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Discontinuity relations for the AdS_4/CFT_3 correspondence

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Abstract

We study in detail the analytic properties of the Thermodynamic Bethe Ansatz (TBA) equations for the anomalous dimensions of composite operators in the planar limit of the 3D $\mathcal{N} = 6$ superconformal Chern–Simons gauge theory and derive functional relations for the jump discontinuities across the branch cuts in the complex rapidity plane. These relations encode the analytic structure of the Y functions and are extremely similar to the ones obtained for the previously-studied AdS_5/CFT_4 case. Together with the Y-system and more basic analyticity conditions, they are completely equivalent to the TBA equations. We expect these results to be useful to derive alternative nonlinear integral equations for the AdS_4/CFT_3 spectrum.

1. Introduction

In recent years, the research in high energy theoretical physics has been characterised by the discovery of deep connections between strings, supersymmetric gauge theories and integrable models. A first link between a quantum integrable model and multicolour reggeised gluon scattering was discovered by Lipatov in [1], see also [2]. More recently, the methods of integrability

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have turned out to be very efficient for the study of some prominent examples of string/gauge theories related by the AdS/CFT correspondence [3–5]. For a review of the rapidly developing field of integrability in AdS/CFT the reader can consult [6].

In this paper we are concerned with the Thermodynamic Bethe Ansatz (TBA) approach to the computation of the planar spectrum of the AdS_4/CFT_3 correspondence [7]. This is the spectrum of the anomalous dimensions of local gauge invariant operators in the $\mathcal{N} = 6$ superconformal Chern–Simons gauge theory, or, equivalently, the energy spectrum of Type IIA string theory on $AdS_4 \times \mathbb{CP}^4$. The development of this subject has been parallel and inspired by the study of the spectrum of the AdS_5/CFT_4 correspondence, and we will often refer to the latter. In fact, it is in the context AdS_5/CFT_4 that integrability was originally discovered.

In the context of AdS_4/CFT_3, Asymptotic Bethe Ansatz (ABA) equations for anomalous dimensions of composite trace operators were proposed in a series of seminal works [8–11], however a crucial limitation has soon emerged as a consequence of the asymptotic character of these equations: the BA equations do not contain information on the finite size contributions that appear when the site-to-site interaction range in the loop expansion of the dilatation operator becomes greater than the number of elementary operators in the trace.

Although, for supersymmetry reasons, these wrapping effects [12–14] do not affect special families of (protected) operators, in general these corrections become particularly relevant in the semiclassical limit of string theory corresponding to the strong coupling regime on the gauge theory side.

This limitation can be surmounted through the use of the Thermodynamic Bethe Ansatz technique [15]; a method originally proposed by Al.B. Zamolodchikov [16] to study the ground state energy of perturbed conformal field theories on a cylinder geometry using the exact knowledge of the scattering data. The method was later adapted to the study of excited states [17–19]. The use of the TBA method to overcome the wrapping problem in AdS/CFT was advocated in [14] and implemented in [21–23] for the AdS_5/CFT_4 case, and in [24,25] for the AdS_4/CFT_3 case (see also the review [26]). As a result of this procedure, the value of anomalous dimensions as function of the coupling constant is represented in terms of the pseudoenergies ε_a, solutions of a set of nonlinear coupled integral equations: the TBA equations. Starting from the latter, sets of finite difference functional relations for $Y_a = e^{\varepsilon_a}$, the Y-systems [33–36], have been derived for the AdS_4/CFT_3 and AdS_5/CFT_4 spectra in [21–25]. Apart for a subtle small difference crucial to describe certain subsectors of the AdS_4/CFT_3 theory, the earlier proposal by Gromov, Kazakov and Vieira coming from symmetry arguments [20] were confirmed.

Y-systems are currently playing an important rôle in Cluster Algebra, gluon scattering amplitudes and other areas of mathematical physics [37]. They are related to discrete Hirota's equations and they are central in the TBA setup since they exhibit a very high degree of universality: the whole set of excited states of a given theory is associated to the same Y-system with different states differing only in the number and positions of the $1+Y_a$ zeros in a certain fundamental strip. In principle, one can then obtain TBA equations describing the excited states by making natural assumptions on the position of these zeroes and reconstructing the TBA from the Y-system. So far, excited state TBA equations have been conjectured only for particular subsectors of the AdS/CFT theories, although a rich picture is already emerging [21,25,27–32].

In relativistic-invariant models the Y functions are, in general, meromorphic in the rapidity u with zeros and poles both linked to $1 + Y_a$ zeros through the Y-system. The situation for the AdS/CFT-related models is further complicated by the presence of square root branch cuts inside and at the border of a certain fundamental strip. According to the known Y-system to TBA transformation procedures, full information on the Y function jump discontinuities across these closest cuts should be independently supplied. The main goal of the paper [44] was to show that, for the AdS_5/CFT_4 case, the relevant analytic structure can be encoded in the Y-system together with a set of state-independent functional relations involving points on different Riemann sheets. Indeed, the results of [44] turned out to be very useful both for finding new families of excited states [46] and for the derivation of important alternative nonlinear integral equations [48,49] for the anomalous dimensions.

In this paper we shall discuss the AdS_4/CFT_3 case using a somehow complementary approach: while in [44] it was shown in detail how to transform the functional relations to the TBA form, here we will start from the AdS_4/CFT_3 TBA and describe, in reasonable detail, how to extract the full set of discontinuity relations. A useful additional identity for the fermionic nodes is derived carefully in Appendix B showing that it is a consequence of the fundamental discontinuity relations and of the Y-system.

The paper is organised as follows. Section 2 contains the TBA equations of [24,25]. The Y-system and the new functional relations are presented in Section 3. In Section 4, we provide a concise derivation of these relations from the TBA equations, while in Section 5 we show how to generate, from the standard Y-system, other non-fundamental identities describing the branch cuts far from the real axis.

There are four appendices. Appendix A contains a list of the kernels entering the TBA equations, in Appendix B we derive a useful additional relation for the fermionic nodes, and in Appendix C we list other identities, which can be used to check that the Y-system supplemented by the branch cut information is equivalent to the TBA. Finally, in Appendix D, we rewrite the fundamental set of relations in terms of the T functions, connecting with the results of [46,49].

2. The TBA equations

The TBA equations for the spectrum of AdS_4/CFT_3 are an infinite set of coupled nonlinear integral equations, depending explicitly on an integer parameter L related to the number of elementary fields in the composite operator under consideration, and depending on the coupling[1] h through the form of the integral kernels $\phi_{a,b}$. Their solutions are a set of pseudoenergies $\varepsilon_a(u)$, of the following species: $\varepsilon_{Q|I}(u)$, $\varepsilon_{Q|II}(u)$, $\varepsilon_{w|M}(u)$, $\varepsilon_{v|N}(u)$, $\varepsilon_{y|-}(u)$, $\varepsilon_{y|+}(u)$, where $Q, M, N \in \mathbb{N}^+$.

It is often useful to consider the so-called Y functions, obtained by exponentiating the pseudoenergies: $Y_a(u) = e^{\varepsilon_a(u)}$. We also set

$$L_a(u) = \ln\big(1 + 1/Y_a(u)\big). \tag{2.1}$$

The TBA equations describing the ground state have been derived in [24,25] and can be written as

$$\varepsilon_{Q|\alpha}(u) = \ln \lambda_{Q|\alpha} + L\tilde{E}_Q(u) \\ - \sum_{\beta} \sum_{Q'=1}^{\infty} L_{Q'|\beta} * \phi_{(Q'|\beta),(Q|\alpha)}(u)$$

[1] In the case of AdS_4/CFT_3, the coupling h entering the S-matrix elements is a so far undetermined function of the t'Hooft coupling $h = h(\lambda)$.

$$+ \sum_{M=1}^{\infty} L_{v|M} * \phi_{(v|M),Q}(u)$$
$$+ \int_{-2}^{2} dz \big[L_{y|-}(z)\phi_{(y|-),Q}(z,u) - L_{y|+}(z)\phi_{(y|+),Q}(z,u) \big], \tag{2.2}$$

$$\varepsilon_{y|\mp}(u) = \ln \lambda_{y|\mp} - \sum_{Q=1}^{\infty} (L_{Q|I} + L_{Q|II}) * \phi_{Q,(y|\mp)}(u)$$
$$+ \sum_{M=1}^{\infty} (L_{v|N} - L_{w|M}) * \phi_M(u), \tag{2.3}$$

$$\varepsilon_{v|K}(u) = \ln \lambda_{v|K} - \sum_{Q=1}^{\infty} (L_{Q|I} + L_{Q|II}) * \phi_{Q,(v|K)}(u)$$
$$+ \sum_{M=1}^{\infty} L_{v|M} * \phi_{M,K}(u) + \int_{-2}^{2} dz \big(L_{y|-}(z) - L_{y|+}(z) \big) \phi_K(z-u), \tag{2.4}$$

$$\varepsilon_{w|K}(u) = \ln \lambda_{v|K} + \sum_{M=1}^{\infty} L_{w|M} * \phi_{M,K}(u)$$
$$+ \int_{-2}^{2} dz \big(L_{y|-}(z) - L_{y|+}(z) \big) \phi_K(z-u), \tag{2.5}$$

for $\alpha \in \{I, II\}$, $K, Q \in \mathbb{N}^+$, where the fugacities λ_a will be specified below. The integral kernels $\phi_{a,b}(z,u)$ appearing in the TBA equations are defined in Appendix A, together with the function $\tilde{E}_Q(u)$, which represents the infinite volume energy of a Q-particle bound state in the mirror theory.

The ground state energy can be computed as

$$\tilde{F}(L) = -\frac{1}{L} \sum_{Q=1}^{\infty} \int_{\mathbb{R}} \frac{du}{2\pi} \frac{d\tilde{p}^Q}{du} \big(L_{Q|I}(u) + L_{Q|II}(u) \big), \tag{2.6}$$

where $\tilde{p}_Q$ is the mirror momentum, also defined in Appendix A. This quantity is exactly zero, as dictated by supersymmetry, as soon as the fugacities reach the values

$$\lambda_{Q|\alpha} = (-1)^Q, \qquad \lambda_{v|K} = \lambda_{w|K} = 1, \qquad \lambda_{y|\pm} = -1 \quad \big(\alpha \in \{I, II\}, K, Q \in \mathbb{N}^+\big). \tag{2.7}$$

This singular limit of the TBA equations can be regularised by taking

$$\lambda_{2Q-1|I} = -e^{i\varphi}, \qquad \lambda_{2Q-1|II} = -e^{-i\varphi}, \qquad \lambda_{2Q|I} = \lambda_{2Q|II} = 1,$$
$$\lambda_{v|K} = \lambda_{w|K} = 1, \qquad \lambda_{y|\pm} = -1, \tag{2.8}$$

such that the TBA equations are regular for $\varphi \neq 0$ and the ground state energy tends to zero as $\varphi \to 0$.

There is an important difference between this system and the TBA equations describing two-dimensional relativistic quantum field theories in finite volume: the S-matrix elements listed in Appendix A contain, in addition to poles and zeroes, also square root branch points in the

Table 1
Square-root branch points for the Y functions.

Function	Singularity position	
$Y_y(u)$ $Y_{w\|M}(u)$	$u = \pm 2 + in_0/h,$	$n_0 = 0, \pm 2, \pm 4, \ldots$
$Y_{v\|M}(u)$ $Y_{M\|I}(u)$, $Y_{M\|II}(u)$	$u = \pm 2 + in_M/h,$	$n_M = \pm M, \pm(M+2), \pm(M+4), \ldots$

rapidity plane. As a consequence, the TBA solutions are multivalued functions with infinitely many square root branch points, whose locations are summarised in Table 1 for the different Y functions.[2] The branch cuts are clearly visible, in the AdS_5/CFT_4 case, in the numerical study presented in [45].

Let us introduce an important convention which becomes relevant when the solutions to the TBA are continued into the complex rapidity plane. We work on sections of the Riemann surface obtained by tracing every branch cut as a horizontal, semi-infinite segment external to the strip $|\operatorname{Re}(u)| < 2$. More explicitly, we draw branch cuts of the form: $(-\infty, -2) \cup (+2, +\infty) + in/h$, where the possible values of $n \in \mathbb{Z}$ are listed in Table 1. Moreover, we denote as the "first" Riemann sheet the one containing the physical values of the Y functions on the real axis. Whenever we need to reach values of the Y functions on another sheet, we will indicate it explicitly.

Finally, let us remind the reader that, according to the methods developed in [17,18] for relativistic invariant models, the nontrivial anomalous dimensions corresponding to excited states can be obtained by considering the same TBA equations (2.2)–(2.5) and (2.6) but with different integration contours enclosing a finite number of zeroes $\{u_{a,j}^{(-1)}\}$ or poles $\{u_{a,j}^{(0)}\}$ of $1 + 1/Y_a(u)$. These contour-trapped pole singularities of $\frac{d}{du} \ln(1 + 1/Y_a(u))$ lead to extra source terms $\mp \ln S_{a,b}(u_{a,j}^{(-1/0)}, u)$ in the TBA equations. The positions of the poles $\{u_{a,j}^{(0)}\}$ are tied, via the Y-system [17,18], to the positions of the roots $\{u_{a,j}^{(-1)}\}$, which are constrained by auxiliary consistency conditions of the form $Y_a(u_{a,j}^{(-1)}) = -1$. The later play the rôle of exact Bethe equations.

The best understood excited state TBA equations in AdS/CFT are the ones describing the $\mathfrak{sl}_2$ sector, initially conjectured in [21] for AdS_5/CFT_4 and generalised to AdS_4/CFT_3 in [25]. In this case the Y functions are symmetric $Y_{n|I} = Y_{n|II}$ and the deformed contour picks the relevant singularities on the second Riemann sheet, so that the exact Bethe equations can be written as $Y_{1|I}((u_j)_{\circlearrowleft}) = Y_{1|II}((u_j)_{\circlearrowleft}) = -1$, where $u_j \in \mathbb{R}$ are the Bethe roots and $u_{\circlearrowleft}$ denotes the image of u reached by analytic continuation through the branch cut with $\operatorname{Im}(u) = +1/h$.[3] We refer the reader to the papers cited above for the explicit form of excited state TBA equations in the $\mathfrak{sl}_2$ sector, and to [27–32] for other very interesting studies on this subject.

