UNIVERSITÀ DEGLI STUDI DI TORINO

A note on a superlinear indefinite Neumann problem with multiple positive solutions

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Abstract

We prove the existence of three positive solutions for the Neumann problem associated to $u'' + a(t)u^{\gamma+1} = 0$, assuming that $a(t)$ has two positive humps and $\int_0^T a^-(t)\, dt$ is large enough. Actually, the result holds true for a more general class of superlinear nonlinearities.

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1. Introduction

In the last two decades, a great interest has been devoted to the study of the existence of *positive* solutions to the nonlinear Neumann problem

$$\begin{cases} u'' + a(t)g(u) = 0 \\ u'(0) = u'(T) = 0 \end{cases} \tag{1.1}$$

and to the PDE's analogous

$$\begin{cases} \Delta u + a(x)g(u) = 0 & x \in \Omega \\ \frac{\partial u}{\partial \nu} = 0 & x \in \partial\Omega, \end{cases} \tag{1.2}$$

being $g : \mathbb{R}^+ := [0, +\infty[\to \mathbb{R}$ a continuous function such that $g(0) = g'(0) = 0$, $g(s) > 0$ for $s > 0$ and superlinear at infinity, in the sense that

$$\lim_{s \to +\infty} \frac{g(s)}{s} = +\infty.$$

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(Of course, in the PDE's case a subcritical growth, with respect to Sobolev embedding theorems, is usually assumed.)

Such a problem is quite particular and interesting, essentially for two reasons. On one hand, the global condition $g(s) > 0$ for $s > 0$ implies that no positive solutions can exist if $a(\cdot)$ is of constant sign. In fact, by the divergence theorem,

$$0 = -\int_{\partial\Omega} \frac{\partial u}{\partial \nu}\, d\sigma = -\int_{\Omega} \Delta u\, dx = \int_{\Omega} a(x) g(u)\, dx.$$

Hence, problem (1.1)-(1.2) is *indefinite* in nature. On the other hand, the local assumption $g'(0) = 0$ makes the problem resonant (near 0) with respect to the principal eigenvalue $\lambda_0 = 0$ of $-\Delta$ with Neumann boundary condition and the interaction of the nonlinearity with such an eigenvalue plays typically a crucial role for the existence of positive solutions.

In the model case $g(s) = s^{\gamma+1}$ with $\gamma > 0$, problem (1.2) has been extensively studied, with variational methods strongly depending on the homogeneity of the nonlinear term, by Berestycki, Capuzzo Dolcetta and Nirenberg [4], who proved the existence of at least one positive solution under the mean value condition

$$\int_{\Omega} a(x)\, dx < 0. \tag{1.3}$$

The same conclusion is achieved in [1, 3], for a more general superlinear nonlinearity satisfying suitable extra assumptions at zero and at infinity. The first paper is based on variational techniques again, while the second relies on some a priori estimates, degree theory and bifurcation arguments. It is worth noticing that condition (1.3) cannot in general be avoided, since it is a necessary one for increasing nonlinearities (see [2] and Remark 4.1). These results have been subsequently extended to more general equations involving the p-Laplacian operator in [17]. However, to the best of our knowledge, no multiplicity results in the general setting are available in literature.

The aim of this brief note is to show that, in the ODE's case (1.1) (or, for problem (1.2), when Ω is an annulus and the weight function is radial, see Corollary 4.2), shooting-type arguments, combining the oscillatory properties of the solutions in the intervals of positivity of the weight function with blow-up phenomena in the intervals of negativity, can be applied in order to prove the existence of multiple positive solutions, depending on the shape of the graph of the weight $a(t)$.
For example, as a corollary of our main result we can prove the following.

Corollary 1.1. *Let $\gamma > 0$ and let $a^+, a^- : [0,T] \to \mathbb{R}$ be continuous functions such that, for some σ, τ with $0 < \sigma < \tau < T$,*

$$a^-(t) = 0, \quad a^+(t) \geq 0, \not\equiv 0 \quad \forall t \in [0, \sigma],$$

$$a^+(t) = 0, \quad a^-(t) \geq 0, \not\equiv 0 \quad \forall t \in [\sigma, \tau],$$

$$a^-(t) = 0, \quad a^+(t) \geq 0, \not\equiv 0 \quad \forall t \in [\tau, T].$$

Then there exists $\mu^* > 0$ such that, for every $\mu > \mu^*$, the Neumann problem

$$\begin{cases} u'' + (a^+(t) - \mu a^-(t))u^{\gamma+1} = 0 \\ u'(0) = u'(T) = 0 \end{cases}$$

has at least three positive solutions.

The use of shooting-type arguments in order to obtain multiplicity results for boundary value problems associated to indefinite nonlinear ODE's (especially in the superlinear case) is classical, starting with the pioneering work of Butler [8]; on the line of this research, we recall the very sharp result of Papini and Zanolin, [16]. However, it is usually proved the existence of solutions with many zeros in the intervals of positivity of the weight $a(t)$; about the existence of positive solutions, fewer results are available.
In particular, a result analogous to Corollary 1.1 has been proved by Gaudenzi, Habets and Zanolin [10] for the Dirichlet problem; it has to be pointed out, however, that, from a functional analytical point of view, the Dirichlet and the Neumann problem are very different (see Remark 4.2).

