View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Institutional Research Information System University of Turin

UNIVERSITÀ DEGLI STUDI DI TORINO

This is an author version of the contribution published on:

Lea TERRACINI. A Weight Independence Result for Quaternionic Hecke Algebras. *International Journal of Number Theory*, Volume 9, Issue 8, 2013, DOI: 10.1142/S1793042113500656

The definitive version is available at:

http://www.worldscientific.com/doi/abs/10.1142/S1793042113500656

A WEIGHT INDEPENDENCE RESULT FOR QUATERNIONIC HECKE ALGEBRAS

LEA TERRACINI

ABSTRACT. Let p be a prime and B be a quaternion algebra indefinite over $\mathbf{Q}$ and ramified at p. We consider the space of quaternionic modular forms of weight k and level p^∞, endowed with the action of Hecke operators. By using cohomological methods, we show that the p-adic topological Hecke algebra does not depend on the weight k. This result provides a quaternionic version of a theorem proved by Hida for classical modular forms; we discuss the relationship of our result to Hida's theorem in terms of Jacquet-Langlands correspondence.

Keywords: modular forms; quaternion algebras; Hecke algebras; cohomology of Shimura curves.

Mathematics Subject Classification 2010: 11F55, 11F75, 11F25

1. Introduction

Le p be an odd prime. We fix an embedding of $\bar{\mathbf{Q}}$ in $\bar{\mathbf{Q}}_p$. Let K_0 be an algebraic extension of $\mathbf{Q}$, K be its closure in $\bar{\mathbf{Q}}_p$, $\mathcal{O}$ its p-adic integer ring. We fix an integer N prime to p.

For every pair of positive integers r, k, let $S_k(\Gamma_1(Np^r), K_0)$ be the space of cuspidal forms of level Np^r, weight k and having q-expansion in K_0; put $S_k((\Gamma_1(Np^r), K) = S_k(\Gamma_1(Np^r), K_0) \otimes_{K_0} K$. Let $\mathcal{H}_k(Np^r, \mathcal{O})$ be the Hecke algebra generated over $\mathcal{O}$ by Hecke operators T_n for all n.

In [5, §1] Hida defines the K-vector space

$$S_k(Np^\infty, K) = \varinjlim_r S_k(\Gamma_1(Np^r), K),$$

obtained by letting the exponent of p vary in the level and keeping the weight fixed. It is equipped with a faithful action of the $\mathcal{O}[[\mathbf{Z}_p^\times]]$ topological Hecke algebra $\mathcal{H}_k(Np^\infty, \mathcal{O}) = \varprojlim_r \mathcal{H}_k(Np^r, \mathcal{O})$; this algebra does not depend on the weight k, [5, (1.7)].

This result is proved in [8, Theorem 3.2] and [9, Theorem 2.3] in a more general situation, for the Hecke algebra acting on the space of automorphic forms with respect to groups arising from Eichler orders in a quaternion algebra B on a totally real field, not ramified at finite places.

In this paper, by using the cohomological methods of [8], we shall prove the independence of weight of the Hecke algebra of level p^∞ in the case where B is a quaternion algebra indefinite over $\mathbf{Q}$ and p is a prime dividing the discriminant of B.

More precisely, let $\Delta = \Delta' p$ be the discriminant of B and let $R = R(N)$ be an Eichler order of level N in B. Let $B_{\mathbf{A}}^{\times,\infty}$ denote the finite part of the adelization of B; we shall consider compact open subgroups of $B_{\mathbf{A}}^{\times,\infty}$, called (p,N)-groups, having the form

$$U(N,U_p)=\prod_{l\neq p} R(N)_l^\times \times U_p$$

where U_p is an open subgroup of $R_p^\times$. Let $S_k^B(U(N,U_p))$ be the complex space of automorphic forms of weight k which are right invariant by $U(N,U_p)$. This space is endowed with a standard action of a Hecke ring $\mathcal{H}_k(U(N,U_p))$, generated over $\mathbf{Z}$ by operators T_q for primes $q\neq p$ and $\langle q\rangle$ for primes $q\not| N\Delta$. It should be noted that the Hecke operator T_p at p is missing.

For a ring A we put $\mathcal{H}_k(U(N,U_p),A)=\mathcal{H}_k(U(N,U_p))\otimes_{\mathbf{Z}} A$. If $U_p'\subseteq U_p$ there are inclusions $S_k^B(U(N,U_p))\to S_k^B(U(N,U_p'))$ and restriction maps

$$\mathcal{H}_k(U(N,U_p))\to\mathcal{H}_k(U(N,U_p')) \tag{1}$$

give rise to a projective system. We use Matshushima-Shimura isomorphism to switch the action of $\mathcal{H}_k(U(N,U_p))$ to the cohomology group $H^1(\mathbf{X}(U(N,U_p)),\mathcal{L}(n,\mathbf{C}))$ where $\mathbf{X}(U(N,U_p))$ is the Shimura curve associated to the group $U(N,U_p))$, and $\mathcal{L}(n,\mathbf{C}))$ is the standard locally constant sheaf associated to weight $k=n+2$. We show that this cohomology group has an integral structure preserved by Hecke action, so that for a suitably p-adic integer ring $\mathcal{O}$ the Hecke algebra $\mathcal{H}_k(U(N,U_p),\mathcal{O})$ faithfully acts on $H^1(\mathbf{X}(U(N,U_p),\mathcal{L}(n,\mathcal{O}))$. The action of diamond operators is compatible with the projective system (1), giving an action of the completed ring $\mathcal{O}[[\mathbf{Z}_p^\times]]$ over $\mathcal{H}_k(p^\infty,N;\mathcal{O})=\varprojlim_{U_p} H^1(\mathcal{H}_k(U(N,U_p),\mathcal{O})$.

Our main result is Theorem 8.1, which states that the topological $\mathcal{O}[[\mathbf{Z}_p^\times]]$-algebra $\mathcal{H}_k(p^\infty,N;\mathcal{O})$ does not depend on the weight $k\geq 2$; more precisely we show that there is an homomorphism of $\mathcal{O}[[\mathbf{Z}_p^\times]]$-algebras

$$\bar{\iota}^*:\mathcal{H}_2(p^\infty,N;\mathcal{O})\longrightarrow\mathcal{H}_k(p^\infty,N;\mathcal{O})$$

sending T_q to T_q for every prime $q\neq p$.
The proof is an adaptation of the Hida's cohomological proof given in [7]. The main idea, explained in Lemma 7.1, consists roughly in saying that for large enough level any torsion coefficient system becomes trivial. The technical difficulties in applying this method are related to the realization of the Hecke algebra on cohomology groups with divisible coefficients; these problems are addressed in the last sections of the paper.

