Chapter 3
Introduction to the Approach of Action, Production, and Communication (APC)

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Abstract By referring to the data presented in Chap. 2, the chapter introduces the theoretical approach of Action, Production, and Communication (APC) and the related tool of the semiotic bundle. APC provides a frame for investigating semiotic resources in the classroom. It addresses the use of semiotic resources from a multimodal perspective including the analysis of gestures as a resource for thinking and communication.

Keywords Theories • Action/Production/Communication • Semiotic bundle

3.1 APC Approach – An Overview

The APC approach focuses on classroom processes of teaching and learning mathematics, on both cognitive and didactic levels. APC means "Action, Production, and Communication," which are considered to be three fundamental components of mathematical activity in the classroom's social context. These components are to be seen as mutually enriching, and inseparable, and are analyzed with a semiotic lens called a "semiotic bundle."

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In this chapter we introduce the main theoretical elements of this theory, and illustrate it by referring to selected episodes from the video of Carlo, Giovanni, and the exponential function (presented in Chap. 2). For further discussion and examples, the reader may refer to Arzarello (2008), Arzarello et al. (2009, 2011), and Sabena (2007, 2008).

3.1.1 Principles

The APC approach has its foundation mainly in two complementary theoretical assumptions: the multimodal perspective on cognition and communication, and the social-cultural characterization of human activity and thinking. Multimodality has its roots in the psychological theories that emphasize the crucial role of the body in thinking and knowledge development: the most recent is the so-called embodied cognition perspective. The relevance of the social-cultural dimension draws on the work of Vygotsky and Vygotskian scholars.

In the next sections, we show how the integration of these theoretical elements allows us to characterize an interpersonal cognitive space, called *Space of Action, Production, and Communication*, suitable for mathematics learning in a social context. We will elaborate on these notions in the next section.

3.1.1.1 Embodiment and Multimodality

Embodiment is a stream in cognitive science that assigns the body a central role in shaping the mind (for an overview, see Wilson 2002). Even if a certain importance to the body was assigned in other relevant pedagogical theories such as those from Montessori (1934) and Piaget (see Overton 2008), in mathematics education the attention to such a theme was prompted by the provocative book *Where Mathematics Comes From* by Lakoff and Núñez (2000), and then applied by researchers in several studies within the field (e.g., Arzarello and Robutti 2001; Nemirovsky 2003; Edwards 2009).

The new stance emphasized sensory and motor functions, as well as their importance for successful interaction with the environment. Criticizing the platonic idealism and the Cartesian mind–body dualism, Lakoff and Núñez (2000) advocated that mathematical ideas are founded on our bodily experiences and develop through metaphorical mechanisms. A typical example is the notion of set, which is based on the grounding metaphor "sets are containers": using this cognitive metaphor without effort, we are able to think and say that an element is IN a set, or OUTSIDE a set, and so on, as it would be IN a container or OUTSIDE of it.

The importance of body experiences was not completely new to the field of education: for instance, Piaget (1952) himself stated the sensory-motor experiences as the first steps in concept formation. Against this background, the embodied perspectives brought two interesting novelties: the claim that bodily experiences intervene beyond a first phase of knowing, and permeate all the process of knowledge production; and the metaphors as cognitive mechanisms for abstract concept formation (see the above example of sets conceptualized as containers). However, we agree with Schiralli and Sinclair (2003) and with Radford et al. (2005) in recognizing several limits to the embodied cognition paradigm, in particular concerning the lack of social, historical, and cultural dimensions in the formation of mathematical concepts; for example, there may be cultural means such as speech and symbols which may shape the way in which a metaphor leads to a concept formation.

More recently, embodied stances seem to receive a certain confirmation by neuroscientific results. Specifically, we refer to results on "mirror neurons" and "multimodal neurons," which are neurons firing when the subject performs an action, when he observes something, as well as when he imagines it (Gallese and Lakoff 2005). On the basis of such neuroscientific results, Gallese and Lakoff use the notion of "multimodality" to highlight the role of the brain's sensory-motor system in conceptual knowledge. This model entails that there is not any central "brain engine" responsible for sense-making, controlling the different brain areas devoted to different sensorial modalities (which would occur if the brain behaved in a modular manner). Instead, there are multiple modalities that work together in an integrated way, overlapping with each other, such as vision, touch, and hearing, but also motor control and planning.