In this paper, we follow closely the arguments of [10], suitably adapting them to the case of Neumann boundary conditions; actually, we are also able to deal with more general nonlinearities $g(s)$, without either convexity or monotonicity assumption. This gives rise to our main multiplicity result, Theorem 2.1. It is worth noticing that, from such a result, we can also deduce the existence of multiple positive solutions for some periodic problems (see Corollary 4.1).

Finally, we point out that, even if, for the sake of simplicity, we consider only a "three-step weight", it is reasonable to expect (on the line of [6, 11, 13]) that some further multiplicity results can be proved also for weights $a(t)$ with k positive humps separated by $k-1$ negative humps, yielding the existence of $2^k - 1$ positive solutions.

2. Statement of the main result

The main result of the paper is the following.

Theorem 2.1. *Let $g : \mathbb{R}^+ \to \mathbb{R}$ be a locally Lipschitz continuous function such that $g(0) = 0$ and $g(s) > 0$ for $s > 0$. Setting $G(s) := \int_0^s g(\xi)\,d\xi$, let us assume that:*

$$(g_0) \qquad \lim_{s\to 0^+} \frac{g(s)}{s} = 0;$$

$$(g_\infty) \qquad \lim_{s\to +\infty} \frac{g(s)}{s} = +\infty \quad \text{and} \quad \int^{+\infty} \frac{d\xi}{\sqrt{G(\xi)}} < +\infty;$$

$$(g_\infty^*) \qquad \limsup_{s\to +\infty} \int_s^{+\infty} \frac{d\xi}{\sqrt{G(\xi) - G(s)}} < +\infty.$$

Moreover, let $a^+, a^- : [0,T] \to \mathbb{R}$ be continuous functions such that, for some σ, τ with $0 < \sigma < \tau < T$,

$$a^-(t) = 0, \quad a^+(t) \geq 0, \not\equiv 0 \quad \forall t \in [0,\sigma],$$

$$a^+(t) = 0, \quad a^-(t) \geq 0, \not\equiv 0 \quad \forall t \in [\sigma,\tau],$$

$$a^-(t) = 0, \quad a^+(t) \geq 0, \not\equiv 0 \quad \forall t \in [\tau,T].$$

Then there exists $\mu^* > 0$ such that, for every $\mu > \mu^*$, the Neumann problem

$$\begin{cases} u'' + (a^+(t) - \mu a^-(t))g(u) = 0 \\ u'(0) = u'(T) = 0 \end{cases} \tag{2.1}$$

has at least three positive solutions.

From now on, by a positive solution of (2.1) (or of some related problems), we mean that $u(t) > 0$ for every $t \in [0,T]$.
A few comments about the assumptions are in order. Hypothesis (g_0) and the first condition in (g_∞) are structural assumptions of the problem.
The second requirement of hypothesis (g_∞) is the well-known Keller-Osserman condition and, according to [7, Theorem 2], it is a necessary condition for the existence of a blowing-up solution of equation $u'' + (a^+(t) - \mu a^-(t))g(u) = 0$.
Finally, hypothesis (g_∞^*) is a time-mapping assumption about the autonomous equation

$$u'' - g(u^+) = 0. \tag{2.2}$$

In fact, it is immediate to check that (g_∞^*) holds true if and only if

$$\limsup_{c\to -\infty} \sqrt{2} \int_{G^{-1}(-c)}^{+\infty} \frac{d\xi}{\sqrt{G(\xi) + c}} < +\infty \tag{2.3}$$

and the integral in (2.3) is just the time for the orbit of (2.2) passing through the point $(G^{-1}(-c), 0)$. As proved in [16, Appendix], hypothesis (g_∞^*) is fulfilled whenever (g_∞) holds true and it is satisfied one of the following conditions:

- for every s large,
$$g(s) \geq h(s),$$
for some continuous and monotone function $h : \mathbb{R}^+ \to \mathbb{R}$ satisfying (g_∞);

- there exists a constant $k > 1$ such that
$$\liminf_{s \to +\infty} \frac{G(ks)}{G(s)} > 1.$$

It is worth noticing that this last condition is related to the Karamata's theory of slowly varying functions, [5]. Clearly, the model nonlinearity $g(s) = s^{\gamma+1}$ (with $\gamma > 0$) satisfies all such assumptions, $(g_0), (g_\infty), (g_\infty^*)$.

3. Proof of the main result

We set $a_\mu(t) := a^+(t) - \mu a^-(t)$; moreover, let us fix $\rho \in]\sigma, \tau[$ and $\epsilon, \delta > 0$ such that, for every $t \in [\rho - \epsilon, \rho]$,

$$a^-(t) \geq \delta. \tag{3.4}$$

Finally, we define the null extension of $g(s)$,

$$g^0(s) := g(s^+) = \begin{cases} g(s) & s \geq 0 \\ 0 & s < 0. \end{cases}$$

We split our proof into three steps.

