UNIVERSITÀ DEGLI STUDI DI TORINO

Smoothness and error bounds of Martensen splines

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Abstract

Martensen splines Mf of degree n interpolate f and its derivatives up to the order $n-1$ at a subset of the knots of the spline space, have local support and exactly reproduce both polynomials and splines of degree $\leq n$. An approximation error estimate has been provided for $f \in \mathbb{C}^{n+1}$.

This paper aims to clarify how well the Martensen splines Mf approximate smooth functions on compact intervals. Assuming that $f \in \mathbb{C}^{n-1}$, approximation error estimates are provided for $D^j f$, $j = 0, 1, \ldots, n-1$, where D^j is the jth derivative operator. Moreover, a set of sufficient conditions on the sequence of meshes are derived for the uniform convergence of $D^j Mf$ to $D^j f$, for $j = 0, 1, \ldots, n-1$.

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1 Introduction

In the construction of spline approximation operators it is desiderable to obtain the three properties of locality, interpolation and optimal polynomial reproduction. However, when the knots of the spline space are chosen to coincide with interpolation points, the properties of locality and interpolation are incompatible for quadratic or higher degree splines [2].

Using a procedure based on the introduction of additional knots, De Villiers and Rohwer [2] constructed, for arbitrary order, an optimal nodal spline interpolation operator possessing the three desired properties.

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This idea was introduced by de Villiers and Rohwer [2] as an alternative to quasi-interpolations methods and generalized in the paper of Dahmen, Goodman and Michelli [1], where the authors studied minimal Hermite spline interpolation which was first investigated by Martensen [11].

Considering the two point Hermite spline interpolation scheme studied by Martensen [11], Siewer [14, 15] constructed the Martensen splines Mf of order $n+1$ (degree $\leq n$) obtaining the properties of locality, interpolation of f and its derivatives up to the order $n-1$ at a subset of the knots of the spline space and optimal polynomials and splines reproduction. Approximation properties of Mf have been considered in [14], where an error estimate has been provided for $f \in \mathbb{C}^{n+1}$. Martensen splines in the case of equidistant knots and bivariate constructions using Boolean methods have been studied respectively in [4] and [5].

In the present paper, we will continue the investigation of Siewer on how well the Martensen splines Mf approximate smooth functions f on compact intervals. Assuming that $f \in \mathbb{C}^{n-1}$, we shall provide approximation error estimates for $D^j f$, $j = 0, 1, \ldots, n-1$, where D^j is the jth derivative operator. Moreover, we shall give a set of sufficient conditions on the sequence of spline knots for the uniform convergence of $D^j Mf$ to $D^j f$ for $j = 0, 1, \ldots, n-1$. In virtue of their approximation properties, Martensen splines can be used for the numerical evaluation of certain finite-part integrals [7, 8, 12].

2 Martensen splines approximation

In this section, we give the necessary background material on Martensen splines based on the works in [14] and [15].

Let $T = \{t_j\}_{j\in\mathbb{Z}}$ be a strictly increasing sequence of points in $\mathbb{R}$. We write $\mathbb{P}^n$ for the set of polynomials of degree n and $S_{n+1}(T)$ for the set of polynomial splines of order $n+1$, with simple knots at the points t_j, so that $S_{n+1}(T) \subset \mathbb{C}^{n-1}(\mathbb{R})$.

We denote by $\{B_{j,m}(x)\}_{j\in\mathbb{Z}}$ the set of normalized B-splines of order m on T, having support $[t_j, t_{j+m}]$ and defined by [13]

$$B_{j,m}(x) := (-1)^m (t_{j+m} - t_j)[t_j, \ldots, t_{j+m}](x - \cdot)_+^{m-1},$$

where the symbol $[t_j, \ldots, t_{j+m}]$ denotes the mth-order divided-difference functional and

$$x_+^r = \begin{cases} x^r, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

The following theorem, stated and proved in [11], specifies the Martensen interpolation scheme

Theorem 1 [11] *Let* α_0^k, α_n^k, *with* $k = 0, \ldots, n-1$, *be arbitrarily given real numbers. For* $n \in \mathbb{N}$ *and the set of knots* $T_n = \{\alpha = t_0 < t_1 < \cdots < t_n = \beta\}$, *there is a uniquely determined spline* $H_n(t), t \in [\alpha, \beta]$, *with* $H_n \in S_{n+1}(T_n)$, *satisfying*

$$D^k H_n(t_0) = \alpha_0^k, \;\; k = 0, \ldots, n-1,$$

$$D^k H_n(t_n) = \alpha_n^k, \;\; k = 0, \ldots, n-1.$$

A generalization of this theorem has been obtained by considering nonsymmetrical interpolation conditions [1].

