The Fano normal function. PREPRINT

THE FANO NORMAL FUNCTION

ALBERTO COLLINO, JUAN CARLOS NARANJO, AND GIAN PIETRO PIROLA

Abstract. The Fano surface F of lines in the cubic threefold V is naturally embedded in the intermediate Jacobian $J(V)$, we call "Fano cycle" the difference $F - F^{-}$, this is homologous to 0 in $J(V)$. We study the normal function on the moduli space which computes the Abel-Jacobi image of the Fano cycle. By means of the related infinitesimal invariant we can prove that the primitive part of the normal function is not of torsion. As a consequence we get that, for a general V, $F - F^{-}$ is not algebraically equivalent to zero in $J(V)$ (proved also by van der Geer and Kouvidakis [GK] with different methods) and, moreover, that there is no divisor in JV containing both F and F^{-} and such that these surfaces are homologically equivalent in the divisor.

Our study of the infinitesimal variation of Hodge structure for V produces intrinsically a threefold $\Xi(V)$ in the Grasmannian of lines $\mathbb{G}$ in $\mathbb{P}^4$. We show that the infinitesimal invariant at V attached to the normal function gives a section of a natural bundle on $\Xi(V)$ and more specifically that this section vanishes exactly on $\Xi \cap F$, which turns out to be the curve in F parameterizing the "double lines" in the threefold. We prove that this curve reconstructs V and hence we get a Torelli-like result: the infinitesimal invariant for the Fano cycle determines V.

Résumé. La surface de Fano F des droites d'un threefold cubique V est naturellement plongée dans la variété Jacobienne intermédiaire $J(V)$, on va appeller "cycle de Fano" la différence $F - F^{-}$, ce cycle est homologue à 0 dans $J(V)$. On étudie l'application normale, définie sur l'éspace des modules, qui compute l'image d'Abel-Jacobi du cycle de Fano. Au moyen du correspondant invariant infinitésimal on prouve que la partie primitive de l'application normale n'est pas de torsion. Par conséquent on a que, si V est générale, $F - F^{-}$ n'est pas algébriquement équivalent à 0 dans $J(V)$ (ce qui a été également prouvé par van der Geer et Kouvidakis [GK] avec une méthode différente); de plus, dans $J(V)$ il n'y a aucun diviseur qui contienne F ainsi que F^{-} et tel que que les deux surfaces soient homologiquement équivalentes dans le diviseur même.

Notre étude de la variation infinitésimale de la structure de Hodge pour V produit, de façon intrinsèque, un threefold $\Xi(V)$ dans la variété de Grassmann G des droites dans $\mathbb{P}^4$. On fait voir que l'invariant infinitésimal en V attaché à l'application normale donne une section du fibré naturel sur $\Xi(V)$; plus précisément, cette section s'annule exactement sur $\Xi(V) \cap F$, ce qui on découvre être la courbe de F qui paramètre les "doubles droites" dans le threefold. On prouve que cette courbe reconstruit V et on obtient, en conclusion, un théorème de type Torelli: l'invariant infinitésimal du cycle de Fano détermine V.

Key words: Algebraic cycle, Fano surface, Intermediate Jacobian, normal function.

1. Introduction

We present some considerations on the classical theme of cubic threefolds [CG] and specifically on geometric properties encoded in the variations of their Hodge structure, [CGGH] and [V3].

The Fano surface F of lines in the cubic threefold V is naturally embedded in the intermediate Jacobian JV. We define the "Fano cycle" to be the difference $F - F^{-}$, which is homologous to 0 in JV. Our main interest is the study of the normal function ν_F on the moduli space of

J.C. Naranjo has been partially supported by the Proyecto de Investigación MTM2009-14163-C02-01.G.P. Pirola has been partially supported by Gnsasa and by MIUR PRIN 2009: Moduli, strutture geometriche e loro applicazioni;

cubic threefolds associated with the Abel-Jacobi class of the Fano cycle, we call it the Fano normal function. By means of a somewhat detailed analysis of the related infinitesimal invariant $\delta_\nu(F)$ we can prove that the primitive part of ν_F is not of torsion. Our result is stronger, it yields further the fact that for V generic there is no map $\mu : W \to JV$ from a smooth variety W of dimension 4, for which there is a cycle Z homologous to 0 in W with $\mu_* Z$ being Abel-Jacobi equivalent to the Fano cycle. We have the consequence that in this case $F - F^-$ is not algebraically equivalent to zero on JV.

More precisely, given a family of Fano surfaces $\mathcal{F} \to B$ parameterized by an open set B of the moduli space of cubic threefolds with a section s, we can define the cycle $\mathcal{F} - \mathcal{F}^-$ in the corresponding family of intermediate Jacobians of the cubic threefolds $\mathcal{J} \to B$. This provides a normal function ν_F which, via the decomposition in primitive cohomology, can be written as $\nu_F = \nu^1 + \nu^3 + \nu^5$ (see §5 for the details). The pieces ν^3, ν^5 are independent of the choice of the section s. We prove (see Proposition 5.4, Theorem 5.11, and Theorem 5.12):

Theorem 1.1. *The following holds:*

- *a) The primitive normal function ν^5 is not of torsion, moreover $\nu^3 = 0$.*
- *b) For a general cubic threefold, F is not algebraically equivalent to F^-.*
- *c) For a general cubic threefold V there is no divisor W on JV which contains F and F^- and is such that $F - F^- \sim_{hom} 0$ in W.*

Part b) is not new. By studing the behaviour of the Fourier-Mukai transform in a rank one degeneration of Abelian varieties, van der Geer and Kouvidakis (see [GK], section 8) proved this statement.

Following Griffiths' lead, see (6.32) of [G3], we show how to reconstruct V from $\delta_\nu(F)$. We came to our proof motivated by the ideas of [CP]; there it was proved, for curves of genus 3, that the infinitesimal invariant $\delta_\nu(C)$ gives back the curve C (recall that the normal function ν_C computes the primitive Abel-Jacobi class of the cycle $C - C^-$ in $J(C)$). In the spirit of [CG] we are able to relate the infinitesimal behaviour of the Fano normal function with the projective geometry of the cubic threefold in $\mathbb{P}^4$. To be concrete, our study of the infinitesimal variations of Hodge structure for V produces intrinsically a threefold $\Sigma(V)$ in the Grassmannian of lines in $\mathbb{P}^4$. These lines appear already in [CG] under the name of *lines of second type*. We show that $\delta_\nu(V)$ yields a section of a natural bundle on $\Sigma(V)$ and more specifically that this section vanishes exactly on $\Sigma(V) \cap F$, which is the curve parameterizing the double lines in V. We notice that the curve of double lines reconstructs V and rephrase this last part of our work as a Torelli-like result (see Theorem 6.5):

Theorem 1.2. *The infinitesimal invariant for the Fano cycle determines V.*

Normal functions have been extensively studied, of special relevance to us here is Nori's work, [N, Section 7.5]. He proved that for $g \geq 4$ there are no interesting normal functions over $\mathcal{A}_g$, the moduli space of Abelian varieties of genus g. On the other hand N. Fakhruddin [Fa], using Nori's connectivity theorem, showed that nevertheless the Griffiths groups of codimension 3 and 4 are of infinite rank on the generic Abelian variety of dimension 5, although they are in the kernel of the Abel-Jacobi mapping. By the Torelli theorem in [CG], the moduli space of cubic threefolds embeds in $\mathcal{A}_5$. Keeping in mind this framework then our result about the normal function might be interpreted as the statement that intermediate jacobians of cubic threefolds lie inside a certain Noether-Lefschetz-like sub-locus of $\mathcal{A}_5$.

Turning instead to the case of Jacobians of curves, consider the symmetric product of a curve C, and specifically its image, the surface $W_2 \subset J(C)$. The normal function associated with the *primitive* Abel-Jacobi class of $W_2 - W_2^-$ is trivial. This statement, which is also proved in the remark (5.14) below, follows from the general results of R. Hain [HL], who proved that over

$\mathcal{M}_g$ all interesting normal function come from the normal function ν_C of the cycle $C - C^-$ in $J(C)$. Now Allcock-Carlson-Toledo (see [ACT]) defined a period map for cubic threefolds which takes values in a ball quotient of dimension 10. A theorem of C. Voisin [V1] implies that this is an open embedding. We think that our results provide some support to the possibility that the structure of the group of normal functions over the moduli space of cubic threefolds should be a cyclic group, as it is the case for the Jacobian situation. We point out that our work shows that ν_F is certainly a truly distinguished element of this group.

The structure of the paper is as follows: in section §3 we give some computations in the Jacobian ring associated with the equation of a cubic threefold. In particular we study the infinitesimal deformations of rank 2 and we characterize them in terms of lines in $\mathbb{P}^4$. Next in section §4 we use the detailed analysis of the differential forms on F given in [CG] and [T] to compute the adjoint class form introduced in [CP] and [PZ]. This is the basic tool we employ in section §5 to compute the Griffiths' infinitesimal invariant of the Fano normal function. The information we get allows us to prove the results summarized in Theorem 1.1 above. In section §6 we do a careful analysis of the geometry of the curve of double lines needed to prove our Torelli-like Theorem 1.2.

Acknowledgments. We would like to thank the referee for many helpful suggestions and careful reading of the manuscript.

2. Notation

- We work over $\mathbb{C}$, the field of the complex numbers,
- V is a smooth cubic threefold in $\mathbb{P}^4$ of equation $E \in S := \mathbb{C}[z_0, \dots, z_4]$,
- JV the intermediate Jacobian of V endowed with the natural principal polarization $\Theta := \Theta_{JV}$ (see [CG], §3),
- $\mathbb{G} := Grass(2,5)$ is the Grassmann variety of lines in $\mathbb{P}^4$. The standard Plücker coordinates $p_{i,j}$ give the embedding $\mathbb{G} \subset \mathbb{P}^9$,
- The hyperplane $p_{i,j} = 0$ cuts on G the Schubert divisor $D_{\pi(i,j)}$, which is the locus of elements $r \in \mathbb{G}$ such that the corresponding line $l_r \subset \mathbb{P}^4$ intersects the fixed plane $\pi(i,j)$ of equation $z_i = 0 = z_j$,
- $F \subset \mathbb{G}$ is the Fano surface of lines contained in V,
- A line $r \in F$ is called *double* if there exists a 2-plane intersecting V in $2l_r + l_s$. Double lines define a curve $\mathcal{D}$ in F.

3. Infinitesimal variations of Hodge structures for cubic threefolds

We are concerned here with infinitesimal variations of Hodge structures for the cubic threefold V and for related objects, namely the intermediate Jacobian JV and the Fano surface F of lines on V. Everything turns around the linear map $\delta : T \to Hom(H^{2,1}(V), H^{1,2}(V))$ where T is the tangent space at $[V]$ to the moduli space of cubic threefolds. This is understood and controlled by means of polynomial computations because of Griffiths' results on the Hodge filtration for hypersurfaces, [G2] and [V3], §6.

We briefly recall the basic facts we need for the smooth cubic threefold V in $\mathbb{P}^4$. Denote by $J \subset \mathbb{C}[z_0, \dots, z_4] = S = \oplus_{i \geq 0} S^i$ the Jacobian ideal generated by the partial derivatives $\frac{\partial E}{\partial z_i} \in S^2$. The Jacobian ring is the graded ring $R := S/J = \oplus_i R^i$.