Step 1: Forward shooting

For every $x \geq 0$ and $\mu \geq 0$, let $u_\mu(\cdot; x)$ be the solution of the Cauchy problem

$$\begin{cases} u'' + a_\mu(t) g^0(u) = 0 \\ u(0) = x, \quad u'(0) = 0 \end{cases} \tag{3.5}$$

and denote by $[0, t_\mu^+(x)[$ its maximal interval of (forward) continuability in $[0, T]$. It is a well known fact in the theory of initial value problems for ODE's that the function $x \mapsto t_\mu^+(x)$ is lower semicontinuous. Set

$$\mathcal{D}_\mu^+ := \{x \geq 0 \mid t_\mu^+(x) > \rho\}$$

and define the translation operator

$$\mathcal{D}_\mu^+ \ni x \mapsto \varphi_\mu(x) := (u_\mu(\rho; x), u'_\mu(\rho; x)).$$

We recall that, for every $S \subset \mathcal{D}_\mu^+$, S and $\varphi_\mu(S)$ are homeomorphic. The final goal of this first step of the proof is to construct two disjoint intervals contained in $\mathcal{D}_\mu^+$, whose images under φ_μ are contained in the right half-plane and connect the y-axis with $(+\infty, +\infty)$. Precisely, we will prove the following.

Lemma 3.1. *There exists $\mu_1^* > 0$ such that, for every $\mu > \mu_1^*$, there exist ξ_1, ξ_2, ξ_3 with $0 < \xi_1 \le \xi_2 < \xi_3$ such that:*

- $[0, \xi_1[\cup]\xi_2, \xi_3] \subset \mathcal{D}_\mu^+$,
- *for every $x \in]0, \xi_1[\cup]\xi_2, \xi_3[$, $u_\mu(t; x) > 0$ for every $t \in [0, \rho]$,*
- $\varphi_\mu(0) = (0, 0)$ *and* $\varphi_\mu(\xi_3) = (0, R)$ *with* $R < 0$,
- $\lim_{x \to \xi_1^-} \varphi_\mu(x) = \lim_{x \to \xi_2^+} \varphi_\mu(x) = (+\infty, +\infty)$.

The proof of such a lemma is based on a careful study of the behavior of the solutions to (3.5) in the time interval $[0, \rho]$. In particular, a special care has to be paid to the blow-up properties of such solutions. Hence, before passing to the proof of Lemma 3.1, we establish the following auxiliary results.

Lemma 3.2. *There exist $x_*, \mu_1^* > 0$ such that:*

i) $t_0^+(x) = T$ *for every* $x \ge 0$ *and* $u_0(t; x) > 0$ *for every* $t \in [0, \rho]$ *and* $x \in]0, x_*]$,

ii) *for every $\mu \ge 0$, it holds that $t_\mu^+(x) > \sigma$ for every $x \ge 0$ and $u_\mu(t; x) > 0$ for every $t \in [0, \rho] \cap [0, t_\mu^+(x)[$ and $x \in]0, x_*]$,*

iii) *for every $\mu > \mu_1^*$, it holds that $t_\mu^+(x_*) \le \rho$.*

Proof. We prove separately the different claims.
i) The fact that $t_0^+(x) = T$ for every $x \ge 0$ follows from the observation that $g^0(s) = 0$ for every $s \le 0$ and, by concavity, $u_0(t; x) \le x$ for every $t \in [0, t_0(x)[$. Next, define x_* small enough such that, for every $s \in]0, x_*]$,

$$\|a^+\|_{L^\infty([0,\sigma])} \frac{g(s)}{s} < \left(\frac{\pi}{2\rho}\right)^2.$$

Let us suppose by contradiction that, for some $x \in]0, x_*]$, there exists $t^* \in [0, \rho]$ such that $u_0(t^*; x) = 0$. Without loss of generality, we can suppose that

$u_0(t;x) > 0$ for every $t \in [0, t^*[$; moreover, as observed above, we have that $u_0(t;x) \le x$ for every $t \in [0, t^*]$. Setting $\omega = \frac{\pi}{2t^*}$ (and supposing $t^* \in [\sigma, \rho]$, the other case being even simpler), we get

$$
\begin{aligned}
0 &= \Big[\omega u_0(t;x)\sin(\omega t) + u_0'(t;x)\cos(\omega t)\Big]_{t=0}^{t^*} \\
&= \int_0^{t^*} \left[\omega^2 - a_0(s)\,\frac{g(u_0(s;x))}{u_0(s;x)}\right] u_0(s;x)\cos(\omega s)\,ds \\
&= \int_0^{\sigma} \left[\omega^2 - a^+(s)\,\frac{g(u_0(s;x))}{u_0(s;x)}\right] u_0(s;x)\cos(\omega s)\,ds + \\
&\quad + \int_\sigma^{t^*} \omega^2 u_0(s;x)\cos(\omega s)\,ds \\
&\ge \int_0^{\sigma} \left[\left(\frac{\pi}{2\rho}\right)^2 - a^+(t)\,\frac{g(u_0(s;x))}{u_0(s;x)}\right] u_0(s;x)\cos(\omega s)\,ds > 0,
\end{aligned}
$$