An important consequence of the methods described above is that the sets of Y functions describing different excited states share the same analytic structure in the rapidity plane. In particular, every analytic functional relation satisfied by the Y functions, such as the ones described in the next section, must be state-independent. This is in turn a very important information as it opens the way to systematic methods to derive excited state equations [46].

2 This has to be considered as a maximal list of branch points. In fact, on some particular Riemann sheets, the branching may disappear at some of the locations listed in the table, at least for the $Y_{w|N}$ functions (see Appendix D). However the branch points appear to be all nontrivial on the first Riemann sheet, on which the Y-system is defined.

3 Notice that, because the $Y_{1|\alpha}$ functions are real on the first sheet, these equations could also be rewritten as $Y_{1|I}((u_j)_{\circlearrowright}) = Y_{1|II}((u_j)_{\circlearrowright}) = -1$, with $u_{\circlearrowright}$ denoting the image of u reached after entering the cut at $\operatorname{Im}(u) = -1/h$.

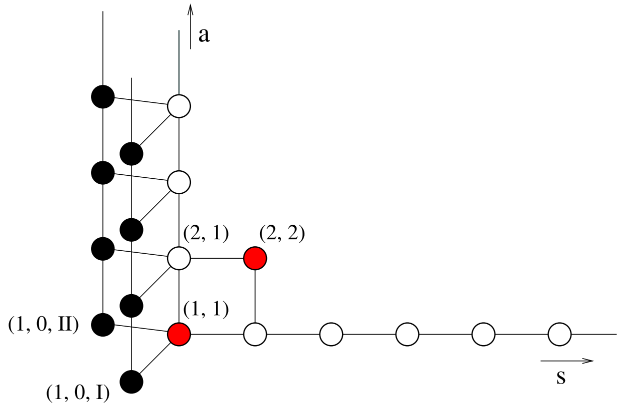

Fig. 1. The lattice associated to the AdS_4/CFT_3 Y-system. The (a, s) labels of the nodes of the diagram are related to the labels of the Y functions by $(Q, 0, \alpha) = (Q|\alpha)$, $(1, 1) = (y|-)$, $(2, 2) = (y|+)$, $(n, 1) = (v|n-1)$, $(1, n) = (w|n-1)$.

3. The extended Y-system

When evaluated on the first Riemann sheet, the solutions to the AdS_4/CFT_3 TBA satisfy the following set of functional relations (the Y-system)

$$Y_{1|I}\left(u+\frac{i}{h}\right)Y_{1|II}\left(u-\frac{i}{h}\right)=\frac{(1+Y_{2|I}(u))}{(1+\frac{1}{Y_{y|-}(u)})}, \tag{3.1}$$

$$Y_{1|II}\left(u+\frac{i}{h}\right)Y_{1|I}\left(u-\frac{i}{h}\right)=\frac{(1+Y_{2|II}(u))}{(1+\frac{1}{Y_{y|-}(u)})}, \tag{3.2}$$

$$Y_{Q|I}\left(u+\frac{i}{h}\right)Y_{Q|II}\left(u-\frac{i}{h}\right)=\frac{(1+Y_{Q-1|II}(u))(1+Y_{Q+1|I}(u))}{(1+\frac{1}{Y_{(v|Q-1)}(u)})}, \quad Q=2,3,\ldots, \tag{3.3}$$

$$Y_{Q|II}\left(u+\frac{i}{h}\right)Y_{Q|I}\left(u-\frac{i}{h}\right)=\frac{(1+Y_{Q-1|I}(u))(1+Y_{Q+1|II}(u))}{(1+\frac{1}{Y_{(v|Q-1)}(u)})}, \quad Q=2,3,\ldots, \tag{3.4}$$

$$Y_{y|-}\left(u+\frac{i}{h}\right)Y_{y|-}\left(u-\frac{i}{h}\right)=\frac{(1+Y_{v|1}(u))}{(1+Y_{w|1}(u))}\prod_{\alpha=I,II}\left(1+\frac{1}{Y_{1|\alpha}(u)}\right)^{-1}, \tag{3.5}$$

$$Y_{w|M}\left(u+\frac{i}{h}\right)Y_{w|M}\left(u-\frac{i}{h}\right)=\prod_{N=1}^{\infty}\left(1+Y_{w|N}(u)\right)^{A_{MN}}\left[\frac{(1+\frac{1}{Y_{y|-}(u)})}{(1+\frac{1}{Y_{y|+}(u)})}\right]^{\delta_{M,1}}, \tag{3.6}$$

$$Y_{v|M}\left(u+\frac{i}{h}\right)Y_{v|M}\left(u-\frac{i}{h}\right)=\frac{\prod_{N=1}^{\infty}(1+Y_{v|N}(u))^{A_{MN}}}{\prod_{\alpha=I,II}(1+\frac{1}{Y_{M+1|\alpha}(u)})}\left[\frac{(1+Y_{y|-}(u))}{(1+Y_{y|+}(u))}\right]^{\delta_{M,1}}, \tag{3.7}$$

where $A_{MN} = \delta_{M,N+1} - \delta_{M,N-1}$. The AdS_4/CFT_3 Y-system has been rigorously derived from the TBA equations in [24,25], where the appropriate choice of Riemann section was discussed. In the special symmetric case $Y_{Q|I} = Y_{Q|II}$, these relations coincide with those originally conjectured in [20] and associated to the diagram in Fig. 1 (see Appendix D for more details). Notice that, contrary to the case in relativistic two-dimensional QFTs, where the Y's are meromorphic functions of the rapidity, the Y-system contains much less information than the TBA: in fact, it

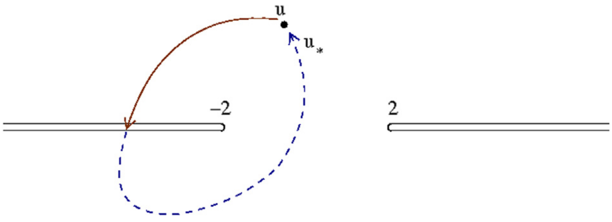

Fig. 2. The second sheet image u_* of u.

hides away completely the presence of the branch points. However, analogously to what was done in [44] for the AdS_5/CFT_4 Y-system, we can encode the missing information in an additional set of local functional relations describing the branching properties of the Y functions. In order to present the result, let us introduce some notation, following [44]. We denote with u_* the image of the point u obtained by analytic continuation along the path represented in Fig. 2, encircling the point -2.[4] Then, let $f(u)$ be a function with a square root branch point at $u = -2 + \frac{i}{h}N$. We describe its local branching properties with a "discontinuity" function, denoted by $[f(u)]_N$. It is defined as the difference between $f(u + \frac{i}{h}N)$ and its image after encircling the branch point $-2 + \frac{i}{h}N$, namely:

$$[f(u)]_N = f\left(u + \frac{i}{h}N\right) - f\left(u_* + \frac{i}{h}N\right). \tag{3.8}$$

We call it a discontinuity function because, if we restrict to $u \in (-\infty, -2) \cup (+2, +\infty) + i0^+$, it returns the value of the jump across the branch cut with imaginary part $+N/h$. However, (3.8) is more general and defines a new complex function living on a multi-sheeted surface. Notice that, in general, $[f(u)]_N$ itself has infinitely many branch points corresponding to the branch points of $f(u)$.

Finally, let us introduce the following important quantities:

$$\Delta^\alpha(u) = \left[\ln Y_{1|\alpha}(u)\right]_{+1}, \quad \alpha = I, II. \tag{3.9}$$

We are now ready to write down the extra analytic information that completes the Y-system. First of all, we assume the knowledge of the position of the branch points inside or on the border of the *physical strip* $|\mathrm{Im}(u)| < 1/h$, namely the fact that $Y_{y|\pm}(u)$ have two branch points at $u = \pm 2$, while $Y_{1|w}(u)$, $Y_{1|v}(u)$ and $Y_{1|\alpha}(u)$ ($\alpha = I, II$) have four branch points at $u = \pm 2 + i/h$, $u = \pm 2 - i/h$.

The first fundamental property is that $Y_{y|\pm}$ are two branches of the same function:

$$Y_{y|-}(u_*) = Y_{y|+}(u). \tag{3.10}$$

The remaining functional relations are:

$$[\ln Y_{w|1}]_1 = \ln\left(\frac{1 + 1/Y_{y|-}}{1 + 1/Y_{y|+}}\right), \qquad [\ln Y_{v|1}]_1 = \ln\left(\frac{1 + Y_{y|-}}{1 + Y_{y|+}}\right), \tag{3.11}$$

$$\left[\ln\left(\frac{Y_{y|-}}{Y_{y|+}}\right)\right]_{\pm 2N} = -\sum_{Q=1}^{N} \sum_{\alpha=I,II} \left[\ln\left(1 + \frac{1}{Y_{Q|\alpha}}\right)\right]_{\pm(2N-Q)}, \tag{3.12}$$

[4] It would be completely equivalent to encircle the branch point $+2$, because the topology of the Riemann surface on which the Y functions are defined is symmetric under reflection across the imaginary axis.

$$\left[\Delta^{\alpha}\right]_{\pm 2N} = \mp\left[\ln\left(1+\frac{1}{Y_{y|\mp}}\right)\right]_{\pm 2N} \mp \sum_{M=1}^{N}\left[\ln\left(1+\frac{1}{Y_{v|M}}\right)\right]_{\pm(2N-M)} \mp \ln\left(\frac{Y_{y|-}}{Y_{y|+}}\right), \tag{3.13}$$

with $\alpha = I, II$, $N = 1, 2, \ldots$. These relations are extremely similar to the ones appearing in the context of the AdS_5/CFT_4 correspondence and discovered in [44]. Together with (3.1)–(3.7), they constitute a fundamental set of local and state-independent equations, which is completely equivalent to the TBA.

In the next section, we show how (3.11)–(3.13) can be extracted from the TBA equations. The reconstruction of the ground state TBA from the Y-system equations (3.1)–(3.7) *extended* by (3.10)–(3.13) is essentially the same as the one contained in [44] for the AdS_5/CFT_4 case and we do not report it here, while the method to turn these conditions into excited state TBA equations can be found in [46].

As a final comment, notice that there is no dependence on α in the rhs of (3.13). Therefore, although we expect that $\varepsilon_{1|I}(u) \neq \varepsilon_{1|II}(u)$ for particular excited state solutions, the related discontinuity functions are equal, as confirmed by the expression for the ground state (4.12) below. In fact, because the two wings $\alpha = I, II$ enter symmetrically in (4.12), any process of analytic continuation will preserve this property, and we conclude that $\Delta^{I}(u) = \Delta^{II}(u)$ for any state.

4. A sketch of the derivation

In this section we provide a concise derivation of (3.11)–(3.13) from the TBA. The idea is to compute the discontinuity functions relative to branch points located in or on the border of the physical strip $|\mathrm{Im}(u)| < 1/h$, which contains the essential analytic information necessary for the reconstruction of the TBA equations. We confine our attention to the following functions: $[\ln Y_{w|1}]_{+1}$, $[\ln Y_{v|1}]_{+1}$, $[\ln Y_{y|-}]_0$, $[\ln Y_{1|\alpha}]_{+1}$ ($\alpha = I, II$). In fact, functions of the form $[\ln Y_a]_{-1}$ can easily be recovered from the ones listed above using the standard Y-system (3.1)–(3.7).

4.0.1. The discontinuity relations for the v and w functions

The easiest quantity to compute is:

$$\left[\ln Y_{w|1}(u)\right]_{+1} = \varepsilon_{w|1}\left(u+\frac{i}{h}\right) - \varepsilon_{w|1}\left(u_*+\frac{i}{h}\right). \tag{4.1}$$

We start from the TBA equation

$$\varepsilon_{w|1}(u) = \sum_{M=1}^{\infty} L_{w|M} * \phi_{M,1}(u) + \int_{-2}^{2} dz\,\big(L_{y|-}(z) - L_{y|+}(z)\big)\phi_1(z-u). \tag{4.2}$$

When analytically continuing equations of TBA type one has to notice that a change in the external variable induces a motion of the poles of the integral kernels. Whenever a pole crosses the integration contour on the real axis, the form of the equation must be modified, either by deforming the contour or by adding a residue term. Notice that all the kernels in (4.2) are meromorphic and therefore no branching can occur as long as the poles stay bounded away from the real axis. Considering the kernels in (4.2), this proves that $\varepsilon_{w|1}$ is analytic for $|\mathrm{Im}(u)| < 1/h$.

To study the analyticity on the border of this strip we can concentrate on two terms in the rhs of (4.2): the convolution

$$I_1(u) = L_{w|2} * \phi_{2,1}(u), \tag{4.3}$$

which is potentially dangerous because $\phi_{2,1}(u) = \phi_1(u) + \phi_3(u)$ and $\phi_1(z-u)$ has two poles at $z = u \pm \frac{i}{h}$, and the integral

$$I_2(u) = \int_{-2}^{2} dz\,\big(L_{y|-}(z) - L_{y|+}(z)\big)\phi_1(z-u). \tag{4.4}$$

In the case of (4.3) the integration contour lies on the real axis. With a slight deformation it is possible to avoid any contact with the poles, so that this term is analytic on the whole line $\mathrm{Im}(u) = 1/h$. In the case of (4.4), deforming the contour we have

$$I_2(u+i/h) = \oint_C dz\,\big(L_{y|-}(z) - L_{y|+}(z)\big)\phi_1(z-u-i/h), \tag{4.5}$$

where the contour C is represented in Fig. 3. However, contrary to the previous case, it is now impossible to avoid trapping the contour when one of the points -2 or $+2$ is encircled. Therefore, taking the residue with the appropriate sign, we find

$$I_2(u_*+i/h) = I_2(u+i/h) - \big(L_{y|-}(z) - L_{y|+}(z)\big). \tag{4.6}$$

Subtracting (4.6) from (4.5), we finally find the first relation in (3.11). Notice that, for simplicity, we have performed the calculation using a clockwise-oriented path for the continuation $u + i/h \to u_* + i/h$ as shown in Fig. 3. The reader can check that following an anticlockwise path would lead to the same result (therefore showing that the branching is of square-root type) using the property $L_{y|+}(u_*) = L_{y|-}(u)$, $L_{y|-}(u_*) = L_{y|+}(u)$. The fact that $\varepsilon_{y|\pm}(u)$ are branches of the same function is a consequence of the TBA equation (2.3) and of the identity (A.7).