One has $\dim R^0 = \dim R^5 = 1$, $\dim R^1 = \dim R^4 = 5$, $\dim R^2 = \dim R^3 = 10$ and the others are zero. Moreover the product of polynomials induces perfect pairings

$$R^i \otimes R^{5-i} \longrightarrow R^5 = \mathbb{C}, \quad i = 0, \dots, 5.$$

Observe that $R^1 = S^1$ is the vector space of equations of hyperplanes on $\mathbb{P}^4$, in other words $\mathbb{P}^4 = \mathbb{P}(R^{1*}) \cong \mathbb{P}(R^4)$. We remark that the tangent space T to the moduli of cubic threefolds is identified as follows:

$$T := H^1(V, T_V) \cong R^3.$$

The canonical isomorphisms of R^i with the pieces of the Hodge decomposition of V are:

$$H^{a,3-a}(V) \cong R^{7-3a}, \quad \forall a.$$

namely:

$$H^{3,0}(V) \cong R^{-2} = 0\,, \quad H^{2,1}(V) \cong R^1\,, \quad H^{1,2}(V) \cong R^4\,, \quad H^{0,3}(V) \cong R^7 = 0.$$

All the cup-product maps correspond via these isomorphisms to products of classes of polynomials. For example the cup product map $H^1(V, \Omega^2_V) \otimes H^1(V, T_V) \xrightarrow{\cup} H^2(V, \Omega^1_V)$ can be seen as the multiplication $R^1 \otimes R^3 \longrightarrow R^4$.

The beautiful results of [CG] show how the Hodge decomposition of V is linked with that of F via the correspondence given by the universal family of lines $\mathcal{L} \subset F \times V$. In particular, there are isomorphisms $H^{2,1}(V) \cong H^{1,0}(F)$ and $H^{1,2}(V) \cong H^{0,1}(F)$. Even more, there is an isomorphism of principally polarized Abelian varieties $JV \cong Alb(F)$. Clemens-Griffiths and Tjurin (cf. [T]) completed Fano's work. In particular, concerning the canonical system of F, they gave modern proofs of the isomorphisms $H^0(F, \Omega^2_F) \cong \Lambda^2 H^0(F, \Omega^1_F) \cong \Lambda^2 R^1$. Furthermore the space of first order deformations of V is the same as that of F, i.e. $H^1(V, T_V) \cong H^1(F, T_F) \cong R^3$.

In order to apply the tool of adjoint forms in a later section we need to have at our disposal elements in $R^1 = H^0(F, \Omega^1_F)$ which are orthogonal to a first order deformation, i.e. to some $\xi \in R^3$. Associated with $\delta : T \to Hom(H^{2,1}(V), H^{1,2}(V))$, i.e. $\delta : R^3 \to Hom(R^1, R^4)$, there are determinantal varieties

$$\Xi_j := \{\xi \in \mathbb{P}R^3 \,|\, rank(\delta(\xi)) \leq j\} \subset \mathbb{P}R^3. \tag{1}$$

Now we have $\Xi_0 = \Xi_1 = \emptyset$, indeed:

Lemma 3.1. *If $0 \neq \xi \in R^3$, then $rank(\delta(\xi)) \geq 2$.*

Proof. Let $K_j(\xi) := \ker(R^j \xrightarrow{\cdot \xi} R^{j+3})$. We want to see that the dimension of $K_1(\xi)$ is ≤ 3. We have $\dim K_2(\xi) = 9$, since $R^2 \otimes R^3 \longrightarrow R^5$ is a perfect pairing and $\xi \neq 0$. Observe that $\dim K_1(\xi) = 5$ is impossible, as $\xi \neq 0$. Recall next that S^2, the vector space of the equations of the quadrics in $\mathbb{P}^4$, has dimension 15. If $\dim K_1(\xi) = 4$, then $K_1(\xi) \cdot S^1 \subset S^2$ is the family of the quadrics vanishing on a point, the base locus of $K_1(\xi)$, and therefore $\dim K_1(\xi) \cdot S^1 = 14$. Since $K_1(\xi) \cdot S^1$ is contained in the hyperplane in S^2 pull back of $K_2(\xi) \subset R^2$, then we see that $K_1(\xi) \cdot S^1$ must contain J_2. In this case the base point should be singular for V, because the partial derivatives vanish on it. This is a contradiction since V is smooth. □

The next natural goal is to determine Ξ_2, i.e. to understand for which $\xi \in R^3$ the dimension of $K_1(\xi)$ is exactly 3. Our answer is in terms of the associated line, cut by the corresponding system of hyperplanes in $\mathbb{P}^4$. Consider the grassmannian $\mathbb{G}' := Grass(3, R^1)$ of 3-dimensional subspaces of R^1 which is isomorphic to the Grasmannian of lines $\mathbb{G} = Grass(2, R^4)$:

$$\begin{aligned} &\mathbb{G} \longrightarrow \mathbb{G}' \\ &r \mapsto W(r) = \{\text{hyperplanes containing the line } l_r\}. \end{aligned}$$

We define

$$\Gamma = \{(r, [\xi]) \in \mathbb{G} \times \mathbb{P}R^3 \,|\, \eta \cdot \xi = 0 \quad \forall \eta \in W(r)\} = \{(r, [\xi]) \in \mathbb{G} \times \mathbb{P}R^3 \,|\, K_1(\xi) = W(r)\}.$$

We write π_1, π_2 the projections of Γ into the factors, thus $\Xi_2 = \pi_2(\Gamma)$ and we set $\Sigma := \pi_1(\Gamma) \subset \mathbb{G}$.

Let $r \in \mathbb{G}$ be an element of the Grassmannian and let l_r be the corresponding line in $\mathbb{P}^4$. Using Macaulay duality and recalling that $W(r) \cdot S^1 \subset S^2$ is the space of quadrics containing l_r (hence of dimension 12), one can easily check the following equivalences:

$$
\begin{aligned}
& r \in \Sigma \iff W(r) \otimes R^1 \to R^2 \text{ is not surjective} \\
& \text{(because the image is contained in the hyperplane } K_2(\xi)) \\
& \iff W(r) \cdot S^1 + J_2 \subsetneq S^2 \iff \dim(W(r) \cdot S^1) \cap J_2 \geq 3,
\end{aligned}
$$

which is the case if and only if the restriction of J_2 to l_r has dimension at most 2. Notice that $\dim(W(r) \cdot S^1) \cap J_2 \geq 4$, is impossible. Indeed in this case the restriction of J_2 to l_r would be of dimension 1, and therefore the zero locus of the ideal $J = (\partial E / \partial z_i)$ would not be empty, hence V would be singular. We have then proved the following:

Lemma 3.2. *Given a line $r \in \mathbb{G}$ the following are equivalent:*

a) $r \in \Sigma$.
b) $W(r) \cdot R^1$ *is a hyperplane in* R^2.
c) $\dim(W(r) \cdot S^1) \cap J_2 = 3$.
d) $\dim(J_2|_{l_r}) = 2$.

Remark 3.3. The lines we have found are classical objects of study, called *special* by Fano and *lines of second type* by [CG]. Since we need to appeal to this property later, we recall here that a line $l \subset \mathbb{P}^4$ is of second type exactly if the dual map

$$
\begin{aligned}
d : \mathbb{P}^4 &\longrightarrow \quad \mathbb{P}^{4\,*} \\
p &\mapsto \left(\frac{\partial E}{\partial z_0}(p) : \cdots : \frac{\partial E}{\partial z_4}(p) \right)
\end{aligned}
$$

maps l to a line l' and $l \longrightarrow l'$ has degree 2. The base locus of the corresponding dual pencil is then a plane.

Next Lemma deals with the geometry of Σ:

Lemma 3.4. *One has:*

a) *The projection* $\pi_1 : \Gamma \to \Sigma$ *is bijective.*
b) $\dim \Gamma = \dim \Sigma = 3$.

Proof. To prove a) we appeal to Lemma (3.2): $r \in \Sigma$ if and only if $W(r) \cdot R^1$ is a hyperplane in R^2. Then there is a unique element, up to constant, $\xi \in (R^2)^* \cong R^3$ corresponding to this hyperplane and by construction $W(r) = K_1(\xi)$.

To show b) we consider the map $\rho : R^3 \to Sym^2 R^4$ induced by the product $R^1 \otimes R^3 \to R^4$ and the isomorphism $R^1 \cong (R^4)^*$. By definition

$$[\xi] \in \Xi_2 \iff rk(\rho(\xi)) \leq 2.$$

In the space of quadrics in $\mathbb{P}^4$, i.e. in $\mathbb{P}(Sym^2 R^4) = \mathbb{P}^{14}$, the variety of the quadrics of rank 2 is of codimension 6, being the space of couples of hyperplanes. Since ρ is injective then we have $\dim \Xi_2 \geq 3$. Recalling that in the Segre product $\mathbb{P}^{4*} \times \mathbb{P}^{4*}$ every effective cycle of dimension 4 intersects the diagonal, we see that if $\dim \Xi_2 \geq 4$, then the variety $\rho(\Xi_2) \subset \mathbb{P}(Sym^2 R^4)$ would intersect the image of the diagonal which corresponds to the locus of rank one quadrics. This contradicts the fact that $rank(\rho(\xi)) > 1$ (Lemma (3.1)). We finally notice that $\pi_2 : \Gamma \to \Xi$ is an isomorphism, therefore the result follows. □

The lines from Σ which are contained in V can be geometrically characterized by their tangency property, see [CG] (6.7) :

Lemma 3.5. *A line $r \in F \cap \Sigma$ if and only if there is a plane tangent to V along l_r, which then we call a double line for V.*

Now we discuss some particular planes contained in Σ. Consider an Eckardt point $p \in V$. This means that through p there is a curve of lines on V, and more precisely the tangent hyperplane $T_p(V)$ cuts V in a cubic cone, with smooth base E_p. These points do not appear on a general cubic threefold. If we take $z_4 = 0$ to be the equation of T_pV, and $p = (1 : 0 : \cdots : 0)$, then the equation for V is of type $K(z_1, z_2, z_3) + z_4Q(z_0, z_1, z_2, z_3, z_4) = 0$. Given a line l such that $p \in l \subset T_pV$ one can assume that l has equations $z_2 = z_3 = z_4 = 0$. In this way it is easy to see that $l \in \Sigma$, and therefore writing π_p for the plane of such lines we have $\pi_p \subset \Sigma \subset \mathbb{G}$ and the intersection $F \cap \pi_p = E_p$.

3.1. **Geometry of the differential forms on F.** In [CG] we find the following important properties of the Fano surface $F \subset \mathbb{G} \subset \mathbb{P}^9$ which are crucial tools for our work.

Theorem 3.6. *(Clemens-Griffiths).*

a) There is a canonical isomorphic between the Albanese variety $Alb(F)$ and the intermediate Jacobian JV of V.
b) The Albanese map $a : F \to JV = Alb(F)$ coincides with the Abel-Jacobi on V and it is an embedding;
c) (Tangent bundle theorem) T_F is the restriction to F of the tautological rank 2 bundle on the Grassmannian $\mathbb{G}$. The Gauss map g attached to a is the tautological embedding $F \hookrightarrow \mathbb{G}$.
d) The canonical embedding $F \hookrightarrow \mathbb{P}^9$ is the composition of g with the Plücker embedding $\mathbb{G} \hookrightarrow \mathbb{P}^9$.

Consider $H \in R^1 \setminus \{0\}$ and the corresponding hyperplane in $\mathbb{P}^4$, and let ω_H be the corresponding holomorphic 1-form on F. The zero locus of ω_H is the set of lines of F contained in H. When H is not tangent to V these are the 27 lines on the cubic surface, this gives $c_2(F) = 27$.

Definition 3.7. Given a plane $\pi = H_1 \cap H_2 \subset \mathbb{P}^4$ we write

$$\Omega_\pi = \omega_{H_1} \wedge \omega_{H_2} \in H^0(F, K_F),$$

which is determined up to a constant.

We denote by $Z(\pi)$ the canonical divisor on F corresponding to Ω_π. Then $Z(\pi)$ is the locus of the lines which intersect π, because it is the restriction to F of the Schubert divisor associated with π.