For $n \in \mathbb{N}$, $j \in \mathbb{Z}$ and $0 \le i < n$, Siewer [14] provides a constructive proof for Theorem 1 by defining recursively the fundamental Hermite splines $G_{i,n,t_j,\ldots,t_{j+n}}(x) \in S_{n+1}(T)$ and $H_{i,n,t_j,\ldots,t_{j+n}}(x) \in S_{n+1}(T)$ satisfying

$$D^k G_{i,n,t_j,\ldots,t_{j+n}}(t_j) = 0, \quad i,k = 0, \ldots, n-1,$$
$$D^k G_{i,n,t_j,\ldots,t_{j+n}}(t_{j+n}) = \delta_{i,k}, \quad i,k = 0, \ldots, n-1,$$

and

$$D^k H_{i,n,t_j,\ldots,t_{j+n}}(t_j) = \delta_{i,k}, \quad i,k = 0, \ldots, n-1,$$
$$D^k H_{i,n,t_j,\ldots,t_{j+n}}(t_{j+n}) = 0, \quad i,k = 0, \ldots, n-1.$$

Here, we are interested in the representation of $G_{i,n,t_j,\ldots,t_{j+n}}$ and $H_{i,n,t_j,\ldots,t_{j+n}}$ as a linear combination of B-splines $B_{s,n+1}$. In order to obtain this expansion, we need the Marsden identity

Theorem 2 [10] *Given any increasing (not necessarily strictly) knot sequence* $T = \{t_j\}_{j \in \mathbb{Z}}$ *and two indices* $l \le r$ *with* $t_l < t_{r+1}$, *for all* $y \in \mathbb{R}$ *and all* $x \in [t_l, t_{r+1})$ *the following identity holds*

$$(y-x)^n = \sum_{s=l-n}^{r} \varphi_{s,n}(y) B_{s,n+1}(x), \tag{1}$$

where

$$\varphi_{s,n}(y) := \prod_{\nu=1}^{n} (y - t_{s+\nu}) \in \mathbb{P}^n \tag{2}$$

is the dual polynomial for $B_{s,n+1}$.

Taking in account of the strict monotonicity of the considered knot sequence T, the following B-Spline expansion for fundamental Hermite splines is provided in [14].

Theorem 3 [14] *Let $n \in \mathbb{N}$ and $i \in \{1, \dots, n\}$. For $x \in [t_j, t_{j+n}]$, $j \in \mathbb{Z}$, the fundamental Hermite splines for the Martensen interpolation allow the B-Spline expansion*

$$G_{n-i,n,t_j,\dots,t_{j+n}}(x) = \sum_{s=j}^{j+n-1} \frac{(-1)^{n-i}}{n!} \frac{d^i}{dy^i} \varphi_{s,n}(t_{j+n}) B_{s,n+1}(x) \tag{3}$$

and

$$H_{n-i,n,t_j,\dots,t_{j+n}}(x) = \sum_{s=j-n}^{j-1} \frac{(-1)^{n-i}}{n!} \frac{d^i}{dy^i} \varphi_{s,n}(t_j) B_{s,n+1}(x). \tag{4}$$

Let $r \in \mathbb{Z}$, $n \in \mathbb{N}$ and $i \in \{0, \dots, n-1\}$. We denote by Hermite-Martensen spline (HM-spline) the uniquely determined spline $F_i^{(rn)} \in S_{n+1}(T)$ satisfying the following conditions [15]

$$supp(F_i^{(rn)}) \subseteq [t_{rn-n}, t_{rn+n}],$$

$$D^k F_i^{(rn)}(t_{rn}) = \delta_{i,k}, \quad k = 0, \dots, n-1.$$

These requirements lead to the problem of finding the spline which satisfies the interpolation conditions

$$D^k F_i^{(rn)}(t_{rn-n}) = 0, \quad k = 0, \dots, n-1,$$

$$D^k F_i^{(rn)}(t_{rn}) = \delta_{i,k}, \quad k = 0, \dots, n-1,$$

$$D^k F_i^{(rn)}(t_{rn+n}) = 0, \quad k = 0, \dots, n-1.$$

By Theorem 1, we can define the HM-spline $F_i^{(rn)}$ in terms of the fundamental Hermite splines [15]

$$F_i^{(rn)}(x) = \chi_{(-\infty,t_{rn}]}(x) G_{i,n,t_{rn-n},\dots,t_{rn}}(x) + \chi_{[t_{rn},\infty)}(x) H_{i,n,t_{rn},\dots,t_{rn+n}}(x).$$

By using the B-spline expansions (3) and (4)

$$H_{i,n,t_{rn},\dots,t_{rn+n}}(x) = \sum_{s=rn-n}^{rn-1} \frac{(-1)^i}{n!} \frac{d^{n-i}}{dy^{n-i}} \varphi_{s,n}(t_{rn}) B_{s,n+1}(x), \quad x \in [t_{rn}, t_{rn+n}]$$

and

$$G_{i,n,t_{rn-n},\dots,t_{rn}}(x) = \sum_{s=rn-n}^{rn-1} \frac{(-1)^i}{n!} \frac{d^{n-i}}{dy^{n-i}} \varphi_{s,n}(t_{rn}) B_{s,n+1}(x), \quad x \in [t_{rn-n}, t_{rn}],$$

we can write the HM-splines in the form

$$F_i^{(rn)}(x) = \sum_{s=rn-n}^{rn-1} \frac{(-1)^i}{n!} \frac{d^{n-i}}{dy^{n-i}} \varphi_{s,n}(t_{rn}) B_{s,n+1}(x), \quad x \in [t_{rn-n}, t_{rn+n}].$$

The Martensen operator M on the space $\mathbb{C}^{n-1}(\mathbb{R})$, introduced in [15], is defined by

$$M : C^{n-1}(\mathbb{R}) \to S_{n+1}(T), \quad f(x) \to \sum_{r\in\mathbb{Z}} \sum_{i=0}^{n-1} D^i f(t_{rn}) F_i^{(rn)}(x). \tag{5}$$

The following theorem [15] summarizes the main properties of M.