which is a contradiction.
ii) Fix $\mu \ge 0$ and $x \ge 0$. Similarly as above, since $u_\mu(t;x) \le x$ for every $t \in [0,\sigma] \cap [0, t_\mu^+(x)[$ and $g^0(s) = 0$ for $s \le 0$, we deduce that $t_\mu^+(x) > \sigma$. Moreover, it is clear that $u_\mu(t;x) = u_0(t;x)$, $u_\mu'(t;x) = u_0'(t;x)$ for every $t \in [0,\sigma]$. Hence, since $u_\mu''(\cdot;x) \ge 0$ on $[\sigma,\rho] \cap [\sigma, t_\mu^+(x)[$, we have that $u_\mu(t;x) \ge u_0(t;x)$ for every $t \in [0,\rho] \cap [0, t_\mu^+(x)[$. The conclusion now follows from point *i*).
iii) Let x_* be as in point *i*) and suppose by contradiction that there exists $\mu_k \to +\infty$ such that $t_{\mu_k}^+(x_*) > \rho$. As observed in the proof of point *ii*), and using the fact that $u_0(\cdot;x_*)$ is non increasing and concave, we get that, for every k,

$$u_{\mu_k}(t;x_*) \ge u_0(t;x_*) \ge u_0(\sigma;x_*), \qquad \forall\, t \in [\sigma,\rho].$$

and

$$u_{\mu_k}'(t;x_*) \ge u_0'(t;x_*) = u_0'(\sigma;x_*), \qquad \forall\, t \in [\sigma,\rho].$$

Set

$$m = \inf_{x \ge u_0(\sigma;x_*)} g(x) > 0;$$

then we have, for every k and for every $t \in [\sigma, \rho-\epsilon]$,

$$
\begin{aligned}
u_{\mu_k}'(t;x_*) &= u_{\mu_k}'(\sigma;x_*) + \int_\sigma^t u_{\mu_k}''(s;x_*)\,ds \\
&\ge u_0'(\sigma;x_*) + \mu_k \int_\sigma^t a^-(s) g(u_{\mu_k}(s;x_*))\,ds \\
&\ge u_0'(\sigma;x_*) + \mu_k m \int_\sigma^t a^-(s)\,ds.
\end{aligned}
$$

Setting $A_-(t) := \int_\sigma^t a^-(s)\,ds$ and integrating on $[\sigma, \rho-\epsilon]$, we have moreover, for every k,

$$\begin{aligned} u_{\mu_k}(\rho-\epsilon; x_*) &\geq u_{\mu_k}(\sigma; x_*) + u_0'(\sigma; x_*)(\rho-\epsilon-\sigma) + \mu_k m \int_\sigma^{\rho-\epsilon} A^-(s)\,ds \\ &\geq u_0(\sigma; x_*) + u_0'(\sigma; x_*)(\rho-\epsilon-\sigma) + \mu_k m \int_\sigma^{\rho-\epsilon} A^-(s)\,ds. \end{aligned}$$

Being, in view of (3.4), $a^-(\rho-\epsilon) \geq \delta$, we have that $\int_\sigma^{\rho-\epsilon} A^-(s)\,ds > 0$ and $\int_\sigma^{\rho-\epsilon} a^-(s)\,ds > 0$; then we can conclude that

$$\lim_{k\to+\infty} u_{\mu_k}(\rho-\epsilon; x_*) = +\infty, \tag{3.6}$$

$$\lim_{k\to+\infty} u'_{\mu_k}(\rho-\epsilon; x_*) = +\infty.$$

In particular, for k large enough and for every $t \in [\rho-\epsilon, \rho]$,

$$u'_{\mu_k}(t; x_*) \geq u'_{\mu_k}(\rho-\epsilon; x_*) \geq 0.$$

Hence, we get, for every $t \in [\rho-\epsilon, \rho]$,

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}u'_{\mu_k}(t; x_*)^2 - \mu_k \delta G(u_{\mu_k}(t; x_*))\right) &= \\ u'_{\mu_k}(t; x_*)(u''_{\mu_k}(t; x_*) - \mu_k \delta g(u_{\mu_k}(t; x_*))) &\geq \\ u'_{\mu_k}(t; x_*)(u''_{\mu_k}(t; x_*) - \mu_k a^-(t) g(u_{\mu_k}(t; x_*))) &= 0; \end{aligned}$$

integrating between $\rho-\epsilon$ and $t \in [\rho-\epsilon, \rho]$, we obtain

$$\frac{1}{2}u'_{\mu_k}(t; x_*)^2 - \mu_k \delta G(u_{\mu_k}(t; x_*)) \geq -\mu_k \delta G(u_{\mu_k}(\rho-\epsilon; x_*)).$$

Then we have

$$\begin{aligned} \epsilon &= \int_{\rho-\epsilon}^{\rho} dt \leq \int_{\rho-\epsilon}^{\rho} \frac{u'_{\mu_k}(t; x_*)}{\sqrt{2\mu_k\delta(G(u_{\mu_k}(t; x_*)) - G(u_{\mu_k}(\rho-\epsilon; x_*))}}\,dt \\ &= \sqrt{\frac{1}{2\mu_k\delta}} \int_{u_{\mu_k}(\rho-\epsilon; x_*)}^{u_{\mu_k}(\rho; x_*)} \frac{d\xi}{\sqrt{G(\xi) - G(u_{\mu_k}(\rho-\epsilon; x_*))}}\,d\xi \\ &\leq \sqrt{\frac{1}{2\mu_k\delta}} \int_{u_{\mu_k}(\rho-\epsilon; x_*)}^{+\infty} \frac{d\xi}{\sqrt{G(\xi) - G(u_{\mu_k}(\rho-\epsilon; x_*))}}\,d\xi. \end{aligned}$$