The second relation in (3.11) can be derived exactly in the same way after rewriting the relevant TBA equation as[5]

$$\varepsilon_{v|1}(u) = -\sum_{Q=2}^{\infty}(L_{Q|I} + L_{Q|II}) * \phi_{Q-1}(u) + \sum_{M=1}^{\infty} L_{v|M} * \phi_{M,1}(u)$$
$$+ \int_{-2}^{2} dz\,\big(\Lambda_{y|-}(z) - \Lambda_{y|+}(z)\big)\phi_1(z-u). \tag{4.8}$$

4.0.2. *The discontinuity relations for the fermionic nodes*

Let us now consider the fermionic excitations ε_y. The value of

$$\big[\ln Y_{y|-}(u)\big]_0 = \varepsilon_{y|-}(u) - \varepsilon_{y|-}(u_*) \tag{4.9}$$

---

5 We have used Eq. (4.10) and the simple kernel identity:

$$\int_{-2}^{+2} ds\,\big(K(z-iQ/h,s) - K(z+iQ/h,s)\big)\phi_1(s-u) + \phi_{Q-1}(z-u) = \phi_{Q,(v|1)}(z,u). \tag{4.7}$$

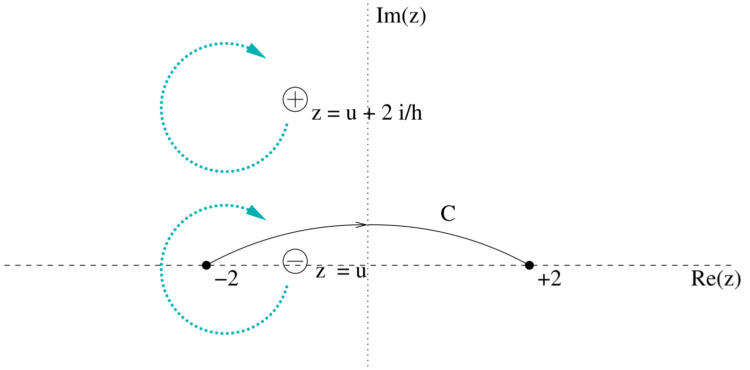

Fig. 3. An illustration of Eqs. (4.5) and (4.6). The "plus" and "minus" circles represent the poles of the kernel $\phi_1(z-u)$ (with residues ± 1, respectively) after the shift $u \to u + i/h$, while the dashed arrows represent the motion of the poles as a result of $u + i/h \to u_* + i/h$.

can be read from the TBA equations:

$$\varepsilon_{y|-}(u) - \varepsilon_{y|+}(u) = -\sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} dv \, \big(L_{Q|I}(v) + L_{Q|II}(v)\big)\big(K(v - iQ/h, u) - K(v + iQ/h, u)\big), \tag{4.10}$$

where we used the identity (A.12).

Notice that, contrary to the case of $[\ln Y_{w|1}(u)]_1$, $[\ln Y_{v|1}(u)]_1$, (4.10) is a non-local expression, and therefore we expect its form to depend on the particular excited state under consideration. However, in analogy to what seen in [44] in the AdS_5/CFT_4 context, we can trade it for an infinite number of local functional relations describing the discontinuity functions $[\ln Y_{y|-}(u)/Y_{y|+}(u)]_{\pm 2N}$, $N = 1, 2, \ldots$. To compute these quantities, it is sufficient to note that, under the analytic continuation $u \to u \pm i2N/h$, (4.10) is modified by a number of residue terms. For example if $u \to u + i2N/h$, $N \in \mathbb{N}$, with $u < \mathrm{Im}(u) < 1/h$, $|\mathrm{Re}(u)| < 2$, we have

$$\ln \frac{Y_{y|-}(u + i2N/h)}{Y_{y|+}(u + i2N/h)} = -\sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} dv \, \big(L_{Q|I}(v) + L_{Q|II}(v)\big)\big(K(v - iQ/h, u + i2N/h) - K(v + iQ/h, u + 2iN/h)\big) - \sum_{Q=1}^{2N} \big(L_{Q|I}\big(u + i(2N - Q)/h\big) + L_{Q|II}\big(u + i(2N - Q)/h\big)\big). \tag{4.11}$$

The analytic continuation $u + i2N/h \to u_* + i2N/h$ has no effect on the convolution in (4.11) and is nontrivial only for half of the residue terms in the last line, because $L_{Q|\alpha}(u)$ is analytic for $|\mathrm{Im}(u)| < Q/h$. This gives precisely (3.12).

Fig. 4. The contour $\bar{\gamma}_{\mathrm{o}}$.

Fig. 5. The contour $\bar{\gamma}_{\mathrm{x}}$.

4.0.3. Discontinuity relations for the functions $\Delta^I = \Delta^{II}$

Finally, let us consider the quantities

$$\Delta^\alpha(u) = \left[\ln Y_{1|\alpha}(u)\right]_{+1} = \varepsilon_{1|\alpha}(u + i/h) - \varepsilon_{1|\alpha}(u_* + i/h).$$

The analytic continuation of the TBA equation (2.2) leads to the following expression, valid for $0 < \mathrm{Im}(u) < \frac{2}{h}$:

$$\begin{aligned}\Delta^I(u) &= \Delta^{II}(u) \\ &= L \ln x^2(u) - L_{y|-}(u) + \oint_{\bar{\gamma}_{\mathrm{o}}} ds\, L_y(s) K(s,u) \\ &\quad + \sum_{N=1}^{\infty} \int_{-\infty}^{\infty} ds\, L_{v|N}(s) K^{\{N\}}(s,u)\, ds + \Delta_4^{\Sigma}(u),\end{aligned} \tag{4.12}$$

where $\bar{\gamma}_{\mathrm{o}}$ is the integration contour shown in Fig. 4,

$$K^{\{N\}}(s,u) \equiv K(s + iN/h, u) + K(s - iN/h, u) \tag{4.13}$$

and

$$\Delta_4^{\Sigma}(u) = \sum_{\alpha=I,II} \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} ds\, L_{Q|\alpha}(s) K_Q^{\Sigma}(s,u). \tag{4.14}$$

The kernel K_Q^{Σ} sums up the contribution of the dressing factor, appeared already in the AdS_5/CFT_4 context and is defined by

$$K_Q^{\Sigma}(s,u) = K_{Q,1}^{\Sigma}(s, u + i/h) - K_{Q,1}^{\Sigma}(s, u_* + i/h). \tag{4.15}$$

Using (A.10), it is easy to show that

$$K_Q^{\Sigma}(s,u) = \oint_{\bar{\gamma}_{\mathrm{x}}} dt\, \phi_{Q,y}(s,t) \oint_{\bar{\gamma}_{\mathrm{x}}} dz\, K_\Gamma^{[2]}(t-z) K(z,u) - \oint_{\bar{\gamma}_{\mathrm{x}}} dt\, \phi_{Q,y}(s,t) K_\Gamma^{[2]}(t-u), \tag{4.16}$$

where $\bar{\gamma}_{\mathrm{x}}$ is the integration contour shown in Fig. 5. Again, Eq. (4.12) is non-local, and in order to express its analytic content in a state-independent way we consider the following quantities:

$$\left[\Delta^I\right]_{\pm 2N} = \left[\Delta^{II}\right]_{\pm 2N}, \quad N = 1, 2, \ldots. \tag{4.17}$$

By analytic continuation of (4.12) using the techniques illustrated in the previous paragraphs, we find

$$\left[\Delta^{\alpha}\right]_{\pm 2N} = \mp[L_{y|\mp}]_{\pm 2N} \mp \sum_{M=1}^{N} [L_{v|M}]_{\pm(2N-M)} + \left[\Delta_4^{\Sigma}\right]_{\pm 2N}, \quad \alpha = I, II. \tag{4.18}$$

The last term can be computed explicitly from Eqs. (4.15)–(4.17). Notice that the first convolution on the rhs of (4.16) has a trivial monodromy for u far from the real axis, and therefore we can discard it. On the contrary, applying the sequence of analytic continuations $u \to (u \pm i2N/h) \to u_* \pm i2N/h$ for $|\mathrm{Im}(u)| < 1/h$, $|\mathrm{Re}(u)| < 2$, the second convolution in (4.16) transforms as follows:

$$\begin{aligned} -\oint_{\bar{\gamma}_x} dt\, \phi_{Q,y}(s,t) K_{\Gamma}^{[2]}(t-u) &\to -\oint_{\bar{\gamma}_x} dt\, \phi_{Q,y}(s,t) K_{\Gamma}^{[2]}\big(t-(u\pm i2N/h)\big) \\ &\to -\oint_{\bar{\gamma}_x} dt\, \phi_{Q,y}(s,t) K_{\Gamma}^{[2]}\big(t-(u\pm i2N/h)\big) \mp \big(\phi_{Q,(y|-)}(s,u) - \phi_{Q,(y|+)}(s,u)\big). \end{aligned} \tag{4.19}$$

Therefore we find

$$\begin{aligned} \left[\Delta_4^{\Sigma}(u)\right]_{\pm 2N} &= \sum_{\alpha=I,II} \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} ds\, L_{Q|\alpha}(s) \big(K_Q^{\Sigma}(s, u \pm i2N/h) - K_Q^{\Sigma}(s, u_* \pm i2N/h)\big) \\ &= \pm \sum_{\alpha=I,II} \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} ds\, L_{Q|\alpha}(s) \big(\phi_{Q,(y|-)}(s,u) - \phi_{Q,(y|+)}(s,u)\big) \\ &= \mp \ln\big(Y_{y|-}(u)/Y_{y|+}(u)\big). \end{aligned} \tag{4.20}$$

Combining the latter with (4.18), we recover (3.13).

5. More discontinuity relations

In this section we show how to deduce further constraints relating branch points lying inside and outside of the physical strip, using only the standard Y-system (3.1)–(3.7). As seen in [44] in the AdS_5/CFT_4 case, these additional discontinuity relations are useful for the purpose of deriving the TBA in its canonical form from the extended Y-system (an alternative inversion procedure is presented in [46]).

We illustrate the strategy in the case of the $Y_{M|w}$ functions. From the Y-system equation

$$\begin{aligned} &\ln Y_{M|w}\big(u + i(K+1)/h\big) + \ln Y_{M|w}\big(u + i(K-1)/h\big) \\ &\quad = (1-\delta_{M,1}) \Lambda_{M-1|w}(u + iK/h) + \Lambda_{M+1|w}(u + iK/h) \\ &\qquad + \delta_{M,1}\big(L_{y|-}(u+iK/h) - L_{y|+}(u+iK/h)\big) \end{aligned} \tag{5.1}$$

it follows that

$$\begin{aligned} &\left[\ln Y_{M|w}(u)\right]_{K+1} + \left[\ln Y_{M|w}(u)\right]_{K-1} \\ &\quad = (1-\delta_{M,1})\left[\Lambda_{M-1|w}(u)\right]_K + \left[\Lambda_{M+1|w}(u)\right]_K + \delta_{M,1}\left[\big(L_{y|-}(u) - L_{y|+}(u)\big)\right]_K, \end{aligned} \tag{5.2}$$

for $K \in \mathbb{Z}$. To derive more complicated identities, it is convenient to introduce a pictorial notation. We represent a shifted Y-system equation such as (5.1) for $M = 2, 3, \ldots$ by a diagram

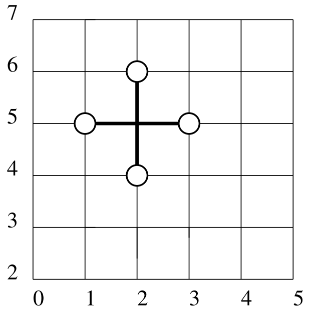

Fig. 6. An illustration of relation (5.1) for $M = 2$ and $K = 5$.

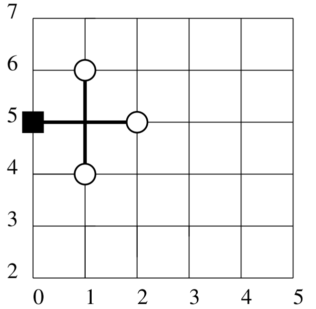

Fig. 7. An illustration of relation (5.1) for $M = 1$ and $K = 5$.

connecting the nodes $(M, K+1)$, $(M, K-1)$, $(M-1, K)$, $(M+1, K)$ on an $\mathbb{N} \times \mathbb{Z}$ grid, see Fig. 6. For $M = 1$, we can employ a different symbol to signal the contribution of the $y|\pm$ functions, as is done by using a black square in Fig. 7. When iterating (5.1), we obtain more complex graphs such as the ones in Figs. 8 and 9. The rule to associate an equation to the graph is very simple. A white circle on the node (M, N) with $M \in \mathbb{N}^+$ gives a term $+\Lambda_{M|w}(u + iN/h)$ for every horizontal link and a term $-\ln Y_{M|w}(u + iN/h)$ for every vertical link departing from it. A black square on the node $(0, N)$ represents the term $L_{y|-}(u + iN/h) - L_{y|+}(u + iN/h)$. For example, Figs. 8 and 9 represent, respectively, the $N = 2$ and $N = 3$ cases of the equation:

$$\begin{aligned} &\ln Y_{w|2}\big(u + i(2+2N)/h\big) \\ &\quad = D^{w|2}_{+(2+2N)}(u) + \Lambda_{w|N+3}\big(u + i(1+N)/h\big) \\ &\qquad + L_{w|N+2}(u + iN/h) - \ln Y_{w|N+1}\big(u + i(N-1)/h\big), \end{aligned} \tag{5.3}$$

where the function $D^{w|2}_{(2+2N)}(u)$ is defined as

$$\begin{aligned} D^{w|2}_{(2+2N)}(u) = {} & L_{y|-}(u + i2N/h) - L_{y|+}(u + i2N/h) + L_{w|1}\big(u + i(2N+1)/h\big) \\ & + \sum_{k=1}^{N} \Big(L_{w|k}\big(u + i(2N-k)/h\big) + 2L_{w|k+1}\big(u + i(2N-k+1)/h\big) \\ & + L_{w|k+2}\big(u + i(2N-k+2)/h\big) \Big). \end{aligned} \tag{5.4}$$

Notice that $D^{w|2}_{(2+2N)}(u)$ sums up the contribution of the nodes that lie on or above the diagonal. In Figs. 8 and 9, this part of the graph is shown enclosed by a rectangular frame. Using the fact

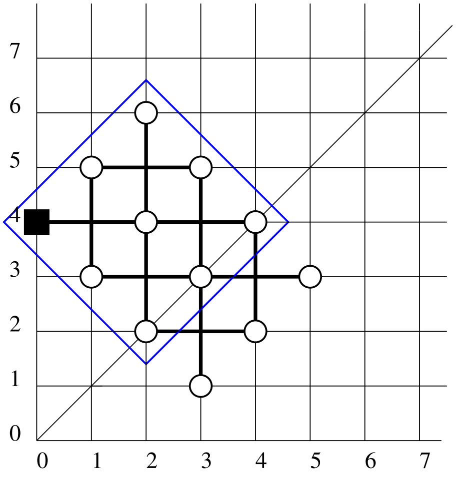

Fig. 8. A representation of Eq. (5.3) with $N = 2$. The nodes in the blue rectangular frame correspond to the function $D_6^{w|2}$.