Theorem 4 [15] *The Martensen operator M, defined by (5), satisfies the properties:*

1. *Mf is uniquely determined by the interpolation conditions*

$$D^k Mf(t_{rn}) = D^k f(t_{rn}), \quad k = 0, \dots, n-1, \ \ r \in \mathbb{Z};$$

2. *$Mf = f$ for all $f \in \mathbb{P}^n$;*
3. *$Mf = f$ for all $f \in S_{n+1}(T)$.*

The interpolation conditions at the knots $\{t_{rn}\}$ suggest a subdivision of the knots in primary and secondary ones. The set of primary knots T_p is defined by

$$T_p := \{t_{rn} | r \in \mathbb{Z}\},$$

whereas the set of secondary knots T_s is given by

$$T_s := \{t_j | j \not\equiv 0 \bmod n, \ j \in \mathbb{Z}\}.$$

Assuming that $f \in C^{n-1}([t_{kn}, t_{kn+n}]), k \in \mathbb{Z}$, and $x \in [t_{kn}, t_{kn+n}]$, we have

$$Mf(x) = \sum_{i=0}^{n-1} (D^i f(t_{kn}) H_{i,n,t_{kn},\dots,t_{kn+n}}(x) + D^i f(t_{kn+n}) G_{i,n,t_{kn},\dots,t_{kn+n}}(x)). \tag{6}$$

We denote by $\overline{M}f$ the approximation error

$$\overline{M}f(x) := f(x) - Mf(x).$$

The following local estimate of the approximation error $\overline{M}f$ holds

Theorem 5 [14] *For $f \in C^{n+1}([t_{kn}, t_{kn+n}])$ and $x \in [t_{kn}, t_{kn+n}]$, it holds*

$$\left|\overline{M}f(x)\right| \leq \left\|f^{(n+1)}\right\|_\infty (t_{kn+n} - t_{kn})^{n+1} \left(\frac{1}{n!} + \sum_{i=0}^{n-1} \frac{2^i}{(n-i)!} \right),$$

where

$$\|g\|_\infty = \max_{x\in[a,b]} |g(x)|, \quad \forall g \in C([a,b]).$$

The Martensen operator M for $f \in \mathbb{C}^{n-1}([a,b])$, with $[a,b] = [t_0, t_{Rn}]$, reduces to a finite sum. For any $x \in [a,b]$, we have [15]

$$M_R f(x) = \sum_{r=0}^{R} \sum_{i=0}^{n-1} D^i f(t_{rn}) F_i^{(rn)}(x). \tag{7}$$

Remark. The representation (7) requires the auxiliary knots $t_{-n} < \cdots < t_{-1}$ at the left of t_0 and $t_{Rn+1} < \cdots < t_{Rn+n}$ at the right of t_{Rn}. These auxiliary knots can be arbitrarily choosen because we don't need values of f at $x \notin [a,b]$. We shall choose the auxiliary knots simmetrically distributed with respect to t_0 and t_{Rn}

$$t_{-i} = 2t_0 - t_i \quad \text{and} \quad t_{Rn+i} = 2t_{Rn} - t_{Rn-i}, \quad i = 1, \ldots, n.$$

Let T_R be a partition of $[a,b]$, $T_R := \{a = t_0 < \cdots < t_{Rn} = b\}$, and $M_R f \in S_{n+1}(T_R)$ the Martensen spline defined by (7) on $[a,b]$. We define

$$h_k := t_{kn+n} - t_{kn}, \ \text{ with } \ k = 0, \ldots, R-1, \tag{8}$$

and we denote by H_R the norm of the primary partition $T_{R,p} := \{t_{rn} \,|\, 0 \le r \le R\}$

$$H_R := \max_{k=0,\ldots,R-1} h_k. \tag{9}$$

We denote by $\overline{M}_R f$ the approximation error $\overline{M}_R f := f - M_R f$. From Theorem 5 we can easily derive a uniform convergence result on $[a,b]$.

Corollary 1 *For any $f \in C^{n+1}([a,b])$ and $x \in [a,b]$ it holds*

$$|\overline{M}_R f(x)| \le \left\| f^{(n+1)} \right\|_\infty H_R^{n+1} \left(\frac{1}{n!} + \sum_{i=0}^{n-1} \frac{2^i}{(n-i)!} \right), \tag{10}$$

with H_R defined by (9).

Moreover, if we assume that the sequence of primary partitions $\{T_{R,p}\}_{R\in\mathbb{N}}$ is such that

$$H_R \to 0, \quad \textit{as} \ \ R \to \infty, \tag{11}$$

then

$$\left\| \overline{M}_R f \right\|_\infty \to 0, \quad \textit{as} \ \ R \to \infty.$$

Proof. From Theorem 5 we obtain the uniform bound (10) since, by (8) and (9), it holds

$$h_k \le H_R.$$

□

3 Smoothness of the operator M_R

In this section, we continue the study on how well the Martensen operator M_R approximates a smooth function f. Assuming that $f \in \mathbb{C}^{n-1}([a,b])$, we derive approximation error estimates for $D^j f$, $j = 0,1,\dots,n-1$, and we provide sufficient conditions on the sequence of spline knots for the uniform convergence of $D^j M_R f$ to $D^j f$ for $j = 0,1,\dots,n-1$.

