}} X$, then $\mathsf{SLO}^{\mathrm{W}}_{\leq X}$ implies that
>
> $$(A+Y)_{\lfloor s^\frown 0 \rfloor} <_{\mathrm{W}} B_{\lfloor t \rfloor} \tag{15}$$
>
> and therefore **II** can easily win the game from this point on.

□

Therefore, by (**??**), if A is self-dual and $\mathsf{SLO}^{\mathrm{W}}_{\leq A+B}$ holds, then

$$\|A+B\|_{\mathrm{W}} = \|A\|_{\mathrm{W}} + \|B\|_{\mathrm{W}} \,. \tag{16}$$

Suppose $\mathsf{SLO}^{\mathrm{W}}_{\leq A}$ holds and that $\|A\|_{\mathrm{W}} \geq \omega$. Let D be clopen. By Lemma 16.6 there is a $C \leq_{\mathrm{W}} A$ such that $A \equiv_{\mathrm{W}} D+C$ and hence $\|A\|_{\mathrm{W}} = 2 + \|C\|_{\mathrm{W}}$. Thus $\|A\|_{\mathrm{W}} = \|C\|_{\mathrm{W}}$, which implies $A \equiv_{\mathrm{W}} C$ and hence $D + A \equiv_{\mathrm{W}} A$. Since $A \leq_{\mathrm{W}} \emptyset + A \leq_{\mathrm{W}} D + A$, then $A \equiv_{\mathrm{W}} \emptyset + A = A^+$. Since $\mathrm{Int}(A^+) = \emptyset$, we have that

$$\mathsf{SLO}^{\mathrm{W}}_{\leq A} \wedge \|A\|_{\mathrm{W}} \geq \omega \Rightarrow \exists B \equiv_{\mathrm{W}} A\,(\mathrm{Int}(B) = \emptyset) \,. \tag{17}$$

Using this and (**??**) we get

$$\mathsf{SLO}^{\mathrm{W}}_{\leq A} \wedge \|A\|_{\mathrm{W}} \geq \omega \Rightarrow A \equiv_{\mathrm{W}} A^+ \,. \tag{18}$$

For the next result we need to recall the definition of **additively indecomposable ordinal**: a non-zero ordinal λ is said to be additively indecomposable iff $\forall \alpha, \beta < \lambda (\alpha + \beta \neq \lambda)$ iff $\forall \alpha, \beta < \lambda (\alpha + \beta < \lambda)$. The additively indecomposable ordinals are exactly those of the form ω^γ, for some γ.

Lemma 16.7. *Assume $\mathsf{SLO}^{\mathrm{W}}_{\leq A} + \mathsf{DC}(\mathbb{R})$ so that $\|A\|_{\mathrm{W}}$ is defined. Let $\delta \leq \|A\|_{\mathrm{W}}$ be an additively indecomposable ordinal. If $C <_{\mathrm{W}} A$ and $\|C\|_{\mathrm{W}} < \delta$, then $C + A \equiv_{\mathrm{W}} A$.*

Proof. Assume first that C is self-dual. Let $D \leq_{\mathrm{W}} A$ be such that $\|D\|_{\mathrm{W}} = \delta$: by Lemma 16.6, there is a $D' \leq_{\mathrm{W}} D$ such that $C + D' \equiv_{\mathrm{W}} D$. By (16) $\|C\|_{\mathrm{W}} + \|D'\|_{\mathrm{W}} = \|C + D'\|_{\mathrm{W}} = \delta$, hence $\|D'\|_{\mathrm{W}} = \|D\|_{\mathrm{W}}$ by indecomposability. Thus $D' \equiv_{\mathrm{W}} D$, hence $D \equiv_{\mathrm{W}} C + D$. If $\delta = \|A\|_{\mathrm{W}}$, then $D \equiv_{\mathrm{W}} A$ and we are done, therefore we may assume $\delta < \|A\|_{\mathrm{W}}$. If D is self-dual then, by Lemma 16.6 again, $D + A' \equiv_{\mathrm{W}} A$ for some A', hence

$$C + A \equiv_{\mathrm{W}} C + (D + A') \equiv_{\mathrm{W}} (C + D) + A' \equiv_{\mathrm{W}} D + A' \equiv_{\mathrm{W}} A\,.$$

If D is non-self-dual, it is enough to show that $C + (D \oplus \neg D) \equiv_{\mathrm{W}} D \oplus \neg D$, and since $D \oplus \neg D$ is self-dual we can proceed as before: by (16) $\|C + (D \oplus \neg D)\|_{\mathrm{W}} = \|C\|_{\mathrm{W}} + \delta + 1 = \delta + 1 = \|D \oplus \neg D\|_{\mathrm{W}}$, hence we are done.

Assume now C is non-self-dual. Then $C + A \leq_{\mathrm{W}} (C \oplus \neg C) + A \equiv_{\mathrm{W}} A$ by the previous paragraph, and since the reverse inequality holds trivially, we are done. □

Lemma 16.8. *Let T be a (non-necessarily pruned) tree on ω such that $[T] \neq \emptyset$, and let $X, Y, Z \subseteq \mathbb{R}$.*

(a) *If $X \cap [T] \leq_{\mathrm{W}} Z$ and $\forall s \in \partial T \left(X_{\lfloor s \rfloor} \leq_{\mathrm{W}} Y\right)$, then $X \leq_{\mathrm{W}} Y + Z$.*

(b) *Suppose $Z \leq_{\mathrm{W}} X \cap [T]$, $\mathrm{Int}([T]) = \emptyset$, and $\forall s \in \partial T \left(Y \leq_{\mathrm{W}} X_{\lfloor s \rfloor}\right)$, then $Y + Z \leq_{\mathrm{W}} X$.*

(c) *Suppose $X = f^{-1}$“$(Y + Z)$ with $f : \mathbb{R} \to \mathbb{R}$ continuous, and $[T] \subseteq f^{-1}$“$\mathbb{R}^+$ and $\mathrm{Int}(Z) = \emptyset$. Then $X \cap [T] \leq_{\mathrm{W}} Z$.*

(d) *Suppose $X = f^{-1}$“$(Y + Z)$ with $f : \mathbb{R} \to \mathbb{R}$ continuous, and suppose T is the pruned tree such that $[T] = f^{-1}$“$\mathbb{R}^+$. Then*

$$\forall s \in \partial T \left(X_{\lfloor s \rfloor} \leq_{\mathrm{W}} Y\right).$$

Proof. (a) Let τ be a winning strategy for **II** in $G_{\mathrm{W}}(X \cap [T], Z)$. Then **II** wins $G_{\mathrm{W}}(X, Y + Z)$ by following $\tau + 1$ as long as **I**'s position is in T, and if **I** ever reaches a position $s \in \partial T$, then **II** plays 0 and then uses the reduction $X_{\lfloor s \rfloor} \leq_{\mathrm{W}} Y$.

(b) Let τ be a winning strategy for **II** in $G_{\mathrm{W}}(Z, X \cap [T])$. Here is a winning strategy for **II** in $G_{\mathrm{W}}(Y + Z, X)$: as long as **I** plays non-0 integers then **II** subtracts 1 and follows τ; if at some round **I** plays 0, then **II** reaches a position $s \in \partial T$ (and this is possible, since $\mathrm{Int}([T]) = \emptyset$) and then follows the reduction $Y \leq_{\mathrm{W}} X_{\lfloor s \rfloor}$.

(c) Let τ be **II**'s winning strategy in $G_{\mathrm{W}}(X, Y+Z)$ defining f. Then **II** wins $G_{\mathrm{W}}(X \cap [T], Z)$ as follows: As long as **I** plays in T, then **II** follows $\tau - 1$. If **I** reaches ∂T, then **II** starts enumerating a real not in Z—here is where we need that $\mathrm{Int}(Z) = \emptyset$.

(d) Let τ be a winning strategy for **II** in $G_{\mathrm{W}}(X, Y+Z)$ and suppose that at round n **I** reaches a position $s \in \partial T$. Since τ can pass, it may happen that at round n **II**'s position be still in $({}^{<\omega}\omega)^+$, although any play extending s will eventually bring **II**'s play outside $\mathbb{R}^+$. Therefore

$$U = \{p \mid \big((s^\frown p) * \tau\big)_{\mathbf{II}} \in ({}^{<\omega}\omega)^+\}$$

is a well-founded (possibly empty) tree. Then **II** wins $G_{\mathrm{W}}(X_{\lfloor s \rfloor}, Y)$ by passing, until **I** reaches a position $s' \supseteq s$, $s' \notin U$, where τ plays 0 for the first time. Then **II** follows τ. □

Theorem 16.9. *Assume* $\mathsf{DC}(\mathbb{R})$ *and suppose* $\mathsf{SLO}^{\mathrm{W}}_{\leq A}$ *holds, with* A *non-self-dual and* $\|A\|_{\mathrm{W}}$ *limit. Then*

$$\|A\|_{\mathrm{W}} \text{ is additively indecomposable } \Leftrightarrow A \equiv_{\mathrm{W}} [\boldsymbol{T}(A)] \cap A\,.$$

Proof. Suppose A is non-self-dual so that $\boldsymbol{T}(A)$ is pruned. If $\|A\|_{\mathrm{W}}$ is limit and additively decomposable, and $B, C <_{\mathrm{W}} A$ are such that $A \equiv_{\mathrm{W}} B + C$, then $\boldsymbol{T}(B+C) = \boldsymbol{T}(C)^+$ and by (**??**)

$$A \cap [\boldsymbol{T}(A)] \equiv_{\mathrm{W}} (B+C) \cap [\boldsymbol{T}(B+C)] = C^+ \cap [\boldsymbol{T}(C)^+] \leq_{\mathrm{W}} C^+.$$

Since $\|C\|_{\mathrm{W}}$ is limit, then $C^+ \equiv_{\mathrm{W}} C$ by (18), hence $A \cap [\boldsymbol{T}(A)] <_{\mathrm{W}} A$ by (**??**). Conversely, assume $\|A\|_{\mathrm{W}}$ is additively indecomposable. For any $s \in \partial \boldsymbol{T}(A)$ (if any such s exist) $A_{\lfloor s \rfloor} \leq_{\mathrm{W}} A$, and since $\mathrm{cof}(\|A\|_{\mathrm{W}}) > \omega$, there is a self-dual $B <_{\mathrm{W}} A$ such that $\forall s \in \partial \boldsymbol{T}(A) \big(A_{\lfloor s \rfloor} \leq_{\mathrm{W}} B\big)$. Lemma 16.8(a) implies $A \leq_{\mathrm{W}} B + ([\boldsymbol{T}(A)] \cap A)$, so that $\|A\|_{\mathrm{W}} \leq \|B\|_{\mathrm{W}} + \|[\boldsymbol{T}(A)] \cap A\|_{\mathrm{W}}$, and therefore $A \equiv_{\mathrm{W}} [\boldsymbol{T}(A)] \cap A$ by (**??**) and case assumption. □

17 Propagation of $\mathsf{SLO}^{\mathrm{W}}$

In this section we will prove that if A is self-dual then $\mathsf{SLO}^{\mathrm{W}}_{\leq A} \Rightarrow \mathsf{SLO}^{\mathrm{W}}_{\leq A+A}$ Since $A+A$ is also self-dual, this argument can be iterated to prove $\mathsf{SLO}^{\mathrm{W}}_{\leq A \cdot n}$ for all n, where

$$\begin{aligned} A \cdot n &= A + (A + (\cdots (A + A) \cdots)) \\ &= \underbrace{A + A + \cdots + A}_{n}\,. \end{aligned} \tag{19}$$

In fact this construction can be pushed through the countable ordinals, but in order to state it in a convenient form it is better to introduce the following notation: since $A \equiv_{\mathrm{W}} B \Rightarrow \left(\mathsf{SLO}^{\mathrm{W}}_{\leq A} \Leftrightarrow \mathsf{SLO}^{\mathrm{W}}_{\leq B}\right)$, let

$$\begin{aligned}\mathsf{SLO}^{\mathrm{W}}_{\leq [A]_{\mathrm{W}}} &\Leftrightarrow \mathsf{SLO}^{\mathrm{W}}_{\leq B} \text{ for some } B \in [A]_{\mathrm{W}} \\ &\Leftrightarrow \mathsf{SLO}^{\mathrm{W}}_{\leq B} \text{ for any } B \in [A]_{\mathrm{W}} .\end{aligned}$$

For $1 \leq \alpha < \omega_1$ and $A \subseteq \mathbb{R}$ define $[A]_{\mathrm{W}} \cdot \alpha$ as follows:

$$[A]_{\mathrm{W}} \cdot 1 = [A]_{\mathrm{W}}$$

and for $1 \leq \alpha, \lambda < \omega_1$

$$\begin{aligned}[A]_{\mathrm{W}} \cdot (\alpha + 1) &= [A]_{\mathrm{W}} \cdot \alpha + [A]_{\mathrm{W}} \\ [A]_{\mathrm{W}} \cdot \lambda &= \sup_n [A]_{\mathrm{W}} \cdot \alpha_n\end{aligned}$$

where λ is limit and $\alpha_n \to \lambda$ is an increasing sequence. The definition of $[A]_{\mathrm{W}} \cdot \lambda$ does not depend on the choice of the α_n's by Exercise 14.4. By an argument as in Remark 5.11 it can be shown that there is no definable map $(A, \alpha) \mapsto A \bullet \alpha$ such that $[A \bullet \alpha]_{\mathrm{W}} = [A]_{\mathrm{W}} \cdot \alpha$.

The main result of this section is

Theorem 17.1. *Assume* $\mathsf{DC}(\mathbb{R}) + \neg\mathsf{FS} + \mathsf{SLO}^{\mathrm{W}}_{\leq [A]_{\mathrm{W}}}$ *where* $[A]_{\mathrm{W}}$ *is self-dual. Then* $\mathsf{SLO}^{\mathrm{W}}_{\leq [A]_{\mathrm{W}} \cdot \alpha}$ *holds, for* $1 \leq \alpha < \omega_1$.

Strictly speaking, the statement of Theorem 17.1 is not quite correct, since the definition of $[A]_{\mathrm{W}} \cdot \alpha$ (with α limit) assumes that $\mathsf{SLO}^{\mathrm{W}}$ holds of all the sets of rank $< \|A\|_{\mathrm{W}} \cdot \alpha$, which is what the theorem aims to show. The correct reformulation is

Theorem 17.2. *Assume* $\mathsf{DC}(\mathbb{R}) + \neg\mathsf{FS} + \mathsf{SLO}^{\mathrm{W}}_{\leq [A]_{\mathrm{W}}}$ *where* $[A]_{\mathrm{W}}$ *is self-dual. Then for each* $1 \leq \gamma < \omega_1$ *there is a set* C *such that* $\mathsf{SLO}^{\mathrm{W}}_{\leq C}$ *holds and* $\|C\|_{\mathrm{W}} = \|A\|_{\mathrm{W}} \cdot \gamma$.

First of all, let us notice that it is enough to prove this when $\|A\|_{\mathrm{W}}$ is an additively indecomposable ordinal, or the successor of an additively indecomposable ordinal. (Recall that a limit ordinal λ is additively indecomposable iff $\forall \xi, \eta < \lambda (\xi + \eta < \lambda)$ and that all such ordinals are of the form ω^{α}.)

Theorem 17.3. *Assume* $\mathsf{DC}(\mathbb{R}) + \neg\mathsf{FS} + \mathsf{SLO}^{\mathrm{W}}_{\leq [A]_{\mathrm{W}}}$ *where* $[A]_{\mathrm{W}}$ *is self-dual and either* $\|A\|_{\mathrm{W}}$ *is additively indecomposable, ore else is the successor of an additively indecomposable. Then for each* $1 \leq \gamma < \omega_1$ *there is a self-dual set* C *such that* $\mathsf{SLO}^{\mathrm{W}}_{\leq [C]_{\mathrm{W}}}$ *holds and* $\|C\|_{\mathrm{W}} = \|A\|_{\mathrm{W}} \cdot \gamma$.

In fact, suppose $[A]_{\mathrm{W}}$ is self-dual, but $\|A\|_{\mathrm{W}}$ is neither additively indecomposable, nor the successor of an additively indecomposable ordinal. Let ω^α be the largest additively indecomposable ordinal below $\|A\|_{\mathrm{W}}$. If $\mathrm{cof}(\alpha) = \omega$, let A' be a set of rank ω^α, and if $\mathrm{cof}(\alpha) > \omega$, let A' be a set of rank $\omega^\alpha + 1$. In either case, $[A']_{\mathrm{W}}$ is self-dual and $\mathsf{SLO}^{\mathrm{W}}_{\leq [A']_{\mathrm{W}}}$ holds. Since

$$\sup_{\gamma<\omega_1} \|A'\|_{\mathrm{W}} \cdot \gamma = \sup_{\gamma<\omega_1} \|A\|_{\mathrm{W}} \cdot \gamma$$

then Theorem 17.2 follows at once from Theorem 17.3.

Towards proving Theorem 17.3, let us start with a few preliminary results.

Lemma 17.4. $\mathsf{SLO}^{\mathrm{W}}_{\leq A} \Rightarrow \mathsf{SLO}^{\mathrm{W}}_{\leq A \oplus \neg A}$.

Proof. Suppose $X = f^{-1}(A \oplus \neg A)$ with $f : \mathbb{R} \to \mathbb{R}$ continuous. For ease of notation let $A_0 = A$ and $A_1 = \neg A$, let

$$D_i = f^{-1}\big(\textstyle\bigcup_n \boldsymbol{N}_{\langle 2n+i \rangle}\big)$$

and let $X_i = X \cap D_i$, for $(i = 0, 1)$. If $A_i \leq_{\mathrm{W}} X_i$ for both $i = 0$ and $i = 1$, then **I** wins $G_{\mathrm{W}}(A \oplus \neg A, X)$ by passing in the first round and then choosing the right reduction. If $A_i \nleq_{\mathrm{W}} X_i$ for some $i = 0, 1$, then $\mathsf{SLO}^{\mathrm{W}}_{\leq B}$ implies that $X_i \leq_{\mathrm{W}} B_{1-i}$ and therefore **II** wins $G_{\mathrm{W}}(X, A_{1-i})$ by passing until **I** has gotten into D_0 or D_1 and then chooses the right reduction. Therefore if $X, Y \leq_{\mathrm{W}} A \oplus \neg A$ then $X \leq_{\mathrm{W}} Y$ or $Y \leq_{\mathrm{W}} \neg X$. □

Lemma 17.5. *Suppose $A_0 <_{\mathrm{W}} A_1 <_{\mathrm{W}} \ldots$ and let $A = \bigoplus_n A_n$. If $\mathsf{SLO}^{\mathrm{W}}_{\leq A_n}$ holds for every n, then $\mathsf{SLO}^{\mathrm{W}}_{\leq A}$ holds.*

Proof. The assumption $\forall n < \omega\ \mathsf{SLO}^{\mathrm{W}}_{\leq A_n}$ is enough to prove that each $A_n <_{\mathrm{W}} \neg A_{n+1}$ hence $A = \bigoplus_n A_n$ is self-dual. Let $X = f^{-1}(A)$ with $f : \mathbb{R} \to \mathbb{R}$ continuous, and let $D_n = f^{-1}(\boldsymbol{N}_{\langle n \rangle})$. If

$$\forall n\, \exists m\ A_n \leq_{\mathrm{W}} X \cap D_m\,,$$

then let $m(n)$ be the least m such that there is an $f_n : \mathbb{R} \to \mathbb{R}$ witnessing $A_n \leq_{\mathrm{W}} X \cap D_m$: since $f_n(\mathrm{Cl}(A_n)) \subseteq D_{m(n)}$ and since D is clopen, we may modify f_n (if needed) so that $\mathrm{ran}(f_n) \subseteq D_{m(n)}$ while preserving the fact that A_n is reducible to $X \cap D_{m(n)}$; therefore **II** can win $G_{\mathrm{W}}(A, X)$ by using the reduction f_n if **I** plays n in the first round. If instead

$$\exists n\, \forall m\ A_n \nleq_{\mathrm{W}} X \cap D_m\,,$$

then since $X \cap D_m \leq_{\mathrm{W}} A_m$, the hypothesis $\mathsf{SLO}^{\mathrm{W}}_{\leq A_m}$ (when $n < m$) implies that $X \cap D_m \leq_{\mathrm{W}} \neg A_n \leq_{\mathrm{W}} A_{n+1}$ and hence **II** wins $G_{\mathrm{W}}(X, A_{n+1})$ by passing until **I** enters in some D_m and then uses the reduction. Therefore we have shown that

$$\forall X \leq_{\mathrm{W}} A\,(X \equiv_{\mathrm{W}} A \;\vee\; \exists n < \omega\,(X \leq_{\mathrm{W}} A_n))\,.$$

Thus if $X, Y \leq_{\mathrm{W}} A$ then either $X \equiv_{\mathrm{W}} A \equiv_{\mathrm{W}} Y$, or $X <_{\mathrm{W}} A \equiv_{\mathrm{W}} Y$, or $Y <_{\mathrm{W}} A \equiv_{\mathrm{W}} X$, or else $X, Y <_{\mathrm{W}} A_n$ for some n. This shows that $\mathsf{SLO}^{\mathrm{W}}_{\leq A}$ holds. □

Lemma 17.6. *Assume A is self-dual and that $\mathsf{SLO}^{\mathrm{W}}_{\leq A\cdot n}$ holds. Suppose that*

$$(\star_{A\cdot n}) \quad \forall X \leq_{\mathrm{W}} A\cdot(n+1)\Big(X \leq_{\mathrm{W}} A \;\vee\; \exists Y \leq_{\mathrm{W}} A\cdot n\,(X \equiv_{\mathrm{W}} A + Y)\Big)\,.$$

Then $\mathsf{SLO}^{\mathrm{W}}_{\leq A\cdot(n+1)}$ holds.

Proof. We must show that for any $X_1, X_2 \leq_{\mathrm{W}} A\cdot(n+1)$

$$X_1 \leq_{\mathrm{W}} X_2 \;\vee\; X_2 \leq_{\mathrm{W}} \neg X_1\,. \tag{20}$$

If both $X_1 \leq_{\mathrm{W}} A$ and $X_2 \leq_{\mathrm{W}} A$ then (20) follows from $\mathsf{SLO}^{\mathrm{W}}_{\leq A}$. If $X_1 \leq_{\mathrm{W}} A$ and $X_2 \equiv_{\mathrm{W}} A + Y_2$, then $X_1 \leq_{\mathrm{W}} X_2$ by (16.3). If $X_i \equiv_{\mathrm{W}} A + Y_i$ (with $Y_i \leq_{\mathrm{W}} A\cdot n$ and $i = 1, 2$) then $Y_1 \leq_{\mathrm{W}} Y_2$ or $Y_2 \leq_{\mathrm{W}} \neg Y_1$ by $\mathsf{SLO}^{\mathrm{W}}_{\leq A\cdot n}$ hence (20) holds. □

Lemma 17.7. *Assume $\mathsf{SLO}^{\mathrm{W}}_{\leq A\cdot n}$ with A self-dual and suppose either $\|A\|_{\mathrm{W}}$ is an additively indecomposable limit ordinal or else $A \equiv_{\mathrm{W}} B \oplus \neg B$ with B non-self-dual, $B + B \equiv_{\mathrm{W}} B$, and $\|B\|_{\mathrm{W}}$ additively indecomposable limit ordinal. Then $(\star_{A\cdot n})$ holds.*

The proof is word-by-word that of Lemma 16.6: there the assumption $\mathsf{SLO}^{\mathrm{W}}_{\leq B}$ was only used to prove (15), while here we replace it with the following

Lemma 17.8. *Assume $\mathsf{SLO}^{\mathrm{W}}_{\leq A\cdot n}$ with A self-dual, and suppose either $\|A\|_{\mathrm{W}}$ is an additively indecomposable limit ordinal or else $A \equiv_{\mathrm{W}} B \oplus \neg B$ with B non-self-dual, $B + B \equiv_{\mathrm{W}} B$, and $\|B\|_{\mathrm{W}}$ additively indecomposable limit ordinal. Suppose also $X \leq_{\mathrm{W}} A\cdot(n+1)$ and $X \nleq_{\mathrm{W}} A$. Then either $A + \emptyset \leq_{\mathrm{W}} X$, or else $A + \mathbb{R} \leq_{\mathrm{W}} X$.*

Proof. Since $\|A\cdot n\|_{\mathrm{W}} \geq \omega$, then by (17) we may assume that $\mathrm{Int}(A\cdot n) = \emptyset$. Let $X = f^{-1}(A\cdot(n+1))$, with $f : \mathbb{R} \to \mathbb{R}$ continuous, and suppose $X \nleq_{\mathrm{W}} A$. Suppose $X \leq_{\mathrm{W}} A\cdot n$: then $n = 1$ implies that $X \leq_{\mathrm{W}} A$ (contradicting our assumption), and $n > 1$ implies $A <_{\mathrm{W}} X$ by $\mathsf{SLO}^{\mathrm{W}}_{\leq A\cdot n}$, and therefore $A + \emptyset \leq_{\mathrm{W}} X$ or $A + \mathbb{R} \leq_{\mathrm{W}} X$. Therefore we may assume that $X \nleq_{\mathrm{W}} A\cdot n$.

The set $f^{-1}(\mathbb{R}^+)$ is closed and let $\hat{T} \supseteq T$ be the pruned trees such that $[\hat{T}] = f^{-1}(\mathbb{R}^+)$ and $[T] = [\hat{T}] \setminus \mathrm{Int}([\hat{T}])$. By Lemma 16.8(c)

$$X \cap [\hat{T}] \leq_{\mathrm{W}} A \cdot n \quad \text{and} \quad X \cap [T] \leq_{\mathrm{W}} A \cdot n\,. \tag{21}$$

Suppose $s \in \partial T$. If $s \in \partial\hat{T}$ then $X_{\lfloor s\rfloor} \leq_{\mathrm{W}} A$ by Lemma 16.8(d), and if $s \in \hat{T}$ then $X_{\lfloor s\rfloor} \subseteq [\hat{T}_{\lfloor s\rfloor}] = \mathbb{R}$ so

$$X_{\lfloor s\rfloor} = (X \cap [T])_{\lfloor s\rfloor} \leq_{\mathrm{W}} X \cap [T] \leq_{\mathrm{W}} A \cdot n$$

Therefore

$$\forall s \in \partial T \left(X_{\lfloor s\rfloor} \leq_{\mathrm{W}} A \cdot n\right) \tag{22}$$

and that For each $t \in T$ let

$$\varphi(t) = \sup\{\left\|X_{\lfloor s\rfloor}\right\|_{\mathrm{W}} \mid s \in \partial T \wedge s \supset t\}$$

and let

$$U = \{t \in T \mid \varphi(t) \geq \|A\|_{\mathrm{W}}\}\,.$$

Since $s \subseteq t \Rightarrow \varphi(s) \geq \varphi(t)$, then U is a (possibly empty) subtree of T. (Notice that the Wadge ranks mentioned above are defined, since the sets involved are reducible to $A \cdot n$ and $\mathsf{SLO}^{\mathrm{W}}_{\leq A\cdot n}$ holds.)

Claim 17.8.1. *Suppose $x_0 \in [U]$. Then $x_0 \in X \Rightarrow A + \mathbb{R} \leq_{\mathrm{W}} X$ and $x_0 \notin X \Rightarrow A + \emptyset \leq_{\mathrm{W}} X$.*

Proof of the Claim. Suppose $x_0 \in X$. Then **II** wins $G_{\mathrm{W}}(A+\mathbb{R}, X)$ as follows

> As long as **I** does not play 0, then **II** follows x_0. If, at round n Player **I** first plays 0, then **II** passes until **I** reaches a position $q = (t+1)^\frown 0^\frown p$ such that $(A+\mathbb{R})_{\lfloor q\rfloor} = A_{\lfloor p\rfloor} <_{\mathrm{W}} A$ (and this is bound to happen, since A is self-dual). Since $[T]$ has empty interior, **II** can reach a position $s \supset x_0 \restriction n$, with $s \in \partial T$, such that $(A+\mathbb{R})_{\lfloor q\rfloor} \leq_{\mathrm{W}} X_{\lfloor s\rfloor}$ and then follows this reduction.

The proof when $x_0 \notin X$ is completely analogous. □

Therefore we may assume that

$$U \text{ is well-founded.} \tag{23}$$

Case 1: $\|A\|_{\mathrm{W}}$ is an additively indecomposable ordinal.

Claim 17.8.2. $\forall u \in \partial U \cap T \left(X_{\lfloor u\rfloor} \leq_{\mathrm{W}} A \cdot n\right)$.

Proof of the Claim. Let $u \in \partial U \cap T$ and let τ be **II**'s winning strategy for $G_{\mathrm{W}}(X_{\lfloor u \rfloor} \cap [T_{\lfloor u \rfloor}], A \cdot n)$, which exists by (21) and by $X_{\lfloor u \rfloor} \cap [T_{\lfloor u \rfloor}] = (X \cap [T])_{\lfloor u \rfloor} \leq_{\mathrm{W}} X \cap [T]$. Let C be such that $\varphi(u) < \|C\|_{\mathrm{W}} < \|A\|_{\mathrm{W}}$ —such a set exists since $\|A\|_{\mathrm{W}}$ is limit. By Lemma 16.7, $C + A \cdot n \equiv_{\mathrm{W}} A \cdot n$, so it is enough to show that **II** wins $G_{\mathrm{W}}(X_{\lfloor u \rfloor}, C + A \cdot n)$:

> **II** follows the reduction τ and adds 1, as long as **I** is in T; if **I** ever reaches a position $s \in \partial T$, then **II** plays 0 and follows a reduction witnessing $X_{\lfloor s \rfloor} \leq_{\mathrm{W}} C$, which exists, since $\left\|X_{\lfloor s \rfloor}\right\|_{\mathrm{W}} \leq \varphi(u) < \|C\|_{\mathrm{W}}$.

□

The Claim yields that $X \leq_{\mathrm{W}} A \cdot n$, contradicting our assumption. In fact **II** wins $G_{\mathrm{W}}(X, A \cdot n)$ as follows:

> **II** passes as long as **I** plays in U. Suppose at round n **I** reaches a position $u \in \partial U$. If $u \in T$ then $X_{\lfloor u \rfloor} \leq_{\mathrm{W}} A \cdot n$ by the Claim. If $u \in \partial T$, then $X_{\lfloor u \rfloor} \leq_{\mathrm{W}} A \cdot n$ by (21). In either case **II** has a winning strategy.

Case 2: $A \equiv_{\mathrm{W}} B \oplus \neg B$ with B non-self-dual, $B \equiv_{\mathrm{W}} B + B$ and $\|B\|_{\mathrm{W}}$ is additively indecomposable.

Let

$$T_i = \{t \in T \mid \exists s \supset t \left(s \in \partial T \wedge B_i \leq_{\mathrm{W}} X_{\lfloor s \rfloor}\right)\},$$

where, for notational simplicity, $B_0 = B$ and $B_1 = \neg B$. Then T_0 and T_1 are (possibly empty) subtrees of T. Notice that our assumption implies that $B_i + B_i \equiv_{\mathrm{W}} B_i$, for $i = 0, 1$, hence by Lemma 16.5

$$B_i + B_{1-i} \equiv_{\mathrm{W}} A + B_{1-i}\,. \tag{24}$$

Suppose $s \in T \setminus (T_0 \cup T_1 \cup U)$. Then $\varphi(s) < \|B\|_{\mathrm{W}}$. Since $[B]_{\mathrm{W}}$ is limit and non-self-dual, then $\mathsf{SLO}^{\mathrm{W}}_{\leq A}$ implies that $\mathrm{cof}(\|B\|_{\mathrm{W}}) > \omega$, hence there is a set C_s such that

$$t \supset s \wedge t \in \partial T \Rightarrow X_{\lfloor t \rfloor} \leq_{\mathrm{W}} C_s\,. \tag{25}$$

Arguing as in Claim 17.8.1 we get

Claim 17.8.3. *Suppose* $x_0 \in [T_0] \cap [T_1]$. *Then* $x_0 \in X \Rightarrow A + \mathbb{R} \leq_{\mathrm{W}} X$ *and* $x_0 \in X \Rightarrow A + \emptyset \leq_{\mathrm{W}} X$.

Therefore we may assume that

$$T_0 \cap T_1 \text{ is pruned.} \tag{26}$$

Suppose τ is a winning strategy for **II** in $G_{\mathrm{W}}(B_{1-i}, X \cap [T_i])$. Then **II** wins $G_{\mathrm{W}}(B_i + B_{1-i}, X)$ by (subtracting 1 and) following τ, as long as **I** does not play 0; if at some round **I** plays 0, then, by the definition of T_i, **II** can reach a position $s \in \partial T$ such that $B_i \leq_{\mathrm{W}} X_{\lfloor s \rfloor}$ and then follow this reduction. Using (24) we get that

$$B_{1-i} \leq_{\mathrm{W}} X \cap [T_i] \Rightarrow A + B_{1-i} \leq_{\mathrm{W}} X .$$

Since $A + \mathbb{R}, A + \emptyset \leq_{\mathrm{W}} A + B_i$ by monotonicity, then the result follows from $B_{1-i} \leq_{\mathrm{W}} X \cap [T_i]$. Therefore we may assume that $B_{1-i} \nleq_{\mathrm{W}} X \cap [T_i]$. Keeping in mind that $B_{1-i}, X \cap [T_i] \leq_{\mathrm{W}} A \cdot n$, $\mathsf{SLO}^{\mathrm{W}}_{\leq A \cdot n}$ implies $X \cap [T_i] \leq_{\mathrm{W}} B_i$. Thus we may assume that

$$X \cap [T_i] \leq_{\mathrm{W}} B_i . \tag{27}$$

We are now going to show that $X \leq_{\mathrm{W}} A \cdot n$, contradicting our initial assumption. Here is a winning strategy for **II** in $G_{\mathrm{W}}(X, A \cdot n)$:

By (26) and (23) **II** passes until **I** reaches a position $s \notin (T_0 \cap T_1) \cup U$, and consider three cases:

Case A $s \notin T$. Then $s \in \partial T$ and therefore $X_{\lfloor s \rfloor} \leq_{\mathrm{W}} A \cdot n$ by (22).

Case B $s \in T \setminus (T_0 \cup T_1)$. Let C_s be as in (25). Then **II** reduces $X_{\lfloor s \rfloor}$ to $C_s + A \cdot n$ by following the reduction $X_{\lfloor s \rfloor} \cap [T] \leq_{\mathrm{W}} A \cdot n$ by (21) as long as **I** plays in T, and then using (25) as soon as **I** reaches ∂T. Since $\|C_s\|_{\mathrm{W}} < \|B\|_{\mathrm{W}}$ and $\|B\|_{\mathrm{W}}$ is additively indecomposable, then $C_s + A \cdot n \equiv_{\mathrm{W}} A \cdot n$, and therefore we are done.

Case C $s \in T_i \setminus T_{1-i}$, for some $i \in \{0, 1\}$. If T_i is wellfounded, then **II** passes until **I** reaches a position $t \in \partial T_i$ and we follow Case A or B depending wether $t \notin T$ or $t \in T$. So we may assume $[T_i] \neq \emptyset$. We will show that $X \leq_{\mathrm{W}} B_i + B_i \equiv_{\mathrm{W}} B_i$, and since $B_i \leq_{\mathrm{W}} A \leq_{\mathrm{W}} A \cdot n$, we are done. The winning strategy for **II** in $G_{\mathrm{W}}(X, B_i + B_i)$ is as follows: as long as **I**'s position is in T_i, use that $X \cap [T_i] \leq B_i$ by (27), if **I** reaches $t \in \partial T_i$, then **II** plays 0 and follows the reduction $X_{\lfloor t \rfloor} <_{\mathrm{W}} B_i$.

This conculdes the proof of Lemma 17.8. □

By Lemmata 17.6 and 17.7 we get

Corollary 17.9. *Suppose A is self-dual and $\mathsf{SLO}^{\mathrm{W}}_{\leq A}$ holds. Suppose also that either $\|A\|_{\mathrm{W}}$ is an additively indecomposable limit ordinal or else $A \equiv_{\mathrm{W}} B \oplus \neg B$ with B non-self-dual, $B + B \equiv_{\mathrm{W}} B$, and $\|B\|_{\mathrm{W}}$ additively indecomposable limit ordinal. Then, for every $n \geq 1$, $\mathsf{SLO}^{\mathrm{W}}_{\leq A \cdot n}$ holds.*

Finally, we prove the "propagation of $\mathsf{SLO}^{\mathrm{W}}$" result mentioned before.

Theorem 17.10. *Suppose A is self-dual and $\mathsf{SLO}^{\mathrm{W}}_{\leq A}$ holds. Then, for every $1 \leq \gamma < \omega_1$ there is a self-dual C such that $A \leq_{\mathrm{W}} C$, $\mathsf{SLO}^{\mathrm{W}}_{\leq C}$ holds, and $\|C\|_{\mathrm{W}} = \|A\|_{\mathrm{W}} \cdot \gamma$.*

Proof. Let us first prove this under the additional assumption ($\star$) that:

> either $\|A\|_{\mathrm{W}}$ is an additively indecomposable ordinal or else $A \equiv_{\mathrm{W}} B \oplus \neg B$ with B non-self-dual, $B + B \equiv_{\mathrm{W}} B$, and $\|B\|_{\mathrm{W}}$ additively indecomposable of uncountable cofinality.

The result clearly holds for $\gamma = 1$ by taking $C = A$, so assume $\gamma > 1$ and that the result holds for ordinals $< \gamma$.

If γ is limit, then choose an increasing sequence $\gamma_n \to \gamma$ and, by inductive hypothesis, choose C_n such that $\mathsf{SLO}^{\mathrm{W}}_{\leq C_n}$ holds and $\|C_n\|_{\mathrm{W}} = \|A\|_{\mathrm{W}} \cdot \gamma_n$. By Lemma **??**, $\bigoplus_n C_n$ witnesses the theorem for γ.

Suppose $\gamma = \gamma' + 1$. If $\gamma' \in \omega$, then we are done by Corollary 17.9, so we may assume that $\gamma' \geq \omega$. Let $\delta \leq \gamma'$ be the largest additively indecomposable ordinal. By inductive hypothesis, $\mathsf{SLO}^{\mathrm{W}}_{\leq D}$ holds and $\|D\|_{\mathrm{W}} = \|A\|_{\mathrm{W}} \cdot \delta$, for some self-dual set D, since $\mathrm{cof}(\delta) = \omega$. Let n be least such that $\delta \cdot n \geq \gamma$ and let $\bar{D} = D \bullet n$: then by Corollary 17.9 $\mathsf{SLO}^{\mathrm{W}}_{\leq \bar{D}}$ holds and therefore there is a self-dual set $C \leq_{\mathrm{W}} \bar{D}$ such that $\mathsf{SLO}^{\mathrm{W}}_{\leq C}$ holds and $\|C\|_{\mathrm{W}} = \|A\|_{\mathrm{W}} \cdot \gamma$.

Now for the general case: Let $\gamma \leq \|A\|_{\mathrm{W}}$ be the largest additively indecomposable ordinal. Suppose that the set A' of Wadge rank γ is self-dual: then by ($\star$) $\mathsf{SLO}^{\mathrm{W}}_{\leq C}$ holds for every C such that $\|C\|_{\mathrm{W}} < \gamma \cdot \omega_1$, and since $\gamma \cdot \omega_1 = \|A\|_{\mathrm{W}} \cdot \omega_1$, we are done. Suppose instead that the set B of Wadge rank γ is non-self-dual, and let $A' = B \oplus \neg B$. By ($\star$) $\mathsf{SLO}^{\mathrm{W}}_{\leq C}$ holds for every C such that $\|C\|_{\mathrm{W}} < (\gamma + 1) \cdot \omega_1 = \|A\|_{\mathrm{W}} \cdot \omega_1$, hence we are done. □

18 Jumping ω_1-many Steps

We will now introduce two new operations on *sets*, $A^\natural$ and $A^\flat$ such that $([A^\natural]_{\mathrm{W}}, [A^\flat]_{\mathrm{W}})$ is the least non-self-dual pair above the $[A]_{\mathrm{W}} \cdot \alpha$'s, $(\alpha < \omega_1)$, when A is self-dual.

Definition 18.1. For $A \subseteq \mathbb{R}$

$$A^{\natural} = \{s^\frown 0^\frown (x+1) \mid s \in {}^{<\omega}\omega \wedge x \in A\} \cup A^{+}$$

and

$$A^{\flat} = A^{\natural} \cup \{x \in \mathbb{R} \mid \exists^{\infty} n \ x(n) = 0\}.$$

$A^{\natural}$ can be seen as some sort of infinite sum of copies of A, and since $A^{\natural} = A^{\natural} + A$, then it is natural to think of it as $A^{\natural} = \cdots + A + A$.

Exercise 18.2. Show that for $A, B \subseteq \mathbb{R}$, $s \in {}^{<\omega}\omega$, $x \in \mathbb{R}$

(i) $A \leq_{\mathrm{W}} A^{\natural}, A^{\flat}$;

(ii) $A \leq_{\mathrm{W}} B \Rightarrow A^{\natural} \leq_{\mathrm{W}} B^{\natural}$ and $A^{\flat} \leq_{\mathrm{W}} B^{\flat}$. Thus we can define the $\natural$ and $\flat$ operations on Wadge degrees by setting

$$[A]_{\mathrm{W}}^{\natural} = [A^{\natural}]_{\mathrm{W}} \qquad \text{and} \qquad [A]_{\mathrm{W}}^{\flat} = [A^{\flat}]_{\mathrm{W}}.$$

(iii) $x \in A^{\natural} \Leftrightarrow s^\frown 0^\frown x \in A^{\natural}$ and $x \in A^{\flat} \Leftrightarrow s^\frown 0^\frown x \in A^{\flat}$;

(iv) $A^{\natural}{}_{\lfloor s^\frown 0 \rfloor} = A^{\natural}$ and $A^{\flat}{}_{\lfloor s^\frown 0 \rfloor} = A^{\flat}$;

(v) $A^{\natural\natural} \equiv_{\mathrm{W}} A^{\natural}$ and $A^{\flat\flat} \equiv_{\mathrm{W}} A^{\flat}$;

Lemma 18.3. *For any $A \subseteq \mathbb{R}$ and $1 \leq \alpha < \omega_1$,*

$$[A]_{\mathrm{W}} \cdot \alpha \leq_{\mathrm{W}} [A^{\natural}]_{\mathrm{W}}, [A^{\flat}]_{\mathrm{W}}$$

Proof. By induction on α. □

Theorem 18.4. *Assume $\neg\mathsf{FS} + \mathsf{AD}^{\mathrm{L}}$. For A self-dual $([A^{\natural}]_{\mathrm{W}}, [A^{\flat}]_{\mathrm{W}})$ is the least non-self-dual pair above the $[A]_{\mathrm{W}} \cdot \alpha$'s, ($1 \leq \alpha < \omega_1$), and therefore, assuming $\mathsf{DC}(\mathbb{R})$,*

$$\left\|A^{\natural}\right\|_{\mathrm{W}} = \left\|A^{\flat}\right\|_{\mathrm{W}} = \|A\|_{\mathrm{W}} \cdot \omega_1.$$

Proof. We first show that $B \in \bigcup_{1 \leq \alpha < \omega_1} [A]_{\mathrm{W}} \cdot \alpha \Leftrightarrow B \leq_{\mathrm{W}} A^{\natural}, A^{\flat}$. One direction follows from 22.15(iii) so it is enough to prove that $B \leq_{\mathrm{W}} A^{\natural}, A^{\flat}$ implies that $B \in \bigcup_{1 \leq \alpha < \omega_1} [A]_{\mathrm{W}} \cdot \alpha$. We need the following

Lemma 18.5. *Assume $\neg\mathsf{FS} + \mathsf{AD}^{\mathrm{L}}$. For any $A, B \subseteq \mathbb{R}$, if $B \leq_{\mathrm{W}} A^{\natural}, A^{\flat}$ then there is a winning strategy τ for **II** in $G_{\mathrm{W}}(B, A^{\natural})$ such that any play for **II** according to τ belongs to $\mathbb{R}^{\natural} = \{x \in \mathbb{R} \mid \forall^{\infty} n \ x(n) \neq 0\}$.*

Proof. Let τ_0, τ_1 be **II**'s winning strategies for $G_{\mathrm{W}}(B, A^\natural)$ and $G_{\mathrm{W}}(B, A^\flat)$, respectively. The plan is to use alternatively τ_0 or τ_1, switching each time the strategy being used requires to play a 0. We dove-tail τ_0 and τ_1 as follows: for any $s \in {}^{<\omega}\omega$ let $u_0, \dots, u_n, v_0, \dots, v_m \in {}^{<\omega}\omega$ be such that

$$(s * \tau_0)_{\mathbf{II}} = (u_0 + 1)^\frown\langle 0\rangle^\frown \cdots {}^\frown\langle 0\rangle^\frown(u_n + 1)$$

and

$$(s * \tau_1)_{\mathbf{II}} = (v_0 + 1)^\frown\langle 0\rangle^\frown \cdots {}^\frown\langle 0\rangle^\frown(v_m + 1)$$

then let

$$(s*\tau)_{\mathbf{II}} = \big((u_0 + 1)^\frown\langle 0\rangle^\frown(v_0 + 1)^\frown\langle 0\rangle^\frown \cdots {}^\frown\langle 0\rangle^\frown(u_k + 1)^\frown\langle 0\rangle^\frown(v_k + 1)\big) \restriction \mathrm{lh}(s)$$

where $k = \min(n, m)$. It is easy to check that this defines a strategy for **II**.

Claim 18.5.1. $\forall x \in \mathbb{R}\,\big((x * \tau)_{\mathbf{II}} \in \mathbb{R}^\natural\big)$.

Proof. Deny. Let $x \in \mathbb{R}$ be such that $(x * \tau)_{\mathbf{II}} \notin \mathbb{R}^\natural$, that is

$$\exists^\infty n\ \tau_0(x \restriction n) = 0 \tag{1}$$

and

$$\exists^\infty n\ \tau_1(x \restriction n) = 0\,. \tag{2}$$

Then by (1) $(x * \tau_0)_{\mathbf{II}} \notin A^\natural$ hence $x \notin B$ and by (2) $(x * \tau_1)_{\mathbf{II}} \in A^\flat$ hence $x \in B$: a contradiction. □

Let us check that τ is winning for **II** in $G_{\mathrm{W}}(B, A^\natural)$. Suppose **I** plays x and let $y = (x * \tau)_{\mathbf{II}}$ be **II**'s response. Then either

(A) τ settles on τ_0, that is

$$y = (u_0{+}1)^\frown\langle 0\rangle^\frown(v_0{+}1)^\frown\langle 0\rangle^\frown \cdots {}^\frown \langle 0\rangle^\frown(u_n{+}1)^\frown\langle 0\rangle^\frown(v_n{+}1)^\frown\langle 0\rangle^\frown(z{+}1)$$

where $(u_0 + 1)^\frown\langle 0\rangle^\frown \cdots {}^\frown \langle 0\rangle^\frown(u_n + 1)^\frown\langle 0\rangle^\frown(z + 1) = (x * \tau_0)_{\mathbf{II}}$, or else

(B) τ settles on τ_1, that is

$$y = (u_0 + 1)^\frown\langle 0\rangle^\frown(v_0 + 1)^\frown\langle 0\rangle^\frown \cdots {}^\frown \langle 0\rangle^\frown(u_n + 1)^\frown\langle 0\rangle^\frown(w + 1)$$

where $(v_0 + 1)^\frown\langle 0\rangle^\frown \cdots {}^\frown \langle 0\rangle^\frown(v_{n-1} + 1)^\frown\langle 0\rangle^\frown(w + 1) = (x * \tau_1)_{\mathbf{II}}$.

Assume (A) holds, i.e., τ settles on τ_0. Then

$$\begin{aligned} x \in B &\Leftrightarrow (x * \tau_0)_{\mathbf{II}} \in A^{\natural} && \text{(by definition of } \tau_0) \\ &\Leftrightarrow z \in A^{\natural} && \text{(by 22.15(i))} \\ &\Leftrightarrow y \in A^{\natural} && \text{(by 22.15(i) again)} \end{aligned}$$

Case (B) when τ settles on τ_1 is completely analogous. □

Let's go back to the proof of 22.16. For any $s \in {}^{<\omega}\omega$ let

$$n(s) = \begin{cases} \text{the largest } i < \mathrm{lh}(s) \text{ such that } s(i) = 0, \\ 0, \text{ if } \forall i < \mathrm{lh}(s)\, (s(i) \neq 0)\,, \end{cases}$$

and let $\prec$ be the strict partial order on ${}^{<\omega}\omega$ defined by

$$s \prec t \Leftrightarrow s \restriction n(s) \subset t \restriction n(t)\,.$$

Fix a strategy τ as in the Lemma and set

$$s \lhd t \Leftrightarrow (s * \tau)_{\mathbf{II}} \prec (t * \tau)_{\mathbf{II}}\,.$$

Then $\lhd$ is a strict partial order on ${}^{<\omega}\omega$, and it is well-founded, since otherwise there would be an $x \in \mathbb{R}$ such that $y \notin \mathbb{R}^{\natural}$, where y is **II**'s response via τ to **I** playing x. We will prove by induction on α that $B \in [A]_{\mathrm{W}} \cdot (\alpha + 1)$, where $\alpha = \|\emptyset\|_{\lhd}$.

Suppose $\alpha = 0$. This corresponds to the case where $\forall s \in {}^{<\omega}\omega\, (\tau(s)$ does not contain any 0 Then **II** wins $G_{\mathrm{W}}(B, A)$ by playing $\tau(s) - 1$, for any s played by **I**: this is indeed a winning strategy since if x and y are the reals played this way by **I** and **II** then

$$x \in B \Leftrightarrow y + 1 \in A^{\natural} \Leftrightarrow y \in A\,.$$

This proves that $B \leq_{\mathrm{W}} A$.

Suppose $\alpha > 1$ and let $C \in [A]_{\mathrm{W}} \cdot \alpha$. Consider the following strategy for **II** in the game $G_{\mathrm{W}}(B, C + A)$:

> As long as **I** stays in a position p such that $\|p\|_{\lhd} = \alpha$ then **II** plays $\tau(p)$. Suppose at some round **I** reaches a position p such that $\|p\|_{\lhd} < \alpha$. Then **II** answers 0 and from now on, to any further move he answers $\tau_p(p^\frown q) + 1$, where τ_p is a strategy witnessing $B_{\lfloor p \rfloor} \leq_{\mathrm{W}} C$. Such a τ_p exists by inductive hypothesis.

Let us check that this strategy is indeed winning for **II**. Let x and y be the reals produced by a complete play. If $\|x \restriction n\|_{\lhd} < \alpha$ for some n, then let x' and

y' be such that $x \restriction n^\frown x' = x$ and $((x \restriction n) * \tau)^\frown\langle 0\rangle^\frown(y'+1) = y$. (Note that $\mathrm{lh}((x \restriction n) * \tau) < n$ is possible since τ may pass.) Then $x' \in B_{\lfloor p \rfloor} \Leftrightarrow y' \in C$ since τ_p witnesses $B_{\lfloor p \rfloor} \leq_{\mathrm{W}} C$ and therefore $x \in B \Leftrightarrow y \in C + A$. If instead $\forall n\,(\|x \restriction n\|_{\lhd} = \alpha)$ then $\forall n\,(\tau(x \restriction n) \neq 0)$ so $y \in \mathbb{R}^\natural$ hence

$$x \in B \Leftrightarrow y \in A^\natural \Leftrightarrow y - 1 \in A \Leftrightarrow y \in C + A\,.$$

Therefore $[B]_{\mathrm{W}} \leq [A]_{\mathrm{W}} \cdot (\alpha + 1)$.

Finally, let us show that $A^\natural \not\equiv_{\mathrm{W}} A^\flat$. Otherwise if $A^\natural \leq_{\mathrm{W}} A^\flat$ then by the argument above, since $A^\natural \leq_{\mathrm{W}} A^\natural$, we should have that $[A^\natural]_{\mathrm{W}} \leq_{\mathrm{W}} [A]_{\mathrm{W}} \cdot \alpha$ for some $\alpha < \omega_1$. Then

$$\begin{aligned} [A]_{\mathrm{W}} \cdot \alpha &<_{\mathrm{W}} [A]_{\mathrm{W}} \cdot (\alpha + 1) && \text{by 22.8(i)} \\ &\leq_{\mathrm{W}} [A^\natural]_{\mathrm{W}} && \text{by 22.15(iii)} \end{aligned}$$

hence $[A^\natural]_{\mathrm{W}} <_{\mathrm{W}} [A^\natural]_{\mathrm{W}}$: a contradiction. □

Corollary 18.6. *Assume* $\neg\mathsf{FS} + \mathsf{DC}(\mathbb{R}) + \mathsf{AD}^{\mathrm{L}}$. *Let A be self-dual. For every B such that* $\|A\|_{\mathrm{W}} < \|B\|_{\mathrm{W}} < \|A\|_{\mathrm{W}} \cdot \omega_1$,

$$[B^\natural]_{\mathrm{W}} = [A^\natural]_{\mathrm{W}} \qquad \text{and} \qquad [B^\flat]_{\mathrm{W}} = [A^\flat]_{\mathrm{W}}\,.$$

Proof. Let α be least such that $\|B\|_{\mathrm{W}} \leq \|A\|_{\mathrm{W}} \cdot \alpha$ and let $C \in [A]_{\mathrm{W}} \cdot \alpha$. By monotonicity (22.14) $A^\natural \leq_{\mathrm{W}} B^\natural \leq_{\mathrm{W}} C^\natural$. The argument for $A^\flat$ is similar. □

Exercise 18.7. (i) Show that

$$\mathbb{R}^\natural = \{x \in \mathbb{R} \mid \forall^\infty n\; x(n) \neq 0\} \in \mathbf{\Sigma}^0_2 \setminus \mathbf{\Delta}^0_2$$

and that

$$\emptyset^\flat = \{x \in \mathbb{R} \mid \exists^\infty n\; x(n) = 0\} \in \mathbf{\Pi}^0_2 \setminus \mathbf{\Delta}^0_2\,.$$

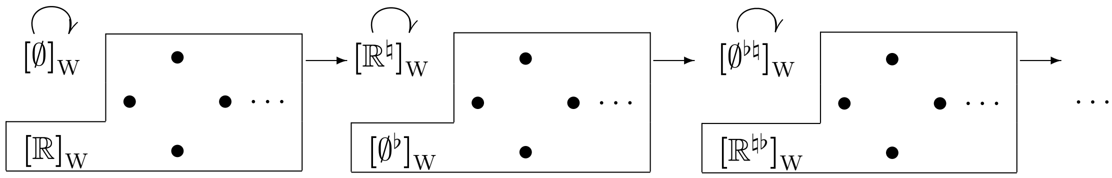
(ii) Show that $\mathbb{R}^\natural = (\mathbb{R} \oplus \emptyset)^\natural$ and $\emptyset^\flat = (\mathbb{R} \oplus \emptyset)^\flat$, $\emptyset^\natural = \emptyset$, $\mathbb{R}^\flat = \mathbb{R}$.