This work takes place in a general program on quaternionic Hecke algebras, which has been sketched in [18]. The idea is to exploit algebraic and geometric properties of quaternion algebras and Shimura curves in order to recover relevant informations about adelic representations of fixed type at a prime p and Galois representations associated to quaternionic automorphic eigenforms. It is well-known that representations of $B_{\mathbf{A}}^\times$ correspond by Jacquet-Langland to representations of GL_2 which are special or supercuspidal at primes dividing the discriminant Δ, [11]. A realization of Jacquet-Langlands correspondence as a Hecke-equivariant map from quaternionic automorphic forms and classical forms is performed in [14] in the case of Eichler orders and in [18] in the case of arbitrary level at a prime p dividing

Δ. However in this last case the absence of new vectors on the quaternionic side prevents to define the operator T_p on the space of quaternionic forms. This poses some limitations to the theory of the quaternionic Hecke algebras; for example the space of quaternionic forms and the Hecke algebra are not in general dual each other (not even on $\mathbf{C}$).
Let us denote by U_s the (p, N)-group $U(N, 1 + p^s R_p)$ and by $U(M, p^r)$ the compact open subgroup of $\mathrm{GL}_2(\mathbf{A})$ defined by $U(M, p^r) = U_0(M) \cap U_1(p^r)$; as it is shown in [18, Théorème 8.0.11] the Hecke algebra $\mathcal{H}_k(U_s, \mathcal{O})$ can be canonically regarded as a quotient of the algebra $\mathcal{H}_k(U(N\Delta', p^{2s+1}), \mathcal{O})$, where the latter is the Hecke algebra generated by operators T_q for $q \neq p$ and diamond operators $\langle q \rangle$ acting on "classical" modular forms: more precisely, $\mathcal{H}_k(U_s, \mathcal{O})$ can be identified to the $\mathcal{O}$-algebra obtained by restricting the Hecke operators in $\mathcal{H}_k(U(N\Delta', p^{2s+1}), \mathcal{O})$ to the space W_k^{2s+1} spanned by eigenforms φ which generate a representation of $\mathrm{GL}_2(\mathbf{A})$ which is special at $q|\Delta'$ and special or supercuspidal at p.

We denote by $\mathcal{H}_k^\sharp(U(M, p^s), \mathcal{O})$ the classical Hecke algebras (including the T_p-operator), as defined for example in [18, Section 9]. By Hida's "classical" weight independence, for every integer M prime to p there is an isomorphism

$$\varprojlim_s \mathcal{H}_k^\sharp(U(M, p^s), \mathcal{O}) \xrightarrow{\sim} \varprojlim_s \mathcal{H}_2^\sharp(U(M, p^s), \mathcal{O}) \tag{2}$$

sending T_q to T_q for $q \in \mathbf{N}$ and $\langle q \rangle$ to $\langle q \rangle$ for q prime to Mp. Of course this isomorphism restricts to an isomorphism of subalgebras "deprived of the T_p operator"

$$\varprojlim_s \mathcal{H}_k(U(M, p^s), \mathcal{O}) \xrightarrow{\sim} \varprojlim_s \mathcal{H}_2(U(M, p^s), \mathcal{O}).$$

Then Theorem 8.1 implies that there is a commutative diagram

$$\begin{array}{ccc} \varprojlim_s \mathcal{H}_k(U(N\Delta', p^s), \mathcal{O}) & \xrightarrow{\sim} & \varprojlim_s \mathcal{H}_2(U(N\Delta', p^s), \mathcal{O}) \\ \pi_k \downarrow & & \downarrow \pi_2 \\ \varprojlim_s \mathcal{H}_k(U_s, \mathcal{O}) & \xrightarrow{\sim} & \varprojlim_s \mathcal{H}_2(U_s, \mathcal{O}) \end{array}$$

where the vertical arrows are surjective and continuous.

Since for every k the space $\mathcal{H}_k(U(N\Delta', p^s), \mathcal{O})$ is compact and the space $\mathcal{H}_k(p^\infty, N; \mathcal{O})$ is Hausdorff, the latter has the quotient topology with respect to the surjection π_k.

Notice that by multiplicity one (see [4, Theorem 5.14]) the spaces W_k^s are also stable by the T_p operator.
Then we could ask whether the isomorphism of Theorem 8.1 could extend to an isomorphism of complete algebras

$$\varprojlim_s \mathcal{H}_2^\sharp(U(N\Delta', p^s), \mathcal{O})|_{W_2^{2s+1}} \xrightarrow{\sim} \varprojlim_s \mathcal{H}_k^\sharp(U(N\Delta', p^s), \mathcal{O})|_{W_k^{2s+1}}.$$

This would mean that the "classical" isomorphism (2) induces an isomorphism on the restrictions of Hecke algebras to modular forms associated to a representation of GL_2 arising by Jacquet-Langlands correspondence from a form over $B_{\mathbf{A}}^\times$ which is invariant by some (p, N)-group (having fixed a maximal order in B). As in [6, §2] the existence of such an isomorphism would be equivalent by duality to the weight independence of the Hecke module $\bar{W}_k$ obtained by completing $\varinjlim_s W_k^s$ with respect to the topology of the uniform convergence in $\mathcal{O}[[q]]$. However, this would contradict the fact that the eigenvalues for the T_p operator associated to newforms in W_k have p-adic valuation $\geq k/2-1$, so that the operator T_p is topologically nilpotent over $\bar{W}_k$; and, on the other hand, special forms of weight 2 are ordinary.

The obstruction seems to arise from special forms. A possible way to avoid them could perhaps be to analyze more deeply the possible Hecke operators at p in the quaternionic setup. This consideration is inspired by the fact that, as remarked for example in [2], there is an action of the whole $G(\mathbf{Q}_p)$ on the p-adically completed cohomology of a reductive group G. Although an operator T_p corresponding to the classical one is not available, there is an action of the unramified p-adic integer ring $\mathbf{Z}_{p^2}^\times$ on quaternionic automorphic forms, as observed in Remark 5.1. This action commutes with Hecke operators outside p, and it is possible that it could help to isolate "supercuspidal components"in the Hecke algebra $\mathcal{H}_k(p^\infty, N, \mathcal{O})$. Since the T_p operator is zero on supercuspidal representation, in this situation the two restricted algebras $\mathcal{H}$ and $\mathcal{H}^\sharp$ would coincide. A step in this direction has been obtained in weight 2 and fixed level in [19, 3], where a Taylor-Wiles system was realized using the cohomology of Shimura curves, in order to show that a suitable local component of a quaternionic Hecke algebra was a universal deformation ring. This result suggest that the action of $\mathbf{Z}_{p^2}^\times$ on the direct limit of cohomology groups of Shimura curves deserves a deeper investigation.
A very interesting perspective for future work would be the study of Galois representations over (local components of) the whole quaternionic Hecke algebra $\mathcal{H}_k(p^\infty, N; \mathcal{O})$; this is a challenging problem, because of the high ramification at p of such representations.
Finally, we mention that the methods used in the present paper could probably work over totally real fields.

The structure of the paper is the following: in Section 2 we introduce the spaces of quaternionic automorphic forms, Hecke algebras, Shimura curves and cohomology groups; the action of Hecke algebra on cohomology in the local and global case is described in Section 3. Section 4 indicates some conditions on the level which assure that quaternionic groups are torsion-free; this is important in order to have locally constant sheaves over Shimura varieties. The Hecke algebra $\mathcal{H}_k(p^\infty, N, \mathcal{O})$ is defined in Section 5, and in Section 6 we show that it faithfully acts on the p-divisible cohomology of Shimura curves in weight k and level p^∞. Section 7 shows that, up to isogenies, this p-divisible Hecke module does not depend on k, which implies our main theorem, stated in Section 8. Sections 9 and 10 are devoted to technical proofs.

Notations and conventions. We shall denote $\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}, \mathbf{C}$ the natural, integer, rational, real and complex numbers respectively. If A is a ring, $A^\times$ denotes the multiplicative group of invertible elements of A; $\mathbf{R}^{\times,+}$ is the subgroup of $\mathbf{R}^\times$ consisting of positive real numbers.

If p is a prime number, the ring of p-adic integers and the field of p-adic numbers are denoted respectively by $\mathbf{Z}_p$ and $\mathbf{Q}_p$. The symbol $\mathbf{A}$ denotes the ring of rational adèles, $\mathbf{A}^\infty$ the finite adèles, $\hat{\mathbf{Z}} = \prod_p \mathbf{Z}_p$. For every prime number p let $v_p : \mathbf{Q}_p \to \mathbf{Z} \cup \{\infty\}$ be the p-adic valuation. If $x \in \mathbf{Q}_p$, $|x|_p = p^{-v_p(x)}$; if $x \in \mathbf{R}$, $|x|_\infty = |x|$ is the absolute value. If $a = (a_v) \in \mathbf{A}^\times$, we shall denote by $||a||$ the idelic norm of a: $||a|| = \prod_v |a|_v$.