We define

$$A_R := \max_{\substack{0\le i,j\le Rn-1\\ |i-j|=1}} \frac{t_{i+1}-t_i}{t_{j+1}-t_j}, \quad A_R \ge 1 \tag{12}$$

and

$$\tilde{A}_R := \max_{\substack{0\le i,j\le R-1\\ |i-j|=1}} \frac{h_i}{h_j}, \quad \tilde{A}_R \ge 1, \tag{13}$$

with h_i and h_j defined by (8). For $\tilde{A}_R$, defined in (13), the following inequality holds [6, Lemma 3.5],

$$\tilde{A}_R \le \sum_{\nu=1}^{n} A_R^{\nu}. \tag{14}$$

We say that the sequence of partitions $\{T_R\}_{R\in\mathbb{N}}$ ($\{T_{R,p}\}$) is locally uniform if there exists a constant $A \ge 1$ ($\tilde{A} \ge 1$) such that $A_R \le A$ ($\tilde{A}_R \le \tilde{A}$) for all $R \in \mathbb{N}$, with A_R and $\tilde{A}_R$ defined by (12) and (13) respectively.

Given $f \in C^{n-1}([a,b])$ and $x \in [a,b]$, we define

$$e_R^{(s)}(x) := D^s \overline{M}_R f(x), \quad \text{with } 0 \le s \le n-1.$$

By considering the Taylor expansion of order $n-1$, we can write

$$f(x) = T_{n-1}(x) + R_T(x).$$

Using property 2. of Theorem 4 and linearity of M_R, we have

$$\begin{aligned} e_R^{(0)}(x) &= f(x) - M_R f(x) = T_{n-1}(x) + R_T(x) - M_R(T_{n-1}+R_T)(x) \\ &= T_{n-1}(x) + R_T(x) - M_R T_{n-1}(x) - M_R R_T(x) \\ &= R_T(x) - M_R R_T(x) \end{aligned} \tag{15}$$

and

$$e_R^{(s)}(x) = D^s(R_T - M_R R_T)(x), \;\text{ with }\; 1 \le s \le n-1. \tag{16}$$

Assuming that $x \in [t_{kn}, t_{kn+n}]$, with $k = 0,\dots,R-1$, from (3), (4), (6) and by differentiating $M_R f(x)$ we get

$$D^s M_R f(x) = \sum_{i=0}^{n-1} \left\{ D^i f(t_{kn}) D^s H_{i,n,t_{kn},\dots,t_{kn+n}}(x) + D^i f(t_{kn+n}) D^s G_{i,n,t_{kn},\dots,t_{kn+n}}(x) \right\}, \tag{17}$$

where

$$D^s H_{i,n,t_{kn},\dots,t_{kn+n}}(x) = \sum_{j=kn-n}^{kn-1} \frac{(-1)^i}{n!} \frac{d^{n-i}}{dy^{n-i}} \varphi_{j,n}(t_{kn}) D^s B_{j,n+1}(x), \tag{18}$$

$$D^s G_{i,n,t_{kn},\dots,t_{kn+n}}(x) = \sum_{j=kn}^{kn+n-1} \frac{(-1)^i}{n!} \frac{d^{n-i}}{dy^{n-i}} \varphi_{j,n}(t_{kn+n}) D^s B_{j,n+1}(x). \tag{19}$$

The derivatives in (18) and (19) are expressed in terms of normalized B-splines derivatives $D^s B_{j,n+1}(x)$. A bound for these derivatives is provided by the following lemma, which can be quoted as a special case of [9, Lemma 2.1].

Lemma 1 *Let $B_{j,n+1}(x)$ be the normalized B-spline of degree n defined on the knots $t_j < \cdots < t_{j+n+1}$. Suppose $x \in [t_l, t_{l+1}]$, with $j \le l < j+n+1$. Fix $0 < s \le n$, then $D^s B_{j,n+1}(x)$ exists and*

$$\left| D^s B_{j,n+1}(x) \right| \le \frac{\Gamma_{n+1,s}}{\delta_{j,l,n} \dots \delta_{j,l,n-s+1}}, \tag{20}$$

where, for $k = n-s+1, \dots, n$, we define $\delta_{j,l,k}$ as the minimum of $t_{r+k} - t_r$, over r such that $t_j \le t_r \le t_l < t_{l+1} \le t_{r+k} \le t_{j+n+1}$, and where

$$\Gamma_{n+1,s} = \frac{n!}{(n-s)!} \binom{s}{[s/2]},$$

with $[s/2]$ = greatest integer less than or equal to $s/2$.

3.1 Local estimates

The purpose of this section is to obtain local estimates of $|e_R^{(s)}(t)|$, $s = 0, 1, \dots, n-1$, for $t \in [t_l, t_{l+1}] \subset [t_{kn}, t_{kn+n}]$, with $0 \le k \le R-1$.

We consider the remainder $R_T(x)$ of the $(n-1)$-order Taylor expansion of f at $t \in [t_l, t_{l+1}] \subset [t_{kn}, t_{kn+n}]$

$$R_T(x) = f(x) - \sum_{j=0}^{n-1} \frac{f^{(j)}(t)}{j!} (x-t)^j. \tag{21}$$

By (15), (16) and considering that $R_T(x)$, defined in (21), and its derivatives up to the order $n-1$ are zero at $x = t$, we can write

$$e_R^{(s)}(t) = D^s M_R R_T(t), \quad 0 \le s \le n-1. \tag{22}$$

From (22), in order to bound $|e_R^{(s)}(t)|$, $0 \le s \le n-1$, we only have to estimate $|D^s M_R R_T(t)|$.

