Perturbative corrections to power suppressed effects in semileptonic B decays

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ABSTRACT: We compute the $O(\alpha_s)$ corrections to the Wilson coefficient of the chromomagnetic operator in inclusive semileptonic B decays. The results are employed to evaluate the complete $\alpha_s \Lambda_{\text{QCD}}^2/m_b^2$ correction to the semileptonic width and to the first moments of the lepton energy distribution.

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Contents

1 Introduction

The model-independent study of inclusive semileptonic B decays, initiated twenty years ago [1–4], is based on an Operator Product Expansion (OPE) in conjunction with the heavy quark expansion. At the B factories, it has allowed for very precise determinations of the CKM matrix elements $|V_{cb}|$ and $|V_{ub}|$, mostly limited by theoretical uncertainties [5]. Further progress therefore requires theoretical improvements, as well as high statistics data from Belle-II in the case of charmless decays. The calculation of higher order corrections in the OPE, in particular, is of crucial importance.

The OPE expresses the widths and the first moments of the kinematic distributions of $B \to X_{u,c}\ell\nu$ as double expansions in α_s and $\Lambda_{\mathrm{QCD}}/m_b$. The leading terms in these double expansions are given by the free b quark decays, while the $O(\alpha_s, \alpha_s^2\beta_0)$ perturbative corrections [6–12] and the $O(\Lambda_{\mathrm{QCD}}^2/m_b^2, \Lambda_{\mathrm{QCD}}^3/m_b^3)$ non-perturbative corrections [3, 4, 13] have been known for a long time. More recently, the complete $O(\alpha_s^2)$ calculation has been completed [14–18], and the $O((\Lambda_{\mathrm{QCD}}/m_Q)^{4,5})$ have been investigated [19]. The parameters of the double expansions are the heavy quark masses m_b and m_c, the strong coupling α_s, and the B-meson matrix elements of local operators of growing dimension. The latter parameterize all the long-distance physics that is relevant for inclusive decays: at $O(\Lambda_{\mathrm{QCD}}^2/m_b^2)$ there are two parameters, μ_π^2 and μ_G^2, at $O(\Lambda_{\mathrm{QCD}}^3/m_b^3)$ two more appear, ρ_D^3 and ρ_{LS}^3, and so on. The non-perturbative parameters are constrained by the experimental data for the moments of the lepton energy and hadron mass distributions of $B \to X_c\ell\nu$ and can be employed to extract $|V_{cb}|$ from the semileptonic width. Recent fits can be found in refs. [5, 20].

The coefficients of the non-perturbative corrections of $O(\Lambda_{\mathrm{QCD}}^n/m_b^n)$ in the double series are Wilson coefficients of power-suppressed local operators and can be computed perturbatively. Only a subset of the $O(\alpha_s\Lambda_{\mathrm{QCD}}^2/m_b^2)$ corrections has been computed so far: the $O(\alpha_s)$ corrections to the coefficient of μ_π^2 [21, 22], which represents the B meson

expectation value of the kinetic operator and is related to the average kinetic energy of the b quark in the B meson. In this paper we present the calculation of the remaining $O(\alpha_s \Lambda_{\mathrm{QCD}}^2/m_b^2)$ corrections, those proportional to μ_G^2, the expectation value of the chromomagnetic operator. We compute the corrections to the triple differential semileptonic B decay width and therefore to the most general moment, in such a way that they can be readily employed to improve the precision of the fits to $|V_{cb}|$.

Our calculation follows the method outlined in ref. [23], where the same corrections were computed in the simpler case of $B \to X_s\gamma$, and in ref. [22]. Here we discuss the matching procedure in greater detail and present analytic results for the $O(\alpha_s \mu_G^2/m_b^2)$ corrections to the three relevant structure functions. In this way the corrections to the triple differential width become available and the corrections to arbitrary moments can be computed. We then present numerical results for the semileptonic width and for the first leptonic moments. The paper is organized as follows: after setting the notation in section 2, we discuss the matching in section 3; the following section presents and discusses the numerical results. Section 5 summarizes our findings. The lengthy analytic results for the structure functions are given in the appendix.

2 Notation

We consider the decay of a B meson of four-momentum $p_B = M_B v$ into a lepton pair with momentum q and a hadronic final state containing a charm quark with momentum $p' = p_B - q$. The hadronic tensor $W^{\mu\nu}$ which determines the hadronic contribution to the differential width is given by the absorptive part of a current correlator in the appropriate kinematic region,

$$W^{\mu\nu}(p_B, q) = \mathrm{Im}\, \frac{2\,i}{\pi M_B} \int d^4x\, e^{-iq\cdot x} \langle \bar{B}|T J_L^{\mu\dagger}(x) J_L^\nu(0)|\bar{B}\rangle, \tag{2.1}$$

where $J_L^\mu = \bar{c}\gamma^\mu P_L b$ is the charged weak current. The correlator admits an OPE in terms of local operators, which at the level of the differential rate takes the form of an expansion in inverse powers of the energy release, whose leading term corresponds to the decay of a free quark.

Our notation follows that of ref. [12, 22]. We express the b-quark decay kinematics in terms of the dimensionless quantities

$$\rho = \frac{m_c^2}{m_b^2}, \qquad \hat{u} = \frac{(p-q)^2 - m_c^2}{m_b^2}, \qquad \hat{q}^2 = \frac{q^2}{m_b^2}, \tag{2.2}$$

where $p = m_b v$ is the momentum of the b quark and

$$0 \le \hat{u} \le \hat{u}_+ = \left(1 - \sqrt{\hat{q}^2}\right)^2 - \rho \qquad \text{and} \qquad 0 \le \hat{q}^2 \le \left(1 - \sqrt{\rho}\right)^2. \tag{2.3}$$

The energy of the hadronic system, normalized to the b mass, is

$$E = \frac{1}{2}(1 + \rho + \hat{u} - \hat{q}^2). \tag{2.4}$$

Tree-level kinematics correspond to $\hat{u} = 0$, in which case we indicate the energy of the hadronic final state as E_0. The normalized total leptonic energy is $\hat{q}_0 = 1 - E$ from which follows $\hat{u} = 2\left(1 - E_0 - \hat{q}_0\right)$. It is customary to decompose the hadronic tensor as follows

$$m_b W^{\mu\nu}(p_B, q) = -W_1 g^{\mu\nu} + W_2 v^\mu v^\nu + i W_3 \epsilon^{\mu\nu\rho\sigma} v_\rho \hat{q}_\sigma + W_4 \hat{q}^\mu \hat{q}^\nu + W_5 \left(v^\mu \hat{q}^\nu + v^\nu \hat{q}^\mu\right), \quad (2.5)$$

where the structure functions W_i are functions of $\hat{q}^2, \hat{q}_0$ or equivalently of $\hat{q}^2, \hat{u}$, v^μ is the four-velocity of the B meson, and $\hat{q}^\mu = q^\mu / m_b$. As only $W_{1,2,3}$ contribute to the decay rate for massless leptons, we will concentrate on these three structure functions.