On the other hand, in the field of communication design, the term "multimodality" is used to refer to the multiple modes we have to communicate and express meanings to our interlocutors: words, sounds, figures, etc. (Kress 2004). With the overwhelming visual richness of our contemporary technology (web, games, tablets, etc.), and the developing possibilities of interaction with it through our body, a multimodal perspective on both thinking and communicating appears to be of increasing relevance.

3.1.1.2 The Importance of Gestures for Communication and Thinking

The multimodal perspective receives confirmation also from the studies on gestures, which have flourished in the last two decades.

Gestures are part of what is called "nonverbal communication," which includes a wide-ranging array of behaviors such as the distance between people in conversation, eye contact, voice prosody, body posture, and so on. In his seminal work, McNeill (1992) defines gestures as "the movements of the hands and arms that we see when people talk" (McNeill 1992, p. 1). This approach comes from the analysis of conversational settings and has been widely adopted in successive research studies in psychology, in which gestures are viewed as distinct but inherently linked with speech utterances. Nowadays, research in a number of disciplines (such as psychology and all its branches, cognitive linguistics, and anthropology) is increasingly showing the importance of gestures not only in communication, but also in cognition (e.g., see Goldin-Meadow 2003; McNeill 1992). Curiously, Kendon (2000) argues that it has been the interest in cognition prompted by Chomsky's view of linguistics as a kind of purely mental science that has led to the vigorous investigation of gestures by those interested in language:

> If language is a cognitive activity, and if, as is clear, gestural expression is intimately involved in acts of spoken linguistic expression, then it seems reasonable to look closely at gesture for the light it may throw on this cognitive activity. (Kendon 2000, p. 49)

Gestures are usually characterized as follows (McNeill 1992): they begin from a position of rest, move away from this position, and then return to rest. The central part of the movement, generally recognized as expressing the conveyed meaning, is called *stroke* or peak; it is preceded by a preparation phase (hand/arm moving from its resting place, and usually to the front away from the speaker), and symmetrically succeeded by a retraction phase (hand/arm back to the quiescence). Speakers of European languages usually perform gestures in a limited space in the frontal plane of the body, called *gesture space*, which goes roughly from the waist to the eyes, and includes the space between the shoulders. However, differences have been detected according to age (the *gesture space* of children is larger) and different cultural settings.

McNeill (1992) provides also an often-quoted classification of gestures, distinguishing the following categories:

- *iconic* gestures bear a relation of resemblance to the semantic content of discourse (object or event);
- *metaphoric* gestures are similar to iconic gestures, but with the pictorial content presenting an abstract idea that has no physical form;
- *deictic* gestures indicate objects, events, or locations in the concrete world;
- *beats* appear when hands move along with the rhythmical pulsation of speech, lending a temporal or emphatic structure to communication.

More recently, the cohesive function of gesture has been further deepened, and theorized with the notion of *catchment* (McNeill 2005). A catchment is recognizable when some gestures' form features are seen to recur in at least two (not necessarily consecutive) gestures. According to McNeill, a catchment indicates discourse cohesion, and it is due to the recurrence of consistent visuospatial imagery in the speaker's thinking. Catchments may, therefore, be of great importance giving us information about the underlying meanings in a discourse and about their dynamics:

> By discovering the catchments created by a given speaker, we can see what this speaker is combining into larger discourse units – what meanings are being regarded as similar or related and grouped together, and what meanings are being put into different catchments or are being isolated, and thus are seen by the speaker as having distinct or less related meanings. (McNeill et al. 2001, p. 10)

In the classroom context, we believe that a catchment can indicate a student expressing concepts he cannot well express in words. In this sense, catchments are also relevant to analyze concept formation (see the examples regarding Carlo in Sect. 3.2 below). Furthermore, catchments may also give clues about the organization of arguments at a logical level (for a discussion applied to mathematics discourses, see Arzarello and Sabena 2014).