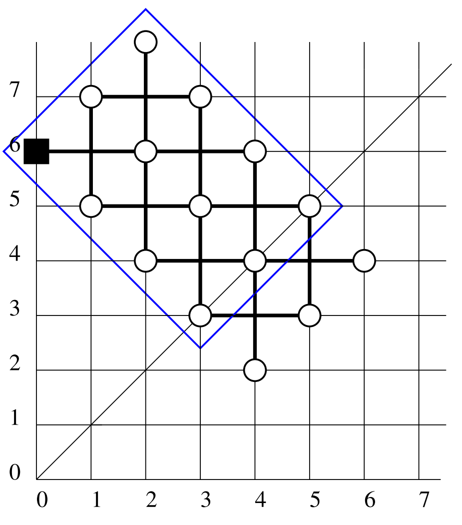

Fig. 9. A representation of Eq. (5.3) with $N = 3$. The nodes in the blue rectangular frame correspond to the function $D_8^{w|2}$.

that the branch points of $\ln Y_{w|M}(u)$ closest to the real axis have $\mathrm{Im}(u) = \pm M/h$, (5.3) implies that

$$\left[\ln Y_{w|2}(u)\right]_{(2+2N)} = \left[D^{w|2}_{(2+2N)}(u)\right]_0 - \delta_{N,0}\left[\ln Y_{1|w}(u)\right]_{-1}. \tag{5.5}$$

Eqs. (5.4) and (5.5) constitute an example of a discontinuity relation derived using only the standard Y-system. Notice that the term $[\ln Y_{1|w}(u)]_{-1}$ in the last line can be determined using the fundamental discontinuity relation (3.13).

As a last comment on the structure of these relations, notice that the part of the graph of Fig. 9 lying strictly above the diagonal coincides with the framed part of the graph in Fig. 8, translated by two units upwards. This property is reflected by the following general identity:

$$D^{w|2}_{(2N+2)}(u) - D^{w|2}_{2N}(u + i2/h)$$
$$= 2L_{w|N+1}\big(u + i(N+1)/h\big) + L_{w|N}(u + iN/h) + L_{w|N+2}\big(u + i(N+2)/h\big), \quad (5.6)$$

for $N = 0, 1, \ldots$, where the terms on the rhs correspond to nodes lying on the diagonal in the subgraph defining $D^{w|2}_{(2N+2)}(u)$. Clearly, the set of identities (5.6) allows one to define the $D^{w|2}_{(2+2N)}(u)$ functions recursively, starting from $D^{w|2}_{2}(u)$. Relations of this form were shown to be very convenient for the purpose of deriving the TBA equations from the extended Y-system in [44]. Although we do not repeat here this calculation, we list all the relevant equations in Appendix C.

6. Conclusions

In this paper we have derived, starting from the Thermodynamic Bethe Ansatz equations describing the AdS_4/CFT_3 spectrum, a set of functional relations that characterise the analytic structure of the Y functions. Extremely similar results have been previously derived for the AdS_5/CFT_4 TBA, and with essentially the same proof as the one contained in [44], one could show that the Y-system (3.1)–(3.7), extended by the new relations (3.10)–(3.13), is equivalent to the TBA. The advantage is that the new functional relations, contrary to the integral TBA equations, take the same form for all the excited states of the theory.

We expect this result to have the same applications as in the AdS_5/CFT_4 case. In particular, in that context the discontinuity relations have been used to derive rigorously excited state TBA equations [46] and also to prove the equivalence between the TBA and a much handier finite set of nonlinear integral equations,[6] the FiNLIE proposed in [49]. The study of the properties of the T-system presented in Appendix D is a preliminary step in this direction.

Very recently a new and much simpler formulation of the AdS_5/CFT_4 spectral problem has appeared, the $\mathbf{P}\mu$-system [50]. We believe that our results will be useful in achieving a similar simplification also for the theory considered here.

Acknowledgements

We thank Dmytro Volin and Fedor Levkovich-Maslyuk for fruitful discussions. This project was partially supported by INFN grants IS FI11, P14, PI11, the Italian MIUR-PRIN contract 2009KHZKRX-007 *“Symmetries of the Universe and of the Fundamental Interactions”*, the UniTo-SanPaolo research grant Nr TO-Call3-2012-0088 *“Modern Applications of String Theory” (MAST)*, the ESF Network *“Holographic methods for strongly coupled systems” (HoloGrav)* (09-RNP-092 (PESC)) and MPNS–COST Action MP1210 *“The String Theory Universe”*. The work of A.C. was supported by a City University Research Fellowship.

[6] For a different finite set of NLIEs (the hybrid-NLIE) see [48], and for an early mixed NLIE-TBA formulation see [47].

Appendix A. Basic definitions

In many of the S-matrix elements as well as in the energy and momentum of elementary excitations in the mirror ABA equations appears the Zhukovsky map defined as

$$x(u) = \left(\frac{u}{2} - i\sqrt{1 - \frac{u^2}{4}}\right). \tag{A.1}$$

Notice that $x(u)$ has two square-root branch points at $u = \pm 2$ and that it behaves as $x(u_*) = 1/x(u)$ under analytic continuation. The first Riemann sheet is defined by the condition $\mathrm{Im}(x) < 0$. The momentum and energy of Q-particle bound states in the mirror ABA can be written as functions of the rapidity u as

$$\tilde{p}^Q(u) = \frac{ih}{2}\left(\sqrt{4 - \left(u + i\frac{Q}{h}\right)^2} - \sqrt{4 - \left(u - i\frac{Q}{h}\right)^2}\right),$$
$$\tilde{E}_Q(u) = \ln \frac{x(u - iQ/h)}{x(u + iQ/h)}. \tag{A.2}$$

The kernels appearing in the TBA equations (2.2)–(2.5) are defined as:

$$\phi_{a,b}(u,z) = \frac{1}{2\pi i}\frac{d}{du}\ln S_{a,b}(u,z).$$

For different values of the indices a, b, the S-matrix elements $S_{a,b}(u,z)$ are defined as

$$S_{(v|M),Q}(u,z) = S_{Q,(v|M)}(z,u) = \left(\frac{x(z - \frac{i}{h}Q) - x(u + \frac{i}{h}M)}{x(z + \frac{i}{h}Q) - x(u + \frac{i}{h}M)}\right)\left(\frac{x(z + \frac{i}{h}Q)}{x(z - \frac{i}{h}Q)}\right)$$
$$\times \left(\frac{x(z - \frac{i}{h}Q) - x(u - \frac{i}{h}M)}{x(z + \frac{i}{h}Q) - x(u - \frac{i}{h}M)}\right)\prod_{j=1}^{M-1}\left(\frac{z - u - \frac{i}{h}(Q - M + 2j)}{z - u + \frac{i}{h}(Q - M + 2j)}\right), \tag{A.3}$$

$$S_M(u) = \left(\frac{u - \frac{i}{h}M}{u + \frac{i}{h}M}\right), \tag{A.4}$$

$$S_{K,M}(u) = \left(\frac{u - \frac{i}{h}|K - M|}{u + \frac{i}{h}|K - M|}\right)\left(\frac{u - \frac{i}{h}(K + M)}{u + \frac{i}{h}(K + M)}\right)\prod_{k=1}^{\min(K,M)-1}\left(\frac{u - \frac{i}{h}(|K - M| + 2k)}{u + \frac{i}{h}(|K - M| + 2k)}\right)^2. \tag{A.5}$$

Moreover, we have

$$S_{(y|\mp),Q}(u,z) = S_{Q,(y|\mp)}(z,u) = \left(\frac{x(z - \frac{i}{h}Q) - (x(u))^{\pm 1}}{x(z + \frac{i}{h}Q) - (x(u))^{\pm 1}}\right)\sqrt{\frac{x(z + \frac{i}{h}Q)}{x(z - \frac{i}{h}Q)}}. \tag{A.6}$$

Notice that we have the important property

$$S_{(y|-),Q}(z,u_*) = S_{(y|+),Q}(z,u). \tag{A.7}$$

The elements $S_{(Q|\alpha),(Q'|\beta)}(u,z)$ are:

$$S_{(Q|\alpha),(Q'|\beta)}(u,z) = S^0_{(Q|\alpha),(Q'|\beta)}(u-z)\big(\Sigma^{Q,Q'}(u,z)\big)^{-1}, \tag{A.8}$$

where $\Sigma^{Q,Q}$ is the improved dressing factor defined in [42]

$$\Sigma^{Q,Q'}(u,z) = \prod_{k=1}^{Q}\prod_{l=1}^{Q'}\left(\frac{1-\frac{1}{x(u+\frac{i}{h}(Q+2-2k))x(z+\frac{i}{h}(Q'-2l))}}{1-\frac{1}{x(u+\frac{i}{h}(Q-2k))x(z+\frac{i}{h}(Q'+2-2l))}}\right)\sigma^{Q,Q'}(u,z), \tag{A.9}$$

and $\sigma^{Q,Q'}(u,z)$ is the dressing factor for the direct theory [38–41,43] with both arguments continued to mirror kinematics. The following equivalent expression was derived in [20]:

$$\begin{aligned} K^{\Sigma}_{Q'Q}(u,v) &= \frac{1}{2\pi i}\frac{d}{du}\ln \Sigma_{Q'Q}(u,v) \\ &= \oint_{\bar{\gamma}_x} ds\,\phi_{Q',y}(u,s)\oint_{\bar{\gamma}_x} dt\,K^{[2]}_{\Gamma}(s-t)\phi_{y,Q}(t,v). \end{aligned} \tag{A.10}$$

The definition of $S^0_{(Q|\alpha),(Q'|\beta)}(u)$ can be found in Appendix A of [24], and we omit it here as it is rather lenghty and does not enter the computations presented in this paper.

Finally, the following kernels appear in the computations of Section 4:

$$K(u,z) = \frac{\sqrt{4-z^2}}{2\pi i\sqrt{4-u^2}(u-z)} \tag{A.11}$$

satisfying

$$K(z-iQ/h,u) - K(z+iQ/h,u) = \phi_{Q,(y|-)}(z,u) - \phi_{Q,(y|+)}(z,u), \tag{A.12}$$

and

$$K^{[2]}_{\Gamma}(z) = \frac{1}{2\pi i}\frac{d}{dz}\ln\frac{\Gamma(1-ihz/2)}{\Gamma(1+ihz/2)}. \tag{A.13}$$

Appendix B. Additional relations for the fermionic Y functions

In this appendix, we show how to derive the following identity:

$$\begin{aligned} &\big[\ln\big((Y_{y|-})(Y_{y|+})\big)\big]_{\pm 2M} \\ &\quad = 2\sum_{j=1}^{M}[L_{v|j} - L_{w|j}]_{\pm(2M-j)} - \sum_{Q=1}^{M}[L_{Q|I} + L_{Q|II}]_{\pm(2M-Q)}, \end{aligned} \tag{B.1}$$

$M = 0, 1, \ldots$ from the extended Y-system (3.1)–(3.7), (3.10)–(3.13).