Due to the OPE, the structure functions can be expanded in series of α_s and $\Lambda_{\rm QCD}/m_b$. There is no term linear in $\Lambda_{\rm QCD}/m_b$ and therefore

$$W_i = W_i^{(0)} + \frac{\mu_\pi^2}{2m_b^2} W_i^{(\pi,0)} + \frac{\mu_G^2}{2m_b^2} W_i^{(G,0)} + \frac{\alpha_s}{\pi} \left[C_F W_i^{(1)} + C_F \frac{\mu_\pi^2}{2m_b^2} W_i^{(\pi,1)} + \frac{\mu_G^2}{2m_b^2} W_i^{(G,1)} \right]$$
$$(2.6)$$

where we have neglected terms of higher order in the expansion parameters. μ_π^2 and μ_G^2 are the B-meson matrix elements of the only gauge-invariant dimension 5 operators that can be formed from the b quark and gluon fields:

$$\mu_\pi^2 = \frac{1}{2M_B} \langle \bar{B} | \bar{b}_v (i\vec{D})^2 b_v | \bar{B} \rangle, \qquad \mu_G^2 = -\frac{1}{2M_B} \langle \bar{B} | \bar{b}_v \frac{g_s}{2} G_{\mu\nu} \sigma^{\mu\nu} b_v | \bar{B} \rangle, \qquad (2.7)$$

where b_v is the static quark field, and $G_{\mu\nu} = G_{\mu\nu}^a T^a$ is the gluon field tensor, which is defined as $g_s G_{\mu\nu}^a T^a = -i[D_\mu, D_\nu]$ with the covariant derivative $D_\mu = \partial_\mu + i g_s G_\mu^a T^a$.[1] The leading order coefficients are given by

$$W_i^{(0)} = w_i^{(0)} \delta(\hat{u}); \qquad w_1^{(0)} = 2E_0, \qquad w_2^{(0)} = 4, \qquad w_3^{(0)} = 2. \qquad (2.8)$$

The tree-level and one-loop coefficients of μ_π^2 can be found in [22]; the tree-level coefficients of μ_G^2 [3, 4], using $\lambda_0 = 4(E_0^2 - \rho)$, can be written as:

$$W_i^{(G,0)} = w_i^{(G,0)} \delta(\hat{u}) + w_i^{(G,1)} \delta'(\hat{u}); \qquad (2.9)$$

$$w_1^{(G,0)} = -\frac{4}{3}(2 - 5E_0), \qquad \qquad w_1^{(G,1)} = -\frac{4}{3}\left(E_0 + 3E_0^2 + \frac{1}{2}\lambda_0\right);$$

$$w_2^{(G,0)} = 0, \qquad \qquad w_2^{(G,1)} = \frac{8}{3}(3 - 5E_0);$$

$$w_3^{(G,0)} = \frac{10}{3}, \qquad \qquad w_3^{(G,1)} = -\frac{4}{3}(1 + 5E_0).$$

3 The matching at $O(\alpha_s)$

Schematically, we can write the OPE in momentum space as

$$\frac{2i}{\pi} \int d^4x \, e^{-iq \cdot x} \, T[J_L^{\dagger\mu}(x) J_L^\nu(0)] = \sum_i c_{\{\alpha\}}^{(i)\mu\nu}(v, q) \, O_i^{\{\alpha\}}(0), \qquad (3.1)$$

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[1]Since we are only interested in $\Lambda_{\rm QCD}^2/m_b^2$ corrections, μ_π^2 and μ_G^2 are here defined in the asymptotic HQET regime, i.e. in the infinite mass limit.

where $O_i^{\{\alpha\}}$ are local operators and $\{\alpha\}$ stands for possible additional Lorentz indices. The number of local operators of dimension $d_i \leq 5$ that contribute to the r.h.s. can be reduced, and their renormalization simplified, by resorting to the Heavy Quark Effective Theory (HQET) and using the relation between the HQET static quark b_v and the QCD b field,

$$b(x) = e^{-im_b v \cdot x} \left(1 + \frac{i\slashed{D}}{2m_b} \right) b_v(x) \,. \tag{3.2}$$

Eventually, we will need the following set:

$$O_b^\mu = \bar{b}\gamma^\mu b, \qquad O_s = \bar{b}\,b \,,$$
$$O_1^\mu = \bar{b}_v iD^\mu b_v, \qquad O_2^{\mu\nu} = \bar{b}_v \frac{1}{2}\{iD^\mu, iD^\nu\}b_v \,, \qquad O_3^{\mu\nu} = \bar{b}_v \frac{g_s}{2} G^\mu{}_\alpha \sigma^{\alpha\nu} b_v \,. \tag{3.3}$$

Notice that $O_{b,s}$ are written in terms of the QCD bottom quark field, while the other operators are constructed in terms of b_v. Up to terms of dimension six, the operator O_s can be expressed in terms of the others:

$$O_s = v_\mu O_b^\mu + \frac{O_{2\,\alpha}^\alpha + O_{3\,\alpha}^\alpha}{2m_b^2} + O\left(\frac{1}{m_b^3}\right) \tag{3.4}$$

but we keep it distinct for reasons that will become clear. We also find operators that include a γ_5, but they can be neglected in our discussion. Indeed, because of the parity invariance of strong interactions, only the operators in (3.3) have non-vanishing matrix elements in the B meson. As we perform an off-shell calculation, we have not used the HQET equation of motion for the b_v field, which would reduce the operator O_1^μ to a linear combination of $O_{2,3}^{\mu\nu}$. The equation of motion will be used only in the last step of the calculation, when we evaluate the matrix elements of the operators in the B meson.

In order to determine the Wilson coefficients $c_{\{\alpha\}}^{(i)\mu\nu}$ we compute renormalized Green's functions of both sides of eq. (3.1) on heavy quark states close to the mass shell. The external heavy quarks have residual momentum k and we Taylor expand the Green's functions for small k up to to second order. To extract $c_{\alpha\beta}^{(3)\mu\nu}$ we also need to consider Green's functions with a soft external gluon. They are Taylor expanded in both k and the gluon virtuality r.

It is convenient to decompose the tensors as in (2.5), writing the l.h.s. of eq. (3.1) as

$$T_{\mu\nu} = \frac{1}{m_b}\left[-g_{\mu\nu}T^{(1)} + v_\mu v_\nu T^{(2)} - i\epsilon_{\mu\nu\alpha\beta}v^\alpha \hat{q}^\beta T^{(3)} + \hat{q}_\mu \hat{q}_\nu T^{(4)} + (v_\mu \hat{q}_\nu + \hat{q}_\mu v_\nu)T^{(5)} \right]. \tag{3.5}$$