Let B be a quaternion algebra over $\mathbf{Q}$ and R be an order in B. For every prime q (included $q = \infty$), we put $B_q = B \otimes_{\mathbf{Q}} \mathbf{Q}_q$, $R_q = R \otimes_{\mathbf{Z}} \mathbf{Z}_q$; we shall denote $B_{\mathbf{A}}$ the adelization of B, $B_{\mathbf{A}}^\times$ the topological group of the invertible elements of $B_{\mathbf{A}}$, and $B_{\mathbf{A}}^{\times,\infty}$ the subgroup of finite idèles. If G is a multiplicative group in a quaternion algebra, we shall denote by $G^{(1)}$ the subgroup consisting of element having reduced norm 1.

In the case where $B = \mathrm{M}_2(\mathbf{Q})$ we put, for every integer N

$$\begin{aligned} U_0(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\hat{\mathbf{Z}}) \mid c \equiv 0 \bmod N \right\} \\ U_1(N) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_0(N) \mid a \equiv 1 \bmod N \right\} \end{aligned}$$

They are compact open subgroups of $\mathrm{GL}_2(\mathbf{A})$.

If $r = p^n$, $\mathbf{F}_r$ denotes the field with r elements, $\mathbf{Q}_r$ is the unramified extension of $\mathbf{Q}_p$ having degree n, $\mathbf{Z}_r$ is its ring of p-adic integers.

If V is a $\mathbf{Q}$-vector space and a lattice $\Lambda \subset V$ we put $\Lambda_p = \Lambda \otimes_{\mathbf{Z}} \mathbf{Z}_p$ and $V_p = V \otimes_{\mathbf{Q}} \mathbf{Q}_p$.

Let $\mathrm{GL}_2^+(\mathbf{R}) = \{g \in \mathrm{GL}_2(\mathbf{R}) \mid \det(g) > 0\}$ and

$$K_\infty = \mathbf{R}^\times \mathrm{O}_2(\mathbf{R}), \qquad K_\infty^+ = \mathbf{R}^\times \mathrm{SO}_2(\mathbf{R}).$$

The upper complex half-plane $\mathcal{H} = \{x + \sqrt{-1}y \in \mathbf{C} \mid y > 0\}$ can be identified to

$$\mathrm{GL}_2(\mathbf{R})/K_\infty = \mathrm{GL}_2^+(\mathbf{R})/K_\infty^+$$

by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{R}) \mapsto \begin{cases} \frac{a\sqrt{-1}+b}{c\sqrt{-1}+d} & \text{if } ad - bc > 0 \\ \frac{-a\sqrt{-1}+b}{-c\sqrt{-1}+d} & \text{if } ad - bc < 0 \end{cases}$$

By this identification, the group $\mathrm{GL}_2^+(\mathbf{R})$ acts on $\mathcal{H}$ by homographies: if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbf{R})$ and $z \in \mathcal{H}$

$$g \cdot z = \frac{az + b}{cz + d}.$$

Let $j(g, z) = cz + d$.

If Γ is a discrete subgroup of $\mathrm{SL}_2(\mathbf{R})$, we shall denote $\overline{\Gamma} = \Gamma/\Gamma \cap \mathbf{R}^\times$ and $\mathcal{H}_\Gamma^* = \mathcal{H} \cup P_\Gamma$, where P_Γ is the set of cusps of Γ (see [17, Chap. I]).

2. Quaternionic automorphic forms, Hecke algebras and cohomology

We shall fix a quaternion algebra B over $\mathbf{Q}$, indefinite, of discriminant Δ, and a maximal order R in B. Let ν and t be the reduced norm and trace of B respectively. Let p be a prime dividing Δ, and put $\Delta = \Delta' p$.

The following lemma will be useful in order to let $B^\times$ act on the standard polynomial representations of GL_2:

LEMMA 2.1. *There exists a totally real Galois extension K_0 of $\mathbf{Q}$ satisfying the following conditions:*

a) *K_0 contains a quadratic extension L_0 in which p is inert;*
b) *there exists an isomorphism of K_0-algebras $\phi_0 : B \otimes_{\mathbf{Q}} K_0 \xrightarrow{\sim} \mathrm{M}_2(K_0)$;*
c) *$\phi_0(R) \subseteq \mathrm{M}_2(\mathcal{O}_0)$ where $\mathcal{O}_0$ is the ring of integers of K_0.*

Proof. Let K_0' be a real quadratic field in which p is inert and splitting B. There is an isomorphism $\phi' : B \otimes_{\mathbf{Q}} K_0' \to \mathrm{M}_2(K_0')$; let R_0' be a maximal order in $M_2(K_0')$ containing $\phi'(R)$. Then R_0' is conjugated by an element β of $\mathrm{GL}_2(K_0')$ to an order of the type $R_0'' = \begin{pmatrix} \mathcal{O}_0' & I^{-1} \\ I & \mathcal{O}_0' \end{pmatrix}$ where $\mathcal{O}_0'$ is the ring of integers of K_0' and I is an ideal of $\mathcal{O}_0'$, see [20, Exercice 5.7]. Then we can assume that $\phi_0'(R) \subseteq R_0''$. Let K_0 be the Hilbert field of K_0', and $\mathcal{O}_0$ be its ring of integers; then K_0 is a totally real extension of $\mathbf{Q}$ and every ideal of K_0' become principal in K_0; we put $I\mathcal{O}_0 = (a)$ with $a \in \mathcal{O}_0$. Extending ϕ' by linearity we obtain an isomorphism $\phi'' : B \otimes_{\mathbf{Q}} K_0 \to \mathrm{M}_2(K_0)$ such that

$$\phi''(R) \subseteq \begin{pmatrix} \mathcal{O}_0 & a^{-1}\mathcal{O}_0 \\ a\mathcal{O}_0 & \mathcal{O}_0 \end{pmatrix} = \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix} \mathrm{M}_2(\mathcal{O}_0) \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

The isomorphism ϕ_0 we are looking for is the composition $\eta \circ \phi''$, where η is the conjugation by the matrix $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$. ■

■

We define the isomorphism $i_\infty : B \otimes_{\mathbf{Q}} \mathbf{R} \to \mathrm{M}_2(\mathbf{R})$ as the isomorphism induced by $\mathbf{R}$-linearity by the isomorphism ϕ_0 of Lemma 2.1.
For every finite prime q not dividing Δ, we fix an isomorphism $i_q : B_q \to \mathrm{M}_2(\mathbf{Q}_q)$ such that $i_q(R_q) = \mathrm{M}_2(\mathbf{Z}_q)$.

If U is a compact open subgroup of $B_{\mathbf{A}}^{\times,\infty}$ we put

$$\Gamma(U) = (\mathrm{GL}_2^+(\mathbf{R}) \times U) \cap B^\times. \tag{3}$$

By the isomorphism i_∞ the group $\Gamma(U)$ can be identified to a discrete subgroup of $\mathrm{SL}_2(\mathbf{R})$ having finite covolume [20, page 104]. Since $\Delta \neq 1$ the quotient space $\Gamma(U)\backslash\mathrm{SL}_2(\mathbf{R})$ is compact and $\Gamma(U)\backslash\mathcal{H} = \Gamma(U)\backslash\mathrm{SL}_2(\mathbf{R})/\mathrm{SO}_2(\mathbf{R})$ is a compact Riemann surface. By [20, page 107], $\Gamma(U)$ contains a torsion-free subgroup Γ of finite index. The group Γ acts properly on $\mathcal{H}$, so it is the fundamental group of the compact surface $\Gamma\backslash\mathcal{H}$. This shows that the groups $\Gamma(U)$ are finitely generated.