3.1.1.3 The Social-Cultural Dimension and the Role of Signs

As mentioned above, the main limit of embodied cognition is in having neglected the social and cultural dimensions in which mathematical concepts arise and evolve, and the fundamental role of signs therein. With this respect, the APC frame takes a Vygotskian perspective. In particular, according to the *genetic law of cultural development*, namely the general law governing the genesis of higher mental functions, there is a passage from *interpsychic* functions, that are shared on the social level, to *intrapsychic* ones, that relate to the person on the individual level:

> Every function in the child's cultural development appears twice: first, on the social level, and later on the individual level; first, between people (interpsychological), and then inside the child (intrapsychological). This applies equally to voluntary attention, to logical memory, and to the formulation of concepts. All the higher functions originate as actual relations between human individuals. (Vygotsky 1978, p. 57).

Furthermore, Vygotsky claims, cultural development is based on the use of signs. Such a general process, accounting for the formation of human consciousness by progressive individualization of inherently social functions, is called *internalization*.

By virtue of the social meaning, signs serve to the individual to exert a voluntary control on his/her behavior, in a way similar to the road sign signaling to the individual the events to regulate his conduct. Without the meaning, words, mnemotechnic signs, mathematical symbols, and all other symbol systems would be nonsense. *Meaning* has therefore a psychological character, rather than a theoretical/abstract one (Leont'ev and Lurija 1973). Meaning allows the human being to produce those *changes to external things* – that are *signs* – that in a second moment express their transformative action on the interior plane of proper psychological processes, thus allowing him to "control," or "appropriate" the criteria to direct his own behavior. Semiotic mediation accomplishes therefore a fundamental role in the formation of the "plane of consciousness" (Wertsch and Addison Stone 1985).

3.1.2 Key Theoretical Constructs and Methodology

On the bases of the presented principles, the APC approach is based on the idea of Space of Action, Production, and Communication, and its analytical tool: the semiotic bundle. The methodology includes the application of fine-grained analysis, carried out with the aid of video-recording tools. We will present theoretical and methodological tools in the following paragraphs.

3.1.2.1 The Space of Action, Production, and Communication

The notion of Space of Action, Production, and Communication (in short, APC-space) has been introduced by Arzarello (2008). It is a model that intends to frame the processes that develop in the classroom among students and the teacher while working together. The main components of the APC-space are:

- the body;
- the physical world;
- the cultural environment.

These components include students' perceptuo-motor experiences, languages, signs, and resources that they use to act in the environment and to socially interact:

> An APC-space is the unitary system of the three main components listed above, amalgamated in a dynamically evolving unit within a concrete learning situation in the classroom, because of the action and mediation of the teacher, who suitably orchestrates their integration. (Arzarello 2008, p. 162)

The APC-space is a theoretical construct aimed at modeling the didactic setting and the teaching–learning process. Considering the classroom context, the APC-space pinpoints the conditions in which the learning process can be fostered:

> The APC-space is built up in the classroom as a dynamic single system, where the different components are integrated with each other into a whole unit. The integration is a product of the interactions among pupils, the mediation of the teacher and possibly the interactions with artifacts. The three letters A, P, C illustrate its dynamic features, namely the fact that three main components characterize learning mathematics: students' actions and interactions, their productions and communication aspects (ibid., p. 162).

"Space" is to be intended not as a physical entity, but rather in an abstract way, as in mathematics theories. Framed in a socio-cultural perspective, the APC-space is an intersubjective space, involving students and the teacher. It is a typical example of a complex system, in which the global result does not derive linearly from the simple superposition of its components. For an APC-space to be active and to work, it is obviously not sufficient that its components are present in the classroom: bodies, physical world, and cultural environment are certainly always there! The teacher is responsible for the construction of the mathematical knowledge in the classroom, and this responsibility realizes first in the setting of didactic activities, and then in the support of the evolution of the personal senses of the students towards the scientific ones. The teacher is hence an active part of the APC-space. Another important dimension is time: the APC-space, gauged at accounting the teaching–learning process, is a complex dynamic system evolving in time.