For massless leptons, only the first three form-factors, $T^{(1-3)}$, contribute to physical quantities. Eq. (3.1) becomes

$$T^{(i)} = c_\alpha^{(i,b)} O_b^\alpha + c^{(i,s)} O_s + c_\alpha^{(i,1)} O_1^\alpha + c_{\alpha\beta}^{(i,2)} O_2^{\alpha\beta} + c_{\alpha\beta}^{(i,3)} O_3^{\alpha\beta} + \ldots, \tag{3.6}$$

where the ellipses stand for contributions of operators of canonical dimension six or higher. All the Wilson coefficients can be expanded in powers of α_s,

$$c_{\{\alpha\}}^{(i,m)} = c_{\{\alpha\}}^{(i,m,0)} + \frac{\alpha_s}{4\pi} c_{\{\alpha\}}^{(i,m,1)} + O(\alpha_s^2)$$

and we are only interested in their imaginary part, cfr. (2.1). We consider the forward matrix element of (3.6) between two b quarks, and between two quarks and a soft gluon:

$$\langle T_i \rangle_{b\bar{b}} = c^{(i,b)}_\alpha \langle O^\alpha_b \rangle_{b\bar{b}} + c^{(i,s)} \langle O_s \rangle_{b\bar{b}} + c^{(i,1)}_\alpha \langle O^\alpha_1 \rangle_{b\bar{b}} + c^{(i,2)}_{\alpha\beta} \langle O^{\alpha\beta}_2 \rangle_{b\bar{b}} + c^{(i,3)}_{\alpha\beta} \langle O^{\alpha\beta}_3 \rangle_{b\bar{b}} + \dots, \quad (3.7)$$

$$\langle T_i \rangle_{b\bar{b}g} = c^{(i,b)}_\alpha \langle O^\alpha_b \rangle_{b\bar{b}g} + c^{(i,s)} \langle O_s \rangle_{b\bar{b}g} + c^{(i,1)}_\alpha \langle O^\alpha_1 \rangle_{b\bar{b}g} + c^{(i,2)}_{\alpha\beta} \langle O^{\alpha\beta}_2 \rangle_{b\bar{b}g} + c^{(i,3)}_{\alpha\beta} \langle O^{\alpha\beta}_3 \rangle_{b\bar{b}g} + \dots. \quad (3.8)$$

Here all the matrix elements should be interpreted as renormalized amputated Green's functions, either in full QCD (the l.h.s. and the matrix elements of O^μ_b and O_s) or in HQET; since the two theories have the same infrared behavior the cancellation of infrared divergences is guaranteed. The matrix elements of a generic operator O_X can be expanded in powers of α_s,

$$\langle O_X \rangle_{b\bar{b}(g)} = \langle O_X \rangle^{(0)}_{b\bar{b}(g)} + \frac{\alpha_s}{4\pi} \langle O_X \rangle^{(1)}_{b\bar{b}(g)} + O(\alpha_s^2).$$

We observe that

$$\langle O_s \rangle^{(0)}_{b\bar{b}} = \langle O^{\alpha\beta}_3 \rangle^{(0)}_{b\bar{b}} = 0, \qquad \langle O^\alpha_b \rangle^{(0)}_{b\bar{b}g} = \langle O_s \rangle^{(0)}_{b\bar{b}g} = 0. \quad (3.9)$$

Therefore, at the tree-level, the expansion in the residual momentum k of the l.h.s. of (3.7) allows for the determination of $c^{(i,b,0)}_\mu$ at $k = 0$, of $c^{(i,1,0)}_\mu$ at $O(k)$, of $c^{(i,2,0)}_{\mu\nu}$ at $O(k^2)$. More precisely, the $O(k)$ term in the l.h.s. of (3.7) is related to the matrix elements of

$$\bar{b}\gamma^\alpha(iD^\beta - m_b v^\beta)b = v^\alpha O^\beta_1 + \frac{1}{m_b}(O^{\alpha\beta}_2 + O^{\alpha\beta}_3) + O\left(\frac{1}{m^2_b}\right). \quad (3.10)$$

The latter equality follows from the relation between b and b_v fields, and therefore the $O(k)$ term in the l.h.s. of (3.7) contributes to the Wilson coefficients of $O_{1,2,3}$.

For what concerns the Taylor expansion in k, r of the l.h.s. of (3.8), the term at $k = r = 0$ allows for the determination of $c^{(i,1,0)}_\mu$, while the term linear in k and r determines $c^{(i,2,0)}_{\mu\nu}$ and $c^{(i,3,0)}_{\mu\nu}$. Gauge invariance guarantees that the same $c^{(i,1,0)}_\mu$ and $c^{(i,2,0)}_{\mu\nu}$ are extracted from the diagrams with and without external gluon. From (3.9) we also have $c^{(i,s,0)} = 0$.

We write down explicitly the tree-level coefficients only in the case of W_1, namely for the first of the tensor structures in (3.5) — the other form factors have the same structure. We work in $d = 4 - 2\epsilon$ dimensions and retain $O(\epsilon)$ terms

$$\text{Im}\, c^{(1,b,0)}_\mu = (1 - \epsilon)(v_\mu - \hat{q}_\mu)\,\delta(\hat{u}) \quad (3.11)$$

$$\text{Im}\, c^{(1,1,0)}_\mu = \frac{1}{m_b}(1 - \epsilon)\big[2(1 - \hat{q}_0)(v_\mu - \hat{q}_\mu)\,\delta'(\hat{u}) + v_\mu\delta(\hat{u})\big] \quad (3.12)$$

$$\text{Im}\, c^{(1,2,0)}_{\mu\nu} = \frac{2}{m^2_b}(1 - \epsilon)(1 - \hat{q}_0)\,\hat{p}'_\mu\,\hat{p}'_\nu\,\delta''(\hat{u})$$

$$+ \frac{2}{m^2_b}(1 - \epsilon)\left[\frac{1 - \hat{q}_0}{2}g_{\mu\nu} + 2v_\mu v_\nu - \frac{3}{2}(\hat{q}_\mu v_\nu + v_\mu\hat{q}_\nu) + \hat{q}_\mu\hat{q}_\nu\right]\delta'(\hat{u})$$

$$- \frac{1}{m^2_b}\left[\epsilon g_{\mu\nu} - \frac{\hat{q}^2 v_\mu v_\nu - \hat{q}_0(\hat{q}_\mu v_\nu + v_\mu\hat{q}_\nu) + \hat{q}_\mu\hat{q}_\nu}{\hat{q}^2 - \hat{q}^2_0}\right]\delta(\hat{u}) \quad (3.13)$$

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$$\text{Im}\, c^{(1,3,0)}_{\mu\nu} = -\frac{2}{m_b^2} \left[\frac{1-\hat{q}_0}{2} g_{\mu\nu}(1+\epsilon) + ((1-\epsilon)\hat{q}_\mu - 2v_\mu)\hat{p}'_\nu + \frac{\hat{q}\cdot\hat{p}'\, v_\mu\hat{q}_\nu - v\cdot\hat{p}'\, \hat{q}_\mu\hat{q}_\nu}{\hat{q}^2 - \hat{q}_0^2} \right] \delta'(\hat{u})$$

$$-\frac{1}{m_b^2} \left[\epsilon g_{\mu\nu} - \frac{\hat{q}^2 v_\mu v_\nu - \hat{q}_0(\hat{q}_\mu v_\nu + v_\mu\hat{q}_\nu) + \hat{q}_\mu\hat{q}_\nu}{\hat{q}^2 - \hat{q}_0^2} \right] \delta(\hat{u}) \qquad (3.14)$$

where $\hat{p}' = v - \hat{q}$. The $O(\epsilon)$ terms depend on whether the tensor decomposition of $T^{\mu\nu}$ is performed in four (as in our case) or d dimensions.