Taking into account assumption (g_∞^*) and (3.6), we get, for k large enough,

$$\sqrt{\frac{1}{2\mu_k\delta}} \int_{u_{\mu_k}(\rho-\epsilon; x_*)}^{+\infty} \frac{d\xi}{\sqrt{G(\xi) - G(u_{\mu_k}(\rho-\epsilon; x_*))}}\,d\xi < \epsilon,$$

a contradiction. □

Lemma 3.3. *There exists a positive solution (i.e, $u(t) > 0$ for every $t \in [0, \sigma[$) of the boundary value problem*

$$\begin{cases} u'' + a^+(t)g(u) = 0 \\ u'(0) = 0, \quad u(\sigma) = 0. \end{cases} \tag{3.7}$$

PROOF. Being $a^+(t)$ continuous, with $a^+ \geq 0$ and $a^+ \not\equiv 0$ on $[0, \sigma]$, [14, Corollary 3.6] implies that there exists a nontrivial solution of (3.7) such that $u(t) > 0$ for every $t \in]0, \sigma[$. The uniqueness for the solutions of the Cauchy problem implies that $u(0) > 0$, too. □

Remark 3.1. Corollary 3.6 of [14] follows from a fixed point theorem for operators acting on the cone of positive functions. Of course, in the model case $g(s) = s^{\gamma+1}$, more classical arguments, like constrained minimization or the Mountain Pass Lemma, can be used to prove the existence of a positive solution of (3.7). For details, we refer the reader to [3].

PROOF OF LEMMA 3.1. Fix $\mu > \mu_1^*$. By point *ii)* of Lemma 3.2, $t_\mu^+(x) > \sigma$ for every $x \geq 0$ and hence the set

$$\mathcal{E}_\mu := \{x > 0 \mid u_\mu(t; y) > 0,\ \forall\, t \in [0, \sigma], \forall\, y \in]0, x]\}$$

is well defined. Moreover, $\mathcal{E}_\mu$ is nonempty, since $x_* \in \mathcal{E}_\mu$, as a consequence of point *ii)* of Lemma 3.2 again. Set $\xi := \sup \mathcal{E}_\mu$; in view of Lemma 3.3, $x_* < \xi < +\infty$. Next define the set

$$\mathcal{F}_\mu := \{x \in [0, \xi] \mid t_\mu^+(x) \leq \rho\};$$

again this set is nonempty since $x_* \in \mathcal{F}_\mu$, as consequence of point *iii)* of Lemma 3.2. Set $\xi_1 := \inf \mathcal{F}_\mu$ and $\xi_2 := \sup \mathcal{F}_\mu$. Since $t_\mu^+(x)$ is lower semicontinuous, $\mathcal{F}_\mu$ is a closed set; being $0, \xi \notin \mathcal{F}_\mu$, we deduce that $0 < \xi_1 \leq x_* \leq \xi_2 < \xi$. Moreover, by construction it holds that $[0, \xi_1[\cup]\xi_2, \xi] \subset \mathcal{D}_\mu^+$ and, in view of point *ii)* of Lemma 3.2, we know that

$$u_\mu(t; x) > 0, \quad \forall\, t \in [0, \rho], \forall\, x \in]0, \xi_1[.$$

We claim that

$$\lim_{x \to \xi_1^-} \varphi_\mu(x) = \lim_{x \to \xi_2^+} \varphi_\mu(x) = (+\infty, +\infty). \tag{3.8}$$

Just to fix the ideas, we prove the relation for $x \to \xi_2^+$. We first verify that $\lim_{x \to \xi_2^+} u_\mu(\rho; x) = +\infty$. Let us suppose by contradiction that, for a subsequence $x_k \to \xi_2^+$, $u_\mu(\rho; x_k) \le M$ and fix $\eta > 0$ small. Since $u_\mu(\cdot; x_k) \to u_\mu(\cdot; \xi_2)$ in $C^1([0, \sigma])$, we have that, for k large enough,

$$|u_\mu(\sigma; x_k) - u_\mu(\sigma; \xi_2)| \le \eta, \quad |u'_\mu(\sigma; x_k) - u'_\mu(\sigma; \xi_2)| \le \eta.$$