PROPOSITION 2.2. *Let U be a compact open subgroup of $B_{\mathbf{A}}^{\times,\infty}$, let q be a finite prime and let $K_q = U \cap B_q^\times$. Then $\Gamma(U)$ is dense in $K_q^{(1)}$ (with the q-adic topology).*

Proof. Let $x \in K_q^{(1)}$, and consider x as an element of $B_{\mathbf{A}}^{(1)}$. Since B is indefinite, by the strong approximation theorem (see [20, page 81]) x can be approximated by elements $x_n = a_n b_n$, where $a_n \in B^{(1)}$, $b_n \in B_\infty^{(1)}$. Therefore the sequence of a_n tends to x in the q-adic topology. Since U is open, $a_n \in U$ for $n \gg 0$, so that $a_n \in \Gamma(U)$. ■

If U is a compact open subgroup of $B_{\mathbf{A}}^{\times,\infty}$, let $S_k^B(U)$ denote the complex space of adelic automorphic forms which are right invariant by U. We refer to [18, §2] for the definition and the basic properties of this space.
If N is an integer prime to Δ, let $R(N)$ be the Eichler order of level N locally defined by:

$$R(N)_l = \begin{cases} R_l & \text{if} \quad l \not| N \\ i_l^{-1}\left(\left\{\begin{pmatrix} a & b \\ Nc & d\end{pmatrix} \mid a,b,c,d \in \mathbf{Z}_l\right\}\right) & \text{if} \quad l|N \end{cases}$$

In what follows we shall consider compact open subgroups of $B_{\mathbf{A}}^{\times,\infty}$ having the form

$$U = U(N, U_p) = \prod_{l \neq p} R(N)_l^{\times} \times U_p$$

where U_p is an open subgroup of $R_p^{\times}$. Such a group will be called a (p,N)-*group*.
In [18, §3] a Hecke abstract ring $\mathcal{R}(N)$ is defined, acting on $S_k^B(U)$ for every (p,N)-group U. It is generated by the operators $\tilde{T}(q)$, for prime $q \neq p$ and $\tilde{T}(q,q)$ for prime $q \ \not| N\Delta$ For a subring A of $\mathbf{C}$ the Hecke algebra $\mathcal{H}_k(U,A)$ is defined as the A-subalgebra of $\mathrm{End}_{\mathbf{C}}(S_k^B(U))$ generated over by the image of $\mathcal{R}(N)$. We denote by $T_q, \langle q\rangle$ the images in $\mathcal{H}_k(U,\mathbf{Z})$ of $\tilde{T}(q), \tilde{T}(q,q)$ respectively. For every positive integer a prime to $N\Delta$, write $a = \prod_i q_i^{e_i}$; then we define $\langle a\rangle = \prod_i < q_i >^{e_i} \in \mathcal{H}_k(U,\mathbf{Z})$.

By strong approximation, if we choose $t_1, ..., t_h \in B_{\mathbf{A}}^{\times,\infty}$ such that $\nu(t_1), ..., \nu(t_h)$ are representatives of $\mathbf{Q}^{\times}\backslash\mathbf{A}^{\times}/\mathbf{R}^{\times,+}\nu(U)$ then $B_{\mathbf{A}}^{\times} = \coprod_{i=1}^h B_{\mathbf{Q}}^{\times}\mathrm{GL}_2^+(\mathbf{R})t_iU$. If we fix such a decomposition, we can define for $i = 1, ..., h$,

$$\Gamma^i(U) = \Gamma(t_iUt_i^{-1}) \tag{4}$$

The complex analytic compact spaces

$$X_i(U) = \Gamma^i(U)\backslash\mathcal{H}$$

are varieties if $\overline{\Gamma^i(U)}$ has no torsion. We also put

$$\mathbf{X}(U) = B_{\mathbf{Q}}^{\times}\backslash B_{\mathbf{A}}^{\times}/K_{\infty}^{+}U.$$

It is well-known (see for example [8, Proposition 6.1 (i)]) that the map $B_{\mathbf{A}}^{\times} \to \coprod_{i=1}^h \mathcal{H}$ sending $g_{\mathbf{Q}}g_{\infty}t_ju$ in the element $g_{\infty}(\sqrt{-1})$ of the j-th component induces an isomorphism

$$\mathbf{X}(U) \simeq \coprod_{i=1}^h X_i(U). \tag{5}$$

If A is a ring, let $L(n,A) = \mathrm{Symm}^n(A^2)$ with an action of $\mathrm{M}_2(A)$ gotten from the standard one in A^2. It can be represented as the A-submodule of $A[X,Y]$ consisting of homogeneous polynomials of degree n; if $P(X,Y) \in L(n,A)$ and $\gamma \in \mathrm{M}_2(A)$ then

$$P|_{\gamma}(X,Y) = P((X,Y)^t\gamma).$$

If M is a A-module, we define $L(n,M) = L(n,A) \otimes_A M$, with the action of $\mathrm{M}_2(A)$ on the first factor. In this way $L(n,M)$ becomes a right module over the multiplicative semigroup $\mathrm{M}_2(A)$. We denote ${}^tL(n,M)$ the left $\mathrm{GL}_2(A)$-module which is $L(n,M)$ as A-module, with the action of $\mathrm{GL}_2(A)$ given by $\gamma \cdot P = P|_{\gamma^{-1}}$.

We follow [8, Section 6] and [10, Chapter 6] in the description of some sheaves on the variety $\mathbf{X}(U)$ arising from a right module over $R_p^\times$ or $\mathrm{GL}_2^+(\mathbf{R})$. We shall refer to the first case as *case* p and to the second as *case* ∞.

Let K be a finite extension of $\mathbf{Q}_p$ in $\bar{\mathbf{Q}}_p$ containing the field K_0 of Lemma 2.1 and let $\mathcal{O}$ be the ring of p-adic integers of K. Since K_0 contains a quadratic extension in which p is inert, K contains $\mathbf{Q}_{p^2}$. The homomorphism ϕ_0 of Lemma 2.1 induces an isomorphism between $R \otimes_{\mathbf{Z}} \mathbf{Z}_p$ and the p-adic closure of R in $M_2(\mathcal{O})$; then we can identify $R_p^\times$ to a subgroup of $\mathrm{GL}_2(\mathcal{O})$; in this way $R_p^\times$ acts in a natural way over $L(n,\mathcal{O})$.

In the case ∞ we assume that M is a real vector space, in the case p that it is an $\mathcal{O}$-module. We define a left action of $B_{\mathbf{Q}}^\times$ over $B_{\mathbf{A}}^\times \times L(n,M)$ by $\alpha(g,v) = (\alpha g, v)$ and a right action of $K_\infty^+ U$ by

$$(g,v)u = \begin{cases} (gu, vu_\infty) & \text{in case } \infty \\ (gu, vu_p) & \text{in case } p. \end{cases}$$

Giving $L(n,M)$ the discrete topology we can consider the covering of $\mathbf{X}(U)$

$$\mathcal{L}(n,M)_{/\mathbf{X}(U)} = B_{\mathbf{Q}}^\times \backslash B_{\mathbf{A}}^\times \times L(n,M)/K_\infty^+ U.$$

The groups $\Gamma^i(U)$ act on ${}^tL(n,M)$ via the natural inclusion $\Gamma^i(U) \hookrightarrow \mathrm{GL}_2^+(\mathbf{R})$ in case ∞ and $\Gamma^i(U) \hookrightarrow R_p^\times$ in case p. Then we can consider the covering of $X_i(U)$

$$\mathcal{L}(n,M)_{/X_i(U)} = \Gamma^i(U)\backslash \mathcal{H} \times^t L(n,M)$$

where $\Gamma^i(U)$ acts diagonally at the left over $\mathcal{H} \times^t L(n,M)$. This covering is unramified if $\overline{\Gamma^i(U)}$ has no torsion.