Following [CG] we define $C_t \subset F$ to be the closure of the set of lines in V which intersect a different fixed line $l_t \subset V$. Then $t \in C_t$ if and only if t is a double line.

Definition 3.8. Three lines $l_{t_1}, l_{t_2}, l_{t_3}$ of V form a *triangle* if they are cut on V by a plane P :

$$V \cdot P = l_{t_1} + l_{t_2} + l_{t_3}.$$

A triangle is called *rare* if the lines of the triangle are incident at the same point.

Remark 3.9. Employing classical terminology [Do] we see that the parabolic points on V coincide with the vertices of rare triangles. The locus of parabolic points is the intersection of V with its Hessian hypersurface, which is of degree 5, and thus a line on V contains 5 vertices of rare triangles, counted with multiplicities. For a parabolic point P there are either 2 or 1 or ∞ rare triangles with vertex P. The first case is when P is a biplanar point. The second if it is uniplanar, as a singular element of the cubic surface cut on V by its tangent hyperplane. The last case goes under the name of Eckardt point. It happens when the tangent section is a cubic cone.

Using the definition of a triangle we can state from [F]:

Theorem 3.10. *(Fano). The following hold:* $C_t^2 = 5$, $p_a(C_t) = 11$ *and for any triangle* $l_{t_1}, l_{t_2}, l_{t_3}$ *the cycle* $C_{t_1} + C_{t_2} + C_{t_3}$ *is a canonical divisor on* F*, so that numerically* $K_F = 3C_t$.

Rare triangles are useful for the proof of the following:

Proposition 3.11. *Let* $r \in R$. *The curve* C_r *is contained in* $Z(\pi)$ *if and only if* $l_r \subset \pi$.

Proof. If $l_r \subset \pi$ it follows immediately that $C_r \subset Z(\pi)$. Now assume that $C_r \subset Z(\pi)$, so that any line $l_t \neq l_r$ of V that intersects l_r intersects also π. First we show that $l_r \cap \pi \neq \emptyset$ which we prove by contradiction. We take any triangle l_r, l_{r_1}, l_{r_2} of which l_r takes part, and we proceed to show first that the vertex $l_{r_1} \cap l_{r_2}$ is in π. Otherwise consider then $P = l_r \vee l_{r_1} \vee l_{r_2}$, the plane of the triangle. By assumption $l_{r_1} \cap \pi \neq \emptyset \neq l_{r_2} \cap \pi$. If the two points of intersection are different then they span a line in $\pi \cap P$, which intersects l_r, and thus $l_r \cap \pi \neq \emptyset$. So π contains all the vertices of the triangles along l_r. Taking now a rare triangle with vertex on l_r, we see that this point belongs to $\pi \cap l_r$.
Having proved that $l_r \cap \pi \neq \emptyset$ we show next that $l_r \subset \pi$. If not, the linear span $H = \pi \vee l_r$ is a hyperplane which satisfies that the general point $s \in C_r$ has the property that $l_s \subset H$, and then this is true in fact for all the points in C_r. On the other hand it is obvious that there are lines on V which intersects l_r but do not lie in the fixed hyperplane H. □

Definition 3.12. Given a line l_s in $\mathbb{P}^4$ we consider the locus of lines in V which intersect it:

$$D_s := \{r \in F \mid l_r \cap l_s \neq \emptyset\} \subset F.$$

Clearly

$$D_s = \{r \in F \mid \Omega_\Pi(r) = 0 \text{ for all 2- planes } \Pi \supset l_s\}.$$

More precisely, D_s is the subscheme cut on F by the net (two dimensional linear system) of Schubert hyperplanes associated with the net of 2-planes which support l_r.

We have

Corollary 3.13. $D_s \supset C_r \iff r = s$.

Proof. Direct consequence of the proposition. □

Consider inside F the set W_p of lines on V which pass through a point p. Out of a finite number of points (which are actually the Eckardt points) this has a natural structure of a finite scheme of length 6. For an Eckardt point, W_p parameterizes the lines in a cone of vertex p and basis the plane smooth cubic E_p. In this situation we shall identify W_p with E_p.

Proposition 3.14. *For* $s \in \Sigma - F$*, one has:*

- *a) If* l_s *does not contain any Eckardt point then the base locus* D_s *is of dimension* 0.
- *b) If* l_s *contains some Eckardt points* $\{p_i\}$ *then* D_s *contains the cubic curves* E_{p_i} *and possibly a residual finite set* Z *.*

Hence, for any element $s \in \Sigma - F$*, the scheme* D_s *is not of dimension zero only when* l_s *meets an Eckardt point.*

4. The method of adjoint forms on the Fano surface

The main ingredient which we employ to compute the infinitesimal invariant of our normal function is the tool of the *adjoint form* due to [PZ]. We begin this section by recalling the relevant definitions and the basic results. Next we derive a sharp result on the properties of the adjoint form in the setting of the Fano surface, which will turn out to be of convenient use in later developments.

4.1. **The adjoint form and the adjoint Theorem.** Let X be a projective smooth surface and let

$$X_\varepsilon \longrightarrow Spec\,\mathbb{C}[\varepsilon]/(\varepsilon^2)$$

be an infinitesimal deformation of the variety X with class $\xi \in H^1(X, T_X)$. The coboundary map $\partial_\xi : H^0(X, \Omega^1_X) \longrightarrow H^1(X, \mathcal{O}_X)$ attached to the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \Omega^1_{X_\varepsilon|X} \longrightarrow \Omega^1_X \longrightarrow 0, \tag{2}$$

corresponds to cupping with ξ. Therefore a global form $\eta \in H^0(X, \Omega^1_X)$ lifts to $\tilde{\eta} \in H^0(X, \Omega^1_{X_\varepsilon|X})$ if and only if $\partial_\xi(\eta) = \xi \cup \eta = 0$.

Assume that there exist 3 global 1-forms on X, η_1, η_2, η_3 such that $\xi \cup \eta_i = 0$ for all i and choose $\tilde{\eta}_i$ some liftings to X_ε. Put $W = \langle \eta_1, \eta_2, \eta_3 \rangle \subset H^0(X, \Omega^1_X)$.

Then $\tilde{\eta}_1 \wedge \tilde{\eta}_2 \wedge \tilde{\eta}_3$ belongs to $\Lambda^3 H^0(X, \Omega^1_{X_\varepsilon|X})$. By taking determinants in the exact sequence (2), one gets that $\Lambda^3 \Omega^1_{X_\varepsilon|X} \cong \omega_X$ and therefore we can define $\omega_{\eta_1,\eta_2,\eta_3,\xi}$ to be the image of $\tilde{\eta}_1 \wedge \tilde{\eta}_2 \wedge \tilde{\eta}_3$ by the natural map:

$$\Lambda^3 H^0(X, \Omega^1_{X_\varepsilon}) \longrightarrow H^0(X, \Lambda^3 \Omega^1_{X_\varepsilon}) \cong H^0(X, \omega_X).$$

We say that this is an *adjoint form* attached to η_i and ξ. It is not hard to see that this form is well-defined up to adding elements of $W^2 := \psi(\Lambda^2 W) \subset H^0(X, \omega_X)$, where

$$\psi : \Lambda^2 H^0(X, \Omega^1_X) \longrightarrow H^0(X, \omega_X).$$

So we can consider the *adjoint class*

$$[\omega_{\eta_1,\eta_2,\eta_3,\xi}] \in H^0(X, \omega_X)/W^2.$$

Finally let B be the fixed base divisor of the linear system $|\,W^2\,| \subset |\,\omega_X\,|$ and let Z be the scheme which is the base locus of the moving part of the system. We recall the basic result on the adjoint class:

Theorem 4.1. *([PZ], 1.5.1. and 3.1.5). Assume that $W^2 \neq 0$. Then:*

a) $[\omega_{\eta_1,\eta_2,\eta_3,\xi}] \in j(H^0(X, \omega_X(-B) \otimes_{\mathcal{O}_X} \mathcal{I}_Z))/W^2$ where

$$j : H^0(X, \omega_X(-B) \otimes_{\mathcal{O}_X} \mathcal{I}_Z) \to H^0(X, \omega_X)$$

is the natural inclusion.

b) If $[\omega_{\eta_1,\eta_2,\eta_3,\xi}] = 0$, then ξ belongs to the kernel of

$$H^1(X, T_X) \longrightarrow H^1(X, T_X(B)).$$

Part b) is a tool to check non-vanishing. Indeed, if $|\,W^2\,|$ is a non empty linear system without base divisor, then $[\omega_{\eta_1,\eta_2,\eta_3,\xi}] \neq 0$.

It is possible and sometimes useful to choose a special representative of the adjoint class; this can be done by the requirement that it is orthogonal to W^2.

Definition 4.2. The standard representative of the class $[\omega_{\eta_1,\eta_2,\eta_3,\xi}]$ is the only form in the given class which is orthogonal to W^2. We write it simply $\omega_{\eta_1,\eta_2,\eta_3,\xi}$, with the understanding that it satisfies the property:

$$\int_X \omega_{\eta_1,\eta_2,\eta_3,\xi} \wedge \overline{\beta_1 \wedge \beta_2} = 0 \tag{3}$$

for all β_1 and β_2 are in W. Clearly $[\omega_{\eta_1,\eta_2,\eta_3,\xi}] = 0 \iff \omega_{\eta_1,\eta_2,\eta_3,\xi} = 0$

4.2. **The adjoint class for the special lines.** We explain how the preceding machinery works in case of the Fano surface F and for a chosen deformation ξ associated with a line r belonging to Σ. In our situation the Hodge pieces $H^{a,b}(F)$ of interest to us can be written in terms of the Jacobian ring (section §3), in particular the space of canonical forms on F is identified with $\wedge^2 R^1$. Having chosen $r \in \Sigma$ we consider next a basis η_1, η_2, η_3 of $W(r)$. The element $\xi \in R^3$, unique up to constant, is an infinitesimal variation with kernel $K_1(\xi) = W(r)$. We set

$$W(r)^2 := \langle \eta_1 \wedge \eta_2, \eta_2 \wedge \eta_3, \eta_3 \wedge \eta_1 \rangle \subset \wedge^2 R^1 \ .$$

Observe that the base locus of the linear system $|W(r)^2|$ is the sublocus D_r defined in (3.12). The goal of this section is to understand for which lines $r \in \Sigma$ the adjoint class $[\omega_{\eta_1,\eta_2,\eta_3,\xi}]$ is zero. In this case we say that "the class vanishes on the line". The vanishing does not depend on the choice of the basis we have just made.

Remark 4.3. Consider the universal rank 3 subbundle on $Grass(3, R^1) =: \mathbb{G}' \simeq \mathbb{G}$, we call $\mathcal{U}$ its restriction to Σ and then we lift it to Γ: $\mathcal{U}_\Gamma := \pi_2^*(\mathcal{U})$. There is a natural map of sheaves on Γ:

$$\Lambda^2 \mathcal{U}_\Gamma \longrightarrow H^0(F, \omega_F) \otimes \mathcal{O}_\Gamma = \Lambda^2 R^1 \otimes \mathcal{O}_\Gamma.$$

Let $\mathcal{E}$ be the cokernel of this map. The adjoint class map amounts to an arrow:

$$\alpha : \Lambda^3 \mathcal{U}_\Gamma \otimes \pi_1^* \mathcal{O}_{\mathbb{P}(R^3)}(-1) \longrightarrow \mathcal{E}.$$

More precisely, the adjoint class is a global section of the sheaf $Im(\alpha)$ supported on Γ. We use the bijection $\Gamma \longrightarrow \Sigma$ to think of the preceding construction as taking place on Σ. We deal next with the vanishing locus of α.