From (6) and (17), we can write for $x = t$ and $0 \le s \le n-1$

$$D^s M_R R_T(t) = \sum_{i=0}^{n-1} \Big(D^i R_T(t_{kn}) D^s H_{i,n,t_{kn},\dots,t_{kn+n}}(t) + D^i R_T(t_{kn+n}) D^s G_{i,n,t_{kn},\dots,t_{kn+n}}(t) \Big), \tag{23}$$

and

$$\left| D^s M_R R_T(t) \right| \le \sum_{i=0}^{n-1} \Big(\left| D^i R_T(t_{kn}) \right| \left| D^s H_{i,n,t_{kn},\dots,t_{kn+n}}(t) \right| + \left| D^i R_T(t_{kn+n}) \right| \left| D^s G_{i,n,t_{kn},\dots,t_{kn+n}}(t) \right| \Big).$$

We first estimate $\left| D^i R_T(t_{kn}) \right|$ and $\left| D^i R_T(t_{kn+n}) \right|$.

Lemma 2 *Let $f \in C^{n-1}([t_{kn}, t_{kn+n}])$. For $i = 0, \dots, n-1$ it holds*

$$\left| D^i R_T(t_{kn}) \right| \le \frac{h_k^{n-i-1}}{(n-i-1)!} \omega(D^{n-1} f; h_k; [t_{kn}, t_{kn+n}]), \tag{24}$$

where, for all continuous function $g \in C(J)$, $\omega(g; \Delta; J)$ is the modulus of continuity of g on J [16]:

$$\omega(g; \Delta; J) = \max_{\substack{x, x+h \in J \\ 0 < h \le \Delta}} |g(x+h) - g(x)|.$$

The estimate (24) is also true for $\left| D^i R_T(t_{kn+n}) \right|$.

Proof. By $(n-2)$–order Taylor expansion of $R_T(x)$, defined in (21), at $t \in [t_l, t_{l+1}] \subset [t_{kn}, t_{kn+n}]$, we can express $R_T(x)$ in the form

$$R_T(x) = \frac{D^{n-1} R_T(\eta)}{(n-1)!} (x-t)^{n-1}, \tag{25}$$

with η between x and t. By differentiating i-times (25), with $0 \le i \le n-1$, we have at $x = t_{kn}$

$$D^i R_T(t_{kn}) = \frac{D^{n-1} R_T(\tilde{\eta})}{(n-i-1)!} (t_{kn} - t)^{n-i-1}, \quad \text{with } \tilde{\eta} \in (t_{kn}, t). \tag{26}$$

Since $t \in [t_l, t_{l+1}] \subset [t_{kn}, t_{kn+n}]$, we can write

$$|t_{kn} - t| \le t_{kn+n} - t_{kn} = h_k. \tag{27}$$

By differentiating $(n-1)$-times (21), we have at $x = \tilde{\eta}$

$$\begin{aligned} \left| D^{n-1} R_T(\tilde{\eta}) \right| &= \left| D^{n-1} f(\tilde{\eta}) - D^{n-1} f(t) \right| \le \\ &\le \max_{\substack{\tilde{\eta}, t \in [t_{kn}, t_{kn+n}] \\ 0 < |t - \tilde{\eta}| \le h_k}} \left| D^{n-1} f(\tilde{\eta}) - D^{n-1} f(t) \right| \\ &= \omega(D^{n-1} f; h_k; [t_{kn}, t_{kn+n}]). \end{aligned} \tag{28}$$

From (26), (27) and (28) we obtain inequality (24). Similarly, we can prove that the estimate (24) holds for $\left|D^i R_T(t_{kn+n})\right|$.

□

The following lemma provides local estimates for the derivatives of fundamental Hermite splines.

Lemma 3 *Suppose $t \in [t_l, t_{l+1}] \subset [t_{kn}, t_{kn+n}]$. Then*

$$\left|D^s G_{i,n,t_{kn},\dots,t_{kn+n}}(t)\right| \le \begin{cases} \dfrac{n\tilde{A}_R^i h_k^i}{i!}, & \textit{for } \ s=0, \\ \dfrac{n!}{i!}\dfrac{\Gamma_{n+1,s}}{(n-1)!}\dfrac{\tilde{A}_R^i h_k^i}{(t_{l+1}-t_l)^s}, & \textit{for } \ 1 \le s \le n. \end{cases} \tag{29}$$

The estimate (29) is also true for $\left|D^s H_{i,n,t_{kn},\dots,t_{kn+n}}(t)\right|$.

Proof. By evaluating (3) at $t \in [t_l, t_{l+1}]$ and considering that $B_{s,n+1}(t) \le 1$, we obtain

$$\left|G_{i,n,t_{kn},\dots,t_{kn+n}}(t)\right| \le \sum_{j=kn}^{kn+n-1} \frac{1}{n!}\left|\frac{d^{n-i}}{dy^{n-i}}\varphi_{j,n}(t_{kn+n})\right|. \tag{30}$$

We first estimate the derivatives of $\varphi_{j,n}$, defined in (2). The ith derivative of $\varphi_{j,n}(y)$ is

$$\frac{d^i}{dy^i}\varphi_{j,n}(y) = \sum_{k_1=1}^{n}\sum_{\substack{k_2=1\\k_2\neq k_1}}^{n}\cdots\sum_{\substack{k_i=1\\k_i\neq k_1,\dots,k_{i-1}}}^{n}\prod_{\substack{\nu=1\\\nu\neq k_1,\dots,k_i}}^{n}(y-t_{j+\nu}).$$