A similar calculation was presented in Appendix F of [44] in the AdS_5/CFT_4 case, but the demonstration contained a logical gap. The amended derivation we present here can be easily adapted to the AdS_5/CFT_4 case.[7]

We start by considering the following relation, which is an immediate consequence of the Y-system equation (3.5) and is valid for any $N \in \mathbb{Z}$:

[7] This answers the question raised in the footnote (43) of [49], and shows that (B.1), and its analogue F.5 of [44], can be derived from the extended Y-system and do not need to be independently postulated.

$$\begin{aligned}
&2\big[\ln(Y_{y|-})\big]_{\pm 2N} + 2\big[\ln(Y_{y|-})\big]_{\pm(2N-2)} \\
&\quad = \big[\ln\big((Y_{y|-})(Y_{y|+})\big) + \ln(Y_{y|-}/Y_{y|+})\big]_{\pm 2N} \\
&\qquad + \big[\ln\big((Y_{y|-})(Y_{y|+})\big) + \ln(Y_{y|-}/Y_{y|+})\big]_{\pm(2N-2)} \\
&\quad = 2\Big[\Lambda_{v|1} - \Lambda_{w|1} - \sum_{\alpha=I,II} L_{1|\alpha}\Big]_{\pm(2N-1)} \\
&\quad = 2\Big[L_{v|1} - L_{w|1} + \ln(Y_{v|1}/Y_{w|1}) - \sum_{\alpha=I,II} L_{1|\alpha}\Big]_{\pm(2N-1)},
\end{aligned} \tag{B.2}$$

where we have set $\Lambda_a(u) = \ln(1 + Y_a(u))$. Next, we use the following identity, which can be derived iterating the Y-system equations (3.6) and (3.7) (for example, one can use the graphical method described in Section 5):

$$\begin{aligned}
\big[\ln(Y_{v|1}/Y_{w|1})\big]_{\pm(2N-1)} &= \big[\ln(Y_{y|-}/Y_{y|+})\big]_{\pm(2N-2)} - \sum_{\alpha=I,II}\sum_{Q=2}^{N}[L_{Q|\alpha}]_{\pm(2N-Q)} \\
&\quad + \sum_{j=1}^{N-1}\sum_{k\geqslant 1} A_{jk}[L_{v|j} - L_{w|j}]_{\pm(2N-1-k)} \\
&\quad + [L_{v|N} - L_{w|N}]_{\pm N} - \delta_{N,1}\big[\ln(Y_{v|1}/Y_{w|1})\big]_{\mp 1},
\end{aligned} \tag{B.3}$$

for $N = 1, 2, \ldots$ and where $A_{jk} = \delta_{j,k+1} + \delta_{j,k-1}$. Combining (B.3) and (B.2) we find:

$$\begin{aligned}
&\big[\ln(Y_{y|-}Y_{y|+})\big]_{\pm 2N} + \big[\ln(Y_{y|-}Y_{y|+})\big]_{\pm(2N-2)} \\
&\quad = 2\sum_{j=1}^{N}[L_{v|j} - L_{w|j}]_{\pm(2N-j)} + 2\sum_{j=1}^{N-1}[L_{v|j} - L_{w|j}]_{\pm(2N-2-j)} \\
&\qquad - 2\sum_{\alpha=I,II}\sum_{Q=1}^{N}[L_{Q|\alpha}]_{\pm(2N-Q)} - \big[\ln(Y_{y|-}/Y_{y|+})\big]_{\pm 2N} \\
&\qquad + \big[\ln(Y_{y|-}/Y_{y|+})\big]_{\pm(2N-2)} - 2\delta_{N,1}\big[\ln(Y_{v|1}/Y_{w|1})\big]_{\mp 1}.
\end{aligned} \tag{B.4}$$

Now we can invoke three of the fundamental relations, namely the two identities in (3.11) (leading to $[\ln(Y_{v|1}/Y_{w|1})]_{\mp 1} = \ln(Y_{y|-}/Y_{y|+})$) and (3.12), to evaluate the terms in the last two lines of (B.4). The result is

$$\begin{aligned}
&\big[\ln(Y_{y|-}Y_{y|+})\big]_{\pm 2N} + \big[\ln(Y_{y|-}Y_{y|+})\big]_{\pm(2N-2)} \\
&\quad = 2\sum_{j=1}^{N}[L_{v|j} - L_{w|j}]_{\pm(2N-j)} + 2\sum_{j=1}^{N-1}[L_{v|j} - L_{w|j}]_{\pm(2N-2-j)} \\
&\qquad - \sum_{\alpha=I,II}\left(\sum_{Q=1}^{N}[L_{Q|\alpha}]_{\pm(2N-Q)} + \sum_{Q=1}^{N-1}[L_{Q|\alpha}]_{\pm(2N-Q-2)}\right), \quad N = 0, 1, \ldots.
\end{aligned} \tag{B.5}$$

Finally, because of (3.11) we have $[\ln(Y_{y|-}Y_{y|+})]_0 = 0$. Using this fact we can solve (B.5) recursively and finally get (B.2).

Appendix C. A list of useful identities

For the sake of completeness, below we list a number of identities that can be derived using only the structure of the basic Y-system (3.1)–(3.7), according to the method of Section 5. As shown in [44] (see Sections 7–9 of that paper), these relations are useful for the purpose of reconstructing the TBA equations.

The discontinuities of the $\ln Y_{v|M}(u)$ functions satisfy:

$$\left[\ln Y_{w|M}(u)\right]_{\pm(M+2N)} = \left[D^{w|M}_{\pm(M+2N)}(u) - \delta_{N,0}\ln Y_{1|w}(u \mp i/h)\right]_0 \tag{C.1}$$

for $N = 0, 1, \ldots$, where the $D^{w|M}_{\pm(M+2N)}(u)$ functions are defined iteratively by

$$\begin{aligned} &D^{w|M}_{\pm(2N+M)}(u) - D^{w|M}_{\pm(2N+M-2)}(u \pm i2/h) \\ &\quad = 2\sum_{k=N+1}^{M+N-1} L_{w|k}(u \pm ik/h) + L_{w|N}(u \pm iN/h) \\ &\qquad + L_{w|M+N}\big(u \pm i(M+N)/h\big), \end{aligned} \tag{C.2}$$

with

$$D^{w|M}_{\pm M}(u) = L_{y|-}(u) - L_{y|+}(u) + \sum_{k=1}^{M-1} L_{w|k}(u \pm ik/h). \tag{C.3}$$

The discontinuities of the $\ln Y_{v|M}(u)$ functions satisfy

$$\left[\ln Y_{v|M}(u)\right]_{\pm(M+2N)} = \left[D^{v|M}_{\pm(M+2N)}(u) - \delta_{N,0}\ln Y_{v|1}(u \mp i/h)\right]_0, \tag{C.4}$$

and the $D^{v|M}_{\pm(M+2N)}(u)$ functions are defined by

$$\begin{aligned} &D^{v|M}_{\pm(M+2N)}(u) - D^{v|M}_{\pm(M+2N-2)}(u \pm i2/h) \\ &\quad = L_{v|N}(u \pm iN/h) + L_{v|M+N}\big(u \pm i(M+N)/h\big) \\ &\qquad + 2\sum_{k=N+1}^{M+N-1} L_{v|k}(u \pm ik/h) - \sum_{Q=N+1}^{M+N}\sum_{\alpha=I,II} L_{Q|\alpha}(u \pm iQ/h), \end{aligned} \tag{C.5}$$

with

$$D^{v|M}_{\pm M}(u) = \Lambda_{y|-}(u) - \Lambda_{y|+}(u) + \sum_{k=1}^{M-1} L_{v|k}(u \pm ik/h). \tag{C.6}$$

The discontinuities of the $\ln Y_{Q|\alpha}$ functions satisfy

$$\left[\ln Y_{Q|\alpha}(u)\right]_{\pm(Q+2P)} = \left[D^{Q|\alpha}_{\pm Q}(u) - \delta_{P,1}\ln Y_{1|\beta}(u \mp i/h)\right]_0 \tag{C.7}$$

for $P = 1, 2, \ldots$, where $\alpha \in \{I, II\}$, $\beta \in \{I, II\}$, $\alpha = \beta$ if Q is odd and $\alpha \neq \beta$ if Q is even.

The $D^{Q|\alpha}_{\pm(Q+2P)}(u)$ functions are defined by[8]

[8] Given a real number $r \in \mathbb{R}$, we denote its integer part as $\lfloor r \rfloor \in \mathbb{Z}$.

$$
\begin{aligned}
&D^{Q|\alpha}_{\pm(Q+2P)}(u) - D^{Q|\alpha}_{\pm(Q+2P-2)}(u \pm i2/h) \\
&\quad = L_{P|\beta}(u \pm iP/h) + L_{Q+P|\alpha}\big(u \pm i(Q+P)/h\big) \\
&\qquad + 2\sum_{k=1}^{\lfloor\frac{Q-1}{2}\rfloor} L_{Q+P-2k|\alpha}\big(u \pm i(Q+P-2k)/h\big) - \sum_{N=P}^{Q+P-1} L_{v|N}(u \pm iN/h) \\
&\qquad + 2\sum_{k=1}^{\lfloor\frac{Q}{2}\rfloor} L_{Q+P+1-2k|\gamma}\big(u \pm i(Q+P+1-2k)/h\big),
\end{aligned} \tag{C.8}
$$

where $\alpha, \beta, \gamma \in \{I, II\}$, $\gamma \neq \alpha$, $\alpha = \beta$ if Q is odd and $\alpha \neq \beta$ if Q is even, and:

$$
\begin{aligned}
D^{Q|\alpha}_{\pm Q}(u) = {} & \sum_{k=1}^{\lfloor\frac{Q-1}{2}\rfloor} L_{Q-2k|\alpha}\big(u \pm i(Q-2k)/h\big) + \sum_{k=1}^{\lfloor\frac{Q}{2}\rfloor} L_{Q+1-2k|\gamma}\big(u \pm i(Q+1-2k)/h\big) \\
& - \left(\sum_{N=1}^{Q-1} L_{v|N}(u \pm iN/h) + L_y(u) \right),
\end{aligned} \tag{C.9}
$$

with $\alpha, \gamma \in \{I, II\}$ and $\gamma \neq \alpha$.

Appendix D. The T-system

In this section we will reconsider the discontinuity relations (3.10)–(3.13), from the point of view of the T-system, as done in [46,49] in the case of AdS_5/CFT_4. In particular, following [49], we show how to encode the analytic content of the TBA equations into a set of very symmetric constraints for the T functions.

The Y-system of AdS_4/CFT_3 is naturally related to the diagram represented in Fig. 1. In fact, let us associate a Y function to every node of the diagram, using an additional index $\alpha \in \{I, II\}$ to distinguish the functions living on the two wings. Then the Y-system relations (3.1)–(3.7) can be written in a universal form using the incidence matrix of the diagram[9]:

$$
\begin{aligned}
&Y_{n,s}(u+i/h)Y_{n,s}(u-i/h) = \frac{(1+Y_{n,s+1}(u))(1+Y_{n,s-1}(u))}{(1+1/Y_{n+1,s}(u))(1+1/Y_{n-1,s}(u))}, \\
&\quad s > 1,\ (n,s) \neq (2,2), \\
&Y_{n,1}(u+i/h)Y_{n,1}(u-i/h) = \frac{(1+Y_{n,2}(u))(1+Y^{I}_{n,0}(u))(1+Y^{II}_{n,0}(u))}{(1+1/Y_{n+1,1}(u))(1+1/Y_{n-1,1}(u))}, \\
&Y^{\alpha}_{n,0}(u+i/h)Y^{\beta}_{n,0}(u-i/h) = \frac{(1+Y_{n,1}(u))}{(1+1/Y^{\alpha}_{n+1,0}(u))(1+1/Y^{\beta}_{n-1,0}(u))}, \\
&\quad \alpha, \beta \in \{I, II\}, \alpha \neq \beta.
\end{aligned} \tag{D.1}
$$

Notice that we have to exclude the node $(2,2)$, as there is no local Y-system equation in this case. The Y functions in the double index notation are related to the ones used in the rest of this paper and in [24] by:

[9] However, we point out that the diagram in Fig. 1 does not capture the crossing between the two wings in the lhs of the Y-system equations for the nodes $(n, 0, \alpha)$, $\alpha = I, II$. For non-symmetric states such that $Y_{n|I} \neq Y_{n|II}$, we need to keep track of this important subtlety.

$$Y^{\alpha}_{Q,0} = 1/Y_{Q|\alpha}, \quad \text{for } Q \geqslant 1, \alpha = I, II,$$
$$Y_{1,1} = 1/Y_{y|-},$$
$$Y_{2,2} = Y_{y|+},$$
$$Y_{n,1} = 1/Y_{v|n-1}, \quad \text{for } n \geqslant 2,$$
$$Y_{1,n} = Y_{w|n-1}, \quad \text{for } n \geqslant 2.$$

The T functions live on a lattice obtained by adding extra nodes to the diagram in Fig. 1. We denote them as

$$T_{n,s}, \quad \text{with } (n,s) \in \mathbb{N} \times \mathbb{N}^+, \quad s \leqslant 2 \text{ or } n \leqslant 2,$$
$$T^{\alpha}_{n,l}, \quad \text{with } (n,l) \in \mathbb{N} \times \{-1, 0\}, \quad \alpha \in \{I, II\},$$

and they are assumed to be zero when the indices are outside the domain indicated above. They are related to the Y functions by:

$$Y_{n,s}(u) = \frac{T_{n,s+1}(u)T_{n,s-1}(u)}{T_{n+1,s}(u)T_{n-1,s}(u)}, \quad \text{for } s \geqslant 2,\ n \geqslant 1, \tag{D.2}$$
$$Y_{n,1}(u) = \frac{T_{n,2}(u)T^{I}_{n,0}(u)T^{II}_{n,0}(u)}{T_{n+1,1}(u)T_{n-1,1}(u)}, \quad \text{for } n \geqslant 1,$$
$$Y^{\alpha}_{n,0}(u) = \frac{T_{n,1}(u)T^{\beta}_{n,-1}(u)}{T^{\alpha}_{n+1,0}(u)T^{\beta}_{n-1,0}(u)}, \quad \text{for } n \geqslant 1,\ \alpha, \beta \in \{I, II\}, \beta \neq \alpha.$$

The Y-system is satisfied provided the T functions obey discrete Hirota equations on the lattice: the T-system. For the AdS_4/CFT_3 diagram, the T-system relations take the usual form for $s \geqslant 2$:

$$T_{n,s}(u + i/h)T_{n,s}(u - i/h) = (1 - \delta_{n,0})T_{n+1,s}(u)T_{n-1,s}(u) + T_{n,s-1}(u)T_{n,s+1}(u), \tag{D.3}$$

while there is a cubic term when $s = 1$

$$T_{n,1}(u + i/h)T_{n,1}(u - i/h)$$
$$= (1 - \delta_{n,0})T_{n+1,1}(u)T_{n-1,1}(u) + T^{I}_{n,0}(u)T^{II}_{n,0}(u)T_{n,2}(u), \tag{D.4}$$

and the equations with $s = -1, 0$ are

$$T^{\alpha}_{n,0}(u + i/h)T^{\beta}_{n,0}(u - i/h) = (1 - \delta_{n,0})T^{\alpha}_{n+1,0}(u)T^{\beta}_{n-1,0}(u) + T^{\beta}_{n,-1}(u)T_{n,+1}(u),$$
$$T^{\alpha}_{n,-1}(u + i/h)T^{\beta}_{n,-1}(u - i/h) = T^{\alpha}_{n+1,-1}(u)T^{\beta}_{n-1,-1}(u) \quad (n \neq 0), \tag{D.5}$$

with $\alpha, \beta \in \{I, II\}$ and $\beta \neq \alpha$.