(iii) ($\mathsf{DC}(\mathbb{R})$) Show that $\mathbf{\Sigma}^0_2 \setminus \mathbf{\Delta}^0_2$ and $\mathbf{\Pi}^0_2 \setminus \mathbf{\Delta}^0_2$ occupy the ω_1-th level of the Wadge hierarchy.

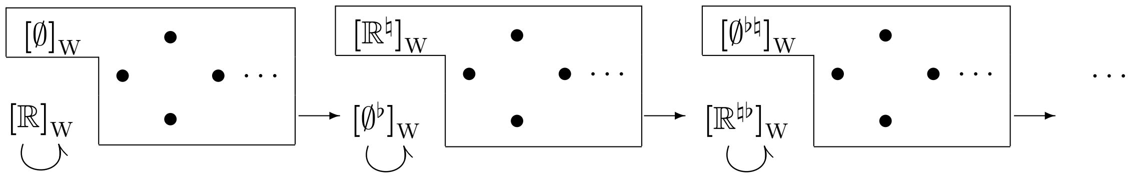
(iv) Show that $\mathbb{R}^\natural + \mathbb{R}^\natural \equiv_{\mathrm{W}} \mathbb{R}^\natural$ and therefore $\forall B \leq_{\mathrm{W}} \mathbb{R}^\natural \left(\mathbb{R}^\natural + B \equiv_{\mathrm{W}} \mathbb{R}^\natural\right)$.

(v) Show that $([\mathbb{R}^{\natural\flat}]_{\mathrm{W}}, [\emptyset^{\flat\natural}]_{\mathrm{W}})$, $([\mathbb{R}^{\natural\flat\natural}]_{\mathrm{W}}, [\emptyset^{\flat\natural\flat}]_{\mathrm{W}})$,... is an increasing sequence of non-self-dual pairs of degrees whose Wadge rank (here we assume $\mathsf{DC}(\mathbb{R})$ so that we can talk about Wadge ranks) are ω_1^2, ω_1^3,....

Therefore the map $[A]_{\mathrm{W}} \mapsto [A]_{\mathrm{W}}^{\natural}$ acts like this



and the map $[A]_{\mathrm{W}} \mapsto [A]_{\mathrm{W}}^{\flat}$ acts like this



Chapter III

Borel reducibilities

In order to prove more results on the $\mathcal{F}$-degrees we must impose further restrictions on the set $\mathcal{F}$.

Definition 18.8. $\mathcal{F} \subseteq {}^{\mathbb{R}}\mathbb{R}$ is *amenable* iff either $\mathcal{F} = \mathrm{Lip}$, or else:

(1) there is a surjection $\mathbb{R} \twoheadrightarrow \mathcal{F}$,

(2) $\mathcal{F} \supseteq \mathrm{Lip}$,

(3) $\mathcal{F}$ is closed under composition,

(4) if each $f_n \in \mathcal{F}$ then $\bigoplus_n f_n \in \mathcal{F}$, where

$$\bigoplus\nolimits_n f_n\,(x) = f_{x(0)}(x^-)\,,$$

and $x^- = \langle x(n+1) \mid n \in \omega \rangle$.

Typical examples of amenable $\mathcal{F}$ are the collections of all Lipschitz functions, all continuous functions, and all Borel functions. The "$\mathcal{F}$" in $\leq_{\mathcal{F}}$, $[A]_{\mathcal{F}}$, $\mathsf{SLO}^{\mathcal{F}}$ etc., will be replaced by "L" in the Lipschitz case, by "W" in the continuous case (after Wadge), and by "**B**" in the Borel case. Notice that Lip satisfies (1), (2), and (3), but not (4).

Lemma 18.9. *Let $\mathcal{F} \neq \mathrm{Lip}$ be amenable and let $A \subseteq \mathbb{R}$.*

(a) *$A \oplus \neg A$ is $\mathcal{F}$-self-dual and $A, \neg A \leq_{\mathcal{F}} A \oplus \neg A$. Moreover if $A, \neg A \leq_{\mathcal{F}} C$, then $A \oplus \neg A \leq_{\mathcal{F}} C$. In particular $[A \oplus \neg A]_{\mathcal{F}}$ is the $\leq_{\mathcal{F}}$-least degree above $[A]_{\mathcal{F}}$ and $[\neg A]_{\mathcal{F}}$.*

(b) *Assume $\mathsf{SLO}^{\mathcal{F}}$ and suppose $[A]_{\mathcal{F}}$ is limit of countable cofinality. Then $[A]_{\mathcal{F}}$ is self-dual.*

Proof. (a) The first part is trivial by (2) of Definition 18.8. For the second part, notice that if $f^{-1}``C = A$ and $g^{-1}``C = \neg A$, then $\bigoplus_n f_n$ witnesses $A \oplus \neg A \leq_{\mathcal{F}} C$, where $f_{2n} = f$ and $f_{2n+1} = g$.

(b) Let $A_0 <_{\mathcal{F}} A_1 <_{\mathcal{F}} \ldots$ witness that $[A]_{\mathcal{F}}$ is limit of countable cofinality. Since $A_{n+1} \not\leq_{\mathcal{F}} A_n$ then there is $f'_n \in \mathcal{F}$ witnessing $\neg A_n \leq_{\mathcal{F}} A_{n+1}$ by $\mathsf{SLO}^{\mathcal{F}}$. The functions $f_n(x) = \langle n+1 \rangle ^\frown f'_n(x)$ belong to $\mathcal{F}$ by (2) and (3) in Definition 18.8. Therefore $\bigoplus_n f_n$ witnesses that $\bigoplus_n \neg A_n \leq_{\mathcal{F}} \bigoplus_n A_n$, that is $\bigoplus_n A_n$ is $\mathcal{F}$-self-dual. Clearly (2) implies that $A_i \leq_{\mathcal{F}} \bigoplus_n A_n$ for each i, and if g_n witnesses $A_n \leq_{\mathcal{F}} C$ then $\bigoplus_n g_n$ witnesses $\bigoplus_n A_n \leq_{\mathcal{F}} C$. In other words $\bigoplus_n A_n$ is a least upper bound of the A_n's. Therefore $\bigoplus_n A_n \equiv_{\mathcal{F}} A$. □

The Lemma is still true if $\mathcal{F} = \mathrm{Lip}$ (and hence it is true for all amenable $\mathcal{F}$) but the argument is more involved and $\mathsf{SLO}^{\mathrm{L}}$ must be assumed also for case (a)—see [**?**].

The Lipschitz game on $A, B \subseteq \mathbb{R}$, $G_{\mathrm{L}}(A, B)$, introduced by Wadge in [Wad83] is the game on ω where **I** plays a real a, **II** plays a real b, and **II** wins iff $a \in A \Leftrightarrow b \in B$. Wadge's Lemma is the simple, but fundamental observation that a winning strategy for **II** yields a Lipschitz map witnessing $A \leq_{\mathrm{L}} B$, while a winning strategy for **I** yields a Lipschitz map (in fact: a contraction) witnessing $\neg B \leq_{\mathrm{L}} A$. Therefore AD implies $\mathsf{SLO}^{\mathrm{L}}$, and since the smaller the $\mathcal{F}$ the stronger the $\mathsf{SLO}^{\mathcal{F}}$,

$$\mathsf{AD} \Rightarrow \mathsf{SLO}^{\mathrm{L}} \Rightarrow \mathsf{SLO}^{\mathrm{W}} \Rightarrow \mathsf{SLO}^{\mathbf{B}} .$$

We do not know whether any of these implications can be reversed—see [**?**] for more on this. In fact, a well-known open problem (probably first formulated by R. Solovay) asks whether $\mathsf{SLO}^{\mathrm{L}}$ or even $\mathsf{SLO}^{\mathrm{W}}$ implies AD, assuming $\mathrm{V} = \mathrm{L}(\mathbb{R})$. A similar question can be asked for $\mathsf{SLO}^{\mathbf{B}}$ or, more boldly, for $\mathsf{SLO}^{\mathcal{F}}$:

Open problem 18.10. *Assume* $\mathrm{V} = \mathrm{L}(\mathbb{R})$*. Does* $\mathsf{SLO}^{\mathbf{B}} \Rightarrow \mathsf{AD}$*? Does* $\mathsf{SLO}^{\mathcal{F}} \Rightarrow \mathsf{AD}$*, if* $\mathcal{F}$ *is amenable?*

A less ambitious goal would be to prove some of the standard consequences of AD (like BP and LM, the assertion that all sets of reals are Lebesgue measurable) from some form of semi-linear ordering principle. This would yield some evidence for positive solutions to these open problems. For example it is known that the perfect set property [Wad83] and the axiom of countable choices for sets of reals [**?**] follow from $\mathsf{SLO}^{\mathrm{W}}$, but the following seems to be open:

Open problem 18.11. *Assume* $\mathrm{V} = \mathrm{L}(\mathbb{R})$ *and let* $\mathcal{F}$ *be amenable. Does* $\mathsf{SLO}^{\mathcal{F}} \Rightarrow \mathsf{BP}$*? Does* $\mathsf{SLO}^{\mathcal{F}} \Rightarrow \mathsf{LM}$*?*

This is open even when $\mathcal{F}$ is the smallest amenable set of functions (and hence $\mathsf{SLO}^{\mathcal{F}}$ is strongest semi-linear ordering principle), that is when $\mathcal{F} =$ Lip, the collection of all Lipschitz functions.

Another partial evidence for the truth of the Open Problem 18.10 would be to prove the equivalence between the various semi-linear ordering principles, say between $\mathsf{SLO}^{\mathrm{L}}$, $\mathsf{SLO}^{\mathrm{W}}$, and $\mathsf{SLO}^{\mathbf{B}}$—again see [**?**].

18.A The Wadge and Lipschitz hierarchies

Assuming $\mathsf{AD} + \mathsf{DC}(\mathbb{R})$ then the following properties hold of the Wadge degrees:

(1) $\leq_{\mathrm{W}}$ is well-founded,

(2) immediately above a self-dual degree there is a non-self-dual pair of degrees, and immediately above a non-self-dual pair of degrees there is a self-dual degree,

(3) at limit levels of countable cofinality there is a single self-dual degree, and at uncountable cofinality there is a non-self-dual pair,

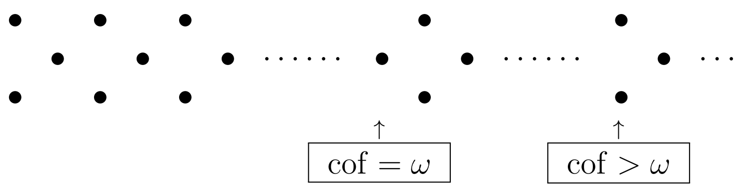
while for the Lipschitz degrees we have the following:

(4) $\leq_{\mathrm{L}}$ is well-founded,

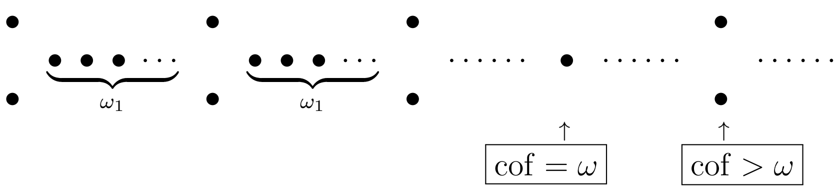
(5) every self-dual Wadge degree is the union of ω_1 consecutive Lipschitz self-dual degrees, while the non-self-dual pairs of Wadge degrees coincide exactly with the non-self-dual pairs of Lipschitz degrees,

(6) at limit levels of countable cofinality there is a single self-dual degree, and at uncountable cofinality there is a non-self-dual pair.

Therefore the Wadge hierarchy looks like this:



and the Lipschitz hierarchy looks like this:



with each ω_1-block of self-dual Lipschitz degrees collapsing to a single self-dual Wadge degree. In [?] it is shown that (1)–(3) follow from $\mathsf{SLO}^{\mathrm{W}} + \mathsf{BP}$, and that (4)–(6) follow from $\mathsf{SLO}^{\mathrm{L}} + \mathsf{BP}$. Therefore, if $\mathcal{F}$ is amenable—so that the $\mathcal{F}$ degrees are coarser than (or equal to) the Lipschitz degrees—and if AD (or even just $\mathsf{SLO}^{\mathrm{L}} + \mathsf{BP}$) is assumed, then $\leq_{\mathcal{F}}$ is well-founded and every non-self-dual $\mathcal{F}$-degree is a non-self-dual Lipschitz degree, i.e., $A \not\equiv_{\mathcal{F}} \neg A \Rightarrow [A]_{\mathcal{F}} = [A]_{\mathrm{L}}$.

19 The Borel-Wadge hierarchy

We now focus on Borel-Wadge degrees. Our first goal is to prove that the well-foundedness of $\leq_{\mathbf{B}}$ follows from $\mathsf{SLO}^{\mathbf{B}} + \mathsf{BP}$. The standard proof of the non-existence of an infinite $<_{\mathrm{L}}$-descending sequence $\langle A_n \mid n \in \omega \rangle$ uses AD to pick winning strategies for $\mathbf{I}$ in $G_{\mathrm{L}}(A_n, A_{n+1})$ and in $G_{\mathrm{L}}(A_n, \neg A_{n+1})$. By pitting them against each other, a flip-set is constructed, contradicting BP. If we start from an infinite $<_{\mathbf{B}}$-descending sequence $\langle A_n \mid n \in \omega \rangle$ we would like to argue, assuming $\mathsf{SLO}^{\mathbf{B}}$, that $\mathbf{I}$ wins $G_{\mathrm{L}}(A_n, A_{n+1})$ and $G_{\mathrm{L}}(A_n, \neg A_{n+1})$ and proceed as before. In order to do this we need a few preliminary results.

A topological space is 0-dimensional if its topology is generated by the clopen sets. A metric space (X, d) is Polish if it is separable and d is complete. The collection of Borel subsets of (X, d) is denoted by $\mathrm{B}(X, d)$.

Lemma 19.1. *Suppose (X, d) is a Polish space and $\langle A_n \mid n \in \omega \rangle$ is a sequence of Borel subsets of (X, d). Then there is metric d' on X such that*

(1) (X, d') is Polish and 0-dimensional;

(2) the new topology is finer than the old one, i.e., every d-open set is also d'-open;

(3) each A_n is d'-clopen;

(4) the two topologies give rise to the same Borel sets, that is $\mathrm{B}(X, d) = \mathrm{B}(X, d')$.

See [Kec95, Theorem 13.1 and Exercise 13.5] for a proof. An easy consequence of this is the following result—see [Kec95, Theorem 13.11].

Lemma 19.2. *Let (X, d) be a Polish space, let $B \in \mathrm{B}(X, d)$, and let $f : B \to B$ be a Borel function. There is a metric d' on B such that*

(1) (B, d') is Polish and 0-dimensional;

(2) *the topology τ' generated by d' on B refines the topology that B inherits from X, i.e., $\tau' \supseteq \{U \cap B \mid U \in \tau\}$, where τ is the topology on X;*

(3) *(B, d') has the same Borel structure as B, that is: for every $C \subseteq B$,*

$$C \in \mathrm{B}(X, d) \Leftrightarrow C \in \mathrm{B}(B, d')\,;$$

(4) *$f : (B, d') \to (B, d)$ is continuous.*

By [Kec95, Theorem 7.8] every 0-dimensional Polish space is homeomorphic to a closed subset of the Baire space, so using Lemmata 19.1 and 19.2 will not take us outside of $\mathscr{P}(\mathbb{R})$.

Lemma 19.3. (a) *If $A \leq_{\mathbf{B}} B$ then there is $A^* \equiv_{\mathbf{B}} A$ such that $A^* \leq_{\mathrm{L}} A$ and $A^* \leq_{\mathrm{L}} B$.*

(b) *Assume $\mathsf{SLO}^{\mathbf{B}}$ and $A <_{\mathbf{B}} B$. Then there is $A^* \equiv_{\mathbf{B}} A$ such that **I** has a winning strategy in $G_{\mathrm{L}}(\neg B, A^*)$ and in $G_{\mathrm{L}}(B, A^*)$.*

Proof. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be Borel such that $f^{-1}``B = A$. By Lemma 19.2 there is a 0-dimensional Polish topology τ on $\mathbb{R}$ that is finer than the standard one and makes f continuous. Let $G : C \to (\mathbb{R}, \tau)$ be a homeomorphism with $C \subseteq \mathbb{R}$ a closed set, and by [Kec95, Proposition 2.8] let $\pi : \mathbb{R} \twoheadrightarrow C$ be Lipschitz and such that $\pi \restriction C$ is the identity. Let

$$A' = (G \circ \pi)^{-1}\, ``A\,.$$

Then $A' \leq_{\mathrm{W}} A$ via $G \circ \pi$, and $A \leq_{\mathbf{B}} A'$ via $G^{-1} : \mathbb{R} \to C \subseteq \mathbb{R}$. Since $f \circ G \circ \pi : \mathbb{R} \to \mathbb{R}$ is continuous and

$$x \in A' \Leftrightarrow G(\pi(x)) \in A \Leftrightarrow f(G(\pi(x))) \in B$$

then $A' \leq_{\mathrm{W}} B$. We need the following result from [**?**, Lemma 19, part (a)]:

Lemma 19.4. *If $A' \leq_{\mathrm{W}} A$ then there is $A'' \equiv_{\mathrm{W}} A'$ such that $A'' \leq_{\mathrm{L}} A'$ and $A'' \leq_{\mathrm{L}} A$.*

Let A'' be as in Lemma 19.4. Since $A'' \leq_{\mathrm{L}} A' \leq_{\mathrm{W}} B$, then $A'' \leq_{\mathrm{W}} B$, so by Lemma 19.4 again there is A^* such that $A^* \leq_{\mathrm{L}} A''$, $A'' \leq_{\mathrm{W}} A^*$, and $A^* \leq_{\mathrm{L}} B$, which is what we had to prove.

(b) If $\forall n \left(A \not\leq_{\mathbf{B}} B_{\lfloor n \rfloor}\right)$ or $\forall n \left(A \not\leq_{\mathbf{B}} \neg B_{\lfloor n \rfloor}\right)$, then by $\mathsf{SLO}^{\mathbf{B}}$ we would have that $\forall n \left(B_{\lfloor n \rfloor} \leq_{\mathbf{B}} \neg A\right)$ or $\forall n \left(B_{\lfloor n \rfloor} \leq_{\mathbf{B}} A\right)$, hence $B \leq_{\mathbf{B}} \neg A$ or $B \leq_{\mathbf{B}} A$, contradicting our assumption in either case. Therefore there are $n_0, m_0 \in \omega$ such that $A \leq_{\mathbf{B}} B_{\lfloor n_0 \rfloor}$ and $A \leq_{\mathbf{B}} \neg B_{\lfloor m_0 \rfloor}$ via Borel functions f and g. By successive applications of Lemma 19.2 there is a 0-dimensional Polish topology on $\mathbb{R}$ that is finer than the standard one and makes f and g continuous. Arguing as in part (a) there is an $A' \leq_{\mathrm{L}} A$ such that $A \leq_{\mathbf{B}} A'$ and $A' \leq_{\mathrm{L}} B_{\lfloor n_0 \rfloor}$, and since $A' \leq_{\mathbf{B}} \neg B_{\lfloor m_0 \rfloor}$, there is $A^* \leq_{\mathrm{L}} A'$ such that $A' \leq_{\mathbf{B}} A^*$ and $A^* \leq_{\mathrm{L}} \neg B_{\lfloor m_0 \rfloor}$. By playing m_0 and then following the reduction witnessing $A^* \leq_{\mathrm{L}} \neg B_{\lfloor m_0 \rfloor}$, **I** wins $G_{\mathrm{L}}(B, A^*)$; similarly **I** has a winning strategy in $G_{\mathrm{L}}(\neg B, A^*)$. □

Corollary 19.5. *Assume* $\mathsf{SLO}^{\mathbf{B}} + \mathsf{BP}$. *Then* $\leq_{\mathbf{B}}$ *is a well-founded relation on* $\mathscr{P}(\mathbb{R})$.

Proof. Suppose $\langle A_n \mid n \in \omega \rangle$ is a $<_{\mathbf{B}}$-descending sequence of sets. Then $A_{n+1} \leq_{\mathbf{B}} A_n$ and $A_n \not\leq_{\mathbf{B}} A_{n+1}$, and hence, by $\mathsf{SLO}^{\mathbf{B}}$, $A_{n+1} <_{\mathbf{B}} A_n$ and $\neg A_{n+1} <_{\mathbf{B}} A_n$. By Lemma 19.3 we can construct inductively $A_0^* = A_0$ and $A_n^* \equiv_{\mathrm{W}} A_n$ such that **I** has a winning strategy σ_n^1 in $G_{\mathrm{L}}(A_n^*, A_{n+1}^*)$ and σ_n^0 in $G_{\mathrm{L}}(\neg A_n^*, A_{n+1}^*)$. For any $z \in {}^{\omega}2$ let $x_n = x_n^z$ be the real in the n-th row of the following diagram where **I** uses $\sigma_n^{z(n)}$ on the n-th row against his opponent on the $(n+1)$-st row:

$$\begin{array}{cccccc} \sigma_0^{z(0)} & x_0(0) & x_0(1) & \cdots & = x_0 \\ \sigma_1^{z(1)} & x_1(0) & x_1(1) & \cdots & = x_1 \\ \vdots & \vdots & \vdots & & \vdots \end{array}$$

Thus x_n^z is the result of applying $\sigma_n^{z(n)}$ to x_{n+1}^z. Then $\{z \in {}^{\omega}2 \mid x_0^z \in A_0^*\}$ is a flip set, contradicting BP.

Lastly, to show that $<_{\mathbf{B}}$ is well-founded on $\mathscr{P}(\mathbb{R})$, it is enough to show that $<_{\mathbf{B}}$ is well-founded on $\{B \in \mathscr{P}(\mathbb{R}) \mid B \leq_{\mathbf{B}} A\}$, for any $A \subseteq \mathbb{R}$. So fix $A \subseteq \mathbb{R}$. Since there is a surjection $\mathbb{R} \twoheadrightarrow \{f \in {}^{\mathbb{R}}\mathbb{R} \mid f \text{ is Borel}\}$, $x \mapsto f_x$, consider the pre-order on $\mathbb{R}$ defined by

$$x \prec y \Leftrightarrow f_x^{-1} “A <_{\mathbf{B}} f_y^{-1} “A\,.$$

Then $<_{\mathbf{B}}$ is well-founded on $\{B \in \mathscr{P}(\mathbb{R}) \mid B \leq_{\mathbf{B}} A\}$ iff $\prec$ is well-founded on $\mathbb{R}$, which, by $\mathsf{DC}(\mathbb{R})$ is equivalent to the non-existence of an infinite $\prec$-descending sequence. But any $\prec$-descending sequence in $\mathbb{R}$ yields a $<_{\mathbf{B}}$-descending sequence in $\{B \in \mathscr{P}(\mathbb{R}) \mid B \leq_{\mathbf{B}} A\}$, hence we are done by the first part of the proof. □

Thus, assuming $\mathsf{SLO}^{\mathbf{B}} + \mathsf{BP}$, the canonical rank function for the well-founded relation $<_{\mathbf{B}}$ on $\mathscr{P}(\mathbb{R})$ can be defined. It is called the *Borel-Wadge rank* and it is denoted by $A \mapsto \|A\|_{\mathbf{B}}$. It is immediate that $[A]_{\mathbf{B}}$ is a limit degree iff $\|A\|_{\mathbf{B}}$ is a limit ordinal, and that $[A]_{\mathbf{B}}$ is of countable cofinality iff $\mathrm{cof}(\|A\|_{\mathbf{B}}) = \omega$. For technical reasons (see [**?**, Proposition 13]) it is convenient to assume that the Wadge rank $\|A\|_{\mathrm{W}}$ of a set is a non-zero ordinal, and hence, by analogy, we make the same assumption of the Borel-Wadge rank. Thus $\|\emptyset\|_{\mathbf{B}} = \|\mathbb{R}\|_{\mathbf{B}} = 1$.

The tree $\boldsymbol{T}(A) = \{s \in {}^{<\omega}\omega \mid A_{\lfloor s \rfloor} \equiv_{\mathrm{W}} A\}$ is a standard tool to investigate the structure of the Wadge degrees. For example $[A]_{\mathrm{W}}$, the Wadge degree of A, is self-dual iff $\boldsymbol{T}(A)$ is well-founded i.e., if the converse of the extension relation on $\boldsymbol{T}(A)$ is well-founded. Notice that if $\boldsymbol{T}(A)$ is well-founded, then

$$\{\boldsymbol{N}_s \mid s \notin \boldsymbol{T}(A) \ \&\ s \restriction \mathrm{lh}(s) - 1 \in \boldsymbol{T}(A)\}$$

is a partition of $\mathbb{R}$ into countably many clopen sets D such that $D \cap A <_{\mathrm{W}} A$. This suggests the correct generalization of $\boldsymbol{T}(A)$ to the Borel context.

Definition 19.6. Let $B \subseteq \mathbb{R}$. A *Borel partition* of B is a family $\{B_n \mid n < N\}$ of non-empty pairwise disjoint Borel sets such that $B = \bigcup_{n<N} B_n$ and $2 \leq N \leq \omega$.

First a trivial but useful fact:

Lemma 19.7. *Let $B \subseteq B'$ be Borel. If $A \cap B' \neq \mathbb{R}$, then $A \cap B \leq_{\mathbf{B}} A \cap B'$. In particular if B is Borel and $A \neq \mathbb{R}$, then $A \cap B \leq_{\mathbf{B}} A$.*

Then:

Lemma 19.8. *Let $\{B_n \mid n < N\}$ be a Borel partition of $\mathbb{R}$, and let $A \neq \mathbb{R}$.*

(a) $\forall n < N\,(A \cap B_n \leq_{\mathbf{B}} A)$, and if C is such that $\forall n < N\,(A \cap B_n \leq_{\mathbf{B}} C)$, then $A \leq_{\mathbf{B}} C$. In other words: $[A]_{\mathbf{B}}$ is the $\leq_{\mathbf{B}}$-least upper bound of $\{[B_n \cap A]_{\mathbf{B}} \mid n < N\}$.

(b) Assume $\mathsf{SLO}^{\mathbf{B}}$. If $\forall n < N\,(A \cap B_n <_{\mathbf{B}} A)$ then $A \leq_{\mathbf{B}} \neg A$. Moreover, if $N < \omega$ then $[A]_{\mathbf{B}}$ is a successor degree.

Proof. (a) The first part follows from Lemma 19.7. If g_n witnesses $B_n \cap A \leq_{\mathbf{B}} C$, then $g = \bigcup_n g_n \restriction B_n$ is Borel and witnesses $A \leq_{\mathbf{B}} C$.

(b) $A \cap B_n <_{\mathbf{B}} A$ implies, by $\mathsf{SLO}^{\mathbf{B}}$, that there are Borel functions $f_n : \mathbb{R} \to \mathbb{R}$ witnessing $A \cap B_n \leq_{\mathbf{B}} \neg A$. Then $f = \bigcup_n f_n \restriction B_n$ is Borel and $f^{-1}``\neg A = A$. Suppose now, towards a contradiction, that $[A]_{\mathbf{B}}$ is a limit degree and $N < \omega$. Let $C_0 = B_0 \cap A$ and, for $n + 1 < N$, let $C_{n+1} = C_n \oplus (B_{n+1} \cap A)$. By induction, using Lemma 18.9(a) and that $[A]_{\mathbf{B}}$ is limit, $\forall n < N\,(C_n <_{\mathbf{B}} A)$. But $B_n \cap A \leq_{\mathbf{B}} C_{N-1}$ for $n < N$, hence $A \leq_{\mathbf{B}} C_{N-1}$ by part (a): a contradiction. □

Definition 19.9. For $A \subseteq \mathbb{R}$, let

$$\mathcal{I}(A) = \{B \mid B \text{ is Borel and } \exists \langle B_n \mid n \in \omega\rangle \text{ Borel sets such that } B = \textstyle\bigcup_n B_n \text{ and } B_n \cap A <_{\mathbf{B}} A\}.$$

By Lemma 19.7, the sets B_n in the definition can be taken to form a partition of B, since $\mathcal{I}(A)$ is empty when $A = \mathbb{R}$ or $A = \emptyset$. The following result can be easily verified.

Lemma 19.10. *Assume* $\mathsf{SLO}^{\mathbf{B}}$*.*

(a) If $B \in \mathcal{I}(A)$ *and* $C \subseteq B$ *is Borel, then* $C \in \mathcal{I}(A)$*.*

(b) If $B_n \in \mathcal{I}(A)$*, then* $\bigcup_n B_n \in \mathcal{I}(A)$*.*

Recall that a *σ-ideal of Borel sets* is a non-empty collection $\mathcal{J}$ of Borel subsets of $\mathbb{R}$, closed under Borel subsets and countable unions. A σ-ideal of Borel sets $\mathcal{J}$ is *proper* if $\mathbb{R} \notin \mathcal{J}$. Then Lemma 19.10 says that $\mathcal{I}(A)$ is a σ-ideal of Borel sets, and Lemma 19.8(b) says that if $\mathcal{I}(A)$ is not proper, then $[A]_{\mathbf{B}}$ is self-dual.

Theorem 19.11. *Assume* BP *and suppose* $A \leq_{\mathbf{B}} \neg A$*. Then there is a Borel partition* $\{B_n \mid n \in \omega\}$ *of* $\mathbb{R}$ *such that* $\forall n < N\, (B_n \cap A <_{\mathbf{B}} A)$*.*

Proof. Towards a contradiction, suppose that for every Borel partition $\{B_n \mid n < N\}$ of $\mathbb{R}$ there is $n_0 < N$ such that $B_{n_0} \cap A <_{\mathbf{B}} A$. For ease of notation, let $\mathcal{I} = \mathcal{I}(A)$.

Claim 19.11.1. *If* B *is Borel and* $B \notin \mathcal{I}$*, then there is a Borel function* $f : B \to B$ *witnessing*

$$\forall x \in B\, (x \in A \cap B \Leftrightarrow f(x) \in \neg A \cap B)\,.$$

Proof of Claim. By case assumption $B \neq \emptyset$, and if $B = \mathbb{R}$ the result follows at once, so we may assume $B \neq \emptyset, \mathbb{R}$. By Lemma 19.7, $A \cap B \leq_{\mathbf{B}} A$ and $\neg A \cap B \leq_{\mathbf{B}} \neg A$. If $A \cap B <_{\mathbf{B}} A$, then, taking $B_n = B$ in Definition 19.9, we would have $B \in \mathcal{I}$: a contradiction. Therefore $\neg A \cap B \leq_{\mathbf{B}} \neg A \leq_{\mathbf{B}} A \equiv_{\mathbf{B}} A \cap B$. Let $h : \mathbb{R} \to \mathbb{R}$ be a Borel function witnessing that $\neg A \cap B \leq_{\mathbf{B}} A \cap B$, and let $k : \mathbb{R} \twoheadrightarrow B$ be defined as

$$k(x) = \begin{cases} x & \text{if } x \in B, \\ b & \text{otherwise,} \end{cases}$$

where b is some fixed element of $A \cap B$. Then $f = (k \circ h) \restriction B$ is the required function. □

We will construct a sequence of Borel sets

$$\mathbb{R} = B_0 \supseteq B_1 \supseteq \cdots$$

such that $B_n \notin \mathcal{I}$. Using the Claim, let $f_n : B_n \to B_n$ be Borel and such that

$$\forall x \in B_n \, (x \in A \cap B_n \Leftrightarrow f_n(x) \in \neg A \cap B_n) \, .$$

We will also choose a separable complete metric d_n on B_n such that d_0 is the usual metric on $\mathbb{R}$, and the topologies τ_n generated by the metrics d_n are all 0-dimensional and τ_{n+1} refines τ_n, that is, $\{U \cap B_{n+1} \mid U \in \tau_n\} \subseteq \tau_{n+1}$. We also require that $f_n \restriction B_{n+1} : (B_{n+1}, d_{n+1}) \to (B_n, d_n)$ be continuous, and that for any $m \leq n$ and every $a, b \in B_{n+1}$

$$d_m \left(g_m \circ \cdots \circ g_n(a), g_m \circ \cdots \circ g_n(b) \right) < 2^{-n} \, , \tag{1}$$

where each g_i is either $f_i \restriction B_{i+1}$ or the identity on B_{i+1}. Then we can apply the Martin-Monk method as follows:

Fix $z \in {}^{\omega}2$ and let

$$g_n = \begin{cases} f_n \restriction B_{n+1} & \text{if } z(n) = 1, \\ \mathrm{id} \restriction B_{n+1} & \text{if } z(n) = 0. \end{cases}$$

For each n, pick $y_{n+1} \in B_{n+1}$ and let

$$x^n_m = g_m \circ \cdots \circ g_n(y_{n+1}) \in B_m \, ,$$

for all $m \leq n$. By construction, for any fixed m

$$\forall n > m \left(g_m(x^n_{m+1}) = x^n_m \right) \tag{2}$$

and $\{x^n_m \mid n \geq m\} \subseteq B_m$ is a Cauchy sequence with respect to d_m, since $d_m(x^n_m, x^k_m) < 2^{-\min(n,k)}$ by (1). Therefore we get an

$$x_m = \lim_{n \to \infty} x^n_m \in B_m$$

and by continuity of $g_m : (B_{m+1}, d_{m+1}) \to (B_m, d_m)$ and by (2)

$$g_m(x_{m+1}) = x_m \, .$$

Naturally x_m really depends on $z \in {}^{\omega}2$, so we should write $x_m = x_m(z)$. By construction, if $\forall n > n_0 \, (z(n) = w(n))$ then

$$\forall n > n_0 \, (x_n(z) = x_n(w))$$

and if $z(n_0) \neq w(n_0)$ then

$$x_{n_0}(z) \in A \cap B_{n_0} \Leftrightarrow x_{n_0}(w) \notin A \cap B_{n_0} .$$

The usual argument yields that $\{z \in {}^{\omega}2 \mid x_0(z) \in A\}$ is a flip-set, contradicting the property of Baire.

Therefore it is enough to construct the B_n's and d_n's. As required, set $B_0 = \mathbb{R}$, d_0 the usual distance on $\mathbb{R}$, and let $f_0 : \mathbb{R} \to \mathbb{R}$ be a Borel function witnessing $A \leq_{\mathbf{B}} \neg A$.

Suppose B_m, f_m, and d_m have been defined for all $m \leq n$. Fix an $s \in {}^{n+1}2$, and, for $i \leq n$, let $g_i^s = g_i$ be f_i or the identity, depending on whether $s(i) = 1$ or $s(i) = 0$. For each $m \leq n$ let $\{C_m^i \mid i \in \omega\}$ be a Borel partition of B_m such that d_m-diam$(C_m^i) < 2^{-n}$. We now inductively construct $B_n \supseteq B^0 \supseteq B^1 \supseteq \cdots \supseteq B^n$ as follows. By the σ-additivity of $\mathcal{I}$ there is $i_0 \in \omega$ such that

$$B^0 = (g_0 \circ \cdots \circ g_n)^{-1} \text{“} C_0^{i_0} \notin \mathcal{I} ,$$

and by σ-additivity again, inductively choose $i_m \in \omega$ such that

$$B^{m+1} = B^m \cap (g_m \circ \cdots \circ g_n)^{-1} \text{“} C_m^{i_m} \notin \mathcal{I} ,$$

for $m < n$. Since the construction above depends on the chosen $s \in {}^{n+1}2$, let $B(s) = B^n$. Now we can repeat the construction above for each element of ${}^{n+1}2$: let $\langle s_i \mid 1 \leq i \leq 2^{n+1} \rangle$ be an enumeration of ${}^{n+1}2$, and construct $B(s_1)$ as above, then construct $B(s_2)$ using $B(s_1)$ instead of B_n, and so on. This gives a sequence of Borel sets not in $\mathcal{I}$

$$B_n \supseteq B(s_1) \supseteq \cdots \supseteq B(s_{2^n}) = B_{n+1}$$

and by construction, for any $a, b \in B_{n+1}$, any $m \leq n$, and any $s \in {}^{n+1}2$

$$d_m \left(g_m^s \circ \cdots \circ g_n^s(a), g_m^s \circ \cdots \circ g_n^s(b) \right) < 2^{-n} .$$

Since $B_{n+1} \notin \mathcal{I}$ then $A \cap B_{n+1} \leq_{\mathbf{B}} \neg A \cap B_{n+1}$ and let $f_{n+1} : B_{n+1} \to B_{n+1}$ witness this. In order to complete the construction we need to prove the existence of d_{n+1} on B_{n+1}. This follows at once from Lemma 19.2 taking $(X, d) = (B_n, d_n)$, $f = f_{n+1}$, and $B = B_{n+1}$. □

Notice that the Property of Baire is used in a “local way” in the proof of Theorem 19.11: if $\prod_n B_n \subset {}^{\omega}\mathbb{R}$ is endowed with the product topology of the τ_n's, rather than with the topology induced by ${}^{\omega}\mathbb{R}$, then the map

$$ {}^{\omega}2 \to \prod_n B_n, \quad z \mapsto \langle x_m(z) \mid m \in \omega \rangle$$

is continuous by (1), hence the map ${}^{\omega}2 \to B_0 = \mathbb{R}$, $z \mapsto x_0(z)$, is continuous, and hence the flip-set is the continuous pre-image of A. Therefore the proof only requires the property of Baire for sets which are Wadge reducible to A.

Suppose $[A]_{\mathbf{B}}$ is limit and self-dual, and let $\{B_n \mid n < \omega\}$ be a Borel partition of $\mathbb{R}$ as in Theorem 19.11. If $C <_{\mathbf{B}} A$ were an upper bound for the $A \cap B_n$'s, i.e., $\forall n\, (A \cap B_n \leq_{\mathbf{B}} C)$, then Lemma 19.8 implies that $A \leq_{\mathbf{B}} C$, a contradiction. Since by Lemma 19.7

$$A \cap B_0 \leq_{\mathbf{B}} A \cap (B_0 \cup B_1) \leq_{\mathbf{B}} A \cap (B_0 \cup B_1 \cup B_2) \leq_{\mathbf{B}} \cdots \leq_{\mathbf{B}} A\,,$$

and $A \cap B_n \leq_{\mathbf{B}} A \cap \bigcup_{i \leq n} B_i$, then $\|A\|_{\mathbf{B}} = \sup_n \left\|A \cap \bigcup_{i \leq n} B_i\right\|_{\mathbf{B}}$. Therefore if $[A]_{\mathbf{B}}$ is limit and $\mathrm{cof}(\|A\|_{\mathbf{B}}) > \omega$, then $[A]_{\mathbf{B}}$ is non-self-dual.

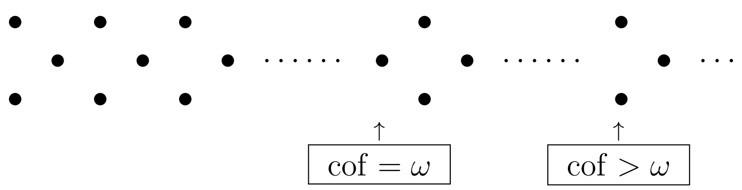
We have already seen that immediately above a non-self-dual pair $([A]_{\mathbf{B}}, [\neg A]_{\mathbf{B}})$ there is a self-dual degree $[A \oplus \neg A]_{\mathbf{B}}$. We will now argue that immediately above a self-dual degree there is a non-self-dual pair. This amounts to proving that if $[A]_{\mathbf{B}} < [B]_{\mathbf{B}}$ are both self-dual then $A <_{\mathbf{B}} C <_{\mathbf{B}} B$ for some C. Let $\{D_n \mid n \in \omega\}$ be a Borel partition of $\mathbb{R}$ such that $\forall n\, (B \cap D_n <_{\mathbf{B}} B)$. If $B \cap D_n \leq_{\mathbf{B}} A$ for all n then $B \leq_{\mathbf{B}} A$, which is absurd, so let $n_0 \in \omega$ be such that $B \cap D_{n_0} \not\leq_{\mathbf{B}} A$. By $\mathsf{SLO}^{\mathbf{B}}$, $\neg A \leq_{\mathbf{B}} B \cap D_{n_0}$ and since $A \equiv_{\mathbf{B}} \neg A$, then $A <_{\mathbf{B}} B \cap D_{n_0} <_{\mathbf{B}} B$. Thus we have proved:

Corollary 19.12. *Assume* $\mathsf{SLO}^{\mathbf{B}} + \mathsf{BP}$.

(a) A limit Borel-Wadge degree of uncountable cofinality is non-self-dual.

(b) Immediately above a self-dual Borel-Wadge degree there is a non-self-dual pair.

Therefore the structure of the Borel degrees is isomorphic to the structure of the Wadge degrees:



At the bottom of the hierarchy there is the non-self-dual pair $([\mathbb{R}]_{\mathbf{B}}, [\emptyset]_{\mathbf{B}})$ which—as already pointed-out in section **??**—is $(\{\mathbb{R}\}, \{\emptyset\})$. Immediately above it there is the least self-dual degree, $\mathbf{\Delta}^1_1 \setminus \{\emptyset, \mathbb{R}\}$. We call these three degrees $[\mathbb{R}]_{\mathbf{B}}$, $[\emptyset]_{\mathbf{B}}$, and $\mathbf{\Delta}^1_1 \setminus \{\emptyset, \mathbb{R}\}$ *trivial*: in other words, $[A]_{\mathbf{B}}$ is non-trivial just in case $\|A\|_{\mathbf{B}} \geq 3$. Lastly we prove the converse to the second half of Lemma 19.8(b).

Proposition 19.13. *Assume* $\mathsf{SLO}^{\mathbf{B}} + \mathsf{BP}$. *If* $[A]_{\mathbf{B}}$ *is a non-trivial self-dual successor degree, then there is a Borel partition of* $\mathbb{R}$, $\{B_0, B_1\}$, *such that* $B_i \cap A <_{\mathbf{B}} A$, *for* $i = 0, 1$.

Proof. Let $[C]_{\mathbf{B}}$ be the immediate predecessor of $[A]_{\mathbf{B}}$. Then $[C]_{\mathbf{B}}$ is non-self-dual and $[C \oplus \neg C]_{\mathbf{B}}$ is its immediate successor, that is $A \equiv_{\mathbf{B}} C \oplus \neg C$. Let $f : \mathbb{R} \to \mathbb{R}$ be Borel witnessing $A \leq_{\mathbf{B}} C \oplus \neg C$, let $D_0 = \{x \in \mathbb{R} \mid x(0) \text{ is even}\}$ and $D_1 = \{x \in \mathbb{R} \mid x(0) \text{ is odd}\}$, and let $B_i = f^{-1}``D_i$. Then $\{B_0, B_1\}$ is a Borel partition of $\mathbb{R}$, and f witnesses $B_i \cap A \leq_{\mathbf{B}} D_i \cap (C \oplus \neg C)$. Since $C \neq \mathbb{R}, \emptyset$, then $D_0 \cap (C \oplus \neg C) \equiv_{\mathbf{B}} C$, and since $C <_{\mathbf{B}} A$ we have that $B_0 \cap A <_{\mathbf{B}} A$. Similarly, $B_1 \cap A \leq_{\mathbf{B}} D_1 \cap (C \oplus \neg C) \equiv_{\mathbf{B}} \neg C <_{\mathbf{B}} A$. □

Corollary 19.14. *Assume* $\mathsf{SLO}^{\mathbf{B}} + \mathsf{BP}$. $[A]_{\mathbf{B}}$ *is a non-trivial self-dual successor degree iff there is a Borel partition of* $\mathbb{R}$, $\{B_n \mid n < N\}$ *such that* $\forall n < N\, (B_n \cap A <_{\mathbf{B}} A)$. *Moreover* $[A]_{\mathbf{B}}$ *is a successor degree iff* N *can be taken to be finite, and in fact* N *can be taken to be* 2.

20 Non-self-dual pointclasses

Assuming AD, Steel and Van Wesep independently showed that $A \leq_{\mathrm{W}} \neg A \Rightarrow A \leq_{\mathrm{L}} \neg A$, hence $[A]_{\mathrm{W}} = [A]_{\mathrm{L}}$ for every non-self-dual $[A]_{\mathrm{L}}$. In fact both statements are provable assuming $\mathsf{SLO}^{\mathrm{L}} + \mathsf{BP}$—see [?]. The analogue of the first statement for $\leq_{\mathbf{B}}$ and $\leq_{\mathrm{W}}$ is clearly false: if U is open but not closed, then $U \leq_{\mathbf{B}} \neg U$ but $U \not\leq_{\mathrm{W}} \neg U$. Nevertheless the second statement can be generalized to the Borel case.

Proposition 20.1. *Assume* $\mathsf{SLO}^{\mathrm{W}} + \mathsf{BP}$. *If* $A \not\equiv_{\mathbf{B}} \neg A$, *then* $[A]_{\mathbf{B}} = [A]_{\mathrm{W}}$.

Proof. If $B \in [A]_{\mathbf{B}}$ and $B \not\equiv_{\mathrm{W}} A$, then, since $A \equiv_{\mathrm{W}} \neg B$ cannot hold, either $B <_{\mathrm{W}} A$ or $A <_{\mathrm{W}} B$. For the sake of definitiveness, assume the former. Then $C \in [A]_{\mathbf{B}}$ for any $B \leq_{\mathrm{W}} C <_{\mathrm{W}} A$. Since by $\mathsf{SLO}^{\mathrm{W}} + \mathsf{BP}$ we certainly can find such a C which is Wadge self-dual, we have that $A \equiv_{\mathbf{B}} C \equiv_{\mathrm{W}} \neg C \equiv_{\mathbf{B}} \neg A$, a contradiction. □

Each non-self-dual pair of Wadge-Borel degrees is a non-self-dual pair of Wadge degrees, but not vice versa. Notice that assuming $\mathsf{SLO}^{\mathrm{L}} + \mathsf{BP}$ the conclusion for non-self-dual $[A]_{\mathbf{B}}$ can be strengthened to $[A]_{\mathbf{B}} = [A]_{\mathrm{L}}$. On the other hand, the self-dual Borel degrees are obtained by glueing together many Wadge degrees. For example, the first self-dual degree is the collection of all Borel sets except for $\mathbb{R}$ and $\emptyset$. For $A \equiv_{\mathbf{B}} \neg A$ let $h([A]_{\mathbf{B}})$ be the length of the interval of Wadge degrees used to construct $[A]_{\mathbf{B}}$. Thus, if $A \in \mathbf{\Delta}^1_1 \setminus \{\emptyset, \mathbb{R}\}$, then $[A]_{\mathbf{B}} = \mathbf{\Delta}^1_1 \setminus \{\emptyset, \mathbb{R}\}$ and

$$\begin{aligned} h([A]_{\mathbf{B}}) &= \delta \\ &= \text{the length of the Wadge hierarchy restricted to } \mathbf{\Delta}^1_1 \\ &= \|B\|_{\mathrm{W}}, \text{ where } B \in \mathbf{\Sigma}^1_1 \cup \mathbf{\Pi}^1_1 \setminus \mathbf{\Delta}^1_1. \end{aligned}$$

In analogy with the case of the Lipschitz-vs-Wadge hierarchies, where each self-dual Wadge degree is the union of ω_1 consecutive Lipschitz degrees, it is tempting to conjecture that $h([A]_{\mathbf{B}}) = \delta$ for any self-dual $[A]_{\mathbf{B}}$. However this is not true. In fact, $h([A]_{\mathbf{B}}) > \|A\|_{\mathrm{W}}$ for all self-dual $[A]_{\mathbf{B}}$, and therefore the Borel-Wadge hierarchy is obtained by collapsing to a single self-dual Borel-Wadge degree larger and larger blocks of the Wadge hierarchy. To see this we need to recall the definition—due to Wadge—of addition of sets of reals. For $A, B \subseteq \mathbb{R}$ let

$$A + B = \{(s+1)^\frown \langle 0 \rangle^\frown (x+1) \mid s \in {}^{<\omega}\omega \ \& \ x \in A\} \cup \{x + 1 \mid x \in B\},$$

where $y + 1 = \langle y(n) + 1 \mid n < \mathrm{lh}(y) \rangle$, for any sequence (finite or infinite) y. If $A \equiv_{\mathrm{W}} \neg A$ and assuming $\mathsf{SLO}^{\mathrm{W}} + \mathsf{BP}$, we have that $\|A + B\|_{\mathrm{W}} = \|A\|_{\mathrm{W}} + \|B\|_{\mathrm{W}}$ (see [Wad83] or [?] for more on this). In particular, if $B \leq_{\mathbf{B}} A$ and f witnesses this, then

$$g(x) = \begin{cases} f(x-1) & \text{if } \forall n\, x(n) \neq 0, \\ y & \text{if } x = (s+1)^\frown \langle 0 \rangle^\frown y, \text{ for some } s \in {}^{<\omega}\omega. \end{cases}$$

is Borel and witnesses $A + B \leq_{\mathbf{B}} A$. Therefore $h([A]_{\mathbf{B}}) \geq \|A\|_{\mathrm{W}} + \|A\|_{\mathrm{W}} > \|A\|_{\mathrm{W}}$.

Assuming $\mathsf{SLO}^{\mathrm{W}}$ we can now describe the first few Borel degrees. Immediately above the trivial degrees $[\mathbb{R}]_{\mathbf{B}}$, $[\emptyset]_{\mathbf{B}}$, and $\mathbf{\Delta}^1_1 \setminus \{\mathbb{R}, \emptyset\}$ there is, by Proposition 20.1, the non-self-dual pair $(\mathbf{\Sigma}^1_1 \setminus \mathbf{\Delta}^1_1, \mathbf{\Pi}^1_1 \setminus \mathbf{\Delta}^1_1)$. At the next level we have a self-dual degree: it is the collection of all Borel-separated-unions of a true $\mathbf{\Sigma}^1_1$ and a true $\mathbf{\Pi}^1_1$,

$$\{A \cup B \mid A \in \mathbf{\Sigma}^1_1 \setminus \mathbf{\Delta}^1_1 \ \& \ B \in \mathbf{\Pi}^1_1 \setminus \mathbf{\Delta}^1_1 \ \& \ \exists C \in \mathbf{\Delta}^1_1 \,(A \subseteq C \ \& \ B \cap C = \emptyset)\}.$$

In order to compute the next non-self-dual pair of degrees it is more convenient to work with pointclasses rather than with degrees. Recall that a collection of sets $\mathbf{\Gamma} \subseteq \mathscr{P}(\mathbb{R})$ is a *boldface pointclass* if it is non-empty and closed under continuous pre-images. It is self-dual if it is closed under complements, otherwise it is non-self-dual. The *dual* of $\mathbf{\Gamma}$ is the pointclass $\breve{\mathbf{\Gamma}} = \{\neg A \mid A \in \mathbf{\Gamma}\}$, and let $\mathbf{\Delta} = \mathbf{\Delta}_{\mathbf{\Gamma}}$ be the pointclass $\mathbf{\Gamma} \cap \breve{\mathbf{\Gamma}}$. Under $\mathsf{SLO}^{\mathrm{W}}$, the non-self-dual boldface pointclasses are of the form $\{X \subseteq \mathbb{R} \mid X \leq_{\mathrm{W}} A\}$ with $[A]_{\mathrm{W}}$ non-self-dual, while the self-dual ones are of the form $\{X \subseteq \mathbb{R} \mid X <_{\mathrm{W}} A\}$, with $A \neq \mathbb{R}, \emptyset$. Conversely, if $\boldsymbol{\Gamma}$ is non-self-dual, then $\boldsymbol{\Gamma} \setminus \breve{\boldsymbol{\Gamma}}$ is a non-self-dual Wadge degree by $\mathsf{SLO}^{\mathrm{W}}$. Therefore $\mathsf{SLO}^{\mathrm{W}}$ yields that boldface pointclasses are (essentially) well-ordered under inclusion: either $\boldsymbol{\Gamma} \subseteq \boldsymbol{\Lambda}$ or $\boldsymbol{\Lambda} \subseteq \breve{\boldsymbol{\Gamma}}$. By Proposition 20.1 and the discussion following its proof, $[A]_{\mathbf{B}}$ is a non-self-dual degree iff $\{X \mid X \leq_{\mathrm{W}} A\}$ is closed under Borel pre-images. A set U is $\boldsymbol{\Gamma}$-*universal* if it $\mathbb{R}$-parametrizes $\boldsymbol{\Gamma}$ and belongs to $\boldsymbol{\Gamma}$, i.e., $U \subseteq \mathbb{R}^2$, $\boldsymbol{\Gamma} = \{U_x \mid x \in \mathbb{R}\}$ where $U_x = \{y \mid (x, y) \in U\}$, and U (or better: its image under the standard homeomorphism $\mathbb{R}^2 \approx \mathbb{R}$) is in $\boldsymbol{\Gamma}$. If $\boldsymbol{\Gamma}$ has a universal set then it is non-self-dual. Conversely, $\mathsf{SLO}^{\mathrm{L}} + \mathsf{BP}$ implies every non-self-dual $\boldsymbol{\Gamma}$ has a universal set: choose an $\mathbb{R}$-parametrization $\langle \boldsymbol{g}_x \mid x \in \mathbb{R}\rangle$ of all Lipschitz functions such that $(x, y) \mapsto \boldsymbol{g}_x(y)$ is continuous; by the theorem of Steel and Van Wesep mentioned at the beginning of this section, $\boldsymbol{\Gamma} = \{X \mid X \leq_{\mathrm{L}} A\}$ for some $A \not\equiv_{\mathrm{L}} \neg A$, and let $U = \{(x, y) \mid \boldsymbol{g}_x(y) \in A\}$. Similarly, by choosing a parametrization $\langle \boldsymbol{f}_x \mid x \in \mathbb{R}\rangle$ of all continuous functions such that $(x, y) \mapsto \boldsymbol{f}_x(y)$ is Borel, $\mathsf{SLO}^{\mathrm{W}}$ implies that if $[A]_{\mathbf{B}}$ is non-self-dual then $\boldsymbol{\Gamma} = \{X \mid X \leq_{\mathrm{W}} A\}$ has a universal set.