By convexity arguments, it is easy to see that, for k large enough and for every $t \in [0, \rho]$,

$$(u'_\mu(\sigma; \xi_2) - \eta)(\rho - \sigma) + u_\mu(\sigma; \xi_2) - \eta \le u_\mu(t; x_k) \le \max(\xi_2 + \eta, M); \quad (3.9)$$

hence, the sequence $u_\mu(\cdot; x_k)$ is uniformly bounded on $[0, \rho]$. Moreover, since $u''_\mu(t; x_k) = a_\mu(t) g(u_\mu(t; x_k))$ and $u'_\mu(t; x_k) = \int_0^t u''_\mu(s; x_k)\, ds$, we conclude that $u_\mu(\cdot; x_k)$ is bounded in $C^2([0, \rho])$. Hence, up to subsequences, $u_\mu(\cdot; x_k)$ converges (weakly in $H^2([0, \rho])$ and in $C^1([0, \rho])$) to a function $u_\mu(\cdot)$ with $u_\mu(0) = \xi_2$, $u'_\mu(0) = 0$ and such that

$$u''_\mu(t) + a_\mu(t) g(u_\mu(t)) = 0, \qquad \forall\, t \in [0, \rho].$$

Since $\xi_2 \in \mathcal{F}_\mu$ (recall that $\mathcal{F}_\mu$ is a closed set), this is a contradiction. To conclude the proof of (3.8), it is sufficient to observe that, by convexity of $u_\mu(\cdot; x)$ on $[\sigma, \rho]$,

$$u'_\mu(\rho; x) \ge \frac{u_\mu(\rho; x) - u_\mu(\sigma; x)}{\rho - \sigma}$$

and that the right-hand side goes to infinity as $x \to \xi_2^+$, since $0 < u_\mu(\sigma; x) \le x$. Notice that for $x \to \xi_1^-$ the proof is even simpler because, in (3.9), we know that $u_\mu(t; x_k) > 0$ for every $t \in [0, \rho]$ and every $x_k < \xi_1$.
Observe now that relation (3.8) implies that $u_\mu(t; x) > 0$ for every $t \in [0, \rho]$ and for x in a right neighborhood of ξ_2. Indeed, if $u_\mu(\tilde{t}; x) \le 0$ for some $\tilde{t} \in [0, \rho]$, then $u_\mu(\rho; x) \le 0$ too. Thus, the set

$$\mathcal{G}_\mu := \{x \in]\xi_2, \xi] \mid\ u_\mu(t; y) > 0,\ \forall\, t \in [0, \rho], \forall\, y \in]\xi_2, x]\}$$

is nonempty and we set $\xi_3 := \sup \mathcal{G}_\mu$. It is easily seen that $\xi_2 < \xi_3 < \xi$ and that $\varphi_\mu(\xi_3) = (0, R)$ with $R < 0$. Moreover, by definition, $u_\mu(t; x) > 0$ for every $t \in [0, \rho]$ and for every $x \in]\xi_2, \xi_3[$. Recalling (3.8), the proof is concluded. □

Step 2: Backward shooting

This second step is completely symmetric to the previous one. Define, for $x \ge 0$ and $\mu \ge 0$, $v_\mu(\cdot; x)$ as the unique backward solution of the Cauchy problem

$$\begin{cases} v'' + a_\mu(t) g^0(v) = 0 \\ v(T) = x, \quad v'(T) = 0 \end{cases}$$

and denote by $]t_\mu^-(x), T]$ its maximal interval of (backward) continuability in $[0, T]$. Then $x \mapsto t_\mu^-(x)$ is upper semicontinuous and we can define the set

$$\mathcal{D}_\mu^- := \{x \geq 0 \mid t_\mu^-(x) < \rho\}$$

and the translation operator

$$\mathcal{D}_\mu^- \ni x \mapsto \psi_\mu(x) := (v_\mu(\rho; x), v_\mu'(\rho; x)).$$

We can prove the following analogous of Lemma 3.1.

Lemma 3.4. *There exists $\mu_2^* > 0$ such that, for every $\mu > \mu_2^*$, there exist η_1, η_2, η_3 with $0 < \eta_1 \leq \eta_2 < \eta_3$ such that:*

- $[0, \eta_1[\cup]\eta_2, \eta_3] \subset \mathcal{D}_\mu^-$,
- *for every $x \in]0, \eta_1[\cup]\eta_2, \eta_3[$, $v_\mu(t; x) > 0$ for every $t \in [\rho, T]$,*
- $\psi_\mu(0) = (0, 0)$ *and* $\psi_\mu(\eta_3) = (0, S)$ *with* $S > 0$,
- $\lim_{x \to \eta_1^-} \psi_\mu(x) = \lim_{x \to \eta_2^+} \psi_\mu(x) = (+\infty, -\infty)$.

Step 3: Conclusion

Define $\mu^* := \max\{\mu_1^*, \mu_2^*\}$ and fix $\mu > \mu^*$; we now can conclude as in [10]. By standard connectivity arguments, the following facts hold true:

- $\varphi_\mu(]0, \xi_1[)$ intersects $\psi_\mu(]\eta_2, \eta_3[)$;
- $\psi_\mu(]0, \eta_1[)$ intersects $\varphi_\mu(]\xi_2, \xi_3[)$;
- $\varphi_\mu(]\xi_2, \xi_3[)$ intersects $\psi_\mu(]\eta_2, \eta_3[)$.

These intersection points are pairwise distinct because of the uniqueness for the solutions to the Cauchy problems and it is clear that each of them corresponds to a value $(u(\rho), u'(\rho))$ of a positive solution of problem (2.1).