Eventually, of course, we need to evaluate eq. (3.1) in the B meson: the corresponding matrix elements of the operators (3.3) are given by

$$\frac{1}{M_B}\langle\bar{B}|O_b^\mu|\bar{B}\rangle = 2\, v^\mu,$$

$$\frac{1}{M_B}\langle\bar{B}|O_s|\bar{B}\rangle = 2 - \frac{\mu_\pi^2 - \mu_G^2}{m_b^2},$$

$$\frac{1}{M_B}\langle\bar{B}|O_1^\mu|\bar{B}\rangle = \frac{\mu_\pi^2 - \eta\,\mu_G^2(\mu)}{m_b}\, v^\mu, \qquad (3.15)$$

$$\frac{1}{M_B}\langle\bar{B}|O_2^{\mu\nu}|\bar{B}\rangle = -\frac{2\,\mu_\pi^2}{d-1}\,(g^{\mu\nu} - v^\mu v^\nu),$$

$$\frac{1}{M_B}\langle\bar{B}|O_3^{\mu\nu}|\bar{B}\rangle = \frac{2\,\mu_G^2}{d-1}\,(g^{\mu\nu} - v^\mu v^\nu),$$

where we have neglected higher order power corrections and introduced the factor

$$\eta = 1 + 2\left[C_F + \left(1 + \ln\frac{\mu}{m_b}\right) C_A \right] \frac{\alpha_s}{4\pi} \qquad (3.16)$$

in order to take into account the $O(\alpha_s)$ corrections to the HQET equation of motion, in the same manner as it has been done in [23]. In the standard tree-level calculation [3, 4], one computes directly the coefficients of μ_π^2 and μ_G^2. However, in order to perform the renormalization properly it is essential to distinguish between the various operators whose matrix elements contain μ_G^2. The evaluation of eq. (3.1) in the B meson leads, through eqs. (3.11)–(3.15), to the well-known $O(\Lambda_{\text{QCD}}^2/m_b^2)$ corrections [3, 4], see also eq. (2.9).

The one-loop calculation of the current correlator requires the imaginary part of the diagrams shown in figure 1. We use dimensional regularization for both ultraviolet and infrared divergences and proceed exactly as described in ref. [22]. The result of the Taylor expansion in k and r is reduced to the master integrals listed in the of appendix of the same paper. We perform the calculation in an arbitrary R_ξ gauge and use the background field gauge for the external gluon. The ultraviolet divergences of the diagrams in figure 1 are removed by standard on-shell quark mass and wave function QCD renormalization, see [23]. Notice that the $b\bar{b}$ one-loop amplitude at $k = 0$ contains terms that lead to $c^{(i,s,1)} \neq 0$; in other words, O_s emerges naturally from the OPE *before* one uses the heavy quark expansion, and its presence is essential to verify that $c^{(i,1,1)}_\mu$ and $c^{(i,2,1)}_{\mu\nu}$ extracted from the diagrams with and without external gluon are the same, as dictated by gauge invariance.

The r.h.s. of (3.6) receives $O(\alpha_s)$ contributions from both one-loop matrix elements of the effective operators and the one-loop Wilson coefficients. However, the unrenormalized one-loop matrix elements of O_{1-3} vanish in dimensional regularization because they reduce
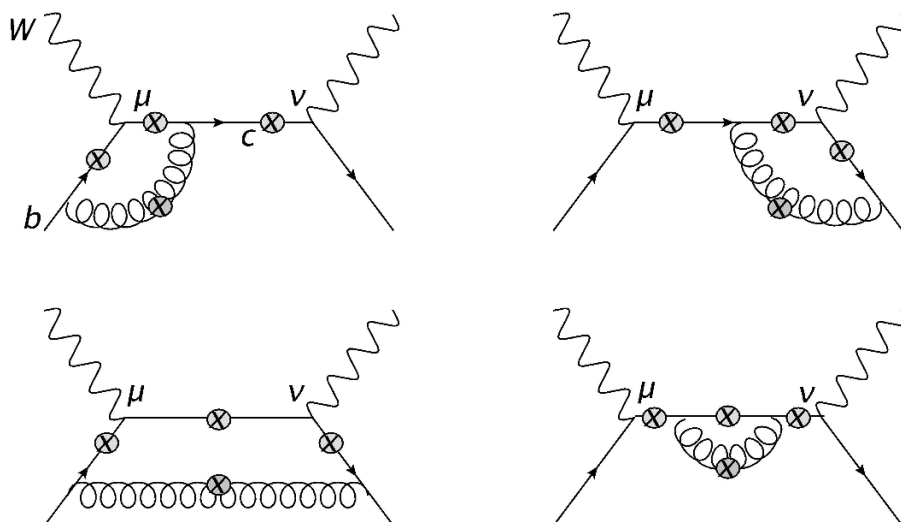
Figure 1. One-loop diagrams contributing to the current correlator. The background gluon can be attached wherever a cross is marked.

to massless one-loop tadpole diagrams . The case of O_b^μ is different and will be explained in a moment. Besides the on-shell wave function renormalization of the b and b_v fields, we need the operator renormalization, which is performed in the $\overline{\text{MS}}$ scheme, see [23]. In particular

$$\left[c_{b\mu}O_b^\mu\right]^{\text{bare}} = Z_b^{\text{OS}}\, c_{b\mu}O_b^\mu\,, \qquad \left[c_{2\mu\nu}O_2^{\mu\nu}\right]^{\text{bare}} = Z_{b_v}^{\text{OS}}\, Z_{\text{kin}}^{\overline{\text{MS}},\mu\nu\alpha\beta}\, c_{2\mu\nu}O_{2\,\alpha\beta}\,,$$

$$\left[c_{1\mu}O_1^\mu\right]^{\text{bare}} = Z_{b_v}^{\text{OS}}\, c_{1\mu}O_1^\mu\,, \qquad \left[c_{3\mu\nu}O_3^{\mu\nu}\right]^{\text{bare}} = Z_{b_v}^{\text{OS}}\, Z_{\text{chromo}}^{\overline{\text{MS}},\mu\nu\alpha\beta}\, c_{3\mu\nu}O_{3\,\alpha\beta}\,. \tag{3.17}$$

where

$$Z_{\text{kin}}^{\overline{\text{MS}},\mu\nu\alpha\beta} = g^{\alpha\mu}g^{\beta\nu} - C_F\frac{3-\xi}{\epsilon}\left(g^{\mu\nu} - 2v^\mu v^\nu\right)v^\alpha v^\beta\,\frac{\alpha_s}{4\pi} + \dots$$

$$Z_{\text{chromo}}^{\overline{\text{MS}},\mu\nu\alpha\beta} = g^{\alpha\mu}g^{\beta\nu} + \frac{C_A}{\epsilon}\left(g^{\mu\alpha} - v^\mu v^\alpha\right)g^{\nu\beta}\,\frac{\alpha_s}{4\pi} + \dots\,. \tag{3.18}$$

The Feynman gauge is obtained by setting $\xi = 1$. It is easy to see that the renormalization of $O_2^{\mu\nu}$ is irrelevant because the matrix element of $Z_{\text{kin}}^{\mu\nu\alpha\beta}O_{2\,\alpha\beta}$ vanishes at the order of the calculation. On the other hand, the B matrix element of $Z_{\text{chromo}}^{\mu\nu\alpha\beta}O_{3\,\alpha\beta}$ is proportional to that of $O_3^{\mu\nu}$, which simplifies the calculation. The operator O_s does not need renormalization because it enters at the loop level only. The one-loop matrix elements of O_b^μ do not vanish: they have to be Taylor expanded in k and r and included in the calculation.

Putting together all pieces we have verified that all infrared and ultraviolet divergences are canceled in the Wilson coefficients and that the latter are independent of the amplitude from which they are extracted. We have also verified that the results, which we express in terms of coefficients of $\alpha_s\mu_{\pi,G}^2$ in eq. (2.6), do not depend on the quantum gauge parameter ξ. The coefficients of μ_π^2, $W_i^{(\pi,1)}$, agree with ref. [22]. The complete analytic results for $W_i^{(G,1)}$ are given in the appendix.