Wadge gave a concrete description of the next non-self-dual pair of boldface pointclasses above a non-self-dual $\boldsymbol{\Gamma}$: Suppose $\boldsymbol{\Gamma} = \{X \mid X \leq_{\mathrm{W}} A\}$ with A non-self-dual and let

$$\boldsymbol{\Gamma}^{\triangledown} = \{(U \cap X) \cup (U' \setminus X') \mid U, U' \in \boldsymbol{\Sigma}^0_1 \;\&\; U \cap U' = \emptyset \;\&\; X, X' \in \boldsymbol{\Gamma}\}\,.$$

Then $\boldsymbol{\Gamma} \cup \breve{\boldsymbol{\Gamma}} \subseteq \boldsymbol{\Gamma}^{\triangledown}$, and $\boldsymbol{\Gamma}^{\triangledown}$ and its dual $(\boldsymbol{\Gamma}^{\triangledown})\breve{}$ are the least non-self-dual pair of pointclasses above $\boldsymbol{\Gamma} \cup \breve{\boldsymbol{\Gamma}}$. Notice that the self-dual pointclass $\boldsymbol{\Delta}_{\boldsymbol{\Gamma}^{\triangledown}} = \{X \mid X \leq_{\mathrm{W}} B \oplus \neg B\}$ is made-up of those $(U \cap X) \cup (U' \setminus X') \in \boldsymbol{\Gamma}^{\triangledown}$ such that $U \subseteq C \subseteq \neg U'$ for some clopen set C. This suggests the following definition.

For $\boldsymbol{\Gamma}$ a boldface pointclass closed under Borel pre-images let

$$\boldsymbol{\Gamma}^{*} = \{(P \cap X) \cup (P' \setminus X') \mid P, P' \in \boldsymbol{\Pi}^1_1 \;\&\; P \cap P' = \emptyset \;\&\; X, X' \in \boldsymbol{\Gamma}\}\,,$$

and let $\boldsymbol{\Delta}^* = \boldsymbol{\Delta}_{\boldsymbol{\Gamma}^*}$. Taking $P = \mathbb{R}$ or $P' = \mathbb{R}$ we have that $\boldsymbol{\Gamma} \subseteq \boldsymbol{\Gamma}^*$ and $\breve{\boldsymbol{\Gamma}} \subseteq \boldsymbol{\Gamma}^*$.

Lemma 20.2. *Assume* $\mathsf{SLO}^{\mathrm{W}}$ *and let* $\boldsymbol{\Gamma}$ *be a non-self-dual pointclass closed under Borel pre-images. Then* $\boldsymbol{\Gamma}^*$ *is non-self-dual and is closed under Borel pre-images.*

Proof. As both $\boldsymbol{\Gamma}$ and $\boldsymbol{\Pi}^1_1$ have universal sets, $\boldsymbol{\Gamma}^*$ also has a universal set, hence it is non-self-dual. Since both $\boldsymbol{\Gamma}$ and $\boldsymbol{\Pi}^1_1$ are closed under Borel pre-images, it follows that $\boldsymbol{\Gamma}^*$ is closed under Borel pre-images. □

Lemma 20.3. *Let $[A]_{\mathbf{B}}$ be non-self-dual and let $\mathbf{\Gamma} = \{X \mid X \leq_{\mathrm{W}} A\}$. Then any set in $\mathbf{\Delta}^*$ is Borel reducible to $A \oplus \neg A$.*

Proof. Let $Y \in \mathbf{\Delta}^*$ and let $P_1, P_1' \in \mathbf{\Pi}^1_1$ and $X_1, X_1' \in \mathbf{\Gamma}$ witness that

$$Y = (P_1 \cap X_1) \cup (P_1' \setminus X_1') \in \mathbf{\Gamma}^*$$

and let $P_2, P_2' \in \mathbf{\Pi}^1_1$, $X_2, X_2' \in \mathbf{\Gamma}$ witness that

$$\neg Y = (P_2 \cap X_2) \cup (P_2' \setminus X_2') \in \mathbf{\Gamma}^*\,.$$

Then $P_1 \cup P_1' \cup P_2 \cup P_2' = \mathbb{R}$. By Reduction for $\mathbf{\Pi}^1_1$, let $\{B_1, B_1', B_2, B_2'\}$ be a Borel partition of $\mathbb{R}$ such that $B_1 \subseteq P_1$, $B_1' \subseteq P_1'$, $B_2 \subseteq P_2$, and $B_2' \subseteq P_2'$. Then

$$\begin{aligned}
&x \in B_1 \Rightarrow (x \in Y \Leftrightarrow x \in X_1)\\
&x \in B_1' \Rightarrow (x \in Y \Leftrightarrow x \notin X_1')\\
&x \in B_2 \Rightarrow (x \in Y \Leftrightarrow x \notin X_2)\\
&x \in B_2' \Rightarrow (x \in Y \Leftrightarrow x \in X_2')\,.
\end{aligned}$$

This implies the desired conclusion. □

Theorem 20.4. *Assume $\mathsf{SLO}^{\mathrm{W}}$. Let $[A]_{\mathbf{B}}$ be non-self-dual and let $\mathbf{\Gamma} = \{X \mid X \leq_{\mathrm{W}} A\}$. Then*

$$\begin{aligned}
\mathbf{\Delta}^* &= \{X \mid X \leq_{\mathbf{B}} A \oplus \neg A\}\\
&= \{X \cup \neg X' \mid \exists B \in \mathbf{\Delta}^1_1\, (X \subseteq B \subseteq \neg X') \;\&\; X, X' \in \mathbf{\Gamma}\}
\end{aligned}$$

and $\mathbf{\Delta}^* \setminus (\mathbf{\Gamma} \cup \breve{\mathbf{\Gamma}}) = [A \oplus \neg A]_{\mathbf{B}}$, i.e., it is the self-dual degree immediately above $([A]_{\mathbf{B}}, [\neg A]_{\mathbf{B}})$ and $(\mathbf{\Gamma}^* \setminus \mathbf{\Delta}^*, (\mathbf{\Gamma}^*)\breve{} \setminus \mathbf{\Delta}^*)$ is the next non-self-dual pair above it.

Proof. It is easy to check that $A \oplus \neg A \in \mathbf{\Delta}^*$ and that the sets which are Borel-reducible to $A \oplus \neg A$ are of the form $X \cup \neg X'$ with $X, X' \in \mathbf{\Gamma}$ Borel separated. Therefore we are done by Lemmata 20.2 and 20.3. □

Wadge's analysis shows that if $\mathbf{\Gamma}_n$ is an increasing sequence of boldface pointclasses, then

$$\mathbf{\Gamma} = \{\textstyle\bigcup_n (U_n \cap X_n) \mid U_n \in \mathbf{\Sigma}^0_1 \text{ are pairwise disjoint and } X_n \in \mathbf{\Gamma}_n\}$$

is non-self-dual, and $\mathbf{\Gamma}$ and its dual are the least non-self-dual pointclasses above the $\mathbf{\Gamma}_n$'s.

Similarly, if $\langle \boldsymbol{\Gamma}_n \mid n \in \omega \rangle$ is a strictly increasing sequence of pointclasses closed under Borel pre-images, then let

$$\boldsymbol{\Delta} = \{\bigcup_n (B_n \cap X_n) \mid B_n \in \boldsymbol{\Delta}^1_1 \text{ are pairwise disjoint and } X_n \in \boldsymbol{\Gamma}_n\}$$

and let

$$\boldsymbol{\Lambda} = \{\bigcup_n (P_n \cap X_n) \mid P_n \in \boldsymbol{\Pi}^1_1 \text{ are pairwise disjoint and } X_n \in \boldsymbol{\Gamma}_n\}\,.$$

If $A_{n+1} \in \boldsymbol{\Gamma}_{n+1} \setminus \boldsymbol{\Gamma}_n$ and there are pairwise disjoint Borel sets B_n such that $A_n \subseteq B_n$, then it is not hard to see that $\bigcup_n A_n$ is Wadge self-dual, that

$$\boldsymbol{\Delta} = \{X \mid X \leq_{\mathrm{W}} \bigcup_n A_n\}$$

is self-dual, that $\bigcup_n \boldsymbol{\Gamma}_n \subset \boldsymbol{\Delta}$, and that there is no boldface pointclass in between. Arguing as above we get:

Theorem 20.5. *Assume* $\mathsf{SLO}^{\mathrm{W}} + \mathsf{BP}$ *and suppose* $\boldsymbol{\Gamma}_n$, $\boldsymbol{\Delta}$ *and* $\boldsymbol{\Lambda}$ *are as above. Then* $\boldsymbol{\Lambda}$ *and* $\breve{\boldsymbol{\Lambda}}$ *are closed under Borel pre-images and are the least non-self-dual pair of boldface pointclasses above the* $\boldsymbol{\Gamma}_n$*'s, and* $\boldsymbol{\Delta} = \boldsymbol{\Delta}_{\boldsymbol{\Lambda}}$.

We can now give a complete description of the first ω_1 levels of the $\leq_{\mathbf{B}}$ hierarchy. By Theorem 20.4 the least non-self-dual pair of pointclasses closed under $\leq_{\mathbf{B}}$ and above $\boldsymbol{\Sigma}^1_1$ and $\boldsymbol{\Pi}^1_1$ is $(\boldsymbol{\Gamma}, \breve{\boldsymbol{\Gamma}})$, where $Y \in \boldsymbol{\Gamma}$ iff $Y = P_1 \cup P_2 \setminus P_3$ with $P_1, P_2, P_3 \in \boldsymbol{\Pi}^1_1$ and $P_1 \cap P_2 = \emptyset$. Without loss of generality we may assume $P_3 \subseteq P_2$ hence $Y = (P_1 \cup P_2) \setminus P_3$, hence $\boldsymbol{\Gamma}$ is the collection $\mathrm{Diff}(2; \boldsymbol{\Pi}^1_1)$ of all differences of $\boldsymbol{\Pi}^1_1$ sets. Inductively, using Theorem 20.5, one can show that the α-th pair of non-self-dual pointclasses closed under Borel reducibility is $(\mathrm{Diff}(\alpha; \boldsymbol{\Pi}^1_1), \breve{\mathrm{Diff}}(\alpha; \boldsymbol{\Pi}^1_1))$.

Chapter IV

Wadge determinacy

Chapter V

The Semi-Linear Ordering Principle

21 $\mathsf{SLO}^{\mathrm{W}}$ and the Structure of the Wadge Hierarchy

We briefly recall the basic facts about the Wadge hierarchy. Unless otherwise indicated, $\mathbb{R}$ denotes the Baire space

$$\mathbb{R} = {}^{\omega}\omega\,.$$

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is **continuous** if there is a $\varphi : {}^{<\omega}\omega \to {}^{<\omega}\omega$ which is

(1) **monotone**, that is $s \subseteq t \Rightarrow \varphi(s) \subseteq \varphi(t)$, and

(2) $\forall x \in \mathbb{R}\,(\lim_{n\to\infty} \mathrm{lh}(\varphi(x \restriction n)) = \infty)$,

and such that $f(x) = \bigcup_n \varphi(x \restriction n)$. We will say that f is induced by φ and write $f = f_\varphi$. With a minor abuse of notation, a φ as above, satisfying (1) and (2) is also said to be continuous. A function $\varphi : {}^{<\omega}\omega \to {}^{<\omega}\omega$ is **tame** if it is continuous and satisfies

(3) $\forall s {}^{<\omega}\omega\, \forall n \left(\varphi(s) \subset \varphi(s^\langle n\rangle) \Rightarrow \mathrm{lh}(\varphi(s^\langle n\rangle)) = \mathrm{lh}(\varphi(s)) + 1\right)$.

Every continuous $f : \mathbb{R} \to \mathbb{R}$ is induced by a tame φ. If condition (2) is strengthened to

(2′) $\mathrm{lh}(\varphi(s)) = \mathrm{lh}(s)$, and φ satisfies (1)+(2′) then it is said to be **Lipschitz**, since

$$x \restriction n = y \restriction n \Rightarrow f_\varphi(x) \restriction n = f_\varphi(y) \restriction n\,,$$

i.e., f_φ is Lipschitz with constant ≥ 1 with respect to the usual metric on $\mathbb{R}$

$$d(x,y) = \begin{cases} 2^{-n} & \text{if } x \restriction n = y \restriction n \;\&\; x(n) = y(n)\,, \\ 0 & \text{if } x = y. \end{cases}$$

Clearly every Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ of constant ≥ 1 is of the form $f = f_\varphi$ with φ Lipschitz. A function $\varphi : {}^{<\omega}\omega \to {}^{<\omega}\omega$ is a **contraction** if it is monotone and satisfies

(4) $\mathrm{lh}(\varphi(s)) = \mathrm{lh}(s) + 1$.

Thus a contraction is continuous and the induced map f_φ has Lipschitz constant $\leq 1/2$.

Definition 21.1. For $A, B \subseteq \mathbb{R}$, say that A is **Wadge reducible** to B, in symbols $A \leq_{\mathrm{W}} B$ just in case $A = f^{-1}$"B for some continuous $f : \mathbb{R} \to \mathbb{R}$. If f^{-1}"B and f is Lipschitz, then A is said to be **Lipschitz reducible** to B, in symbols $A \leq_{\mathrm{L}} B$. $A <_{\mathrm{W}} B$ stands for $A \leq_{\mathrm{W}} B$ and $\not\leq_{\mathrm{W}} A$, and similarly for $A <_{\mathrm{L}} B$.

It is clear that $\leq_{\mathrm{W}}$ and $\leq_{\mathrm{L}}$ are reflexive and transitive, so the relations

$$A \equiv_{\mathrm{W}} B \Leftrightarrow A \leq_{\mathrm{W}} B \;\&\; B \leq_{\mathrm{W}} A$$
$$A \equiv_{\mathrm{L}} B \Leftrightarrow A \leq_{\mathrm{L}} B \;\&\; B \leq_{\mathrm{L}} A$$

are equivalence relations, and the equivalence classes are called, respectively, **Wadge degrees** and **Lipschitz degrees**. $[A]_{\mathrm{W}}$ and $[A]_{\mathrm{L}}$ are, respectively, the Wadge degree and the Lipschitz degree of A. It is immediate to check that the relations $\leq_{\mathrm{W}}$ and $\leq_{\mathrm{W}}$ can be defined on the set of all Wadge degrees, and similarly for $\leq_{\mathrm{L}}$ and $<_{\mathrm{L}}$. The **dual** of the Wadge/Lipschitz degree of a set A is the degree of $\neg A$; a degree is **self-dual** if it coincides with its dual; otherwise it is called **non-self-dual**.

In order to prove non-trivial results about the ordering $A \leq_{\mathrm{W}} B$ we need two games invented by Wadge. The first one, called the **Lipschitz game** for (A, B), in symbols $G_{\mathrm{L}}(A, B)$, is the game:

$$\begin{array}{llllll} \mathbf{I} & a_0 & & a_1 & & \\ & & & & & \cdots \\ \mathbf{II} & & b_0 & & b_1 & \end{array}$$

where **I** and **II** alternate and play $a_i, b_i \in \omega$. Let $a = \langle a_i \mid i \in \omega\rangle$ and $b = \langle b_i \mid i \in \omega\rangle$. Then **II** wins iff $(a \in A \Leftrightarrow b \in B)$.

The second game $G_{\mathrm{W}}(A, B)$, which we will call the **Wadge game**, is similar to $G_{\mathrm{L}}(A, B)$, but **II** has the option of passing, with the proviso that if he does not play infinitely often then he loses:

I	a_0	a_1	$\dots$	a_n	a_{n+1}	a_{n+2}	$\dots$	a_m	$\dots\dots$
II		p	$\mathsf{p}\dots\mathsf{p}$	b_0	p	$\mathsf{p}\dots\mathsf{p}$	b_1		

As before **II** wins iff $(a \in A \Leftrightarrow b \in B)$.

For $s \in {}^{<\omega}\omega$, let

$$A_{\lfloor s\rfloor} = \{x \in \mathbb{R} \mid s^\frown x \in A\}\,.$$

(When $s = \langle n\rangle$ we will write $A_{\lfloor n\rfloor}$ rather than$A_{\lfloor\langle n\rangle\rfloor}$.)

Exercise 21.2. Show that

(i) **II** wins $G_{\mathrm{W}}(A, B)$ iff $A \leq_{\mathrm{W}} B$;

(ii) **II** wins $G_{\mathrm{L}}(A, B)$ iff $A \leq_{\mathrm{L}} B$;

(iii) **I** wins $G_{\mathrm{L}}(A, B)$ iff $\exists n \left(\neg B \leq_{\mathrm{L}} A_{\lfloor n\rfloor}\right)$;

(iv) if **I** wins $G_{\mathrm{W}}(A, B)$ then **I** wins $G_{\mathrm{L}}(A, B)$;

(v) if **II** wins $G_{\mathrm{L}}(A, B)$ then **I** wins $G_{\mathrm{W}}(A, B)$.

Since $A_{\lfloor n\rfloor} \leq_{\mathrm{L}} A$ for all $n \in \omega$, (iii) implies that **I** wins $G_{\mathrm{L}}(A, B) \Rightarrow \neg B \leq_{\mathrm{L}} A$, hence we have shown:

Theorem 21.3 (Wadge's Lemma)**.** *If $G_{\mathrm{L}}(A, B)$ is determined, then $A \leq_{\mathrm{L}} B$ or $\neg B \leq_{\mathrm{L}} A$. If $G_{\mathrm{W}}(A, B)$ is determined, then $A \leq_{\mathrm{W}} B$ or $\neg B \leq_{\mathrm{W}} A$.*

The **Semi-Linear-Ordering Principle**, $\mathsf{SLO}^{\mathrm{W}}$ for short, is the statement

$$\forall A, B \subseteq \mathbb{R}\,(A \leq_{\mathrm{W}} B\ \vee \neg B \leq_{\mathrm{W}} A)\,.$$

$\mathsf{SLO}^{\mathrm{L}}$ is the analogous statement where the maps are taken to be Lipschitz rather than continuous. Therefore

$$\mathsf{AD} \Rightarrow \mathsf{SLO}^{\mathrm{L}} \Rightarrow \mathsf{SLO}^{\mathrm{W}}\,.$$

It is open whether these implications can be reversed. It is known that $\mathsf{SLO}^{\mathrm{W}}$ implies certain simple consequences of AD, for example $\mathsf{AC}_\omega(\mathbb{R})$ and the perfect set property—see [**?**, 11.16, 11.10]. $\mathsf{SLO}^{\mathrm{W}}$ implies that if $[A]_{\mathrm{W}}$ and $[B]_{\mathrm{W}}$ are not $\leq_{\mathrm{W}}$-comparable, then $A \equiv_{\mathrm{W}} \neg B$. Hence the antichains in the $\leq_{\mathrm{W}}$ order on degrees have size at most 2, and every antichain of size 2 is of the form $\{[A]_{\mathrm{W}}, [\neg A]_{\mathrm{W}}\}$ with $A \not\equiv_{\mathrm{W}} \neg A$. We will call such a pair $([A]_{\mathrm{W}}, [\neg A]_{\mathrm{W}})$ a non-self-dual pair. If $A_n \subseteq \mathbb{R}$ let

$$\bigoplus_n A_n = \bigcup_n \langle n \rangle^\frown A_n \,,$$

and let $A \oplus \neg A = \bigoplus_n A_n$ with $A_{2n} = A$ and $A_{2n+1} = \neg A$. Clearly

- $A \leq_{\mathrm{L}} A \oplus \neg A$ and $\neg A \leq_{\mathrm{L}} A \oplus \neg A$,
- $\neg(A \oplus \neg A) = \neg A \oplus A \leq_{\mathrm{L}} A \oplus \neg A$,
- $A, \neg A \leq_{\mathrm{W}} C \Rightarrow A \oplus \neg A \leq_{\mathrm{W}} C$.

Therefore $[A \oplus \neg A]_{\mathrm{W}}$ is self-dual, and it is the least degree above $[A]_{\mathrm{W}}$ and $[\neg A]_{\mathrm{W}}$. Here is a characterization of successor self-dual degrees.

Exercise 21.4. Assume $\mathsf{SLO}^{\mathrm{W}}$. A successor degree $[B]_{\mathrm{W}}$ is self-dual iff there is a clopen partition $C \cup D = \mathbb{R}$, $C \cap D = \emptyset$, such that $B \cap C, B \cap D <_{\mathrm{W}} B$.

If every degree is coalesced with its dual, then $\mathsf{SLO}^{\mathrm{W}}$ implies that $\leq_{\mathrm{W}}$ is a linear order. In fact Martin, extending a partial result of Leonard Monk, showed that $\leq_{\mathrm{W}}$ is well-founded, assuming AD. The technique used in this proof (the so-called Martin–Monk method) is a standard tool in the theory of Wadge degrees.

Theorem 21.5 (Martin)**.** AD *implies there is no infinite* $<_{\mathrm{W}}$*-descending sequence*

$$\dots A_2 <_{\mathrm{W}} A_1 <_{\mathrm{W}} A_0 \,.$$

Proof. Suppose otherwise and let $\langle A_n \mid n \in \omega \rangle$ be the offending sequence. By $\mathsf{SLO}^{\mathrm{W}}$, $\neg A_{n+1} <_{\mathrm{W}} A_n$ hence $A_n \not\leq_{\mathrm{W}} \neg A_{n+1}$, so **II** does not win $G_{\mathrm{W}}(A_n, \neg A_{n+1})$. Since **II** does not win $G_{\mathrm{W}}(A_n, A_{n+1})$ as well, then, by AD, fix winning strategies σ_n^0 and σ_n^1 for **I** in $G_{\mathrm{W}}(A_n, A_{n+1})$ and $G_{\mathrm{W}}(A_n, \neg A_{n+1})$, respectively. For any fixed $z \in {}^\omega 2$ we pit the strategies $\sigma_n^{z(n)}$ against each other, with

$\sigma_{n+1}^{z(n+1)}$ playing as player **I** in $G_{\mathrm{W}}(A_{n+1}, A_{n+2})$ and as player **II** in $G_{\mathrm{W}}(A_n, A_{n+1})$,

$$
\begin{array}{cccccccccc}
\boxed{\sigma_0^{z(0)}} & a_0^z(0) & & a_0^z(1) & & a_0^z(2) & & a_0^z(3) & \dots & a_0^z \\
 & & \nearrow_{\sigma_0^{z(0)}} & & \nearrow_{\sigma_0^{z(0)}} & & \nearrow_{\sigma_0^{z(0)}} & & & \\
\boxed{\sigma_1^{z(1)}} & a_1^z(0) & & a_1^z(1) & & a_1^z(2) & & & \dots & a_1^z \\
 & & \nearrow_{\sigma_1^{z(1)}} & & \nearrow_{\sigma_1^{z(1)}} & & & & & \\
\boxed{\sigma_2^{z(1)}} & a_2^z(0) & & a_2^z(1) & & & & & \dots & a_2^z \\
 & & \nearrow_{\sigma_2^{z(2)}} & & & & & & & \\
\boxed{\sigma_3^{z(1)}} & a_3^z(0) & & & & & & & \dots & a_3^z \\
\vdots & & & & & & & & & \vdots
\end{array}
$$

a_n^z is the real constructed in the n-th line of the diagram above. If z, w differ exactly in one coordinate n_0, that is $\forall n \neq n_0\,(z(n) = w(n))$ and $z(n_0) \neq w(n_0)$, then the two diagrams obtained using z and w agree below the n_0-th line, hence $n > n_0 \Rightarrow a_n^z = a_n^w$. On the other hand

$$a_{n_0}^z \in A_{n_0} \Leftrightarrow a_{n_0}^w \notin A_{n_0}$$

and by induction on n_0 it is easy to check that $a_0^z \in A_0 \Leftrightarrow a_0^w \notin A_0$. We need the following

Definition 21.6. A set $Z \subseteq {}^{\omega}2$ is a **flip set** if for all $z, w \in {}^{\omega}2$ if $\{n \in \omega \mid z(n) \neq w(n)\}$ is finite, then

$$(z \in Z \Leftrightarrow w \in Z) \Leftrightarrow |\{n \in \omega \mid z(n) \neq w(n)\}| \text{ is even.}$$

A flip set cannot be measurable nor can have the Baire property (see [**?**, 5.23, 6.41]). In particular, AD implies that flip sets do not exist. Going back to Martin's proof, it is easy to check that

$$Z = \{z \in {}^{\omega}2 \mid a_0^z \in A_0\}$$

is a flip set, contradicting AD. □

Note that AD was used twice, once to get winning strategies for **I** in the Wadge games, and the second time to obtain a contradiction from the existence of a flip set. We will now re-prove Theorem 21.5 under the weaker assumption SLO$^{\mathrm{W}}$ + BP, where BP stands for "All sets have the Baire property". BP is needed to obtain the contradiction from the existence of a flip set. In order to avoid the first use of determinacy in Martin's proof we need to prove a preliminary lemma.

Lemma 21.7. *(a) If $A \leq_{\mathrm{W}} B$ then there is $A' \equiv_{\mathrm{W}} A$ such that $A' \leq_{\mathrm{L}} A$ and $A' \leq_{\mathrm{L}} B$.*

*(b) Assume $\mathsf{SLO}^{\mathrm{W}}$ and $A <_{\mathrm{W}} B$. Then there is $A' \equiv_{\mathrm{W}} A$ such that $A' \leq_{\mathrm{L}} B$ and $A' \leq_{\mathrm{L}} \neg B$ via a contraction, i.e., **I** wins $G_{\mathrm{L}}(\neg B, A')$ and $G_{\mathrm{L}}(B, A')$.*

Proof. (a) Let $\varphi : {}^{<\omega}\omega \to {}^{<\omega}\omega$ be a tame map whose induced function witnesses $A \leq_{\mathrm{W}} B$. The set

$$S = \{s \in {}^{<\omega}\omega \mid \forall t \subset s \ \mathrm{lh}(\varphi(t)) < \mathrm{lh}(\varphi(s))\}$$

is the set of all sequences where φ properly extends its previous values. The idea is to replace S with ${}^{<\omega}\omega$ via the standard enumeration $\langle \boldsymbol{s}_i \mid i \in \omega \rangle$ of ${}^{<\omega}\omega$, so that the resulting map will be Lipschitz. Let $\gamma : {}^{<\omega}\omega \to {}^{<\omega}\omega$ be the continuous map

$$\gamma(t) = \begin{cases} \boldsymbol{s}_{k_0+1}{}^\frown \cdots {}^\frown \boldsymbol{s}_{k_n+1} & \text{if } t = \langle k_0, \ldots, k_n \rangle\,, \\ \emptyset & \text{if } t = \emptyset\,, \end{cases}$$

and let $G = f_\gamma : \mathbb{R} \to \mathbb{R}$ be its induced function. Let

$$\begin{aligned} C &= \bigl\{z \in \mathbb{R} \mid \forall n \ \mathrm{lh}\bigl(\varphi(\gamma(z \restriction n))\bigr) = n\bigr\} \\ &= \{z \in \mathbb{R} \mid \forall n \ \gamma(z \restriction n) \in S\}\,. \end{aligned}$$

Given a real $x \in \mathbb{R}$ we can construct, continuously in x, a real $F(x) = z \in C$ such that $x = \boldsymbol{s}_{z(0)+1}{}^\frown \boldsymbol{s}_{z(1)+1}{}^\frown \cdots$: if $z \restriction n$ has been defined, let s be of least length (which exists since φ is continuous) such that $\boldsymbol{s}_{z(0)+1}{}^\frown \cdots {}^\frown \boldsymbol{s}_{z(n-1)+1}{}^\frown s \subset x$ and $\mathrm{lh}(\varphi(\boldsymbol{s}_{z(0)+1}{}^\frown \cdots {}^\frown \boldsymbol{s}_{z(n-1)+1}{}^\frown s)) = n+1$, and set $z(n) =$ the unique k such that $\boldsymbol{s}_{k+1} = s$. Clearly C is closed and $G \restriction C : C \to \mathbb{R}$, $z \mapsto \boldsymbol{s}_{z(0)+1}{}^\frown \boldsymbol{s}_{z(1)+1}{}^\frown \cdots$ is Lipschitz. It is easy to check (see e.g. [**?**, 2.23]) that there is a Lipschitz $\pi : \mathbb{R} \twoheadrightarrow C$ such that $\pi \restriction C$ is the identity. Let

$$A' = (G \circ \pi)^{-1}\text{“}A\,.$$

Then $A' \leq_{\mathrm{L}} A$, and $A \leq_{\mathrm{W}} A'$ since $A = F^{-1}\text{“}A'$. We are only left to check that $A' \leq_{\mathrm{L}} B$. Since the map $t \mapsto \varphi\bigl(\gamma(\pi(t))\bigr)$ is Lipschitz, its induced function witnesses $A' \leq_{\mathrm{L}} B$.

(b) If $\forall n \left(A \not\leq_{\mathrm{W}} B_{\lfloor n \rfloor}\right)$ then, by $\mathsf{SLO}^{\mathrm{W}}$, $\forall n \left(B_{\lfloor n \rfloor} \leq_{\mathrm{W}} \neg A\right)$ hence $B \leq_{\mathrm{W}} \neg A <_{\mathrm{W}} \neg B$: a contradiction. Similarly $\forall m \left(A \not\leq_{\mathrm{W}} \neg B_{\lfloor m \rfloor}\right)$ does not hold, so fix $n_0, m_0 \in \omega$ such that $A \leq_{\mathrm{W}} B_{\lfloor n_0 \rfloor}, \neg B_{\lfloor m_0 \rfloor}$. Fix tame φ and ψ such that ${f_\varphi}^{-1}\text{“}B_{\lfloor n_0 \rfloor} = A$ and ${f_\psi}^{-1}\text{“}\neg B_{\lfloor m_0 \rfloor} = A$. Define

$$C = \{z \in \mathbb{R} \mid \forall n \ \min\{\mathrm{lh}(\varphi(\gamma(z \restriction n))), \mathrm{lh}(\psi(\gamma(z \restriction n)))\} = n\}\,.$$

Arguing as in part (a), C is closed and fix $\pi : \mathbb{R} \twoheadrightarrow C$, $F : \mathbb{R} \to C$ and $A' = (G \circ \pi)^{-1}$“A as above. Then $A' \leq_{\mathrm{L}} A$ via $G \circ \pi$ and $A \leq_{\mathrm{W}} A'$ via F, and $t \mapsto \varphi\big(\gamma(\pi(t))\big)$ and $t \mapsto \psi\big(\gamma(\pi(t))\big)$ witness $A' \leq_{\mathrm{L}} B_{\lfloor n_0 \rfloor}$ and $A' \leq_{\mathrm{L}} \neg B_{\lfloor m_0 \rfloor}$, respectively. Therefore $A' \leq_{\mathrm{L}} B, \neg B$ via contractions. □

Theorem 21.8. *Assume* $\mathsf{BP} + \mathsf{SLO}^{\mathrm{W}}$. *There is no sequence* $\langle A_n \mid n \in \omega \rangle$ *such that*

$$\cdots <_{\mathrm{W}} A_2 <_{\mathrm{W}} A_1 <_{\mathrm{W}} A_0 \,.$$

Proof. Using 21.7(b), by induction on n define $A'_n \in [A_n]_{\mathrm{W}}$ such that **I** wins $G_{\mathrm{L}}(A'_n, A'_{n+1})$ and $G_{\mathrm{L}}(\neg A'_n, A'_{n+1})$. We can now apply the Martin–Monk method of Theorem 21.5 and reach the desired contradiction:

Let σ^0_n be **I**'s winning strategy in $G_{\mathrm{L}}(A'_n, A'_{n+1})$, and σ^1_n be **I**'s winning strategy in $G_{\mathrm{L}}(\neg A'_n, A'_{n+1})$. For any $z \in {}^{\omega}2$ let $x_n = x^z_n$ be the real in the n-th row of the following diagram where **I** uses $\sigma^{z(n)}_n$ on the n-th row against his opponent on the $(n+1)$-st row:

$$\begin{array}{ccccc} \sigma^{z(0)}_0 & x_0(0) & x_0(1) & \cdots & = x_0 \\ \sigma^{z(1)}_1 & x_1(0) & x_1(1) & \cdots & = x_1 \\ \vdots & \vdots & \vdots & & \vdots \end{array}$$

Thus x^z_n is the result of applying $\sigma^{z(n)}_n$ to x^z_{n+1}. Then $\{z \in {}^{\omega}2 \mid x^z_0 \in A'_0\}$ is a flip set, contradicting BP. □

Exercise 21.9. Assume $\mathsf{BP} + \mathsf{SLO}^{\mathrm{W}} + \mathsf{DC}(\mathbb{R})$. Show that $\leq_{\mathrm{W}}$ is a well-founded pre-order on $\mathscr{P}(\mathbb{R})$.

Therefore $\mathsf{BP} + \mathsf{SLO}^{\mathrm{W}} + \mathsf{DC}(\mathbb{R})$ implies that the Wadge degrees are essentially well-ordered, each level consisting of a single self-dual degree or of a non-self-dual pair. Let $\|A\|_{\mathrm{W}}$ be the rank of A in the well-founded pre-order $<_{\mathrm{W}}$. A degree $[A]_{\mathrm{W}}$ is a successor/limit degree iff $\|A\|_{\mathrm{W}}$ is a successor/limit ordinal.

In order to show that above a self-dual degree there is a non-self-dual pair we need the following tree introduced by Wadge:.

Definition 21.10. For $A \subseteq \mathbb{R}$ let

$$\boldsymbol{T}(A) = \{s \in {}^{<\omega}\omega \mid A_{\lfloor s \rfloor} \equiv_{\mathrm{W}} A\} \,.$$

Lemma 21.11. *Assume* $\mathsf{AC}_\omega(\mathbb{R}) + \mathsf{BP}$. *If* $A \leq_{\mathrm{W}} \neg A$ *then* $\boldsymbol{T}(A)$ *is well-founded.*

Proof. The proof uses the Martin–Monk method and an idea as in [VW78, 3.1]—see also [**?**, 11.31].

Towards a contradiction, let $x \in [\boldsymbol{T}(A)]$ and $A \leq_{\mathrm{W}} \neg A$. Clearly for every $n \in \omega$, **II** wins $G_{\mathrm{W}}(A_{\lfloor x \restriction n \rfloor}, \neg A_{\lfloor x \restriction n \rfloor})$ via some strategy τ_n, and **I** wins $G_{\mathrm{L}}(A_{\lfloor x \restriction n \rfloor}, \neg A_{\lfloor x \restriction n+1 \rfloor})$ via the strategy σ_n that plays $x(n)$ in the first round and then copies **II**'s moves. Let τ be **II**'s copying strategy, i.e., $\tau(s) =$ the last element of s. We are now going to apply the Martin–Monk method. The basic idea is to pit τ against the τ_n's so that a flip set is constructed, contradicting BP. The problem is that τ and τ_n are strategies for **II**, while we need strategies for **I** in order to start the filling-in procedure of the Martin–Monk diagram. This is where the σ_n's come into the picture: an increasing sequence of natural numbers $0 = M_0 < M_1 < M_2 < \cdots$ is defined so that $M_k + 1 < M_{k+1}$ and on the M_k-th row **II** plays using either τ or τ_n for some n, and on the m-th row with $m \notin \{M_k \mid k \in \omega\}$ **I** plays using σ_n for an appropriate n. More precisely, letting $\langle N_j \mid j \in \omega \rangle$ be the enumeration of $\omega \setminus \{M_k \mid k \in \omega\}$

- on the M_k-th row **II** is playing $G_{\mathrm{W}}(A_{\lfloor x \restriction i \rfloor}, \neg A_{\lfloor x \restriction i \rfloor})$ using τ_i or τ, where $N_i = M_k + 1$;
- on the N_j-th row **I** is playing $G_{\mathrm{L}}(A_{\lfloor x \restriction j \rfloor}, \neg A_{\lfloor x \restriction j+1 \rfloor})$ using σ_j.

E.g., if the first few M_k's are $\langle 0, 3, 5, \dots \rangle$, then the table below summarizes the games and strategies used.

Row	Game	Strategy
$M_0 = 0$	$G_{\mathrm{W}}(A, \neg A)$	τ or τ_0
$N_0 = 1$	$G_{\mathrm{L}}(A, \neg A_{\lfloor x \restriction 1 \rfloor})$	σ_0
$N_1 = 2$	$G_{\mathrm{L}}(A_{\lfloor x \restriction 1 \rfloor}, \neg A_{\lfloor x \restriction 2 \rfloor})$	σ_1
$M_1 = 3$	$G_{\mathrm{W}}(A_{\lfloor x \restriction 2 \rfloor}, \neg A_{\lfloor x \restriction 2 \rfloor})$	τ or τ_2
$N_2 = 4$	$G_{\mathrm{L}}(A_{\lfloor x \restriction 2 \rfloor}, \neg A_{\lfloor x \restriction 3 \rfloor})$	σ_2
$M_2 = 5$	$G_{\mathrm{W}}(A_{\lfloor x \restriction 3 \rfloor}, \neg A_{\lfloor x \restriction 3 \rfloor})$	τ or τ_3
$N_3 = 6$	$G_{\mathrm{L}}(A_{\lfloor x \restriction 3 \rfloor}, \neg A_{\lfloor x \restriction 4 \rfloor})$	σ_3
$\vdots$	$\vdots$	$\vdots$

The choice of using τ versus τ_i on the M_k-th row, where i is such that $M_k + 1 = N_i$, will be given by $z(k)$ where $z \in {}^{\omega}2$: τ will be used when $z(k) = 0$, and τ_i will be used when $z(k) = 1$. For example, if the first few M_k's are, as before, $\langle 0, 3, 5, \dots \rangle$ and $z = \langle 0, 1, 1, 0, \dots \rangle$ then the strategies in the Martin–Monk diagram are: $\langle \tau, \sigma_0, \sigma_1, \tau_2, \sigma_2, \tau_3, \sigma_3, \dots \rangle$. Since τ is the copying strategy, for any $s \in {}^{<\omega}\omega$, τ's response to s is a sequence of the same length as s, that is $\mathrm{lh}((s * \tau)_{\mathbf{II}}) = \mathrm{lh}(s)$. On the other hand, the τ_n's might pass, so it can happen that τ_n's response to s is a sequence of shorter length, $\mathrm{lh}((s * \tau_n)_{\mathbf{II}}) < \mathrm{lh}(s)$. Therefore the M_k's are to be taken sparse enough so that the filling-in procedure does not come to a grinding halt. Suppose $\langle M_k \mid k \leq n \rangle$ has been defined, and suppose that for every $s \in {}^{n+1}2$ the finite diagram with M_n-many rows and with τ is used on the M_k-th row when $s(k) = 0$, has the first n entries of the 0-th row are filled-in. Fix any $s \in {}^{n+1}2$. Suppose the $M_n + 1$-st row is filled-in with $\langle x(m + i) \mid m \in \omega \rangle$, where i is such that $N_i = M_n + 1$. Then the 0-th row is completely determined, so by continuity of the strategies there is a $j = j(s) > 0$ such that $\langle x(i), x(i + 1), \dots x(i + j - 1) \rangle$ is enough to fill-in the first $n + 1$ entries of the 0-th row. Therefore if the σ's are used in the rows $M_n + 1, \dots, M_n + j$, then the first $n + 1$ entries of the 0-th row are filled-in. Let

$$M_{n+1} = \sup\{M_n + j(s) + 1 \mid s \in {}^{n+1}2\}\,.$$

By construction, for any $s \in {}^{n+1}2$ the finite diagram with M_{n+1}-many rows and with τ used on the M_k-th row when $s(k) = 1$ has the first $n+1$ entries of the 0-th row filled-in, and therefore for any $z \in {}^{\omega}2$ the Martin–Monk diagram relative to z can be filled-in. Arguing as in Theorem 21.5, $\{z \in {}^{\omega}2 \mid$ the real in the 0-th row of the Martin–Monk diagram relative to z is in $A\}$ is a flip set, contradicting BP. □

Lemma 21.11 admits a converse. Given $\langle A_n \mid n \in \omega \rangle$, then $\forall m\, (A_m \leq_{\mathrm{W}} \bigoplus_n A_n)$ and if $\forall m\, (A_m \leq_{\mathrm{W}} C)$ then **II** wins $G_{\mathrm{W}}(\bigoplus_n A_n, C)$: if **I** plays n_0 then **II** passes and then applies the winning strategy witnessing $A_{n_0} \leq_{\mathrm{W}} C$. In particular if $A_m <_{\mathrm{W}} \bigoplus_n A_n$ for each m, then by $\mathsf{SLO}^{\mathrm{W}}$ $\forall m\, (\neg A_m <_{\mathrm{W}} \bigoplus_n A_n)$ hence

$$\neg \bigoplus_n A_n = \bigoplus_n \neg A_n \leq_{\mathrm{W}} \bigoplus_n A_n\,,$$

i.e., $\bigoplus_n A_n$ is Wadge self-dual. Therefore if A is a $\leq_{\mathrm{W}}$-least upper bound of $A_n <_{\mathrm{W}} A$ (i.e., $A_n \leq_{\mathrm{W}} A$ for all n, and there is no C such that $\forall n\, (A_n <_{\mathrm{W}} C <_{\mathrm{W}} A)$) then $A \equiv_{\mathrm{W}} \bigoplus_n A_n$ is Wadge self-dual.

Lemma 21.12. *Assume* $\mathsf{SLO}^{\mathrm{W}}$. *If A is Wadge non-self-dual then $\boldsymbol{T}(A)$ is pruned and therefore ill-founded.*

Proof. Suppose $s \in \mathbf{tn}(\boldsymbol{T}(A))$. Then

$$A_{\lfloor s^\frown\langle m\rangle\rfloor} <_{\mathrm{W}} A_{\lfloor s\rfloor} = \bigoplus_n A_{\lfloor s^\frown\langle n\rangle\rfloor}$$

and clearly $A_{\lfloor s\rfloor}$ is the $\leq_{\mathrm{W}}$-least upper bound of the $A_{\lfloor s^\frown\langle m\rangle\rfloor}$'s. Therefore $A \equiv_{\mathrm{W}} A_{\lfloor s\rfloor}$ is Wadge self-dual. □

Corollary 21.13. *Assume* $\mathsf{SLO}^{\mathrm{W}}$. *A is Wadge self-dual iff there is a partition of* $\mathbb{R}$ *into clopen sets* $\mathbb{R} = \bigcup_n D_n$*, such that* $\forall n\,(A \cap D_n <_{\mathrm{W}} A)$.

Proof. If $\boldsymbol{T}(A)$ is well-founded then

$$A = \bigoplus_{s\in\partial\boldsymbol{T}(A)} A_{\lfloor s\rfloor} = \bigcup_{s\in\partial\boldsymbol{T}(A)} \boldsymbol{N}_s \cap A\,.$$

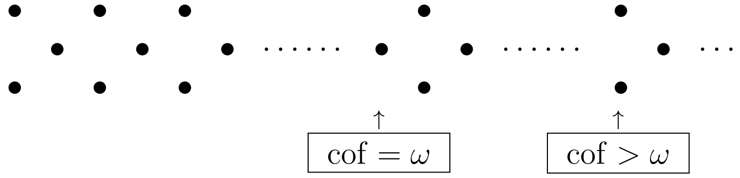
Since $\langle \boldsymbol{N}_s \mid s \in \partial\boldsymbol{T}(A)\rangle$ is a clopen partition of $\mathbb{R}$ we are done.

Conversely, suppose $\langle D_n \mid n \in \omega\rangle$ be a partition of $\mathbb{R}$ into clopen sets such that $A \cap D_n <_{\mathrm{W}} A$. By $\mathsf{SLO}^{\mathrm{W}}$ $A \cap D_n \leq_{\mathrm{W}} \neg A$ via some f_n, hence $A \leq_{\mathrm{W}} \neg A$ via $\bigcup_n f_n \restriction D_n$. □

We can now show that immediately above a self-dual degree there is a non-self-dual pair. Suppose $[A]_{\mathrm{W}} <_{\mathrm{W}} [B]_{\mathrm{W}}$ are both self-dual and let $s \in \mathbf{tn}(\boldsymbol{T}(B))$, which exists by the well-foundedness of $\boldsymbol{T}(B)$. If $\forall n\,\big(B_{\lfloor s^\frown\langle n\rangle\rfloor} \leq_{\mathrm{W}} A\big)$ then $B_{\lfloor s\rfloor} \leq_{\mathrm{W}} A$, which is absurd, so $\exists n_0\,\big(A <_{\mathrm{W}} B_{\lfloor s^\frown\langle n_0\rangle\rfloor} <_{\mathrm{W}} B_{\lfloor s\rfloor}\big)$. Thus $[B]_{\mathrm{W}}$ cannot be the immediate successor of $[A]_{\mathrm{W}}$, and therefore immediately above a self-dual degree there is a non-self-dual pair.

We have already seen that if $[A]_{\mathrm{W}}$ is the least upper bound of an increasing ω sequence of degrees then it is self-dual. Conversely, if $[A]_{\mathrm{W}}$ is limit and self-dual, then $A = \bigoplus_{s\in\partial\boldsymbol{T}(A)} A_{\lfloor s\rfloor}$, and $[A]_{\mathrm{W}}$ is the $\leq_{\mathrm{W}}$-least upper bound of $\{A_{\lfloor s\rfloor} \mid s \in \partial\boldsymbol{T}(A)\}$. Since $\boldsymbol{T}(A)$ is countable, $\mathrm{cof}(\|A\|_{\mathrm{W}}) = \omega$.

Therefore $\mathsf{BP} + \mathsf{SLO}^{\mathrm{W}} + \mathsf{DC}(\mathbb{R})$ yield the usual picture for the Wadge hierarchy: at levels of countable cofinality there is a single self-dual degree, at levels of uncountable cofinality there is a non-self-dual pair.



Exercise 21.14. If $s^\frown A \leq_{\mathrm{L}} B$ for some $s \in {}^{<\omega}\omega\setminus\{\emptyset\}$, then $\mathbf{I}$ wins $G_{\mathrm{L}}(\neg B, A)$, i.e., A is reducible to B via a contraction.

Exercise 21.15. Show that if $\mathrm{Res}_2(\neg B) \neq \emptyset$ then **II** wins $G_{\mathrm{W}}(A, B)$ for any $A \in \mathrm{Diff}(2; \boldsymbol{\Pi}^0_1)$.

Conclude that $\Delta(\mathrm{Diff}(2; \boldsymbol{\Pi}^0_1)) \setminus (\boldsymbol{\Pi}^0_1 \cup \boldsymbol{\Sigma}^0_1)$ is the $\leq_{\mathrm{W}}$ successor of $(\boldsymbol{\Pi}^0_1 \setminus \boldsymbol{\Sigma}^0_1, \boldsymbol{\Sigma}^0_1 \setminus \boldsymbol{\Pi}^0_1)$ and the $\leq_{\mathrm{W}}$ predecessor of $(\mathrm{Diff}(2; \boldsymbol{\Pi}^0_1) \setminus \breve{\mathrm{Diff}}(2; \boldsymbol{\Pi}^0_1), \breve{\mathrm{Diff}}(2; \boldsymbol{\Pi}^0_1) \setminus \mathrm{Diff}(2; \boldsymbol{\Pi}^0_1))$.

Exercise 21.16. Assume SLO$^{\mathrm{W}}$.

(i) Show that if $[A]_{\mathrm{W}}$ is self-dual then $[A \oplus \neg A]_{\mathrm{W}}$ is the immediate successor of $([A]_{\mathrm{W}}, [\neg A]_{\mathrm{W}})$.

(ii) Show that $[A_0 \oplus (A_1 \oplus \cdots (A_{n-1} \oplus A_n) \cdots)]_{\mathrm{W}}$ is the least upper bound of $[A_0]_{\mathrm{W}}, \ldots, [A_n]_{\mathrm{W}}$.

Lemma **??** admits a converse.

Lemma 21.17. *Assume* SLO$^{\mathrm{W}}$.

(a) If $\boldsymbol{T}(A)$ is well-founded then $A \equiv_{\mathrm{W}} \neg A$.

(b) If $[A]_{\mathrm{W}}$ is self-dual and limit then it is a join-degree, i.e., it is the least upper bound of an ω-sequence $[B_0]_{\mathrm{W}} <_{\mathrm{W}} [B_1]_{\mathrm{W}} <_{\mathrm{W}} \ldots$

Proof. (a) Replacing A with $A_{\lfloor s \rfloor}$ with $s \in \mathbf{tn}\boldsymbol{T}(A)$, we may assume that $\boldsymbol{T}(A) = \{\emptyset\}$. Then $\forall n \left(A_{\lfloor n \rfloor} <_{\mathrm{W}} A\right)$ hence by SLO$^{\mathrm{W}}$, $\forall n \left(A_{\lfloor n \rfloor} \leq_{\mathrm{W}} \neg A\right)$ and therefore $A = \bigoplus_n A_{\lfloor n \rfloor} \leq_{\mathrm{W}} \neg A$.

(b) As before we may assume $\boldsymbol{T}(A) = \{\emptyset\}$ so that $\forall n \left(A_{\lfloor n \rfloor} <_{\mathrm{W}} A\right)$ and we can take $B_n = A_0 \oplus \cdots \oplus A_n$. $B_n <_{\mathrm{W}} A$, since $[A]_{\mathrm{W}}$ is limit, and if the $[B_n]_{\mathrm{W}}$'s were bounded by $[C]_{\mathrm{W}} <_{\mathrm{W}} [A]_{\mathrm{W}}$, then $A = \bigoplus_n A_n \leq_{\mathrm{W}} \bigoplus_n B_n \leq_{\mathrm{W}} C$, a contradiction. Lastly, by thinning-down the sequence of the B_n's we may assume that they are strictly increasing. $\square$

Exercise 21.18. For $A \subseteq \mathbb{R}$ let $\langle \mathrm{Res}_\alpha(A) \mid \alpha < \omega_1 \rangle$ be the sequence of sets defined by:

$$\begin{aligned} \mathrm{Res}_0(A) &= A \\ \mathrm{Res}_\alpha(A) &= A \cap \bigcap_{\beta < \alpha} \mathrm{Cl}\left(\mathrm{Res}_\beta(\neg A)\right). \end{aligned}$$

(i) Show that

$$\begin{aligned} &\alpha < \beta \Rightarrow \mathrm{Res}_\alpha(A) \supseteq \mathrm{Res}_\beta(A)\,, \\ &\mathrm{Res}_{\alpha+1}(A) = X \cap \mathrm{Cl}\left(\mathrm{Res}_\alpha(\neg A)\right), \\ &\mathrm{Res}_\lambda(A) = \bigcap_{\alpha < \lambda} \mathrm{Res}_\alpha(A) \qquad (\lambda \text{ limit}), \\ &\mathrm{Res}_\alpha(A) = \emptyset \Rightarrow \mathrm{Res}_{\alpha+1}(\neg A) = \emptyset\,. \end{aligned}$$

Thus, for example,

$$\begin{aligned}\mathrm{Res}_1(A) &= A \cap \mathrm{Cl}\,(\neg A)\\ &= \{a \in A \mid \exists \langle b_n \mid n \in \omega\rangle \in {}^{\omega}(\neg A)\ \lim_n b_n = a\}\end{aligned}$$

is the set of points in A which are limit of points in $\neg A$, and $\mathrm{Res}_2(A)$ is the set of points in A which are limit of points in $\neg A$ which are limit of points in A. $\mathrm{Res}_2(A)$ is sometimes called, after Hausdorff, the **first residue of** A.

(ii) Show that

$$A \in \mathbf{\Sigma}^0_1 \Leftrightarrow \mathrm{Res}_1(A) = \emptyset$$

and that if $\mathrm{Res}_2(A) = \emptyset$ then $\neg A \cap \mathrm{Cl}(A)$ is closed, and therefore $A \in \mathrm{Diff}(2; \mathbf{\Pi}^0_1)$. Conversely show that for U, V open sets $\mathrm{Res}(U \setminus V) = \emptyset$ and therefore

$$A \in \mathrm{Diff}(2; \mathbf{\Pi}^0_1) \Leftrightarrow \mathrm{Res}_2(A) = \emptyset\,.$$

Conclude that if $A \in \Delta(\mathrm{Diff}(2; \mathbf{\Pi}^0_1))$ then

$$\forall x \in \mathbb{R}\, \exists k \in \omega\, \big(A_{\lfloor x \restriction k \rfloor} \text{ is open or closed}\big)\,.$$

(The converse implication will be proved in 23.7.)

(iii) Show that if $\mathrm{Res}_{2\alpha}(A) = \emptyset$ then

$$A = \bigcup_{\beta < \alpha} \mathrm{Cl}(\mathrm{Res}_{2\beta}(A)) \setminus \mathrm{Cl}(\mathrm{Res}_{2\beta+1}(\neg A)) \in \mathrm{Diff}(2\alpha; \mathbf{\Pi}^0_1)\,,$$

and that if $\mathrm{Res}_{2\alpha+1}(A) = \emptyset$ then

$$\neg A = \bigcup_{\beta \leq \alpha} \mathrm{Cl}(\mathrm{Res}_{2\beta}(\neg A)) \setminus \mathrm{Cl}(\mathrm{Res}_{2\beta+1}(A)) \in \mathrm{Diff}(2\alpha + 1; \mathbf{\Pi}^0_1)\,,$$

Therefore

$$\{A \subseteq \mathbb{R} \mid \exists \alpha < \omega_1\ \mathrm{Res}_\alpha(A) = \emptyset\} \subseteq \mathbf{\Delta}^0_2\,.$$

(The other inclusion will be proved in 23.3.)

(iv) Show that $x \in \mathrm{Res}_\alpha(A)$ implies $\langle x(n+i) \mid i \in \omega\rangle \in \mathrm{Res}_\alpha(A_{\lfloor x \restriction n \rfloor}) \neq \emptyset$.

22 Operations on the Degrees

This and the next section are devoted to the detailed analysis of the Wadge hierarchy. They safely can (and probably should) be skipped on first reading. We will develop the theory under $\mathsf{BP} + \mathsf{AD}^{\mathrm{L}}$. The assumption that all sets have the property of Baire (or even that there are no flip sets) comes from the Martin–Monk theorem **??** and the Steel–Van Wesep theorem **??**.

22.A The Next Non-Self-Dual Pair

We have seen how the maps

$$A \mapsto A \oplus \neg A$$
$$\langle A_n \mid n \in \omega \rangle \mapsto \bigoplus_n A_n$$

allow us to construct the immediate successor of a non-self-dual degree, and the supremum of an ω-sequence of degrees. There is a uniform way to construct from a self-dual A a pair of non-self-dual sets immediately above A.