By evaluating the derivative at $y = t_{kn+n}$, we obtain

$$\left|\frac{d^i}{dy^i}\varphi_{j,n}(t_{kn+n})\right| \le \sum_{k_1=1}^{n}\sum_{\substack{k_2=1\\k_2\neq k_1}}^{n}\cdots\sum_{\substack{k_i=1\\k_i\neq k_1,\dots,k_{i-1}}}^{n}\prod_{\substack{\nu=1\\\nu\neq k_1,\dots,k_i}}^{n}\left|t_{kn+n}-t_{j+\nu}\right|.$$

For all $j = kn, \dots, kn+n-1$ and $\nu = 1, \dots, n$, it holds $t_{j+\nu} \in [t_{kn}, t_{(k+1)n}]$ or $t_{j+\nu} \in (t_{(k+1)n}, t_{(k+2)n}]$ as the case may be. By (13) it holds

$$\left|t_{kn+n}-t_{j+\nu}\right| \le \tilde{A}_R h_k$$

and we can write

$$\left|\frac{d^i}{dy^i}\varphi_{j,n}(t_{kn+n})\right| \le \frac{n!}{(n-i)!}\tilde{A}_R^{n-i}h_k^{n-i}. \tag{31}$$

By inserting (31) into (30) we obtain

$$\left|G_{i,n,t_{kn},\dots,t_{kn+n}}(t)\right| \le \sum_{j=kn}^{kn+n-1}\frac{1}{n!}\frac{n!}{i!}\tilde{A}_R^i h_k^i = \frac{n\tilde{A}_R^i h_k^i}{i!}.$$

From (19), we can write for $x = t \in [t_l, t_{l+1}]$ and $1 \le s \le n$

$$\left|D^s G_{i,n,t_{kn},\dots,t_{kn+n}}(t)\right| \le \frac{1}{n!} \sum_{j=kn}^{kn+n-1} \left|\frac{d^{n-i}}{dy^{n-i}} \varphi_{j,n}(t_{kn+n})\right| \left|D^s B_{j,n+1}(t)\right|. \tag{32}$$

By inserting inequalities (31) and (20) into (32), we obtain

$$\left|D^s G_{i,n,t_{kn},\dots,t_{kn+n}}(t)\right| \le \frac{n!\Gamma_{n+1,s}}{(n-1)!i!\delta_{j,l,n}\dots\delta_{j,l,n-s+1}} \tilde{A}_R^i h_k^i.$$

The thesis (29) follows from inequalities

$$\delta_{j,l,k} \ge t_{l+1} - t_l \quad \text{for all} \quad k = n-s+1,\dots,n.$$

Since the inequality (31) holds for $\left|\frac{d^i}{dy^i}\varphi_{j,n}(t_{kn})\right|$, $j = kn-n,\dots,kn-1$, the estimate (29) is true for $\left|D^s H_{i,n,t_{kn},\dots,t_{kn+n}}(t)\right|$.

□

The following theorem provides a local estimate for $e_R^{(s)}(t)$ defined by (22).

Theorem 6 *Suppose $t \in [t_l, t_{l+1}] \subset [t_{kn}, t_{kn+n}]$ and let $f \in C^{n-1}([t_{kn}, t_{kn+n}])$. Then*

$$\left|e_R^{(s)}(t)\right| \le \begin{cases} K_0 h_k^{n-1} \omega(D^{n-1}f; h_k; [t_{kn}, t_{kn+n}]), & \text{for } s = 0, \\ K_{l,s} h_k^{n-1-s} \omega(D^{n-1}f; h_k; [t_{kn}, t_{kn+n}]), & \text{for } 1 \le s \le n-1, \end{cases} \tag{33}$$

where

$$K_0 := \frac{2}{(n-1)!} \sum_{i=0}^{n-1} (n-i) \binom{n}{i} \tilde{A}_R^i \tag{34}$$

and

$$K_{l,s} := \frac{2\Gamma_{n+1,s}}{(n-1)!} \frac{h_k^s}{(t_{l+1}-t_l)^s} \sum_{i=0}^{n-1} (n-i) \binom{n}{i} \tilde{A}_R^i. \tag{35}$$

Proof. The first inequality in (33) follows from Lemma 2 and from (22), (23) and (29) with $s = 0$,

$$\begin{aligned} |M_R R_T(t)| &\le 2 \sum_{i=0}^{n-1} \left(\frac{h_k^{n-i-1}}{(n-i-1)!} \omega(D^{n-1}f; h_k; [t_{kn}, t_{kn+n}]) \frac{n h_k^i}{i!} \tilde{A}_R^i \right) \\ &= \frac{2h_k^{n-1}\omega(D^{n-1}f; h_k; [t_{kn}, t_{kn+n}])}{(n-1)!} \sum_{i=0}^{n-1} (n-i) \binom{n}{i} \tilde{A}_R^i \end{aligned}$$