It is well known that the same solution to the Y-system is parametrised by a large family of equivalent solutions to the T-system, connected by gauge transformations. A generic gauge transformation preserving the validity of the T-system (D.3)–(D.6) and leaving invariant the Y functions can be written as follows:

$$T^{\alpha}_{n,0} \to f^{[n]}_{\alpha} g^{[n]}_{\alpha} h^{[-n]} j^{[-n]} T^{\alpha}_{n,0}, \quad \text{for } \alpha \in \{I, II\}, n \in \mathbb{N},$$
$$T^{\alpha}_{n,-1} \to \frac{(f_{\alpha})^{[n-1]}}{(f_{\alpha})^{[n+1]}} \frac{(g_{\alpha})^{[n+1]}}{(g_{\alpha})^{[n-1]}} \frac{(h)^{[-n-1]}}{(h)^{[-n+1]}} \frac{(j)^{[-n+1]}}{(j)^{[-n-1]}} T^{\alpha}_{n,-1},$$
for $\alpha, \beta \in \{I, II\}, \alpha \neq \beta$ and $n \in \mathbb{N}$,
$$T_{n,s} \to (f_I f_{II})^{[n+s]} (g_I g_{II})^{[n-s]} \big(h^2\big)^{[-n+s]} \big(j^2\big)^{[-n-s]} T_{n,s}, \quad \text{for } s \in \mathbb{N}^+, n \in \mathbb{N}, \tag{D.6}$$

where $f_I, f_{II}, g_I, g_{II}, h, j$ are arbitrary functions and we have adopted the notation: $A^{[a]}(u) \equiv A(u + ia/h)$ to denote imaginary shifts in the rapidity.

Let us now translate the discontinuity relations in terms of the T functions. A straightforward calculation shows that the two equations in (3.11) can be rewritten in the following form:

$$\frac{T_{1,1}(u - i/h)T_{1,1}(u_* + i/h)}{T_{2,2}(u - i/h)T_{2,2}(u_* + i/h)} \frac{T_{2,3}(u)}{T_{0,1}(u)} = \frac{E(u)}{E(u_*)} \tag{D.7}$$

and

$$\frac{T_{2,2}(u - i/h)T_{2,2}(u_* + i/h)}{T_{1,1}(u - i/h)T_{1,1}(u_* + i/h)} \frac{T^I_{1,0}(u)T^{II}_{1,0}(u)}{T_{3,2}(u)} = \frac{F(u)}{F(u_*)}, \tag{D.8}$$

where we have defined

$$\begin{aligned} E(u) &= T_{1,3}(u + i/h)T_{0,2}(u_* + i/h), \\ F(u) &= T_{3,1}(u_* + i/h) \prod_{\alpha=I,II} T^{\alpha}_{2,0}(u + i/h). \end{aligned} \tag{D.9}$$

It was discovered in [46] in the AdS_5/CFT_4 context that the remaining, infinitely many discontinuity relations can be greatly simplified by making appropriate assumptions on the analyticity strips of the T functions. Moreover, in [49] it was shown that there exist two special gauges where the analytic information contained in the discontinuity relations can be recast in a very symmetric form. They were denoted with the fonts $\mathbf{T}$ and $\mathbb{T}$, with the $\mathbf{T}_{n,s}$ functions having particularly convenient analytic properties in the upper band of the diagram defined by $n \geqslant s$ and the $\mathbb{T}_{n,s}$ functions being particularly well behaved in the right band defined by $s \geqslant n$. We conclude this section by showing how the same can be achieved in the present case.

A closed set of properties of the T functions in these two special gauges is given below in (i), (ii), (iii), (iv), (a), (b), (c), (d). This set of conditions, together with the transformation (D.23), encodes the same analytic information as the original TBA equations, or, equivalently, the extended Y-system of Section 3. In the following we provide an almost complete proof and make further observations.

We follow very closely Appendix C of [49], where similar calculations are presented. Let us borrow a useful notation: we denote as $\mathcal{A}_n$ the class of functions meromorphic in the strip $|\operatorname{Im}(u)| < n/h$. In general, we expect T functions in $\mathcal{A}_n$ to have branch points at $\pm 2 + in/h$, $\pm 2 - in/h$. Then, we start by considering a gauge, which we denote generically with the font $T \equiv t$, such that the t functions are real[10] and satisfy

$$t^{\alpha}_{n,-1} = 1, \qquad t^{\alpha}_{n,0} \in \mathcal{A}_{n+1}, \qquad t_{n,1} \in \mathcal{A}_n, \tag{D.10}$$

for $n \in \mathbb{N}, \alpha \in \{I, II\}$.

Let us now consider the discontinuity relation (3.12). When expanding the factors $(1 + 1/Y_{v|n})$ in terms of the t functions, their jump discontinuities cancel pairwise (see similar calculations in [46,49]) and we find that this condition is equivalent to[11]

$$\left[\ln \mathbf{b}(u)\right]_{2N} = 0, \quad N = 1, 2, \ldots, \tag{D.11}$$

10 In the case of non-symmetric states such that $Y^I_{n,0} \neq Y^{II}_{n,0}$, the requirement of reality has to be replaced with $\bar{t}^I_{n,0} = t^{II}_{n,0}$.

11 To be precise, the requirement $t^{\alpha}_{n,0} \in \mathcal{A}_{n+1}$ is enough to prove (D.11). The other conditions in (D.10) have been added for future convenience.

where

$$\mathbf{b}(u) = Y_{1,1}(u)Y_{2,2}(u) \prod_{\alpha=I,II} \frac{t^{\alpha}_{0,0}(u-i/h)}{t^{\alpha}_{1,0}(u)} = \frac{t_{2,3}(u)}{t_{3,2}(u)} \frac{t^{I}_{0,0}(u-i/h)t^{II}_{0,0}(u-i/h)}{t_{0,1}(u)}. \tag{D.12}$$

Therefore the function $\mathbf{b}(u)$ defined above is meromorphic in the upper half plane. Notice that $\mathbf{b}(u)$ is still a gauge-dependent quantity. However, following [49], let us define a transition function f, analytic for $\mathrm{Im}(u) > -1/h$, such that

$$\mathbf{b}(u) \equiv \frac{f^2(u+i/h)}{f^2(u-i/h)}. \tag{D.13}$$

Then, making a gauge transformation of the form[12]

$$\begin{aligned}
&\mathbf{T}_{n,s} = f^{[n+s]} f^{[n-s]} \bar{f}^{[-n+s]} \bar{f}^{[-n-s]} t_{n,s}, \quad \text{for } s \in \mathbb{N}^+,\ n \in \mathbb{N},\\
&\mathbf{T}^{\alpha}_{n,0} = f^{[n]} \bar{f}^{[-n]} t^{\alpha}_{n,0}, \quad \text{for } \alpha \in \{I, II\},\ n \in \mathbb{N},\\
&\mathbf{T}^{\alpha}_{n,-1} = t^{\alpha}_{a,-1} = 1, \quad \text{for } \alpha \in \{I, II\},\ n \in \mathbb{N},
\end{aligned} \tag{D.14}$$

we find a gauge satisfying

$$\frac{\mathbf{T}_{2,3}(u)}{\mathbf{T}_{3,2}(u)} \frac{\mathbf{T}^{I}_{0,0}(u-i/h)\mathbf{T}^{II}_{0,0}(u-i/h)}{\mathbf{T}_{0,1}(u)} = 1. \tag{D.15}$$

Following [49], we can now prove the following analytic properties of the **T** gauge:

(i) The **T** functions are real, and we have

$$\mathbf{T}^{\alpha}_{n,-1} = 1, \quad \mathbf{T}^{\alpha}_{n,0} \in \mathcal{A}_{n+1}, \quad \mathbf{T}_{n,1} \in \mathcal{A}_{n}, \quad \mathbf{T}_{n,2} \in \mathcal{A}_{n-1},$$
$$n \in \mathbb{N},\ \alpha \in \{I, II\}. \tag{D.16}$$

(ii) The two quantities $\mathbf{T}^{I}_{0,0}$, $\mathbf{T}^{II}_{0,0}$ are periodic:

$$\mathbf{T}^{\alpha}_{0,0}(u+i/h) = \mathbf{T}^{\alpha}_{0,0}(u-i/h), \quad \alpha \in \{I, II\}. \tag{D.17}$$

(iii) Finally, the **T** functions enjoy the following "group-theoretical" properties:

$$\mathbf{T}_{0,n} = \left(\mathbf{T}^{I}_{0,0}\mathbf{T}^{II}_{0,0}\right)^{[n]} \quad n = 1, 2, \ldots, \tag{D.18}$$

$$\mathbf{T}_{n,2} = \mathbf{T}_{2,n}, \quad n = 2, 3, \ldots. \tag{D.19}$$

Here is a brief summary of the proof. Property (i) follows from the fact the transformation is real and does not change the analyticity domains of the t functions. Then, the complex conjugate of (D.15) implies that the product $\mathbf{T}^{I}_{0,0}\mathbf{T}^{II}_{0,0}$ is periodic, and from $\mathbf{T}^{I}_{0,-1} = \mathbf{T}^{II}_{0,-1} = 1$ and the product of the T-system equations for the $(0, 0, I)$ and $(0, 0, II)$ nodes we find $\mathbf{T}_{0,1} = (\mathbf{T}^{I}_{0,0}\mathbf{T}^{II}_{0,0})^{[+1]}$. Comparing this equation with the T-system at one of the above mentioned nodes, we find that

[12] Notice that we denote with $\bar{f}$ the complex conjugate function such that $\bar{f}(u) = (f(u^*))^*$.

not only their product, but each of the functions $\mathbf{T}^{I}_{0,0}$ and $\mathbf{T}^{II}_{0,0}$ is periodic, thus establishing property (ii). Moreover, (D.15) now implies $\mathbf{T}_{3,2} = \mathbf{T}_{2,3}$. Eqs. (D.18)–(D.19) for general n can be demonstrated by iterating the T-system and using (D.17).

In the rest of this section we also make the crucial hypothesis that it is possible to choose

$$\text{(iv)} \qquad \mathbf{T}^{I}_{0,0} = \mathbf{T}^{II}_{0,0}. \tag{D.20}$$

Because many of the following results depend on this assumption, it is worth making a comment. Condition (D.20) is certainly true in the important subsector of the symmetric states such that $Y_{n|I} = Y_{n|II}\ \forall n$, which includes the best-studied case of the $\mathfrak{sl}_2$ states [25,32]. Moreover we argue that this choice can be made even for some non-symmetric subsectors and possibly for all states. We reason as follows. Even if $\mathbf{T}^{I}_{0,0} \neq \mathbf{T}^{II}_{0,0}$, the ratio $w(u) = \mathbf{T}^{I}_{0,0}/\mathbf{T}^{II}_{0,0}$ is necessarily meromorphic, because the property $\Delta^{I} = \Delta^{II}$ proved in Section 3 implies that $\ln \mathbf{T}^{I}_{0,0}$ and $\ln \mathbf{T}^{II}_{0,0}$ have the same discontinuities.[13] Therefore, we can define a gauge transformation that sets $\mathbf{T}^{I}_{0,0} = \mathbf{T}^{II}_{0,0}$ by taking $g_I = \sqrt{\mathbf{T}^{I}_{0,0}/\mathbf{T}^{II}_{0,0}}$ and $g_{II} = 1/g_I$ in (D.6), with all the other transition functions being unity. Notice that this transformation does not spoil any other property of the $\mathbf{T}$ gauge, on the condition that $g_I(u) = \sqrt{w(u)}$ is still meromorphic and no new branch cuts are introduced by the square root. We believe that this is indeed the case for the physical solutions to the AdS_4/CFT_3 T-system.

Finally, here we leave open the problem of proving the uniqueness of the $\mathbf{T}$ gauge. However, by analogy with the AdS_5/CFT_4 case it is natural to expect that it can be fixed completely (modulo a constant rescaling of the form $\mathbf{T}^{\alpha}_{n,0} \to k\mathbf{T}^{\alpha}_{n,0}$, $\mathbf{T}_{n,s} \to k^2\mathbf{T}_{n,s}$, with $k \in \mathbb{R}$) by adding the further requirement that the $\mathbf{T}$ functions do not have poles and have the minimal amount of zeroes in their analyticity strips.

To further underline the analogy with the AdS_5/CFT_4 case treated in [49], it is useful to introduce the notation

$$\mathcal{F} = \mathbf{T}^{I}_{0,0} = \mathbf{T}^{II}_{0,0}. \tag{D.21}$$

The absence of the square root in this definition, as compared to the AdS_5/CFT_4 case, is simply due to the different structure of the Y-system, but we will see that $\mathcal{F}$ plays the same rôle in many respects.

An interesting observation is that, as already noticed in [49], $\mathcal{F}$ is strictly related to the dressing factor. In fact, from the regularity strips of $\mathbf{T}$ functions we expect $\mathcal{F}$ to have the closest branch points at distance $\pm i/h$ from the real axis. From (D.18) and the identity $Y_{1,1}Y_{2,2} = (T^{I}_{1,0}T^{II}_{1,0}T_{2,3})/(T_{0,1}T_{3,2})$ it is possible to prove

$$Y_{1,1}(u)Y_{2,2}(u) = \frac{\mathcal{F}(u_* + i/h)}{\mathcal{F}(u + i/h)} \tag{D.22}$$

[13] In fact, using the identity $\mathbf{T}^{\alpha}_{0,0} = \mathbf{T}_{1,1}/(Y^{\beta}_{1,0}\mathbf{T}^{\beta}_{2,0})$ ($\alpha \neq \beta$) we get $[\ln \mathbf{T}^{\alpha}_{0,0}]_1 = [\ln \mathbf{T}_{1,1}]_1 - \Delta$, $\alpha = I, II$, where $\Delta = \Delta^{I} = \Delta^{II}$. Therefore $[\ln \mathbf{T}^{I}_{0,0}]_1 = [\ln \mathbf{T}^{II}_{0,0}]_1$ and because of the periodicity this is sufficient to prove that $\mathbf{T}^{I}_{0,0}/\mathbf{T}^{II}_{0,0}$ is meromorphic.

and, because of the periodicity (D.17), $[\ln\mathcal{F}]_1 = [\ln\mathcal{F}]_{(1+2n)} = -\ln Y_{1,1}Y_{2,2}$, $\forall n \in \mathbb{Z}$. According to (4.20), a periodic jump discontinuity equal to $\pm\ln Y_{1,1}Y_{2,2}$ characterises precisely the contribution of the dressing factor to the TBA equations.