The main result of this section is:

Theorem 4.4. *The adjoint class map vanishes exactly on $\Sigma \cap F$, i.e. on the locus of double lines on V.*

Proof. We divide the proof into three parts.
First case: We assume that $r \in F \cap \Sigma$. We denote $\Omega_{ij} = \eta_i \wedge \eta_j$. This is the basis of $W(r)^2$ given above. By Corollary (3.13), the three corresponding canonical divisors contain the curve C_r. Hence:

$$W(r)^2 :=< \Omega_{12}, \Omega_{23}, \Omega_{31} >\subset H^0(F, \omega_F(-C_r)). \tag{4}$$

Now we use the following result of Tjurin:

Proposition 4.5. *(cf. [T], Corollary (2.2)). For any $r \in F$, $h^0(F, \omega_F(-C_r)) = 3$ and the linear system $|\omega_F(-C_r)|$ has no base divisor.*

Now $C_r \subset D_r$, and then $\omega_{\eta_1,\eta_2,\eta_3,\xi} \in H^0(\omega_F - (C_r))$. But $H^0(\omega_F(-C_r)) = W(r)^2$ by the previous proposition. Therefore we have the requested vanishing.
Second case: We assume that $r \in \Sigma - F$ and we require that $l_r \cap V$ contains no Eckardt points. By Proposition (3.14) the linear system $|W(r)^2|$ has no base divisor and then from Theorem (4.1), b) we see that the adjoint class $[\omega_{\eta_1,\eta_2,\eta_3,\xi}]$ is non trivial in this situation.
Third case: Finally we assume that $r \in \Sigma - F$ and $l_r \cap V$ is a finite set, which contains some Eckardt points, to be called p_i. Consider the elliptic curves $E_i \subset F$, which parametrize the lines through our points p_i. Clearly $E_i \cap E_j = \emptyset$, because otherwise $l_r \cap V = l_r$. We need the following result.

Lemma 4.6. *In this case the base divisor B of the linear system $W(r)^2$ is reduced and supported on the union of the curves E_i's.*

Proof. Consider a line $l_r \not\subset V$ which we assume to intersect V in an Eckardt point. Our aim here is to prove the statement that the linear system $W(r)^2$ has reduced base divisor along the Eckardt curve $E \subset F$. In order to prove reducedness it is enough to find a point $s \in E$ where the tangent plane $T_s(F)$ is not contained in one of the hyperplanes which cut on F our linear system $W(r)^2$. Those are the Schubert hyperplanes which give the condition of non empty intersection with a plane containing the line l_r. If the plane is $x_1 = x_3 = 0$, then the Schubert hyperplane has equation $p_{1,3} = 0$. Our V has equation $x_4 Q + K(x_1, x_2, x_3) = 0$. Up to a linear change of the coordinates, we may assume that on the plane $x_4 = 0 = x_0$ the line $x_2 = 0$ is tangent to the cubic $K = 0$ at the point $[1 : 0 : 0]$. We choose our line l_s to be $(*, *, 0, 0, 0)$, then l_s is in V and it contains the Eckard point $[1 : 0 \cdots : 0]$. By setting $p_{0,1} = 1$, one can pass to an affine chart in the Grassmannian, using the other Plücker coordinates $p_{0,j}$ and $p_{1,j}$, and s is the origin. In our situation it is a matter of routine to check that the tangent plane $T_s(F)$ is given in the said chart as the coordinate plane $(p_{0,3}, p_{1,3})$, and therefore it is not contained in the Schubert hyperplane. $\square$

The next two propositions complete the proof of our theorem. Consider

$$0 \to T_F \to T_F(B) \to T_F \otimes \mathcal{O}_B(B) \to 0$$

so that one has the exact sequence:

$$\cdots \to H^0(B, T_F \otimes \mathcal{O}_B(B)) \to H^1(F, T_F) \to H^1(F, T_F(B)) \to \dots$$

Proposition 4.7. *The arrow $H^1(F, T_F) \to H^1(F, T_F(B))$ is injective, because*

$$H^0(B, T_F \otimes \mathcal{O}_B(B)) = 0.$$

We remark that $H^0(B, T_F \otimes \mathcal{O}_B(B)) = \oplus H^0(E_i, T_F \otimes \mathcal{O}_{E_i}(E_i))$. This is due to B being the disjoint union of the E_i. Then the proof of the proposition descends from:

Proposition 4.8. *For any Eckardt curve $E \subset F$*

$$H^0(E, T_F \otimes \mathcal{O}_E(E)) = 0\,.$$

Proof. We recall that T_F is the restriction to F of the universal rank 2 sub-bundle on $\mathbb{G}$. Moreover $\mathcal{O}_E(E) \simeq \mathcal{O}_E(-1)$, by the adjunction formula on F, since $\omega_F \simeq \mathcal{O}_F(1)$. Our elliptic curve E represents the lines passing through a fixed point p contained in V, and therefore they are contained in the tangent space $T_p(V)$. The lines through p and contained in $T_p(V)$ are parameterized by any plane $\pi \subset T_p(V)$ not containing p. Over our plane, seen as a subvariety of $\mathbb{G}$, there is a bundle $\mathcal{B}$ which is the restriction of the universal rank 2 subbundle. This bundle $\mathcal{B}$ comes with a subbundle $\mathcal{L}$ of rank 1, corresponding to the geometric condition of passing through p, and further it has a second subbundle, $\mathcal{G}$, which gives the intersection of the line with the plane $\pi \subset T_p(V)$. In other words there is a splitting: $\mathcal{L} \oplus \mathcal{G} \simeq \mathcal{B}$. Remember that the universal subbundle is in fact a subbundle of the trivial bundle of rank 5, and therefore both line bundles come with a non trivial morphism to the trivial bundle. Since $det(\mathcal{B}) = \mathcal{O}_{\mathbb{P}^2}(-1)$, we must have $\mathcal{G} \simeq \mathcal{O}_{\mathbb{P}^2}$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(-1)$. In conclusion: $\mathcal{O}(E, T_F \otimes \mathcal{O}_E(E)) \simeq \mathcal{O}_E(-2) \oplus \mathcal{O}_E(-1)$, hence, $H^0(E, T_F \otimes \mathcal{O}_E(E)) = 0$. $\square$

The proof of our Theorem is thus completed. $\square$

4.3. **Explicit computation of the adjoint form for a transverse line.** In this section we compute the adjoint form when the first order deformation ξ is associated with a special line l_r which intersects V in a least two different points. We begin with a cohomological computation, in the proof of which we need to appeal to a simple fact: the scheme $W_p \subset F$ which parameterizes the lines through a point $p \in V$ is in fact a planar scheme but it is not supported on any line in $\mathbb{P}^9$.

Lemma 4.9. *For any line l_r as above one has:*

$$\dim H^0(F, \omega_F \otimes \mathcal{I}_{D_r}) = 4.$$

Proof. Let A and B be two points in $V \cap l_r$. The lines through A in V are contained in the projective tangent space $T_A(V)$. Similarly for B and the remaing point C (if any) on $V \cap l_r$. Since $r \in \Sigma$, then the image of l_r under the dual map is a line (see Remark (3.3)), thus we have that $T_A(V) \cap T_B(V) \cap T_C(V)$ is a 2-plane π, the base of the dual pencil. The lines in V through A intersect π, and the same is true for the lines passing through B and for C. Hence the Schubert hyperplane associated with the base π provides a canonical divisor K_π which supports our scheme D_r. This is true for a general line r of second type, and then the same holds for every $r \in \Sigma$, by continuity. Now the assumption $l_r \not\subset V$ gives that $r \not\subset \pi$, and therefore $K_\pi \notin |W(r)^2|$, which is the system cut on F by the Schubert divisors associated with the 2-planes containing l_r. Therefore $h^0(F, \omega_F \otimes \mathcal{I}_{D_r}) \geq 4$, for $r \notin F$.

On the other hand consider the plane $\pi_A \subset \mathbb{G}$ parameterizing the lines through A and contained in the tangent hyperplane $T_A(V)$. We have recalled above that the scheme W_p parameterizing the lines through p is a subscheme of π_A, but it is not supported on any line of π_A. A hyperplane of $\mathbb{P}^9$ supporting D_r must then contain π_A and for the same reason contains π_B. Now $\pi_A \cap \pi_B = \emptyset$ and therefore the linear space they span is of dimension 5. This shows that $h^0(F, \omega_F \otimes \mathcal{I}_{D_r}) \leq 4$ hence the proof is completed. □

The useful outcome is then the property that the adjoint class takes value in a space of dimension 1 which is $H^0(F, \omega_F \otimes \mathcal{I}_{D_r})/W(r)^2$.

We can write down explicitily a generator of this space in terms of a basis in R^1: let l_r be the line $z_0 = z_1 = z_2 = 0$ and $A = (0:0:0:1:0)$, $B = (0:0:0:0:1)$, i.e. $W(r) = \langle z_0, z_1, z_2 \rangle \subset R^1 = S^1$. We can assume that $T_A(V), T_B(V)$ are given by $z_4 = 0$ and $z_3 = 0$ respectively. Identifying R^1 with the space of 1 forms on F we put ω_{i+1} for the form corresponding to the hyperplane $z_i = 0$. Then the canonical divisors attached to the planes $z_0 = z_1 = 0$, $z_0 = z_2 = 0$ and $z_1 = z_2 = 0$ containing l_r generate $W(r)^2$ and correspond to $\omega_1 \wedge \omega_2, \omega_1 \wedge \omega_3, \omega_2 \wedge \omega_3$. On the other hand the lines intersecting $T_A(V) \cap T_B(V) : z_3 = z_4 = 0$ give the $(2,0)$ form $\omega_4 \wedge \omega_5$. We get that:

Proposition 4.10. *For this deformation ξ the adjunction class $[\omega_{\omega_1,\omega_2,\omega_3,\xi}]$ is represented in*

$$H^0(F, \omega_F \otimes \mathcal{I}_{D_r})/W(r)^2$$

by $\rho\, \omega_4 \wedge \omega_5$, $\rho \neq 0$.

Remark 4.11. It is easy to see that a special line $r \notin F$ intersects V in either 3 points or else $V \cdot l_r = 3P$. Indeed if l_r of equation $z_0 = z_1 = z_2 = 0$, is simply tangent to V at the point $P := [0 : \cdots : 0 : 1]$ then we may assume that the equation for V is of type $E = Az_0 + Bz_1 + Cz_2 + z_3^2 z_4$. We have that the restriction of the quadrics $\partial E/\partial z_3$ e $\partial E/\partial z_4$ are independent on l_r. Still, they vanish on P, and since l_r is special, then it must be the case that all $\partial E/\partial z_j$ vanish on P, which then should be singular on V. One can check that the points P as above, where a special line is $3-$tangent to V, are the points on the intersection with the Hessian hypersurface associated to V. These points have the property that the tangent space cuts V in a surface with singularity either of biplanal or uniplanar type (or else P could be an Eckardt point). The special line in the biplanar situation is the line through P which is the intersection of the two components on the tangent cone, while in the uniplanar case any line of the pencil of base P is special.

It is apparent from the preceding discussion that in general the adjoint class is represented by the canonical divisor K_π corresponding to the plane π, base of the dual pencil. By continuity therefore the adjoint class for $r \in \Sigma$ should always be represented by the same type of Schubert divisor.

5. Infinitesimal invariants

First we recall briefly the definition of the infinitesimal invariant of a normal function associated with a varying family of cycles. This is a theme of Griffiths, investigated by him, M. Green and C. Voisin [G3], [Gr1], [Gr] and [V2]. Next we turn to [PZ] so to compute this invariant by means of the method of the adjunction form. Finally we can apply our preceding results to detect the properties of the normal function associated with the cycle $F - F^{-}$ in JV, and draw as a consequence the statements about the Abel-Jacobi class of this cycle.