The map $B_{\mathbf{A}}^\times \times L(n,M) \to \coprod_{i=1}^h \mathcal{H} \times^t L(n,M)$ defined by

$$(g_{\mathbf{Q}} g_\infty t_i u, m) \mapsto \begin{cases} (g_\infty(\sqrt{-1}), g_\infty m)_i & \text{in the case } \infty \\ (g_\infty(\sqrt{-1}), t_i u_p m)_i & \text{in the case } p \end{cases}$$

induces an isomorphism of coverings

$$\mathcal{L}(n,M) \simeq \coprod_{i=1}^h \mathcal{L}_i(n,M) \tag{6}$$

which is compatible with the isomorphism (5), see [8, Proposition 6.1 (ii)]. If $\overline{\Gamma^i(U)}$ has no torsion, then $\mathcal{L}_i(n,M)$ is locally isomorphic to the variety $X_i(U)$ and we can consider the sheaf of continuous sections of $\mathcal{L}_i(n,M)$ over $X_i(U)$; we shall denote it with the same symbol $\mathcal{L}_i(n,M)_{/X_i(U)}$. There is also a sheaf $\mathcal{L}(n,M)_{/\mathbf{X}(U)}$.

We need to compare the cohomology of sheaves defined in case ∞ and in case p by means of a common K_0-structure. We define a sheaf $\mathcal{L}(n,A)$ for a global $\mathcal{O}_0$-algebra A. Let A be a ring such that $\mathcal{O}_0 \subseteq A \subseteq \mathbf{C}$. For $i = 1, ..., h$ the group $\Gamma^i(U)$ is contained in $R^\times$, so $\Gamma^i(U)$ acts on the right on $L(n,A)$. We shall denote $L_i(n,A)$ the right $\Gamma^i(U)$-module $L(n,A)$ and ${}^tL_i(n,A)$ the coresponding left $\Gamma^i(U)$-module. If $\overline{\Gamma^i(U)}$ has no torsion, we can consider the sheaves

$$\begin{aligned}\mathcal{L}_i(n,A)_{/X_i(U)} &= \Gamma^i(U)\backslash\mathcal{H}\times^t L_i(n,A)\\ \mathcal{L}(n,A)_{/\mathbf{X}(U)} &= \coprod_{i=1}^h \mathcal{L}_i(n,A).\end{aligned}$$

We shall refer to the latter situation as "global case", and to cases ∞ and p as "local cases".

Isomorphisms (5) and (6) assure that if A is local or global, there is a fonctorial isomorphism

$$H^*(\mathbf{X}(U),\mathcal{L}(n,A)) \simeq \bigoplus_{i=1}^h H^*(X_i(U),\mathcal{L}_i(n,A)). \tag{7}$$

Moreover, if $\overline{\Gamma^i(U)}$ has no torsion

$$H^*(X_i(U),\mathcal{L}_i(n,A)) \simeq H^*(\overline{\Gamma^i(U)},{}^t L_i(n,A)), \tag{8}$$

(cf. for example [16, 1.5] and [10, Appendix, Th.1 and Cor. 1]. For every $\mathcal{O}_0$-algebra D we define the sheaf $\mathcal{L}(n,D)$ over $\mathbf{X}(U)$ by

$$\mathcal{L}(n,D)_{/\mathbf{X}(U)} = \mathcal{L}(n,\mathcal{O}_0)\otimes_{\mathcal{O}_0} D.$$

Remark 2.3. If D is a $\mathcal{O}$-algebra, then the action of $\Gamma^i(U)$ on ${}^tL(n,D)$ via its inclusion in $\mathrm{GL}_2(\mathcal{O}_0)$ coincides with the action of $\Gamma^i(U)$ via its inclusion in $R_p^\times \subseteq \mathrm{GL}_2(\mathcal{O})$, by Lemma 2.1. Therefore the sheaf $\mathcal{L}(n,\mathcal{O}_0)\otimes_{\mathcal{O}_0} D_{/\mathbf{X}(U)}$ is isomorphic to the sheaf $\mathcal{L}(n,D)_{/\mathbf{X}(U)}$ obtained in case p. In particular there are injective homomorphisms

$$\begin{aligned}\mathcal{L}(n,\mathcal{O}_0)_{/\mathbf{X}(U)} &\hookrightarrow \mathcal{L}(n,\mathbf{C})_{/\mathbf{X}(U)}\\ \mathcal{L}(n,\mathcal{O}_0)_{/\mathbf{X}(U)} &\hookrightarrow \mathcal{L}(n,\mathcal{O})_{/\mathbf{X}(U)}\end{aligned}$$

The following theorem is well-known [13, §4]

Theorem 2.4 (Matsushima-Shimura). *There is a canonical isomorphism*

$$S_k^B(U)\oplus\overline{S_k^B(U)} \simeq H^1(\mathbf{X}(U),\mathcal{L}(n,\mathbf{C})).$$

3. Hecke operators on cohomology groups

In [8, Section 7] Hida defines the action of double cosets UxU, where $x\in B_{\mathbf{A}}^{\times,\infty}$ over the cohomology groups

$$H^*(\mathbf{X}(U),\mathcal{L}(n,M)),\bigoplus_{i=0}^h H^*(X_i(U),\mathcal{L}_i(n,M)),\bigoplus_{i=0}^h H^*(\overline{\Gamma^i(U)},{}^t L_i(n,M))$$

in local and global cases. We briefly recall the construction.

3.1. **Hecke operators over $H^*(\mathbf{X}(U), \mathcal{L}(n, M))$ in the local case.** Let U, U' be compact open subgroups of $B_{\mathbf{A}}^{\times,\infty}$, and M be an $\mathbf{R}$-vector space in case ∞ or an $\mathcal{O}$-module in case p. We put $\sigma = \infty$ or p. If $x \in B_{\mathbf{A}}^{\times,\infty}$ the map

$$\begin{array}{rcl} [x] : B_{\mathbf{A}}^{\times} \times L(n, M) & \longrightarrow & B_{\mathbf{A}}^{\times} \times L(n, M) \\ (g, m) & \longmapsto & (gx, m \cdot x_\sigma) \end{array}$$

induce an homomorphism

$$B_{\mathbf{Q}}^{\times} \backslash B_{\mathbf{A}}^{\times} \times L(n, M)/K_\infty^+ V \longrightarrow B_{\mathbf{Q}}^{\times} \backslash B_{\mathbf{A}}^{\times} \times L(n, M)/K_\infty^+ V^x$$

where $V = U \cap xU'x^{-1}$ and $V^x = x^{-1}Vx$; therefore there is an homomorphism of sheaves

$$[x] : \mathcal{L}(n, M)_{/\mathbf{X}(V)} \longrightarrow \mathcal{L}(n, M)_{/\mathbf{X}(V^x)}$$

which induces a map in cohomology

$$[x] : H^*(\mathbf{X}(V), \mathcal{L}(n, M)) \longrightarrow H^*(\mathbf{X}(V^x), \mathcal{L}(n, M)).$$

We define the operator

$$[UxU'] : H^*(\mathbf{X}(U), \mathcal{L}(n, M)) \longrightarrow H^*(\mathbf{X}(U'), \mathcal{L}(n, M))$$

by

$$[UxU'] = \mathrm{Tr}_{U'/V^x} \circ [x] \circ \mathrm{res}_{U/V}$$

where $\mathrm{res}_{U/V} : H^*(\mathbf{X}(U), \mathcal{L}(n, M) \to H^*(\mathbf{X}(V), \mathcal{L}(n, M))$ is the restriction map, and $\mathrm{Tr}_{U'/V^x} : H^*(\mathbf{X}(V^x), \mathcal{L}(n, M)) \to H^*(\mathbf{X}(U'), \mathcal{L}(n, M))$ is the trace map [8, Section 7]. It is easy to see that the operator $[UxU']$ depends only on the double coset UxU'.