Remark 3.2. It is worth noticing that the proof of Theorem 2.1 is quite constructive and, as a consequence, one can imagine the behavior in the phase plane of the three Neumann solutions produced. A very naive numerical experiment is plotted (with MAPLE® software) in Figure 1, with $T = 3$, $\mu = 15$, $g(x) = x^2$ and

$$a^+(t) = \begin{cases} 0.9\sin(\pi t) & 0 \leq t < 1 \\ 0 & 1 \leq t < 2 \\ \sin(\pi t) & 2 \leq t < 3 \end{cases}, \qquad a^-(t) = \begin{cases} 0 & 0 \leq t < 1 \\ -\sin(\pi t) & 1 \leq t < 2 \\ 0 & 2 \leq t < 3 \end{cases}.$$

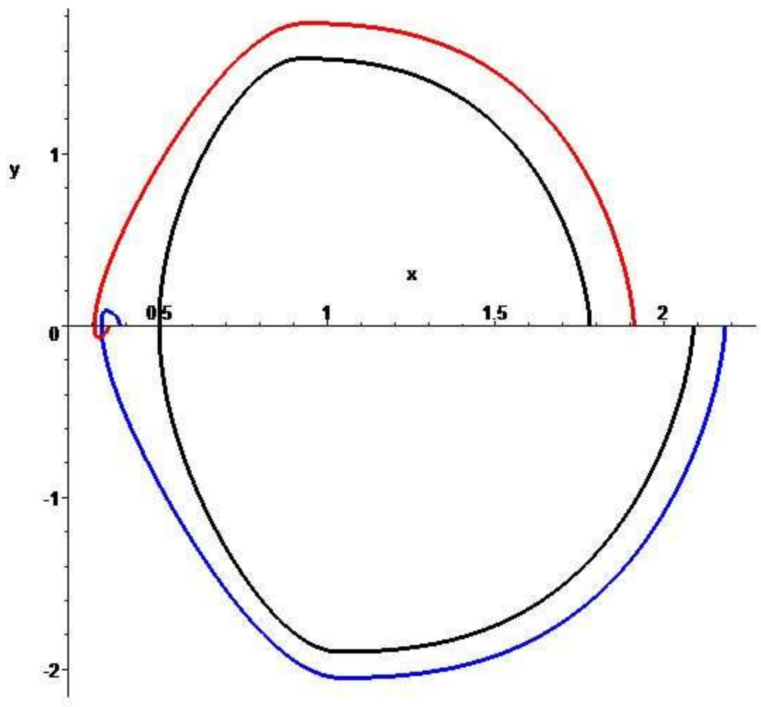

Figure 1: The three Neumann solutions in the phase plane $\{(u, u')\}$.

4. Corollaries and final remarks

For the sake of completeness, we state two further multiplicity results (for positive solutions), which follow in a standard way from Theorem 2.1. The first one deals with a periodic problem with even-symmetric (indefinite) weight. We point out that, in spite of a large number of results in literature about positive periodic solutions, only very few references concerning *multiple* positive periodic solutions can be quoted.

Corollary 4.1. *Let $g(s)$ be as in Theorem 2.1 and let $a^+, a^- : \mathbb{R} \to \mathbb{R}$ be continuous, even-symmetric and T-periodic functions such that, for some σ, τ with $0 < \sigma < \tau < \frac{T}{2}$,*

$$a^-(t) = 0, \quad a^+(t) \geq 0, \not\equiv 0 \quad \forall t \in [0, \sigma],$$

$$a^+(t) = 0, \quad a^-(t) \geq 0, \not\equiv 0 \quad \forall t \in [\sigma, \tau],$$

$$a^-(t) = 0, \quad a^+(t) \geq 0, \not\equiv 0 \quad \forall t \in [\tau, \tfrac{T}{2}].$$

Then there exists $\mu^* > 0$ such that, for every $\mu > \mu^*$, the equation

$$u'' + (a^+(t) - \mu a^-(t))g(u) = 0 \tag{4.1}$$

has at least three positive, even-symmetric, T-periodic solutions.

PROOF. It is enough to observe that, if $v(t)$ is a positive solution to the Neumann problem

$$\begin{cases} v'' + (a^+(t) - \mu a^-(t))g(v) = 0 \\ v'(0) = v'(\frac{T}{2}) = 0, \end{cases}$$

then the function defined by $u(t) := v(|t|)$ for $t \in [-\frac{T}{2}, \frac{T}{2}]$ and extended by T-periodicity is a positive and even-symmetric T-periodic solution of (4.1). The conclusion follows from Theorem 2.1. □

The second corollary deals with the PDE's problem (1.2) when Ω is an annulus and the weight $a(x)$ is radially symmetric. Precisely, we have the following result; notice that no growth restrictions on $g(s)$ are imposed.