The second inequality in (33) follows from (22), (23), Lemma 2 and (29) with $1 \le s \le n-1$,

$$
\begin{aligned}
\left|D^s M_R R_T(t)\right| &\le 2\sum_{i=0}^{n-1}\left(\frac{h_k^{n-i-1}}{(n-i-1)!}\omega(D^{n-1}f;h_k;[t_{kn},t_{kn+n}])\frac{n!\Gamma_{n+1,s}}{(n-1)!i!}\frac{h_k^i}{(t_{l+1}-t_l)^s}\tilde{A}_R^i\right) \\
&= \frac{2h_k^{n-1}\Gamma_{n+1,s}\omega(D^{n-1}f;h_k;[t_{kn},t_{kn+n}])}{(n-1)!(t_{l+1}-t_l)^s}\sum_{i=0}^{n-1}(n-i)\binom{n}{i}\tilde{A}_R^i.
\end{aligned}
$$

□

3.2 Uniform bounds

The following uniform bounds can be derived from the local estimates of Theorem 6

Theorem 7 *Let $f \in C^{n-1}([a,b])$, then*

$$
\left\|e_R^{(s)}\right\|_\infty \le \begin{cases} K_0 H_R^{n-1}\omega(D^{n-1}f;H_R;[a,b]), & \text{for } s=0, \\ \overline{K}_s H_R^{n-1-s}\omega(D^{n-1}f;H_R;[a,b]), & \text{for } 1 \le s \le n-1, \end{cases} \tag{36}
$$

where K_0 is defined in (34) and

$$
\overline{K}_s := \frac{2\Gamma_{n+1,s}}{(n-1)!}\left(\sum_{j=kn}^{l}A_R^{l-j}+\sum_{j=l+1}^{kn+n-1}A_R^{j-l}\right)^s\sum_{i=0}^{n-1}(n-i)\binom{n}{i}\left(\sum_{\nu=1}^{n}A_R^{\nu}\right)^i.
$$

Proof.

For $s=0$, the constant K_0, defined in (34), does not depend on $[t_{kn},t_{kn+n}]$. Moreover, since $h_k \le H_R$, from monotonicity of modulus of continuity

$$
\omega(D^{n-1}f;h_k;[t_{kn},t_{kn+n}]) \le \omega(D^{n-1}f;h_k;[a,b]) \le \omega(D^{n-1}f;H_R;[a,b]). \tag{37}
$$

The first inequality in (36) follows immediately from (37).

For $1 \le s \le n-1$, by (8) we can write for $[t_l,t_{l+1}] \subset [t_{kn},t_{kn+n}]$

$$
h_k = \sum_{j=kn}^{l-1}(t_{j+1}-t_j)+(t_{l+1}-t_l)+\sum_{j=l+1}^{kn+n-1}(t_{j+1}-t_j).
$$

For $j<l$ we have

$$
t_{j+1}-t_j \le A_R(t_{j+2}-t_{j+1}) \le \cdots \le A_R^{l-j}(t_{l+1}-t_l). \tag{38}
$$

Similarly, for $j > l$,

$$t_{j+1} - t_j \le A_R(t_j - t_{j-1}) \le \cdots \le A_R^{j-l}(t_{l+1} - t_l). \tag{39}$$

From (38) and (39) we can write

$$h_k \le (t_{l+1} - t_l)\left(\sum_{j=kn}^{l-1} A_R^{l-j} + 1 + \sum_{j=l+1}^{kn+n-1} A_R^{j-l}\right) = (t_{l+1} - t_l)\left(\sum_{j=kn}^{l} A_R^{l-j} + \sum_{j=l+1}^{kn+n-1} A_R^{j-l}\right). \tag{40}$$

By (40), it holds for $1 \le s \le n-1$

$$\frac{(t_{kn+n} - t_{kn})^s}{(t_{l+1} - t_l)^s} \le \left(\sum_{j=kn}^{l} A_R^{l-j} + \sum_{j=l+1}^{kn+n-1} A_R^{j-l}\right)^s, \tag{41}$$

by (14) and inserting (41) into (35), we obtain

$$K_{l,s} \le \frac{2\Gamma_{n+1,s}}{(n-1)!}\left(\sum_{j=kn}^{l} A_R^{l-j} + \sum_{j=l+1}^{kn+n-1} A_R^{j-l}\right)^s \sum_{i=0}^{n-1}(n-i)\binom{n}{i}\left(\sum_{\nu=1}^{n} A_R^{\nu}\right)^i =: \overline{K}_s.$$

The second inequality in (36) follows from (37)

□

3.3 Uniform convergence

The following uniform convergence results follow immediately from Theorem 7

Corollary 2 *Assume that $f \in C^{n-1}([a,b])$ and (11) holds.*
If the sequence of primary partitions $\{T_{R,p}\}_{R\in\mathbb{N}}$ is locally uniform, then

$$\left\|e_R^{(0)}\right\|_\infty \to 0 \quad as \quad R \to \infty. \tag{42}$$

If the sequence of partitions $\{T_R\}_{R\in\mathbb{N}}$ is locally uniform, then for $1 \le s \le n-1$

$$\left\|e_R^{(s)}\right\|_\infty \to 0 \quad as \quad R \to \infty. \tag{43}$$

Proof. The theses (42) and (43) follows immediately from Theorem 7.

□

4 Numerical examples

In this section, we present some numerical examples to illustrate the results given in the above sections.