3.2. **Hecke operators over $\bigoplus_{i=1}^h H^*(X_i(U), \mathcal{L}_i(n, M))$ in local and global cases.** For sake of simplicity we shall assume that $U = U'$ is a (p, N)-group and that $x \in B_{\mathbf{A}}^{\times,\infty}$ is such that $x_p = 1$. Then $(xUx^{-1})_p = U_p$, so that

$$B_{\mathbf{A}}^{\times} = \coprod_{j=1}^{h} B_{\mathbf{Q}}^{\times} \mathrm{GL}_2^+(\mathbf{R}) t_j U = \coprod_{j=1}^{h} B_{\mathbf{Q}}^{\times} \mathrm{GL}_2^+(\mathbf{R}) t_j V = \coprod_{j=1}^{h} B_{\mathbf{Q}}^{\times} \mathrm{GL}_2(\mathbf{R}) t_j V^x.$$

By strong approximation for every j we can write $t_j x = \gamma_j t_i (x^{-1} u x)$ with $u \in V$ and $\gamma_j \in B_{\mathbf{Q}}^{\times}$ ($j \mapsto i$ is a permutation of $\{1, ..., h\}$). There is a map

$$\begin{array}{rcl} {}^t L_j(n, M) & \longrightarrow & {}^t L_i(n, M) \\ m & \longmapsto & \gamma_j^{-1} m \end{array}$$

and $\Gamma^i(V^x) = \gamma_j^{-1} \Gamma^j(V) \gamma_j$. Therefore for every j there is a sheaves homomorphism

$$[x]_j : \mathcal{L}_j(n, M)_{/X_j(V)} \to \mathcal{L}_i(n, M)_{/X_i(V^x)}$$

which induces maps in cohomology

$$\begin{array}{rcl} [x]_j : H^*(X_j(V), \mathcal{L}_j(n, M)) & \longrightarrow & H^*(X_i(V^x), \mathcal{L}_i(n, M)) \\ [x]_j : H^*(\overline{\Gamma^j(V)}, {}^t L_j(n, M)) & \longrightarrow & H^*(\overline{\Gamma^i(V^x)}, {}^t L_i(n, M)). \end{array}$$

Then $[x]$ corresponds to $\bigoplus_{j=1}^{h}[x]_j$ via the isomorphism (7). We define operators

$$
\begin{aligned}
[UxU] : \bigoplus_{j=1}^{h} H^*(X_j(U), \mathcal{L}_j(n,M)) &\longrightarrow \bigoplus_{j=1}^{h} H^*(X_j(U), \mathcal{L}_j(m,M)) \\
[UxU] : \bigoplus_{j=1}^{h} H^*(\overline{\Gamma^j(U)}, {}^t L_j(n,M)) &\longrightarrow \bigoplus_{j=1}^{h} H^*(\overline{\Gamma^j(U)}, {}^t L_j(n,M))
\end{aligned}
$$

by

$$[UxU] = \mathrm{Tr}_{U/V^x} \circ \bigoplus_{j=1}^{h} [x]_j \circ \mathrm{res}_{U/V}$$

where $\mathrm{res}_{U/V} = \bigoplus_{j=1}^{h} \mathrm{res}_{X_j(U)/X_j(V)}$ and $\mathrm{Tr}_{U/V^x} = \bigoplus_{j=1}^{h} \mathrm{Tr}_{X_j(U)/X_j(V^x)}$ (and analogously in group cohomology). When A is local, these operators correspond, by isomorphism (7), to the operator $[UxU]$ defined in the local case.

3.3. **Action of the Hecke algebra on cohomology.** Let U be a (p,N)-group. The isomorphism of Theorem 2.4 is compatible by the action of the algebra $\mathcal{R}(N)$. Consequently, if A is a subring of $\mathbf{C}$, the algebra $\mathcal{H}_k(U,A)$ is isomorphic to the subalgebra of $\mathrm{End}_{\mathbf{C}}(H^1(\mathbf{X}(U), \mathcal{L}(n,\mathbf{C})))$ generated over A by the images of operators in $\mathcal{R}(N)$.

By the universal coefficient theorem (see for example [1, II, Theorem 15.3] if A is a principal ring contained in $\mathbf{C}$ and B is a flat A-algebra then there is a Hecke-equivariant canonical isomorphism

$$H^1(\mathbf{X}(U), \mathcal{L}(n,A)) \otimes_A B \simeq H^1(\mathbf{X}(U), \mathcal{L}(n,B)).$$

In particular

$$H^1(\mathbf{X}(U), \mathcal{L}(n,K_0)) \otimes_{K_0} \mathbf{C} \simeq H^1(\mathbf{X}(U), \mathcal{L}(n,\mathbf{C})) \tag{9}$$

$$H^1(\mathbf{X}(U), \mathcal{L}(n,K_0)) \otimes_{K_0} K \simeq H^1(\mathbf{X}(U), \mathcal{L}(n,K)) \tag{10}$$

By (9)the algebra $\mathcal{H}_k(U,\mathcal{O}_0)$ faithfully acts on $H^1(\mathbf{X}(U), \mathcal{L}(n,K_0))$. Let $\mathcal{O}_0' = \mathcal{O} \cap K_0$; then $\mathcal{O}_0'$ is a discrete valuation ring and $K_0 \otimes_{\mathcal{O}_0'} \mathcal{O} = K$. Then we can consider $\mathcal{H}_k(U,\mathcal{O}) = \mathcal{H}_k(U,\mathcal{O}_0') \otimes_{\mathcal{O}_0'} \mathcal{O}$ as a subalgebra of $\mathrm{End}_K(H^1(\mathbf{X}(U), \mathcal{L}(n,K)))$, by (10), and obtain the following:

Proposition 3.1. *Let U be a (p,N)-group. The Hecke algebra $\mathcal{H}_k(U,\mathcal{O})$ acts faithfully on $H^1(\mathbf{X}(U), \mathcal{L}(n,K))$.*

4. Torsion freeness conditions for quaternionic groups

Let B, Δ, R be as in Section 2, with $\Delta = \Delta' p$. Let $R(N)$ be an Eichler order of level N in R. For every positive integer s let $V_s(N)$ be the (p,N)-group whose component at p is $U_p = 1 + u_p^s R_p$, where u_p is a uniformizer of $B_p^\times$.

In order to have locally constant sheaves over Shimura varieties we have to deal with adelic groups U such that $\overline{\Gamma(U)}$ has no torsion. The following proposition gives some conditions over N, Δ, U which assure this property. Analogous conditions for modular groups are stated for example in [10, §6.1].

As in the remaining of this paper, p is an odd prime; we put $U_{\max} = \prod_l R_l^\times$.