4 Numerical results

In this section we present a preliminary investigation of the numerical relevance of the $O(\alpha_s \Lambda_{\text{QCD}}^2/m_b^2)$ corrections, using for the heavy quark masses the reference values $m_b = 4.6\,\text{GeV}$ and $m_c = 1.15\,\text{GeV}$. First, we consider on-shell quark masses; in this case the phase space integration of the triple differential width (see e.g. eq. (2.10) of ref. [22]) leads to the total semileptonic width

$$\Gamma_{B \to X_c \ell \nu} = \Gamma_0 \left[\left(1 - 1.78\,\frac{\alpha_s}{\pi}\right) \left(1 - \frac{\mu_\pi^2}{2m_b^2}\right) - \left(1.94 + 2.42\,\frac{\alpha_s}{\pi}\right) \frac{\mu_G^2(m_b)}{m_b^2} \right],$$

where $\Gamma_0 = G_F^2 m_b^5 (1 - 8\rho + 8\rho^3 - \rho^4 - 12\rho^2 \ln\rho)/192\pi^3$ is the tree level width, $\rho = m_c^2/m_b^2$, and we have neglected higher order terms of $O(\alpha_s^2)$ and $O(1/m_b^3)$. The parameter μ_G^2 is renormalized at the scale $\mu = m_b$. It is advisable to evaluate the QCD coupling constant at a scale lower than m_b. Here and in the following we adopt $\alpha_s = 0.25$, which implies that the $O(\alpha_s)$ correction increases the μ_G^2 coefficient by about 10%. Neglecting again higher order effects, the mean lepton energy is given by

$$\langle E_\ell \rangle = 1.41\text{GeV} \left[\left(1 - 0.02\,\frac{\alpha_s}{\pi}\right) \left(1 + \frac{\mu_\pi^2}{2m_b^2}\right) - \left(1.19 + 4.20\,\frac{\alpha_s}{\pi}\right) \frac{\mu_G^2(m_b)}{m_b^2} \right],$$

while the variance of the lepton energy distribution is $\ell_2 = \langle E_\ell^2 \rangle - \langle E_\ell \rangle^2$,

$$\ell_2 = 0.183\,\text{GeV}^2 \left[1 - 0.16\,\frac{\alpha_s}{\pi} + \left(4.98 - 0.37\,\frac{\alpha_s}{\pi}\right) \frac{\mu_\pi^2}{m_b^2} - \left(2.89 + 8.44\,\frac{\alpha_s}{\pi}\right) \frac{\mu_G^2(m_b)}{m_b^2} \right].$$

In the two above leptonic moments the NLO corrections to the coefficients of μ_G^2 are larger than in the total rate: they amount to +28% and +23%, respectively. They have therefore the same sign and size of the corrections to the width and photon energy moments in $b \to s\gamma$ [23]. Of course, the coefficients of the $O(\alpha_s)$ corrections depend on the perturbative scheme and on the renormalization scale of μ_G^2. In the kinetic scheme with cutoff $\mu_{kin} = 1\text{GeV}$, which is often employed in semileptonic fits [5, 20], the width becomes

$$\Gamma_{B \to X_c \ell \nu} = \Gamma_0 \left[1 - 1.11\,\frac{\alpha_s}{\pi} - \left(\frac{1}{2} - 0.99\,\frac{\alpha_s}{\pi}\right) \frac{\mu_\pi^2}{m_b^2} - \left(1.94 + 3.46\,\frac{\alpha_s}{\pi}\right) \frac{\mu_G^2(m_b)}{m_b^2} \right], \quad (4.1)$$

where the NLO corrections to the coefficients of μ_π^2, μ_G^2 are both close to 15% but have different signs.[2] Overall, the $O(\alpha_s \Lambda_{\text{QCD}}^2/m_b^2)$ contributions decrease the total width by about 0.3%. However, NLO corrections also modify the coefficients of μ_π^2, μ_G^2 in the moments which are fitted to extract the non-perturbative parameters, and will ultimately shift the values of μ_π^2, μ_G^2 to be employed in (4.1). Therefore, in order to quantify the eventual numerical impact of the new corrections on the semileptonic width and on $|V_{cb}|$, a new global fit has to be performed.

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[2]In the kinetic scheme the $O(1/m_b^3)$ corrections (here neglected) contribute to the determination of the perturbative corrections and slightly modify the numerical values reported in eqs. (4.1)–(4.3).
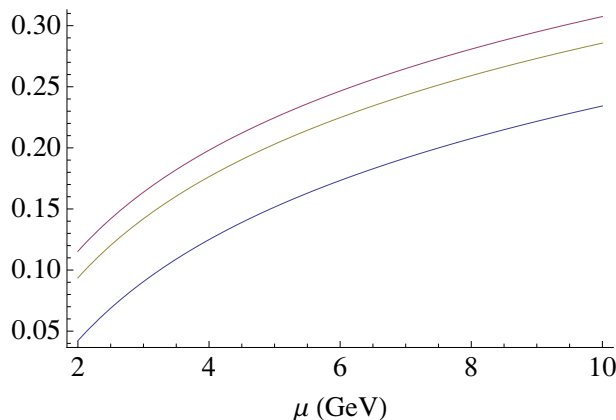
Figure 2. Relative NLO correction to the μ_G^2 coefficients in the width (blue), first (red) and second central (yellow) leptonic moments as a function of the renormalization scale μ of μ_G^2.

For what concerns the first leptonic moment in the kinetic scheme we find

$$\langle E_\ell \rangle = 1.41\,\text{GeV} \left[1 - 0.01\,\frac{\alpha_s}{\pi} + \left(\frac{1}{2} - 0.44\,\frac{\alpha_s}{\pi} \right) \frac{\mu_\pi^2}{m_b^2} - \left(1.19 + 3.21\,\frac{\alpha_s}{\pi} \right) \frac{\mu_G^2(m_b)}{m_b^2} \right], \quad (4.2)$$

where the new corrections lead to a $\approx 0.5\%$ suppression. In practice, experiments measure this observable applying a lower cut on the lepton energy and the typical experimental error is lower than 0.5%. We postpone the consideration of cuts to a future publication. In eq. (4.2) the $O(\alpha_s \Lambda_{\text{QCD}}^2/m_b^2)$ correction is dominated by the term proportional to μ_G^2, corresponding to a 20% increase of the μ_G^2 coefficient. Finally, the second central moment in the kinetic scheme is given by

$$\ell_2 = 0.183\,\text{GeV}^2 \left[1 - 0.24\,\frac{\alpha_s}{\pi} + \left(4.98 - 3.89\,\frac{\alpha_s}{\pi} \right) \frac{\mu_\pi^2}{m_b^2} - \left(2.89 + 7.01\,\frac{\alpha_s}{\pi} \right) \frac{\mu_G^2(m_b)}{m_b^2} \right]. \quad (4.3)$$

Here the new corrections lead to a 1.5% suppression, again of the same order of the experimental error. The NLO correction to the μ_G^2 coefficient is also about 20%.