Following [49], let us now introduce a new gauge $\mathbb{T}$ by[14]

$$\begin{aligned} &\mathbb{T}_{n,s} = (-1)^{n(s+1)}\mathbf{T}_{n,s}\big(\mathcal{F}^{[n+s]}\big)^{n-2}, \quad s \geqslant 1, \\ &\mathbb{T}^{\alpha}_{n,0} = (-1)^{n}\mathbf{T}^{\alpha}_{n,0}\big(\sqrt{\mathcal{F}^{[n]}}\big)^{n-2}, \\ &\mathbb{T}^{\alpha}_{n,-1} = \mathbf{T}^{\alpha}_{n,-1} = 1. \end{aligned} \tag{D.23}$$

From (D.23), it follows immediately that the $\mathbb{T}$ functions are real and satisfy

$$\begin{aligned} &\mathbb{T}_{0,s} = 1 \quad \text{for } s \geqslant -1, \\ &\mathbb{T}_{2,s} = \mathbf{T}_{2,s} \in \mathcal{A}_s \quad \text{for } s \geqslant 2, \qquad \mathbb{T}_{1,1} \in \mathcal{A}_1. \end{aligned} \tag{D.24}$$

Moreover, it is possible to show that

$$\mathbb{T}_{1,s} \in \mathcal{A}_s, \quad s \geqslant 1. \tag{D.25}$$

In fact, using (D.18)–(D.19) and the transformation (D.23), one finds

$$\frac{Y_{11}}{Y_{22}} = \frac{T_{3,2}}{T_{2,3}}\frac{T^2_{1,2}T^{I}_{1,0}T^{II}_{1,0}}{T_{0,1}T^2_{2,1}} = \frac{\mathbb{T}^2_{1,2}\mathbf{T}^{I}_{1,0}\mathbf{T}^{II}_{1,0}}{\mathbf{T}^2_{2,1}}. \tag{D.26}$$

Recalling the analyticity strips of the $\mathbf{T}$ functions and remembering that $Y_{11}/Y_{22} = 1/(Y_{y|-}Y_{y|+}) \in \mathcal{A}_2$, this shows that $\mathbb{T}_{1,2} \in \mathcal{A}_2$. The analyticity strips for $\mathbb{T}_{1,s}$, $s \geqslant 3$ can be established, for example, by considering the various identities (B.1),[15] which were derived in Appendix B.

Now let us consider the set of discontinuity relations (3.13). Repeating the derivation of Section D.3 in [49], one can show that these relations can be rewritten as

$$\left[\Delta^{\alpha}(u) + \ln\frac{\mathbf{T}_{0,1}(u)}{\mathbf{T}_{1,1}(u+i/h)}\right]_{2N} = -\ln Y_{1,1}(u)Y_{2,2}(u), \quad N \in \mathbb{N}^{+}, \tag{D.27}$$

and using the identity

$$\begin{aligned} \Delta^{\alpha}(u) &= \ln\mathbb{T}_{1,1}(u+i/h) - \ln\mathbb{T}_{1,1}(u_* + i/h) \\ &= \ln\mathbf{T}_{1,1}(u+i/h) - \ln\mathbb{T}_{1,1}(u_* + i/h) - \mathcal{F}(u+i/h) \end{aligned}$$

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[14] Although this transformation has a quite unusual form, it defines a new solution to the T-system thanks to the periodicity of $\mathcal{F}$.

[15] Let us exemplify the derivation by showing that $\mathbb{T}_{1,3} \in \mathcal{A}_3$. The first subcase of (B.1) can be written as

$$\left[\ln\frac{Y_{11}}{Y_{22}}\right]_{\pm 2} = 2\left[\ln\frac{(1+1/Y_{1,2})}{(1+Y_{2,1})}\right]_{\pm 1} + \sum_{\alpha}\big[\ln\big(1+Y^{\alpha}_{1,0}\big)\big]_{\pm 1}.$$

Expressing $(1+1/Y_{1,2})$ in the $\mathbb{T}$ gauge and $(1+Y_{2,1})$, $(1+Y_{1,0})$ in the $\mathbf{T}$ gauge, the above expression becomes

$$\left[\ln\frac{Y_{11}}{Y_{22}}\right]_{\pm 2} = \left[\ln\frac{\mathbb{T}^2_{1,2}\mathbf{T}^{I}_{1,0}\mathbf{T}^{II}_{1,0}}{\mathbf{T}^2_{2,1}}\right]_{\pm 2} - \big[\ln\mathbb{T}^2_{1,3}\big]_{\pm 1},$$

and comparing this result with (D.26) we deduce that the last term on the rhs vanishes. Therefore $\mathbb{T}_{1,3}$ has no branch points with $\mathrm{Im}(u) = \pm 1/h$ and using $\mathbb{T}_{1,2} \in \mathcal{A}_2$ the T-system implies $\mathbb{T}_{1,3} \in \mathcal{A}_3$.

we find

$$\left[\ln \mathbb{T}_{1,1}(u_* \pm i/h)\right]_{2N} = 0, \quad N = 1, 2, \ldots. \tag{D.28}$$

This condition tells us that, when evaluated on a Riemann section defined with only "short" cuts of the form $(-2, +2) + in/h$, the function $\mathbb{T}_{1,1}$ has only two cuts with $\mathrm{Im}(u) = \pm 1/h$. Because of this surprising property, this Riemann section was called the "magic sheet" in [49]. Borrowing a further notation, we will denote with a hat $\hat{\mathrm{T}}$ the analytic continuation of the T functions on a sheet with only short cuts, starting from their real values. Notice that, as discussed in Section 3, the Y-system and the T-system are naturally defined on a Riemann sheet with "long" branch cuts of the form $(-\infty, -2) \cup (+2, +\infty) + in/h$, a convention that is precisely the opposite of the "magic sheet" prescription.

The $\mathbb{T}$ gauge enjoys precisely the same analytic properties as the gauge denoted with the same font in [49], describing the right band of the AdS_5/CFT_4 diagram. Let us list the relevant properties, postponing the proof.

The first three properties state that, when evaluated on the magic sheet, the $\hat{\mathbb{T}}_{a,s}$ functions with $s \geqslant a$ have at most two branch cuts[16] each:

(a) For $n \geqslant 1$, $\hat{\mathbb{T}}_{1,n}$ has only two branch cuts on the magic sheet: $(-2, 2) \pm in/h$.
(b) For $m \geqslant 2$, $\hat{\mathbb{T}}_{2,m}$ has only four branch cuts on the magic sheet:
$(-2, 2) \pm i(m-1)/h$, $(-2, 2) \pm i(m+1)/h$.
(c) For $n \geqslant -1$, $\hat{\mathbb{T}}_{0,n} = 1$.

Moreover, the $\hat{\mathbb{T}}$ functions possess a special discrete symmetry, which in [49] was identified with a quantum version of the $\mathbb{Z}_4$ symmetry of the AdS_5/CFT_4 sigma model. To discover this special symmetry one has to consider an extended domain given by the infinite horizontal band:

$$\mathfrak{B} = \left\{(a, s) \in \mathbb{Z}^2 \mid 0 \leqslant a \leqslant 2\right\}. \tag{D.29}$$

The solution on $\mathfrak{B}$ is constructed by assigning the values of $\hat{\mathbb{T}}$ to the nodes with $s \geqslant a$ and using the T-system on the magic sheet to populate the rest of the domain. We use the font $\underline{\hat{\mathbb{T}}}$ for the nodes with $s < a$ in $\mathfrak{B}$, to emphasise that they may differ from the values of the $\hat{\mathbb{T}}_{a,s}$ functions on corresponding nodes of the original AdS_4/CFT_3 diagram.[17] With these notations, the special discrete symmetry can be written as follows:

[16] Incidentally, they also show that, in the magic sheet kinematics, each of the Y-functions $\hat{Y}_{w|n}$ with $n \geqslant 1$ has only two short branch cuts, a property that is very hard to guess from the TBA.

[17] Notice that, for $s > a$, the analyticity strips of the $\mathbb{T}$ functions are wide enough that the T-system equations hold even if we change $\mathbb{T} \to \hat{\mathbb{T}}$. However, this is no longer true at the nodes with $s = a$ and we expect that, for $s < a$, the T functions on $\mathfrak{B}$ bear no resemblance with the T functions on the corresponding nodes of the original AdS_4/CFT_3 diagram. A good example is provided by the node $(1, 0)$. In $\mathfrak{B}$ we find $\underline{\hat{\mathbb{T}}}_{1,0} = 0$, while in the original diagram there are *two* functions $\hat{\mathbb{T}}^{I}_{1,0}$, $\hat{\mathbb{T}}^{II}_{1,0}$, and they are both different from zero according to (D.33).

$$\text{(d)} \qquad \hat{\underline{\mathbb{T}}}_{a,-s} = (-1)^a \hat{\mathbb{T}}_{a,s}, \quad s \geqslant a, \qquad \hat{\underline{\mathbb{T}}}_{a,-s} = (-1)^a \hat{\underline{\mathbb{T}}}_{a,s}, \quad s < a. \tag{D.30}$$

Although a complete proof of these properties can already be found in [49] in the similar AdS_5/CFT_4 case, we find it useful to provide a partially alternative argument. We start by showing that

$$\hat{\mathbb{T}}_{2,2}(u) = \hat{\mathbb{T}}_{1,1}(u + 2i/h)\hat{\mathbb{T}}_{1,1}(u - 2i/h). \tag{D.31}$$

One possible way to establish this result is to notice that, as shown in the following Section D.1, the ratio $G(u) = (\hat{\mathbb{T}}_{1,1}(u + 2i/h)\hat{\mathbb{T}}_{1,1}(u - 2i/h))/\hat{\mathbb{T}}_{2,2}(u)$ can be rewritten as the following combination of Y functions:

$$G(u) = \frac{(1 + 1/Y_{2,2}(u + i/h))(1 + 1/Y_{2,2}(u - i/h))(1 + 1/Y_{2,1}(u))}{Y^{I}_{1,0}(u_{\circlearrowleft})Y^{II}_{1,0}(u_{\circlearrowleft})}. \tag{D.32}$$

Here, $u_{\circlearrowleft}$ denotes the image of the point u on the Riemann sheet reached by analytic continuation through the branch cut with $\mathrm{Im}(u) = +1/h$ and $u_{\circlearrowright}$, conversely, is the image of u reached after following a path that encircles one of the branch points with $\mathrm{Im}(u) = -1/h$. As we show in Appendix D.2 below, from the TBA it is possible to prove that this quantity is precisely one, and this establishes (D.31).

Notice that, using the T-system on the nodes $(2, s)$ together with condition (D.31), this result can be generalised to $\hat{\mathbb{T}}_{2,s}(u) = \hat{\mathbb{T}}_{1,1}(u + si/h)\hat{\mathbb{T}}_{1,1}(u - si/h)$ for $s \geqslant 1$. Since $\hat{\mathbb{T}}_{1,1}$ has only one pair of branch cuts, this proves property (b).

To establish the remaining properties of the $\hat{\mathbb{T}}$ functions, let us derive some preliminary useful relations. Notice that (D.19) implies $\mathbb{T}_{3,2} = \mathbb{T}_{2,3}\mathcal{F}$, so that in the $\mathbb{T}$ gauge the identity $Y_{1,1}Y_{2,2} = (T^{I}_{1,0}T^{II}_{1,0}T_{2,3})/(T_{0,1}T_{3,2})$ takes the form $Y_{1,1}Y_{2,2} = (\mathbb{T}^{I}_{1,0}\mathbb{T}^{II}_{1,0})/(\mathcal{F})$. Comparing this result with (D.22), we find the important expression

$$\mathbb{T}^{I}_{1,0}(u)\mathbb{T}^{II}_{1,0}(u) = \mathcal{F}(u_* + i/h). \tag{D.33}$$

Finally, let us rewrite the discontinuity relations (D.7)–(D.8) in the $\mathbb{T}$ gauge. From the properties of the $\mathbb{T}$ functions listed above and using (D.33), it is possible to show that (D.8) is equivalent to

$$\hat{\mathbb{T}}_{2,2}(u - i/h)\hat{\mathbb{T}}_{2,2}(u + i/h) = \hat{\mathbb{T}}_{1,1}(u - i/h)\hat{\mathbb{T}}_{1,1}(u + i/h)\hat{\mathbb{T}}_{2,3}(u). \tag{D.34}$$

The constraint (D.7) leads to the same equation but multiplied by a factor $\mathbb{T}_{1,3}(u + i/h)/(\mathbb{T}_{1,3}(u_* + i/h))$ that we must set to one, therefore this provides another confirmation that $\mathbb{T}_{1,3} \in \mathcal{A}_3$.

Condition (D.34) is crucial to prove the symmetry property (d). In fact, it contains precisely the information needed to extend the solution from the right band into the left part of $\mathfrak{B}$. Consider the T-system equation in $\mathfrak{B}$: $\hat{\mathbb{T}}_{2,2}(u + i/h)\hat{\mathbb{T}}_{2,2}(u - i/h) = \hat{\mathbb{T}}_{2,3}(u)\hat{\underline{\mathbb{T}}}_{2,1}(u)$, where $\hat{\underline{\mathbb{T}}}_{2,1}(u)$ is so far unknown and defined by the previous relation. Comparing this equation with (D.34) we find

$$\hat{\underline{\mathbb{T}}}_{2,1}(u) = \hat{\mathbb{T}}_{1,1}(u - i/h)\hat{\mathbb{T}}_{1,1}(u + i/h), \tag{D.35}$$

and matching (D.35) with $\hat{\mathbb{T}}_{1,1}(u-i/h)\hat{\mathbb{T}}_{1,1}(u+i/h) = \hat{\underline{\mathbb{T}}}_{1,0}(u)\hat{\mathbb{T}}_{1,2}(u) + \hat{\underline{\mathbb{T}}}_{2,1}(u)\hat{\mathbb{T}}_{0,1}(u)$ we get

$$\hat{\underline{\mathbb{T}}}_{1,0}(u) = 0. \tag{D.36}$$

Moreover, from (D.35) we can compute $\hat{\underline{\mathbb{T}}}_{2,1}(u+i/h)\hat{\underline{\mathbb{T}}}_{2,1}(u-i/h) = \hat{\mathbb{T}}^2_{1,1}(u)\hat{\mathbb{T}}_{1,1}(u+2i/h)\hat{\mathbb{T}}_{1,1}(u-2i/h)$. Using condition (D.31), we find that in order to match the T-system equation $\hat{\underline{\mathbb{T}}}_{2,1}(u+i/h)\hat{\underline{\mathbb{T}}}_{2,1}(u-i/h) = \hat{\underline{\mathbb{T}}}_{2,0}(u)\hat{\mathbb{T}}_{2,2}(u)$ we have to take

$$\hat{\underline{\mathbb{T}}}_{2,0} = \hat{\mathbb{T}}^2_{1,1}. \tag{D.37}$$

Using (D.37) and the T-system at the nodes $(1,0)$ and $(2,0)$ it is now easy to check that a solution constructed using the symmetry (D.30) satisfies the T-system at all nodes of $\mathfrak{B}$.