3.1.2.2 The Semiotic Bundle

When the students interact (with each other and with the teacher) in the APC-space, the result is not a linear development, but a complex interplay of multimodal actions, productions, and communications. Within the Vygotskian frame outlined above, the semiotic lens can be considered a good tool for observing such an interplay (see also Bartolini Bussi and Mariotti 2008). The semiotic bundle notion is elaborated in the next paragraphs in order to consider, besides linguistic and mathematical semiotic systems, also embodied ones, such as gestures.

The notion is based on Peirce's theorization, according to which a sign is a triad constituted by the sign or *representamen* (that represents), the *object* (that is represented), and the *interpretant* (specifying in which respect the representamen is representing the object). In Peirce's words, anything that can be interpreted by somebody in some respect can be considered as a sign (Peirce 1931–1958, vol. 2, par. 228). The interpretant is the most delicate element, since it constitutes a new sign (conceived in the triadic way), generating a new interpretant, and so on.

Such a characterization of "signs" provides us with two features apt for our needs: the first one regards the generality of the definition of sign, and the second one the dynamicity of the semiotic processes, framed with the idea of the "interpretant."

Basing on this approach to signs, the semiotic bundle notion considers both static and dynamic aspects. It consists in:

> a *system of signs* […] that is produced by one or more interacting subjects and that evolves in time. Typically, a semiotic bundle is made of the signs that are produced by a student or by a group of students while solving a problem and/or discussing a mathematical question. Possibly the teacher too participates to this production and so the semiotic bundle may include also the signs produced by the teacher. (Arzarello et al. 2009, p. 100)

As an example, we can consider the set of gestures and the set of words that are produced during a certain problem-solving activity. The two sets are intertwined, because they are used simultaneously during the activity: so they constitute the elements of the semiotic bundle for that activity.

Differently from other semiotic approaches in mathematics education (e.g., Duval 2006; Ernest 2006), the semiotic bundle includes all the bodily means of expression, such as gestures, gazes, sketches, and so on, as semiotic resources in teaching and learning. Such an approach widens the notion of a semiotic system, so that signs can include gestural and segmented forms of language, which we consider as fundamental components of the multimodal activities in the classroom.

The semiotic enlargement described has also been favored by a refinement of the tools used for the observation of relevant activities in the classroom. Video-recordings play a crucial role, in that they can be examined in detail, in order to carefully analyze the observed processes.

Based on these videos, a transcript including information about gestures is produced, and used for the a posteriori analysis. The analysis, however, is carried out not only by relying on the transcript, but also by looking constantly and repeatedly to the videos. Specifically, the analysis of the semiotic bundle considers dynamics along two dimensions:

- a *diachronic analysis*, focused on the evolution of signs over time, and the transformation of their relationships (in periods with variable length, from a few minutes to years);
- a *synchronic analysis*, focused on the relations among the signs used in a certain moment.

In this way, the focus of the analysis is on the ongoing dynamic contextual teaching and learning processes where the cognitive aspects intertwine with the didactic and communicative ones.

3.1.2.3 The Semiotic Game

Analyzing the teaching and learning activities with the above-mentioned enlarged semiotic lens, general and specific results have been detected (for a general account, see Sabena et al. 2012).

The most important result regards the role of the teacher in the multimodal perspective: the so-called "semiotic game" between teacher and students (Arzarello and Paola 2007; Arzarello et al. 2009). A semiotic game may occur when the teacher is interacting with the students, as in classroom discussions or during group-work. In a semiotic game, the teacher tunes with the students' semiotic resources (e.g., words and gestures), and uses them to make the mathematical knowledge evolve towards scientifically shared meanings. More specifically, the teacher uses one kind of sign (typically, gestures) to tune with the students' discourse, and another one to support the evolution of new meanings (typically, language). For instance, the teacher repeats a gesture that one or more students have just made, and accompanies it with appropriate linguistic expressions and explanations. Such semiotic games can develop if the students produce something meaningful with respect to the problem at hand, using some signs (words, gestures, drawings, etc.). It is apt for the teacher to seize these moments to enact her/his semiotic game. Even a vague gesture of the student can really indicate a certain comprehension level, even when the student has not yet the words to express himself at this level. In a Vygotskian frame, the semiotic game is likely to "work," that is, be useful to the student, if the student is in a zone of proximal development for a certain concept (Vygotsky 1978), so that the teacher may have the chance to intervene in its cognitive development. The intervention is imitative-based, that is, the teacher imitates the students' gestures and accompanies them with certain scientific meanings (expressed in appropriated words), in order that in the following, the students will be able to imitate the teacher's words. At the same time, the teacher encourages the student, signaling that his idea, though not fully or correctly expressed, is on the right way on learning.