Definition 22.1. Let $T^{\triangledown}$ be the tree $\{0^{(n)} \mid n \in \omega\}$ given by the initial segments of $\vec{0}$. For $A \subseteq \mathbb{R}$ let

$$\begin{aligned} A^{\triangledown} &= \boldsymbol{S}(A; T^{\triangledown}) \\ &= \bigcup_{n,m} 0^{(n)} {}^\frown \langle m+1 \rangle {}^\frown A \end{aligned}$$

and

$$\begin{aligned} A^{\circ} &= \boldsymbol{S}(A; T^{\triangledown}) \cup [T^{\triangledown}] \\ &= A^{\triangledown} \cup \{\vec{0}\} \end{aligned}$$

Exercise 22.2. Show that:

(i) $A \leq_{\mathrm{W}} A^{\triangledown}, A^{\circ}$,

(ii) if $A \leq_{\mathrm{W}} B$ (or $A \leq_{\mathrm{L}} B$) then $A^{\triangledown} \leq_{\mathrm{W}} B^{\triangledown}$ and $A^{\circ} \leq_{\mathrm{W}} B^{\circ}$ (respectively: $A^{\triangledown} \leq_{\mathrm{L}} B^{\triangledown}$ and $A^{\circ} \leq_{\mathrm{L}} B^{\circ}$),

(iii) $A^{\triangledown\triangledown} \equiv_{\mathrm{W}} A^{\triangledown}$ and $A^{\circ\circ} \equiv_{\mathrm{W}} A^{\circ}$.

By (ii) we can define the operations $^{\triangledown}$ and $^{\circ}$ on the Wadge degrees by letting $[A]_{\mathrm{W}}^{\triangledown} = [A^{\triangledown}]_{\mathrm{W}}$ and $[A]_{\mathrm{W}}^{\circ} = [A^{\circ}]_{\mathrm{W}}$, and similarly for the Lipschitz degrees.

Proposition 22.3. *Assume* $\mathsf{BP}+\mathsf{AD}^{\mathrm{L}}$. *Suppose A is self-dual. Then* $([A^{\triangledown}]_{\mathrm{W}}, [A^{\circ}]_{\mathrm{W}})$ *is the least non-self-dual pair above* $[A]_{\mathrm{W}}$.

Proof. We first show that $A <_{\mathrm{W}} A^{\triangledown}$ and that $A^{\circ} \nleq_{\mathrm{W}} A^{\triangledown}$ and $A^{\triangledown} \nleq_{\mathrm{W}} A^{\circ}$. By 22.2(i) **II** has a winning strategy in $G_{\mathrm{W}}(A, A^{\triangledown})$ so it is enough to show that **II** does not have a winning strategy in $G_{\mathrm{W}}(A^{\triangledown}, A)$. Towards a contradiction let τ be a winning strategy for **II** and let n be least such that τ's response to $0^{(n)}$ is in $\partial\boldsymbol{T}(A)$, i.e.,

$$(0^{(n)} * \tau)_{\mathbf{II}} = s \in \partial\boldsymbol{T}(A)\,.$$

Thus

$$A^{\triangledown} = A^{\triangledown}_{\lfloor 0^{(n)}\rfloor} \leq_{\mathrm{W}} A_{\lfloor s\rfloor} <_{\mathrm{W}} A\,,$$

a contradiction. Therefore $A <_{\mathrm{W}} A^{\triangledown}$.

The proof that no strategy τ for **II** in $G_{\mathrm{W}}(A^{\triangledown}, A^{\circ})$ can be winning is analogous. If $\vec{0} * \tau = \vec{0}$ then **II** would lose since $\vec{0} \in A^{\circ} \setminus A^{\triangledown}$, so let n be least such that $\tau(0^{(n)}) = m \neq 0$. Then **I**'s position is $0^{(n)}$ while **II**'s position is $0^{(k)}{}^\frown\langle m\rangle$ with $k \leq n$. Then

$$A^{\triangledown} = A^{\triangledown}_{\lfloor 0^{(n)}\rfloor} \leq_{\mathrm{W}} A^{\circ}_{\lfloor 0^{(k)}{}^\frown\langle m\rangle\rfloor} = A$$

which is impossible.

Similarly $A^{\circ} \not\leq_{\mathrm{W}} A^{\triangledown}$.

Therefore $([A^{\triangledown}]_{\mathrm{W}}, [A^{\circ}]_{\mathrm{W}})$ is a non-self-dual pair above $[A]_{\mathrm{W}}$. We now show it is the least such.

Let B be non-self-dual so that by **??** $\boldsymbol{T}(B)$ is pruned, and such that $A <_{\mathrm{W}} B$.

Claim 22.3.1.

$$[\boldsymbol{T}(B)] \cap \neg B \neq \emptyset \;\;\Rightarrow\;\; A^{\triangledown} \leq_{\mathrm{L}} B$$
$$[\boldsymbol{T}(B)] \cap B \neq \emptyset \;\;\Rightarrow\;\; A^{\circ} \leq_{\mathrm{L}} B$$

Proof. Suppose $b \in [\boldsymbol{T}(B)] \setminus B$. Then **II** wins $G_{\mathrm{L}}(A^{\triangledown}, B)$ as follows:

> As long as **I** plays $0^{(n)}$ then **II** plays $b \restriction n$, so that if $\vec{0}, b$ are the final outcome then **II** has won. If at some stage **I** first plays $m \neq 0$ then **II** still follows b so that after this round **I** and **II** have reached positions $0^{(n-1)}{}^\frown\langle m\rangle$ and $b \restriction n$, respectively. Then $A^{\triangledown}_{\lfloor 0^{(n-1)}{}^\frown\langle m\rangle\rfloor} = A$ and $B_{\lfloor b\restriction n\rfloor} \equiv_{\mathrm{L}} B$, since $b \restriction n \in \boldsymbol{T}(B)$ and by **??**(c), so **II** can now use his winning strategy in $G_{\mathrm{L}}(A, B)$.

The proof that **II** wins $G_{\mathrm{L}}(A^{\circ}, B_{\lfloor b\restriction n\rfloor})$ is analogous. □

This proves that $([A^{\triangledown}]_{\mathrm{W}}, [A^{\circ}]_{\mathrm{W}})$ is the immediate successor of $[A]_{\mathrm{W}}$. □

We can now give a characterization of the limit non-self-dual sets in terms of $\boldsymbol{T}$.

Proposition 22.4. *Assume* $\mathsf{BP} + \mathsf{AD}^{\mathrm{L}}$. *Let* A *be non-self-dual. Then*

$$[A]_{\mathrm{W}} \text{ is limit} \quad \Leftrightarrow \quad [\boldsymbol{T}(A)] \cap A \neq \emptyset \;\&\; [\boldsymbol{T}(A)] \cap \neg A \neq \emptyset \,.$$

Proof. ($\Longleftarrow$) It is clear that $A \neq \emptyset, \mathbb{R}$, so it is enough to show that $[A]_{\mathrm{W}}$ is not a successor. Let $C <_{\mathrm{W}} A$ be self-dual. Then by Claim 22.3.1 $C^{\triangledown}, C^{\circ} \leq_{\mathrm{W}} A$ and since $C^{\triangledown} \not\equiv_{\mathrm{W}} C^{\circ}$, then $C^{\triangledown}, C^{\circ} <_{\mathrm{W}} A$.

($\Longrightarrow$) Suppose instead $[A]_{\mathrm{W}}$ is limit and, say, that $[\boldsymbol{T}(A)] \subseteq A$. Fix an enumeration $\langle A_n \mid n < \omega \rangle$ of $\{A_{\lfloor s \rfloor} \mid s \notin \boldsymbol{T}(A)\}$. Pick

$$A_n <_{\mathrm{W}} B_n <_{\mathrm{W}} A$$

such that the B_n's are self-dual. This can be done as follows: if A_n is non-self-dual, let $B_n = A_n \oplus \neg A_n$, if A_n is self-dual let $B_n = A_n^{\triangledown} \oplus A_n^{\circ}$. Then $B = \bigoplus_n B_n \leq_{\mathrm{W}} A, \neg A$ is self-dual by **??**(iv), and therefore $B <_{\mathrm{W}} A$. Let $C = B^{\circ}$ so that C is non-self-dual, $C <_{\mathrm{W}} A$ and $[\boldsymbol{T}(C)] \subseteq C$. (Had we assumed $[\boldsymbol{T}(A)] \cap A = \emptyset$ we would let C be $B^{\triangledown}$.) Then **II** wins $G_{\mathrm{L}}(A, C)$ as follows:

> As long as **I**'s position is in $\boldsymbol{T}(A)$ then **II** enumerates some fixed $c \in [\boldsymbol{T}(C)]$. (Although irrelevant for the proof, notice that c must be $\vec{0}$.) If at some round n **I** reaches a position $s \in \partial \boldsymbol{T}(A)$ then **II** plays $c(n)$: at this point the two positions of the players are s and $c \restriction n{+}1$ and $A_{\lfloor s \rfloor} <_{\mathrm{W}} A$ and $C_{\lfloor c \restriction n+1 \rfloor} \equiv_{\mathrm{W}} C$. Since $A_{\lfloor s \rfloor} = A_k$ for some k, and $A_k <_{\mathrm{W}} B_k \leq_{\mathrm{W}} B <_{\mathrm{W}} C$, then $A_{\lfloor s \rfloor} <_{\mathrm{W}} C$, so **II** can now use his winning strategy in $G_{\mathrm{L}}(A_{\lfloor s \rfloor}, C)$.

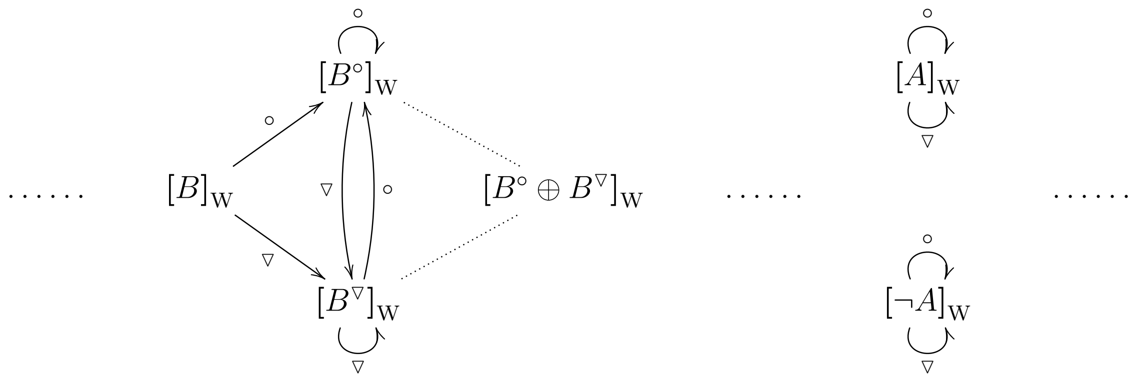
Therefore $A \leq_{\mathrm{W}} C$, hence A cannot be limit: a contradiction. □

Exercise 22.5. Assume $\mathsf{BP} + \mathsf{AD}^{\mathrm{L}}$.

(i) Show that if A is successor and non-self-dual then $A \equiv_{\mathrm{W}} A^{\triangledown}$ or $A \equiv_{\mathrm{W}} A^{\circ}$ depending on whether $[\boldsymbol{T}(A)] \subseteq \neg A$ or $[\boldsymbol{T}(A)] \subseteq A$. Conclude that for B self-dual $B^{\triangledown\circ} \equiv_{\mathrm{W}} B^{\circ}$ and $B^{\circ\triangledown} \equiv_{\mathrm{W}} B^{\triangledown}$.

(ii) Show that for A limit and non-self-dual then $A \equiv_{\mathrm{W}} A^{\triangledown} \equiv_{\mathrm{W}} A^{\circ}$.

The following diagram summarizes the situation when $[B]_{\mathrm{W}}$ is self-dual and $\operatorname{cof}(\|A\|_{\mathrm{W}}) > \omega$:



22.B Addition of Degrees

Using the operations $A \mapsto A^{\triangledown}, A^{\circ}$ and $\langle A_n \mid n \in \omega\rangle \mapsto \bigoplus_n A_n$ it is easy to construct the next ω_1 degrees above any $[A]_{\mathrm{W}}$. We will now define the **sum of two sets** $A + B$ in such a way that $\|A + B\|_{\mathrm{W}} = \|A\|_{\mathrm{W}} + 1 + \|B\|_{\mathrm{W}}$, whenever A is self-dual.

Definition 22.6. (i) For $x \in {}^{\omega}\omega$ or $x \in {}^{<\omega}\omega$ let

$$x + 1 = \langle x(n) + 1 \mid n \in \operatorname{lh}(x)\rangle$$

and for $X \subseteq \mathbb{R}$ or $X \subseteq {}^{<\omega}\omega$ let

$$X^{+} = \{x + 1 \mid x \in X\}\,.$$

(ii) For $A, B \subseteq \mathbb{R}$ let

$$A + B = \{(s + 1)^\frown\langle 0\rangle^\frown x \mid s \in {}^{<\omega}\omega \ \&\ x \in A\} \cup B^{+}\,.$$

Therefore $B^{+} = \emptyset + B$.

Exercise 22.7. Show that $A_0 \leq_{\mathrm{L}} A_1 \ \&\ B_0 \leq_{\mathrm{L}} B_1 \Rightarrow A_0 + B_0 \leq_{\mathrm{L}} A_1 + B_1$, and similarly for $\leq_{\mathrm{W}}$.

Therefore we can define the **addition of two Lipschitz/Wadge degrees** by

$$[A] + [B] = [A + B]\,.$$

Note that the sum of two degrees is far from being commutative, that is $A + B \not\equiv_{\mathrm{W}} B + A$. On the other hand part (v) below shows it is associative.

Exercise 22.8. (i) Show that $A^{\triangledown} \equiv_{\mathrm{L}} A+\emptyset$ and $A^{\circ} \equiv_{\mathrm{L}} A+\mathbb{R}$. Conclude that (assuming $\mathsf{BP}+\mathsf{AD}^{\mathrm{L}}$) if A is self-dual and B is arbitrary, then $A <_{\mathrm{W}} A+B$.

(ii) Show that

$$(A+B)_{\lfloor s+1 \rfloor} = A + B_{\lfloor s \rfloor}$$

and

$$(A+B)_{\lfloor (s+1)^\frown \langle 0 \rangle \rfloor} = A\,.$$

(iii) Show that $\neg(A+B) = \neg A + \neg B$. In particular, for A self-dual, $\neg(A+B) \equiv_{\mathrm{W}} A + \neg B$.

(iv) Show that $\bigoplus_n (A+B_n) \equiv_{\mathrm{W}} A + (\bigoplus_n B_n)$.

(v) Show that $(A+B)+C \equiv_{\mathrm{W}} A+(B+C)$.

(vi) Show that $\emptyset + \emptyset = \emptyset$ and $\mathbb{R}+\mathbb{R} = \mathbb{R}$.

(vii) Let C be closed and U be open. Show that $\emptyset + C$ is closed and $\mathbb{R}+U$ is open. Show that if $C \neq \mathbb{R}$ and $U \neq \emptyset$ then $\emptyset + U$ is not open and $\mathbb{R}+C$ is not closed.

Lemma 22.9. *Assume* $\mathsf{BP}+\mathsf{AD}^{\mathrm{L}}$. *For* $A, B, B' \subseteq \mathbb{R}$ *and* A *self-dual*

$$A+B \leq_{\mathrm{W}} A+B' \Leftrightarrow B \leq_{\mathrm{W}} B'\,.$$

Proof. ($\Longleftarrow$) follows from 22.7 so it is enough to prove ($\Longrightarrow$).

Suppose τ is a winning strategy for **II** in $G_{\mathrm{W}}(A+B, A+B')$. It is enough to show that **II** is not the first player to play 0, i.e.,

$$\neg\exists s,t \in {}^{<\omega}\omega \left(\tau(s+1) = (t+1)^\frown\langle 0\rangle\right)$$

since then **II** wins $G_{\mathrm{W}}(B, B')$ using τ' defined by

$$\tau'(s) = t \Leftrightarrow \tau(s+1) = t+1\,.$$

Towards a contradiction suppose $\tau(s+1) = (t+1)^\frown\langle 0\rangle$, for some $s,t \in {}^{<\omega}\omega$. Then

$$\begin{aligned} A &<_{\mathrm{W}} A+B_{\lfloor s\rfloor} && \text{(by 22.8(i))}\\ &= (A+B)_{\lfloor s+1\rfloor} && \text{(by 22.8(ii))}\\ &\leq_{\mathrm{W}} (A+B')_{\lfloor (t+1)^\frown\langle 0\rangle\rfloor} && \text{(since } \tau \text{ is winning)}\\ &= A && \text{(by 22.8(ii))}\end{aligned}$$

a contradiction. $\square$

The lemma fails if A is non-self-dual—see 22.19. Therefore for A self-dual the map $A \mapsto A+B$ is $\leq_{\mathrm{W}}$-preserving. The next result shows that it induces an isomorphism between the whole collection of Wadge degrees and the ones above $[A]_{\mathrm{W}}$. Again 22.19 shows that the assumption that A is self-dual is crucial.

Lemma 22.10. *Assume* $\mathsf{BP} + \mathsf{AD}^{\mathrm{L}}$. *Suppose* A *is self-dual and* $A <_{\mathrm{W}} B$. *Then*

$$\exists C \subseteq \mathbb{R}\,(A + C \equiv_{\mathrm{W}} B)\ .$$

Proof. Let

$$T = \{s \in {}^{<\omega}\omega \mid A <_{\mathrm{W}} B_{\lfloor s \rfloor}\}\ .$$

T is a tree since $\emptyset \in T$ by hypothesis and $t \subset s \Rightarrow B_{\lfloor s \rfloor} \leq_{\mathrm{W}} B_{\lfloor t \rfloor}$. Also if $\forall n \left(B_{\lfloor s^\frown\langle n\rangle \rfloor} \leq_{\mathrm{W}} A\right)$ then $B_{\lfloor s \rfloor} \leq_{\mathrm{W}} A$, hence T is pruned. Let $\varphi : {}^{<\omega}\omega \twoheadrightarrow T$ be Lipschitz and such that $\varphi \restriction T = \mathrm{id}$, and let $f : \mathbb{R} \to \mathbb{R}$ be the map induced by φ (see **??**). Let $C = f^{-1}``B$ so that $B \cap [T] = C \cap [T]$. We claim that

$$B \equiv_{\mathrm{W}} A + C\ .$$

II wins $G_{\mathrm{W}}(B, A + C)$ as follows:

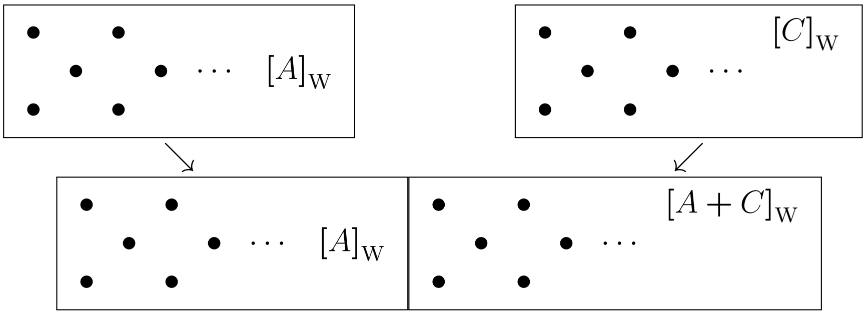
> As long as **I** plays $s \in T$ then **II** plays $s + 1$: if **I**'s play x belongs to $[T]$ then
>
> $$x \in B \Leftrightarrow x \in B \cap [T] = C \cap [T] \Leftrightarrow x + 1 \in A + C$$
>
> so **II** wins. If at some point **I** reaches a position $s \notin T$, by $\mathsf{SLO}^{\mathrm{W}}$, then $B_{\lfloor s \rfloor} \leq_{\mathrm{W}} A$ so **II** plays 0 and then uses his strategy in $G_{\mathrm{W}}(B_{\lfloor s \rfloor}, A)$: in this case **I**'s play will be $s^\frown x \notin [T]$ and **II**'s play will be $(s+1)^\frown\langle 0\rangle^\frown y$ so that
>
> $$s^\frown x \in B \Leftrightarrow x \in B_{\lfloor s \rfloor} \Leftrightarrow y \in A \Leftrightarrow (s+1)^\frown\langle 0\rangle^\frown y \in A + C\ ,$$
>
> hence **II** wins.

And here is a winning strategy for **II** in $G_{\mathrm{W}}(A + C, B)$:

> As long as **I** plays $s + 1$, then **II** plays $\varphi(s)$: if $x + 1, y$ are the reals constructed by **I** and **II** then
>
> $$x + 1 \in A + C \Leftrightarrow x \in C \Leftrightarrow y = f(x) \in B\ .$$
>
> Suppose instead there is a first stage when **I** plays 0. Then **II** plays an extension $t \in T$ of $\varphi(s)$ (which exists since T is pruned) so that the positions of the two players are $(s + 1)^\frown\langle 0\rangle$ and $t \in T$. Thus $A <_{\mathrm{W}} B_{\lfloor t \rfloor}$, so that **II** has a winning strategy in $G_{\mathrm{W}}(A, B_{\lfloor t \rfloor})$. Let **II** use this strategy.

□

The following diagram summarizes the behavior of the map $C \mapsto A + C$ with A self-dual. (For the sake of definiteness C is taken to be non-self-dual.)



Corollary 22.11. *Assume* $\mathsf{BP} + \mathsf{AD}^{\mathrm{L}} + \mathsf{DC}(\mathbb{R})$ *so that the Wadge rank is defined, and let A be self-dual. Then*

$$\|A + B\|_{\mathrm{W}} = \|A\|_{\mathrm{W}} + 1 + \|B\|_{\mathrm{W}} \ .$$

Exercise 22.12. Assume $\mathsf{BP} + \mathsf{AD}^{\mathrm{L}} + \mathsf{DC}(\mathbb{R})$. Show that if $\|C\|_{\mathrm{W}} \geq \omega$ or $\|C\|_{\mathrm{W}} = 0$ then

$$[A]^{\triangledown}_{\mathrm{W}} + [C]_{\mathrm{W}} = [A]^{\circ}_{\mathrm{W}} + [C]_{\mathrm{W}} = [A]_{\mathrm{W}} + [C]_{\mathrm{W}} \, .$$

Notice that $\mathsf{DC}(\mathbb{R})$ was used only to state 22.12 in a more compact form, using the Wadge rank, but can easily be eliminated.

What about $B + C$ when B is non-self-dual? Exercise 22.19(iii) shows that for B of limit Wadge rank and small C, $B + C \equiv_{\mathrm{W}} B$ can hold. By 22.12 if $B \equiv_{\mathrm{W}} A^{\triangledown}$ or $B \equiv_{\mathrm{W}} A^{\circ}$ and $\|C\|_{\mathrm{W}} \geq \omega$ or $\|C\|_{\mathrm{W}} = 0$ then $A + C \equiv_{\mathrm{W}} B + C$. The assumption that $\|C\|_{\mathrm{W}} \geq \omega$ cannot be removed: if $U \neq \mathbb{R}$ is open then $A + U <_{\mathrm{W}} B + U$, since otherwise by 22.8(v) and 22.9

$$A + U \equiv_{\mathrm{W}} (A + \emptyset) + U \equiv_{\mathrm{W}} A + (\emptyset + U) \Rightarrow U \equiv_{\mathrm{W}} \emptyset + U$$

contradicting 22.8(viii). Yet we will see in 23.20 that if C is non-self-dual and $0 < \|C\|_{\mathrm{W}} < \omega$ and A is self-dual, then either $A^{\triangledown} + C \equiv_{\mathrm{W}} A + C$ or $A^{\circ} + C \equiv_{\mathrm{W}} A + C$.

22.C Jumping ω_1-many Steps

Iterating the $+$ operation one can construct

$$A \cdot n = \underbrace{A + \cdots + A}_{n}$$

for any $n > 0$ and therefore by 22.9 we can define $[A]_{\mathrm{W}} \cdot n = [A \cdot n]_{\mathrm{W}}$. We can also set

$$A \cdot \omega = \bigoplus_n A \cdot (n+1) = \bigcup_n \langle n \rangle^\frown (A \cdot (n+1)) \, ,$$

and $[A]_{\mathrm{W}} \cdot \omega = [A \cdot \omega]_{\mathrm{W}}$. The definition of **multiplication of a degree by an ordinal** can be easily extended to all countable ordinals by letting

$$[A]_{\mathrm{W}} \cdot 1 = [A]_{\mathrm{W}}$$

and for $1 \leq \alpha, \lambda < \omega_1$

$$[A]_{\mathrm{W}} \cdot (\alpha + 1) = [A]_{\mathrm{W}} \cdot \alpha + [A]_{\mathrm{W}}$$
$$[A]_{\mathrm{W}} \cdot \lambda = \sup_n [A]_{\mathrm{W}} \cdot \alpha_n$$

where λ is limit and $\alpha_n \to \lambda$. By **??**(iv) the definition of $[A]_{\mathrm{W}} \cdot \lambda$ does not depend on the choice of the α_n's.

By an argument as in Remark **??** it can be shown that there is no definable map $(A, \alpha) \mapsto A \cdot \alpha$ such that $[A \cdot \alpha]_{\mathrm{W}} = [A]_{\mathrm{W}} \cdot \alpha$.

We will now introduce two new operations on *sets*, $A^\natural$ and $A^\flat$ such that $([A^\natural]_{\mathrm{W}}, [A^\flat]_{\mathrm{W}})$ is the least non-self-dual pair above the $[A]_{\mathrm{W}} \cdot \alpha$'s, $(\alpha < \omega_1)$, when A is self-dual.

Definition 22.13. For $A \subseteq \mathbb{R}$

$$A^\natural = \{ s^\frown \langle 0 \rangle^\frown (x+1) \mid s \in {}^{<\omega}\omega \ \& \ x \in A \} \cup A^+$$

and

$$A^\flat = A^\natural \cup \{ x \in \mathbb{R} \mid \exists^\infty n \; x(n) = 0 \} \, .$$

$A^\natural$ can be seen as some sort of infinite sum of copies of A, and since $A^\natural = A^\natural + A$, then it is natural to think of it as $A^\natural = \cdots + A + A$.

Exercise 22.14. Show that for any $A_0, A_1 \subseteq \mathbb{R}$

$$A_0 \leq_{\mathrm{W}} A_0^\natural, A_0^\flat$$

and

$$A_0 \leq_{\mathrm{W}} A_1 \Rightarrow A_0^\natural \leq_{\mathrm{W}} A_1^\natural \ \& \ A_0^\flat \leq_{\mathrm{W}} A_1^\flat \, .$$

Thus we can define the $\natural$ and $\flat$ operations on degrees by setting

$$[A]_{\mathrm{W}}^{\natural} = [A^{\natural}]_{\mathrm{W}} \qquad \text{and} \qquad [A]_{\mathrm{W}}^{\flat} = [A^{\flat}]_{\mathrm{W}} .$$

Exercise 22.15. (i) Show that for any $s \in {}^{<\omega}\omega$:

$$\begin{aligned} x \in A^{\natural} &\Leftrightarrow s^\frown\langle 0\rangle^\frown x \in A^{\natural} \\ x \in A^{\flat} &\Leftrightarrow s^\frown\langle 0\rangle^\frown x \in A^{\flat} \\ A^{\natural}_{\lfloor s^\frown\langle 0\rangle\rfloor} &= A^{\natural} \\ A^{\flat}_{\lfloor s^\frown\langle 0\rangle\rfloor} &= A^{\flat} . \end{aligned}$$

(ii) Show that $A^{\natural\natural} \equiv_{\mathrm{W}} A^{\natural}$ and $A^{\flat\flat} \equiv_{\mathrm{W}} A^{\flat}$.

(iii) Show that $[A]_{\mathrm{W}} \cdot \alpha \leq_{\mathrm{W}} [A^{\natural}]_{\mathrm{W}}, [A^{\flat}]_{\mathrm{W}}$, for any self-dual $A \subseteq \mathbb{R}$ and any $1 \leq \alpha < \omega_1$.

Theorem 22.16. *Assume* $\mathsf{BP} + \mathsf{AD}^{\mathrm{L}}$*. For A self-dual* $([A^{\natural}]_{\mathrm{W}}, [A^{\flat}]_{\mathrm{W}})$ *is the least non-self-dual pair above the* $[A]_{\mathrm{W}} \cdot \alpha$*'s,* $(1 \leq \alpha < \omega_1)$*, and therefore, assuming* $\mathsf{DC}(\mathbb{R})$*,*

$$\left\|A^{\natural}\right\|_{\mathrm{W}} = \left\|A^{\flat}\right\|_{\mathrm{W}} = \|A\|_{\mathrm{W}} \cdot \omega_1 .$$

Proof. We first show that $B \in \bigcup_{1\leq\alpha<\omega_1} [A]_{\mathrm{W}} \cdot \alpha \Leftrightarrow B \leq_{\mathrm{W}} A^{\natural}, A^{\flat}$. One direction follows from 22.15(iii) so it is enough to prove that $B \leq_{\mathrm{W}} A^{\natural}, A^{\flat}$ implies that $B \in \bigcup_{1\leq\alpha<\omega_1} [A]_{\mathrm{W}} \cdot \alpha$. We need the following

Lemma 22.17. *Assume* $\mathsf{BP} + \mathsf{AD}^{\mathrm{L}}$*. For any* $A, B \subseteq \mathbb{R}$*, if* $B \leq_{\mathrm{W}} A^{\natural}, A^{\flat}$ *then there is a winning strategy* τ *for* **II** *in* $G_{\mathrm{W}}(B, A^{\natural})$ *such that any play for* **II** *according to* τ *belongs to* $\mathbb{R}^{\natural} = \{x \in \mathbb{R} \mid \forall^{\infty} n\ x(n) \neq 0\}$.

Proof. Let τ_0, τ_1 be **II**'s winning strategies for $G_{\mathrm{W}}(B, A^{\natural})$ and $G_{\mathrm{W}}(B, A^{\flat})$, respectively. The plan is to use alternatively τ_0 or τ_1, switching each time the strategy being used requires to play a 0. We dove-tail τ_0 and τ_1 as follows: for any $s \in {}^{<\omega}\omega$ let $u_0, \dots, u_n, v_0, \dots, v_m \in {}^{<\omega}\omega$ be such that

$$(s * \tau_0)_{\mathbf{II}} = (u_0 + 1)^\frown\langle 0\rangle^\frown \cdots {}^\frown\langle 0\rangle^\frown(u_n + 1)$$

and

$$(s * \tau_1)_{\mathbf{II}} = (v_0 + 1)^\frown\langle 0\rangle^\frown \cdots {}^\frown\langle 0\rangle^\frown(v_m + 1)$$

then let

$$(s*\tau)_{\mathbf{II}} = \left((u_0 + 1)^\frown\langle 0\rangle^\frown(v_0 + 1)^\frown\langle 0\rangle^\frown \cdots {}^\frown\langle 0\rangle^\frown(u_k + 1)^\frown\langle 0\rangle^\frown(v_k + 1)\right) \restriction \mathrm{lh}(s)$$

where $k = \min(n, m)$. It is easy to check that this defines a strategy for **II**.

Claim 22.17.1. $\forall x \in \mathbb{R}\left((x * \tau)_{\mathbf{II}} \in \mathbb{R}^{\natural}\right)$.

Proof. Deny. Let $x \in \mathbb{R}$ be such that $(x * \tau)_{\mathbf{II}} \notin \mathbb{R}^{\natural}$, that is

$$\exists^{\infty} n \; \tau_0(x \restriction n) = 0 \tag{1}$$

and

$$\exists^{\infty} n \; \tau_1(x \restriction n) = 0\,. \tag{2}$$

Then by (1) $(x * \tau_0)_{\mathbf{II}} \notin A^{\natural}$ hence $x \notin B$ and by (2) $(x * \tau_1)_{\mathbf{II}} \in A^{\flat}$ hence $x \in B$: a contradiction. $\square$

Let us check that τ is winning for **II** in $G_{\mathrm{W}}(B, A^{\natural})$. Suppose **I** plays x and let $y = (x * \tau)_{\mathbf{II}}$ be **II**'s response. Then either

(A) τ settles on τ_0, that is

$$y = (u_0{+}1)^\frown\langle 0\rangle^\frown(v_0{+}1)^\frown\langle 0\rangle^\frown \cdots {}^\frown \langle 0\rangle^\frown(u_n{+}1)^\frown\langle 0\rangle^\frown(v_n{+}1)^\frown\langle 0\rangle^\frown(z{+}1)$$

where $(u_0 + 1)^\frown\langle 0\rangle^\frown \cdots {}^\frown \langle 0\rangle^\frown(u_n + 1)^\frown\langle 0\rangle^\frown(z + 1) = (x * \tau_0)_{\mathbf{II}}$, or else

(B) τ settles on τ_1, that is

$$y = (u_0 + 1)^\frown\langle 0\rangle^\frown(v_0 + 1)^\frown\langle 0\rangle^\frown \cdots {}^\frown \langle 0\rangle^\frown(u_n + 1)^\frown\langle 0\rangle^\frown(w + 1)$$

where $(v_0 + 1)^\frown\langle 0\rangle^\frown \cdots {}^\frown \langle 0\rangle^\frown(v_{n-1} + 1)^\frown\langle 0\rangle^\frown(w + 1) = (x * \tau_1)_{\mathbf{II}}$.

Assume (A) holds, i.e., τ settles on τ_0. Then

$$\begin{aligned} x \in B &\Leftrightarrow (x * \tau_0)_{\mathbf{II}} \in A^{\natural} && \text{(by definition of } \tau_0\text{)} \\ &\Leftrightarrow z \in A^{\natural} && \text{(by 22.15(i))} \\ &\Leftrightarrow y \in A^{\natural} && \text{(by 22.15(i) again)} \end{aligned}$$

Case (B) when τ settles on τ_1 is completely analogous. $\square$

Let's go back to the proof of 22.16. For any $s \in {}^{<\omega}\omega$ let

$$n(s) = \begin{cases} \text{the largest } i < \mathrm{lh}(s) \text{ such that } s(i) = 0, \\ 0, \text{ if } \forall i < \mathrm{lh}(s)\,(s(i) \neq 0)\,, \end{cases}$$

and let $\prec$ be the strict partial order on ${}^{<\omega}\omega$ defined by

$$s \prec t \Leftrightarrow s \restriction n(s) \subset t \restriction n(t)\,.$$

Fix a strategy τ as in the Lemma and set

$$s \lhd t \Leftrightarrow (s * \tau)_{\mathbf{II}} \prec (t * \tau)_{\mathbf{II}}\,.$$

Then $\lhd$ is a strict partial order on ${}^{<\omega}\omega$, and it is well-founded, since otherwise there would be an $x \in \mathbb{R}$ such that $y \notin \mathbb{R}^{\natural}$, where y is **II**'s response via τ to **I** playing x. We will prove by induction on α that $B \in [A]_{\mathrm{W}} \cdot (\alpha + 1)$, where $\alpha = \|\emptyset\|_{\lhd}$.

Suppose $\alpha = 0$. This corresponds to the case where $\forall s \in {}^{<\omega}\omega\, (\tau(s)$ does not contain any 0). Then **II** wins $G_{\mathrm{W}}(B, A)$ by playing $\tau(s) - 1$, for any s played by **I**: this is indeed a winning strategy since if x and y are the reals played this way by **I** and **II** then

$$x \in B \Leftrightarrow y + 1 \in A^{\natural} \Leftrightarrow y \in A\,.$$

This proves that $B \leq_{\mathrm{W}} A$.

Suppose $\alpha > 1$ and let $C \in [A]_{\mathrm{W}} \cdot \alpha$. Consider the following strategy for **II** in the game $G_{\mathrm{W}}(B, C + A)$:

> As long as **I** stays in a position p such that $\|p\|_{\lhd} = \alpha$ then **II** plays $\tau(p)$. Suppose at some round **I** reaches a position p such that $\|p\|_{\lhd} < \alpha$. Then **II** answers 0 and from now on, to any further move he answers $\tau_p(p^\frown q) + 1$, where τ_p is a strategy witnessing $B_{\lfloor p \rfloor} \leq_{\mathrm{W}} C$. Such a τ_p exists by inductive hypothesis.

Let us check that this strategy is indeed winning for **II**. Let x and y be the reals produced by a complete play. If $\|x \restriction n\|_{\lhd} < \alpha$ for some n, then let x' and y' be such that $x \restriction n^\frown x' = x$ and $((x \restriction n) * \tau)^\frown \langle 0 \rangle^\frown (y' + 1) = y$. (Note that $\mathrm{lh}((x \restriction n) * \tau) < n$ is possible since τ may pass.) Then $x' \in B_{\lfloor p \rfloor} \Leftrightarrow y' \in C$ since τ_p witnesses $B_{\lfloor p \rfloor} \leq_{\mathrm{W}} C$ and therefore $x \in B \Leftrightarrow y \in C + A$. If instead $\forall n\, (\|x \restriction n\|_{\lhd} = \alpha)$ then $\forall n\, (\tau(x \restriction n) \neq 0)$ so $y \in \mathbb{R}^{\natural}$ hence

$$x \in B \Leftrightarrow y \in A^{\natural} \Leftrightarrow y - 1 \in A \Leftrightarrow y \in C + A\,.$$

Therefore $[B]_{\mathrm{W}} \leq [A]_{\mathrm{W}} \cdot (\alpha + 1)$.

Finally, let us show that $A^{\natural} \not\equiv_{\mathrm{W}} A^{\flat}$. Otherwise if $A^{\natural} \leq_{\mathrm{W}} A^{\flat}$ then by the argument above, since $A^{\natural} \leq_{\mathrm{W}} A^{\natural}$, we should have that $[A^{\natural}]_{\mathrm{W}} \leq_{\mathrm{W}} [A]_{\mathrm{W}} \cdot \alpha$ for some $\alpha < \omega_1$. Then

$$\begin{aligned} [A]_{\mathrm{W}} \cdot \alpha &<_{\mathrm{W}} [A]_{\mathrm{W}} \cdot (\alpha + 1) && \text{by 22.8(i)} \\ &\leq_{\mathrm{W}} [A^{\natural}]_{\mathrm{W}} && \text{by 22.15(iii)} \end{aligned}$$

hence $[A^{\natural}]_{\mathrm{W}} <_{\mathrm{W}} [A^{\natural}]_{\mathrm{W}}$: a contradiction. □

Corollary 22.18. *Assume* $\mathsf{BP}+\mathsf{DC}(\mathbb{R})+\mathsf{AD}^{\mathrm{L}}$. *Let* A *be self-dual. For every* B *such that* $\|A\|_{\mathrm{W}} < \|B\|_{\mathrm{W}} < \|A\|_{\mathrm{W}} \cdot \omega_1$,

$$[B^\natural]_{\mathrm{W}} = [A^\natural]_{\mathrm{W}} \qquad \textit{and} \qquad [B^\flat]_{\mathrm{W}} = [A^\flat]_{\mathrm{W}}\,.$$

Proof. Let α be least such that $\|B\|_{\mathrm{W}} \leq \|A\|_{\mathrm{W}} \cdot \alpha$ and let $C \in [A]_{\mathrm{W}} \cdot \alpha$. By monotonicity (22.14) $A^\natural \leq_{\mathrm{W}} B^\natural \leq_{\mathrm{W}} C^\natural$. The argument for $A^\flat$ is similar. □

Exercise 22.19. (i) Show that

$$\mathbb{R}^\natural = \{x \in \mathbb{R} \mid \forall^\infty n\ x(n) \neq 0\} \in \mathbf{\Sigma}^0_2 \setminus \mathbf{\Delta}^0_2$$

and that

$$\emptyset^\flat = \{x \in \mathbb{R} \mid \exists^\infty n\ x(n) = 0\} \in \mathbf{\Pi}^0_2 \setminus \mathbf{\Delta}^0_2\,.$$

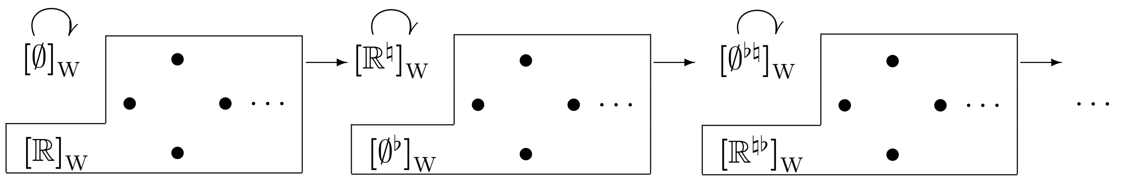
(ii) Show that $\mathbb{R}^\natural = (\mathbb{R} \oplus \emptyset)^\natural$ and $\emptyset^\flat = (\mathbb{R} \oplus \emptyset)^\flat$, $\emptyset^\natural = \emptyset$, $\mathbb{R}^\flat = \mathbb{R}$.

(iii) ($\mathsf{DC}(\mathbb{R})$) Show that $\mathbf{\Sigma}^0_2 \setminus \mathbf{\Delta}^0_2$ and $\mathbf{\Pi}^0_2 \setminus \mathbf{\Delta}^0_2$ occupy the ω_1-th level of the Wadge hierarchy.

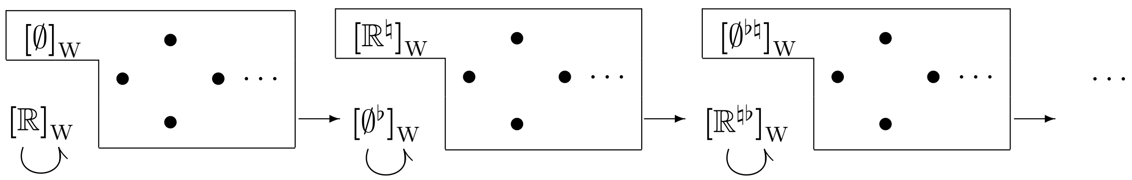
(iv) Show that $\mathbb{R}^\natural + \mathbb{R}^\natural \equiv_{\mathrm{W}} \mathbb{R}^\natural$ and therefore $\forall B \leq_{\mathrm{W}} \mathbb{R}^\natural \left(\mathbb{R}^\natural + B \equiv_{\mathrm{W}} \mathbb{R}^\natural\right)$.

(v) Show that $([\mathbb{R}^{\natural\flat}]_{\mathrm{W}}, [\emptyset^{\flat\natural}]_{\mathrm{W}})$, $([\mathbb{R}^{\natural\flat\natural}]_{\mathrm{W}}, [\emptyset^{\flat\natural\flat}]_{\mathrm{W}})$,... is an increasing sequence of non-self-dual pairs of degrees whose Wadge rank (here we assume $\mathsf{DC}(\mathbb{R})$ so that we can talk about Wadge ranks) are ω_1^2, ω_1^3,....

Therefore the map $[A]_{\mathrm{W}} \mapsto [A]^\natural_{\mathrm{W}}$ acts like this



and the map $[A]_{\mathrm{W}} \mapsto [A]^\flat_{\mathrm{W}}$ acts like this



22.D Back-track Reducibility

There is a coarser notion of reducibility which is quite useful. It is defined in terms of games.

Definition 22.20 (Van Wesep). $G_{\text{bt}}(A, B)$ is a game like G_{W} where **I** and **II** alternate in playing integers. Just as before **II** has the option of passing at any round (with the proviso that he must play infinitely often, otherwise he loses), but moreover **II** has a further advantage: at any round he can back-track (whence the subscript in G_{bt}) and erase his moves (but not changing **I**'s moves) starting his board anew. The only restriction is that he cannot backtrack infinitely often, otherwise he loses. Let x be **I**'s play and let y be the sequence of moves that **II** plays after his final back-tracking (if he ever used this option). Then **II** wins $G_{\text{bt}}(A, B)$ iff

$$x \in A \Leftrightarrow y \in B\,.$$

In Exercise 22.50 we will look at similar games where various form of back-tracking is allowed.

For $A, B \subseteq \mathbb{R}$, A is **back-track reducible to** B, in symbols $A \leq_{\text{bt}} B$ iff

$$\textbf{II} \text{ has a winning strategy in } G_{\text{bt}}(A, B)\,.$$

Exercise 22.21. (i) Show that $\leq_{\text{bt}}$ is reflexive and transitive so that

$$A \equiv_{\text{bt}} B \Leftrightarrow A \leq_{\text{bt}} B \;\&\; B \leq_{\text{bt}} A$$

is an equivalence relation. Let $[A]_{\text{bt}}$ be the $\equiv_{\text{bt}}$ equivalence class of A, called the **bt-degree** of A.

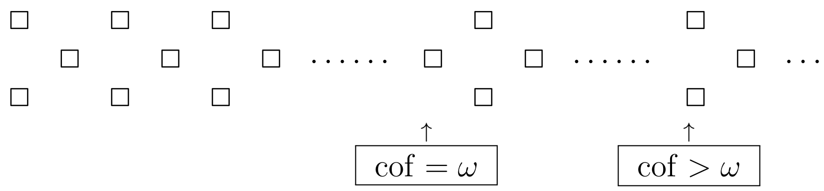
(ii) Show that $A \leq_{\text{W}} B \Rightarrow A \leq_{\text{bt}} B$. Conclude that $\leq_{\text{bt}}$ is well-founded on the bt-degrees.

(iii) Show that $A \leq_{\text{bt}} B \Leftrightarrow A \leq_{\text{W}} B^\natural, B^\flat$.

(iv) Show that if $A \not\equiv_{\text{bt}} \neg A$ and then $[A]_{\text{bt}} = [A]_{\text{W}}$.

(v) Show that if $A \equiv_{\text{W}} \bigoplus_n A_n$ and $\forall n\,(A_n \equiv_{\text{bt}} \neg A_n)$ then $A \equiv_{\text{bt}} \neg A$.

By (ii) each bt-degree is union of Wadge degrees and $[A]_{\text{bt}} \supseteq [A]_{\text{W}}$, and (iii) implies that $[A]_{\text{bt}} \supset [A]_{\text{W}}$ for A self-dual. As with the Wadge degrees $\{\emptyset\} = [\emptyset]_{\text{bt}}$ and $\{\mathbb{R}\} = [\mathbb{R}]_{\text{bt}}$ are at the bottom of the bt-hierarchy. If $[A]_{\text{bt}}$ self-dual i.e., $[A]_{\text{bt}} = [\neg A]_{\text{bt}}$ then for any $B \in [A]_{\text{bt}}$ such that $B \equiv_{\text{W}} \neg B$ the degrees $[B^\natural]_{\text{bt}}$ and $[B^\flat]_{\text{bt}}$ are dual to each other and are minimal above $[A]_{\text{bt}}$. Therefore above each self-dual bt-degree there is a non-self-dual pair of bt-degrees, and above each non-self-dual pair of bt-degrees there is a single self-dual bt-degree. By (v) at limit levels of cofinality ω there is a single self-dual bt-degree. Proposition 22.24 will imply that at levels of uncountable cofinality there is a non-self-dual pair of bt-degrees, just like in the Wadge hierarchy. Therefore the bt-hierarchy looks like this:



In order to analyze the bt-degrees we need the following result. It can be used to characterize those Wadge degrees that contain *homogeneous sets* i.e., sets B which are Wadge-equivalent to any of their localizations, i.e., $\boldsymbol{T}(B) = {}^{<\omega}\omega$.

Proposition 22.22 (Van Wesep). *Assume* $\mathsf{BP} + \mathsf{AD}^{\mathrm{L}}$. *For* $A \notin \boldsymbol{\Pi}^0_1$ *the following are equivalent.*

(1) $A \not\leq_{\mathrm{bt}} \neg A$,

(2) $\exists B \in [A]_{\mathrm{L}}\, (\boldsymbol{T}(B) = {}^{<\omega}\omega)$,

(3) $\exists B \in [A]_{\mathrm{L}}\, (B \cap [\boldsymbol{T}(B)] \equiv_{\mathrm{L}} B)$,

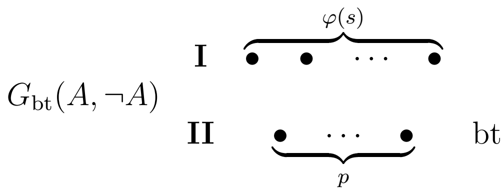
(4) $\forall B \in [A]_{\mathrm{L}}\, (B \cap [\boldsymbol{T}(B)] \equiv_{\mathrm{L}} B)$.

Proof. (1) $\Rightarrow$ (2): By 22.21(ii) $A \not\leq_{\mathrm{W}} \neg A$ so $\forall B \in [A]_{\mathrm{L}} = [A]_{\mathrm{W}}\, (\boldsymbol{T}(B)$ is pruned$)$ by **??**. Let σ be a winning strategy for **I** in $G_{\mathrm{bt}}(A, \neg A)$. Then $T = \mathrm{ran}(\sigma) \cup \{\emptyset\}$ is a pruned tree so by **??** there is a Lipschitz map $f : \mathbb{R} \twoheadrightarrow [T]$ such that $f \restriction [T] = \mathrm{id}$. Let also $\varphi : {}^{<\omega}\omega \twoheadrightarrow T$ be the Lipschitz map inducing f. Let $B = f^{-1}``A$. Since f is Lipschitz $B \leq_{\mathrm{L}} A$, hence in order to prove that $A \leq_{\mathrm{L}} A$ it is enough to show (by AD^{L}) that $B \not\leq_{\mathrm{L}} \neg A$, i.e., that **I** wins $G_{\mathrm{L}}(B, \neg A)$: As $\mathrm{ran}(\sigma) \subset [T]$ and $[T] \cap A \subseteq B$ it is easy to see that σ is winning for **I** in $G_{\mathrm{L}}(B, \neg A)$. Therefore $B \in [A]_{\mathrm{L}}$. We will show that this B works, that is that $B_{\lfloor s \rfloor} \equiv_{\mathrm{W}} B$ for every $s \in {}^{<\omega}\omega$.

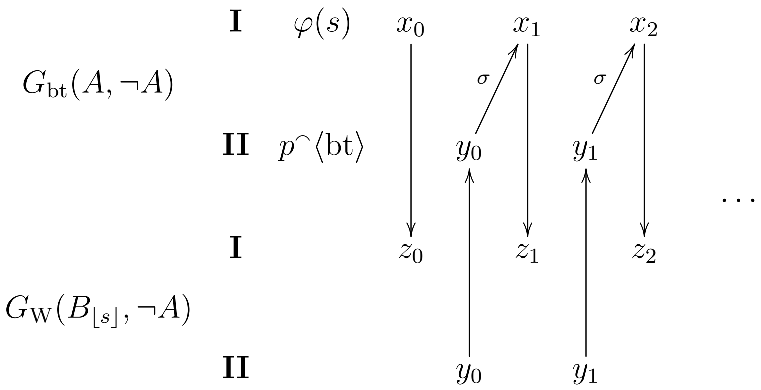
Fix $s \in {}^{<\omega}\omega$. As $B_{\lfloor \emptyset \rfloor} = B$ we may assume that $\mathrm{lh}(s) > 0$. Since $B_{\lfloor s \rfloor} \leq_{\mathrm{W}} B$ comes for free, by AD^{L} it is enough to show that **I** has a winning strategy $\tilde{\sigma}$ in $G_{\mathrm{W}}(B_{\lfloor s \rfloor}, \neg A)$:

> Since $\varphi(s) \in \mathrm{ran}(\sigma)$ let p be a position for **II** in $G_{\mathrm{bt}}(A, \neg A)$ such that $\varphi(s)$ is σ's answer to p. (Notice that p may contain some back-tracking or passing.) It is **II**'s turn to move in this game

and let's have him back-track.



Then **I** using σ plays some $x_0 \in \omega$ in $G_{\mathrm{bt}}(A, \neg A)$ and let $z_0 \in \omega$ be such that $\varphi(s^\frown\langle z_0\rangle) = \varphi(s)^\frown\langle x_0\rangle$ —such a z_0 certainly exists since $\varphi(s)^\frown\langle x_0\rangle \in T$. Define $\tilde{\sigma}(\emptyset) = z_0$, i.e., this is **I**'s first move in $G_{\mathrm{W}}(B, \neg A)$. From now on we simply copy **II**'s moves from $G_{\mathrm{W}}(B, \neg A)$ to $G_{\mathrm{bt}}(A, \neg A)$ and transfer-back **I**'s moves via φ: if **II** plays y_0 in $G_{\mathrm{W}}(B, \neg A)$ let x_1 be σ's response to this move in this game and let z_1 be such that $\varphi(s^\frown\langle z_0, z_1\rangle) = \varphi(s)^\frown\langle x_0, x_1\rangle$, and so on:



Thus we end up with reals $\varphi(s)^\frown x$ and y in $G_{\mathrm{bt}}(A, \neg A)$, z and y in $G_{\mathrm{W}}(B_{\lfloor s\rfloor}, \neg A)$. Notice that $\varphi(s)^\frown x \in [T]$ and since σ is winning

$$\varphi(s)^\frown x \in A \cap [T] \Leftrightarrow y \notin \neg A\,.$$

Since $f(s^\frown x) = x$ then

$$s^\frown z \in B \Leftrightarrow y \notin \neg A$$

that is $\tilde{\sigma}$ is winning for **I** in $G_{\mathrm{W}}(B_{\lfloor s\rfloor}, \neg A)$.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (4) follows from **??**.