5.1. **Definition of the infinitesimal invariant.** We state some facts, mostly for our notational convenience. A complete reference is [V3].

Let $\rho : \mathcal{X} \longrightarrow B$ be a smooth projective morphism onto a smooth base B. Let $\mathcal{H}^k$ be the sheaf $R^k\rho_*\mathbb{C} \otimes \mathcal{O}_B$ with the holomorphic subbundle $F^i\mathcal{H}^k$ determined by the Hodge filtration on each fiber $H^k(X_b, \mathbb{C})$. The $(2p-1)-th$ intermediate Jacobian of X_b is defined as:

$$J^{2p-1}(X_b) := H^{2p-1}(X_b, \mathbb{C})/(F^pH^{2p-1}(X_b, \mathbb{C}) + H^{2p-1}(X_b, \mathbb{Z})).$$

All these tori fit in a family over B. The sheaf of holomorphic sections of this family is

$$\mathcal{J}^{2p-1} := \mathcal{H}^{2p-1}/(F^p\mathcal{H}^{2p-1} + \mathcal{H}^{2p-1}_{\mathbb{Z}}),$$

where $\mathcal{H}^{2p-1}_{\mathbb{Z}}$ is the locally constant sheaf on B with fiber $H^{2p-1}(X_b, \mathbb{Z})/(\text{torsion})$ over b.

A codimension p cycle $\mathcal{Z}$ on $\mathcal{X}$ such that $\mathcal{Z}_b \equiv_{hom} 0$, $\forall b \in B$, defines a *normal function* $\nu_{\mathcal{Z}} : B \longrightarrow \mathcal{J}^{2p-1}$ sending b to the Abel-Jacobi image of $\mathcal{Z}_b$. The Gauss-Manin connection

$$\nabla : \mathcal{H}^{2p-1} \longrightarrow \mathcal{H}^{2p-1} \otimes_{\mathcal{O}_B} \Omega^1_B$$

satisfies the property that $\nabla(F^i\mathcal{H}^{2p-1}) \subset F^{i-1}\mathcal{H}^{2p-1} \otimes \Omega^1_B$ (Griffiths' transversality theorem). Hence it induces

$$\nabla^{p-1}_p : \frac{F^p\mathcal{H}^{2p-1}}{F^{p+1}\mathcal{H}^{2p-1}} \longrightarrow \left(\frac{F^{p-1}\mathcal{H}^{2p-1}}{F^p\mathcal{H}^{2p-1}}\right) \otimes_{\mathcal{O}_B} \Omega^1_B.$$

Let $\tilde{\nu}_{\mathcal{Z}} : B \to \mathcal{H}^{2p-1}$ be a a local lifting of $\nu_{\mathcal{Z}}$. The infinitesimal invariant $\delta_{\mathcal{Z}}$ is defined as the section of $Coker(\nabla^{p-1}_p) \to B$ given by the composition $\nabla \circ \tilde{\nu}_{\mathcal{Z}}$. The non-triviality of $\delta_{\mathcal{Z}}$ implies that $\nu_{\mathcal{Z}}$ is not a torsion section.

5.2. **The Fano surface in its Albanese variety.** We return to our cubic threefold $V \subset \mathbb{P}^4$ and to the Fano surface F of the lines on V. There is a universal morphism from the Albanese variety $Alb(F)$ to the intermediate Jacobian JV, due to the universal properties of the Albanese map and the use of the restriction of the Abel-Jacobi map to the $1-$cycles represented by the lines. More precisely, the two maps, which we obtain by fixing a point $r \in F$, in fact coincide [CG]:

$$a_r : F \to Alb(F) = JV.$$

Moreover in this way F is embedded in $J(V)$. By abuse of language we call $F \subset JV$ the cycle $a_r(F)$. Composing the Albanese map with the -1 involution on JV we get another embedding $a_r^- : F \to JV$ and we let F^- be the image. The -1 involution acts trivially on the even cohomology, hence we see that the cycle $F - F^-$ is homologically equivalent to zero. This is completely analogous to the Ceresa cycle, see [Ce], where it is dealt with the curve in its Jacobian. Let

$$J^5(JV) = \frac{H^{0,5}(JV) \oplus H^{1,4}(JV) \oplus H^{2,3}(JV)}{H^5(JV, \mathbb{Z})}$$

be the intermediate Jacobian associated to the fifth cohomology of JV. In dealing with the Abel-Jacobi map, AJ, it is convenient to use the equivalent definition:

$$J^5(JV) = \frac{(H^{5,0}(JV) \oplus H^{4,1}(JV) \oplus H^{3,2}(JV))^*}{H_5(JV, \mathbb{Z})}.$$

The Abel-Jacobi invariant of the Fano cycle is defined by us to be:

$$\nu_r(V) = AJ(F - F^-) = \int_{F^-}^{F} \in J^5(JV). \tag{5}$$

5.3. **The Primitive Intermediate Jacobian.** The natural principal polarization θ of JV determines the Lefschetz splitting

$$H^5(JV,\mathbb{Z}) = \theta^2 H^1(JV,\mathbb{Z}) \oplus \theta P^3(JV,\mathbb{Z}) \oplus P^5(JV,\mathbb{Z}),$$

where

$$P^3(JV,\mathbb{Z}) = \ker(\theta^3 : H^3(JV,\mathbb{Z}) \to H^9(JV,\mathbb{Z})),$$
$$P^5(JV,\mathbb{Z}) = \ker(\theta : H^5(JV,\mathbb{Z}) \to H^7(JV,\mathbb{Z}))$$

are the primitive cohomology groups.

On the complex primitive cohomology there is the Hodge decomposition:

$$P^5(JV,\mathbb{C}) = \sum_{i+j=5} P^{i,j}$$

where $P^{i,j} = \ker\theta : H^{i,j}(JV) \to H^{i+1,j+1}(JV)$. In particular $P^{5,0} = H^{5,0}(JV)$. Let $W \subset H^{1,0}(JV)$ be a $3-$ dimensional space and $L \subset H^{0,1}(JV)$ be the annihilator of W, so $\dim L = 2$. Consider the inclusion $\bigwedge^3 W \otimes H^{0,2} \subset H^{3,2}(JV)$.

Lemma 5.1. *We have the equality:*

$$\bigwedge^3 W \otimes \bigwedge^2 L = (\bigwedge^3 W \otimes H^{0,2}(JV)) \cap P^{3,2}.$$

Proof. This is a standard computation in linear algebra. We give some details for the reader's convenience. A polarization on $H^1(JV,\mathbb{C})$ amounts to an alternating form $Q(\alpha,\beta)$ such that the Hermitian form $H(\alpha,\beta) := iQ(\alpha,\bar\beta)$ is positive definite on $H^{1,0}$. We can fix an orthonormal basis $\{\omega_i\}_{i=1\dots5}$ of $H^{1,0}(JV)$ such that $W =<\omega_1,\omega_2,\omega_3>$. By construction we have that the polarization can be represented by an element of the form:

$$\Theta = -i\sum \omega_i \wedge \overline{\omega}_i.$$

Now $P^{3,2} = \ker(\theta : H^{3,2}(JV) \to H^{4,3}(JV))$, and the map is given by the cup-product with θ. An easy count yields that $\ker(\theta) \cap (\bigwedge^3 W \otimes H^{0,2})$ is generated by the decomposable form

$$\Omega_W = \omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \overline{\omega}_4 \wedge \overline{\omega}_5, \tag{6}$$

where $\{\omega_1,\omega_2,\omega_3\}$ is a basis of W and $\{\overline{\omega_4},\overline{\omega_5}\}$ a basis of L. This gives the equality. $\square$

We can define the primitive intermediate Jacobian

$$P^5(V) = \frac{H^{0,5}(JV) \oplus P^{1,4} \oplus P^{2,3}}{P^5(JV,\mathbb{Z})}$$

and similarly for $P^3(V)$. From the Lefschetz decomposition above we get then a corresponding decomposition for our intermediate Jacobian

$$J^5(JV) = \theta^2 JV \oplus \theta P^3(V) \oplus P^5(V). \tag{7}$$

Hence the definition (5) becomes:

$$\nu_r(V) = \nu^1_r(V) + \nu^3_r(V) + \nu^5_r(V). \tag{8}$$

In the Proposition 5.4 below we prove that $\nu^3_r(V) = 0$ and that $\nu^5_r(V) = \nu^5(V)$ does not depend on r. We call $\nu^5(V)$ the primitive Abel-Jacobi invariant of the cycle. To prove this Proposition we need the following result:

Proposition 5.2. *Consider in JV an algebraic cycle Z of class $[Z] = c\,\theta^p$, with $p = 3$ (resp. $p = 4$), and take a translate Z_a. Then the projection inside of $P^5(V)$ (resp. inside $\theta^2 P^3(V)$) of the Abel-Jacobi invariant of $Z_a - Z$ vanishes.*

Proof. Let $*$ be the Pontrjagin product. We first remark that

$$[Z]*H^9(JV, \mathbb{C}) \subset \theta^{p-1} H^1(JV, \mathbb{C}). \tag{9}$$

This is a consequence of the following facts, cf. [L]: (i) the hypothesis yields $[Z] = k(\theta^4)^{*\,(5-p)}$ (ii) the action on $H^*(JV, \mathbb{C})$ given by $*$ with θ^4 is a multiple of the operator Λ dual to the Lefschetz operator $L(z) = \theta \wedge z$. Then (9) holds because $H^9(JV, \mathbb{C}) = \theta^4 H^1(JV, \mathbb{C})$. Given a form ω of type $11 - 2p$ we look at the product $\mu^*[\omega] \wedge ([\alpha] \times [Z])$ where $\mu : JV \times JV \to JV$ is the sum and $[\alpha] \in H^9(JV, \mathbb{C})$.

Lemma 5.3. *The Künneth component of type $(1, 10)$ of $\mu^*[\omega] \wedge (1_{JV} \times [Z])$ vanishes if ω is primitive and $p \neq 5$.*

Proof. One has $\mu_*(\mu^*[\omega] \wedge ([\alpha] \times [Z])) = [\omega] \wedge ([\alpha] * [Z]) \overset{(9)}{=} ([\omega] \wedge \theta^{p-1}) \wedge \ldots = 0$, since $[\omega]$, being primitive, is by definition in the kernel of θ^{2p-5}. $\square$

To finish the proof of the proposition we take a path γ from the origin to a so that $Z * \gamma$ is a chain with boundary our cycle $Z_a - Z$. We show that $\int_{Z*\gamma} \omega = 0$ for any ω harmonic form representing a primitive cohomology class from $F^{6-p} H^{11-2p}$. Indeed:

$$\int_{Z*\gamma} \omega = \int_{\gamma \times Z} \mu^*(\omega),$$

which we compute by means of Fubini's theorem integrating first along Z. Then we note that the remaining integral along γ vanishes because we should integrate the first part from the Künneth component of type $(1, 10)$ of $\mu^*[\omega] \wedge (1_{JV} \times [Z])$, which is null in force of the preceding lemma. $\square$

Now we can prove:

Proposition 5.4. *For the Fano normal function one has $\nu_r^3(V) = 0$. Moreover $\nu_r^5(V)$ does not depend on the choice of the line r.*

Proof. It is enough to show $\theta \cdot \nu_r^3(V) = 0$. By [V3], proposition 9.23, this is the same thing as to check that the cycle $\Theta \cdot F - \Theta \cdot F^-$ has the property that its Abel-Jacobi invariant has null component in $\theta^2 P^3(V)$. We use

$$\Theta \cdot F - \Theta \cdot F^- = (\Theta \cdot F - 2C_t) + 2(C_t - C_t^-) + (2C_t^- - \Theta \cdot F^-),$$

where C_t is the curve in F of the lines that intersect a given line $l_t \in V$. We claim that the Abel-Jacobi invariant of the first summand belongs to $\theta^3 \cdot Pic^o(J)$, which then implies the vanishing of its second component. Indeed (i) $\Theta \cdot F$ and $2C_t$ are homologous divisors in F , (ii) the Albanese embedding of F in J induces an isomorphism of the Picard varieties and (iii) the cohomology class of F is $\theta^3/3!$. The same thing holds for the last summand. On the other hand C_t (which is a "Prym curve" for JV) is known to be invariant by $(-1)^*$ up to translation by some point $\lambda \in J$, so that $C_t^- = t_\lambda^*(C_t)$. Then Proposition 5.2 above applies to $(C_t - C_t^-)$. The statement about $\nu_r^5(V)$ is also proved using the same proposition, because a change in the base line determines a translation on the Albanese image of F. $\square$

Our next aim is to show that on a generic cubic threefold V the value $\nu^5(V)$ is not a torsion point in the corresponding intermediate Jacobian $P^5(V)$.