3.1.3 Questions

The typical research questions asked within the APC approach are the following:

- What is the role of gestures in the development of mathematical concepts?
- What are the roots of the mathematics representations in students' activities?
- What is the role of the teacher, considering the multimodal perspective?
- How do the different components of the semiotic bundle concur to the conceptualization processes in students?
- What are the different relationships between the components and their evolution in time?

3.2 Illustrating the APC Approach Through Analysis of the Video of Carlo, Giovanni, and the Exponential Function

In the following, we will illustrate the APC approach by selecting excerpts from the video of Carlo, Giovanni, and the exponential function (see Sect. 2.1) and analyzing them in accordance with the notion of a semiotic bundle.

3.2.1 Exploring $\mathbf{y = a^x}$

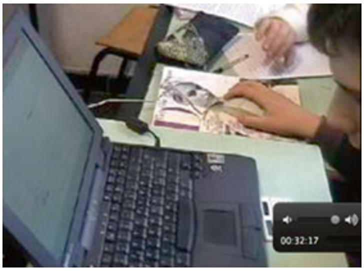
The first episode refers to the students facing Task 2 (see Sect. 2.1.3). After reading the text of the task, Carlo and Giovanni construct $y = a^x$ with the Dynamic Geometry Software. They start exploring the function, according to the task request:

> Then moving the point A changes the base of the exponential. Moving the point P, you run along the graph of an exponential function with a fixed base. Explore, share your impressions (is there something which is not clear and we were not expecting or that is clear and you were expecting). Describe briefly your exploration on the sheet. (See Fig. 2.2 in Sect. 2.1.3)

They decide to consider the case in which the base *a* is very big. Let us analyze how they start to explore the function in the transcript lines 160–165 (see Appendix for full transcript. In the transcript, underlining designates the part of an utterance during which the speaker gestured.)

160 G we try to move *A*
161 C try to put the *a* very high [*moving his hand upwards, at the top of the screen*]… when we have seen to happen that chaos *[meaning: in a previous lesson]*
162 G no, it always gets… because here it is interrupted… because here it is interrupted
163 C wouldn't it do like this? [*Gesture a*]
wouldn't it do like this? [*Gesture b*]

Gesture in 163 (a): C's quick gesture with right hand



Gesture in 163 (b): like Gesture a with more visible hand, going upwards very steeply

164 G what?

165 C to do like this [*gesture*]

Gesture in 165: C's similar gesture, more evident, with the hand moving very steeply upwards

In this episode, we can observe a semiotic bundle composed of three different kinds of semiotic resources: spoken words, graphical representations on the screen, and gestures. They are strictly interrelated: using words and gestures the students are discussing the behavior of the exponential function and its graph on the screen, when the base is "very big." By using words and gestures, Carlo is making a conjecture of the graph (line 163–165): through words he is indicating the case he is considering ("the *a* very high", line 161) and with gestures he is showing how he is imagining the graph will be (screenshots (a) and (b) in line 163 and in line 165).

While speaking very few words, Carlo performs three gestures, which show similar features: the shape of the hand, and the dynamic movement going upwards (although the concavity changes). This is a case of catchment (McNeill et al. 2001). In the repetition, the gesture becomes bigger and bigger, being performed in a greater space and longer time. Even if we cannot see Carlo's gaze to confirm this, our interpretation is that the student is performing the second and third gesture to show it to Giovanni, who is looking at him (the video shows that Giovanni is turning his head towards Carlo). In the evolving APC-space, Carlo's semiotic resources are used first as thinking tools, in order to produce a conjecture, and then as a communicative means.