Corollary 4.2. *Let $g(s)$ be as in Theorem 2.1. Moreover, let $0 < r_1 < r_2$ and let $a^+, a^- : [r_1, r_2] \to \mathbb{R}$ be continuous functions such that, for some σ, τ with $r_1 < \sigma < \tau < r_2$,*

$$a^-(r) = 0, \quad a^+(r) \geq 0, \not\equiv 0 \quad \forall r \in [r_1, \sigma],$$

$$a^+(r) = 0, \quad a^-(r) \geq 0, \not\equiv 0 \quad \forall r \in [\sigma, \tau],$$

$$a^-(r) = 0, \quad a^+(r) \geq 0, \not\equiv 0 \quad \forall r \in [\tau, r_2].$$

Finally, define $\Omega := \{x \in \mathbb{R}^N \mid r_1 < |x| < r_2\}$. Then there exists $\mu^* > 0$ such that, for every $\mu > \mu^*$, the Neumann problem

$$\begin{cases} \Delta u + (a^+(|x|) - \mu a^-(|x|))g(u) = 0 & x \in \Omega \\ \frac{\partial u}{\partial \nu} = 0 & x \in \partial\Omega \end{cases} \tag{4.2}$$

has at least three positive, radially symmetric, (classical) solutions.

PROOF. It is well known that $u(x)$ is a radially symmetric classical solution of (4.2) if and only if $u(r) = u(|x|)$ satisfies

$$\begin{cases} u'' + \frac{N-1}{r}u'(r) + (a^+(r) - \mu a^-(r))g(u) = 0 \\ u'(r_1) = u'(r_2) = 0. \end{cases} \tag{4.3}$$

Setting

$$[r_1, r_2] \ni r \mapsto h(r) := \frac{\int_{r_1}^{r} \xi^{1-N}\, d\xi}{\int_{r_1}^{r_2} \xi^{1-N}\, d\xi} \in [0, 1],$$

standard calculations show that $u(r)$ is a solution of (4.3) if and only if $v(t) := u(h^{-1}(t))$ solves

$$\begin{cases} v'' + \left(\int_{r_1}^{r_2} \xi^{1-N}\, d\xi\right)^2 h^{-1}(t)^{2(N-1)} \Big(a^+(h^{-1}(t)) - \mu a^-(h^{-1}(t))\Big) g(v) = 0 \\ v'(0) = v'(1) = 0. \end{cases}$$

Since $h^{-1}(t)$ is bounded away from zero on $[0,1]$, the conclusion follows from Theorem 2.1. $\square$

We conclude the paper with some further remarks about problem (1.1) and our result.

Remark 4.1. On the line of [2, 3], if $g : [0,+\infty[\to \mathbb{R}$ is of class C^1 with $g'(s) > 0$ for every $s > 0$, we can give a necessary condition for the existence of a positive solution to (1.1). In fact, writing the equation in the equivalent form

$$\frac{u''(t)}{g(u(t))} = -a(t),$$

and since

$$\int_0^T \frac{u''(t)}{g(u(t))}\, dt = \left[\frac{u'(t)}{g(u(t))}\right]_0^T + \int_0^T g'(u(t)) \left(\frac{u'(t)}{g(u(t))}\right)^2 dt,$$

we get, taking into account the boundary conditions $u'(0) = u'(T) = 0$ as well as the fact that $u' \not\equiv 0$,

$$\int_0^T a(t)\, dt < 0.$$

In this situation, the lower bound

$$\mu^* > \frac{\int_0^T a^+(t)\, dt}{\int_0^T a^-(t)\, dt}$$

can be given in Theorem 2.1 (and in particular in Corollary 1.1).

Remark 4.2. As already observed in the Introduction, the Neumann problem

$$\begin{cases} u'' + a(t)u^{\gamma+1} = 0 \\ u'(0) = u'(T) = 0 \end{cases} \tag{4.4}$$

is really different from the analogous Dirichlet one. In fact, the nonlinearity $g(s) = s^{\gamma+1}$ is no more interacting (near 0) with the principal eigenvalue

$\lambda_1(-u'') = \left(\frac{\pi}{T}\right)^2$ of $-u''$ with Dirichlet boundary conditions. We also notice that the indefiniteness of the weight is not an intrinsic feature of the problem; to the contrary, it is well known that a positive solution always exists in the definite-sign case $a(t) \geq 0$ and never exists in the definite-sign case $a(t) \leq 0$. From both points of view, the natural Dirichlet analogous of problem (4.4) is given by

$$\begin{cases} u'' + \left(\frac{\pi}{T}\right)^2 u + a(t)u^{\gamma+1} = 0 \\ u(0) = u(T) = 0. \end{cases}$$

A detailed bifurcation analysis of the nonlinear eigenvalue problem

$$\begin{cases} \Delta u + \lambda u + a(x)u^{\gamma+1} = 0 & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases} \tag{4.5}$$

is contained in [15]. We stress, however, that all the results about multiple positive solutions of (4.5) we know (see, for instance, [1, 6, 9, 11, 12]) are for $\lambda \neq \lambda_1(-\Delta)$.

Remark 4.3. It can be interesting to compare the superlinear case with the sublinear one, namely

$$\begin{cases} u'' + a(t)u^{\delta} = 0 \\ u'(0) = u'(T) = 0 \end{cases}$$

with $0 < \delta < 1$. This problem has been studied (in the PDE's case) by Bandle, Pozio and Tesei in [2]; again, it turns out that the mean value condition $\int_0^T a(t)\,dt < 0$ is sufficient for the existence of a positive solution, which, in this case, is *unique*. Only nonnegative multiple solutions can exist; of course, this is strictly related to the lack of uniqueness at zero for the Cauchy problems. Our result, hence, shows that the situation in the superlinear case is very different.

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