PROPOSITION 4.1. a) *For every positive integer N, if $s \geq 1$ then $\overline{\Gamma(V_s(N))} = \Gamma(V_s(N)$ does not contain elements of finite order.*

b) *If there exist prime numbers q, r dividing Δ such that $q \equiv 1 \bmod 3$ and $r \equiv 1 \bmod 4$ then $\overline{\Gamma(U_{\max})}$ does not contain elements of finite order.*

c) *Suppose that either 36 divides N or there exist two prime numbers q, r dividing N such that $q \equiv 3 \bmod 4$ and $r \equiv 2 \bmod 3$. Then $\overline{\Gamma(U(N, R_p^\times))}$ does not contain elements of finite order.*

Proof. If U is a compact open subgroup of $B_{\mathbf{A}}^{\times,\infty}$ then the group $\Gamma(U)$ is a discrete subgroup of $\mathrm{SL}_2(\mathbf{R})$ whose elements have integer trace. By [17, Chap. I], an element $\alpha \neq \pm 1$ of $\Gamma(U)$ has finite order if, and only if, $|\mathrm{t}(\alpha)|_\infty \leq 1$. The possible characteristic polynomials for such an element are X^2+1, X^2+X+1, X^2-X+1. The roots of X^2+1 have order 2 in $\Gamma(U)$, those of $X^2 \pm X + 1$ have order 3. If $\alpha \in V_s(N)$ then $\mathrm{t}(\alpha) \equiv 2 \bmod p$; this proves a), since we are assuming $p \neq 2$. An element of order 2 exists in $\overline{\Gamma(U_{\max})}$ if and only if $\mathbf{Q}(\sqrt{-1})$ splits B, and the primes that split in $\mathbf{Q}(\sqrt{-1})$ are exacly those congruent to 1 mod 4. Analogously, an element of order 3 exists in $\overline{\Gamma(U_{\max})}$ if and only if $\mathbf{Q}(\sqrt{-3})$ splits B, and the primes that split in $\mathbf{Q}(\sqrt{-3})$ are exacly those congruent to 1 mod 3. Assertion b) follows. Let $\alpha \in \Gamma(U(N, R_p^\times))$. For every q dividing N, $i_q(\alpha) = \begin{pmatrix} a_q & b_q \\ Nc_q & d_q \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z}_q)$. If α has order 2 in $\overline{\Gamma(U(N, R_p^\times))}$ then $\mathrm{t}(\alpha) = 0$, thus $a_q^2 + 1 \in N\mathbf{Z}_q$ for every q dividing N, so that either $q = 2$ and 4 does not divide N, or $q \equiv 1 \bmod 4$. Analogously, if α has order 3 in $\overline{\Gamma(U(N, R_p^\times))}$ then $\mathrm{t}(\alpha) = \pm 1$, thus $a_q^2 \pm a_q + 1 \in N\mathbf{Z}_q$ for every q dividing N, so that either $q = 3$ and 9 does not divide N, or $q \equiv 1 \bmod 3$. ∎

5. THE HECKE ALGEBRA OF LEVEL p^∞

If U and V are (p, N)-groups, and $U \subseteq V$, there is a natural inclusion

$$H^1(\mathbf{X}(V), \mathcal{L}(n, K)) \hookrightarrow H^1(\mathbf{X}(U), \mathcal{L}(n, K))$$

which is invariant for the operators $T_q, T_{q,q}$. Correspondingly, the restriction of operators in $\mathcal{H}_k(U, \mathcal{O})$ to the image of $H^1(\mathbf{X}(V), \mathcal{L}(n, K))$ gives a surjective homomorphism

$$\mathcal{H}_k(U, \mathcal{O}) \to \mathcal{H}_k(V, \mathcal{O})$$

mapping T_q on T_q and $T_{q,q}$ on $T_{q,q}$. Then we can consider the injective limit

$$\varinjlim_{U_p \subseteq R_p^\times} H^1(U(N, U_p), \mathcal{L}(n, K))$$

which is endowed with the faithful action of the algebra

$$\mathcal{H}_k(p^\infty, N; \mathcal{O}) = \varprojlim_{U_p \subseteq R_p^\times} \mathcal{H}_k(U(N, U_p), \mathcal{O}).$$

Since every (p,N)-group contains a subgroup of the type $V_s(N)$ for some s, the algebra $\mathcal{H}_k(p^\infty,N;\mathcal{O})$ is also equal to $\varprojlim_s \mathcal{H}_k(V_s(N),\mathcal{O})$.

If U is a (p,N)-group, we give to $\mathcal{H}_k(U,\mathcal{O})$ the p-adic topology, and to $\mathcal{H}_k(p^\infty,N;\mathcal{O})$ the topology arising from the projective limit: namely, the weakest topology for which the projections

$$\mathcal{H}_k(p^\infty,N;\mathcal{O}) \to \mathcal{H}_k(V_s(N),\mathcal{O})$$

are continuous. Then the algebra $\mathcal{H}_k(p^\infty,N;\mathcal{O})$ becomes a topological compact algebra.

For every integer s we put

$$\begin{aligned} U_s &= V_{2s}(N) = U(N, 1+p^sR_p),\\ \Phi_s &= \Gamma(U_s),\\ \mathbf{X}_s &= \mathbf{X}(U_s),\\ X_s &= X(U_s) = \Phi_s\backslash\mathcal{H}. \end{aligned}$$

We want to give the Hecke algebra $\mathcal{H}_k(p^\infty,N;\mathcal{O})$ a structure of continue topological algebra over $\mathcal{O}[[\mathbf{Z}_p^\times]]$.

It is easily verify that if a is prime to $N\Delta$ and $a\equiv 1 \bmod p^s$ then $\langle a\rangle$ trivially acts on $S_k^B(U_s)$; moreover for every integer b prime to p there exists an element a prime to $N\Delta$ such that $a\equiv b \bmod p^s$. Therefore the action of diamond operators induce an action of the finite group $(\mathbf{Z}/p^s\mathbf{Z})^\times$ on $S_k^B(U_s)$ by $\varphi\mapsto\langle a\rangle\varphi$. There is a commutative diagram

$$\begin{tikzcd} (\mathbf{Z}/p^s\mathbf{Z})^\times \arrow[r, "\langle\ \rangle"] & \mathcal{H}_k(U_s,\mathcal{O}) \\ (\mathbf{Z}/p^{s+1}\mathbf{Z})^\times \arrow[r, "\langle\ \rangle"] \arrow[u] & \mathcal{H}_k(U_{s+1},\mathcal{O}) \arrow[u, "res"'] \end{tikzcd}$$

Therefore the map $a\mapsto\langle\ \rangle$ gives a continue homomorphism $\mathbf{Z}_p^\times\to\mathcal{H}_k(p^\infty,N,\mathcal{O})$, which can be extended by linearity and continuity to obtain a continue homomorphism of algebras

$$\langle\ \rangle : \mathcal{O}[[\mathbf{Z}_p^\times]] \longrightarrow \mathcal{H}_k(p^\infty,N,\mathcal{O}),$$

where we put $\mathcal{O}[[\mathbf{Z}_p^\times]] = \varprojlim_s \mathcal{O}[(\mathbf{Z}/p^s\mathbf{Z})^\times]$. In this way we give $\mathcal{H}_k(p^\infty,N,\mathcal{O})$ the structure of $\mathcal{O}[[\mathbf{Z}_p^\times]]$-algebra.

Remark 5.1. If we fix an embedding of $\mathbf{Z}_{p^2}^\times$ in $R_p^\times$ we also have an action of $\mathbf{Z}_{p^2}^\times$ over $\varinjlim_{U_p} S_k^B(U(N,U_p))$. We decided in this work to not include these operators in the Hecke algebra, because we are primarily interested in the the action of the classical Hecke algebra over the forms which are special or supercuspidal at p. However, as discussed in the Introduction, the $\mathbf{Z}_{p^2}^\times$-action seems to be of interest in order to isolate in the cohomology of Shimura curves some Galois representations the having a fixed type at p.