The size of the $O(\alpha_s \mu_G^2/m_b^2)$ corrections depends on the renormalization scale μ of the chromomagnetic operator. This is illustrated in figure 2, where the size of the NLO correction relative to the tree level results is shown for the width and the first two leptonic moments at different values of μ. The NLO corrections are quite small for $\mu \approx 2\,\text{GeV}$ and, as expected, increase with μ. For $\mu \gtrsim m_b$ the running of μ_G^2 appears to dominate the NLO corrections.

5 Summary

We have calculated the $O(\alpha_s)$ corrections to the Wilson coefficients of the chromomagnetic operator in inclusive semileptonic B decays, employing the techniques developed in refs. [23] and [22]. This calculation turned out to be significantly more demanding than that of [23], motivating us to explain the matching procedure in greater detail. We have also studied the numerical relevance of the new contributions in the absence of cuts: the perturbative

$O(\alpha_s)$ corrections increase the μ_G^2 coefficients in the total semileptonic rate and in the first two leptonic moments by 15% to 20% if μ_G^2 is renormalized at $\mu = m_b$. For $\mu = 2\mathrm{GeV}$ the corrections are in the 5-10% range. The complete $O(\alpha_s \Lambda_{\mathrm{QCD}}^2/m_b^2)$ correction to the width is a few per mill, but the corrections to the first two leptonic moments are of the same order of the experimental errors. A complete estimate of the effect of these corrections on the width and on $|V_{cb}|$ therefore requires their inclusion in the global fit to the moments, which will be the subject of a future publication.

Acknowledgments

A   Analytic results

We provide results for the contributions proportional to either C_F or C_A

$$W_i^{(G,1)} = W_{i,u}^{(G,1)} + \frac{2}{3} C_F W_{i,F}^{(G,1)} + \frac{2}{3} C_A W_{i,A}^{(G,1)}. \tag{A.1}$$

The term $W_{i,u}^{(G,1)}$ contains a few recurring structures

$$\begin{aligned}
W_{i,u}^{(G,1)} = {} & \left[2C_F(1 - E_0 I_{1,0})\, w_i^{(G,1)} + \frac{C_A}{3}\, p_i^{(G,1)} \right] \left[\frac{1}{\hat{u}^2} \right]_+ + \frac{1}{2} C_A W_i^{(G,0)} \ln \frac{\mu}{m_b} \\
& + \frac{2d_i^{(G,1)}}{3\hat{y}^2} (2C_F - C_A)(1 - E_0) \big[2(1 - E_0)\, I_{1,0} + \ln \rho \big] \delta(\hat{u})
\end{aligned} \tag{A.2}$$

where we have set $d_1^{(G,1)} = 1 - E_0$, $d_2^{(G,1)} = 0$, $d_3^{(G,1)} = 1$, $p_1^{(G,1)} = -\lambda_0$, $p_2^{(G,1)} = 8(1 - E_0)$ and $p_3^{(G,1)} = -4E_0$. The μ dependence originates in the $\overline{\mathrm{MS}}$ renormalization of μ_G^2. The remaining expressions are

$$\begin{aligned}
W_{1,F}^{(G,1)} = {} & \left[\left(\frac{8\rho}{\lambda_0}(1 + 2E_0 - 3\rho) + 5\lambda_0 + 2(1 - 2E_0 + 5\rho) \right) I_{1,0} \right. \\
& \left. - \frac{8}{\lambda_0}(2\rho + E_0(1 - 3\rho)) - \frac{2E_0}{\rho}(1 + 5\rho) \right] \left[\frac{1}{\hat{u}} \right]_+ + D_{1,F}^{(G,1)} \delta(\hat{u}) \\
& - \left[\frac{2}{\hat{y}} \left(20E_0^4 - \rho + E_0 \rho(2 + \rho) - 5E_0^3(1 + 2\rho) - E_0^2(2 + 5\rho) \right) I_{1,0} \right. \\
& \left. - (8\rho - 5E_0 - 23E_0^2) + 2S(5E_0^2 + E_0 - 2\rho) + \frac{\lambda_0}{4\hat{y}} \ln \rho \right] \delta'(\hat{u}) + R_{1,F}^{(G,1)}
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
W_{1,A}^{(G,1)} = {} & \left[\frac{1}{2}(1 + 8E_0 - 3\rho)I_{1,0} - 1 - E_0 \left(\frac{3}{2\rho} - \frac{5}{2} \right) \right] \left[\frac{1}{\hat{u}} \right]_+ + D_{1,A}^{(G,1)} \delta(\hat{u}) \\
& + \left[\frac{\lambda_0}{2} - \frac{1}{2} \left(\frac{\lambda_0}{2} - E_0 \right) \ln \rho - E_0(E_0 + 2E_0^2 + \rho)I_{1,0} \right] \delta'(\hat{u}) + R_{1,A}^{(G,1)}
\end{aligned} \tag{A.4}$$

$-10-$

$$W_{2,F}^{(G,1)} = \left[\frac{8}{\lambda_0}\left(\frac{E_0}{\rho}+4-5E_0\right)(1-2E_0)+8\left(1-\frac{1-13\rho+2E_0(1+5\rho)}{\lambda_0}\right)I_{1,0}\right]\left[\frac{1}{\hat{u}}\right]_+$$
$$+\left[8E_0(3-5E_0)I_\Delta-2(5E_0^2-2E_0-3\rho)I_{1,0}+(17-30E_0)\ln\rho+14-26E_0\right]\delta'(\hat{u})$$
$$+D_{2,F}^{(G,1)}\delta(\hat{u})+R_{2,F}^{(G,1)} \tag{A.5}$$

$$W_{2,A}^{(G,1)} = \left[\frac{4E_0}{\lambda_0\rho}+\frac{4}{\lambda_0}(3+7\rho-11E_0)-\frac{3}{\rho}(1-3\rho)-4\left(\frac{1-11\rho+E_0(3+7\rho)}{\lambda_0}-2\right)I_{1,0}\right]\left[\frac{1}{\hat{u}}\right]_+$$
$$+\left[2(1-2E_0)(1+E_0)I_{1,0}-4(1-E_0)+(1-2E_0)\ln\rho\right]\delta'(\hat{u})+D_{2,A}^{(G,1)}\delta(\hat{u})+R_{2,A}^{(G,1)}$$
$$\tag{A.6}$$

$$W_{3,F}^{(G,1)} = \left[2\left(\frac{4}{\lambda_0}(E_0(1-5\rho)+3\rho)+5E_0+2\right)I_{1,0}-\frac{2}{\rho}-\frac{8}{\lambda_0}(1+3E_0-5\rho)\right]\left[\frac{1}{\hat{u}}\right]_+ \tag{A.7}$$
$$+\left[20E_0^2I_{1,0}-10E_0S-\frac{5}{2}\ln\rho-4E_0I_\Delta-3-25E_0\right]\delta'(\hat{u})+D_{3,F}^{(G,1)}\delta(\hat{u})+R_{3,F}^{(G,1)}$$

$$W_{3,A}^{(G,1)} = \left[2\left(\frac{E_0+\rho(4-3E_0)}{\lambda_0}+2\right)I_{1,0}-\frac{2}{\lambda_0}(1+4E_0-3\rho)-\frac{3-7\rho}{2\rho}\right]\left[\frac{1}{\hat{u}}\right]_++R_{3,A}^{(G,1)}$$
$$+\left[E_0-(E_0+2E_0^2+\rho)I_{1,0}-\left(E_0-\frac{1}{2}\right)\ln\rho\right]\delta'(\hat{u})+D_{3,A}^{(G,1)}\delta(\hat{u}) \tag{A.8}$$