(4) $\Rightarrow$ (1). Assume (4) so that in particular $[\boldsymbol{T}(A)] \cap A \equiv_{\mathrm{L}} A$. Since $[\boldsymbol{T}(A)] \neq \emptyset$ then A is non-self-dual by **??**. If $[A]_{\mathrm{W}}$ were successor then either

$[\boldsymbol{T}(A)] \cap A = \emptyset$ or else $[\boldsymbol{T}(A)] \subseteq A$ by 22.4 and therefore either $A \equiv_{\mathrm{L}} \emptyset$ or else $A \equiv_{\mathrm{L}} [\boldsymbol{T}(A)]$, contradicting the assumption that A is not closed. Since

$$\forall p \in \boldsymbol{T}(A) \left(\boldsymbol{T}(A_{\lfloor p \rfloor}) = \boldsymbol{T}(A)_{\lfloor p \rfloor} \qquad \text{and} \qquad A_{\lfloor p \rfloor} \equiv_{\mathrm{L}} A \right)$$

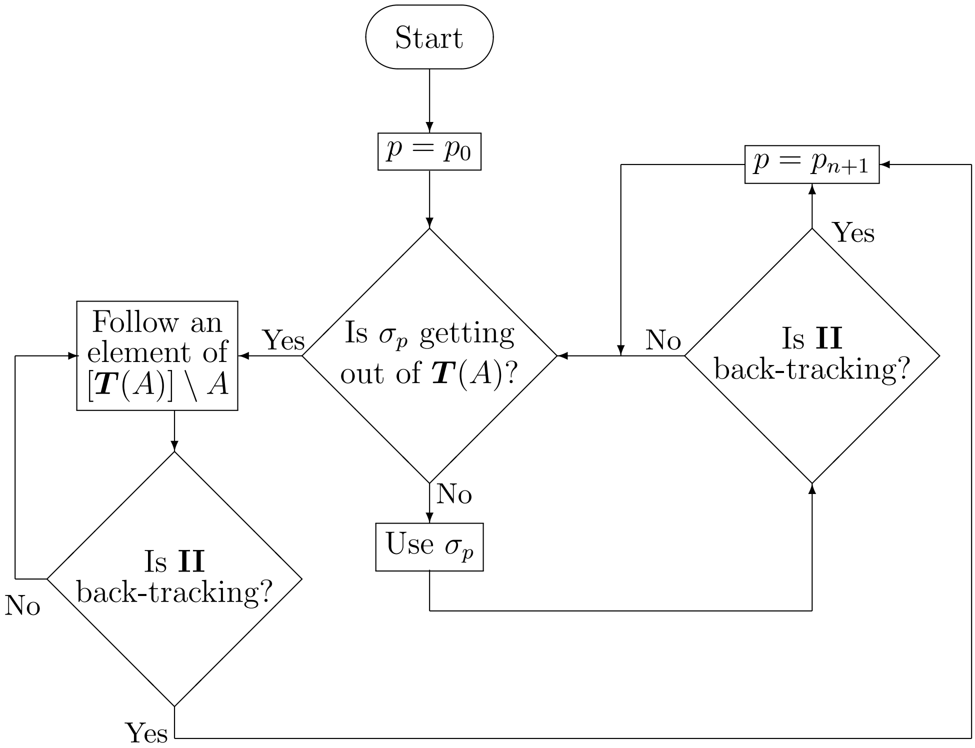
the argument above shows that

$$(*) \qquad \forall p \in \boldsymbol{T}(A) \left([\boldsymbol{T}(A)_{\lfloor p \rfloor}] \nsubseteq A_{\lfloor p \rfloor} \right) .$$

Since $\neg A_{\lfloor p \rfloor} \equiv_{\mathrm{L}} \neg A$ for any $p \in \boldsymbol{T}(A)$, fix σ_p a winning strategy for **I** in $G_{\mathrm{W}}(A_{\lfloor p \rfloor} \cap [\boldsymbol{T}(A)_{\lfloor p \rfloor}], \neg A)$. Consider the following strategy for **I** in $G_{\mathrm{bt}}(A, \neg A)$:

> As the play develops we inductively define a sequence of positions for the two Players $(p_0, q_0), (p_1, q_1), \ldots, (p_n, q_n)$ with $\emptyset = p_0 \subset \cdots \subset p_n$ and $\emptyset = q_0 \subset \cdots \subset q_n$, such that $\langle q_1, \ldots, q_n \rangle$ lists all the positions where **II** back-tracks (i.e., if q is a position of the current play such that $q(\mathrm{lh}(q) - 1) = \mathrm{bt}$ then $q = q_i$ for some $1 \leq i \leq n$. We will maintain that $p_i \in \boldsymbol{T}(A)$ so that $A_{\lfloor p_i \rfloor} \equiv_{\mathrm{W}} A$. Let **I** follow $\sigma = \sigma_{p_0} = \sigma_{\emptyset}$ as long as **II** does not back-track or σ does not force **I** out of $\boldsymbol{T}(A)$. Suppose at some round we are at positions $p \in \boldsymbol{T}(A)$ for **I** and q for II, but $p^\frown\langle \sigma(q) \rangle \notin \boldsymbol{T}(A)$. Notice that $\boldsymbol{N}_q \cap A = \emptyset$ since otherwise **I** could defeat σ in $G_{\mathrm{W}}(A \cap [\boldsymbol{T}(A)], \neg A)$ by playing the elements of some $y \in \boldsymbol{N}_q \cap A$ from this point on. Then **I** picks a branch $b \in [\boldsymbol{T}(A)_{\lfloor p \rfloor}] \setminus A_{\lfloor p \rfloor}$ which exists by $(*)$ above, and follows b from now on. Suppose we reach a round when **II** decides to back-track and let p_1, q_1 be the positions of the players at this round: then in order for **I** to win it is enough to show that he has a winning strategy for $G_{\mathrm{bt}}(A_{\lfloor p_1 \rfloor}, \neg A)$. Since $A \equiv_{\mathrm{W}} A_{\lfloor p_1 \rfloor}$ we can repeat the argument above: **I** follows $\sigma = \sigma_{p_1}$ until—if ever—either **II** back-tracks or σ forces us out of $\boldsymbol{T}(A)$. And so on.

The following flow-chart gives a dynamic description of σ:



Let us check that this strategy is winning: consider a play according to it and let n be the latest round when **II** did back-track, if this ever happened, or $n = 0$ otherwise. Let $p = p_n$ be **I**'s position at round n. If σ_p never takes us out of $\boldsymbol{T}(A)_{\lfloor p \rfloor}$, then we kept on using it forever. Let x and y be the reals played by **I** and **II** after round n. Since σ_p is winning for **I** in $G_{\mathrm{W}}(A_{\lfloor p \rfloor}, \neg A)$ then $x \in A_{\lfloor p \rfloor} \Leftrightarrow y \in A$ so $p^\frown x \in A \Leftrightarrow y \in A$, hence **I** wins $G_{\mathrm{bt}}(A, \neg A)$. If at some round $n' \geq n$, σ_p forces us out of $\boldsymbol{T}(A)$ then we follow a $b \in [\boldsymbol{T}(A)] \setminus A$ passing through our (i.e., Player **I**'s) current position $p' \supseteq p$. Since **II** does not back-track any more, we stick to b from now until the end of the game. Since $b \notin A$, in order for **I** to win $G_{\mathrm{bt}}(A, \neg A)$ we must check that y, the real played by **II** after round n, belongs to $\neg A$. Let q' be **II**'s position at round n': since σ_p is winning for **I** in $G_{\mathrm{W}}(A_{\lfloor p \rfloor} \cap [\boldsymbol{T}(A)_{\lfloor p \rfloor}], \neg A)$ and as $\sigma_p(q') \notin \boldsymbol{T}(A)$ then $\boldsymbol{N}_{q'} \cap A = \emptyset$, since otherwise **II** could defeat σ_p by playing any $y' \in \boldsymbol{N}_{q'} \cap A$. Therefore $y \notin A$. □

Exercise 22.23. Assume $\mathsf{BP} + \mathsf{AD}^{\mathrm{L}}$.

(i) Show that if $A \notin \boldsymbol{\Pi}^0_1$ and $A \equiv_{\mathrm{bt}} \neg A$ then for any $C \in [A]_{\mathrm{W}}$

$$\forall s \in {}^{<\omega}\omega \, \exists t \supset s \, (t \notin \boldsymbol{T}(C)) \ ,$$

that is ${}^{<\omega}\omega \setminus \boldsymbol{T}(C)$ is dense in ${}^{<\omega}\omega$.

(ii) Let $A \not\equiv_{\mathrm{bt}} \neg A$ with $A \notin \boldsymbol{\Pi}^0_1$ and let $[B]_{\mathrm{bt}}$ be the $\leq_{\mathrm{bt}}$-successor of $[A]_{\mathrm{bt}}$ and $[\neg A]_{\mathrm{bt}}$. Show that $\{s \in {}^{<\omega}\omega \mid B_{\lfloor s \rfloor} \leq_{\mathrm{W}} A \ \vee \ B_{\lfloor s \rfloor} \leq_{\mathrm{W}} \neg A\}$ is dense in ${}^{<\omega}\omega$.

Proposition 22.24 (Steel)**.** *Assume* $\mathsf{BP} + \mathsf{AD}^{\mathrm{L}}$. *If* A *is non-self-dual and* $\forall B <_{\mathrm{W}} A \left(B^\natural <_{\mathrm{W}} A\right)$ *then* $A \not\leq_{\mathrm{bt}} \neg A$.

Proof. It is enough to show that $A \cap [\boldsymbol{T}(A)] \equiv_{\mathrm{W}} A$ and apply 22.22. Since $A \cap [\boldsymbol{T}(A)] \leq_{\mathrm{W}} A$ follows from A being non-self-dual, we may assume, towards a contradiction, that $A \cap [\boldsymbol{T}(A)] <_{\mathrm{W}} A$. If $[A]_{\mathrm{W}}$ were the successor of a self-dual $[B]_{\mathrm{W}}$ then $A <_{\mathrm{W}} B^\natural$, contrarily to our assumption, so $[A]_{\mathrm{W}}$ is a limit degree. Also $[A]_{\mathrm{W}}$ is not a join degree, since otherwise it would be self-dual. Therefore $\bigoplus_n A_n <_{\mathrm{W}} A$ where $\langle A_n \mid n \in \omega\rangle$ is an enumeration of $\{A_{\lfloor s \rfloor} \mid s \notin \boldsymbol{T}(A)\}$. Thus we can find a $B <_{\mathrm{W}} A$ such that $\forall s \notin \boldsymbol{T}(A) \left(A_{\lfloor s \rfloor} \leq_{\mathrm{W}} B\right)$ and $A \cap [\boldsymbol{T}(A)] \leq_{\mathrm{W}} B$. Since $B \leq_{\mathrm{W}} B^\natural$ and $A_{\lfloor s \rfloor} \leq_{\mathrm{W}} B$ for $s \notin \boldsymbol{T}(A)$, we can fix winning strategies for **II**: τ_0 in $G_{\mathrm{W}}(A \cap [\boldsymbol{T}(A)], B^\natural)$ and τ_s in $G_{\mathrm{W}}(A_{\lfloor s \rfloor}, B)$. Then **II** wins $G_{\mathrm{W}}(A, B^\natural)$ as follows:

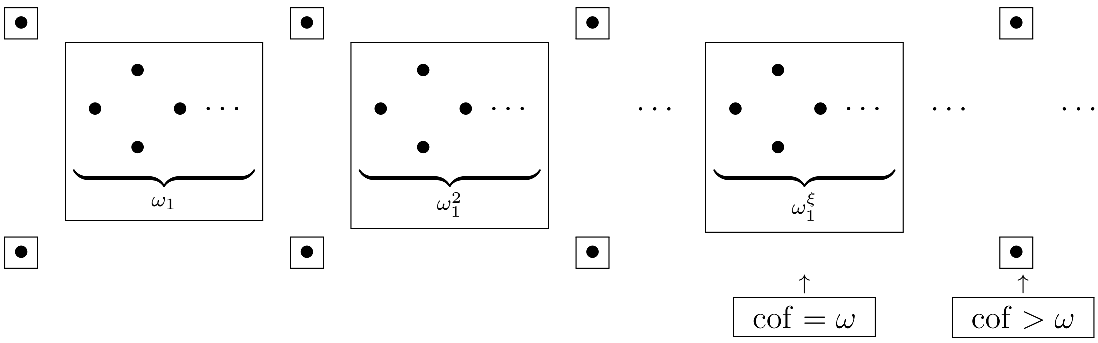
> As long as **I**'s position is inside $\boldsymbol{T}(A)$ then **II** uses τ_0. If **I** ever moves into a position $s \notin \boldsymbol{T}(A)$ then **II** plays 0 and then uses $\tau_s + 1$.

Therefore $A \leq_{\mathrm{W}} B^\natural$ and since $B^\natural \leq_{\mathrm{W}} \neg A$ by hypothesis, $A \leq_{\mathrm{W}} \neg A$: a contradiction. □

Corollary 22.25. *Assume* $\mathsf{BP} + \mathsf{AD}^{\mathrm{L}} + \mathsf{DC}(\mathbb{R})$.

$$A \not\equiv_{\mathrm{bt}} \neg A \Leftrightarrow \Big(\|A\|_{\mathrm{W}} = 0 \vee \exists \xi > 0 \left(\|A\|_{\mathrm{W}} = \omega_1^\xi \ \& \ \operatorname{cof}(\xi) \neq \omega\right)\Big) .$$

Below is the picture of the bt-hierarchy versus the Wadge hierarchy (a dot • denotes a Wadge degree and a box □ denotes a bt-degree):



Using Wadge's analysis of the Borel hierarchy (see §23**??**), it can be shown that the sets of Wadge rank ω_1^ξ with $\xi < \omega_1$ are exactly the sets in $\mathrm{D}_\xi = \mathrm{Diff}(\xi; \mathbf{\Pi}^0_2) \setminus \breve{\mathrm{Diff}}(\xi; \mathbf{\Pi}^0_2)$. Therefore the first ω_1 levels of the $\leq_{\mathrm{bt}}$-hierarchy are:

$$\begin{matrix} [\emptyset]_{\mathrm{W}} & & \mathrm{D}_1 & & \mathrm{D}_2 & & & \mathrm{D}_\xi & & \\ & \Delta_{\mathrm{D}_1} & & \Delta_{\mathrm{D}_2} & & \cdots\cdots & \Delta_{\mathrm{D}_\xi} & & \Delta_{\mathrm{D}_{\xi+1}} & \cdots\cdots \\ [\mathbb{R}]_{\mathrm{W}} & & \breve{\mathrm{D}}_1 & & \breve{\mathrm{D}}_2 & & & \breve{\mathrm{D}}_\xi & & \end{matrix}$$

22.E The Stretch Operation

Here is another useful operation on sets and degrees.

Definition 22.26. (i) For $x, y \in \mathbb{R}$ y is the **stretch** of x iff there is a strictly increasing $f : \omega \to \omega$ such that for all $m \in \omega$

$$y(m) = \begin{cases} x(n) + 1 & \text{if } f(n) = m \\ 0 & \text{otherwise.} \end{cases}$$

Notice that y can be the stretch of at most one real. In this case we say that x is the **unstretch** of y.

(ii) For $A \subseteq \mathbb{R}$ the stretch of A is the set

$$A^{\mathrm{stretch}} = \{y \in \mathbb{R} \mid \exists x \in A\, (y \text{ is a stretch of } x)\}$$

Here are the basic properties of the stretch operation.

Exercise 22.27. Assume $\mathsf{BP} + \mathsf{AD}^{\mathrm{L}}$.

(i) $A \leq_{\mathrm{L}} A^{\mathrm{stretch}}$;

(ii) $A^{\mathrm{stretch}} \equiv_{\mathrm{L}} (A^{\mathrm{stretch}})^{\mathrm{stretch}}$;

(iii) $\forall s \in {}^{<\omega}\omega \left((A_{\lfloor s \rfloor})^{\mathrm{stretch}} = (A^{\mathrm{stretch}})_{\lfloor s+1 \rfloor}\right)$;

(iv) A^{stretch} is non-self-dual;

(v) $A \leq_{\mathrm{L}} B \Rightarrow A^{\mathrm{stretch}} \leq_{\mathrm{L}} B^{\mathrm{stretch}}$.

Therefore we can define the stretch operation on the Lipschitz/Wadge degrees by

$$[A]_{\mathrm{L}}^{\mathrm{stretch}} = [A^{\mathrm{stretch}}]_{\mathrm{L}} = [A^{\mathrm{stretch}}]_{\mathrm{W}} = [A]_{\mathrm{W}}^{\mathrm{stretch}},$$

the middle equality being true by (iv).

(vi) Show that $\mathbb{R}^{\text{stretch}} = \{x \in \mathbb{R} \mid \exists^\infty n\, x(n) \neq 0\}$ is $\mathbf{\Pi}^0_2$-complete and

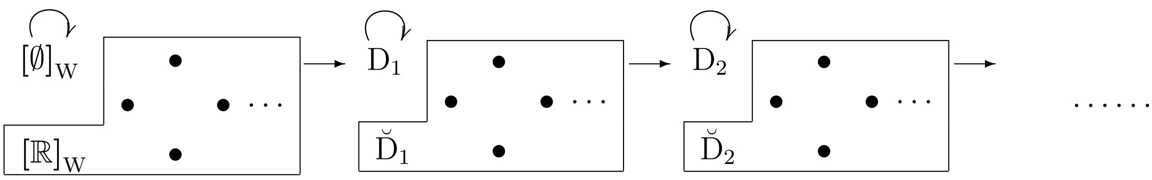
$$\forall A \in \mathbf{\Delta}^0_2 \left(A \neq \emptyset \Rightarrow A^{\text{stretch}} \text{ is } \mathbf{\Pi}^0_2\text{-complete}\right) .$$

(vii) $[\boldsymbol{T}(A)] \cap A \neq \emptyset \Rightarrow [A]^{\text{stretch}}_{\text{W}}$ is limit non-self-dual;

(viii) $(A + B)^{\text{stretch}} \equiv_{\text{L}} A^{\text{stretch}} + B^{\text{stretch}}$.

22.27 yields a complete description of the stretch operation on degrees of rank $< \omega_1^\omega$. If $A = \{x \in \mathbb{R} \mid \forall^\infty n\, x(n) = 0\} \in \mathbf{\Sigma}^0_2 \setminus \mathbf{\Pi}^0_2$ then $A^{\text{stretch}} = \{x \in \mathbb{R}^{\text{stretch}} \mid \forall^\infty n\,(x(n) \neq 0 \Rightarrow x(n) = 1)\} \in \text{Diff}(2; \mathbf{\Pi}^0_2)$. More generally, if $A \in \breve{\text{Diff}}(n; \mathbf{\Pi}^0_2) \setminus \text{Diff}(n; \mathbf{\Pi}^0_2)$, then $A^{\text{stretch}} \in \text{Diff}(n+1; \mathbf{\Pi}^0_2)$ and $\neg A \leq_{\text{W}} A^{\text{stretch}}$ (see **??**). Since $\text{D}_n = \text{Diff}(n; \mathbf{\Pi}^0_2) \setminus \breve{\text{Diff}}(n; \mathbf{\Pi}^0_2)$ occupies the ω_1^n-th level of the Wadge hierarchy (see §23**??**), then by 22.4 and part (vii) above, $A \in \breve{\text{D}}_n$ implies $\text{cof}(\|A^{\text{stretch}}\|_{\text{W}}) > \omega$ and therefore $A^{\text{stretch}} \in \text{D}_{n+1}$.

In other words the first ω degrees of the form $[A^{\text{stretch}}]_{\text{W}}$ are $[\emptyset]_{\text{W}}, \text{D}_1, \text{D}_2, \ldots$ and their Wadge ranks are $0, \omega_1, \omega_1^2, \ldots$. And this is how the stretch operator acts on the first ω_1^ω degrees:



Next we want to see how the stretch operates on a degree $[A]_{\text{W}}$ which is the upper bound of stretched degrees. If $\text{cof}(\|A\|_{\text{W}}) = \omega$ then $A <_{\text{W}} A^{\text{stretch}}$ by 22.27(iv) and in fact $\|A^{\text{stretch}}\|_{\text{W}}$ is an immediate successor of $\|A\|_{\text{W}}$.

Proposition 22.28 (Van Wesep)**.** *Assume* $\mathsf{BP}+\mathsf{AD}^{\mathsf{L}}$. *Suppose A is self-dual and $\forall B\,(B <_{\text{W}} A \Rightarrow B^{\text{stretch}} <_{\text{W}} A)$. Then*

(a) $$[A]^{\text{stretch}}_{\text{W}} = [A]^{\triangledown}_{\text{W}} = [A]_{\text{W}} + [\emptyset]_{\text{W}}$$

and

(b) $$\forall C \subseteq \mathbb{R}\left([A + C]^{\text{stretch}}_{\text{W}} = [A]_{\text{W}} + [C]^{\text{stretch}}_{\text{W}}\right)$$

Proof. We start proving part (a). $A <_{\text{W}} A^{\text{stretch}}$ since A is self-dual and 22.27(iv), hence either $A^{\triangledown} \leq_{\text{W}} A^{\text{stretch}}$, or else $A^{\circ} \leq_{\text{W}} A^{\text{stretch}}$. We prove that the former holds by showing that **II** wins $G_{\text{L}}(A^{\triangledown}, A^{\text{stretch}})$:

> **II** plays 0 until, if ever, **I** plays a non-zero integer: then **II** plays 0 again and from now on he plays $i + 1$, where i is **I**'s move.

In order to show that **II** wins $G_{\mathrm{W}}(A^{\mathrm{stretch}}, A^{\triangledown})$ keep in mind that $\boldsymbol{T}(A)$ is well-founded (**??**):

> **II** plays 0 until, if ever, **I** reaches a position p of the form
>
> $$p = 0^{(k_0)}{}^\frown\langle m_0+1\rangle^\frown 0^{(k_1)}{}^\frown\langle m_1+1\rangle^\frown\cdots{}^\frown 0^{(k_n)}{}^\frown\langle m_n+1\rangle$$
>
> and $A_{\lfloor s\rfloor} <_{\mathrm{W}} A$, where $s = \langle m_0,\ldots,m_n\rangle \in {}^{<\omega}\omega$. At this round **II** plays 1, and since $(A^{\mathrm{stretch}})_{\lfloor p\rfloor} = (A_{\lfloor s\rfloor})^{\mathrm{stretch}}$ by 22.27(iii), from now on he uses the strategy to reduce $(A_{\lfloor s\rfloor})^{\mathrm{stretch}}$ to A.

Therefore $[A]_{\mathrm{W}}^{\mathrm{stretch}} = [A^{\triangledown}]_{\mathrm{W}}$.

Now for (b): If $C=\emptyset$ then by (a) $[A+C]_{\mathrm{W}} = [A]_{\mathrm{W}}^{\mathrm{stretch}}$ hence by 22.27(ii) and by $\emptyset^{\mathrm{stretch}} = \emptyset$, then

$$[A+C]_{\mathrm{W}}^{\mathrm{stretch}} = [A]_{\mathrm{W}}^{\mathrm{stretch}} = [A]_{\mathrm{W}} + [C]_{\mathrm{W}}^{\mathrm{stretch}}.$$

If $C \neq \emptyset$ then $[C]_{\mathrm{W}}^{\mathrm{stretch}} = [\emptyset]_{\mathrm{W}} + [C]_{\mathrm{W}}^{\mathrm{stretch}}$ by 22.27 (v) and (vii), hence

$$\begin{aligned}
[A]_{\mathrm{W}} + [C]_{\mathrm{W}}^{\mathrm{stretch}} &\leq [A]_{\mathrm{W}}^{\mathrm{stretch}} + [C]_{\mathrm{W}}^{\mathrm{stretch}} && \text{by 22.7}\\
&= ([A]_{\mathrm{W}} + [\emptyset]_{\mathrm{W}}) + [C]_{\mathrm{W}}^{\mathrm{stretch}} && \text{by (a)}\\
&= [A]_{\mathrm{W}} + [C]_{\mathrm{W}}^{\mathrm{stretch}} && \text{by 22.12.}
\end{aligned}$$

Since the stretch operation commutes with addition the result follows at once. □

Notice that the appeal to $\mathsf{DC}(\mathbb{R})$ in the proof of part (b) could be easily eliminated at the expense of clarity. Assuming the results in 23**??** the least A as in 22.28 is of rank ω_1^{ω}, e.g. $A = \bigoplus_n A_n$ with $A_n \in \mathrm{D}_{n+1}$.

Exercise 22.29. Assume AD^{L} and $A \not\leq_{\mathrm{L}} B^{\mathrm{stretch}}$. Then **I** wins $G_{\mathrm{W}}(A,B)$.

22.28 shows that the stretch operation is periodic with period ω_1^{ω} and enables us to compute A^{stretch} when $\|A\|_{\mathrm{W}} < \omega_1^{\omega}\cdot\omega_1 = \omega_1^{\omega+1}$. The next result settles what happens at later stages: if $[A]_{\mathrm{bt}}$ is non-self-dual and has infinitely many $\leq_{\mathrm{bt}}$-predecessors, then $A \equiv_{\mathrm{W}} A^{\mathrm{stretch}}$.

Proposition 22.30 (Steel)**.** *Assume* $\mathsf{BP}{+}\mathsf{AD}^{\mathrm{L}}$*. If* $A \not\equiv_{\mathrm{bt}} \neg A$ *and* $A \notin \mathrm{Diff}(<\omega;\boldsymbol{\Pi}_2^0)$ = *the smallest Boolean algebra containing* $\boldsymbol{\Pi}_2^0$*, then* $[A]_{\mathrm{W}} = [A]_{\mathrm{W}}^{\mathrm{stretch}}$*.*

Proof. In order to motivate the technicalities that follow, let us see where the naive attempt to prove the result breaks down. Towards a contradiction, suppose $A^{\mathrm{stretch}} \not\leq_{\mathrm{L}} A$ and let σ be a winning strategy for **I** in $G_{\mathrm{L}}(A^{\mathrm{stretch}}, A)$.

We now apply the Martin–Monk method to σ and τ, the copying strategy for **II**. For any $z \in {}^{\omega}2$ let $\Sigma_n^z = \sigma$ if $z(n) = 0$ and $\Sigma_n^z = \tau$ if $z(n) = 1$, and consider the diagram generated by $\langle \Sigma_n^z \mid n \in \omega \rangle$ and constructed according to the following rule: take $x^{n+1}(z)$, the real in the $n+1$-st row, unstretch it and feed it into Σ_n^z to construct the n-th row $x^n(z)$. If z and w differ for exactly one entry, then $H(z) \in A^{\text{stretch}} \Leftrightarrow H(w) \notin A^{\text{stretch}}$, where $H : {}^{\omega}2 \to \mathbb{R}$ is the map $z \mapsto x^0(z)$. Therefore $\{z \in {}^{\omega}2 \mid H(z) \in A^{\text{stretch}}\}$ is a flip-set, contradicting the property of Baire. The only problem is that for some z and n the real $x^{n+1}(z)$ may not be the stretch of anything, that is $x_i^{n+1}(z) = 0$ for all sufficiently large i, so that the diagram does not admit a global play. As in the proof of **??** we need some "padding" in the Martin–Monk diagram, i.e., we need to introduce enough well-behaved strategies that guarantee the existence of a global play for every $z \in {}^{\omega}2$. Here are the details:

Recall from 22.6 that $X^+ = \{x + 1 \mid x \in X\}$. Rather than working with A we shall work with A^+ since it is technically more convenient not to subtract 1 when unstretching a real. Let $\pi : {}^{<\omega}\omega \to {}^{<\omega}\omega$ be the map that erases all 0's from a sequence, i.e., $\pi(\emptyset) = \emptyset$ and $\pi(s^\frown\langle 0 \rangle) = \pi(s)$. Therefore π induces a continuous function $g : \mathbb{R}^{\text{stretch}} \twoheadrightarrow \mathbb{R}^+$.

By 22.22 we may assume that $\boldsymbol{T}(A) = {}^{<\omega}\omega$ so that $\boldsymbol{T}(A^+) = \{t + 1 \mid t \in {}^{<\omega}\omega\}$. **II** wins $G_{\text{L}}(A^+, A)$ as follows:

> As long as **I** plays integers of the form $n + 1$ then **II** answers n. Suppose at some round **I** plays 0, and let p be **II**'s position before this round. Since $A_{\lfloor p \rfloor} \equiv_{\text{W}} A \neq \mathbb{R}$, let $x \notin A_{\lfloor p \rfloor}$. Then **II** follows x from now on.

Since $A \leq_{\text{L}} A^+$ is immediate, we have that $A \equiv_{\text{L}} A^+$ and since A is non-self-dual, so is A^+. In other words **I** wins $G_{\text{L}}(A^+, \neg A^+)$.

Towards a contradiction suppose that $A^{\text{stretch}} \not\leq_{\text{L}} A \equiv_{\text{L}} A^+$, so fix σ a winning strategy for **I** in $G_{\text{L}}(A^{\text{stretch}}, A^+)$.

Exercise 22.31. (i) Show that for any $x \in \mathbb{R}$

$$\sigma * (x + 1) \in \mathbb{R}^{\text{stretch}} = \{y \mid \exists^{\infty} m \, (y(m) \neq 0)\} .$$

(ii) Show that $\forall s \in \boldsymbol{T}(A^+) = \{t+1 \mid t \in {}^{<\omega}\omega\}$ there is a winning strategy σ_s for **I** in $G_{\text{L}}(A^+, \neg A^+)$ that always plays non-zero natural numbers (i.e., $\operatorname{ran}(\sigma_s) \subseteq \omega \setminus 1$) and such that plays s in the first $\operatorname{lh}(s)$ moves, regardless of **II**'s moves.

Also let τ be the copying strategy for **II**. We now apply the Martin–Monk method. Let $\langle \Sigma_n \mid n < \omega \rangle$ be a sequence of strategies such that

$$\forall n \left(\Sigma_{2n} \in \{\sigma_s \mid s \in \boldsymbol{T}(A^+)\} \;\&\; \Sigma_{2n+1} \in \{\sigma, \tau\} \right) .$$

Let also

$$\mathcal{G}_n = \begin{cases} G_{\mathrm{L}}(A^+, \neg A^+) & \text{if } n \text{ is even,} \\ G_{\mathrm{L}}(A^{\mathrm{stretch}}, A^+) & \text{if } \Sigma_n = \sigma\,, \\ G_{\mathrm{L}}(A^+, A^{\mathrm{stretch}}) & \text{if } \Sigma_n = \tau\,, \end{cases}$$

so that Σ_n is winning in $\mathcal{G}_n$. The Σ_n's construct together with π the following diagram:

$$\begin{array}{lllll l}
 & x_0^0 & x_1^0 & x_2^0 & \cdots\cdots & = x^0 \\
\mathcal{G}_0, \Sigma_0 & & & & & \\
 & x_0^1 & x_1^1 & x_2^1 & \cdots\cdots & = x^1 \\
 & \uparrow & \uparrow & \uparrow & & \\
 & x_0^2 & x_1^2 & x_2^2 & \cdots\cdots & = x^2 \\
\mathcal{G}_1, \Sigma_1 & & & & & \\
 & x_0^3 & x_1^3 & x_2^3 & \cdots\cdots & = x^3 \\
 & \uparrow & \uparrow & \uparrow & & \\
 & x_0^4 & x_1^4 & x_2^4 & \cdots\cdots & = x^4 \\
\mathcal{G}_2, \Sigma_2 & & & & & \\
 & x_0^5 & x_1^5 & x_2^5 & \cdots\cdots & = x^5 \\
\quad\vdots & & & & &
\end{array}$$

where the vertical arrows represent the application of the π map: if in the $(2n)$-th row we have filled-in a sequence $\langle x_0^{2n}, \ldots, x_k^{2n} \rangle$ then in the $(2n-1)$-st row we copy $\pi(\langle x_0^{2n}, \ldots, x_k^{2n} \rangle) = \langle x_0^{2n-1}, \ldots, x_h^{2n-1} \rangle$, the subsequence of all non-zero elements of $\langle x_0^{2n}, \ldots, x_k^{2n} \rangle$. Notice that if $\langle x_0^{2n}, \ldots, x_k^{2n} \rangle = 0^{(k+1)}$ then $\pi(\langle x_0^{2n}, \ldots, x_k^{2n} \rangle) = \emptyset$, i.e., we do not copy anything at this point in the $(2n-1)$-st row. Thus if for some $n > 0$, x^{2n} is definitively equal to 0 then we cannot complete the $(2n-1)$-st row and hence $\langle \Sigma_n \mid n \in \omega \rangle$ does not admit a global play. If instead $\exists^\infty m\,(x_m^{2n} \neq 0)$ then $x^{2n+1} = g(x^{2n})$. Since τ and the σ_s's always yield elements of $\mathbb{R}^+$, problems can only occur between rows with indices of the form $4n+2$ and $4n+1$ and such that $\Sigma_{2n+1} = \sigma$.

We will construct $\langle s_n \mid n \in \omega \rangle \in {}^\omega\boldsymbol{T}(A^+)$ such that for any $z \in {}^\omega 2$ the sequence $\langle \Sigma_n^z \mid n \in \omega \rangle$ defined by

$$\Sigma_n^z = \begin{cases} \sigma_{s_m} & \text{if } n = 2m, \\ \sigma & \text{if } n = 2m+1 \text{ and } z(m) = 0, \\ \tau & \text{if } n = 2m+1 \text{ and } z(m) = 1, \end{cases}$$

admits a global play, which amounts to say that $\forall n > 0\, \exists^\infty m\,(x_m^{2n} \neq 0)$, or equivalently (by the exercise), that $\exists^\infty m\,(x_m^0 \neq 0)$. Notice that if n is odd and $\Sigma_n = \sigma$ then the $(2n)$-th and $(2n+1)$-st rows in the diagram are

$$G_{\mathrm{L}}(A^{\mathrm{stretch}}, A^+), \sigma \quad \begin{array}{llllll} \mathbf{I} & x_0^{2n} & & x_1^{2n} & \cdots\cdots & = x^{2n} \\ \mathbf{II} & & x_0^{2n+1} & & x_1^{2n+1} \cdots\cdots & = x^{2n+1} \end{array}$$

and $x^{2n} \in A^{\mathrm{stretch}} \Leftrightarrow x^{2n+1} \notin A^+$. Vice versa is n is odd but $\Sigma_n = \tau$ then the $(2n)$-th and $(2n+1)$-st rows are

$$G_{\mathrm{L}}(A^+, A^{\mathrm{stretch}}), \tau \quad \begin{array}{llllll} \mathbf{II} & & x_0^{2n} & & x_1^{2n} \cdots\cdots & = x^{2n} \\ \mathbf{I} & x_0^{2n+1} & & x_1^{2n+1} & \cdots\cdots & = x^{2n+1} \end{array}$$

and $x^{2n} \in A^{\mathrm{stretch}} \Leftrightarrow x^{2n+1} \in A^+$, since $x^{2n} = x^{2n+1}$ and $x^{2n+1} = g(x^{2n+2}) \in \mathbb{R}^+$.

Given $\langle s_n \mid n \in \omega \rangle$ as above we reach a contradiction as in **??**: let $H : {}^{\omega}2 \to \mathbb{R}$, $z \mapsto x^0(z) =$ the real in the 0th row of the Martin–Monk diagram when $\langle \Sigma_n^z \mid n \in \omega \rangle$ is used. As before H is continuous and $H^{-1}(A^+)$ is a flip-set, i.e.,

$$\exists! n \, (z(n) \neq w(n)) \Rightarrow \left(H(z) \in A^+ \Leftrightarrow H(w) \notin A^+ \right),$$

contradicting **??**. Therefore we are only left to construct the s_n's.

We need the following

Lemma 22.32. *Let $\langle \Sigma_i \mid i < 2n \rangle$ be such that $\Sigma_i \in \{\sigma_s \mid s \in \boldsymbol{T}(A^+)\}$ if i is even, and $\Sigma_i \in \{\sigma, \tau\}$ if i is odd. Then the set of all t such that $\langle \Sigma_i \mid i < 2n \rangle^\frown \langle \sigma_t \rangle$ admits a finite play with its 0-th row of length at least n is dense in $\boldsymbol{T}(A^+)$, that is to say*

$$\forall s \in \boldsymbol{T}(A^+) \, \exists u \in \boldsymbol{T}(A^+) \, (s \subseteq u \,\&\, \forall t \in \boldsymbol{T}(A^+) \, \exists (D, f) \text{ finite play} \\ \text{for } \langle \Sigma_i \mid i < 2n \rangle^\frown \langle \sigma_t \rangle \text{ and } |D \cap \{0\} \times \omega| \geq n).$$

Proof. Let $\langle \Sigma_i \mid i < 2n \rangle$ and $s \in \boldsymbol{T}(A^+)$ be as in the hypothesis. It is enough to show that for some $\bar{x} \in \mathbb{R}^+ \cap \boldsymbol{N}_s$, letting $x^{4n-1} = \bar{x}$, then the reals in the first $4n-1$ rows $x^{4n-2}, x^{4n-3}, \ldots, x^0$ are defined, since, by continuity, we can take $u = \bar{x} \restriction k$ for k sufficiently large. How do we get such an $\bar{x}$? Since the only possible source of troubles is when $\Sigma_i = \sigma$, we may assume that $\{i < 2n \mid \Sigma_i = \sigma\} \neq \emptyset$ and since the x^m's with larger labels are constructed first, let $\langle J_0, \ldots, J_M \rangle$ be the enumeration of $\{i < 2n \mid \Sigma_i = \sigma\}$ in *decreasing* order. We must find an $\bar{x} \in \mathbb{R}^+ \cap \boldsymbol{N}_s$ such that the rows $x^{2J_0}, x^{2J_1}, \ldots, x^{2J_M}$

are in $\mathbb{R}^{\text{stretch}}$ so that the filling-in procedure of the diagram does not come to a grinding halt. For any $z \in A^{\text{stretch}}$ then $g(z) \in A^+$ and

$$(\dagger) \qquad \begin{aligned} &(\sigma_s * g(z))_{\mathbf{I}} \in A^+\,, \\ &(g(z) * \tau)_{\mathbf{II}} \in A^+\,, \\ &(\sigma * g(z))_{\mathbf{I}} \notin A^{\text{stretch}}\,. \end{aligned}$$

In other words, applying σ_s's or τ's do not change membership in A^{stretch} while σ *does* change membership in A^{stretch}. Using this

$$\bar{x} \notin A^+ \Leftrightarrow x^{2J_0+1} \notin A^+ \Leftrightarrow x^{2J_0} = \sigma * x^{2J_0+1} \in A^{\text{stretch}}\,.$$

So if $x \notin A^+ \Rightarrow x^{2J_0} \in \mathbb{R}^{\text{stretch}}$. However the converse implication does not hold since by continuity of $\bar{x} \mapsto x^{2J_0}$ it would imply that A^+ (and hence A) is $\mathbf{\Pi}^0_2$, contrarily to our hypothesis. Thus we can find $Y \subseteq \boldsymbol{N}_s \cap A^+$ such that $x^{2J_0} \in \mathbb{R}^{\text{stretch}}$ for all $\bar{x} \in Y$, and therefore $x^{2J_0-1}, \ldots, x^{2J_1}$ are defined. By ($\dagger$) again, for $\bar{x} \in Y$,

$$\bar{x} \in A^+ \Leftrightarrow x^{2J_0} \notin A^{\text{stretch}} \Leftrightarrow x^{2J_1} \in A^{\text{stretch}}$$

so that x^{2J_1-1} (and hence x^{2J_1}) is defined. Trying to enforce that $g(x^{2J_2})$ is defined presents the same problem again, so that we must find a $Y' \subseteq \boldsymbol{N}_s \setminus A^+$ such that for every $\bar{x} \in Y'$ all the rows up to x^{2J_2} are defined. By iterating the argument M-many times, each time changing our minds on whether or not $\bar{x} \in A^+$, we can determine $\bar{x}$.

The informal argument above can be given a sleeker presentation so that the real $\bar{x}$ will be obtained as a play in a game which we now describe.

For every $m \in \omega$ we say that $x \in \mathbb{R}$ codes a real in the m-th coordinate iff

$$\forall i \in \omega\, \exists! j \in \omega\, \exists k \in \omega\, (x(k) = \langle m, \langle i, j \rangle \rangle)\,,$$

that is $\{((n)_0, (n)_1) \mid \langle m, n \rangle \in \operatorname{ran}(x)\}$ is the graph of a function $\omega \to \omega$. In this case let $c_{(x,m)}(i)$ be the unique j such that $\langle m, \langle i, j \rangle \rangle \in \operatorname{ran}(x)$. If x codes $c_{(x,m)}$ in the m-th coordinate and $c_{(x,m)} \in \mathbb{R}^+$ then $g(c_{(x,m)})$ is the real obtained from $c_{(x,m)}$ by deleting all 0 entries. It is easy to check that

$$\{x \in \mathbb{R} \mid g(c_{(x,m)}) \text{ is defined}\} = \{x \in \mathbb{R} \mid c_{(x,m)} \in \mathbb{R}^{\text{stretch}}\} \in \mathbf{\Pi}^0_2\,.$$

As above, let $\langle J_0, \ldots, J_M \rangle$ be the enumeration in decreasing order of $\{i < 2n \mid \varSigma_i = \sigma\}$ and let

$$B = \{y \in \mathbb{R} \mid \exists k \leq M \left(g(c_{(y,J_k)}) \text{ is undefined and } k \text{ is odd } \& \ \forall i < k \left(g(c_{(y,J_i)}) \text{ is defined}\right)\right)\}\,.$$

Then B is a Boolean combination of $\mathbf{\Pi}^0_2$ sets and hence, by our assumption, $B <_{\mathrm{W}} A$. Therefore $\mathbf{I}$ wins $G_{\mathrm{L}}(A^+, \neg B)$. In fact by an argument as before we may pick a winning strategy $\hat{\sigma}$ for **I** in $G_{\mathrm{L}}(A^+, \neg B)$ that plays s in the first $\mathrm{lh}(s)$ moves and such that never plays a 0. The real $\bar{x}$ we are looking for will be given by $\hat{\sigma}$ played against the strategy for **II** $\hat{\tau}$ defined as follows:

> Suppose p and q represent the positions of **I** and **II** at some round in $G_{\mathrm{L}}(A^+, \neg B)$. If for some $i \leq M$ there are k, h such that $\langle J_i, \langle k, h\rangle\rangle \notin \mathrm{ran}(q)$ and h is in the m-th row and the k-th column of the partial play of $\langle \Sigma_i \mid i < 2n\rangle$ generated by p, where $m = 2J_i + 1$, i.e., $x^m_k = h$, then $\hat{\tau}$ plays the least such $\langle J_i, \langle k, h\rangle\rangle$. Otherwise $\hat{\tau}$ passes, if there is no such a triple.

Let $\bar{x}$ and $\bar{y}$ be the reals played by $\hat{\sigma}$ and $\hat{\tau}$. The following Claim shows that $\bar{y}$ is indeed defined.

Claim 22.32.1. *$\hat{\tau}$ does not pass forever.*

Proof. Since $2n > k > J_0 \Rightarrow \Sigma_k \neq \sigma$ then $x^{4n-1}, x^{4n-2}, \ldots, x^{2J_0}$ are defined, and so is $x^{2J_0} = \sigma * x^{2J_0+1}$. Therefore by waiting long enough **II** can always find triples of the form $\langle J_0, \langle k, h\rangle\rangle$ which he has not yet played. $\square$

As $\hat{\sigma}$ is winning in $G_{\mathrm{L}}(A^+, \neg B)$, then $\bar{x} \in A^+ \Leftrightarrow \bar{y} \in B$.

Claim 22.32.2. *$\bar{x} \notin A^+$.*

Proof. Deny. Then $\bar{y} \in B$, and if k is least such that $g(c_{(\bar{y}, J_k)})$ is undefined, that is $c_{(\bar{y}, J_k)} \notin \mathbb{R}^{\mathrm{stretch}}$, then k is odd. As $\bar{y}$ encodes the rows $x^{2J_0+1}, x^{2J_1+1}, \ldots, x^{2J_{k+1}-1}$ in the sense that

$$\forall i < k \left(c_{(\bar{y}, J_i)} = x^{2J_i+1}\right),$$

then, by (†) and since k is odd,

$$\begin{aligned} \bar{x} = x^{4n-1} \in A^+ &\Leftrightarrow x^{2J_0+1} = c_{(\bar{y}, J_0)} \notin A^{\mathrm{stretch}} \\ &\Leftrightarrow x^{2J_1+1} = c_{(\bar{y}, J_1)} \in A^{\mathrm{stretch}} \\ &\quad\vdots \\ &\Leftrightarrow x^{2J_{k-1}+1} = c_{(\bar{y}, J_{k-1})} \notin A^{\mathrm{stretch}}. \end{aligned}$$

Therefore $g(c_{(\bar{y}, J_{k-1})})$, which is defined by the minimality of k, does not belong to A^+. As the σ_s's and τ always yield elements of $\mathbb{R}^+$, then $x^{2J_k+1} = c_{(\bar{y}, J_k)}$ is defined and since $x^{2J_{k-1}} = g(c_{(\bar{y}, J_{k-1})}) \notin A^+$, then $x^{2J_k+1} = c_{(\bar{y}, J_k)} \in A^{\mathrm{stretch}}$ and therefore $g(c_{(\bar{y}, J_k)})$ is defined: a contradiction. $\square$

Thus $\bar{x} \in A^+$ and $\bar{y} \notin B$. Suppose $g(c_{(\bar{y},J_k)})$ is undefined for some $k \leq M$. If k is the least such, k must be even, since otherwise $\bar{y} \in B$. Arguing as above

$$\bar{x} \in A^+ \Leftrightarrow c_{(\bar{y},J_0)} \in A^{\text{stretch}} \Leftrightarrow c_{(\bar{y},J_1)} \notin A^{\text{stretch}} \Leftrightarrow \cdots \Leftrightarrow c_{(\bar{y},J_{k-1})} \notin A^{\text{stretch}},$$

hence $g(c_{(\bar{y},J_k)})$ is defined: a contradiction. Therefore $g(c_{(\bar{y},J_k)})$ is defined for all $k \leq M$, that is

$$\forall k \leq 4n-1 \left(x^k \text{ is defined}\right)$$

and in particular the 0th row x^0 is defined, and this is what we had to prove. □

Let's go back to the proof of 22.30. Suppose the $\langle s_i \mid i < n\rangle$ has been defined: considering the 2^n many $\langle \Sigma_i \mid i < 2n\rangle$'s such that $\Sigma_{2i} = s_i$ and $\Sigma_{2i+1} \in \{\sigma, \tau\}$, and applying consecutively 2^n many times the Lemma we get a s_n such that $\langle \Sigma_i \mid i < 2n\rangle^\frown\langle \sigma_{s_n}\rangle$ admits a finite play with at least $2n$ non-zero entries in the 0th row.

This completes the proof of 22.30. □

Using 22.25 and the fact that any set in $\text{Diff}(<\omega; \mathbf{\Pi}^0_2)$ has Wadge rank $< \omega_1^\omega$ (see §23**??**) we obtain at once that

$$\|A\|_{\text{W}} = \omega_1^\xi \ \& \ \operatorname{cof}(\xi) > \omega \Rightarrow A^{\text{stretch}} \equiv_{\text{L}} A\,.$$

This result will be generalized in 22.41 to the case when $\|A\|_{\text{W}} = \omega_1^\omega \cdot \alpha$ and $\operatorname{cof}(\alpha) > \omega$.

22.F Degree Multiplication

We have seen several ways to construct new sets of higher Wadge rank from old sets. For example $\left\|A^\natural\right\|_{\text{W}} = \left\|A^\flat\right\|_{\text{W}} = \|A\|_{\text{W}} \cdot \omega_1$, for A non-self-dual. We now introduce a new operation on sets that will allow to take even larger leaps on the Wadge hierarchy.

Definition 22.33 (Steel)**.** For $A, B \subseteq \mathbb{R}$ the **multiplication of A and B**, in symbols $A \cdot B \subseteq \mathbb{R}$ is defined by

$$
\begin{aligned}
x \in A \cdot B \Leftrightarrow & [\exists n\, \exists s_0, \dots, s_{2n} \in {}^{<\omega}\omega \setminus \{\emptyset\}\, \exists a \in A \\
& \quad \big(x = (s_0+1)^\frown\langle 0\rangle^\frown(s_1+1)^\frown\langle 0\rangle^\frown \cdots {}^\frown\langle 0\rangle^\frown(s_{2n}+1)^\frown\langle 0\rangle^\frown(a+1)\big)] \vee \\
& [\exists n\, \exists s_0, \dots, s_{2n+1} \in {}^{<\omega}\omega \setminus \{\emptyset\}\, \exists y \in \mathbb{R} \\
& \quad (x = (s_0+1)^\frown\langle 0\rangle^\frown(s_1+1)^\frown\langle 0\rangle^\frown \cdots {}^\frown\langle 0\rangle^\frown(s_{2n+1}+1)^\frown\langle 0\rangle^\frown(y+1)\ \& \\
& \qquad s_0{}^\frown s_2{}^\frown \cdots {}^\frown s_{2n}{}^\frown y \in B)] \vee \\
& [\exists\langle s_i \mid i \in \omega\rangle \in {}^\omega({}^{<\omega}\omega \setminus \{\emptyset\}) \\
& \quad (x = (s_0+1)^\frown\langle 0\rangle^\frown(s_1+1)^\frown\langle 0\rangle^\frown \cdots {}^\frown\langle 0\rangle^\frown(s_n+1)^\frown\langle 0\rangle^\frown \cdots\ \& \\
& \qquad s_0{}^\frown s_2{}^\frown s_4{}^\frown \cdots \in B)]
\end{aligned}
$$

The intuition behind this definition is that for any real x not containing two consecutive 0's and not starting with 0, i.e.,

$$x = (s_0+1)^\frown\langle 0\rangle^\frown(s_1+1)^\frown\langle 0\rangle^\frown \cdots$$

we can think of x as being entered on two different rows, the s_{2n}'s on the B-row, the s_{2n+1}'s on the A-row:

$$
\begin{array}{lccccc}
B\text{-row} & s_0 & & s_2 & & \\
 & & & & & \cdots \\
A\text{-row} & & s_1 & & s_3 &
\end{array}
$$

with the understanding that if $x(m)$ is definitively non-zero, that is if

$$x = (s_0+1)^\frown\langle 0\rangle^\frown \cdots {}^\frown(s_n+1)^\frown\langle 0\rangle^\frown(y+1)\,,$$

then y is recorded on the A-row, if n is even, or on the B-row, if n is odd. Call an x such that it does not contain two consecutive 0s, a **product real**. Thus $x \in A \cdot B$ just in case x is a product real and *either* we write infinitely often on the B-row and $b \in B$, where b is the real recorded there, *or else* from some point on we only write on the A-row and $a \in A$, where a is the real recorded there from that point on.

Just like with $A + B$ or $A^\natural$, the game $G_{\mathrm{W}}(C; A \cdot B)$ can be given an alternate description.

Exercise 22.34. For $A, B, C \subseteq \mathbb{R}$ the game $G_{\cdot}(C; A, B)$ is a Wadge-style game where **I** and **II** play natural numbers and **II** has the option of passing at any stage, but must play infinitely many times. Player **II**'s moves are recorded on two different rows, the A-row and the B-row. His first move must be on the B-row, but after that he can flip between the two rows. Let x be the real played by **I**. If **II** plays infinitely often on the B-row (i.e., if he flips infinitely often between the two rows or if he eventually settles on the B-row) let y be the real recorded on it:

$$\mathbf{II}\begin{array}{llllll} B\text{-row} & y_0,\ldots,y_{n_0}, & & y_{n_0+1},\ldots,y_{n_1}, & & \\ A\text{-row} & & z_0,\ldots,z_{m_0}, & & z_{m_0+1},\ldots,z_{m_1}, & \ldots\ldots \end{array}$$

Then **II** wins $G.(C;A,B)$ iff

$$x \in C \Leftrightarrow y \in B\,.$$

If instead after a certain stage **II** always plays on the A-row let y be the real on the A-row recorded after this stage:

$$\mathbf{II}\begin{array}{llllll} B\text{-row} & s_0 & & \ldots & s_{2n} & \\ A\text{-row} & & s_1 & & & y_0, y_1, \ldots \end{array}$$

Then **II** wins $G.(C;A,B)$ iff

$$x \in C \Leftrightarrow y \in A\,.$$

Show that $G.(C;A,B)$ is equivalent to $G_{\mathrm{W}}(C, A \cdot B)$.

The following exercise summarizes the main properties of multiplication which will be used in the sequel.

Exercise 22.35. (i) $A_0 \leq_{\mathrm{W}} A_1$ & $B_0 \leq_{\mathrm{L}} B_1 \Rightarrow A_0 \cdot B_0 \leq_{\mathrm{W}} A_1 \cdot B_1$.
Thus if B is non-self-dual we can define

$$[A]_{\mathrm{W}} \cdot [B]_{\mathrm{W}} = [A \cdot B]_{\mathrm{W}}\,.$$

(ii) $A \cdot \emptyset \equiv_{\mathrm{W}} A^{\natural}$ and $A \cdot \mathbb{R} \equiv_{\mathrm{W}} A^{\flat}$.

(iii) If $B \neq \emptyset$, then $A \cdot (\langle 0 \rangle ^\frown B) \equiv_{\mathrm{W}} A \cdot B + A$.
Therefore $B_0 \leq_{\mathrm{L}} B_1$ in (i) cannot be replaced by $B_0 \leq_{\mathrm{W}} B_1$.

(iv) $A \cdot (\bigoplus_n B_n) \equiv_{\mathrm{W}} \bigoplus_n (A \cdot B_n + A)$.

(v) $A \cdot B^{\natural} \equiv_{\mathrm{W}} (A \cdot B)^{\natural}$ and $A \cdot B^{\flat} \equiv_{\mathrm{W}} (A \cdot B)^{\flat}$.

(vi) $A \cdot (B + C) \equiv_{\mathrm{W}} (A \cdot B) + (A \cdot C)$.

(vii) If A and B are self-dual, so is $A \cdot B$.

(viii) $A^{\mathrm{stretch}} \leq_{\mathrm{L}} \emptyset \cdot (\langle 0 \rangle ^\frown A)$.

The main result on multiplication of sets is the following

Theorem 22.36 (Steel)**.** *Assume* $\mathsf{BP} + \mathsf{AD}^{\mathrm{L}} + \mathsf{DC}(\mathbb{R})$. *Let* A *be self-dual. If* $\operatorname{cof}(\|B\|_{\mathrm{W}}) > \omega$ *then*

$$\|A \bullet B\|_{\mathrm{W}} = \|A\|_{\mathrm{W}} \cdot \omega_1 \cdot \|B\|_{\mathrm{W}} \; .$$

This result will be proved through a series of lemmata.