5.4. **The Fano Normal Function.** We deal now with a family of smooth cubic threefolds $\pi_{\mathcal{V}} : \mathcal{V} \to B$, the corresponding Fano surfaces family $\pi_{\mathcal{F}} : \mathcal{F} \to B$, and the family of their Albanese varieties $\pi_{\mathcal{A}} : \mathcal{A} \to B$ which coincides with the family of intermediate Jacobians $\pi_{\mathcal{J}} : \mathcal{J} \to B$. We then consider the family of intermediate Jacobians which are built using the fifth cohomology of the fibres of $\mathcal{J}$ (i.e. they are the intermediate Jacobians of the intermediate Jacobians):

$$\pi_{\mathcal{J}} : \mathcal{J}^5 \to B, \tag{10}$$

where the fiber over $b \in B$ is $J^5(JV_b)$.

The family of intermediate Jacobians of cubic threefolds $\pi_{\mathcal{V}}$ depends on the local system $R^3\pi_{\mathcal{V}*}\mathbb{Z} = \{H^3(V_b, \mathbb{Z})\}_{b\in B}$, while $\pi_{\mathcal{J}}$ depends on the local system $R^5\pi_{\mathcal{A}*}\mathbb{Z}$ with fibres

$$\{H^5(J(V_b), \mathbb{Z})\}_{b\in B} \simeq \{\bigwedge^5 H^3(V_b, \mathbb{Z})\}_{b\in B}.$$

From (7) we obtain then a decomposition of the fibration:

$$\mathcal{J}^5 \simeq \mathcal{J} \oplus \mathcal{P}^3 \oplus \mathcal{P}^5 \tag{11}$$

where $\pi_3 : \mathcal{P}^3 \to B$ and $\pi_5 : \mathcal{P}^5 \to B$ denote the obvious families of complex tori.

We assume for a while that we are given a section $s : B \longrightarrow \mathcal{F}$ of $\pi_{\mathcal{F}} : \mathcal{F} \to B$. Note that locally such section always exists. The section s allows us to define the family of Albanese maps $a_b : F_b \to Alb(F_b) = JV_b$ and then it allows us to construct two embeddings of families $\mathcal{F} \hookrightarrow \mathcal{J}$ and $\mathcal{F}^- \hookrightarrow \mathcal{J}$. Now $\mathcal{J}$ plays here the rôle of $\mathcal{X}$ in (5.1), the cycle $\mathcal{Z}$ being then $\mathcal{F} - \mathcal{F}^-$, with fibre the cycle $F_b - F_b^-$ which is homologically equivalent to zero. In this way the use of the Abel-Jacobi mapping provides a section ν_s of $\mathcal{J}^5$, which is a normal function according to Griffiths ([G3]). In view of Proposition (5.4) we have from (8):

$$\nu_s = \nu_s^1 + \nu^3 + \nu^5 = \nu_s^1 + \nu^5 \tag{12}$$

Definition 5.5. We call $\nu = \nu^5$ the normal function associated to the Fano cycle. Note that it does not depend on the choice of the section s.

5.5. **Forms, deformations and the computation of the infinitesimal invariant.** In [PZ] the infinitesimal invariant attached to an irregular surface in its Albanese variety is computed. We take from there that the infinitesimal invariant can be seen as a functional on $\mathrm{Ker}(\gamma)$, where

$$\gamma : T_{B,b} \otimes H^2(JV_b, \Omega^3_{JV_b}) \longrightarrow H^3(JV_b, \Omega^2_{JV_b}) \tag{13}$$

is induced as usual by the cup-product and the Kodaira-Spencer map. We will use presently the main formula [PZ, Corollary 5.2.4] (see Theorem 5.8 below). In preparation we need to present some new considerations on differential forms.

First we observe that given a deformation $\zeta \in H^1(T_F) = H^1(T_V) = R^3$ there is a natural primitive form $\Omega_\zeta \in P^{3,2}(A)$, defined up to a constant, which is given as follows:

$$\Omega_\zeta = \zeta \cdot (\zeta \cdot \Omega), \tag{14}$$

where Ω is a generator of $H^{5,0}(JV) = P^{5,0}$. Notice that Ω_ζ is primitive since $\zeta \cdot \theta = 0$.

Given a line $r \in \Sigma$ we have the corresponding first order deformation of rank 2, $\xi \in R^3$, and we write $K_1(\xi) = W(r)$ (see §3).

Lemma 5.6. *For any r and ξ as above, the form Ω_ξ is a non trivial decomposable primitive form. More precisely*

$$\langle \Omega_\xi \rangle = \langle \Omega_{W(r)} \rangle = \langle \omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \overline{\omega_4} \wedge \overline{\omega_5} \rangle,$$

where $\{\omega_1, \ldots, \omega_5\}$ is a basis of $H^{1,0}(JV)$ orhogonal with respect to the polarization and such that $\omega_1, \omega_2, \omega_3$ is a basis of $W(r)$ and $\overline{\omega_4}, \overline{\omega_5}$ is a basis of the annihilator L of $W(r)$.

Proof. We complete a basis $\omega_1, \omega_2, \omega_3$ of $W(r)$ by choosing a convenient basis for $\bar{L}$, where $L \subset H^{0,1}(F)$ is the annihilator of W. One has:

$$\Omega_\xi = \xi \cdot (\xi \cdot \omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4 \wedge \omega_5) = 2\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \xi\omega_4 \wedge \xi\omega_5.$$

Then $\Omega_\xi \in (\bigwedge^3 W \otimes H^{0,2}(JV)) \cap P^{3,2}$. From (5.1) we see that $\langle \Omega_\xi \rangle = \langle \Omega_{W(r)} \rangle$. □

Given r as above we define $\Phi_\xi := \omega_1 \wedge \omega_2 \wedge \omega_3$. Note that the distinguished primitive decomposable form that appear in Lemma (5.6) can be written

$$\Omega_\xi = \omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \overline{\omega_4} \wedge \overline{\omega_5} = \Phi_\xi \wedge \overline{\omega_4} \wedge \overline{\omega_5}.$$

By recalling that $\xi \cdot \Phi_\xi = 0$ we have:

Lemma 5.7. *Let $\gamma : H^1(T_F) \otimes H^{3,2}(JV) \to H^{2,3}(JV)$ be the map defined in (13), then for all $\beta \in H^{0,2}(JV)$*

$$\xi \otimes \Phi_\xi \wedge \beta \in \ker(\gamma).$$

Now we evaluate the infinitesimal invariant functional on these tensors, i.e. we compute

$$\delta\nu(\xi \otimes \Phi_\xi \wedge \beta).$$

As it was explained above, our normal function is associated with the codimension 3 cycle $\mathcal{Z} = \mathcal{F} - \mathcal{F}^-$ which lives in the Albanese fibration $\mathcal{J}$ over some suitable base B. Fixing a point $b \in B$ we write $F := \mathcal{F}_b$, $JV := \mathcal{J}_b$. Let then $\xi \in H^1(F, T_F)$ be the class of the infinitesimal deformation

$$F_\varepsilon \longrightarrow Spec\, \mathbb{C}[\varepsilon]/(\varepsilon^2)$$

corresponding to a local parameter t on B around b.

In the present situation the main result of [PZ] gives:

Theorem 5.8. *With the preceding notations:*

$$\delta_\nu(b)(\xi \otimes \Phi_\xi \wedge \beta) = 2 \int_F \omega_{\omega_1,\omega_2,\omega_3,\xi} \wedge a_b^*(\beta),$$

where $\omega_{\omega_1,\omega_2,\omega_3,\xi}$ is the standard representative (see (3)) of the adjoint form defined above in section 4 and a_b is the Albanese map.

Proof. [PZ](Corollary 5.2.4). □

We are interested in the following straightforward consequence of the theorems 4.1 and 5.8. We use the identifications between the Hodge components and the graded pieces of the Jacobian ring (section §3):

Proposition 5.9. *Let $r \in \Sigma$ and let $\xi \in R^3$ such that $K_1(\xi) = W(r) = \langle \omega_1, \omega_2, \omega_3 \rangle \subset R^1$.*

a) If $r \in F$ then for all $\beta \in H^{2,0}(JV)$, $\delta\nu(\xi \otimes (\omega_1 \wedge \omega_2 \wedge \omega_3) \wedge \beta) = 0$.

b) If $r \in \Sigma \setminus F$ then there exists $\beta \in H^{2,0}(JV)$ such that $\delta\nu(\xi \otimes (\omega_1 \wedge \omega_2 \wedge \omega_3) \wedge \beta) \neq 0$.

c) If $r \in \Sigma \setminus F$ intersects in 3 distinct points then $\delta\nu^5(\xi \otimes \Omega_\xi) \neq 0$. Therefore $\delta\nu^5 \neq 0$.

Proof. As usual we identify $H^{i,j}(JV)$ with $H^{i,j}(F)$ when $0 \leq i + j \leq 1$. Then we can rewrite the formula of the infinitesimal invariant in the form

$$\delta\nu(\xi \otimes (\omega_1 \wedge \omega_2 \wedge \omega_3) \wedge \beta) = 2 \int_F \omega_{\omega_1,\omega_2,\omega_3,\xi} \wedge \beta.$$

From Theorem 4.4 we know that the adjoint class vanishes exactly on the locus of the double lines.

a) If $r \in F$, $\delta\nu(\xi \otimes (\omega_1 \wedge \omega_2 \wedge \omega_3) \wedge \beta) = 0$.

b) If $r \notin F$ then take $\beta = \overline{\omega_{\omega_1,\omega_2,\omega_3,\xi}}$ we get $\delta(\xi \otimes (\omega_1 \wedge \omega_2 \wedge \omega_3) \wedge \beta) = -\int \beta \wedge \overline{\beta} \neq 0$.

c) If r intersects transversally V, by (4.10) we have that $[\omega_{\omega_1,\omega_2,\omega_3,\xi}]$ is equivalent to $\eta_4 \wedge \eta_5$ where $\{\omega_1, \omega_2, \omega_3, \eta_4, \eta_5\}$ is a basis of $H^{1,0}(F)$. Then we have

$$\eta_4 \wedge \eta_5 = \sum_{ij} a_{i,j} \omega_i \wedge \omega_j$$

and we get that $a_{4,5} \neq 0$. Now $\delta\nu^5(\xi \otimes \Omega_\xi) = \delta\nu(\xi \otimes \Omega_\xi)$ since the form is primitive. We compute

$$\delta\nu^5(\xi \otimes \Omega_\xi) = 2 \int_F \eta_4 \wedge \eta_5 \wedge \overline{\omega_4} \wedge \overline{\omega_5} = 2a_{4,5} \int_F \omega_4 \wedge \omega_5 \wedge \overline{\omega_4} \wedge \overline{\omega_5} \neq 0.$$

□

Remark 5.10. Consider the locus of special deformation $\Xi_2 \subset \mathbb{P}R^3$ (see (1) for the definition) and let $\pi : \mathcal{O}(-1) \to \Xi_2$ be the tautological line bundle Set $U = \mathcal{O}(-1) \setminus Z$ where Z is the zero section, U is a fourfold and the map $\pi : U \to \Xi_2$ is a $\mathbb{C}^*$ bundle. By fixing a generator Ω of $H^{5,0}(A)$, we obtain a holomorphic map $f : U \to \mathbb{C}$ defined by

$$f(\xi) = \delta\nu(\xi \otimes \Omega_\xi) = \delta\nu^5(\xi \otimes \Omega_\xi).$$

We know from 5.9 a) that f vanishes on $\pi^{-1}(\mathcal{D})$ where $\mathcal{D} \subset F$ is the curve of double lines of the Fano surface. Now $\pi^{-1}(\mathcal{D})$ is a surface, but f must vanish on a divisor. The corresponding lines are precisely the non-transverse ones by 5.9 c).