As confirmed by the teacher (personal communication), Carlo has some difficulties in expressing his ideas in oral and written language; here we can see that the gesture is co-timed with deictic terms ("like this") that point to the gesture itself: the gesture is indeed part of his thinking and communication means, and in the semiotic bundle, words and gestures complete each other (with reference to the shown screen).

3.2.2 Formulating the Written Answer

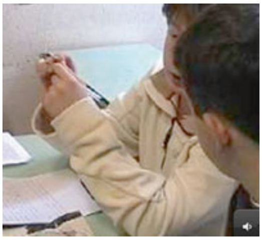
A gesture similar to those discussed in the previous paragraph appears again some minutes later, when the students are about to write the solution:

189 C well… so we write that… let's say: the point A… we put that one thing we had said… [*Gesture a*], we had said that…
I'm still thinking if… [*not understandable*] how I can say… but… also for a same space of the *x* [*Gesture* b], the *y* increases a lot [*Gesture c*]

Gestures in 189:

(a) Carlo's quick gesture in the air

(b) C's fingers close to each other



(c) C's right hand moving upwards

Carlo is offering to write the answer for the teacher. The answer has to be in written language, and still we can observe his difficulty in finding the right words to express what he is proposing: his words have little semantic content ("we put that one thing we had said…"), whereas on the contrary his gesture (screenshot (a) in line 189) offers again the pictorial image he was proposing some minutes before, that is, a graph with a high slope. This is another case of catchment, which can be detected by looking at the semiotic bundle in a diachronic way.

In the second part of line 189, Carlo is connecting the very inclined graph (see the hand moving almost vertically in screenshot b) to the incremental ratio of the function: his fingers are indicating a very small interval on the x-axes, and his words relate this fixed interval of abscissas with increasing increments of the ordinates ("for a <u>same space of the</u> <u>x</u>, the y increases a lot", line 189). Let us notice that the information that x-increments are considered small is expressed only in the gesture (which is therefore non-redundant, in the sense of Kita 2000); however, it is the co-timing with the words that allows the student to connect this information with the variation of the corresponding y-increments: this kind of analysis is typical of the semiotic bundle lens, and witnesses the potential of such an analytical tool.

While Carlo is talking-gesturing, Giovanni is looking at him and following his argument. He immediately agrees, and helps Carlo to find the right words for expressing his ideas in the following lines of the transcript:

190 G yes
191 C eh… how do I say that?
192 G or you can say that with the <u>differences</u> [*in parallel, Carlo gestures*]

Gesture in 192: Carlo is gesturing with two hands parallel to each other; Giovanni is performing a beat gesture

193 C <u>for an</u> [*gesture*] interval [*…inaudible*]
194 G the differences are bigger and bigger
194 C the differences, right?
195 G yes
Carlo finishes writing the answer.

Gesture in 193: C's gesture with two parallel hands is anticipating the word "interval"

The students are now going about producing the written answer. However, this formulation moment is not purely a communicative moment. As a matter of fact, Carlo interrupts himself many times while writing, with seconds of silence, gestures, and words. According to a Vygotskian perspective, the writing act, fostered by the social dimension (the teacher asking for a written answer), has deep influence on the thinking processes.

Giovanni is enriching the semiotic bundle with the word "differences," which Carlo could not find (line 192). While Giovanni is pronouncing it, Carlo is performing a gesture with two parallel open palms (gesture in line 192). Carlo's gesture with the two hands represents the ends of an interval on the *x*-axes: however, the word "interval" appears only later in his speech (line 193), with respect to which the gesture is anticipatory.

By contrast, there is a perfect interpersonal synchrony between Giovanni's words and Carlo's gesture: such synchrony, which can be detected with a synchronic analysis of the semiotic bundle, is an indication that the students are sharing an active APC-space (Sabena 2007). Another clue in the same direction is provided by the fact that Giovanni is completing Carlo's sentence (lines 193–194), with a perfect timing (there is no time left between the two sentences in lines 193 and 194): due to the close coordination, a careful listening of the video-recording is necessary in order to identify which student is speaking.