Lemma 22.37. *Assume* $\mathsf{BP} + \mathsf{AD}^{\mathrm{L}}$. *For* $A, B, C \subseteq \mathbb{R}$ *with* A *self-dual*

$$A \bullet B \leq_{\mathrm{L}} A \bullet C \Rightarrow B \leq_{\mathrm{L}} C$$

and

$$A \bullet B \leq_{\mathrm{W}} A \bullet C \Rightarrow B \leq_{\mathrm{W}} C \, .$$

Proof. We shall prove the second implication only (i.e., the case of $\leq_{\mathrm{W}}$) since the other one is similar. Fix a winning strategy τ for **II** in $G_{\mathrm{W}}(A \bullet B, A \bullet C)$. We will define a winning strategy $\tilde{\tau}$ for **II** in $G_{\mathrm{W}}(B, C)$ such that if b and $c = b * \tilde{\tau}$ are the reals played by **I** and **II** in this game, then there are reals x and $y = x * \tau$ played by **I** and **II** in $G_{\mathrm{W}}(A \bullet B, A \bullet C)$ so that b is the component of x on the B-row and c is the component of y on the C-row. Here is the definition of $\tilde{\tau}$:

> Suppose **I** plays $b_0, b_1, \dots$ in $\tilde{\mathcal{G}} = G_{\mathrm{W}}(B, C)$. These values are copied on the B-row of **I** in $\mathcal{G} = G_{\mathrm{W}}(A \bullet B, A \bullet C)$. As long as τ passes or plays $c_0, c_1, \dots$ in the C-row, the $\tilde{\tau}$ passes or copies $c_0, c_1, \dots$ in $G_{\mathrm{W}}(B, C)$. Suppose we reach a round $n - 1$ when **II** first decides to play in the A-row: then τ responds with 0 and a_1 to b_n and b_{n+1}
>
> | | | | | | |
> |---|---|---|---|---|---|
> | $\tilde{\mathcal{G}}$ | **I** | | $b_0, \dots, b_{n-1},$ | $b_n,$ | b_{n+1} |
> | | **II** | | $c_0, \dots, c_{n'-1}$ | $-$ | |
> | $\mathcal{G}$ | **I** | B-row | $b_0, \dots, b_{n-1},$ | $b_n,$ | b_{n+1} |
> | | | A-row | | | |
> | | **II** | C-row | $c_0, \dots, c_{n'-1},$ | | |
> | | | A-row | | | a_1 |
>
> (Keep in mind that $n' \leq n$ since τ may pass.) $\tilde{\tau}$ passes at round n: this is denoted by the "$-$" in the diagram above. If $\langle a_1 \rangle \in \boldsymbol{T}(A)$ then $\tilde{\tau}$ passes at this round. Suppose **I** plays b_{n+2}. If τ announces

that he is going to play in the C-row, i.e., τ plays 0 then $\tilde{\tau}$ passes this turn and then copies τ's moves:

$$\begin{array}{cclllllll}
 & \mathbf{I} & & b_0,\dots,b_{n-1}, & b_n, & b_{n+1}, & b_{n+2}, & b_{n+3},\dots \\
\tilde{\mathcal{G}} & & & & & & & \\
 & \mathbf{II} & & c_0,\dots,c_{n'-1}, & - & - & - & c_{n'} \\
\hline
 & \mathbf{I} & B\text{-row} & b_0,\dots,b_{n-1}, & b_n, & b_{n+1}, & b_{n+2}, & b_{n+3} \\
 & & A\text{-row} & & & & & \\
\mathcal{G} & & & & & & & \\
 & \mathbf{II} & C\text{-row} & c_0,\dots,c_{n'-1}, & & & & c_{n'},\dots \\
 & & A\text{-row} & & & a_1 & &
\end{array}$$

If instead τ passes or plays a_2 in the A-row such that $\langle a_1, a_2\rangle \in \boldsymbol{T}(A)$ then $\tilde{\tau}$ passes again. And so on. Since A is self-dual $\boldsymbol{T}(A)$ is well-founded, hence there is a least $i \geq 0$ such that for every $j < i - 1$, at round $n + j$ the strategy τ has either passed or has played $a_{j'}$ in the A-row (with $j' \leq j$ since τ might have taken naps) such that $\langle a_1, \dots, a_{j'}\rangle \in \boldsymbol{T}(A)$, but at round $n + i$ either

(1) τ decides to play in the C-row, or else

(2) τ plays $a_{i'}$ in the A-row such that $t = \langle a_1, \dots, a_{i'}\rangle \notin \boldsymbol{T}(A)$.

If (1) holds then $\tilde{\tau}$ first passes and then copies τ.

$$\begin{array}{ccllllll}
 & \mathbf{I} & & b_0,\dots,b_{n-1}, & b_n, & b_{n+1},\dots,b_{n+i-1}, & b_{n+i}, & b_{n+i+1},\dots \\
\tilde{\mathcal{G}} & & & & & & & \\
 & \mathbf{II} & & c_0,\dots,c_{n'-1}, & - & -\cdots\cdots & - & c_{n'},\dots \\
\hline
 & \mathbf{I} & B\text{-row} & b_0,\dots,b_{n-1}, & b_n, & b_{n+1},\dots,b_{n+i-1}, & b_{n+i}, & b_{n+i+1},\dots \\
 & & A\text{-row} & & & & & \\
\mathcal{G} & & & & & & & \\
 & \mathbf{II} & C\text{-row} & c_0,\dots,c_{n'-1}, & & & & c_{n'},\dots \\
 & & A\text{-row} & & & a_1,\dots,a_{i'-1}, & &
\end{array}$$

Otherwise (2) holds: since $A_{\lfloor t\rfloor} <_{\mathrm{W}} A$, let σ be a winning strategy for **I** in $G_{\mathrm{W}}(A, A_{\lfloor t\rfloor})$ and let Player **I** use σ to obtain values in his A-row of $G_{\mathrm{W}}(A\boldsymbol{\cdot}B, A\boldsymbol{\cdot}C)$ by pitting σ against τ. (In the diagram below the output of σ is $\bar{a} = \langle \bar{a}_0, \bar{a}_1, \ldots \rangle$ and $a_{i'+1}$ is τ' response to **I** playing 0, i.e., having announced his entering the A-row.)

$$\mathcal{G} \quad \begin{array}{llllll} \mathbf{I} & B\text{-row} & b_0, \ldots, b_{n-1}, & b_n, & b_{n+1}, \ldots, b_{n+i}, & \\ & A\text{-row} & & & & \bar{a}_0, \bar{a}_1, \ldots \\ \mathbf{II} & C\text{-row} & c_0, \ldots, c_{n'-1}, & & & \\ & A\text{-row} & & & a_1, \ldots, a_{i'}, \quad a_{i'+1}, & a_{i'+2}, a_{i'+3}, \ldots \end{array}$$

As σ is winning, τ's response cannot always be on the A-row, since otherwise

$$\bar{a} \in A \Leftrightarrow \langle a_m \mid m > i' \rangle \notin A_{\lfloor t \rfloor} \Leftrightarrow a \notin A$$

which would imply that **II** loses this run of $G_{\mathrm{W}}(A \cdot B, A \cdot C)$, contrarily to the assumption that τ is winning. Therefore at some round τ must decide to switch-back to the C-row; then $\tilde{\tau}$ passes until then, and after that copies τ's moves on his C-row:

$$\tilde{\mathcal{G}} \quad \begin{array}{lll} \mathbf{I} & b_0, \ldots, b_{n-1}, b_n, b_{n+1}, \ldots, & b_{n+i} \\ \mathbf{II} & c_0, \ldots, c_{n'-1}, \quad - \cdots\cdots - & c_{n'} \end{array}$$

$$\mathcal{G} \quad \begin{array}{lllll} \mathbf{I} & B\text{-row} & b_0, \ldots, b_{n-1}, b_n, b_{n+1}, \ldots, b_{n+i}, & & \\ & A\text{-row} & & \bar{a}_0, \bar{a}_1 \ldots & \\ \mathbf{II} & C\text{-row} & c_0, \ldots, c_{n'-1}, & & c_{n'} \\ & A\text{-row} & \quad a_1, \quad \ldots\ldots, a_{i'+1}, a_{i'+2}, \ldots & & \end{array}$$

The argument can now be repeated.

By construction we have insured that $\tilde{\tau}$ plays infinitely often so we end up with two reals b, c which are a play of $G_{\mathrm{W}}(B, C)$ and x, y which are a play of $G_{\mathrm{W}}(A \cdot B, A \cdot C)$. Since b and c are the components of x and y on the B-row of **I** and on the C-row of **II**, respectively, and since τ is winning

$$b \in B \Leftrightarrow x \in A \cdot B \Leftrightarrow y \in A \cdot C \Leftrightarrow c \in C,$$

hence τ is a winning strategy for **II** in $G_{\mathrm{W}}(B, C)$. $\square$

Exercise 22.38. Assume $\mathsf{BP} + \mathsf{AD}^{\mathrm{L}}$. Show that if A is self-dual and B is non-self-dual then

(i) $A \cdot B$ is non-self-dual and $\neg(A \cdot B) \equiv_{\mathrm{W}} A \cdot (\neg B)$, and

(ii) $A \cdot (B \oplus \neg B) \equiv_{\mathrm{W}} A \cdot B + A$.

The heart of the proof of 22.36 is the fact that the map $B \mapsto A \cdot B$ is—in some sense—continuous when A is self-dual.

Lemma 22.39. *($\mathsf{AD}^{\mathrm{L}} + \mathsf{DC}(\mathbb{R})$) If A is self-dual and B non-self-dual, $B \neq \emptyset$, then*

$$\|A \cdot B\|_{\mathrm{W}} = \sup\{\|A \cdot C\|_{\mathrm{W}} \mid C <_{\mathrm{W}} B\}.$$

Proof. $\|A \cdot B\|_{\mathrm{W}} \geq \sup\{\|A \cdot C\|_{\mathrm{W}} \mid C <_{\mathrm{W}} B\}$ follows from 22.35 i) so it is enough to prove by induction on $\|B\|_{\mathrm{W}}$ the other inequality.

Suppose first $\|B\|_{\mathrm{W}}$ is a successor ordinal. Pick C self-dual such that $[B]_{\mathrm{W}}$ is the immediate successor of $[C]_{\mathrm{W}}$. Then either $B \equiv_{\mathrm{W}} C + \emptyset$ or else $B \equiv_{\mathrm{W}} C + \mathbb{R}$. If the former holds, using 22.35

$$\begin{aligned} A \cdot B &\equiv_{\mathrm{W}} A \cdot (C + \emptyset) && \text{(by (i))} \\ &\equiv_{\mathrm{W}} A \cdot C + A \cdot \emptyset && \text{(by (vi))} \\ &\equiv_{\mathrm{W}} A \cdot C + A^{\natural}. && \text{(by (ii))} \end{aligned}$$

Analogously, if $B \equiv_{\mathrm{W}} C + \mathbb{R}$ then $A \cdot B \equiv_{\mathrm{W}} A \cdot C + A^{\flat}$. In both cases, using 22.35(vii) and 22.16, $\|A \cdot B\|_{\mathrm{W}} = \|A \cdot C\|_{\mathrm{W}} + \|A\|_{\mathrm{W}} \cdot \omega_1$. If $\|C'\|_{\mathrm{L}} = \|C\|_{\mathrm{L}} + \alpha$ for some $\alpha < \omega_1$, then 22.35 (iii) and (iv) imply that $\|A \cdot C'\|_{\mathrm{W}} = \|A \cdot C\|_{\mathrm{W}} + \|A\|_{\mathrm{W}} \cdot \alpha$, and since $[C]_{\mathrm{W}}$ is the union of ω_1 Lipschitz degrees,

$$\sup\{\|A \cdot C'\|_{\mathrm{W}} \mid C' \in [C]_{\mathrm{W}}\} = \|A \cdot C\|_{\mathrm{W}} + \|A\|_{\mathrm{W}} \cdot \omega_1.$$

Therefore $\sup\{\|A \cdot C'\|_{\mathrm{W}} \mid C' <_{\mathrm{W}} C\} = \|A \cdot B\|_{\mathrm{W}}$.

Suppose now $\|B\|_{\mathrm{W}}$ is limit. We distinguish three cases.

- $\exists \nu < \|B\|_{\mathrm{W}} \, (\|B\|_{\mathrm{W}} = \nu \cdot \omega_1)$.

Then by changing ν to $\nu + 1$ if needed we may assume that there is a self dual C of Wadge rank ν so that $B \equiv_{\mathrm{W}} C^{\natural}$ or $B \equiv_{\mathrm{W}} C^{\flat}$. Then using 22.35 (vii) and (v) we see that $A \cdot C$ is self-dual hence, in either case,

$$\|A \cdot B\|_{\mathrm{W}} = \|A \cdot C\|_{\mathrm{W}} \cdot \omega_1.$$

- $\exists \nu < \|B\|_{\mathrm{W}} \, (\|B\|_{\mathrm{W}} < \nu \cdot \omega_1)$.

Fix ν as above and let $\xi < \omega_1$ be least such that $\|B\|_{\mathrm{W}} \leq \nu \cdot \xi$. Since B is non-self-dual then $\mathrm{cof}(\|B\|_{\mathrm{W}}) > \omega$ hence ξ must be a successor, $\xi = \alpha + 1$.

Thus $\nu \cdot \alpha + \beta$ with β limit, $\operatorname{cof}(\beta) > \omega$, and $\beta < \nu < \|B\|_{\mathrm{W}}$. Let C be self-dual of Wadge rank $\nu \cdot \alpha$ or $\nu \cdot \alpha + 1$, and let D be such that $\|D\|_{\mathrm{W}} = \beta$. Then $B \equiv_{\mathrm{W}} C + D$ and $A \bullet B \equiv_{\mathrm{W}} A \bullet C + A \bullet D$. Since $\|D\|_{\mathrm{W}} < \|B\|_{\mathrm{W}}$, by inductive hypothesis

$$\begin{aligned}\|A \bullet B\|_{\mathrm{W}} &= \|A \bullet C\|_{\mathrm{W}} + \|A \bullet D\|_{\mathrm{W}} \\ &= \|A \bullet C\|_{\mathrm{W}} + \sup\{\|A \bullet D'\|_{\mathrm{W}} \mid D' <_{\mathrm{W}} D\} \\ &= \sup\{\|A \bullet (C + D')\|_{\mathrm{W}} \mid D' <_{\mathrm{W}} D\} \\ &\leq \sup\{\|A \bullet B'\|_{\mathrm{W}} \mid B' <_{\mathrm{W}} B\}.\end{aligned}$$

- $\forall \nu < \|B\|_{\mathrm{W}} \, (\nu \cdot \omega_1 < \|B\|_{\mathrm{W}})$.

Then $\|B\|_{\mathrm{W}} = \omega_1^{\xi}$ with $\operatorname{cof}(\xi) > \omega$, so by 22.30 and the remarks after its proof on page 137, $B^{\mathrm{stretch}} \equiv_{\mathrm{W}} B$ and $(\neg B)^{\mathrm{stretch}} \equiv_{\mathrm{W}} \neg B$. Let $C <_{\mathrm{W}} A \bullet B$. We must show that $C \leq_{\mathrm{W}} A \bullet D$ for some $D <_{\mathrm{W}} B$. Since $C <_{\mathrm{W}} \neg(A \bullet B) \equiv_{\mathrm{W}} A \bullet (\neg B)$ by 22.38, fix τ_0, τ_1 winning strategies for **II** in $G_{\mathrm{W}}(C, A \bullet B)$ and $G_{\mathrm{W}}(C, A \bullet (\neg B))$, respectively. Let

$$\begin{aligned} Z = \{x \in \mathbb{R} \mid\ & \exists^{\infty} j\, ((x * \tau_0)_{\mathbf{II}}(j) \text{ is entered in the } B\text{-row}) \ \& \\ & \exists^{\infty} j\, ((x * \tau_1)_{\mathbf{II}}(j) \text{ is entered in the } \neg B\text{-row})\}\end{aligned}$$

Exercise 22.40. $C \cap Z \leq_{\mathrm{W}} B^{\mathrm{stretch}}$ and $C \cap Z \leq_{\mathrm{W}} (\neg B)^{\mathrm{stretch}}$.

Since $(\neg B)^{\mathrm{stretch}} \equiv_{\mathrm{W}} \neg(B^{\mathrm{stretch}}) \equiv_{\mathrm{W}} \neg B$, it follows that $C \cap Z <_{\mathrm{W}} B$, so it is enough to show that $C \leq_{\mathrm{W}} A \bullet (C \cap Z)$. We will define a winning strategy τ for **II** in $G_{\mathrm{W}}(C, A \bullet (C \cap Z))$ such that any play of **II** according to τ is a product real, and if $y = (c * \tau)_{\mathbf{II}}$ then either

(1) y was entered infinitely often on the $C \cap Z$-row, and c is the real recorded there, i.e., $y = (s_0 + 1)^\frown\langle 0\rangle^\frown(s_1 + 1)^\frown\langle 0\rangle^\frown \cdots$ and $c = s_0{}^\frown s_2{}^\frown \cdots$; or else

(2) from some point on y was only entered on the A-row, i.e.,

$$y = (s_0 + 1)^\frown\langle 0\rangle^\frown \cdots ^\frown \langle 0\rangle^\frown(s_{2n} + 1)^\frown(a + 1)$$

and for some $i \in \{0, 1\}$, $\exists m \leq n\, \exists t_0, \ldots, t_{2m} \in {}^{<\omega}\omega \setminus \{\emptyset\}$ such that $(c * \tau_i)_{\mathbf{II}} = (t_0 + 1)^\frown\langle 0\rangle^\frown \cdots ^\frown \langle 0\rangle^\frown(t_{2m} + 1)^\frown(a + 1)$.

Here is the definition of τ:

Let **I** play the real c in the three games

$$\begin{aligned}\mathcal{G} &= G_{\mathrm{W}}(C, A \bullet (C \cap Z)) \\ \mathcal{G}_0 &= G_{\mathrm{W}}(C, A \bullet B) \\ \mathcal{G}_1 &= G_{\mathrm{W}}(C, A \bullet (\neg B))\,.\end{aligned}$$

As long as τ_0 and τ_1 pass or enter elements in the B-row and the $\neg B$-row, respectively, when presented with $c_0, c_1, \ldots$. then τ simply copies them on the $C \cap Z$-row. If this happens forever then $c \in Z$ so $c \in C \Leftrightarrow c \in C \cap Z$. So suppose n is least such that, say, τ_0 responds to c_n with entering the A-row. We temporarily forget about τ_1 and $\mathcal{G}_1$ and concentrate on τ_0 and $\mathcal{G}_0$. As long as τ_0 keeps on playing on the A-row of $\mathcal{G}_0$ then τ simply copies these values in the A-row of G:

$\mathcal{G}_0$	**I**		$c_0, \ldots, c_{n-1},$	$c_n, c_{n+1}, \ldots$
	II	B-row	$b_0, \ldots, b_{n'-1},$	
		A-row		$a^0_0, a^0_1, \ldots$

$\mathcal{G}$	**I**		$c_0, \ldots, c_{n-1},$	$c_n, c_{n+1}, \ldots$
	II	$C \cap Z$-row	$c_0, \ldots, c_{n-1},$	
		A-row		$a^0_0, a^1_1, \ldots$

(As usual $n' < n$ can happen since τ_0 may pass.)

Suppose τ_0 stays in the A-row forever. If $\langle b_0, \ldots, b_{n'-1} \rangle \neq \emptyset$ then

$$\begin{aligned} c \in C &\Leftrightarrow a^0 \in A && \text{(since } \tau_0 \text{ is winning)} \\ &\Leftrightarrow ((c \restriction n) + 1)^\frown \langle 0 \rangle^\frown (a^0 + 1) \in A \cdot (C \cap Z) . \end{aligned}$$

If $\langle b_0, \ldots, b_{n'-1} \rangle = \emptyset$ then $(c * \tau_0)_{\mathbf{II}} = \mathbf{II}$'s play in G_0 is not a product real, hence it does not belong to $A \cdot B$, hence $c \notin C$ and $c \notin C \cap Z$. Therefore in either case **II** wins G.

Suppose instead τ_0 comes back to the B-row at some round $m > n$. Let $s_0 = c \restriction n$ and $s_1 = \langle a^0_0, \ldots \rangle$ be the values entered by τ_0 in the A-row between rounds n and $m-1$. First τ goes back to the $C \cap Z$ row and there he plays c_n

$\mathcal{G}$	**I**		s_0	$c_n, c_{n+1}, \ldots, c_m,$	c_{m+1}
	II	$C \cap Z$-row	s_0		c_n
		A-row		s_1	

and now he turns his attention to τ_1.

If at round $m+1$ the strategy τ_1 is acting (i.e., playing or passing) in the A-row, then we set $s_2 = \langle c_n \rangle$ and have τ enter the A-row of $\mathcal{G}$ and copy τ_1's values from round $m+1$ on:

$\mathcal{G}_1$	**I**		$c_0, \ldots$	$\ldots, c_m,$	$c_{m+1},$	$c_{m+2}, \ldots$
	II	$\neg B$-row	$b'_0, \ldots$			
		A-row		$\ldots$	$a^1_0,$	$a^1_1, \ldots$

$\mathcal{G}$	**I**		$c_0, \ldots$	$\ldots, c_m,$	$c_{m+1},$	$c_{m+2}, \ldots$
	II	$C \cap Z$-row	s_0		s_2	
		A-row		s_1		$a^1_0, \ldots$

where $s_3 = \langle a^1_0, \ldots \rangle$ are the values played by τ_1 after round $m+1$. Thus we now turn the attention again to τ_0 and repeat the argument.

If instead τ_1 at round $m+1$ was already in the $\neg B$-row then we skip the paragraph above and focus the attention on τ_0 right away.

If both τ_0 and τ_1 are in the B-row and $\neg B$-row, then we are in the same situation as in the beginning of the game, so the arguments above can be repeated.

And so on: we keep dove-tailing τ_0 and τ_1 unless one of the two settles on its A-row.

It is immediate to check that τ is winning for **II** in $G_{\mathrm{W}}(C, A \cdot (C \cap Z))$. □

We are now ready to show that if A is non-self-dual then $\mathrm{cof}(\|B\|_{\mathrm{W}}) > \omega \Rightarrow \|A \cdot B\|_{\mathrm{W}} = \|A\|_{\mathrm{W}} \cdot \omega_1 \cdot \|B\|_{\mathrm{W}}$.

Proof. 22.36 By induction on $\|B\|_{\mathrm{W}}$.

If $\|B\|_{\mathrm{W}} = \omega_1$ then either $B \equiv_{\mathrm{W}} \mathbb{R}^\natural$ or $B \equiv_{\mathrm{W}} \emptyset^\flat$. Assume the former. Then keeping in mind that $\left\|A^{\flat\natural}\right\|_{\mathrm{W}} = \|A\|_{\mathrm{W}} \cdot \omega_1 \cdot \omega_1$ and using 22.35

$$\begin{aligned} \|A \cdot B\|_{\mathrm{W}} &= \left\|A \cdot \mathbb{R}^\natural\right\|_{\mathrm{W}} && \text{(by (i))} \\ &= \left\|(A \cdot \mathbb{R})^\natural\right\|_{\mathrm{W}} && \text{(by (v))} \\ &= \left\|A^{\flat\natural}\right\|_{\mathrm{W}} && \text{(by (ii))} \\ &= \|A\|_{\mathrm{W}} \cdot \omega_1 \cdot \|B\|_{\mathrm{W}} \ . \end{aligned}$$

Similarly if $B \equiv_{\mathrm{W}} \emptyset^\flat$.

So we may assume that $\|B\|_{\mathrm{W}} > \omega_1$. By Cantor's normal form theorem

$$\|B\|_{\mathrm{W}} = \omega_1^{\xi_0} \cdot \eta_0 + \cdots + \omega_1^{\xi_k} \cdot \eta_k$$

with $\xi_0 > \cdots > \xi_k$ and $0 < \eta_0, \ldots, \eta_k < \omega_1$.

Suppose first $k > 0$. Then pick C of least possible Lipschitz rank and such that $\|C\|_{\mathrm{W}} = \omega_1^{\xi_0} \cdot \eta_0$. We distinguish two cases:

C **is self-dual.** Then by 22.10 there is a D such that $C + D \equiv_{\mathrm{W}} B$ and $\|D\|_{\mathrm{W}} = \omega_1^{\xi_1} \cdot \eta_1 + \cdots + \omega_1^{\xi_k} \cdot \eta_k$ is of cofinality $> \omega$. By inductive hypothesis $\|A \cdot D\|_{\mathrm{W}} = \|A\|_{\mathrm{W}} \cdot \omega_1 \cdot \|D\|_{\mathrm{W}}$. Fix $\langle \nu_n \mid n < \omega \rangle$ increasing and converging to $\omega_1^{\xi_0} \cdot \eta_0$ such that $\mathrm{cof}(\nu_n) > \omega_1$, and let $\|C_n\|_{\mathrm{W}} = \nu_n$. Then by inductive hypothesis $\|A \cdot C_n\|_{\mathrm{W}} = \|A\|_{\mathrm{W}} \cdot \omega_1 \cdot \|C_n\|_{\mathrm{W}}$, and $\bigoplus_n C_n \equiv_{\mathrm{L}} C$. Thus by 22.35

$$\begin{aligned}
\|A \cdot C\|_{\mathrm{W}} &= \Big\| A \cdot \bigoplus_n C_n \Big\|_{\mathrm{W}} && \text{(by (i))} \\
&= \Big\| \bigoplus_n (A \cdot C_n + A) \Big\|_{\mathrm{W}} && \text{(by (iv))} \\
&= \sup\{\|A \cdot C_n + A\|_{\mathrm{W}} \mid n < \omega\}\,.
\end{aligned}$$

Since $A \cdot C_n$ is non-self-dual we cannot apply 22.11 and we only have

$$\begin{aligned}
\|A\|_{\mathrm{W}} \cdot \omega_1 \cdot \|C_n\|_{\mathrm{W}} &\le \|A \cdot C_n + A\|_{\mathrm{W}} \\
&\le \|A \cdot C_n\|_{\mathrm{W}} + \|A\|_{\mathrm{W}} \\
&= \|A\|_{\mathrm{W}} \cdot \omega_1 \cdot \|C_n\|_{\mathrm{W}} + \|A\|_{\mathrm{W}} \\
&< \|A\|_{\mathrm{W}} \cdot \omega_1 \cdot \|C_{n+1}\|_{\mathrm{W}}
\end{aligned}$$

hence

$$\begin{aligned}
\sup\{\|A \cdot C_n + A\|_{\mathrm{W}} \mid n < \omega\} &= \sup\{\|A\|_{\mathrm{W}} \cdot \omega_1 \cdot \|C_n\|_{\mathrm{W}} \mid n < \omega\} \\
&= \|A\|_{\mathrm{W}} \cdot \omega_1 \cdot \|C\|_{\mathrm{W}}\ ,
\end{aligned}$$

and therefore

$$\begin{aligned}
\|A \cdot B\|_{\mathrm{W}} &= \|A \cdot (C + D)\|_{\mathrm{W}} && \text{(by (i))} \\
&= \|A \cdot C\|_{\mathrm{W}} + \|A \cdot D\|_{\mathrm{W}} && \text{(by (vi))} \\
&= \|A\|_{\mathrm{W}} \cdot \omega_1 \cdot \|C\|_{\mathrm{W}} + \|A\|_{\mathrm{W}} \cdot \omega_1 \cdot \|D\|_{\mathrm{W}} \\
&= \|A\|_{\mathrm{W}} \cdot \omega_1 \cdot \|B\|_{\mathrm{W}}\ .
\end{aligned}$$

C **is non-self-dual.** As before choose D such that $(C \oplus \neg C) + D \equiv_{\mathrm{W}} B$ and $\|D\|_{\mathrm{W}} = \omega_1^{\xi_1} \cdot \eta_1 + \cdots + \omega_1^{\xi_k} \cdot \eta_k$. Then by 22.38 and the inductive hypothesis

$$\begin{aligned}\|A \cdot B\|_{\mathrm{W}} &= \|A \cdot (C \oplus \neg C)\|_{\mathrm{W}} + \|A \cdot D\|_{\mathrm{W}} \\ &= \|A \cdot C + A\|_{\mathrm{W}} + \|A\|_{\mathrm{W}} \cdot \omega_1 \cdot \|D\|_{\mathrm{W}} \ .\end{aligned}$$

As $\|A \cdot C\|_{\mathrm{W}} \leq \|A \cdot C + A\|_{\mathrm{W}} \leq \|A \cdot C\|_{\mathrm{W}} + \|A\|_{\mathrm{W}}$,

$$\begin{aligned}\|A \cdot C + A\|_{\mathrm{W}} + \|A\|_{\mathrm{W}} \cdot \omega_1 \cdot \|D\|_{\mathrm{W}} &= \|A \cdot C\|_{\mathrm{W}} + \|A\|_{\mathrm{W}} \cdot \omega_1 \cdot \|D\|_{\mathrm{W}} \\ &= \|A\|_{\mathrm{W}} \cdot \omega_1 \cdot (\|C\|_{\mathrm{W}} + \|D\|_{\mathrm{W}}) \\ &= \|A\|_{\mathrm{W}} \cdot \omega_1 \cdot \|B\|_{\mathrm{W}} \ ,\end{aligned}$$

which is what we had to prove.

So we may assume $k = 0$, that is

$$\|B\|_{\mathrm{W}} = \omega_1^{\xi} \cdot \eta \, .$$

Since this is an ordinal of uncountable cofinality, η must be a successor and ξ either a successor or $\mathrm{cof}(\xi) > \omega$. If $\eta = \gamma + 1$ with $\gamma > 0$, then $\|B\|_{\mathrm{W}} = \omega_1^{\xi} \cdot \gamma + \omega_1^{\xi}$, we can choose $\|C\|_{\mathrm{W}} = \omega_1^{\xi} \cdot \gamma$ and $\|C\|_{\mathrm{W}} = \omega_1^{\xi}$, and proceed as before. So we may assume $\eta = 1$, i.e.,

$$\|B\|_{\mathrm{W}} = \omega_1^{\xi} \, .$$

If ξ is a successor, $\xi = \delta + 1$, then choose C of least Lipschitz degree such that $B \equiv_{\mathrm{W}} C^{\natural}$ or $B \equiv_{\mathrm{W}} C^{\flat}$ and $\|C\|_{\mathrm{W}} = \omega_1^{\delta}$. If C is non-self-dual then we are done by 22.35 and the inductive hypothesis, so we may assume C is self-dual, i.e., $\mathrm{cof}(\delta) = \omega$. Let $\langle \nu_n \mid n < \omega \rangle$ be increasing and cofinal in ω_1^{δ} and such that $\mathrm{cof}(\nu_n) > \omega$. Pick C_n such that $\|C_n\|_{\mathrm{W}} = \nu_n$ so that $\bigoplus_n C_n \equiv_{\mathrm{L}} C$. Then again we are done by 22.35 and the inductive hypothesis.

Finally, suppose $\mathrm{cof}(\xi) > \omega$. Then by 22.39

$$\begin{aligned}\|A \cdot B\|_{\mathrm{W}} &= \sup\{\|A \cdot C\|_{\mathrm{W}} \mid C <_{\mathrm{W}} B\} \\ &= \sup\{\|A \cdot C\|_{\mathrm{W}} \mid C <_{\mathrm{W}} B \ \& \ \mathrm{cof}(\|C\|_{\mathrm{W}}) > \omega\} \\ &= \sup\{\|A\|_{\mathrm{W}} \cdot \omega_1 \cdot \|C\|_{\mathrm{W}} \mid C <_{\mathrm{W}} B \ \& \ \mathrm{cof}(\|C\|_{\mathrm{W}}) > \omega\} \\ &= \|A\|_{\mathrm{W}} \cdot \omega_1 \cdot \|B\|_{\mathrm{W}} \ .\end{aligned}$$

□

Corollary 22.41. *Assume* $\mathsf{BP} + \mathsf{AD}^{\mathrm{L}} + \mathsf{DC}(\mathbb{R})$.

(a) If $\|A\|_{\mathrm{W}} = \omega_1 \cdot \|A\|_{\mathrm{W}}$ *is of uncountable cofinality, then* $A^{\mathrm{stretch}} \equiv_{\mathrm{W}} A$.

(b) If $\|A\|_{\mathrm{W}} = \omega_1^{\omega} \cdot \alpha$ *and* $B <_{\mathrm{W}} A$ *then* $B^{\mathrm{stretch}} <_{\mathrm{W}} A$.

Proof. (a) Let D be clopen and non-trivial, i.e., $D \neq \emptyset, \mathbb{R}$. By 22.35

$$A^{\text{stretch}} \leq_{\mathrm{W}} \emptyset \cdot (\langle 0 \rangle^\frown A) \equiv_{\mathrm{W}} \emptyset \cdot A + \emptyset \leq_{\mathrm{W}} D \cdot A + \emptyset\,.$$

Then by 22.36, $\|D \cdot A + \emptyset\|_{\mathrm{W}} \leq \|A\|_{\mathrm{W}} + 1$ and since A^{stretch} is non-self-dual (22.27) then either $A^{\text{stretch}} \leq_{\mathrm{W}} A$ or else $A^{\text{stretch}} \leq_{\mathrm{W}} \neg A$. The latter is impossible, since $A \leq_{\mathrm{W}} A^{\text{stretch}}$, hence $A \equiv_{\mathrm{W}} A^{\text{stretch}}$.

(b) If α is limit, then $\|B\|_{\mathrm{W}} < \omega_1^\omega \cdot \gamma$ for some $\gamma < \alpha$. Let C be of least Lipschitz degree and such that $B <_{\mathrm{W}} C <_{\mathrm{W}} A$ and $\|C\|_{\mathrm{W}} = \omega_1^\omega \cdot \gamma$. If $\operatorname{cof}(\gamma) > \omega$ then by part (a) $B^{\text{stretch}} \leq_{\mathrm{W}} C^{\text{stretch}} \equiv_{\mathrm{W}} C <_{\mathrm{W}} A$, so we may assume $\operatorname{cof}(\gamma) = \omega$. Let $\langle \nu_n \mid n \in \omega \rangle$ be increasing and cofinal in $\omega_1^\omega \cdot \gamma$, and such that $\operatorname{cof}(\nu_n) > \omega$. Choose C_n such that $\|C_n\|_{\mathrm{W}} = \nu_n$. Then $\bigoplus_n C_n \equiv_{\mathrm{L}} C$ and, as before,

$$B^{\text{stretch}} \leq_{\mathrm{W}} C^{\text{stretch}} \leq_{\mathrm{W}} D \cdot C + \emptyset \equiv_{\mathrm{W}} \bigoplus_n (D \cdot C_n + D) + \emptyset$$

where $D \in \mathbf{\Delta}^0_1 \setminus \{\emptyset, \mathbb{R}\}$. Arguing as in the proof of 22.36

$$\left\| \bigoplus_n (D \cdot C_n + D) \right\|_{\mathrm{W}} = \sup\{\|D \cdot C_n\|_{\mathrm{W}} \mid n \in \omega\} = \sup\{\omega_1 \cdot \nu_n \mid n \in \omega\} < \omega_1^\omega \cdot \alpha$$

and therefore $B^{\text{stretch}} <_{\mathrm{W}} A$. □

22.G Wadge Determinacy and the Semi-Linear-Ordering Principle

We are now going to prove the equivalence between AD^{W} and AD^{L}.

Theorem 22.42. $\mathsf{BP} + \mathsf{DC}(\mathbb{R}) + \mathsf{AD}^{\mathrm{L}} \Rightarrow \mathsf{AD}^{\mathrm{W}}$.

Proof. We will show that

(a) $\neg A \equiv_{\mathrm{W}} B$ and A is non-self-dual, or

(b) $B <_{\mathrm{W}} \neg A$.

then **I** wins $G_{\mathrm{W}}(A, B)$. This implies that every game $G_{\mathrm{W}}(A, B)$ is determined: if $A \leq_{\mathrm{W}} B$ then **II** wins and if $A \nleq_{\mathrm{W}} B$ then by $\mathsf{SLO}^{\mathrm{W}}$ (which follows from AD^{L}) either (a) or (b) holds, hence **I** wins. The result will be proved by induction on $(\|A\|_{\mathrm{W}}, \|B\|_{\mathrm{W}})$ using Gödel's well-ordering of $\mathrm{Ord} \times \mathrm{Ord}$

$$\begin{aligned}(\alpha, \beta) \lhd (\gamma, \delta) \Leftrightarrow {} & \max(\alpha, \beta) < \max(\gamma, \delta) \vee \\ & \big(\max(\alpha, \beta) = \max(\gamma, \delta) \;\&\; (\alpha, \beta) <_{\mathrm{lex}} (\gamma, \delta)\big)\end{aligned}$$

• Suppose (a) holds.

We will show that **I** wins $G_W(A, B)$. Let's agree on the following notation: If q is a sequence of non-zero integers and p's let $q - 1$ be the sequence obtained from q by subtracting 1 whenever possible, i.e., for all $i < \mathrm{lh}(q)$

$$(q-1)(i) = \begin{cases} q(i) - 1 & \text{if } q(i) \in \omega\,, \\ \mathsf{p} & \text{if } q(i) = \mathsf{p}\,. \end{cases}$$

There are four cases:

(1) $[A]_W$ is a successor degree.

Then either $[\boldsymbol{T}(A)] \subseteq A$ or else $[\boldsymbol{T}(A)] \cap A = \emptyset$. For the sake of definitiveness suppose the former, so that $[\boldsymbol{T}(B)] \cap B = \emptyset$. Let **I** enumerate some fixed real $a \in [\boldsymbol{T}(A)]$ as long as **II** plays inside $\boldsymbol{T}(B)$ or passes. If **II** never plays outside $\boldsymbol{T}(B)$ then he loses so let's assume that at some round we have positions $p \in \boldsymbol{T}(A)$ for **I** and $q \in \partial\boldsymbol{T}(B)$ for **II**. Then $B_{\lfloor q \rfloor} <_W B \equiv_W \neg A$ and $A_{\lfloor p \rfloor} \equiv_W A$ so $B_{\lfloor q \rfloor} <_W A_{\lfloor p \rfloor}$. By inductive hypothesis **I** wins $G_W(A_{\lfloor p \rfloor}, B_{\lfloor q \rfloor})$.

(2) $[A]_W$ is a limit degree and $\|A\|_W = \alpha \cdot \omega_1$, with $\alpha < \|A\|_W$.

Pick a self-dual $C <_W A$ such that $\{[C^\natural]_W, [C^\flat]_W\} = \{[A]_W, [B]_W\}$, for the sake of definitiveness say $C^\natural \equiv_W A$ and $C^\flat \equiv_W B$. Since by **??**(c) there are Lipschitz reductions $C^\natural \leq_L A$ and $C^\flat \leq_L B$, any winning strategy for **I** in $G_W(C^\natural, C^\flat)$ induces a winning strategy for **I** in $G_W(A, B)$. Since $C <_W \neg C^\natural$ by 22.16, then $(\|C^\natural\|_W, \|C\|_W) \lhd (\|C^\natural\|_W, \|C^\flat\|_W)$ hence by inductive hypothesis there is a winning strategy σ for **I** in $G_W(C^\natural, \neg C)$. Then **I** wins $G_W(C^\natural, C^\flat)$ as follows:

At round 0 **I** plays $\sigma(\emptyset) + 1$.

If at any round n **II** plays 0 then **I** plays 0 at round $n + 1$.

If after having played a 0 **II** plays non-zero integers or passes then **I** applies σ to **II**'s inputs -1 as if the game had just restarted after the 0. In other words if **II**'s true play is $q_0{}^\frown\langle 0\rangle^\frown q_1{}^\frown\langle 0\rangle^\frown \ldots$ with $q_i \in {}^{<\omega}(\{\mathsf{p}\} \cup \omega \setminus \{0\})$ then

I	$\sigma(\emptyset)$	p_0		0 p_1		0	$\ldots$
II		q_0	0		q_1	0	

where

$$p_i = \begin{cases} \sigma * (q_i - 1) & \text{if } q_i \neq \emptyset\,, \\ \emptyset & \text{otherwise.} \end{cases}$$

If **II** plays 0 infinitely often then so does **I**, and **I** wins by definition of $C^\natural$ and $C^\flat$. Thus we may assume that the play of **I** is of the form $s^\frown\langle 0\rangle^\frown(x+1)$ and that the true play of **II** is of the form $t^\frown\langle 0\rangle^\frown z$, with $s \in {}^{<\omega}\omega$, $t \in {}^{<\omega}(\{\mathsf{p}\} \cup \omega)$, $\mathrm{lh}(s) = \mathrm{lh}(t)$, $x \in \mathbb{R}$, $z \in {}^{\omega}(\{\mathsf{p}\} \cup \omega \setminus \{0\})$, and $x+1 = \sigma * (z-1)$. If z is definitively $= \mathsf{p}$ then **II** loses so we may assume otherwise. Let $y+1$ be the real obtained from z by dropping all p's and, similarly, let $\tilde{t}$ be the sequence obtained from t by deleting p. Then

$$s^\frown\langle 0\rangle^\frown(x+1) \in C^\natural \Leftrightarrow x+1 \in C^\natural \Leftrightarrow y \notin C \Leftrightarrow \tilde{t}^\frown\langle 0\rangle^\frown(y+1) \in C^\flat .$$

Therefore **I** wins $G_{\mathrm{W}}(C^\natural, C^\flat)$.

(3) $[A]_{\mathrm{W}}$ is a limit degree and $\|A\|_{\mathrm{W}} = \alpha + \omega_1$ and (2) does not hold.

Choose A', B', C with C is non-self-dual, $A' \equiv_{\mathrm{L}} \neg B'$, $A \equiv_{\mathrm{L}} C + A'$, $B \equiv_{\mathrm{L}} C + B'$, and $\omega_1 = \|A'\|_{\mathrm{W}} = \|B'\|_{\mathrm{W}} < \|C\|_{\mathrm{W}} < \|A\|_{\mathrm{W}}$. By (2) and by inductive hypothesis **I** has winning strategies σ and τ for $G_{\mathrm{W}}(A', B')$ and $G_{\mathrm{W}}(C, \neg C)$ respectively. It is enough to show that **I** wins $G_{\mathrm{W}}(C + A', C + B')$.

> As long as **II** does not play 0 **I** can use σ against $q-1$, where q is the true position of player **II**. If at some round n **II** plays 0 then at round $n+1$ **I** plays 0 and then switches to τ.

It is immediate to check that this is a winning strategy for **I** in $G_{\mathrm{W}}(C + A', C + B')$.

(4) $[A]_{\mathrm{W}}$ is a limit degree and (2) and (3) fail.

Hence $\|A\|_{\mathrm{W}} = \omega_1^\omega \cdot \alpha$ with $\mathrm{cof}(\alpha) > \omega$. Then by 22.41 $B \equiv_{\mathrm{L}} B^{\mathrm{stretch}} \not\leq_{\mathrm{L}} A$ and we are done by 22.29.

- Suppose now (b) holds. Again we must examine various cases.

(5) $[A]_{\mathrm{W}}$ is self-dual.

Then **I** plays along $\boldsymbol{T}(A)$ until at some point he hits a terminal node $p \in \mathbf{tn}(\boldsymbol{T}(A))$. Then he can play i such that $B \leq_{\mathrm{W}} \neg A_{\lfloor p^\frown\langle i\rangle\rfloor}$. Let q be **II**'s position at this round. (Note that $q = \emptyset$ is possible, since he can pass.) In any case $B_{\lfloor q\rfloor} \leq_{\mathrm{W}} B$, so by inductive hypothesis **I** wins $G_{\mathrm{W}}(A_{\lfloor p^\frown\langle i\rangle\rfloor}, B_{\lfloor q\rfloor})$, and hence **I** wins $G_{\mathrm{W}}(A, B)$.

(6) $[A]_{\mathrm{W}}$ is non-self-dual and $[B]_{\mathrm{W}}$ is self-dual.

Then **I** plays in $\boldsymbol{T}(A)$ as long as **II** plays in $\boldsymbol{T}(B)$. Since $\boldsymbol{T}(B)$ is well-founded and **II** cannot pass forever there is a stage after which the positions are $p \in \boldsymbol{T}(A)$ and $q \in \partial\boldsymbol{T}(B)$. Since $B_{\lfloor q \rfloor} <_{\mathrm{W}} B$ then, by inductive hypothesis, there is a winning strategy σ in $G_{\mathrm{W}}(A_{\lfloor p \rfloor}, B_{\lfloor q \rfloor})$, hence **I** can follow σ from now on.

Therefore we may assume

$$A \text{ and } B \text{ are non-self-dual.}$$

(7) $[A]_{\mathrm{W}}$ is a successor degree.

Then either $A \equiv_{\mathrm{L}} C^{\triangledown}$ or else $A \equiv_{\mathrm{L}} C^{\circ}$ where C is self-dual and $\|C\|_{\mathrm{W}} + 1 = \|A\|_{\mathrm{W}}$, say $A \equiv_{\mathrm{L}} C^{\triangledown}$. Then it is enough to show that **I** wins $G_{\mathrm{W}}(C^{\triangledown}, B)$: **I** plays 1 (or any non-zero integer, for that matter) so that after the first round the two Players are facing the sets $C^{\triangledown}_{\lfloor 1 \rfloor} = C$ and $B_{\lfloor q \rfloor}$ where $\mathrm{lh}(q) \leq 1$. In any case, since $B \leq_{\mathrm{W}} \neg C$, then $B_{\lfloor q \rfloor} \leq_{\mathrm{W}} \neg C$ so by inductive hypothesis **I** wins.

(8) $[A]_{\mathrm{W}}$ is a limit degree.

Let C be non-self-dual and such that $\|C\|_{\mathrm{W}} = \|B\|_{\mathrm{W}} + 2$. By (7) **I** wins $G_{\mathrm{W}}(C, B)$ via some σ, and since $C \leq_{\mathrm{L}} A$ σ's moves can be copied, obtaining a winning strategy of **I** in $G_{\mathrm{W}}(A, B)$.

□

We now turn to the other implications, that is $\mathsf{AD}^{\mathrm{W}} \Rightarrow \mathsf{AD}^{\mathrm{L}}$ and $\mathsf{SLO}^{\mathrm{L}} \Rightarrow \mathsf{AD}^{\mathrm{L}}$. These are somewhat trickier than 22.42 since the theory in the previous sections was developed under the assumption of AD^{L}, and hence many of the basic constructions are not immediately available. For example, special care must be paid when dealing with (non)-self-dual sets, since the equivalence of the Lipschitz and Wadge version depends on **??** which used AD^{L} in an essential way. We will develop the theory as much as we can under the weakest assumption, namely $\mathsf{SLO}^{\mathrm{W}}$.

Since the proof of 22.3 used only that $\boldsymbol{T}(A)$ is well-founded iff $[A]_{\mathrm{W}}$ is self-dual, then $\mathsf{SLO}^{\mathrm{W}}$ implies that immediately above each self-dual $[A]_{\mathrm{W}}$ there is a non-self-dual pair $([A^{\circ}]_{\mathrm{W}}, [A^{\triangledown}]_{\mathrm{W}})$.

Lemma 22.43. *Assume* $\mathsf{BP} + \mathsf{DC}(\mathbb{R})$ *and either* $\mathsf{SLO}^{\mathrm{L}}$ *or* AD^{W}.

(a) $\|A\|_{\mathrm{W}} < \|B\|_{\mathrm{W}} \Rightarrow$ **II** *wins* $G_{\mathrm{L}}(A, B)$ *and* $\|A\|_{\mathrm{W}} > \|B\|_{\mathrm{W}} \Rightarrow$ **I** *wins* $G_{\mathrm{L}}(A, B)$.

(b) If $A \equiv_{\mathrm{W}} B$ *are non-self-dual then* $A \equiv_{\mathrm{L}} B$.

Proof. (a) Assume first $\mathsf{SLO}^{\mathrm{L}}$. If $\|A\|_{\mathrm{W}} < \|B\|_{\mathrm{W}}$ then $\neg B \not\leq_{\mathrm{L}} A$ so, by $\mathsf{SLO}^{\mathrm{L}}$, $A \leq_{\mathrm{L}} B$ hence **II** wins $G_{\mathrm{L}}(A, B)$. Vice versa, if $\|A\|_{\mathrm{W}} > \|B\|_{\mathrm{W}}$ then $\langle 0\rangle^\frown B \equiv_{\mathrm{W}} B <_{\mathrm{W}} \neg A$ so by 21.14 **I** wins $G_{\mathrm{L}}(A, B)$. The argument under AD^{W} is simpler, since a winning strategy for **I** in G_{W} is also winning in the Lipschitz game.

(b) Since $A \not\leq_{\mathrm{W}} \neg B$ and $B \not\leq_{\mathrm{W}} \neg A$, then $A \not\leq_{\mathrm{L}} \neg B$ and $B \not\leq_{\mathrm{L}} \neg A$. Hence $A \equiv_{\mathrm{L}} B$ by $\mathsf{SLO}^{\mathrm{L}}$. The simpler case under AD^{W} is left to the reader. □

We can now re-prove the Steel–Van Wesep theorem.

Theorem 22.44 (same as **??**)**.** *Assume* $\mathsf{BP} + \mathsf{DC}(\mathbb{R})$ *and either* $\mathsf{SLO}^{\mathrm{L}}$ *or* AD^{W}. *Then*

$$A \leq_{\mathrm{W}} \neg A \Rightarrow A \leq_{\mathrm{L}} \neg A\,.$$

Proof. By induction on $\|A\|_{\mathrm{W}} \cdot \|\boldsymbol{T}(A)\|$ let us show that **II** wins $G_{\mathrm{L}}(A, \neg A)$. As long as **I** plays inside $\boldsymbol{T}(A)$ then **II** copies **I**'s moves. When **I** reaches a position $p^\frown\langle n\rangle \in \partial\boldsymbol{T}(A)$ (and this is bound to happen since $\boldsymbol{T}(A)$ is well-founded) then $A_{\lfloor p^\frown\langle n\rangle\rfloor} <_{\mathrm{W}} A \equiv_{\mathrm{W}} A_{\lfloor p\rfloor}$ hence by 22.43(a) **I** wins $G_{\mathrm{L}}(A_{\lfloor p\rfloor}, A_{\lfloor p^\frown\langle n\rangle\rfloor})$ via some σ. Then **II** can use σ to answer **I**'s moves in $G_{\mathrm{L}}(A, \neg A)$. □

Since the proofs of 22.3 and 22.4 used only that $\boldsymbol{T}(A)$ is well-founded when $A \equiv_{\mathrm{W}} \neg A$, we can conclude, under $\mathsf{BP} + \mathsf{DC}(\mathbb{R})$ and either $\mathsf{SLO}^{\mathrm{L}}$ or AD^{W}, that for A non-self-dual

- $[A]_{\mathrm{W}}$ is a successor degree iff either $[\boldsymbol{T}(A)] \subseteq A$ or else $[\boldsymbol{T}(A)] \cap A = \emptyset$, but not both; and
- $[A]_{\mathrm{W}}$ is a limit degree iff $[\boldsymbol{T}(A)] \cap A \neq \emptyset$ and $[\boldsymbol{T}(A)] \cap \neg A \neq \emptyset$.

Finally:

Theorem 22.45. *Assume* $\mathsf{BP} + \mathsf{DC}(\mathbb{R})$ *and either* $\mathsf{SLO}^{\mathrm{L}}$ *or* AD^{W}. *Then* AD^{L} *holds.*

Proof. By induction on $(\|A\|_{\mathrm{W}}, \|B\|_{\mathrm{W}})$ we show that if $[A]_{\mathrm{W}} = [B]_{\mathrm{W}}$ is self-dual then

$$\|\boldsymbol{T}(A)\| \leq \|\boldsymbol{T}(B)\| \Rightarrow \textbf{ II} \text{ wins } G_{\mathrm{L}}(A, B)$$
$$\|\boldsymbol{T}(A)\| > \|\boldsymbol{T}(B)\| \Rightarrow \textbf{ I} \text{ wins } G_{\mathrm{L}}(A, B)\,.$$

Since 22.43(a) takes care of the cases when $\|A\|_{\mathrm{W}} \neq \|B\|_{\mathrm{W}}$, and 22.43(b) takes care of the cases when $A \equiv_{\mathrm{W}} \neg B$ are non-self-dual, this will prove the theorem.

Let $\alpha = \|\boldsymbol{T}(A)\|$ and $\beta = \|\boldsymbol{T}(B)\|$. If $\alpha \leq \beta$ then **II** wins $G_{\mathrm{L}}(A, B)$ as follows

As long as **I** plays so that his position p is inside $\boldsymbol{T}(A)$ then **II** plays so that his position q is inside $\boldsymbol{T}(B)$ and $\|p\|_{\boldsymbol{T}(A)} \leq \|q\|_{\boldsymbol{T}(B)}$. Suppose at some round **I** plays outside $\boldsymbol{T}(A)$ (and this is going to happen sooner or later, since $\boldsymbol{T}(A)$ is well-founded), say **I** plays i so that $p^\frown\langle i\rangle \in \partial\boldsymbol{T}(A)$, $A_{\lfloor p^\frown\langle i\rangle\rfloor} <_{\mathrm{W}} A_{\lfloor p\rfloor} \equiv_{\mathrm{W}} A$. Let q be **II**'s position before starting this round, i.e., $\mathrm{lh}(p) = \mathrm{lh}(q)$ and $q \in \boldsymbol{T}(B)$. If $\|q\|_{\boldsymbol{T}(B)} > \|p\|_{\boldsymbol{T}(A)} = 0$, then **II** can play a j such that $q^\frown\langle j\rangle \in \boldsymbol{T}(B)$. Then $A_{\lfloor p^\frown\langle i\rangle\rfloor} <_{\mathrm{W}} B_{\lfloor q^\frown\langle j\rangle\rfloor}$, so $(\|A_{\lfloor p^\frown\langle i\rangle\rfloor}\|_{\mathrm{W}}, \|B_{\lfloor q^\frown\langle j\rangle\rfloor}\|_{\mathrm{W}}) \lhd (\|A\|_{\mathrm{W}}, \|B\|_{\mathrm{W}})$, hence we are done by inductive hypothesis.

If instead $\|q\|_{\boldsymbol{T}(B)} = \|p\|_{\boldsymbol{T}(A)} = 0$ we distinguish two cases.

- $[A]_{\mathrm{W}}$ is a limit degree. Then $[A]_{\mathrm{W}}$ is a join degree and since $A_{\lfloor p\rfloor} = \bigoplus_n A_{\lfloor p^\frown\langle n\rangle\rfloor}$ and $B_{\lfloor q\rfloor} = \bigoplus_n B_{\lfloor q^\frown\langle n\rangle\rfloor}$, then **II** can choose j such that $A_{\lfloor p^\frown\langle i\rangle\rfloor} \leq_{\mathrm{W}} B_{\lfloor q^\frown\langle j\rangle\rfloor}$. Since $(\|A_{\lfloor p^\frown\langle i\rangle\rfloor}\|_{\mathrm{W}}, \|B_{\lfloor q^\frown\langle j\rangle\rfloor}\|_{\mathrm{W}}) \lhd (\|A\|_{\mathrm{W}}, \|B\|_{\mathrm{W}})$ again we are done by inductive hypothesis.
- $[A]_{\mathrm{W}}$ is a successor degree. Then $A \equiv_{\mathrm{W}} C \oplus \neg C$ for some non-self-dual C. If $A_{\lfloor p^\frown\langle i\rangle\rfloor} \leq_{\mathrm{W}} C$, since $B_{\lfloor q\rfloor} \equiv_{\mathrm{W}} C \oplus \neg C$ then **II** can choose j such that $B_{\lfloor q^\frown\langle j\rangle\rfloor} \equiv_{\mathrm{W}} C$, and similarly if $A_{\lfloor p^\frown\langle i\rangle\rfloor} \leq_{\mathrm{W}} \neg C$. Again we are done by inductive hypothesis.