We have called $D_{i,F/A}^{(G,1)}$ the various coefficients of the $\delta(\hat{u})$ distribution

$$D_{1,F}^{(G,1)} = \left[1+4E_0+5E_0^2(1-4E_0)-(9-8E_0)\rho+\frac{2}{\hat{y}}(1-E_0)(5E_0^2-4E_0-2)\right.$$
$$\left.+\frac{12E_0^2}{\lambda_0}(1-E_0)(1+3E_0)\right]I_{1,0}+\frac{2}{\hat{y}}(1-E_0)-\frac{E_0}{2\rho}(1-20E_0)-\frac{1}{2}(8-27E_0-40E_0^2)$$
$$+\left[\frac{2E_0}{\rho}-\frac{1}{2}(4-31E_0)-(1-E_0)\frac{1+5E_0}{\hat{y}}+\frac{4E_0}{\lambda_0}(2-E_0)(1+3E_0)\right]\ln\rho \tag{A.9}$$
$$-8\rho+\frac{4E_0}{\lambda_0}(1-E_0)(1+3E_0)+\left(2-4E_0+5\lambda_0+\frac{2\rho}{\lambda_0}(4+8E_0+5\lambda_0)-24\frac{\rho^2}{\lambda_0}\right)I_\Delta$$

$$D_{1,A}^{(G,1)} = \left[\frac{1-E_0}{\hat{y}}(4-5E_0)-\frac{1}{2}(3-18E_0+8E_0^2-3\rho)-\frac{2E_0^2}{\lambda_0}(1-E_0)(1+3E_0)\right]I_{1,0}$$
$$+\left(\frac{1}{4}(4-5E_0)+\frac{3E_0}{2\rho}+\frac{3-5E_0}{2\hat{y}}\right)\ln\rho+\frac{1}{2}(1+8E_0-3\rho)I_\Delta$$
$$+\frac{E_0}{2}(5+4E_0)-\frac{E_0}{2\rho}-2\rho-\frac{1-E_0}{\hat{y}}+\frac{2E_0}{\lambda_0}(1-E_0)(1+3E_0) \tag{A.10}$$

$$D_{2,F}^{(G,1)} = 2\left(21E_0-9-20E_0^2-\frac{12}{\lambda_0}(1-E_0)(1-E_0-3E_0^2)\right)I_{1,0}$$
$$+\frac{8}{E_0\lambda_0}(1-E_0)(1-9E_0+11E_0^2)-\frac{4(1-2E_0)}{\lambda_0\rho}(9\rho+E_0(2-5\rho))\ln\rho$$
$$+\frac{2-17E_0+20E_0^2}{E_0\rho}-5(3-8E_0)-\frac{8}{\lambda_0}(1-\lambda_0-13\rho+2E_0(1+5\rho))I_\Delta \tag{A.11}$$

$$D_{2,A}^{(G,1)} = \left(10E_0 - 3 - \frac{4}{\lambda_0}(1-E_0)(2 + 12E_0 - 19E_0^2)\right)I_{1,0} - 3 + 4E_0$$

$$-\frac{1}{\rho} - \left(1 + \frac{1-3E_0}{E_0\rho} + \frac{2}{E_0\lambda_0}(2-E_0)(1+4E_0-7E_0^2)\right)\ln\rho$$

$$+\frac{4}{\lambda_0}(1-E_0)(4-5E_0) - \frac{4}{\lambda_0}(1 - 2\lambda_0 - 11\rho + E_0(3+7\rho))I_\Delta \qquad\text{(A.12)}$$

$$D_{3,F}^{(G,1)} = \frac{8}{\lambda_0}(1 + 4E_0 - 4E_0^2) + \left(\frac{35}{4} + \frac{2}{\rho} - \frac{6 - 9E_0 + 5E_0^2}{2\hat{y}(1-E_0)} + 4(2-E_0)\frac{1 + 3E_0 - 5E_0^2}{\lambda_0(1-E_0)}\right)\ln\rho$$

$$+\frac{2}{\hat{y}} + 2 - (1 - 20E_0)\left(1 + \frac{1}{2\rho}\right) + \frac{2}{\lambda_0}(E_0(4 + 5\lambda_0 - 20\rho) + 2(\lambda_0 + 6\rho))I_\Delta$$

$$+ \left(2(1 + 4E_0 - 10E_0^2) - \frac{1}{\hat{y}}(8 - 9E_0 + 5E_0^2) + \frac{8E_0}{\lambda_0}(1 + 2E_0 - 6E_0^2)\right)I_{1,0} \quad\text{(A.13)}$$

$$D_{3,A}^{(G,1)} = \frac{2}{\lambda_0}(E_0(1 + 8E_0) - \rho(4 + 3E_0))I_\Delta + 2E_0 - \frac{1}{2\rho} - \frac{1}{\hat{y}} - \frac{4E_0^2}{\lambda_0}$$

$$- \left(1 - \frac{3}{2\rho} + \frac{2 - 3E_0}{2\hat{y}(1-E_0)} - (2-E_0)\frac{1 + 4E_0 - 3E_0^2}{\lambda_0(1-E_0)}\right)\ln\rho$$

$$+ \left(\frac{3}{2}(3 - E_0) - \frac{1 - 3E_0}{\hat{y}} + \frac{4E_0}{\lambda_0}(1 + 4E_0 - 2E_0^2)\right)I_{1,0} \qquad\text{(A.14)}$$

The terms labelled as $R_{i,F/A}^{(G,1)}$ stand for the regular contributions

$$R_{1,F}^{(G,1)} = \left[\frac{4}{\lambda}(1 - 3E + \rho) - \frac{2 - 15E + 5\hat{u}}{2} - \frac{24E_0 - 15\lambda_0 - 52\rho}{2\hat{u}} + \frac{\hat{u}}{\lambda}(11 - 13E) + \frac{5\hat{u}^2}{\lambda}\right]I_1$$

$$+\frac{2E_0}{\hat{u}\rho}(1 + 5E_0 - 5\rho) - \frac{\rho}{4z^3}(5\lambda + 7z) + \frac{12 - 11E - 13\rho + 10\rho E}{\lambda} + \frac{13}{4}\left(1 + \frac{1}{z}\right)$$

$$-\frac{5}{2\rho z}(\lambda + 2E\rho + \rho^2) - \left[\frac{1}{\hat{u}}\left(2(1 - 2E_0) + 5(\lambda_0 + 2\rho)\right) + \frac{8\rho}{\lambda_0\hat{u}}(1 + 2E_0 - 3\rho)\right]I_{1,0}$$

$$+\frac{8E_0}{\lambda_0\hat{u}}(1 + 2E_0 - 3\rho) - \frac{5}{2\rho}(z + 4(1-E)) + \frac{5}{8z^2}(4E + \lambda - 4E\rho - 2\rho^2)$$

$$-\frac{E}{\lambda z}(4 - 7\rho + 5\rho^2) - \frac{z}{\lambda}(5E - 13) + \frac{\lambda_0(1 + 5E_0) + 4\rho(1 + 3E_0)}{\hat{u}^2}(I_1 - I_{1,0}) \quad\text{(A.15)}$$