We can draw the consequences of our computation.

Theorem 5.11. *The following hold.*

a) The Fano normal function ν determines $\mathcal{D}$, the curve of double lines on the cubic threefold.

b) The Fano normal function ν is not a torsion section.

5.6. **Algebraic equivalence.** In the next section we will see that 5.11 i) yields a Torelli-like theorem. Here we give our first application of the work done:

Theorem 5.12. *The following hold.*

a) For a generic V there is no proper map $\mu : W \to JV$ from a smooth variety W of dimension ≤ 4, for which there is a cycle Z homologous to 0 in W with $\mu_* Z$ Abel-Jacobi equivalent to $F - F^-$.

b) (Van der Geer-Kouvidakis,[GK]*) For such a V the cycle $F - F^-$ is not algebraically equivalent to zero on JV.*

Proof. We work locally, so we will consider a Kuranishi family of cubic threefolds $\rho : \mathcal{V} \to B$. Let $R^5\rho_*\mathbb{C} \otimes \mathcal{O}_B = \mathcal{H}^5 \to B$ be the variation of polarized Hodge structures, with fibre $H^5(JV_b, \mathbb{C})$ at b. Assume by contradiction that there exists W as described in (a) for a generic JV_b. Via an argument using relative Hilbert schemes, cf.[V2, Remark 11.16], after shrinking B if necessary, we can find a family $\rho' : \mathcal{W} \to B$ with a compatible map over B: $\tilde{\mu} : \mathcal{W} \to \mathcal{J}V$. There is also a cycle $\mathcal{Z}$ on $\mathcal{W}$, flat over B such that $AJ(\tilde{\mu}_{b*}(\mathcal{Z}_b)) = AJ(F_b - F_b^-)$. The map $\tilde{\mu}$ defines for any b a Hodge substructure and we can assume that this provides an inclusion of local systems $\mathcal{L} \subset \mathcal{H}^5$, where $\mathcal{L} = \tilde{\mu}_* R^5\rho'_*\mathbb{C} \otimes \mathcal{O}_B$. Observe that $\mathcal{L}_b$ is orthogonal to $H^{5,0}(JV_b)$ in $H^5(JV_b, \mathbb{C}), \forall b \in B$. We fix a generator Ω in $H^{5,0}(JV_b)$. Since $\mathcal{L}$ in $\mathcal{H}^5$ is preserved by the Gauss-Manin connection, $\xi \otimes \xi \cdot \Omega = \Omega_\xi$ must be orthogonal to $\mathcal{L}_b$. Because of our hypothesis, we remark that every computation one may need to deal with in regard to ν and $\delta\nu$ can in fact be performed assuming that the lift of ν takes value in $\mathcal{L}$, but then $\delta\nu(\xi \otimes \Omega_\xi) = 0$. This is the required contradiction with 5.9 i). So we have proved (a), and then (b) is an immediate corollary. □

Remark 5.13. Note that because the Hodge structure on $H^5(J(V), \mathbb{Q})_{prim}$ is simple for very general V by monodromy arguments, the nonvanishing of the primitive Abel-Jacobi invariant also implies that the cycle $F - F^-$ is not homologous to 0 in a divisor of $J(V)$, since otherwise its Abel-Jacobi invariant would belong to a proper subtorus of $J^5(J(V))_{prim}$ corresponding to a sub-Hodge structure of level 3.

Remark 5.14. Turning to the case of a curve C consider the surface $C^{(2)}$ which is its second symmetric product, and specifically look at its image, the surface $W_2 \subset J(C)$. The same procedure as above constructs a normal fuction γ which splits as $\gamma^3 + \gamma^5$ over the moduli space of Jacobian varieties; it is associated with the Abel-Jacobi class of the cycle $W_2 - W_2^-$. The results of Hain, Thm 7.13 in [HL] imply $\gamma^5 = 0$ (but γ^3 is not of torsion in general). In genus 5 one can prove directly that the preceding statement (a) fails for $W_2 - W_2^-$. Indeed we show below that up to a translation the cycle is homologically equivalent to zero on a suitable theta divisor of $J(C)$. This is a contrast with our situation where ν^3 is 0 (see Lemma 5.4) but ν^5 is not torsion. We find very interesting this difference between the two families of Abelian varieties of dimension 5. Moreover the above result suggests that the theory of the representation for the *Torelli* group, of the orbifold fundamental group of the moduli space of the cubic threefolds should be very rich.

We restrict our attention to a a general curve C of genus 5, and consider $C^{(4)}$, the standard symmetric product. There is a rational involutive correspondence $j : C^{(4)} \dashrightarrow C^{(4)}$ which maps an element $D \in C^{(4)}$ to its Riemann-Roch adjoint $K_C - D$, and therefore an involution $\rho : H^*(C^{(4)}, \mathbb{C}) \to H^*(C^{(4)}, \mathbb{C})$. By choosing a base point $p \in C$ we embed $C^{(2)}$ in $C^{(4)}$ and define $G(2) := \overline{jC^{(2)}}$. With the notations of [ACGH] we consider the map $u : C^{(4)} \to J(C)$ and recall the result written there at p. 326:

Proposition 5.15. *Let* $C_4^1 = \{D \in C^{(4)} \mid \dim |D| \geq 1\}$. *Then* $-\theta x + \theta^2/2 = [C_4^1]$ *in* $H^4(C^{(4)}, \mathbb{C})$.

Here x is the class of $C^{(d-1)}$ in $C^{(d)}$, while θ is the restriction of the class of the Θ divisor on JC, which turns out to be a translation of W_4, where we use the notation $W_d := u(C^{(d)})$. Now $u(G(2))$ is a translation of W_2^-. We have:

Proposition 5.16. *The class* $[W_2 - u(G(2))]$ *vanishes in* $H_4(W_4, \mathbb{C})$.

Proof. The map $u : C^{(4)} \to W_4$ contracts the surface C_4^1 to a curve, so that $0 = u_*[C_4^1]$ in $H_4(W_4)$. On the other hand [BB] contains the formula for the Riemann Roch involution ρ. This was a result contributed by the first author. What it says in our case is that $[C^{(2)}] - [G(2)] = \theta x - \theta^2/2 = -[C_4^1]$. □

Proposition 5.17. *The primitive Abel-Jacobi image of* $W_2 - W_2^-$ *vanishes in* $P^5(J(C))$.

Proof. This is because W_2^- and $G(2)$ are translations of each other, so that the primitive Abel-Jacobi image of their difference vanishes. Moreover $W_2 - u(G(2))$ comes from a cycle homologous to zero on $C^{(4)}$, and therefore we also have for it the vanishing of the primitive Abel-Jacobi image. □

Remark 5.18. An extension of the same argument yields a proof for the fact that $C^{(2)}$ and $G(2)$ are cohomologous in W_{g-1} for any $g \geq 5$.

Remark 5.19. Let $\mathcal{M}_g$ be the moduli space of curves of genus g and $\mathcal{H}_g \subset \mathcal{M}_g$ be the hyperelliptic locus. The Ceresa normal function vanishes on $\mathcal{H}_g$, and according to a conjecture of Clemens it should be non zero everywhere on $\mathcal{M}_g \setminus \mathcal{H}_g$. It is a difficult problem to determine where the Fano normal function vanishes. Here we give an answer to a simpler, but important, question of G. Pearlstein. He asked, in analogy with the genus 3 case, if there is a divisor $\mathcal{D}$ in the moduli space $\mathcal{M}$ of cubic threefolds where the Fano normal function vanishes. Our results implies that this divisor does not exist. Indeed take $[V] \in \mathcal{D} \subset \mathcal{M}$ the general point of any divisor, and identify the tangent space T' to $[V]$ at $\mathcal{D}$ with an hyperplane of $H^1(T_V)$. It follows that the projective space $\mathbb{P}(T')$ intersects the threefold Σ of the special deformations, in a surface. Then we can find in T' a special deformation that does not correspond to a double line of F. It follows that the infinitesimal invariant is not zero along $\mathcal{D}$, therefore the normal function cannot vanish. On the other hand let $\overline{\mathcal{M}}$ be the closure of $\mathcal{M}$ in $\mathcal{A}_5$, the moduli space of principally polarized Abelian varieties. From [Co] (see also [CL]) we see that $\overline{\mathcal{M}}$ contains the locus H_5 of hyperelliptic Jacobians. Note that H_5 is a divisor of $\overline{\mathcal{M}}$. The flat limits of the Fano surfaces in the intermediate jacobians when they approach an hyperelliptic Jacobian is the Abel-Jacobi image W_2 of the $2-$symmetric product of the hyperelliptic curve. This follows, for instance, from a theorem of Olivier Debarre [De]. In fact the homology class of the limit is a minimal class and Debarre prove that in a Jacobian the minimal classes are given only by the Abel-Jacobi image of symmetric products. It follows that the limits of the primitive normal function is zero on H_5.

6. Recovering the cubic threefold from the infinitesimal invariant

The goal of this section is to show how one can reconstruct the threefold V from the infinitesimal invariant studied above. We consider the first order deformations associated with the elements of Σ. Working as we did for (5.9) we find that the infinitesimal invariant provides a functional over Σ which vanishes exactly where the adjoint class is trivial. In (4.3) we noticed that the adjoint class can be considered as a section of a sheaf on $\Sigma \subset \mathbb{G}$ and we proved in Theorem (4.4) that this section vanishes exactly on $\mathcal{D} = \Sigma \cap F$, the curve of double lines on V. So the title of this section amounts to the statement that one can reconstructs V from $\mathcal{D}$, or equivalently the question is whether the ruled surface $S_{\mathcal{D}}$ covered by double lines is contained in a unique cubic threefold. It is enough to verify that $S_{\mathcal{D}}$ has degree larger than 9. Fano has studied this ruled surface in great detail, see [F], and he determined all the relevant numerical data, for instance he proved that $S_{\mathcal{D}}$ has degree 90 in general. We need on the other hand to check that the degree of $S_{\mathcal{D}}$ is always big enough to make it contained in only one cubic threefold. To this aim we prove first that every component of $\Sigma \cap F$ is reduced (see (6.2) below) and then we find in (6.4) that the surface covered by double lines has degree at least 15. We expect, but do not need, that indeed it is always of degree 90.