Suppose now $\alpha > \beta$. Then **I** plays a_0 such that $\langle a_0\rangle \in \boldsymbol{T}(A)$ and $\|\langle a_0\rangle\| \geq \beta$. Then both players can switch roles and since $\boldsymbol{T}(\neg(A_{\lfloor a_0\rfloor})) = \boldsymbol{T}(A)_{\lfloor a_0\rfloor}$ has rank $\geq \beta$ we can apply the argument above and conclude that **II** wins $G_{\mathrm{L}}(B, \neg(A_{\lfloor a_0\rfloor}))$, hence **I** wins $G_{\mathrm{L}}(A, B)$. □

Additional Exercises

Exercise 22.46. Assume $\mathsf{BP} + \mathsf{AD}^{\mathrm{L}}$ and let $A \subseteq \mathbb{R}$ be non-self-dual.

(i) Suppose T is an ill-founded tree on ω such that ${}^{<\omega}\omega \setminus T$ is dense, i.e.,

$$\forall s \in {}^{<\omega}\omega\, \exists t \supseteq s\, (t \notin T)\ .$$

Show that $\boldsymbol{S}(A;T) \equiv_{\mathrm{W}} A^{\triangledown}$ and $\boldsymbol{S}(A;T) \equiv_{\mathrm{W}} A^{\triangledown} \cup [\boldsymbol{T}(A)] \equiv_{\mathrm{W}} A^{\circ}$.

(ii) Show that

$$\begin{aligned} B \in [A^{\triangledown}]_{\mathrm{W}} &\ \Rightarrow\ [\boldsymbol{T}(B)] \cap B = \emptyset\,, \\ B \in [A^{\circ}]_{\mathrm{W}} &\ \Rightarrow\ [\boldsymbol{T}(B)] \subseteq B\,. \end{aligned}$$

Exercise 22.47. For Γ and Λ pointclasses let

$$\bigvee_{\Lambda}(\Gamma) = \{\bigcup_n (U_n \cap A_n) \mid \langle U_n \mid n < \omega\rangle \in {}^{\omega}\Lambda \ \& \ \langle A_n \mid n < \omega\rangle \in {}^{\omega}\Gamma \\ \& \ \forall n < m < \omega \, (U_n \cap U_m = \emptyset)\}$$

and let

$$\bigwedge_{\Lambda}(\Gamma) = (\bigvee_{\Lambda}(\breve{\Gamma}))\breve{}\,.$$

Similarly

$$\bigsqcup_{\Lambda}(\Gamma) = \{\bigcup_n (U_n \cap A_n) \mid \langle U_n \mid n < \omega\rangle \in {}^{\omega}\Lambda \ \& \ \langle A_n \mid n < \omega\rangle \in {}^{\omega}\Gamma \ \& \\ \forall n < m < \omega \, (U_n \cap U_m = \emptyset) \ \& \ \bigcup_n U_n = \mathbb{R}\}\,.$$

Thus, for example, the elements of $\bigvee_{\boldsymbol{\Sigma}^0_1}(\Gamma)$ are countable disjoint unions of sets in Γ each of which is contained in an open set so that these open sets are pairwise disjoint. If the U_n's form a partition, i.e., $\bigcup_n U_n = \mathbb{R}$, then they must be clopen so we simply obtain the pointclass $\bigsqcup_{\boldsymbol{\Delta}^0_1}(\Gamma) = \bigsqcup_{\boldsymbol{\Sigma}^0_1}(\Gamma)$ described in **??**.

(i) Show that if $\boldsymbol{\Gamma}$ and $\boldsymbol{\Lambda}$ are boldface pointclasses then $\bigvee_{\boldsymbol{\Lambda}}(\boldsymbol{\Gamma})$ is a boldface pointclass.

(ii) Show that for any $A, B \subset \mathbb{R}$

$$B \leq_{\mathrm{W}} A^{\triangledown} \Leftrightarrow B \in \bigvee_{\boldsymbol{\Sigma}^0_1}(\{C \subset \mathbb{R} \mid C \leq_{\mathrm{W}} A\})$$

and

$$B \leq_{\mathrm{W}} A^{\circ} \Leftrightarrow B \in \bigwedge_{\boldsymbol{\Sigma}^0_1}(\{C \subset \mathbb{R} \mid C \leq_{\mathrm{W}} A\})$$

Fix $A \subseteq \mathbb{R}$.

(iii) Let $F_n = \boldsymbol{s}_n{}^\frown\langle 0\rangle^\frown(\mathbb{R}+1)$, and let $A_n = \{u^\frown x \mid \mathrm{lh}(u) = \mathrm{lh}(\boldsymbol{s}_n) + 1 \ \& \ \boldsymbol{s}_n{}^\frown\langle 0\rangle^\frown(x \dot{-} 1) \in A\}$, where $n \dot{-} m = n - m$ if $n \geq m$ and $= 0$ otherwise, and $\langle \boldsymbol{s}_n \mid n < \omega\rangle$ was defined on page **??**.
Show that $A_n \leq_{\mathrm{W}} A$ and that $A_n \cap F_n = A^{\natural} \cap F_n$, for any n.

(iv) Show that $\{B \mid B \leq_{\mathrm{W}} A^{\natural}\} \subseteq \bigvee_{\boldsymbol{\Pi}^0_1}(\{B \mid B \leq_{\mathrm{W}} A\}) \subseteq \bigvee_{\boldsymbol{\Sigma}^0_2}(\{B \mid B \leq_{\mathrm{W}} A\})$.

(v)

(vi) Conclude that for any $A \subseteq \mathbb{R}$

$$\{B \mid B \leq_{\mathrm{W}} A^\natural\} = \bigvee_{\mathbf{\Pi}^0_1}(\{B \mid B \leq_{\mathrm{W}} A\}) = \bigvee_{\mathbf{\Sigma}^0_2}(\{B \mid B \leq_{\mathrm{W}} A\})$$

and

$$\{B \mid B \leq_{\mathrm{W}} A^\flat\} = \bigwedge_{\mathbf{\Pi}^0_1}(\{B \mid B \leq_{\mathrm{W}} A\}) .$$

Exercise 22.48. Show that

(i) $1 \leq \alpha \leq \beta < \omega_1 \Rightarrow [A]_{\mathrm{W}} \cdot \alpha \leq [A]_{\mathrm{W}} \cdot \beta$, and if $A \neq \mathbb{R}, \emptyset$ then $\alpha < \beta \Rightarrow [A]_{\mathrm{W}} \cdot \alpha < [A]_{\mathrm{W}} \cdot \beta$.

(ii) $[A]_{\mathrm{W}} \cdot (\sup_n \alpha_n) = \sup_n [A]_{\mathrm{W}} \cdot \alpha_n$.

(iii) $[A]_{\mathrm{W}} \cdot (\alpha + \beta) = [A]_{\mathrm{W}} \cdot \alpha + [A]_{\mathrm{W}} \cdot \beta$.

(iv) $[A]_{\mathrm{W}} \cdot (\alpha \cdot \beta) = ([A]_{\mathrm{W}} \cdot \alpha) \cdot \beta$.

Exercise 22.49. We now describe a few Wadge-style games (like G_{bt}) that model the relations $A \leq_{\mathrm{W}} B + C$, $A \leq_{\mathrm{W}} B \cdot n$, $A \leq_{\mathrm{W}} B^\natural$, etc. In all of them **I** and **II** alternatively play natural numbers $x(0), x(1), \ldots$ and $y(0), y(1), \ldots$, and, just like in G_{W}, **II** has the option of passing at any round, with the proviso that if he does not play infinitely often then he loses by default. Player **II** will also have the opportunity at any round to erase *his* moves and start his play all over again.

- In $G_+(A, B, C)$ Player **II** can take back his moves at any round, but can do so at most once. If **II** never took back his moves then he wins iff

  $$x \in A \Leftrightarrow y \in B .$$

  Suppose instead **II** erased his moves at round n, that is first he plays $y(0), \ldots, y(n-1)$ and then he plays $y'(0), y'(1), \ldots$: then **II** wins iff

  $$x \in A \Leftrightarrow y' \in C .$$

- In $G_{\cdot n}(A, B)$ Player **II** can erase his board at most n times, so that a typical play looks like $\langle s_0, \ldots, s_k, y\rangle$ where $k < n$ and $s_0, \ldots, s_k \in {}^{<\omega}\omega$ are his initial plays before he changed his mind. **II** wins iff

  $$x \in A \Leftrightarrow y \in B .$$

- $G_{\cdot\omega}(A, B)$ is similar, but **II** must play some $n \in \omega$ together with his first (real, non-passing) move, and then he can take-back his moves at most n times. Therefore a play of **II** looks like $\langle n, s_0, \ldots, s_k, y\rangle$ where $k < n$, $s_0, \ldots, s_k \in {}^{<\omega}\omega$ and $y \in \mathbb{R}$. The winning condition is as before.
- In $G_\natural(A, B)$ **II** can take-back his moves as many times as he wants—even infinitely often. If **II** erases the board n many times, i.e., his play is $\langle s_0, \ldots, s_{n-1}, y\rangle$ then **II** wins iff
$$x \in A \Leftrightarrow y \in B\,.$$
If instead he erases his back infinitely often, then **II** wins iff
$$x \notin A\,.$$
- $G_\flat(A, B)$ is similar, but when **II** takes back his moves infinitely often then he wins iff
$$x \in A\,.$$

Show that

$$\begin{aligned}
\textbf{II} \text{ wins } G_+(A, B, C) &\Leftrightarrow A \leq_{\mathrm{W}} B + C\,,\\
\textbf{II} \text{ wins } G_{\cdot n}(A, B) &\Leftrightarrow A \leq_{\mathrm{W}} B \cdot n\,,\\
\textbf{II} \text{ wins } G_{\cdot\omega}(A, B) &\Leftrightarrow A \leq_{\mathrm{W}} B \cdot \omega\,,\\
\textbf{II} \text{ wins } G_\natural(A, B) &\Leftrightarrow A \leq_{\mathrm{W}} B^\natural\,,\\
\textbf{II} \text{ wins } G_\flat(A, B) &\Leftrightarrow A \leq_{\mathrm{W}} B^\flat\,.
\end{aligned}$$

The following exercise asks to develop a division algorithm for degrees below $A^\natural$.

Exercise 22.50. Assume AD^{L}. Show that if A is self-dual and $B <_{\mathrm{W}} A^\natural$ then exactly one of the following holds:

1. $[B]_{\mathrm{W}} < [A]_{\mathrm{W}}$,
2. $[B]_{\mathrm{W}} = [A]_{\mathrm{W}} \cdot \alpha$, for some $1 \leq \alpha < \omega_1$,
3. $[B]_{\mathrm{W}} = [A]_{\mathrm{W}} \cdot \alpha + [C]_{\mathrm{W}}$, for some $1 \leq \alpha < \omega_1$ and some $C <_{\mathrm{W}} A$.

Notes and References

Everything in §§A–D and exercises 22.46–22.50 are from [Wad83]. The material in §§E–G is from the union of [Ste77] and [VW77]. The material in §22.G is, to the best of our knowledge, due to the author.

23 Borel-Wadge degrees

23.A The Structure of $\mathbf{\Delta}^0_2$

Recall from 21.18 the definition of $\mathrm{Res}_\alpha(A)$, and from **??** that $\mathbf{\Delta}^0_2 = \bigcup_{\alpha<\omega_1} \mathrm{Diff}(\alpha; \mathbf{\Pi}^0_1)$. Also for simplicity of notation let

$$\mathbf{D}_\alpha = \begin{cases} \mathrm{Diff}(\alpha; \mathbf{\Pi}^0_1) & \text{if } \alpha \text{ is even,} \\ \breve{\mathrm{Diff}}(\alpha; \mathbf{\Pi}^0_1) & \text{if } \alpha \text{ is odd.} \end{cases}$$

Note that $\mathbf{D}_\alpha$ is a non-self-dual pointclass.

Lemma 23.1. *($\mathsf{AC}_\omega(\mathbb{R})$) Suppose $\alpha < \omega_1$ and $A, B \subseteq \mathbb{R}$. Then*

$$A \in \mathbf{D}_\alpha \;\&\; \mathrm{Res}_\alpha(\neg B) \neq \emptyset \Rightarrow A \leq_{\mathrm{L}} B\,.$$

Proof. By induction on α. If $\alpha = 0$ then $\mathbf{D}_\alpha = \mathrm{Diff}(0; \mathbf{\Pi}^0_1) = \{\emptyset\}$ so $A = \emptyset$. Since $\neg B = \mathrm{Res}_0(\neg B) \neq \emptyset$ then $B \neq \mathbb{R}$ so $A \leq_{\mathrm{L}} B$.

Suppose $\alpha > 0$ and assume the result for all $\beta < \alpha$. We assume first that α is even and hence $\mathbf{D}_\alpha = \mathrm{Diff}(\alpha; \mathbf{\Pi}^0_1)$. Then

$$A = \bigcup_{2\nu<\alpha} (C_{2\nu} \setminus C_{2\nu+1})$$

with $C_0 \supseteq C_1 \supseteq \cdots$ closed sets, and let

$$U = \bigcup_{2\nu<\alpha} \neg C_{2\nu+1} \in \mathbf{\Sigma}^0_1\,.$$

Pick $b \in \mathrm{Res}_\alpha(\neg B) \neq \emptyset$. Then **II** wins $G_{\mathrm{L}}(A, B)$ via the following strategy:

> **II** plays $b(0), b(1), \ldots$ while **I** plays $a(0), a(1), \ldots$ until, if ever, $\boldsymbol{N}_{a\restriction n} \subseteq U$. If this never happens then **II**'s play will be b. If instead n_0 is least such that $\boldsymbol{N}_{a\restriction n} \subseteq U$ let ν be least such that $\boldsymbol{N}_s \subseteq \neg C_{2\nu+1}$, were $s = a \restriction n$. Then
>
> $$A_{\lfloor s\rfloor} = \bigcup_{2\gamma<2\nu+1} C'_{2\gamma} \setminus C'_{2\gamma+1} \in \mathrm{Diff}(2\nu+1; \mathbf{\Pi}^0_1)$$
>
> where $C'_\gamma = (C_\gamma)_{\lfloor s\rfloor}$. Thus $\neg A_{\lfloor s\rfloor} \in \mathbf{D}_{\alpha'}$ where $\alpha' = 2\nu+1 < \alpha$. Let $t = b \restriction n_0$. By definition of $\mathrm{Res}_\alpha(\neg B)$ there is a $b' \in \mathrm{Res}_{\alpha'}(B)$ such that $t \subset b'$, and since the "Res" operation is defined in terms of Boolean and topological operations by **??**
>
> $$\mathrm{Res}_{\alpha'}(B_{\lfloor t\rfloor}) = (\mathrm{Res}_{\alpha'}(B))_{\lfloor t\rfloor} \neq \emptyset\,.$$
>
> Therefore by inductive hypothesis **II** has a winning strategy in $G_{\mathrm{L}}(\neg A_{\lfloor s\rfloor}, \neg B_{\lfloor t\rfloor})$. Let **II** follow this strategy from now on in the game $G_{\mathrm{L}}(A, B)$.

It is easy to check that this is indeed a winning strategy for **II**.

The case when α is odd is similar: since

$$\neg A = \bigcup_{2\nu<\alpha} C_{2\nu} \setminus C_{2\nu+1} \, ,$$

then set

$$U = \bigcup_{2\nu<\alpha} \neg C_{2\nu}$$

and proceed as before. □

This implies

Proposition 23.2. *($\mathsf{AC}_\omega(\mathbb{R})$) For all $\alpha < \omega_1$ and $B \subseteq \mathbb{R}$ the following conditions are equivalent:*

(1) $\mathrm{Res}_\alpha(\neg B) \neq \emptyset$*,*

(2) $\neg B \notin \mathbf{D}_\alpha$*,*

(3) B *is* $\mathbf{D}_\alpha$*-hard.*

Proof. (2) $\Rightarrow$ (1) follows from 21.18(iii), and (1) $\Rightarrow$ (3) follows from the lemma, so it is enough to prove (3) $\Rightarrow$ (2).

Since $\mathrm{Diff}(\alpha; \mathbf{\Pi}^0_1)$ is non-self-dual (**??**(iv)) pick $A \in \mathbf{D}_\alpha \setminus \breve{\mathbf{D}}_\alpha$. Then $A \leq_{\mathrm{W}} B$ by hypothesis hence $\neg B \in \mathbf{D}_\alpha$ would imply $\neg A \in \mathbf{D}_\alpha$: a contradiction. □

Corollary 23.3. $\{A \subseteq \mathbb{R} \mid \exists \alpha < \omega_1 \ \mathrm{Res}_\alpha(A) = \emptyset\} = \mathbf{\Delta}^0_2$.

Proof. From 21.18(iii) we already know $\subseteq$. If $\forall \alpha < \omega_1 (\mathrm{Res}_\alpha(A) \neq \emptyset)$ then $A \notin \bigcup_{\alpha<\omega_1} \mathbf{D}_\alpha$, and by **??**(iii) and **??** $\bigcup_{\alpha<\omega_1} \mathbf{D}_\alpha = \mathbf{\Delta}^0_2$, so $A \notin \mathbf{\Delta}^0_2$. □

Proposition 23.4. *For any $1 \leq \alpha < \omega_1$ and any $B \subseteq \mathbb{R}$ the following are equivalent:*

(1) $\forall \beta < \alpha \, (\mathrm{Res}_\beta(B) \neq \emptyset \ \& \ \mathrm{Res}_\beta(\neg B) \neq \emptyset)$*,*

(2) $B \notin \bigcup_{\beta<\alpha} \mathbf{D}_\beta$ *and* $\neg B \notin \bigcup_{\beta<\alpha} \mathbf{D}_\beta$*,*

(3) B *is* $\Delta(\mathbf{D}_\alpha)$*-hard.*

Proof. (1) $\Leftrightarrow$ (2) follows by 23.2.

Since $\mathbf{D}_\beta \neq \mathbf{D}_\alpha$ for all $\beta < \alpha$ by **??** then (3) $\Rightarrow$ (2).

Suppose $B, \neg B \notin \bigcup_{\beta<\alpha} \mathbf{D}_\beta$. Let $A \in \mathbf{D}_\beta \cup \breve{\mathbf{D}}_\beta$ for some $\beta < \alpha$. By 23.2

$$\emptyset = \mathrm{Res}_\alpha(A) = A \cap \bigcap_{\beta<\alpha} \mathrm{Cl}(\mathrm{Res}_\alpha(\neg A))$$

and

$$\emptyset = \mathrm{Res}_\alpha(\neg A) = A \cap \bigcap_{\beta<\alpha} \mathrm{Cl}(\mathrm{Res}_\alpha(A))$$

Fix $a \in \mathbb{R}$: since either $a \in A$ or $a \in \neg A$, there is a $\beta < \alpha$ such that

$$a \notin \mathrm{Cl}(\mathrm{Res}_\beta(\neg A)) \qquad \text{or} \qquad a \notin \mathrm{Cl}(\mathrm{Res}_\beta(A))$$

hence there is n such that

$$\boldsymbol{N}_{a\restriction n} \cap \mathrm{Res}_\beta(\neg A) = \emptyset \qquad \text{or} \qquad \boldsymbol{N}_{a\restriction n} \cap \mathrm{Res}_\beta(A) = \emptyset\,.$$

Then **II** wins $G_{\mathrm{W}}(A, B)$ as follows:

> **II** passes until **I** reaches a position $a \restriction n$ such that either $\boldsymbol{N}_{a\restriction n} \cap \mathrm{Res}_\beta(\neg A) = \emptyset$ or $\boldsymbol{N}_{a\restriction n} \cap \mathrm{Res}_\beta(A) = \emptyset$ for some $\beta < \alpha$. This means that either $\mathrm{Res}_\beta(\neg A_{\lfloor a\restriction n\rfloor}) = \emptyset$ or $\mathrm{Res}_\beta(A_{\lfloor a\restriction n\rfloor}) = \emptyset$, which by 23.2 means that
>
> $$A_{\lfloor a\restriction n\rfloor} \in \mathbf{D}_\beta \cup \breve{\mathbf{D}}_\beta\,.$$
>
> Since $B, \neg B \notin \bigcup_{\gamma<\alpha} \mathbf{D}_\gamma$ then by 23.2 B is $\mathbf{D}_\beta \cup \breve{\mathbf{D}}_\beta$-hard hence **II** has a winning strategy in $G_{\mathrm{W}}(A_{\lfloor a\restriction n\rfloor}, B)$. Let **II** use this strategy in $G_{\mathrm{W}}(A, B)$ from now on.

It is easy to verify that this is a winning strategy for **II**, hence $A \leq_{\mathrm{W}} B$. Since $A \in \Delta(\mathbf{D}_\alpha)$ was arbitrary this establishes (3). □

Corollary 23.5. *($\mathsf{AC}_\omega(\mathbb{R})$) For any $B \subseteq \mathbb{R}$, either $B \in \boldsymbol{\Delta}^0_2$ or $\forall A \in \boldsymbol{\Delta}^0_2\,(A \leq_{\mathrm{W}} B)$.*

Proof. If $\mathrm{Res}_\alpha(B) = \emptyset$ for some $\alpha < \omega_1$ then $B \in \boldsymbol{\Delta}^0_2$, so by 23.2 we may assume that B is $\mathbf{D}_\alpha$-hard for any $\alpha < \omega_1$, which is the same as saying that $\forall A \in \boldsymbol{\Delta}^0_2\,(A \leq_{\mathrm{W}} B)$. □

Exercise 23.6. ($\mathsf{AC}_\omega(\mathbb{R})$) Show that the only Wadge degrees below $\boldsymbol{\Delta}^0_2$ are

$$\mathrm{Diff}(\alpha; \boldsymbol{\Pi}^0_1)\setminus\Delta(\mathrm{Diff}(\alpha; \boldsymbol{\Pi}^0_1)), \qquad \breve{\mathrm{Diff}}(\alpha; \boldsymbol{\Pi}^0_1)\setminus\Delta(\mathrm{Diff}(\alpha; \boldsymbol{\Pi}^0_1)), \qquad \Delta(\mathrm{Diff}(\alpha; \boldsymbol{\Pi}^0_1))\,.$$

Exercise 23.7. ($\mathsf{AC}_\omega(\mathbb{R})$) Show that

$$\Delta(\mathrm{Diff}(2; \boldsymbol{\Pi}^0_1)) = \{A \subseteq \mathbb{R} \mid \forall a \in \mathbb{R}\, \exists k\, (A \cap \boldsymbol{N}_{a\restriction k} \text{ is open or closed})\}$$

?? shows that if $A \in \mathbf{\Pi}^1_1 \setminus \mathbf{\Delta}^0_2$ then we cannot prove in ZFC that A is $\mathbf{\Pi}^0_2$-hard. We shall see shortly that the pair $\mathbf{\Pi}^0_2, \mathbf{\Pi}^1_1$ is—in a sense—the least potential counterexample to AD^{L}, in that if $A \in \mathbf{\Pi}^0_2$ and $B \in \mathbf{\Sigma}^1_1$ then either

$$A \leq_{\mathrm{W}} B \ \vee \ \neg B \leq_{\mathrm{W}} A$$

or

$$A \leq_{\mathrm{W}} \neg B \ \vee \ B \leq_{\mathrm{W}} A\,.$$

Before we prove this we turn to a generalization (due again to W. Wadge) of the reducibility $\leq_{\mathrm{W}}$ to pairs of sets:

Definition 23.8. For $A_0, A_1, B_0, B_1 \subseteq \mathbb{R}$

$$(A_0, A_1) \leq_{\mathrm{sep}} (B_0, B_1)$$

just in case $f^{-1}[B_0] \supseteq A_0$ and $f^{-1}[B_1] \supseteq A_1$, for some continuous $f : \mathbb{R} \to \mathbb{R}$.

The intuition behind $\leq_{\mathrm{sep}}$ is that A_0 and A_1 are no harder to *separate* (i.e., to tell-apart) than B_0 and B_1 are. The reason why we do not require $f^{-1}[B_i] = A_i$ $(i = 0, 1)$ is that even if $A_0 \cap A_1 = B_0 \cap B_1 = \emptyset$ and A_0 and A_1 may be very complex but lying far apart (and hence easy to separate) while B_0 and B_1 are simpler but tightly entwined (and hence hard to separate). We shall see that AD implies an AD^{L} principle for pairs of sets

$$(A_0, A_1) \leq_{\mathrm{sep}} (B_0, B_1)\,.$$

Exercise 23.9. For $A_0, A_1, B_0, B_1 \subseteq \mathbb{R}$ show that

(i) $A_0 \leq_{\mathrm{W}} B_0 \Leftrightarrow (A, \neg A) \leq_{\mathrm{sep}} (B, \neg B)$.

(ii) $(A_0, A_1) \leq_{\mathrm{sep}} (B_0, B_1) \Leftrightarrow (A_1, A_0) \leq_{\mathrm{sep}} (B_1, B_0)$.

(iii) $(A_0, A_1) \leq_{\mathrm{sep}} (A_0, \neg A_0)$ and $(A_0, A_1) \leq_{\mathrm{sep}} (\neg A_1, A_1)$.

(iv) $A_0 \subseteq B_0$ & $A_1 \subseteq B_1 \Rightarrow (A_0, A_1) \leq_{\mathrm{sep}} (B_0, B_1)$.

(v) $A_0 = A_1 = \emptyset \Rightarrow (A_0, A_1) \leq_{\mathrm{sep}} (B_0, B_1)$.

(vi) $B_0 = B_1 = \emptyset \Rightarrow (A_0, A_1) \leq_{\mathrm{sep}} (B_0, B_1)$.

(vii) $\leq_{\mathrm{sep}}$ is reflexive and transitive.

Thus every pair of non-disjoint sets is $\leq_{\mathrm{sep}}$-maximal and $\equiv_{\mathrm{sep}}$-equivalent to one another. (Here $\equiv_{\mathrm{sep}}$ is the equivalence relation induced by $\leq_{\mathrm{sep}}$.)

Exercise 23.10. For $A_0, A_1, B_0, B_1 \subseteq \mathbb{R}$ and $\mathbf{\Gamma}$ a boldface pointclass, if $(A_0, A_1) \leq_{\mathrm{sep}} (B_0, B_1)$ and B_0, B_1 can be separated by a set in $\mathbf{\Gamma}$ (i.e., $\exists C \in \mathbf{\Gamma}\,(B_0 \subseteq C \ \&\ B_1 \cap C = \emptyset)$) then A_0, A_1 can be separated by a set in $\mathbf{\Gamma}$.

Just like in the case of $\leq_W$ the preorder $\leq_{sep}$ can be characterized in terms of games.

Definition 23.11. Let $G_{sep}(A_0, A_1, B_0, B_1)$ be the game where **I** plays a, **II** plays b with the understanding that he can "pass" infinitely often as long as he plays infinitely often, and **II** wins iff

$$(a \in A_0 \ \& \ b \in B_0) \ \vee \ (a \in A_1 \ \& \ b \in B_1) \ \vee \ (a \notin A_0 \cup A_1) \ .$$

If $A_0 \cap A_1 = B_0 \cap B_1 = \emptyset$ then the winning conditions of G_{sep} can be expressed via the following table

	$b \in B_0$	$b \in B_1$	$b \notin B_0 \cup B_1$
$a \in A_0$	**II**	**I**	**I**
$a \in A_1$	**I**	**II**	**I**
$a \notin A_0 \cup A_1$	**II**	**II**	**II**

Exercise 23.12. Show that in the game $G_{sep}(A_0, A_1, B_0, B_1)$ Player **II** wins iff $(A_0, A_1) \leq_{sep} (B_0, B_1)$, and Player **I** wins iff $(B_1, B_0) \leq_{sep} (A_0, A_1)$.

Therefore AD implies that

$$(A_0, A_1) \leq_{sep} (B_0, B_1) \ \vee \ (B_1, B_0) \leq_{sep} (A_0, A_1) \, .$$

As a simple application let us show the following lemma.

Lemma 23.13. *($\mathsf{AC}_\omega(\mathbb{R})$) For $A, B \in \mathbf{\Sigma}^0_2$ disjoint sets*

$$(A, B) \leq_{sep} (E^*, O^*)$$

where

$$E^* = \{x \in \mathbb{R} \mid \forall^\infty n \ x(n) \text{ is even}\}$$
$$O^* = \{x \in \mathbb{R} \mid \forall^\infty n \ x(n) \text{ is odd}\} \, .$$

Proof. Let $A = \bigcup_n [T_{2n}]$ and $B = \bigcup_n [T_{2n+1}]$, with $T_n \subseteq T_{n+2}$ pruned trees. The **II** wins $G_{sep}(A, B, E^*, O^*)$ by playing at round n the least m such that $a \restriction n + 1 \in T_m$. □

Definition 23.14. A set $A \subseteq \mathbb{R}$ is **guessable** iff there are disjoint subsets Y (for yes) and N (for no) of ${}^{<\omega}\omega$ such that for all $a \in \mathbb{R}$

$$a \in A \ \Leftrightarrow \ \exists n_0 \, \forall m \geq n_0 \ a \restriction m \in Y$$
$$a \notin A \ \Leftrightarrow \ \exists n_0 \, \forall m \geq n_0 \ a \restriction m \in N$$

The pair (Y, N) is called a pair of guessing sets for A.

Definition 23.15 (Hausdorff)**.** For $A \subseteq \mathbb{R}$ let $\mathrm{Rm}(A)$ the **remainder of** A be the set

$$\mathrm{Rm}(A) = \left(A \cap \bigcap_{\alpha<\omega_1} \mathrm{Cl}\,(\mathrm{Res}(\neg A)) \right) \cup \left(\neg A \cap \bigcap_{\alpha<\omega_1} \mathrm{Cl}\,(\mathrm{Res}(A)) \right)$$

Lemma 23.16. *$A \cap \mathrm{Rm}(A)$ and $\neg A \cap \mathrm{Rm}(A)$ are dense in $\mathrm{Rm}(A)$.*

Proposition 23.17. *If A is guessable and $\mathrm{Rm}(A) \neq \emptyset$ then **I** wins $G_{\mathrm{W}}(B, \neg A)$.*

Proof. Let (Y, N) be guessing sets for A and let $z \in \mathrm{Rm}(B) \cap B$. **I** wins by playing $z(0), z(1), \ldots$ until **II**'s sequence is guessed to be in $\neg A$. If we reach such a place then since $z \restriction k$ has an extension in $\mathrm{Rm}(B) \cap B$ (i.e., z) there is an extension in $\mathrm{Rm}(B) \cap \neg B$, thus let z' be such an extension and let **I** play z' until **II** guesses A. And so on. □

Proposition 23.18. *A is guessable $\Leftrightarrow A \in \mathbf{\Delta}^0_2$.*

Proof. Suppose A is guessable and let Y, N be guessing sets for A. Notice that N, Y are guessing sets for $\neg A$. Then

$$(A, \neg A) \leq_{\mathrm{sep}} (E^*, O^*)$$

where E^* and O^* be as in 23.13, since **II** wins $G_{\mathrm{sep}}(\neg A, A, E^*, O^*)$ by playing at stage n

$$\begin{array}{ll} 0 & \text{if } a \restriction n+1 \in Y \\ 1 & \text{if } a \restriction n+1 \in N \end{array}$$

where $a \restriction n+1$ is **I**'s play at stage n. Since $E^*, O^* \in \mathbf{\Sigma}^0_2$ and $\neg A = f^{-1}[E^*]$ and $A = f^{-1}[O^*]$ then $A \in \mathbf{\Delta}^0_2$.

Conversely suppose $A \in \mathbf{\Delta}^0_2$. Since by 23.13 (E^*, O^*) is complete for pairs of disjoint $\mathbf{\Sigma}^0_2$ sets, then $(A, \neg A) \leq_{\mathrm{sep}} (E^*, O^*)$. Let τ be **II**'s winning strategy in $G_{\mathrm{sep}}(A, \neg A, E^*, O^*)$. Then

$$\begin{aligned} Y &= \{s \mid \tau(s) \text{ ends with an even number}\} \\ N &= \{s \mid \tau(s) \text{ ends with an odd number}\} \end{aligned}$$

are guessing sets for A. □

Proposition 23.19. *For any $A \subseteq \mathbb{R}$*

$$A \in \mathbf{\Delta}^0_2 \Leftrightarrow \mathrm{Rm}(A) = \emptyset\,.$$

Proof. If $A \in \mathbf{\Delta}^0_2$ and $\mathrm{Rm}(A) \neq \emptyset$ then **I** wins $G_{\mathrm{L}}(A, A)$: a contradiction.
Suppose $\mathrm{Rm}(A) = 0$. Let

$$Y = \{s \mid \exists \nu < \omega_1 \,(s \text{ has an extension in } \mathrm{Res}_\nu(A), \text{ but none in } \mathrm{Res}_\nu(\neg A))\}$$
$$N = \{s \mid \exists \nu < \omega_1 \,(s \text{ has an extension in } \mathrm{Res}_\nu(\neg A), \text{ but none in } \mathrm{Res}_\nu(A))\}$$

□

Proposition 23.20 (Van Wesep). *Assume* AD^{L} *and* $\mathsf{DC}(\mathbb{R})$*. Suppose A is self-dual and C is non-self-dual and* $0 < \|C\|_{\mathrm{W}} < \omega_1$*. Then*

$$C \in \mathrm{Diff}(n; \mathbf{\Pi}^0_1) \setminus \breve{\mathrm{Diff}}(n; \mathbf{\Pi}^0_1) \Rightarrow A^\triangledown + C \equiv_{\mathrm{W}} A + C$$

and

$$C \in \breve{\mathrm{Diff}}(n; \mathbf{\Pi}^0_1) \setminus \mathrm{Diff}(n; \mathbf{\Pi}^0_1) \Rightarrow A^\triangledown + C \equiv_{\mathrm{W}} A + C.$$

Proof. By 22.8(iii) it is enough to prove the result when $C \in \mathrm{Diff}(n; \mathbf{\Pi}^0_1) \setminus \breve{\mathrm{Diff}}(n; \mathbf{\Pi}^0_1)$. Let $C = C_0 \setminus (C_1 \setminus (\cdots (C_{n-1} \setminus C_n) \cdots)$, with $C_0 \supseteq C_1 \supseteq \cdots \supseteq C_n = \emptyset$ closed and let

$$\begin{aligned} D_0 &= \neg C \\ D_{2j+1} &= \mathrm{Cl}(D_{2j}) \cap C \\ D_{2j+2} &= \mathrm{Cl}(D_{2j+1}) \setminus C\,. \end{aligned}$$

since $C \notin \breve{\mathrm{Diff}}(n; \mathbf{\Pi}^0_1)$ then $D_n \neq \emptyset$. **II** wins $G_{\mathrm{W}}((A^\triangledown + \emptyset) + C, A + C)$ as follows:

> Pick $x_n \in D_n$ and let **II** play $x_n + 1$ until, if ever, **I** plays outside $C_{n-1} + 1$, i.e., reaches a position $p \notin T_{C_{n-1}} + 1$, where $T_{C_{n-1}}$ is the tree associated to C_{n-1}, i.e., $[T_{C_{n-1}}] = C_{n-1}$. If this happens at some round k then **II** chooses $x_{n-1} \in D_{n-1}$ such that $x_{n-1} \supset x_n \restriction k$ and now follows $x_{n-1} + 1$ until, if ever, **I** plays out of $C_{n-2} + 1$. And so on. If at some stage **I** plays 0 then **II** starts playing (if he is not already doing so) some $x_0 + 1$, with $x_0 \in D_0 = \neg C$: **II** follows $x_0 + 1$ as long as **I** stays in $\boldsymbol{T}(A^\triangledown + \emptyset)$. If at some round **I** reaches a position $p \notin \boldsymbol{T}(A^\triangledown + \emptyset)$ then $(A^\triangledown)_{\lfloor p \rfloor} \leq_{\mathrm{W}} A$ so **II** plays 0 and follows a winning strategy for the game $G_{\mathrm{W}}((A^\triangledown)_{\lfloor p \rfloor}, A)$.

□

23.B Generalized Boolean Operations

For every natural boldface pointclasses $\mathbf{\Gamma}$ there is a "natural construction principle" that builds elements of $\mathbf{\Gamma}$ from countable sequences of open sets: for example the map $\langle U_n \mid n < \omega\rangle \mapsto \bigcap_n U_n$ constructs all $\mathbf{\Pi}^0_2$ sets, the map $\langle U_n \mid n < \omega\rangle \mapsto \bigcup_i \bigcap_j U_{\langle i,j\rangle}$ constructs all $\mathbf{\Sigma}^0_3$ sets, while Suslin's operation $\boldsymbol{\mathcal{A}}$ (see **??**) yields (modulo a bijection between ${}^{<\omega}\omega$ and ω) a map ${}^{\omega}\mathbf{\Sigma}^0_1 \twoheadrightarrow \mathbf{\Sigma}^1_1$. We need a general definition of "natural construction principle."

Definition 23.21. (i) Let $\mathcal{F} : {}^{I}\mathscr{P}(\mathbb{R}) \to \mathscr{P}(\mathbb{R})$ and $T \subseteq \mathscr{P}(I)$. Then T is a **truth-table** for $\mathcal{F}$ iff

$$\forall A \in {}^{I}\mathscr{P}(\mathbb{R})\, \forall x \in \mathbb{R}\, (x \in \mathcal{F}(A) \Leftrightarrow \{i \in I \mid x \in A_i\} \in T)\ .$$

(ii) $\mathcal{F} : {}^{I}\mathscr{P}(\mathbb{R}) \to \mathscr{P}(\mathbb{R})$ is a **generalized Boolean I-ary operation** iff it admits a truth-table $T \subseteq \mathscr{P}(I)$.

(iii) Let Γ, Λ be pointclasses. Then Γ is Λ ω-ary Boolean iff $\exists \mathcal{F}\, (\mathcal{F} : {}^{\omega}\mathscr{P}(\mathbb{R}) \to \mathscr{P}(\mathbb{R}))$ is a generalized ω-ary Boolean operation and $\Gamma = \mathcal{F}``{}^{\omega}\Lambda$. If $\Lambda = \mathbf{\Sigma}^0_1$ we will simply say that Γ is open-Boolean.

Therefore $\mathcal{F}$ is a generalized I-ary Boolean operation just in case the answer to "$x \in \mathcal{F}(A)$" depends only on the answers to "$x \in A_i$" for $i \in I$. In other words: $\mathcal{F} : {}^{I}\mathscr{P}(\mathbb{R}) \to \mathscr{P}(\mathbb{R})$ is a generalized Boolean I-ary operation iff $\forall A, B \in {}^{I}\mathscr{P}(\mathbb{R})$, $\forall x, y \in \mathbb{R}$

$$\forall i \in I\, (x \in A(i) \Leftrightarrow y \in B(i)) \Rightarrow (x \in \mathcal{F}(A) \Leftrightarrow y \in \mathcal{F}(B))\, .$$

Exercise 23.22. Show that $\bigcap_n A_n$, $\bigcup_n A_n$, $\bigcup_n \bigcap_m A_{\langle n,m\rangle}$ are examples of generalized ω-ary Boolean operations. Show that operation $\boldsymbol{\mathcal{A}}$ is generalized ${}^{<\omega}\omega$-ary Boolean.

If $f : I \to J$ is a bijection, then we can copy any generalized I-ary Boolean operation $\mathcal{F} : {}^{I}\mathscr{P}(\mathbb{R}) \to \mathscr{P}(\mathbb{R})$ to a generalized J-ary Boolean operation $\mathcal{G} : {}^{J}\mathscr{P}(\mathbb{R}) \to \mathscr{P}(\mathbb{R})$ by letting

$$\mathcal{G}(\langle A_j \mid j \in J\rangle) = \mathcal{F}(\langle A_{f(i)} \mid i \in I\rangle)\, .$$

Therefore, with a minor blurring of vision, any generalized ${}^{<\omega}\omega$-ary Boolean operation (like Suslin's operation $\boldsymbol{\mathcal{A}}$) will be considered to be ω-ary.

Exercise 23.23. (i) Show that if Γ is open ω-ary Boolean, then so is $\breve{\Gamma}$.

(ii) Show that if Γ is open ω-ary Boolean, then Γ is a non-self-dual boldface pointclass.

The main point of this section is to prove the converse of this namely:

Theorem 23.24. *Assume* $\mathsf{AD} + \mathsf{DC}(\mathbb{R})$*. Every non-self-dual boldface pointclass* $\boldsymbol{\Gamma}$ *is open* ω*-ary Boolean.*

Let us prove first a particular case of the theorem.

Lemma 23.25. *(*$\mathsf{DC}(\mathbb{R})$*) Suppose* $\boldsymbol{\Gamma}$ *is non-self-dual of rank* $< \omega_1^\omega$*, that is* $\forall A \in \boldsymbol{\Gamma}\, (\|A\|_{\mathrm{W}} < \omega_1^\omega)$*. Then* $\boldsymbol{\Gamma}$ *is open* ω*-ary Boolean.*

Proof. Later. □

We also need the following lemma due to Lon Radin: it says that the theorem is true for many $\boldsymbol{\Gamma}$'s.

Lemma 23.26 (Radin). *(*AD^{L}*) If* $[A]_{\mathrm{W}} = [A]_{\mathrm{W}}^{\mathrm{stretch}}$ *then* $\{B \subseteq \mathbb{R} \mid B \leq_{\mathrm{W}} A\}$ *is open* ω*-ary Boolean.*

Proof. Let $\mathcal{F} : {}^{<\omega\omega}\mathscr{P}(\mathbb{R}) \to \mathscr{P}(\mathbb{R})$ be defined by

$$\mathcal{F}(\langle U_s \mid s \in {}^{<\omega}\omega\rangle) = \{x \in \mathbb{R} \mid \forall n \in \omega\, \exists! s \in {}^{n}\omega\, (x \in U_s)\ \&\ \exists z \in A\, \forall n \in \omega\, (x \in U_{z\restriction n})\}\,.$$

It is easy to check that $\mathcal{F}$ is ${}^{<\omega}\omega$-ary (and hence ω-ary) generalized Boolean. Suppose $\forall s \in {}^{<\omega}\omega\, \left(U_s \in \boldsymbol{\Sigma}^0_1\right)$ and let $B = \mathcal{F}(\langle U_s \mid s \in {}^{<\omega}\omega\rangle)$. We will show that $B \leq_{\mathrm{L}} A^{\mathrm{stretch}}$ by exhibiting a winning strategy for **II** in $G_{\mathrm{L}}(B, A^{\mathrm{stretch}})$:

> Let **I** play $x(0), x(1), \ldots$. **II** plays 0 until, if ever, a round m_0 is reached when $\boldsymbol{N}_{x\restriction m_0} \subseteq U_\emptyset$. If this happens then **II** still keeps playing 0 until, if ever, a round $m_1 > m_0$ is reached when $\boldsymbol{N}_{x\restriction m_1} \subseteq U_{s_1}$ for some unique $s_1 \in {}^{1}\omega$. Then **II** plays $s_1(0) + 1$. This takes care of the sequences of length 1. Now **II** plays 0 until, if ever, a round $m_2 > m_1$ is reached when $\boldsymbol{N}_{x\restriction m_2} \subseteq U_{s_2}$ for some unique $s_2 \in {}^{2}\omega$. Then **II** plays $s_2(1) + 1$. And so on.

If $x \in B$ then $x \in \bigcap_n U_{z\restriction n}$ for some $z \in A$, and since the U's are open, $\forall n\, \exists m_n\, (\boldsymbol{N}_{x\restriction m_n} \subseteq U_{z\restriction n})$, so that y, the real constructed by the strategy, is a stretch of z. Vice versa suppose $x \notin B$. If $|\{s \in {}^{n}\omega \mid x \in U_s\}| \neq 1$ for some $n \in \omega$, then **II** plays a sequence eventually equal to 0 hence not in A^{stretch}. If $\forall n\, \exists! s \in {}^{n}\omega\, (x \in U_s)$ then $\neg\exists z \in A\, (x \in \bigcap_n U_{z\restriction n})$, hence $y \notin A^{\mathrm{stretch}}$, where y is the real played by **II**. Therefore $\mathcal{F}``\,{}^{\omega}\boldsymbol{\Sigma}^0_1 \subseteq \{B \subseteq \mathbb{R} \mid B \leq_{\mathrm{W}} A^{\mathrm{stretch}} \equiv_{\mathrm{W}} A\}$.

On the other hand suppose $B = f^{-1}[A]$ for some continuous f. For $s \in {}^{<\omega}\omega$ let $U_s = f^{-1}[\boldsymbol{N}_s]$. Then

$$\forall x \in {}^{\omega}\omega\, \forall n \in \omega\, \exists! s \in {}^{n}\omega\, (x \in U_s)$$

and thus clearly $B = \mathcal{F}(\langle U_s \mid s \in {}^{<\omega}\omega\rangle)$. Therefore $\{B \mid B \leq_{\mathrm{W}} A\} = \mathcal{F}``\,{}^{\omega}\boldsymbol{\Sigma}^0_1$. □

Proof. 23.25 Let $B \in \mathbf{\Gamma} \setminus \breve{\mathbf{\Gamma}}$ and let α be largest such that $\omega_1^\omega \cdot \alpha \leq \|B\|_{\mathrm{W}}$. If equality holds then $\mathrm{cof}(\alpha) > \omega$ so by 22.41(a) and 23.26 we are done. Hence we may assume $\omega_1^\omega \cdot \alpha < \|B\|_{\mathrm{W}}$. Choose A such that

$$\|A\|_{\mathrm{W}} = \begin{cases} \omega_1^\omega \cdot \alpha + 1 & \text{if } \mathrm{cof}(\alpha) > \omega, \\ \omega_1^\omega \cdot \alpha & \text{otherwise.} \end{cases}$$

Then A is self-dual and by 22.10 there is C such that $A + C \equiv_{\mathrm{W}} B$ and $\|C\|_{\mathrm{W}} < \omega_1^\omega$.

Suppose first $\|C\|_{\mathrm{W}} \geq \omega$. (In fact, since C is non-self-dual, $\|C\|_{\mathrm{W}} > \omega$.) Then we have the following:

Claim 23.26.1. $\mathbf{\Gamma} = \{(F \cap U) \cup (G \setminus U) \mid F \leq_{\mathrm{W}} A + \emptyset \;\&\; G \leq_{\mathrm{W}} C \;\&\; U \in \mathbf{\Sigma}_1^0\}$.

Proof. Let $F \leq_{\mathrm{W}} A + \emptyset$, $G \leq_{\mathrm{W}} C$, and U be open. Fix τ_0 and τ_1 winning strategies for **II** in $G_{\mathrm{W}}(G, C)$ and $G_{\mathrm{W}}(F, A + \emptyset)$, respectively. Then **II** wins $G_{\mathrm{W}}((F \cap U) \cup (G \setminus U), (A + \emptyset) + C)$ as follows:

> As long as **I** has not reached a position p such that $\boldsymbol{N}_p \subseteq U$ (in other words: as long as **I** has not entered U), then **II** follows $\tau_0 + 1$, that is for any position q of **I** he plays $\tau_0(q) + 1$. If **I** ever enters U at some round n then **II** plays 0 and then uses τ_1 as if the game just started now:

I	$x_0, \ldots, x_{n-1}$	x_n	x_{n+1}	x_{n+2}	$\ldots$
II	$\tau_0(\langle x_0, \ldots, x_{n-1} \rangle) + 1$	0	$\tau_1(\langle x_0 \rangle)$	$\tau_1(\langle x_0, x_1 \rangle)$	$\ldots$

Since $\|C\|_{\mathrm{W}} \geq \omega$ then $(A + \emptyset) + C \equiv_{\mathrm{W}} A + C \equiv_{\mathrm{W}} B$ and therefore $(F \cap U) \cup (G \setminus U) \in \mathbf{\Gamma}$.

For the other inclusion it is enough to show that

$$A + C \in \{(F \cap U) \cup (G \setminus U) \mid F \leq_{\mathrm{W}} A + \emptyset \;\&\; G \leq_{\mathrm{W}} C \;\&\; U \in \mathbf{\Sigma}_1^0\},$$

since the collection of sets on the right is obviously a boldface pointclass. Let $F = A + \emptyset$, let $G = \{x \mid x \dot{-} 1 \in C\}$, and let $U = \{x \mid \exists n\; x(n) = 0\}$. Then $F = F \cap U$ and

$$A + C = A + \emptyset \cup C^+ = (F \cap U) \cup (G \setminus U),$$

which is what we had to prove. □

Since

$$(A+\emptyset)^{\text{stretch}} \equiv_{\text{W}} (A^{\text{stretch}})^{\text{stretch}} \equiv_{\text{W}} A^{\text{stretch}} \equiv_{\text{W}} A+\emptyset$$

by 22.27(ii), 22.28(a), and 22.41(b), then by Radin's lemma $\{F \mid F \leq_{\text{W}} A+\emptyset\}$ is open ω-ary Boolean, and let $\mathcal{F}$ witness this. Using 23.25 then $\{G \mid G \leq_{\text{W}} C\}$ is also open ω-ary Boolean via some $\mathcal{G}$. Then $\boldsymbol{\Gamma}$ is open ω-ary Boolean via $\mathcal{H}$, where

$$\mathcal{H}(\langle U_n \mid n \in \omega\rangle) = (\mathcal{F}(\langle U_{2n} \mid n \geq 1\rangle) \cap U_0) \cup (\mathcal{G}(\langle U_{2n+1} \mid n \geq 0\rangle) \setminus U_0)\,.$$

Suppose now $\|C\|_{\text{W}} = n < \omega$. Then by §2323.A either $C \in \text{Diff}(n;\boldsymbol{\Pi}^0_1) \setminus \breve{\text{Diff}}(n;\boldsymbol{\Pi}^0_1)$, or else $C \in \breve{\text{Diff}}(n;\boldsymbol{\Pi}^0_1) \setminus \text{Diff}(n;\boldsymbol{\Pi}^0_1)$. By 23.23(i) we may assume the former. If $n = 0$ then $C = \emptyset$ and by 22.28(a)

$$B \equiv_{\text{W}} A+\emptyset \equiv_{\text{W}} A^{\text{stretch}}$$

hence we are done by Radin's lemma, so we may assume $n \geq 1$. Then by 23.20 $(A+\emptyset)+C \equiv_{\text{W}} A+C$ hence the Claim holds also in this case and we are done by the same argument. □

Additional Exercises

Notes and References

23.23(ii) is due to Addison.

24 Boldface Pointclasses

Recall that a non-empty $\boldsymbol{\Gamma} \subseteq \mathscr{P}(\mathbb{R})$ is a boldface pointclass if it is downward closed under $\leq_{\text{W}}$. Under AD any non-self-dual boldface pointclass is of the form $\{B \subseteq \mathbb{R} \mid B \leq_{\text{W}} A\}$ for some non-self-dual $[A]_{\text{W}}$, and under $\mathsf{AD} + \mathsf{DC}(\mathbb{R})$ any self-dual boldface pointclass is of the form $\mathscr{P}^{(\alpha)}(\mathbb{R})$, for some $0 < \alpha < \Theta$.

Exercise 24.1 (Wadge's Lemma for boldface pointclasses)**.** (AD) Show that if $\boldsymbol{\Gamma}$, $\boldsymbol{\Lambda}$ are boldface pointclasses then $\boldsymbol{\Gamma} \subseteq \boldsymbol{\Lambda}$ or $\breve{\boldsymbol{\Lambda}} \subseteq \boldsymbol{\Gamma}$.

We now investigate some structural properties of pointclasses.

24.A Separation

Definition 24.2. A pointclass Γ has the separation property, $\mathsf{Sep}(\Gamma)$, iff

$$\forall A, B \in \Gamma\, (A \cap B = \emptyset \Rightarrow \exists C \in \Delta_\Gamma\, (A \subseteq C \;\&\; C \cap B = \emptyset))\;.$$

By **??** $\mathsf{Sep}(\boldsymbol{\Sigma}^1_1)$ and by **??** $\neg\mathsf{Sep}(\boldsymbol{\Pi}^1_1)$. We will show

Theorem 24.3 (Steel–Van Wesep). *(*AD*) Let* $\boldsymbol{\Gamma}$ *be a non-self-dual boldface pointclass. Then exactly one of* $\mathsf{Sep}(\boldsymbol{\Gamma})$ *and* $\mathsf{Sep}(\breve{\boldsymbol{\Gamma}})$ *holds.*

The theorem is a corollary of

Theorem 24.4 (Steel). *(*AD*) Let* $\boldsymbol{\Gamma}$ *be non-self-dual. Then* $\mathsf{Sep}(\boldsymbol{\Gamma})$ *or* $\mathsf{Sep}(\breve{\boldsymbol{\Gamma}})$*.*

and

Theorem 24.5 (Van Wesep). *(*AD*) Let* $\boldsymbol{\Gamma}$ *be non-self-dual. Then* $\neg\mathsf{Sep}(\boldsymbol{\Gamma})$ *or* $\neg\mathsf{Sep}(\breve{\boldsymbol{\Gamma}})$*.*

Given $A_0, A_1, B_0, B_1 \subseteq \mathbb{R}$, the separation game $G^*_{\text{sep}}(A_0, A_1, B_0, B_1)$ is the game where **I** plays x, **II** plays y, and **I** wins iff

$$(y \in A_0 \setminus A_1 \Rightarrow x \in B_0) \;\&\; (y \in A_1 \setminus A_0 \Rightarrow x \in B_1)\,.$$

The next exercise explains the reason for the name G^*_{sep}.

Exercise 24.6. Let $A_0, A_1, B_0, B_1 \subseteq \mathbb{R}$.

(i) Show that if **I** wins $G^*_{\text{sep}}(A_0, A_1, B_0, B_1)$ then there is a continuous $f : \mathbb{R} \to \mathbb{R}$ such that

$$f[A_0 \setminus A_1] \subseteq B_0 \quad \text{and} \quad f[A_1 \setminus A_0] \subseteq B_1\,.$$

In particular if $A_0 \cap A_1 = \emptyset$ then $f[A_i] \subseteq B_i$, for $i = 0, 1$.