$$R_{1,A}^{(G,1)} = \frac{E}{2z^2} - \frac{3E\rho}{\lambda z} - \frac{6z}{\lambda} + \left[1 + \frac{1 - 2E}{\lambda}(8E + 3\rho) + \frac{3z}{\lambda}(1 + 2E)\right]I_1$$

$$-\frac{3 - \rho}{4\rho} + \frac{3E - \rho}{2\rho z} - \frac{8 - 13E - 6\rho}{\lambda} + \frac{1 + 8E_0 - 3\rho}{2\hat{u}}(I_1 - I_{1,0}) \qquad\text{(A.16)}$$

$$R_{2,F}^{(G,1)} = -\left[25 + \frac{48}{\lambda^2}(1 - 5E + 8\rho - 5E\rho + \rho^2) + \frac{166 - 152E + 74\rho}{\lambda}\right.$$

$$+\frac{8}{\lambda\hat{u}}(1 - 4E + 3\rho) + \frac{4}{\hat{u}}(6 - 5E) + \frac{10\hat{u}}{\lambda}(19 - 5E + \hat{u}) + \frac{60\hat{u}^3}{\lambda^2}$$

$$\left. -\frac{12\hat{u}}{\lambda^2}(-39 + 47E - 41\rho + 13E\rho) + \frac{12\hat{u}^2}{\lambda^2}(42 - 23E + 5\rho)\right]I_1$$

$$-\frac{45}{2z} - \frac{4}{\rho\hat{u}}(3 - 5E + 5\rho) + \frac{\rho}{z^3}(8 - 10E - 5\rho) + \frac{2 + 10E - 15\rho}{2z^2}$$

$$+\frac{12}{\lambda^2}(E(39 - \rho) - 20(1 + \rho)) - \frac{8}{\lambda_0\hat{u}}(2 - 13E_0 + 10\rho)$$

$$-\frac{12\hat{u}^2}{\lambda^2}(23-5E) + \frac{4}{\rho z}(4-5E) - \frac{2E}{\lambda z}(4-7\rho+5\rho^2)\left(\frac{1}{z}-\frac{6}{\lambda}\right)$$

$$+\frac{106E-199+10\rho-73\hat{u}}{\lambda} - \frac{8}{\lambda\hat{u}}(4-3E) - \frac{12\hat{u}}{\lambda^2}(47-42E+13\rho)$$

$$-\frac{4(5-16E)-3\rho(9-10E)+10\rho^2}{\lambda z} - \frac{8E_0}{\lambda_0\rho\hat{u}} - \frac{8E}{\lambda\rho}\left(\frac{1}{z}-\frac{1}{\hat{u}}\right) \tag{A.17}$$

$$+\frac{8}{\lambda_0\hat{u}}(1+2E_0-\lambda_0-13\rho+10E_0\rho)I_{1,0} - \frac{8E_0}{\hat{u}^2}(3-5E_0)(I_1-I_{1,0})$$

$$R_{2,A}^{(G,1)} = \left[\frac{2}{\lambda}(61-52E+25\rho+40\hat{u}) - \frac{24}{\lambda^2}\left(2E(2+7\rho)-\rho(13+5\rho)\right) + \frac{12\hat{u}^2}{\lambda^2}(25-6E)\right.$$

$$\left.+\frac{8}{\hat{u}} - \frac{4}{\lambda\hat{u}}(1+3E-11\rho+7E\rho) + \frac{12\hat{u}}{\lambda^2}(29-40E+35\rho-6E\rho)\right]I_1$$

$$+\frac{4}{\hat{u}\lambda_0}\left(1+E_0(3-8E_0)-\rho(3-7E_0)\right)I_{1,0} + \frac{6\hat{u}}{\lambda}\left(1+\frac{12\hat{u}}{\lambda}\right)$$

$$+\frac{4}{\lambda z}(2-5E-3\rho) + \frac{4}{\lambda\hat{u}}(3-11E+7\rho) + \frac{12\hat{u}}{\lambda^2}(40-25E+6\rho)$$

$$+\frac{1}{z^2} - \frac{4E}{\lambda\rho z} + \frac{12}{\lambda^2}\left(8-29E+2\rho(14-5E)\right) - \frac{4}{\rho\hat{u}}\left(\frac{E_0}{\lambda_0}-\frac{E}{\lambda}\right)$$

$$-\frac{4}{\lambda_0\hat{u}}(3+7\rho-11E_0) + \frac{6}{\lambda}(24-5E) - \frac{6E\rho}{\lambda z}\left(\frac{1}{z}-\frac{6}{\lambda}\right) + \frac{3}{\rho z} \tag{A.18}$$

$$R_{3,F}^{(G,1)} = -\left[\frac{25}{2} + \frac{6}{\lambda}(2-4E+3\rho) + \frac{8\rho}{\lambda\hat{u}} + \frac{4}{\hat{u}}(1-5E) + \frac{2\hat{u}}{\lambda}(14-5E) - \frac{4E_0}{\hat{u}^2}(1+5E_0)\right]I_1$$

$$-\left[\frac{4E_0+5\lambda_0+20\rho}{\hat{u}^2} + \frac{8}{\lambda_0\hat{u}}(E_0+3\rho-5E_0\rho) + \frac{2}{\hat{u}}(2+5E_0)\right]I_{1,0}$$

$$+\frac{8}{\lambda_0\hat{u}}(1+3E_0-5\rho) - \frac{4}{\lambda}(6-7E) + \frac{2}{\rho\hat{u}}(5E-5\rho+1) - \frac{10\hat{u}}{\lambda}$$

$$+\frac{8E}{\lambda\hat{u}} - \frac{5E\rho}{z^3} + \frac{5}{2z^2}(1+E-\rho) - \frac{5}{z} + \frac{2E}{\lambda z}(2-5\rho) - \frac{10E}{\rho z} \tag{A.19}$$

$$R_{3,A}^{(G,1)} = \left[\frac{2}{\lambda}\left(3z+5(1-2E)\right) + \frac{2}{\lambda\hat{u}}(E+4\rho-3E\rho) + \frac{4}{\hat{u}}\right]I_1$$

$$+\frac{2}{\lambda_0\hat{u}}(1+4E_0-3\rho) - \frac{2}{\lambda_0\hat{u}}(E_0+2\lambda_0+4\rho-3E_0\rho)I_{1,0}$$

$$+\frac{3}{2\rho z} - \frac{2}{\lambda\hat{u}}(1+4E-3\rho) + \frac{2}{\lambda}(10-3E) + \frac{1}{2z^2} - \frac{2E}{\lambda z} \tag{A.20}$$

where we have introduced $z = \hat{u}+\rho$ and $\lambda = 4(E^2-\rho-\hat{u})$. The integrals I_1, $I_{1,0}$, $I_{2,0}$, and $I_{4,0}$ are given in the appendix of [22], and $I_\Delta = I_{2,0}-I_{4,0}$. The plus distributions are defined by their action on a test function $f(\hat{u})$:

$$\int d\hat{u}\left[\frac{1}{\hat{u}^m}\right]_+ f(u) = \int_0^1 d\hat{u}\,\frac{1}{u^m}\left[f(u) - \sum_{p=0}^{m-1}\frac{u^p}{p!}f^{(p)}(0)\right] \tag{A.21}$$

with $f^{(p)}(u) = \frac{d^p f(u)}{du^p}$.

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