Finally, we refer the reader to a complex proof contained in [49], Section 4.2. The authors show that a solution of the T-system on $\mathfrak{B}$ with the above mentioned properties including the discrete symmetry also has to satisfy

$$[\ln \hat{\mathbb{T}}_{1,n}]_{\pm(n+2m)} = 0, \quad n, m = 1, 2, \ldots. \tag{D.38}$$

Therefore, all $\hat{\mathbb{T}}_{1,s}$ have only two branch cuts, establishing (a).

This concludes the proof of the properties of the $\mathbb{T}$ functions. Summarising, the set of conditions (i), (ii), (iii), (iv), (a), (b), (c), (d), together with the transformation (D.23), is completely equivalent to the discontinuity relations.

Similarly to the AdS_5/CFT_4 case, a discrete symmetry can also be derived for the $\hat{\mathbf{T}}$ functions. This symmetry emerges when considering the $\mathbf{T}$ functions on the magic sheet and extending them from the upper band to the following vertical domain:

$$\mathfrak{C} = \{(n, s, \alpha) \mid n \in \mathbb{Z},\ s \in \{-1, 0\},\ \alpha \in \{I, II\}\} \cup \{(n, s) \mid n \in \mathbb{Z},\ s \in \{1, 2\}\}. \tag{D.39}$$

The original values $\hat{\mathbf{T}}_{n,s}$ are assigned to the nodes with $n \geqslant s$ and the remaining T functions are computed by enforcing the validity of the T-system in the magic sheet kinematics in $\mathfrak{C}$. By very similar calculations as the ones reported above, one can construct a solution with the following symmetry:

$$\begin{aligned} &\hat{\mathbf{T}}^{\gamma}_{n,-1} = -\hat{\mathbf{T}}^{\gamma}_{-n,-1} = 1, \qquad \hat{\mathbf{T}}^{\alpha}_{n,0} = \hat{\mathbf{T}}^{\beta}_{-n,0}, \\ &\hat{\mathbf{T}}_{n,1} = -\hat{\mathbf{T}}_{-n,1}, \qquad \hat{\mathbf{T}}_{n,2} = \hat{\mathbf{T}}_{-n,2}, \end{aligned} \tag{D.40}$$

with $n \in \mathbb{Z}$, $\alpha, \beta, \gamma \in \{I, II\}$, $\alpha \neq \beta$. For simplicity of notation, we have used the same font $\hat{\mathbf{T}}_{n,s}$ for all the T functions in (D.40). The reader should be aware that they differ from the T functions on the original diagram when $n < s$.

Notice that there is a discontinuity in the first relation of (D.40), and that the value of $\hat{\mathbf{T}}^{\gamma}_{0,-1}$ appears double valued. In fact, the attentive reader will notice that T-system equations hold everywhere in $\mathfrak{C}$ apart from the nodes $(0, -1, \gamma)$, $\gamma \in \{I, II\}$. We can give an interpretation of this fact by viewing the extension of the T functions from the upper band to the whole of $\mathfrak{C}$ as an analytic continuation in their discrete indices. In this case the analytic continuation introduces a branch point on each of the wings at the index value $(0, 0, \gamma)$, $\gamma = I, II$, and we trace the branch cuts to the left of these points, so that they cross the nodes $(0, -1, \gamma)$.

As a last comment, let us compare the structure of these constraints with the ones found in [49] for AdS_5/CFT_4. While the $\mathbb{T}$ gauge has precisely the same properties in the two systems, an important difference lies in the shape of the vertical domain on which the $\hat{\mathbf{T}}$ functions are endowed of their version of the discrete symmetry. In the AdS_5/CFT_4 case, this was a strip $\mathbb{Z} \times \{-2, -1, 0, 1, 2\}$, while in the present case it is given by $\mathfrak{C}$ defined in (D.39). It should be possible to derive FiNLIEs for the present case by adapting straightforwardly the methods of [49]. However one ingredient is still missing, namely finding a parametrisation of the T-system on $\mathfrak{C}$ in terms of a finite number of Q functions.

Finally, while the quantity $\mathcal{F}$ plays in many respects the same rôle here as in the AdS_5/CFT_4 case, there is an important difference: in AdS_5/CFT_4, single zeroes of $\mathcal{F}$ are interpreted as Bethe roots of the $\mathfrak{sl}_2$ sector, while in the present case we expect the Bethe roots to correspond to double zeroes of $\mathcal{F}$. In fact, one can derive the expression

$$\prod_{\alpha=I,II}\left(1+Y^{\alpha}_{1,0}(u)\right)=\frac{\prod_{\alpha=I,II}\mathbb{T}^{\alpha}_{1,0}(u+i/h)\mathbb{T}^{\alpha}_{1,0}(u-i/h)}{\mathbb{T}^{I}_{2,0}(u)\mathbb{T}^{II}_{2,0}(u)}=\frac{\mathcal{F}(u_{\circlearrowleft})\mathcal{F}(u_{\circlearrowright})}{\mathbb{T}^{I}_{2,0}(u)\mathbb{T}^{II}_{2,0}(u)} \tag{D.41}$$

and by the analytic continuation $u \to u_{\circlearrowleft}$ we find

$$\prod_{\alpha=I,II}\left(1+Y^{\alpha}_{1,0}(u_{\circlearrowleft})\right)=\frac{\mathcal{F}(u)\mathcal{F}((u_{\circlearrowright})_{\circlearrowleft})}{\prod_{\beta=I,II}\mathbb{T}^{\beta}_{2,0}(u)}. \tag{D.42}$$

Excited state TBA equations for the $\mathfrak{sl}_2$ subsector have been conjectured in [20,21] for AdS_5/CFT_4 and [25] for AdS_4/CFT_3. In our notation, the Bethe roots u_j are described by the condition $Y_{1|\alpha}((u_j)_{\circlearrowleft})=-1$, $\alpha\in\{I,II\}$, therefore they are zeroes of the lhs of (D.42). Because of the symmetry $Y^{I}_{1,0}=Y^{II}_{1,0}$ in this sector and since $\mathcal{F}(u)\neq\mathcal{F}((u_{\circlearrowright})_{\circlearrowleft})$, we expect that $\mathcal{F}$ exhibits a double zero. In AdS_5/CFT_4, we would have the same expression (D.42) but without the products over the α, β indices, thus leading to a single zero.

D.1. Proof of Eq. (D.32)

Consider the identity

$$\left(1+1/Y_{1,1}(u)\right)=\frac{\mathbb{T}_{1,1}(u+i/h)\mathbb{T}_{1,1}(u-i/h)}{\mathbb{T}_{1,0}(u)\mathbb{T}_{1,2}(u)}.$$

After the analytic continuation $u \to u_*$ we get (using the fact that $\mathbb{T}_{1,2}(u)$ has no branch points on the real axis and identity (D.33))

$$\left(1+Y_{2,2}(u)\right)=\frac{\mathbb{T}_{1,1}(u_*+i/h)\mathbb{T}_{1,1}(u_*-i/h)}{\mathcal{F}(u+i/h)\mathbb{T}_{1,2}(u)}.$$

Shifting the previous expression starting from real u allows us to reconstruct the product of $\hat{\mathbb{T}}_{1,1}(u+2i/h)\hat{\mathbb{T}}_{1,1}(u-2i/h)$. Using $\mathbb{T}_{1,2}(u+i/h)\mathbb{T}_{1,2}(u-i/h)=(1+Y_{1,2}(u))\mathbb{T}_{2,2}(u)$ we arrive at (for real u)

$$\left(1+Y_{2,2}(u+i/h)\right)\left(1+Y_{2,2}(u-i/h)\right)\left(1+Y_{1,2}(u)\right)=G(u)\frac{\mathbb{T}_{1,1}(u_{\circlearrowleft})\mathbb{T}_{1,1}(u_{\circlearrowright})}{\mathcal{F}^2(u)},$$

where $u_{\circlearrowleft}$ (or $u_{\circlearrowright}$, respectively) is the image of the point u reached by analytic continuation through the branch cut with $\mathrm{Im}(u)=+1/h$ (resp. $\mathrm{Im}(u)=-1/h$). Moreover using $Y^{\alpha}_{1,0}=\frac{\mathbb{T}_{1,1}}{\mathbb{T}^{\alpha}_{2,0}}$ we can rewrite the above identity as

$$\begin{aligned}&\big(1+Y_{2,2}(u+i/h)\big)\big(1+Y_{2,2}(u-i/h)\big)\big(1+Y_{1,2}(u)\big)\\&\quad=G(u)\frac{Y^{I}_{1,0}(u_{\circlearrowleft})}{Y^{I}_{1,0}(u)}\frac{Y^{II}_{1,0}(u_{\circlearrowright})}{Y^{II}_{1,0}(u)}\frac{\mathbb{T}^{2}_{1,1}(u)}{\mathcal{F}^{2}(u)}.\end{aligned}$$

At the same time we have:

$$\prod_{\alpha}\big(1+1/Y^{\alpha}_{1,0}(u)\big)=\frac{\prod_{\alpha}\mathbb{T}^{\alpha}_{1,0}(u+i/h)\mathbb{T}^{\alpha}_{1,0}(u-i/h)}{\mathbb{T}^{2}_{1,1}(u)}=\frac{\mathcal{F}(u_{\circlearrowleft})\mathcal{F}(u_{\circlearrowright})}{\mathbb{T}^{2}_{1,1}(u)},$$

where the last equality follows from (D.33) and the periodicity of $\mathcal{F}$, while using (D.22) we find

$$\frac{\mathcal{F}(u_{\circlearrowleft})\mathcal{F}(u_{\circlearrowright})}{\mathcal{F}^{2}(u)}=\big(Y_{1,1}Y_{2,2}(u+i/h)\big)\big(Y_{1,1}Y_{2,2}(u-i/h)\big).$$

Putting all together and using the Y-system relation (3.5) we arrive at (D.32).

D.2. *Proof that* $G(u)=1$

We prove this relation starting from the TBA. We start by noticing that the relevant TBA kernels and the driving term satisfy the following identities:

$$\begin{aligned}&\tilde{E}_{1}(u_{\circlearrowright})+\tilde{E}_{1}(u_{\circlearrowleft})=0,\\&\phi_{(y|\pm),1}(z,u_{\circlearrowright})+\phi_{(y|\pm),1}(z,u_{\circlearrowleft})=-\phi_{1}(z-u),\\&\phi_{(v|M),1}(z,u_{\circlearrowright})+\phi_{(v|M),1}(z,u_{\circlearrowleft})=-\phi_{M,1}(z-u),\\&\phi_{(Q'|\alpha),(1|I)}(z,u_{\circlearrowright})+\phi_{(Q'|\alpha),(1|II)}(z,u_{\circlearrowleft})=-\phi_{Q',(v|1)}(z,u),\quad\forall\alpha\in\{I,II\}.\end{aligned}\tag{D.43}$$

When applying the above analytic continuation to the convolutions in the TBA equations describing $\varepsilon_{1|\alpha}(u)$, some residue terms need to be taken into account. To list the relevant properties, let us give the following definitions:

$$\begin{aligned}&A(u)=\int_{-2}^{2}dz\,\big(a(z)\phi_{(y|-),1}(z,u)-a(z_{*})\phi_{(y|+),Q}(z,u)\big),\\&B_{M}(u)=\int_{-\infty}^{\infty}dz\,b(z)\phi_{(v|M),Q}(u),\\&C_{Q,\alpha,\beta}(u)=\int_{-\infty}^{\infty}dz\,c(z)\phi_{(Q|\alpha),(1|\beta)}(z,u).\end{aligned}$$

where a denotes a function with two square root branch points at $u=\pm2$ and b, c are two functions regular on the real axis. Then a careful monitoring of the movement of singularities leads to the following properties:

$$\begin{aligned}&A(u_{\circlearrowright})+A(u_{\circlearrowleft})=-\int_{-2}^{2}dz\,\big(a(z)-a(z_{*})\big)\phi_{1}(z-u)-a_{+}(u+i/h)-a_{+}(u-i/h),\\&B_{M}(u_{\circlearrowright})+B_{M}(u_{\circlearrowleft})=-b*\phi_{M,1}(u)-\delta_{M,1}b(u),\end{aligned}$$

$$C_{Q,\alpha,I}(u_{\circlearrowright}) + C_{Q,\alpha,II}(u_{\circlearrowleft}) = -\int_{-\infty}^{\infty} dz\, c(z)\phi_{Q,(v|1)}(z,u),$$

where $a_+(u) = a(u_*)$.

Using these relations, from the TBA equation describing $\varepsilon_{1|\alpha}(u)$ we obtain

$$\begin{aligned}\varepsilon_{1|I}(u_{\circlearrowright}) + \varepsilon_{1|II}(u_{\circlearrowleft}) = {} & \sum_{\beta}\sum_{Q=1}^{\infty} L_{Q|\beta} * \phi_{Q,(v|1)}(u) - \sum_{M=1}^{\infty} L_{v|M} * \phi_{M,1}(u) \\ & - \int_{-2}^{2} dz \big[L_{y|-}(z) - L_{y|+}(z)\big]\phi_1(z-u) \\ & - L_{y|+}(u+i/h) - L_{y|+}(u-i/h) - L_{v|1}(u)\end{aligned} \qquad \text{(D.44)}$$

and by comparison with the TBA equation for $\varepsilon_{v|1}(u)$ this can be rewritten as

$$\varepsilon_{1|I}(u_{\circlearrowright}) + \varepsilon_{1|II}(u_{\circlearrowleft}) = -L_{y|+}(u+i/h) - L_{y|+}(u-i/h) - \ln\big(1 + Y_{v|1}(u)\big). \qquad \text{(D.45)}$$

This is precisely the statement that $G(u) = 1$.

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