For increasing values of R, we construct the cubic Martensen splines $M_R f$ on uniform partitions T_R of $[a,b] = [-5,5]$. We denote by $E_R(f)$ the maximum norm of the error $e_R^{(0)} = f - M_R f$

$$E_R(f) := \max_{0 \le j \le 3R-1} \left| e_R^{(0)} \left(\frac{t_j + t_{j+1}}{2} \right) \right|.$$

We consider the C^2 function $f = f_1$ and the smooth functions $f = f_j$, $j = 2,3$, where

$$f_1(x) = x^4 + |x|^{\frac{5}{2}},$$
$$f_2(x) = \tanh(x) + 1,$$
$$f_3(x) = \frac{1}{1+x^2}.$$

The results in Table 1 confirm the error bound (36) with $s = 0$.

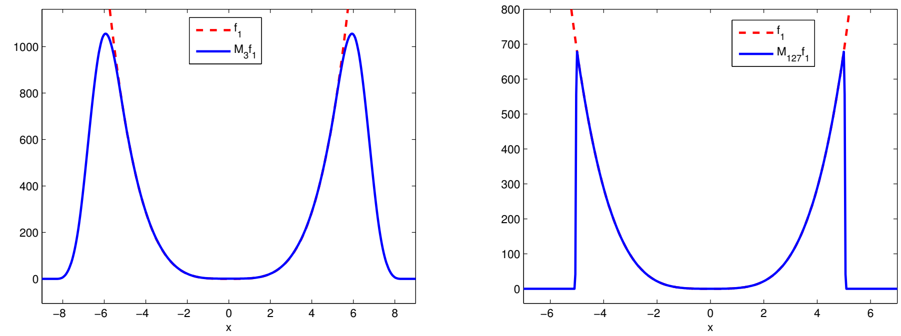
R	$\lvert E_R(f_1)\rvert$	$\lvert E_R(f_2)\rvert$	$\lvert E_R(f_3)\rvert$
3	8.49e-001	9.08e-003	3.82e-001
7	2.88e-002	1.03e-003	3.02e-002
15	3.53e-003	1.41e-004	2.65e-004
31	7.23e-004	1.04e-005	5.42e-005
63	1.31e-004	7.20e-007	4.08e-006
127	2.32e-005	4.49e-008	2.62e-007

Table 1: Maximum norm of error

Finally, to illustrate the behaviour of $M_R f$, the functions f_j and the corresponding Martensen splines $M_R f_j$, $j = 1,2,3$, are represented in Figures 1, 2 and 3 for $R = 3$ and $R = 127$.

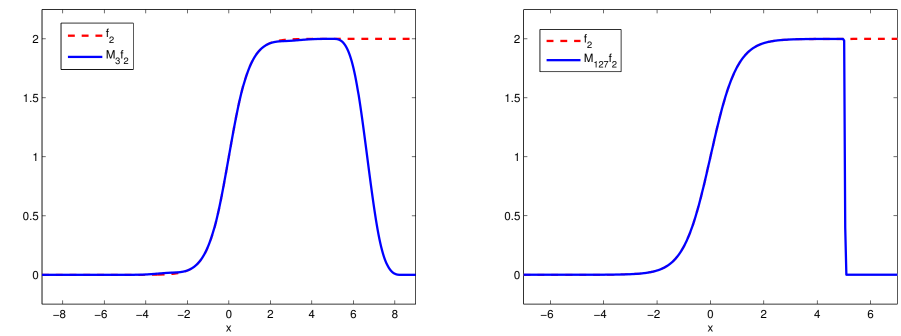
5 Conclusions

The paper studies how well the Martensen spline operator M_R approximates a smooth function $f \in \mathbb{C}^{n-1}([t_0, t_{Rn}])$. Approximation error estimates for f and its derivatives and a set of sufficient conditions for the uniform convergence of $D^j M_R f$ to $D^j f$, $j = 0,1,\ldots,n-1$, are provided.



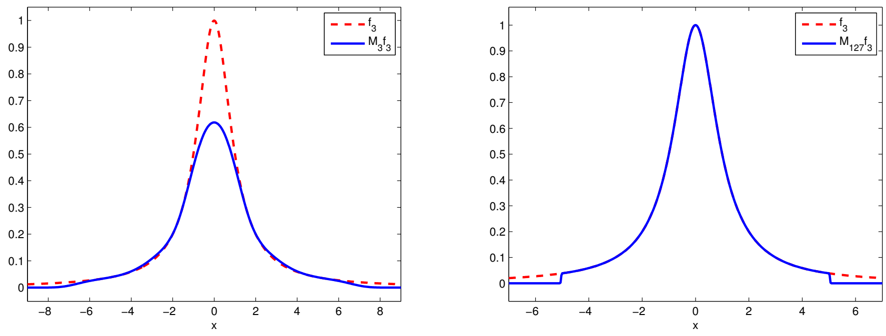
(a) (b)

Figure 1: Graphical representation of f_1 and $M_R f_1$, with $R = 3$ (a) and $R = 127$ (b).



(a) (b)

Figure 2: Graphical representation of f_2 and $M_R f_2$, with $R = 3$ (a) and $R = 127$ (b).



(a) (b)

Figure 3: Graphical representation of f_3 and $M_R f_3$, with $R = 3$ (a) and $R = 127$ (b).

The approximation error estimate (36) generalizes the classical result for piecewise linear interpolation to the smoother situation of piecewise Hermite Martensen interpolation. A natural application is the approximation of spline functions of the same order but with different interpolation conditions such as nodal splines.

The obtained uniform convergence results (42) and (43), together with interpolation conditions 1. of Theorem 4 at t_0 and t_{Rn}, allow the use of Martensen splines $M_R f$ for the numerical evaluation of certain finite-part integrals [7, 8, 12].

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