(ii) Show that if **II** wins $G^*_{\text{sep}}(A_0, A_1, B_0, B_1)$ then there is a continuous $f : \mathbb{R} \to A_0 \triangle A_1 \subseteq \mathbb{R}$. Moreover if $B_0 \cap B_1 = \emptyset$ and $A_0, A_1 \in \boldsymbol{\Gamma}$ or $A_0, A_1 \in \breve{\boldsymbol{\Gamma}}$, where $\boldsymbol{\Gamma}$ is a non-self-dual boldface pointclass, then B_0, B_1 are $\Delta_{\boldsymbol{\Gamma}}$-separable, that is

$$\exists C \in \Delta_{\boldsymbol{\Gamma}}(B_0 \subseteq C \;\&\; B_1 \cap C = \emptyset)\,.$$

Proof. 24.4 Suppose, towards a contradiction, that $A_0, A_1 \in \boldsymbol{\Gamma}$ and $C_0, C_1 \in \breve{\boldsymbol{\Gamma}}$ are $\Delta_{\boldsymbol{\Gamma}}$-inseparable, i.e.,

$$\begin{aligned} &A_0 \cap A_1 = \emptyset \;\&\; \neg\exists D \in \Delta_{\boldsymbol{\Gamma}}(A_0 \subseteq D \;\&\; A_1 \cap D = \emptyset)\,,\\ &C_0 \cap C_1 = \emptyset \;\&\; \neg\exists D \in \Delta_{\boldsymbol{\Gamma}}(C_0 \subseteq D \;\&\; C_1 \cap D = \emptyset)\,. \end{aligned}$$

By 24.6 there is a continuous $f : \mathbb{R} \to \mathbb{R}$ such that $f[A_i] \subseteq C_i$, $i \in 2$. Let $B_i = f^{-1}[C_i]$. Therefore

- $A_0 \subseteq B_0$, $A_1 \subseteq B_1$, and $B_0 \cap B_1 = \emptyset$,
- $A_0, A_1 \in \mathbf{\Gamma}$,
- $B_0, B_1 \in \breve{\mathbf{\Gamma}}$.

Consider the games

$$\begin{aligned} \mathcal{G}_0 &= G^*_{\text{sep}}(A_0, A_1, A_1, A_0)\,, \\ \mathcal{G}_1 &= G^*_{\text{sep}}(A_0, \neg B_0, A_0, A_1)\,, \\ \mathcal{G}_2 &= G^*_{\text{sep}}(A_1, \neg B_1, A_1, A_0)\,. \end{aligned}$$

By 24.6 again, let σ_i be $\mathbf{I}$'s winning strategy in $\mathcal{G}_i$, for $i \in 3$. Let also f_i be the continuous function given by σ_i. Thus

$$\begin{array}{lcl} f_0[A_0] \subseteq A_1 & \& & f_0[A_1] \subseteq A_0\,, \\ f_1[A_0] \subseteq A_0 & \& & f_1[\neg B_0] \subseteq A_1\,, \\ f_2[A_1] \subseteq A_1 & \& & f_2[\neg B_1] \subseteq A_0\,. \end{array}$$

We will apply the Martin–Monk technique.

For $z \in {}^{\omega}3$, $\langle(\mathcal{G}_{z(i)}, \sigma_{z(i)}) \mid i \in \omega\rangle$ admits a global play $x(z)$ as each Player$(\sigma_{z(i)}) = \mathbf{I}$. Let $x(z)_i$ be the i-th row of $x(z)$, i.e., the real on the i-th row of the global board so that

$$x(z)_i = f_{z(i)}(x(z)_{i+1})\,.$$

Let

$$X = \{z \in {}^{\omega}3 \mid x(z)_0 \notin A_0 \cup A_1\}\,.$$

By the definition of the f_i's, it is easy to see that

$$z \in X \Rightarrow \forall n \in \omega\, (x(z)_n \notin A_0 \cup A_1)$$

hence if $s \in {}^{n}3$

$$s^\frown z \in X \Rightarrow x(s^\frown z)_n = x(z)_0 \notin A_0 \cup A_1 \Rightarrow z \in X\,.$$

By AD the set X has the property of Baire.

Suppose first X is non-meager. Let $s \in {}^{<\omega}3$ be such that X is comeager in $\boldsymbol{N}_s$, that is: $\{z \in {}^{\omega}3 \mid s^\frown z \in X\}$ is comeager. By the remark above

$$X \supseteq \{z \in {}^{\omega}3 \mid s^\frown z \in X\}$$

so X is comeager. As $\neg B_0 \cup \neg B_1 = \mathbb{R}$ pick $i \in 2$ such that

$$Y = \{z \in {}^\omega 3 \mid x(z)_0 \notin B_i\}$$

is non-meager. Let $z \in Y$. If $i = 0$ then $f_1(x(z)_0) \in A_1$, and if $i = 1$ then $f_2(x(z)_0) \in A_0$. Therefore

$$z \in Y \Rightarrow x(\langle i+1\rangle^\frown z)_0 \in A_0 \cup A_1 .$$

Let $Y' = \{\langle i+1\rangle^\frown z \mid z \in Y\}$. Then Y' is non-meager in $\boldsymbol{N}_{\langle i+1\rangle}$, hence in ${}^\omega 3$, and $Y' \cap X = \emptyset$, contradicting the fact that X is comeager.

Suppose now X is meager, i.e.,

$$\neg X = \{z \in {}^\omega 3 \mid x(z)_0 \in A_0 \cup A_1\}$$

is comeager. Pick $i \in 2$ such that

$$Y = \{z \in {}^\omega 3 \mid x(z)_0 \in A_i\}$$

is non-meager. Then Y is comeager in $\boldsymbol{N}_s$, for some $s \in {}^{<\omega}3$. By extending s, if needed, we can assume that s has an even number of 0's. As f_1 and f_2 map A_0 into A_0 and A_1 into A_1, while $f_0[A_0] \subseteq A_1$ and $f_0[A_1] \subseteq A_0$, it is easy to check that

$$x(z)_0 \in A_i \Rightarrow x(s^\frown z)_0 \in A_i .$$

For $z \in Y$

$$\begin{aligned} x(z)_0 \in A_i &\Rightarrow x(\langle 0\rangle^\frown z)_0 = f_0(x(z)_0) \in A_{1-i} \\ &\Rightarrow x(s^\frown\langle 0\rangle^\frown z) \in A_{1-i} \\ &\Rightarrow s^\frown\langle 0\rangle^\frown z \notin Y . \end{aligned}$$

therefore $\{s^\frown\langle 0\rangle^\frown z \mid z \in Y\}$ is non-meager and disjoint from Y which is comeager: a contradiction. □

We now turn to Theorem 24.5. Let $\mathsf{Sep}^*(\boldsymbol{\Gamma})$ be the following assertion:

$$\forall A, B \in \boldsymbol{\Gamma}\ \exists A', B' \in \breve{\boldsymbol{\Gamma}}\ (A \setminus B \subseteq A' \ \&\ B \setminus A \subseteq B' \ \&\ A' \cap B' = \emptyset) .$$

Theorem 24.5 will follow from the following two lemmata:

Lemma 24.7. *(*AD*) Let* $\boldsymbol{\Gamma}$ *be a non-self-dual pointclass. Then*

$$\mathsf{Sep}(\boldsymbol{\Gamma}) \Leftrightarrow \neg\mathsf{Sep}^*(\breve{\boldsymbol{\Gamma}}) .$$

Lemma 24.8. *(*AD*) Let* $\boldsymbol{\Gamma}$ *be a non-self-dual pointclass. Then*

$$\mathsf{Sep}^*(\boldsymbol{\Gamma}) \vee \mathsf{Sep}(\breve{\boldsymbol{\Gamma}}) .$$

Assume, towards a contradiction, that $\mathsf{Sep}(\boldsymbol{\Gamma})$ and $\mathsf{Sep}(\breve{\boldsymbol{\Gamma}})$. The by 24.7 $\neg\mathsf{Sep}^*(\breve{\boldsymbol{\Gamma}})$ and $\neg\mathsf{Sep}^*(\boldsymbol{\Gamma})$, contradicting 24.7. Therefore we will be done once we prove 24.7 and 24.7.

Proof. 24.7 ($\Longleftarrow$) Suppose $\neg\mathsf{Sep}(\boldsymbol{\Gamma})$ and let B_0, B_1 witness this, i.e., $B_0, B_1 \in \boldsymbol{\Gamma}$, $B_0 \cap B_1 = \emptyset$, and $\neg\exists C \in \Delta_{\boldsymbol{\Gamma}} \, (B_0 \subseteq C \ \& \ B_1 \cap C = \emptyset)$. We must show that

$$\forall A_0, A_1 \in \breve{\boldsymbol{\Gamma}} \ \exists A_0', A_1' \in \boldsymbol{\Gamma} \, (A_0 \setminus A_1 \subseteq A_0' \ \& \ A_1 \setminus A_0 \subseteq A_1' \ \& \ A_0' \cap A_1' = \emptyset) \ .$$

Fix $A_0, A_1 \in \breve{\boldsymbol{\Gamma}}$ and consider the game $G^*_{\mathrm{sep}}(A_0, A_1, B_0, B_1)$ defined on page 169. By Exercise 24.6 **II** cannot have a winning strategy, so by determinacy, there is a continuous $f : \mathbb{R} \to \mathbb{R}$ such that

$$f[A_0 \setminus A_1] \subseteq B_0 \qquad \text{and} \qquad f[A_1 \setminus A_0] \subseteq B_1 \, .$$

Take $A_0' = f^{-1}[B_0]$ and $A_1' = f^{-1}[B_1]$.

($\Longrightarrow$) Suppose $\mathsf{Sep}^*(\breve{\boldsymbol{\Gamma}})$. We must find a pair of inseparable $\boldsymbol{\Gamma}$-sets. Let $A \in \boldsymbol{\Gamma} \setminus \breve{\boldsymbol{\Gamma}}$ and let

$$\begin{aligned} U &= \{(x \oplus y, z) \mid \boldsymbol{\ell}_x(z) \in A\} \, , \\ V &= \{(x \oplus y, z) \mid \boldsymbol{\ell}_y(z) \in A\} \, . \end{aligned}$$

Then (U, V) is a universal pair for $\boldsymbol{\Gamma}$, in the sense of **??**.

By $\mathsf{Sep}^*(\breve{\boldsymbol{\Gamma}})$ there are $B_0, B_1 \in \boldsymbol{\Gamma}$ such that

- $B_0, B_1 \subseteq \mathbb{R} \times \mathbb{R}$, $B_0 \cap B_1 = \emptyset$,
- $U \setminus V = \neg V \setminus \neg U \subseteq B_0$,
- $V \setminus U = \neg U \setminus \neg V \subseteq B_1$.

Let $A_i = \{z \mid (z, z) \in B_i\}$ with $i \in 2$. As B_0 and B_1 are disjoint, so are A_0 and A_1, and as $z \mapsto (z, z)$ is continuous, then $A_0, A_1 \in \boldsymbol{\Gamma}$. We will show that they cannot be separated by a set in $\Delta_{\boldsymbol{\Gamma}}$. Suppose, towards a contradiction, that $A_0 \subseteq C$ and $A_1 \cap C = \emptyset$ for some $C \in \Delta_{\boldsymbol{\Gamma}}$. As $\neg C, C \in \boldsymbol{\Gamma}$ let $x, y \in \mathbb{R}$ be such that $\neg C = \boldsymbol{\ell}_x^{-1}[A]$ and $C = \boldsymbol{\ell}_y^{-1}[A]$. Let $w = x \oplus y$. Then $\neg C = U_w$ and $C = V_w$.

$$\begin{aligned} w \in C &\Rightarrow (w,w) \in V \setminus U \subseteq B_1 \\ &\Rightarrow w \in A_1 \\ &\Rightarrow w \notin C \end{aligned}$$

and

$$\begin{aligned} w \in \neg C &\Rightarrow (w,w) \in U \setminus V \subseteq B_0 \\ &\Rightarrow w \in A_0 \\ &\Rightarrow w \in C \end{aligned}$$

in both cases reaching a contradiction. □

We now turn to 24.7. Let U and V be a universal pair for $\mathbf{\Gamma}$ as in the proof of 24.7 on page 172. Define the following two games $\mathcal{G}_0$ and $\mathcal{G}_1$ on ω: let x be the real played by **I** and y be the real played by **II**

II wins iff $\mathcal{G}_0$

$$(y \notin U \cap V) \ \& \ (x \in U \setminus V \Rightarrow y \in V \setminus U) \ \& \ (x \in V \setminus U \Rightarrow y \in U \setminus V) \ .$$

II wins iff $\mathcal{G}_1$

$$(y \in U \cup V) \ \& \ (x \in U \setminus V \Rightarrow y \in U \setminus V) \ \& \ (x \in V \setminus U \Rightarrow y \in V \setminus U) \ .$$

Lemma 24.9. *(a) If* **II** *wins* $\mathcal{G}_0$ *then* $\mathsf{Sep}^*(\breve{\mathbf{\Gamma}})$.

(b) If **II** *wins* $\mathcal{G}_1$ *then* $\mathsf{Sep}^*(\mathbf{\Gamma})$.

Proof. Suppose τ is a winning strategy in $\mathcal{G}_0$ and let $f : \mathbb{R} \to \mathbb{R}$ be the induced function. Let $U' = f^{-1}[V]$ and $V' = f^{-1}[U]$.

Claim 24.9.1. $U \setminus V \subseteq U'$, $V \setminus U \subseteq V'$ *and* $U' \cap V' = \emptyset$.

Proof. Suppose $x \in U \setminus V$. Then $f(x) \in V \setminus U \subseteq V$ so $x \in U' = f^{-1}[V]$. Similarly $V \setminus U \subseteq V'$. If $x \in U' \cap V'$ then $f(x) \in V \cap U$, contradicting the fact that τ is winning. □

Let $A, B \in \breve{\mathbf{\Gamma}}$ and let $a, b \in \mathbb{R}$ such that $A = \neg (U_{a\oplus b}) = (\neg U)_{a\oplus b}$ and $B = \neg (V_{a\oplus b}) = (\neg V)_{a\oplus b}$. Then

$$\begin{gathered} A \setminus B = (\neg U \setminus \neg V)_{a\oplus b} = (V \setminus U)_{a\oplus b} \subseteq (U')_{a\oplus b} \\ B \setminus A = (\neg V \setminus \neg U)_{a\oplus b} = (U \setminus V)_{a\oplus b} \subseteq (V')_{a\oplus b} \\ (U')_{a\oplus b} \cap (V')_{a\oplus b} = \emptyset \end{gathered}$$

and $(U')_{a\oplus b}, (V')_{a\oplus b} \in \boldsymbol{\Gamma}$. Therefore $\mathsf{Sep}^*(\breve{\boldsymbol{\Gamma}})$.

(b) As in (a) let τ be a winning strategy for **II** in $\mathcal{G}_1$ and let $f : \mathbb{R} \to \mathbb{R}$ be the induced function. Let $U' = f^{-1}[\neg V]$ and $V' = f^{-1}[\neg U]$. The proof proceeds as before. □

Proof. 24.8 By the Lemma and by determinacy it is enough to show that **I** cannot win both $\mathcal{G}_0$ and $\mathcal{G}_1$. Suppose, towards a contradiction, that σ_0 and σ_1 are winning strategies for **I** in $\mathcal{G}_0$ and $\mathcal{G}_1$, and let f_0 and f_1 be the induced functions. Let $y \in \mathbb{R}$ and $x = f_0(y) = \sigma_0 * y$. As σ_0 is winning then

$$(x \in U \setminus V \ \& \ y \notin V \setminus U) \ \vee \ (x \in V \setminus U \ \& \ y \notin U \setminus V) \ \vee \ (y \in U \cap V)$$

hence, by a tedious but straightforward computation,

$$\begin{aligned}&(y \in U \setminus V \Rightarrow f(y) \in U \setminus V) \ \& \\ &\quad (y \in V \setminus U \Rightarrow f(y) \in V \setminus U) \ \& \ (y \in \neg\,(U \cup V) \Rightarrow f(y) \in U \triangle V)\ .\end{aligned}$$

Similarly for every $y \in \mathbb{R}$,

$$\begin{aligned}&(y \in V \setminus U \Rightarrow f(y) \in V \setminus U) \ \& \\ &\quad (y \in U \setminus V \Rightarrow f(y) \in U \setminus V) \ \& \ (y \in U \cap V \Rightarrow f(y) \in U \triangle V)\ .\end{aligned}$$

To summarize:

(1) $f_0[U \setminus V] \subseteq U \setminus V$ and $f_0[V \setminus U] \subseteq V \setminus U$,

(2) $f_0[U \setminus V] \subseteq V \setminus U$ and $f_0[V \setminus U] \subseteq U \setminus V$,

(3) $f_i[U \triangle V] \subseteq U \triangle V$, for $i = 0, 1$,

(4) $f_0\left[\neg(U \cap V)\right] \subseteq U \triangle V$,

(5) $f_1[U \cap V] \subseteq U \triangle V$.

We now apply the Martin–Monk technique. For any $z \in {}^{\omega}2$, $\langle(\mathcal{G}_{z(i)}, \sigma_{z(i)}) \mid i \in \omega\rangle$ admits a global play $x(z)$, as Player$(\sigma_{z(i)}) = \mathbf{I}$ for all $i \in \omega$. As usual $x(z)_i$ is the real played on the i-th row and $x(z)_i = f_{z(i)}(x(z)_{i+1})$

Claim 24.9.2. *$\{z \in {}^{\omega}2 \mid x(z)_0 \in \neg\,(U \triangle V)\}$ is meager.*

Proof. Suppose the set in question is non-meager. Then it is comeager in some $\boldsymbol{N}_s$, with $s \in {}^{<\omega}2$, hence

$$\{z \in {}^{\omega}2 \mid x(s^\frown z)_0 \in \neg\,(U \triangle V)\}$$

is comeager. Notice that

$$x(s^\frown z)_0 = x(z)_{\mathrm{lh}(s)} = f_{s(0)} \circ \cdots \circ f_{s(\mathrm{lh}(s)-1)}(x(z)_0)$$

so by (3)

$$x(s^\frown z)_0 \in \neg\,(U \triangle V) \Rightarrow x(z)_0 \in \neg\,(U \triangle V)$$

hence $Z = \{z \in {}^\omega 2 \mid x(z)_0 \in \neg\,(U \triangle V)\}$ is comeager, and therefore $\{z \in {}^\omega 2 \mid x(z)_0 \in U \triangle V\}$ is meager. Then $Z = A \cup B$, where

$$A = \{z \in Z \mid x(z)_0 \in U \cap V\}, \qquad B = \{z \in Z \mid x(z)_0 \in \neg\,(U \cup V)\},$$

so A or B (or both) is non-meager. We will finish the proof of the claim by showing that both A and B are meager, reaching thus a contradiction.

Assume A is non-meager. By (5)

$$z \in A \Rightarrow x(\langle 1\rangle^\frown z)_0 = f_1(x(z)_0) \in U \triangle V$$

hence $\{z \mid x(z)_0 \in U \triangle V\}$ is non-meager, as it contains $\langle 1\rangle^\frown A$: a contradiction.

Similarly if B is non-meager then (4) implies

$$z \in B \Rightarrow x(\langle 0\rangle^\frown z)_0 = f_0(x(z)_0) \in U \triangle V$$

hence $\{z \mid x(z)_0 \in U \triangle V\}$ is non-meager, as it contains $\langle 0\rangle^\frown B$: again a contradiction. □

Therefore $C = \{z \mid x(z) \in U \triangle V\}$ is comeager. Suppose without loss of generality that $\{z \mid x(z) \in U \setminus V\}$ is non-meager, hence comeager in some $\boldsymbol{N}_s$. Then

$$D = \{z \mid x(s^\frown z)_0 \in U \setminus V\}$$

is comeager. Let $k = |\{i < \mathrm{lh}(s) \mid s(i) = 1\}|$ and recall that $x(s^\frown z)_0 = f_{s(0)} \circ \cdots \circ f_{s(n-1)}(x(z)_0)$.

If k is odd then by (1) and (2)

$$x(z)_0 \in U \setminus V \Rightarrow x(s^\frown z)_0 \in V \setminus U$$
$$x(z)_0 \in V \setminus U \Rightarrow x(s^\frown z)_0 \in U \setminus V$$

so $E = \{z \mid x(z)_0 \notin U \setminus V\}$ is comeager as $E \cap C \supseteq \{z \mid x(z)_0 \in V \setminus U\} \supseteq D$. But this contradicts our assumption that $\neg E$ is non-meager.

Suppose now k is even. By (1) and (2) again

$$x(z)_0 \in U \setminus V \Rightarrow x(s^\frown z)_0 \in U \setminus V$$
$$x(z)_0 \in V \setminus U \Rightarrow x(s^\frown z)_0 \in V \setminus U$$

so $F = \{z \mid x(z)_0 \notin V \setminus U\}$ is comeager. By (1) and (2)

$$z \in F \cap C \Rightarrow x(\langle 1 \rangle^\frown z)_0 \in V \setminus U\,.$$

As the map ${}^\omega 2 \to \boldsymbol{N}_{\langle 1 \rangle}$, $z \mapsto \langle 1 \rangle^\frown z$, is a homeomorphism, then $\{z \mid x(z)_0 \in V \setminus U\}$ is comeager in $\boldsymbol{N}_{\langle 1 \rangle}$, hence it is non-meager, contradicting the comeagerness of F. This concludes the proof of 24.8. □

The separation property allows to "transfer" closure property of pointclasses from $\Delta_{\boldsymbol{\Gamma}}$ to $\boldsymbol{\Gamma}$.

Theorem 24.10 (Kechris–Steel)**.** *(*AD*) Suppose* $\boldsymbol{\Gamma}$ *is non-self-dual and* $\mathsf{Sep}(\boldsymbol{\Gamma})$ *holds. Let* $\boldsymbol{\Delta} = \Delta_{\boldsymbol{\Gamma}}$*.*

(a) If $\boldsymbol{\Delta}$ *is closed under finite (countable) unions then so is* $\boldsymbol{\Gamma}$*, that is*

$$\bigcup(\alpha; \boldsymbol{\Delta}) \subseteq \boldsymbol{\Delta} \Rightarrow \bigcup(\alpha; \boldsymbol{\Gamma}) \subseteq \boldsymbol{\Gamma}$$

for $\alpha = 2, \omega$*.*

(b) If $\boldsymbol{\Delta}$ *is closed under projections then so is* $\boldsymbol{\Gamma}$*, and hence* $\breve{\boldsymbol{\Gamma}}$ *is closed under co-projections, that is*

$$\exists^{\mathbb{R}} \boldsymbol{\Delta} \subseteq \boldsymbol{\Delta} \quad \Rightarrow \quad \left(\exists^{\mathbb{R}} \boldsymbol{\Gamma} \subseteq \boldsymbol{\Gamma} \;\&\; \forall^{\mathbb{R}} \breve{\boldsymbol{\Gamma}} \subseteq \boldsymbol{\Gamma}\right)\,.$$

Proof. (a) Let us assume $\boldsymbol{\Delta}$ is closed under countable unions, the finite case being similar. Towards a contradiction suppose $A_n \in \boldsymbol{\Gamma}$ and $\bigcup_n A_n \notin \boldsymbol{\Gamma}$. Let $B \in \breve{\boldsymbol{\Gamma}} \setminus \boldsymbol{\Gamma}$. Then $\bigcup_n A_n \not\leq_{\mathrm{W}} \neg B$ so by Wadge's Lemma $B \leq_{\mathrm{W}} \bigcup_n A_n$ via some f. Then $B = f^{-1}\left[\bigcup_n A_n\right] = \bigcup_n f^{-1}\left[A_n\right]$. Each $f^{-1}\left[A_n\right] \in \boldsymbol{\Gamma}$ and it is disjoint from $\neg B \in \boldsymbol{\Gamma}$, hence there are $C_n \in \boldsymbol{\Delta}$ such that $B \supseteq C_n \supseteq f^{-1}\left[A_n\right]$. Therefore $B = \bigcup_n C_n \in \boldsymbol{\Delta}$, contradicting our choice of B.

(b) Suppose, towards a contradiction, that $\exists^{\mathbb{R}} \boldsymbol{\Gamma} \not\subseteq \boldsymbol{\Gamma}$. By Wadge's Lemma $\breve{\boldsymbol{\Gamma}} \subseteq \exists^{\mathbb{R}} \boldsymbol{\Gamma}$. Therefore it is enough to show that any pair of disjoint sets in $\exists^{\mathbb{R}} \boldsymbol{\Gamma}$ can be separated by a set in $\boldsymbol{\Delta}$, as this would imply $\mathsf{Sep}(\breve{\boldsymbol{\Gamma}})$, contradicting $\mathsf{Sep}(\boldsymbol{\Gamma})$ and 24.4. So let $A, B \in \exists^{\mathbb{R}} \boldsymbol{\Gamma}$ with $A \cap B = \emptyset$. Then

$$\begin{aligned} A &= \{x \in \mathbb{R} \mid \exists y \in \mathbb{R} (x, y) \in P\} \\ B &= \{x \in \mathbb{R} \mid \exists z \in \mathbb{R} (x, z) \in Q\} \end{aligned}$$

for some $P, Q \in \boldsymbol{\Gamma}$. Let $P', Q' \subseteq \mathbb{R}^3$,

$$\begin{aligned} P' &= \{(x, y, z) \mid (x, y) \in P\} \\ Q' &= \{(x, y, z) \mid (x, z) \in Q\}\,. \end{aligned}$$

Then $P', Q' \in \mathbf{\Gamma}$ and $P' \cap Q' = \emptyset$, by the fact that $A \cap B = \emptyset$. Let $D \subseteq \mathbb{R}^3$, $D \in \mathbf{\Delta}$, such that $P' \subseteq D$ & $D \cap Q' = \emptyset$, and let

$$\begin{aligned} C &= \exists^{\mathbb{R}} \forall^{\mathbb{R}} D \\ &= \{x \mid \exists y \forall z\ (x, y, z) \in D\}. \end{aligned}$$

Then $C \in \mathbf{\Delta}$ since for self-dual pointclasses closure under $\exists^{\mathbb{R}}$ is equivalent to closure under $\forall^{\mathbb{R}}$. If $x \in A$ then $(x, y) \in P$ for some y and hence $(x, y, z) \in P' \subseteq D$ for every z, and therefore $x \in C$. Vice versa if $x_0 \in B \cap C$ then pick $y_0, z_0 \in \mathbb{R}$ such that $(x_0, z_0) \in Q$ and $(x_0, y_0) \in \forall^{\mathbb{R}} D = \{(x, y) \mid \forall z\, (x, y, z) \in D\}$: then $(x_0, y_0, z_0) \in D \cap Q' = \emptyset$, a contradiction. Therefore $B \cap C = \emptyset$. □

Theorem 24.11 (Steel)**.** *(*AD*) Let* $\mathbf{\Gamma}$ *be non-self-dual and suppose* $\neg\mathsf{Sep}(\mathbf{\Gamma})$*. If* $\mathbf{\Gamma}$ *is closed under finite unions then* $\mathbf{\Gamma}$ *is closed under countable unions, that is*

$$\bigcup(2; \mathbf{\Gamma}) \subseteq \mathbf{\Gamma} \Rightarrow \bigcup(\omega; \mathbf{\Gamma}) \subseteq \mathbf{\Gamma}.$$

Proof. Towards a contradiction, suppose $\bigcup(\omega; \mathbf{\Gamma}) \nsubseteq \mathbf{\Gamma}$. By Wadge's Lemma 24.1 $\breve{\mathbf{\Gamma}} \subseteq \bigcup(\omega; \mathbf{\Gamma})$. Let $A_0, A_1 \in \mathbf{\Gamma}$, $A_0 \cup A_1 = \emptyset$ and $\Delta_{\mathbf{\Gamma}}$-inseparable. As $\mathbf{\Gamma}$ is closed under finite unions $\neg(A_0 \cup A_1) \in \breve{\mathbf{\Gamma}}$ and therefore $\neg(A_0 \cup A_1) = \bigcup_n C_n$, for some $C_n \in \mathbf{\Gamma}$. Notice that $A_0 \cup C_n, A \in \mathbf{\Gamma}$ are also $\Delta_{\mathbf{\Gamma}}$-inseparable so by 24.6 let

$$\begin{aligned} \sigma_{2n} &\text{ a winning strategy for } \mathbf{I} \text{ in } \mathcal{G}_{2n} = G^*_{\text{sep}}(A_0 \cup C_n, A_1, A_0, A_1) \\ \sigma_{2n+1} &\text{ a winning strategy for } \mathbf{I} \text{ in } \mathcal{G}_{2n+1} = G^*_{\text{sep}}(A_0 \cup C_n, A_1, A_1, A_0). \end{aligned}$$

Now apply the Martin–Monk method: as Player$(\sigma_n) = \mathbf{I}$ for all n, $\langle(\mathcal{G}_{z(n)}, \sigma_{z(n)}) \mid n \in \omega\rangle$ admits a global play for any $z \in {}^{\omega}\omega = \mathbb{R}$. Let $x(z)_n$ be the real on the n-th row of the Martin–Monk diagram, so that $x(z)_n = f_{z(n)}(x(z)_{n+1})$, where f_i is the function induced by σ_i. By 24.6

$$\begin{aligned} f_{2n}[A_0 \cup C_n] &\subseteq A_0 & f_{2n}[A_1] &\subseteq A_1 \\ f_{2n+1}[A_0 \cup C_n] &\subseteq A_1 & f_{2n+1}[A_1] &\subseteq A_0 \end{aligned}$$

hence $f_{s_0} \circ \cdots \circ f_{s_{k-1}}(x) \in A_0 \cup A_1$ for any $s \in {}^k\omega$, any $i < 2$ and any $x \in A_i$, and

$$f_{s_0} \circ \cdots \circ f_{s_{k-1}}(x) \in A_i \quad \Leftrightarrow \quad |\{j < k \mid s_j \text{ is odd }\}| \text{ is even.}$$

As $\mathbb{R} = A_0 \cup A_1 \cup \bigcup_n C_n$, to reach a contradiction it is enough to show that

$$\{z \in \mathbb{R} \mid x(z)_0 \in A_0 \cup A_1\}$$

and

$$\{z \in \mathbb{R} \mid x(z)_0 \in C_n\} \qquad\qquad (n < \omega)$$

are meager.

Suppose $\{z \in \mathbb{R} \mid x(z)_0 \in A_0 \cup A_1\}$ is non-meager and pick $i < 2$ and $s \in {}^{<\omega}\omega$ such that $\{z \in \mathbb{R} \mid x(z)_0 \in A_i\}$ is comeager in $\boldsymbol{N}_s$. By extending s, if needed, we may assume that $|\{j < k = \mathrm{lh}(s) \mid s_j \text{ is odd }\}|$ is even. Notice that

$$\begin{aligned} x(s^\frown z)_0 &= f_{s_0} \circ \cdots \circ f_{s_{k-1}}(x(s^\frown z)_k) \\ &= f_{s_0} \circ \cdots \circ f_{s_{k-1}}(x(z)_0) \end{aligned}$$

so $x(s^\frown z)_0 \in A_i \Leftrightarrow x(z)_0 \in A_i$ and therefore $X = \{z \in \mathbb{R} \mid x(z)_0 \in A_i\}$ is comeager. But

$$z \in X \Rightarrow x(\langle 1\rangle^\frown z)_0 = f_1(x(z)_0) \in A_{1-i}\,,$$

hence

$$\{z \in \mathbb{R} \mid x(z)_0 \in A_{1-i}\} \supseteq \langle 1\rangle^\frown X$$

is non-meager: a contradiction.

Suppose now $M_n = \{z \in \mathbb{R} \mid x(z)_0 \in C_n\}$ is non-meager and hence comeager in some $\boldsymbol{N}_s$, with $s \in {}^k\omega$. If $x(z)_0 \in C_n$ then $f_{2n}(x(z)_0) \in A_0$ hence

$$x(s^\frown\langle 2n\rangle^\frown z)_0 = f_{s_0} \circ \cdots \circ f_{s_{k-1}}(f_{2n}(x(z)_0)) \in A_0 \cup A_1$$

so

$$\{z \in \mathbb{R} \mid x(z)_0 \notin C_n\} \supseteq s^\frown\langle 2n\rangle^\frown M_n$$

hence it is comeager in $\boldsymbol{N}_{s^\frown\langle 2n\rangle^\frown s}$, and therefore non-meager in $\boldsymbol{N}_s$: a contradiction. □

Open problem 24.12 (Steel)**.** *Assume* AD *and suppose* $\boldsymbol{\Gamma}$ *is a non-self-dual pointclass closed under countable unions and intersections. Does* $\mathcal{A}\boldsymbol{\Gamma} \subseteq \boldsymbol{\Gamma}$ *or* $\mathcal{A}\breve{\boldsymbol{\Gamma}} \subseteq \breve{\boldsymbol{\Gamma}}$*?*

24.B Reduction

Definition 24.13. A pointclass Γ has the reduction property, $\mathsf{Red}(\Gamma)$, iff

$$\forall A, B \in \Gamma\, \exists A', B' \in \Gamma\, (A' \subseteq A \;\&\; B' \subseteq B \;\&\; A \cup B = A' \cup B' \;\&\; A' \cap B' = \emptyset)\ .$$

We say that the pair (A', B') reduces the pair (A, B).

Exercise 24.14. (i) Show that $\mathsf{Unif}(\mathbf{\Gamma}) \Rightarrow \mathsf{Red}(\mathbf{\Gamma})$.

(ii) Show that $\mathsf{Red}(\mathbf{\Gamma}) \Rightarrow \mathsf{Sep}(\breve{\mathbf{\Gamma}})$. Therefore under AD, if $\mathbf{\Gamma}$ is a non-self-dual pointclass then $\mathsf{Red}(\mathbf{\Gamma}) \Rightarrow \neg\mathsf{Sep}(\mathbf{\Gamma})$.

The second part of (ii) can also be proved by quoting 24.1 and using

Proposition 24.15 (Novikov)**.** *Suppose* $\mathbf{\Gamma}$ *has a universal set. Then* $\mathsf{Red}(\mathbf{\Gamma}) \Rightarrow \neg\mathsf{Sep}(\mathbf{\Gamma})$.

Proof. Deny. Let $U \subseteq \mathbb{R}^2$ be $\mathbf{\Gamma}$-universal and let

$$\begin{aligned} A &= \{(x,y) \in \mathbb{R}^2 \mid (x, y_{\mathbf{I}}) \in U\} \\ B &= \{(x,y) \in \mathbb{R}^2 \mid (x, y_{\mathbf{II}}) \in U\}. \end{aligned}$$

Then $A, B \in \mathbf{\Gamma}$ so there are $A', B' \in \mathbf{\Gamma}$ such that $A' \subseteq A$, $B' \subseteq B$, and $A' \cap B' = \emptyset$. Suppose $A' \subseteq C$ and $B' \cap C = \emptyset$ for some $C \in \Delta_{\mathbf{\Gamma}}$. Then C is $\Delta_{\mathbf{\Gamma}}$-universal, contradicting **??**(iii): if $D \in \Delta_{\mathbf{\Gamma}}$ let $a, b \in \mathbb{R}$ such that $D = U_a$ and $\neg D = U_b$, and let $d = a \oplus b$. Then

$$\begin{aligned} (x,d) \in A &\Leftrightarrow (x,a) \in U \\ &\Leftrightarrow x \in D \\ &\Leftrightarrow x \notin \neg D \\ &\Leftrightarrow (x,b) \notin U \\ &\Leftrightarrow (x,d) \notin B \end{aligned}$$

so $(x,d) \in A' \Leftrightarrow (x,d) \notin B'$ hence $x \in D \Leftrightarrow (x,d) \in C$. □

Corollary 24.16. *(*AD*)* $\neg\mathsf{Red}(\mathbf{\Gamma}) \vee \neg\mathsf{Red}(\breve{\mathbf{\Gamma}})$*, for any non-self-dual* $\mathbf{\Gamma}$*.*

By Kondo's theorem **??** then $\mathsf{Red}(\mathbf{\Pi}^1_1)$, but one can give a direct proof of this fact.

Theorem 24.17 (Kuratowski)**.** $\mathsf{Red}(\mathbf{\Pi}^1_1)$.

Proof. Blackwell Let $A, B \in \mathbf{\Pi}^1_1$ and let T, U be trees on $\omega \times \omega$ such that $\neg A = \mathrm{p}[T]$ and $\neg B = \mathrm{p}[U]$. For $z \in \mathbb{R}$ consider the game G_z on ω where **I** plays x, **II** plays y and **I** wins iff $\exists n\ (x \restriction n \in U_z \ \&\ y \restriction n \notin T_z)$.

Exercise 24.18. Show that G_z is a clopen game and hence determined.

Let

$$\begin{aligned} A' &= \{z \in A \mid \mathbf{II} \text{ does not win } G_z\} \\ B' &= \{z \in B \mid \mathbf{I} \text{ does not win } G_z\}. \end{aligned}$$

Claim 24.18.1. *$A \setminus B \subseteq A'$ and $B \setminus A \subseteq B'$.*

Proof. Let $z \in A \setminus B = \mathrm{p}[U] \setminus \mathrm{p}[T]$. Then U_z is ill-founded while T_z is well-founded so **I** wins G_z by playing any $x \in [U_z]$. Therefore $z \in A'$.

The other inclusion is similar. □

If $z \in A \cap B$ then $z \in A' \cup B'$ as otherwise both **I** and **II** would have been winning strategies for G_z. As all G_z's are determined, A' and B' are disjoint. Summarizing

$$A' \subseteq A \ \&\ B' \subseteq B \ \&\ A \cup B = A' \cup B' \ \&\ A' \cap B' = \emptyset\,.$$

Moreover

$$z \in A' \Leftrightarrow \underbrace{z \in A}_{\mathbf{\Pi}^1_1} \ \&\ \underbrace{\forall \tau\ \underbrace{\exists n\, \exists s \in {}^n\omega\, ((z \restriction n, s) \in U \ \&\ (z \restriction n, (s * \tau)_{\mathbf{II}}) \notin T)}_{\mathbf{\Sigma}^0_1}}_{\mathbf{\Pi}^1_1}$$

shows that $A' \in \mathbf{\Pi}^1_1$ and similarly $B' \in \mathbf{\Pi}^1_1$. □

Theorem 24.19 (Steel–Van Wesep)**.** *Let $\mathbf{\Gamma}$ be non-self-dual such that $\mathbf{\Delta} = \Delta_{\mathbf{\Gamma}}$ is closed under finite unions. Then $\mathsf{Red}(\mathbf{\Gamma}) \vee \mathsf{Red}(\breve{\mathbf{\Gamma}})$.*

Proof. By 24.3 $\mathsf{Sep}(\mathbf{\Gamma})$ or $\mathsf{Sep}(\breve{\mathbf{\Gamma}})$ but not both. Assume for the sake of definiteness that $\mathsf{Sep}(\breve{\mathbf{\Gamma}})$. By 24.14 or 24.15 $\neg\mathsf{Red}(\breve{\mathbf{\Gamma}})$, as otherwise $\mathsf{Sep}(\mathbf{\Gamma})$, so we are left to prove $\mathsf{Red}(\mathbf{\Gamma})$. Suppose that $\neg\mathsf{Red}(\mathbf{\Gamma})$. We will reach a contradiction by proving $\mathsf{Sep}(\mathbf{\Gamma})$. Let $A, B \in \mathbf{\Gamma}$ witness $\neg\mathsf{Red}(\mathbf{\Gamma})$ and let $C, D \in \mathbf{\Gamma}$ be disjoint. Consider the game where **I** and **II** play x and y and **II** wins iff

$$y \in A \cup B \ \&\ (x \in C \Rightarrow y \in A \setminus B) \ \&\ (x \in D \Rightarrow y \in B \setminus A)\ .$$

Claim 24.19.1. **I** *does not win this game.*

Proof. Let σ be a winning strategy for **I** and let f be the induced function. Let $A' = f^{-1}[D] \cap A$ and $B' = f^{-1}[C] \cap B$. By 24.10 $\bigcup(2; \breve{\mathbf{\Gamma}}) \subseteq \breve{\mathbf{\Gamma}}$ hence $\bigcap(2; \mathbf{\Gamma}) \subseteq \mathbf{\Gamma}$ and therefore $A', B' \in \mathbf{\Gamma}$. By construction $A' \subseteq A$, $B' \subseteq B$ and $A' \cap B' = \emptyset$. Let $y \in A \cup B$ and let $x = f(y)$: as σ is winning then

$$(x \in C \ \&\ y \in B) \ \vee\ (x \in D \ \&\ y \in A)$$

so that $(y \in f^{-1}[C] \cap B = B') \ \vee\ (y \in f^{-1}[D] \cap A = A')$. In other words (A', B') reduces (A, B) contrarily to our assumption. □

Therefore **II** wins and let f be the map induced by the winning strategy. Then $\operatorname{ran}(f) \subseteq A \cup B$, $f[C] \subseteq A \setminus B$, and $f[D] \subseteq B \setminus A$. Let $C' = f^{-1}[\neg B] = f^{-1}[A \setminus B]$ and $D' = f^{-1}[\neg A] = f^{-1}[B \setminus A]$. Then $C', D' \in \breve{\boldsymbol{\Gamma}}$, $C \subseteq C'$, $D \subseteq D'$ and $C' \cap D' = \emptyset$. By $\mathsf{Sep}(\breve{\boldsymbol{\Gamma}})$ there is an $E \in \boldsymbol{\Delta}$ which separates C' from D' and hence C from D. As C, D are arbitrary disjoint sets in $\boldsymbol{\Gamma}$, this proves $\mathsf{Sep}(\boldsymbol{\Gamma})$. □

Exercise 24.20. (AD) Show that if $\boldsymbol{\Gamma}$ is non-self-dual then

$$\bigcap(2; \boldsymbol{\Gamma}) \subseteq \boldsymbol{\Gamma} \ \& \ \mathsf{Sep}(\breve{\boldsymbol{\Gamma}}) \Rightarrow \mathsf{Red}(\boldsymbol{\Gamma}) \, .$$

The next result shows that there are plenty of non-self-dual boldface pointclasses $\boldsymbol{\Gamma}$ for which reduction fails for $\boldsymbol{\Gamma}$ and $\breve{\boldsymbol{\Gamma}}$. In order to state it in a convenient form we need the notion of backtrack reducibility—see 22.20.

Theorem 24.21 (Van Wesep)**.** *Assume* AD^{L}*. Suppose* $A \not\equiv_{\mathrm{bt}} \neg A$*,* $A \notin \{\emptyset, \mathbb{R}\}$*, and let* B *be non-self-dual such that* $[B]_{\mathrm{bt}}$ *is the* $\leq_{\mathrm{bt}}$*-successor of* $[A]_{\mathrm{bt}}$ *and* $[\neg A]_{\mathrm{bt}}$*. Let*

$$\boldsymbol{\Gamma} = \{C \subseteq \mathbb{R} \mid C \leq_{\mathrm{W}} \neg B\}$$

and

$$\breve{\boldsymbol{\Gamma}} = \{C \subseteq \mathbb{R} \mid C \leq_{\mathrm{W}} B\} \, .$$

Then reduction fails for $\boldsymbol{\Gamma}$ *and* $\breve{\boldsymbol{\Gamma}}$*, that is* $\neg\mathsf{Red}(\boldsymbol{\Gamma}) \ \& \ \neg\mathsf{Red}(\breve{\boldsymbol{\Gamma}})$*.*

Proof. Let $\boldsymbol{\Lambda} = \{C \subseteq \mathbb{R} \mid C \leq_{\mathrm{W}} A\}$. By 24.14 reduction must fail for $\boldsymbol{\Lambda}$ or $\breve{\boldsymbol{\Lambda}}$, say $\neg\mathsf{Red}(\breve{\boldsymbol{\Lambda}})$. Let

$$C = \{x \oplus y \mid x \in A\}$$

and

$$D = \{x \oplus y \mid y \in A\} \, .$$

We shall prove that the pair (C, D) cannot be reduced by a pair of sets in $\boldsymbol{\Gamma}$ nor by a pair of sets in $\breve{\boldsymbol{\Gamma}}$.

Claim 24.21.1. (C, D) *cannot be reduced by a pair of sets in* $\boldsymbol{\Lambda} \cup \breve{\boldsymbol{\Lambda}}$*.*

Proof. Towards a contradiction, let (C^*, D^*) be a pair of sets in $\mathbf{\Lambda} \cup \breve{\mathbf{\Lambda}}$ reducing (C, D). If $C^*, D^* \leq_W A$ then for any $C', D' \leq_W A$ pick continuous functions f and g such that $C' = f^{-1}[A]$ and $D' = g^{-1}[A]$ and let $h(x) = (f(x), g(x))$. Then $(h^{-1}[C^*], h^{-1}[D^*])$ reduces (C', D'), contradicting $\neg\mathsf{Red}(\mathbf{\Lambda})$. Therefore at least one among C^* and D^* is not in $\mathbf{\Lambda}$, say $C^* \in \breve{\mathbf{\Lambda}} \setminus \mathbf{\Lambda}$. Pick $y_0 \notin A$ —here is where $\emptyset \subset A \subset \mathbb{R}$ is used. Then

$$x \in A \Rightarrow x \oplus y_0 \in C \;\&\; x \oplus y_0 \notin D \Rightarrow x \oplus y_0 \in C^*$$

that is to say: the map $x \mapsto x \oplus y_0$ witnesses $A \leq_W C^*$ and therefore $\mathbf{\Lambda} \subseteq \breve{\mathbf{\Lambda}}$: a contradiction. □

By 22.22 we may assume $\boldsymbol{T}(A) = {}^{<\omega}\omega$ and thus $\forall s \in {}^{<\omega}\omega \left(C_{\lfloor s \rfloor} \equiv_W C \;\&\; D_{\lfloor s \rfloor} \equiv_W D\right)$. Therefore by the arguments above

$$\forall s \in {}^{<\omega}\omega \; (C_{\lfloor s \rfloor}, D_{\lfloor s \rfloor}) \text{ cannot be reduced by a pair of sets in } \mathbf{\Lambda} \cup \breve{\mathbf{\Lambda}}\,.$$

So suppose (C^*, D^*) is a pair of sets in $\mathbf{\Gamma}$ reducing (C, D) —the case when $C^*, D^* \in \breve{\mathbf{\Gamma}}$ is completely analogous and it is left to the reader. By the arguments above we may assume that $\{C^*, D^*\} \not\subseteq \mathbf{\Lambda} \cup \breve{\mathbf{\Lambda}}$, say $C^* \in [B]_{\mathrm{bt}}$. Since $[B]_{\mathrm{bt}}$ is the $\leq_{\mathrm{bt}}$-successor of $[A]_{\mathrm{bt}}$, by 22.23(ii) there is an $s_0 \in {}^{<\omega}\omega$ such that $C^*_{\lfloor s_0 \rfloor} \in \mathbf{\Lambda} \cup \breve{\mathbf{\Lambda}}$. If $D^* \in \mathbf{\Lambda} \cup \breve{\mathbf{\Lambda}}$ then set $s = s_0$; otherwise $D^* \equiv_{\mathrm{bt}} B$ so again by 22.23(ii) there is an $s \supseteq s_0$ such that $D^*_{\lfloor s \rfloor} \in \mathbf{\Lambda} \cup \breve{\mathbf{\Lambda}}$. In both cases $(C^*_{\lfloor s \rfloor}, D^*_{\lfloor s \rfloor})$ is a pair of sets in $\mathbf{\Lambda} \cup \breve{\mathbf{\Lambda}}$ reducing (C, D): a contradiction. □

Exercise 24.22. Show in $\mathsf{ZF} + \mathsf{AC}_\omega(\mathbb{R})$ that the least non-self-dual $\mathbf{\Gamma} \supset \mathbf{\Sigma}^0_2 \cup \mathbf{\Pi}^0_2$ witnesses $\neg\mathsf{Red}(\mathbf{\Gamma})$ & $\neg\mathsf{Red}(\breve{\mathbf{\Gamma}})$

Notes and References

25 Scattered results

Suppose μ is a countably complete ultrafilter on some $\kappa > \Theta^{\mathrm{L}(\mathbb{R})}$ and let $\mathrm{Ult}(\mathrm{L}(\mathbb{R}), \mu) = {}^{\kappa}\mathrm{L}(\mathbb{R})/\mu$ be the ultrapower formed using functions in $\mathrm{L}(\mathbb{R})$.

Lemma 25.1. $\mathrm{Ult}(\mathrm{L}(\mathbb{R}), \mu)$ *satisfies Los' theorem.*

Proof. It is enough to show that if

$$S = \{\alpha < \kappa \mid \mathrm{L}(\mathbb{R}) \models \exists v_0\, \varphi[v_0, f_1(\alpha), \dots, f_n(\alpha)]\} \in \mu$$

where $f_1, \dots, f_n \in \mathrm{L}(\mathbb{R})$, then there is an $f_0 \in \mathrm{L}(\mathbb{R})$ such that

$$\{\alpha \in S \mid \mathrm{L}(\mathbb{R}) \models \varphi[f_0(\alpha), f_1(\alpha), \dots, f_n(\alpha)]\} \in \mu.$$

Let $\Phi : \mathbb{R} \times \mathrm{Ord} \twoheadrightarrow \mathrm{L}(\mathbb{R})$ and let $g : S \to \mathrm{Ord}$,

$$g(\alpha) = \text{ the least } \beta \left(\exists r \in \mathbb{R}\ \mathrm{L}(\mathbb{R}) \models \varphi[\Phi(r,\beta), f_1(\alpha), \ldots f_n(\alpha)]\ \right)$$

For $\alpha \in S$ let

$$A_\alpha = \{r \in \mathbb{R} \mid \mathrm{L}(\mathbb{R}) \models \varphi[F(r, g(\alpha)), f_1(\alpha), \ldots f_n(\alpha)]\} \neq \emptyset$$

so there is $S' \subset S$, $S' \in \mu$ such that

$$\forall \alpha, \beta \in S' \,(A_\alpha = A_\beta)\,.$$

Fix $\bar{r} \in A_\alpha$, with $\alpha \in S'$. Set $f_0 : S \to \mathrm{L}(\mathbb{R})$, $f_0(\alpha) = F(\bar{r}, g(\alpha))$. As $g \in \mathrm{L}(\mathbb{R})$ by definability, $f_0 \in \mathrm{L}(\mathbb{R})$ too. Hence

$$S' = \{\alpha < \kappa \mid \mathrm{L}(\mathbb{R}) \models \varphi[f_0(\alpha), f_1(\alpha), \ldots f_n(\alpha)]\} \in \mu.$$

□

Notice that $\mathrm{Ult}(\mathrm{L}(\mathbb{R}), \mu)$ is well-founded, otherwise, by DC in $\mathrm{L}(\mathbb{R})$, there would be $f_n \in \mathrm{L}(\mathbb{R})$ such that $[f_{n+1}]_\mu \in^{\mathrm{Ult}(\mathrm{L}(\mathbb{R}),\mu)} [f_n]_\mu$. Therefore

$$S_n = \{\alpha < \kappa \mid f_{n+1}(\alpha) \in f_n(\alpha)\} \in \mu$$

hence $S = \bigcap_n S_n \in \mu$ and therefore $\bar{\alpha} \in S \Rightarrow \forall n\,(f_{n+1}(\alpha) \in f_n(\alpha))$: a contradiction.

Also $j : \mathrm{L}(\mathbb{R}) \to \mathrm{Ult}(\mathrm{L}(\mathbb{R}), \mu)$, the canonical embedding is elementary. Thus $\mathrm{Ult}(\mathrm{L}(\mathbb{R}), \mu) \models \mathrm{V} = \mathrm{L}(j(\mathbb{R}))$. As $j(\mathbb{R}) = \mathbb{R}$, $\mathrm{Ult}(\mathrm{L}(\mathbb{R}), \mu) = \mathrm{L}(\mathbb{R})$, hence $\mathbb{R}^\sharp$ exists, Therefore we have shown that

Lemma 25.2. *In* $\mathrm{L}(\mathbb{R})$ *there is no measurable cardinal above* Θ.

Θ is the supremum of measurable cardinals, so if $\mathscr{P} \not\subseteq \mathrm{L}(\mathbb{R})$, then $\Theta^{\mathrm{L}(\mathbb{R})} < \Theta$ and there is a measurable $\Theta^{\mathrm{L}(\mathbb{R})} < \kappa < \Theta$.

Some remarks.

Suppose M and N are proper class models with divergent Wadge hierarchies. Let $\mathbb{R} = \mathbb{R}^M$. Then $(\mathscr{P}(\mathbb{R}))^M \not\subseteq \mathrm{L}(\mathbb{R})$ so $\mathbb{R}^\sharp$ exists in M. By absoluteness of sharps, $\mathbb{R}^\sharp$ exists in V.

Proposition 25.3. *Assume* $\mathsf{AD} + \Theta$ *singular. Then* $\mathbb{R}^\sharp$ *exists and hence* $\mathsf{Con}(\mathsf{ZF} + \mathsf{AD})$.

Proof. In $\mathrm{L}(\mathbb{R})$ there is a Θ-sequence cofinal in $\mathscr{P}(\mathbb{R})$ and this is equivalent to the regularity of Θ. Therefore $\Theta^{\mathrm{L}(\mathbb{R})} < \Theta$. Now proceed as before. □

Recall the following conjecture of Martin's in recursion theory:

Open problem 25.4 (Martin's Conjecture)**.** *Assume* $\mathsf{AD}+\mathsf{DC}(\mathbb{R})$*. Suppose* $f : \boldsymbol{\mathcal{D}} \to \boldsymbol{\mathcal{D}}$ *is such that* $\mu_{\mathrm{M}}(\{\mathbf{d} \in \boldsymbol{\mathcal{D}} \mid f(\mathbf{d}) \not\leq \mathbf{d}\}) = 1$*. Then*

$$\exists \mathbf{c} \in \boldsymbol{\mathcal{D}}\, \mu_{\mathrm{M}}(\{\mathbf{d} \in \boldsymbol{\mathcal{D}} \mid f(\mathbf{d}) = \mathbf{c}\}) = 1$$

i.e. f *is constant on a cone.*

Proposition 25.5. *Assume* $\mathsf{AD}+\mathsf{DC}(\mathbb{R})$*. Then Martin's conjecture implies that* $|\boldsymbol{\mathcal{D}} \times 2| \neq |\boldsymbol{\mathcal{D}}|$*.*

Proof. Let $f : \boldsymbol{\mathcal{D}} \times 2 \to \boldsymbol{\mathcal{D}}$ be a bijection. For $i = 0, 1$ let $f_i(\mathbf{d}) = f(\mathbf{d}, i)$ so that $f_i : \boldsymbol{\mathcal{D}} \rightarrowtail \boldsymbol{\mathcal{D}}$. By Martin's conjecture $f_i(\mathbf{d}) \geq_{\mathrm{T}} \mathbf{d}$ on a cone of $\mathbf{d}$'s hence $\operatorname{ran}(f_i)$ cannot be disjoint from a cone. But $\operatorname{ran}(f_0) \cup \operatorname{ran}(f_1) = \boldsymbol{\mathcal{D}}$ and $\operatorname{ran}(f_0) \cap \operatorname{ran}(f_1) = \emptyset$, so one the two must be of μ_{M}-measure 0, hence disjoint from a cone. □

Theorem 25.6. *The perfect set property for* $\boldsymbol{\Pi}^1_1$ *implies* ω_1 *is inaccessible in* L*.*

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