Such kinds of semiotic acts are accessible to the researcher only by means of video-recordings and a careful micro-analysis of video and screenshots. They have been observed in students' joint activity also in other contexts (e.g., for algebraic context, Radford et al. 2007), and appear most likely after students have developed a fruitful cooperation in group-work and are deeply engaged in the problem at hand. In the perspective of APC-space, the semiotic bundle analysis provides a suitable lens to seize them, and to study their role in mathematics learning.

3.3 Conclusion

In the chapter we have illustrated how the theoretical construct of APC-space can be useful to properly underline the way mathematical concepts are built up by students. The example of Carlo and Giovanni illustrates how three different kinds of semiotic resources intertwine in this complex dynamic process: spoken words, graphical representations on the screen, and gestures. These are typical inhabitants of the APC-space and as such they embody the actions, productions, and communications of students; from its side, the semiotic bundle lens allows pointing out how such components concretely intertwine and evolve in time. We can use a metaphor from physics to point out the differences between the two notions. In dynamics there is the second law of Newton, $\boldsymbol{F} = m\boldsymbol{a}$; to completely understand it, one must operatively define what are force, mass and acceleration. But this is only half of the story – the other half consists in understanding how the three quantities relate each other in expressing a law of physics; the law is a lens that allows the modeling of the motions of classical mechanics. In our case the three components of the APC-space (action, production, communication) are pointed out as basic components of the didactical phenomena in the classroom; the semiotic bundle describes the mutual relationships between them in time. Of course, didactics is not an exact science and the metaphor must be considered *cum grano salis*: the semiotic bundle is not like a physical law but is a construct that qualitatively describes the way the three components of APC relate each other in the classroom, because of the interactions between the students or between the student(s) and the teacher. This phenomenological description possibly points out some didactical phenomena that systematically happen, for example the semiotic game: in the metaphor it corresponds to the use that one can make of the second principle of dynamics to design the trajectory of a rocket. In the same way, a teacher, who is aware of how the components of the APC-space interact in the semiotic bundle, can play a semiotic game to support a student towards a better understanding and formulation of a mathematical concept.

The main result of this approach consists in pointing out not only that more variables than the purely discursive ones are important in the didactical processes, but also defining suitable observation methods in order to give reasons to them. This issue shows the partiality of all those descriptions, which limit to comment only the protocols of the speech or written productions of students. In fact our model aims at better giving account of learning processes as dynamic phenomena, so overcoming the limits pointed out by Freudenthal, when he wrote:

> Indeed, didactics itself is concerned with processes. Most educational research, however, and almost all of it that is based on or related to empirical evidence, focuses on states (or time sequences of states when education is to be viewed as development). States are *products* of previous *processes*. As a matter of fact, *products* of learning are more easily accessible to observation and analysis than are learning *processes* which, on the one hand, explains why researchers prefer to deal with states (or sequences of states), and on the other hand why much of this educational research is didactically pointless. (Freudenthal 1991, p. 87, emphasis in the original)

We have been able to point out a wider range of observables to look at in order to understand the life in the classroom. Of course this does not mean to say that the discursive productions are useless, but only that they must be integrated within the more complex picture given by the semiotic bundle, according to a multimodal perspective. In other chapters of the book (see Chaps. 9 and 11) we will show how this approach can usefully be integrated with other approaches, more based on discursive analysis.

A further by-product of this research consists in indicating a clear position in respect of the complex intertwining between culture and nature in students' performances. The debate about the relationships between the two components has been a must in psychology (McLeod 2007) and has generated considerable discussion also in mathematical education: indeed, the National Association of Mathematics Advisers (http://www.nama.org.uk/index.php) held its 2013 Conference on this issue. For example, the gesture–speech unity (McNeill 1992) of our productions is a typical construct that shows the two aspects to be deeply intertwined: biological and cultural aspects are inextricably bound together in all our performances within the APC-space when we as students (teachers) are learning (teaching) mathematics.

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