Consider the variety $K(V)$ which parametrizes the set of flags (L, π), where L is a double line on V and $\pi \cdot V = 2L + R$. Next lemma yields by a standard argument (we have in mind here the use of local coordinates) the consequence that $K(V)$ is a non singular curve at the points where $R \neq L$. Moreover, since $K(V)$ is a fibre in a universal fibration, all such curves are in the same homology class. By projection we have the map $K(V) \to \mathcal{D}(V)$, which is in fact injective, because a plane can cut V with multiplicity 2 at most along one line. This says that all curves $\mathcal{D}(V)$ are reduced, with the same homology class in $\mathbb{G}$.

We choose coordinates such that $z_0 = z_1 = z_2 = 0$ is a double line of V with residual line $z_0 = z_1 = z_3 = 0$. So the equation of our fixed chosen cubic threefold V_0 is of the form

$$E_0 = z_0 Q_0 + z_1 Q_1 + z_2^2 z_3,$$

where $Q_0, Q_1 \in S^2$, the space of homogeneous polynomials of degree 2 in $z_0, \ldots, z_4$.

Consider the map of affine spaces:

$$\Phi : (S^1)^4 \times (S^2)^2 \longrightarrow S^3$$
$$(L_0, L_1, L_2, L_3, A_0, A_1) \mapsto L_0 A_0 + L_1 A_1 + L_2^2 L_3.$$

Lemma 6.1. *Assume that V_0 is smooth. Then the differential $d\Phi$ is surjective at $p_0 := (z_0, z_1, z_2, z_3, Q_0, Q_1)$.*

Proof. We compute $d\Phi$ at p_0 in the obvious way.

Let $u = (L_0, L_1, L_2, L_3, A_0, A_1) \in T_{p_0}((S^1)^4 \times (S^2)^2)$, then

$$\frac{d\Phi(p_0 + \varepsilon u)}{d\varepsilon}(0) = z_0A_0 + L_0Q_0 + z_1A_1 + L_1Q_1 + z_2^2L_3 + 2z_2z_3L_2 \in S^3 = T_{E_0}(S^3).$$

We need to show that any element in S^3 is of the form

$$z_0A_0 + L_0Q_0 + z_1A_1 + L_1Q_1 + z_2^2L_3 + 2z_2z_3L_2$$

when the L_i's move in S^1 and A_i in S^2. Due to the summands $z_0A_0 + z_1A_1$ we can reduce modulo z_0 and z_1. Denote by $C_i = Q_i(0, 0, z_2, z_3, z_4), i = 0, 1$ the corresponding conics. We are reduced to prove that the combinations

$$M_0C_0 + M_1C_1 + M_3z_2^2 + M_4z_2z_3$$

run over all $\mathbb{C}[z_2, z_3, z_4]_3$ when the M_i's are linear forms in the variables z_2, z_3, z_4.

We claim that:

a) There is no common solution of the three quadratic equations C_0, C_1, z_2^2

b) The conic z_2z_3 is linearly independent of C_0, C_1, z_2^2.

The claim is a consequence of the smoothness of the given cubic threefold. Indeed, by computing the 5 partial derivatives of E_0 and reducing modulo z_0, z_1 we get the system:

$$C_0 = C_1 = z_2^2 = z_2z_3 = 0.$$

A solution of $C_0 = C_1 = z_2^2 = 0$ would give a singularity of the threefold over the line $z_0 = z_1 = z_2 = 0$. On the other hand a linear combination

$$z_2z_3 = \alpha z_2^2 + \beta C_0 + \gamma C_1$$

would imply the existence of a denerate conic of the form $z_2 \cdot (a_3z_3 + a_4z_4 + a_5z_5)$ in the linear pencil generated by C_0, C_1. Again this produces a singularity over the line $z_0 = z_1 = z_2 = 0$.

Part a) of the claim allows to use Macaulay's Theorem for the ideal sheaf $I = \langle C_0, C_1, z_2^2\rangle \subset \mathbb{C}[z_2, z_3, z_4]$. Put $R_I = \mathbb{C}[z_2, z_3, z_4]/I$. Then R_I^3 has dimension 1 and there are perfect pairings:

$$R_I^i \otimes R_I^{3-i} \longrightarrow R_I^3 = \mathbb{C}$$

(cf. for instance, section (6.2.2) in [V3]). Part b) of the claim says that z_2z_3 is not zero in R_I^2, thus $z_2z_3 \cdot R_I^1 = R_I^3$ and we are done. □

Corollary 6.2. *Every component of $\mathcal{D}(V)$ is reduced if V is non singular.*

Consider now the incidence variety

$$T = \{(l, x) \in \mathbb{G} \times \mathbb{P}^4 \,|\, x \in l\},$$

with its projections π_1, π_2. Given a curve $X \subset \mathbb{G}$ we denote by $S_X \subset \mathbb{P}^4$ the surface covered by the lines parameterized by X. In other words $S_X = \pi_2(\pi_1^{-1}(X))$. Standard techniques from intersection theory, [Fu], give:

Lemma 6.3. *The degree of the cycle $\pi_{2*}(\pi_1^*(X))$ is equal to $d(X)$, the degree of $X \subset \mathbb{G} \subset \mathbb{P}^9$. The degree of S_X is then found by dividing $d(X)$ by the degree of the map $\pi_1^{-1}(X) \to S_X$.*

Lemma 6.4. *The ruled surface $S_{\mathcal{D}} \subset \mathbb{P}^4$ of double lines is always of degree at least* 15.

Proof. Since the degree of $\pi_1^{-1}(S_{\mathcal{D}}) \to V$ is at most 6, it is then enough to compute that the degree of $\mathcal{D}$ is 90. By recalling that the tautological embedding $F \subset \mathbb{G} \subset \mathbb{P}^9$ is the canonical map of F, we get $deg(\mathcal{D}) = \mathcal{D} \cdot K_F$. On the other hand $\mathcal{D} \in |2\,K_F\,|$ (cf. (10.20) in [CG]), and then $deg(\mathcal{D}) = 2K_F \cdot K_F = 2 \cdot K_F^2 = 2 \cdot 45 = 90$. □

Theorem 6.5. *There is only one cubic threefold containing $S_{\mathcal{D}}$. In particular the infinitesimal invariant determines the cubic threefold.*

References

[ACT] D. Allcock, J. Carlson, D. Toledo, *The moduli space of cubic threefolds as a ball quotient*, Memoirs of the AMS **209** (2011).

[ACGH] E. Arbarello, M. Cornalba, P. A. Griffiths and J. Harris, *Geometry of Algebraic Curves. Volume I*, Grundlehren der mathematischen Wissenschaften, No. 267. Springer-Verlag, New York-Heidelberg, 1984.

[BB] P. Bressler and J.L. Brylinski, *On the singularities of theta divisors on Jacobians*, J. Algebraic Geom. **7** (1998), pp. 781-796.

[CGGH] J. Carlson, M. Green, P. Griffiths, J. Harris, *Infinitesimal variation of Hodge structures I*, Compositio Math. **50** (1983) pp. 109–205.

[CL] S. Casalaina-Martin, R. Laza, *The moduli space of cubic threefolds via degenerations of the intermediate Jacobian*, J. reine angew. Math. **633** (2009), pp. 29–65.

[Ce] G. Ceresa, *C is not algebraic equivalent to C^- in its Jacobian*, Ann. of Math. **117** (1983), pp. 285–291.

[CG] C.H. Clemens, P.A. Griffiths, *The intermediate Jacobian of the cubic threefold*, Ann. of Math. **95** (1972), pp. 281–356.

[Co] A. Collino, *The fundamental group of the Fano surface. I, II*, Algebraic threefolds (Varenna, 1981), Lecture Notes in Math., vol. 947, Springer, Berlin, 1982, pp. 209–218, 219–220.

[CP] A. Collino and G. Pirola, *Griffiths' infinitesimal invariant for a curve in its jacobian*, Duke Math. J. **78** (1995), pp. 59–88.

[De] O. Debarre *Minimal Cohomology Classes and Jacobians*, J. Algebraic Geom. **4** (1995), pp. 321–335.

[Do] I. Dolgachev, *Topics in Classical Algebraic Geometry, Part I*, Lecture Notes, 2010, available online at `www.math.lsa.umich.edu/~idolga/topics.pdf`.

[Fa] N. Fakhruddin, *Algebraic cycles on generic Abelian varieties*, Compos. Math. **100** (1996), pp. 101-119.

[F] G. Fano *Ricerche sulla varietà cubica generale dello spazio a quattro dimensioni e sopra i suoi spazi pluritangenti*, Ann. Mat. Pur. Appl. **10** (1904), pp. 251-285.

[Fu] W. Fulton, *Intersection Theory*, Ergebnisse, 3. Folge, Band 2, Springer Verlag (1984).

[GK] G. van der Geer, A. Kouvidakis, *The rank-one limit of the Fourier-Mukai transform*, Doc. Math. **15** (2010), 747–763.

[Gr1] M. Green, *Griffiths' infinitesimal invariant and the Abel-Jacobi map*, J. Differ. Geom. **29** (1989), pp. 545–555.

[Gr] M. Green, *Infinitesimal methods in Hodge theory. Algebraic cycles and Hodge theory (Torino, 1993)*, L.N.M., 1594, Springer, Berlin (1994), pp. 1–92.

[GH] P.A. Griffiths, J. Harris, *Principles of Algebraic Geometry*, New York, Wiley and Sons, 1978.

[G1] P.A. Griffiths, *Some results on algebraic cycles on algebraic manifolds*, Algebr. Geom., Bombay Colloq. 1968,(1969), pp. 93-191.

[G2] P.A. Griffiths, *On the periods of rational integrals, I and II* , Ann. of Math. **90** (1969), pp. 460-541.

[G3] P.A. Griffiths, *Infinitesimal variations of Hodge structures III: determinantal varieties and the infinitesimal invariant of normal functions*, Compos. Math. **50** (1983), pp. 267–324.

[HL] R. Hain, E. Looijenga, *Mapping class groups and moduli spaces of curves*, Algebraic Geometry, Santa Cruz, 1995: Proc. Symp. Pure Math 62.2 (1997), pp. 97-142

[L] Lieberman, D. I. : *Numerical and homological equivalence of algebraic cycles on Hodge manifolds.* Amer. J. Math. **90** (1968), 366-374.

[N] M.V. Nori, *Algebraic Cycles and Hodge Theoretic Connectivity*, Invent. Math. **111** (1993), pp. 349–373.

[PZ] G.P. Pirola, F. Zucconi, *Variations of the Albanese morphisms*, J. Algebraic Geom. **12** (2003), pp. 535–572.

[T] A.N. Tjurin, *The Geometry of the Fano surface of a non singular cubic threefold $F \subset \mathbb{P}^4$ and Torelli Theorems for Fano surfaces and cubics*, Izv. Akad. Nauk SSSR Ser. Mat. **35** (1971), pp. 498–529.

[V1] C. Voisin, *Théorème de Torelli pour les cubiques de $\mathbb{P}^5$*, Invent. Math. **86** (1986), pp. 577–601.

[V2] C. Voisin, *Une remarque sur l'invariant infinitesimal des functions normal*, C. R.Acad. Sci. Paris Ser. I **307** (1988), pp. 157-160.

[V3] C. Voisin, *Hodge theory and complex algebraic geometry. I-II*, Cambridge Studies in Advanced Mathematics, 7. Cambridge University Press, Cambridge, 2007.

Dipartimento di Matematica, Universita di Torino, Via Carlo Alberto 10, 10123, Torino, Italy
E-mail address: alberto.collino@unito.it

Departament d'Àlgebra i Geometria, Facultat de Matemàtiques, Universitat de Barcelona, Gran Via 585, 08007 Barcelona, Spain
E-mail address: jcnaranjo@ub.edu

Dipartimento di Matematica "F. Casorati", Università di Pavia, Via Ferrata 1, 27100 Pavia, Italy
E-mail address: gianpietro.pirola@unipv.it