6. The action of Hecke operators over some p-divisible modules

The subgroup U_s is normal in $U(N,R_p^\times)$ and $\nu(U_s)=\prod_{q\neq p}\mathbf{Z}_q^\times\times(1+p^s\mathbf{Z}_p)$. For every positive integer s and every $i\in(\mathbf{Z}/p^s\mathbf{Z})^\times$ let us choose a representative $t_i^s \in R_p^\times$ such that $\nu(t_i^s) \equiv i \bmod p^s$ and such that

$$\text{if } j \in (\mathbf{Z}/p^{s+1}\mathbf{Z})^\times \text{ and } j \equiv i \bmod p^s \text{ then } t_j^{s+1} \equiv t_i^s \bmod p^s R_p. \tag{11}$$

Such a choice is possible: having chosen the representatives at level s, for every $j \in (\mathbf{Z}/p^{s+1}\mathbf{Z})^\times$ such that $j \equiv i \bmod p^s$ we choose an $u \in R_p^\times$ such that $\nu(u) \equiv j \bmod p^{s+1}$. Then $\nu(u(t_i^s)^{-1}) \in 1 + p^s\mathbf{Z}_p$. We choose $\alpha \in 1 + p^s R_p$ such that $\nu(\alpha) = \nu(u(t_i^s)^{-1})$ and put $t_j^{s+1} = t_i^s\alpha$. Then $t_j^{s+1} \equiv t_i^s \bmod p^s R_p$ and $\nu(t_j^{s+1}) = \nu(u) \equiv j \bmod p^{s+1}$.

Remark 6.1. a) Since U_s is normal in $U(N, R_p^\times)$, the groups $\Gamma^i(U_s)$ defined in 4 are all equal to Φ_s. Consequently, isomorphisms (7) and (8) establish an isomorphism between $H^1(\mathbf{X}_s, \mathcal{L}(n, M))$ and the direct sum of $\varphi(p^s)$ copies of $H^1(\Phi_s, {}^tL(n, M))$ for every $\mathcal{O}$-module M.

b) Having chosen the representatives t_i^s as in 11, if $r \geq s$ the restriction map

$$H^1(\mathbf{X}_s, \mathcal{L}(n, M)) \longrightarrow H^1(\mathbf{X}_r, \mathcal{L}(n, M))$$

corresponds by (7) and (8) to $\bigoplus_{i \in (\mathbf{Z}/p^s\mathbf{Z})^\times} \mathrm{res}_i$ where

$$\mathrm{res}_i : H^1(\Gamma^i(U_s), {}^tL(n, M)) \longrightarrow \bigoplus_{\substack{i \in (\mathbf{Z}/p^s\mathbf{Z})^\times \\ j \equiv i \bmod p^s}} H^1(\Gamma^j(U_r), {}^tL(n, M))$$

sends x in $(\mathrm{res}(x), ..., \mathrm{res}(x))$.

For every positive integer n, let us consider the exact sequence of $\mathcal{O}[R_p^\times]$-modules

$$0 \longrightarrow L(n, \mathcal{O}) \longrightarrow L(n, K) \longrightarrow L(n, K/\mathcal{O}) \longrightarrow 0. \tag{12}$$

For every positive integer s, sequence (12) induces a long exact sequence in sheaves cohomology

$$\begin{array}{lllll} 0 & \to H^0(\mathbf{X}_s, \mathcal{L}(n, \mathcal{O})) & \to H^0(\mathbf{X}_s, \mathcal{L}(n, K)) & \to H^0(\mathbf{X}_s, \mathcal{L}(n, K/\mathcal{O})) & \to \\ & \to H^1(\mathbf{X}_s, \mathcal{L}(n, \mathcal{O})) & \to H^1(\mathbf{X}_s, \mathcal{L}(n, K)) & \to H^1(\mathbf{X}_s, \mathcal{L}(n, K/\mathcal{O})) & \to \\ & \to H^2(\mathbf{X}_s, \mathcal{L}(n, \mathcal{O})) & \to H^2(\mathbf{X}_s, \mathcal{L}(n, K)) & \to H^2(\mathbf{X}_s, \mathcal{L}(n, K/\mathcal{O})) & \to 0 \end{array}$$

because $\mathbf{X}_s$ has cohomological dimension 2, being a complete curve. We also have an analogous exact sequence in group cohomology.

Let $\mathcal{V}_s^n$ be the image of $H^1(\mathbf{X}_s, \mathcal{L}(n, K))$ in $H^1(\mathbf{X}_s, \mathcal{L}(n, K/\mathcal{O}))$ and V_s^n be the image of $H^1(\Phi_s, {}^tL(n, K))$ in $H^1(\Phi_s, {}^tL(n, K/\mathcal{O}))$. If $n = 0$ then $\mathcal{L}(0, A)$ is a constant sheaf for every $\mathcal{O}$-module A; therefore

$$\mathcal{V}_s^0 = H^1(\mathbf{X}_s, \mathcal{L}(0, K/\mathcal{O})) \quad \text{and} \quad V_s^0 = H^1(\Phi_s, {}^tL(0, K/\mathcal{O})). \tag{13}$$

Let $\mathbf{\Lambda}_s^n(\mathcal{O})$ be the image of $H^1(\mathbf{X}_s, \mathcal{L}(n, \mathcal{O}))$ in $H^1(\mathbf{X}_s, \mathcal{L}(n, K))$ and let $\Lambda_s^n(\mathcal{O})$ be the image of $H^1(\Phi_s, {}^tL(n, \mathcal{O}))$ in $H^1(\Phi_s, {}^tL(n, K))$. By the universal coefficient theorem, since K is $\mathcal{O}$-flat,

$$H^1(\mathbf{X}_s, \mathcal{L}(n, \mathcal{O})) \otimes_{\mathcal{O}} K \simeq H^1(\mathbf{X}_s, \mathcal{L}(n, K)).$$

Therefore

$$\begin{array}{ll} \mathbf{\Lambda}^n(\mathcal{O}) & \text{is a lattice in } H^1(\mathbf{X}_s, \mathcal{L}(n, K)), \text{ and} \\ \Lambda_s^n(\mathcal{O}) & \text{is a lattice in } H^1(\Phi_s, {}^tL(n, K)). \end{array} \tag{14}$$

By definition of Hecke operators $\mathbf{\Lambda}_s^n(\mathcal{O})$ is stable under the algebra $\mathcal{H}_k(U_s,\mathcal{O})$; by linearity this algebra faithfully acts on $\mathcal{V}_s^n$.

LEMMA 6.2. *If $n \neq 0$ then the Φ_s-module ${}^tL(n,K)$ is simple.*

Proof. It is enough to proof the claim for $\Phi = \Gamma(U_{\max})$ where $U_{\max} = \prod_q R_q^\times$, because Φ_s has finite index in Φ. Let Φ' be the p-adic closure of Φ in $\mathrm{M}_2(\mathcal{O})$. By continuity, it is enough to show the irreducibility of the action of Φ' over ${}^tL(n,K)$. Since p divides Δ, B_p is isomorphic to

$$B'_p = \left\{ \begin{pmatrix} a & b \\ pb^\sigma & a^\sigma \end{pmatrix} \mid a,b \in \mathbf{Q}_{p^2} \right\},$$

where σ is the non-trivial element of $\mathrm{Gal}(\mathbf{Q}_{p^2}/\mathbf{Q}_p)$. The maximal order of B'_p is $S_p = B'_p \cap \mathrm{M}_2(\mathbf{Z}_{p^2})$. An isomorphism between B_p and B'_p can be extended by K-linearity to an automorphism of the K-algebra $\mathrm{M}_2(K)$, which is the conjugation by an element of $\mathrm{GL}_2(K)$, [20, Théorème 2.1]. Thus it suffices to show that the action of the group $S_p^{(1)}$ over ${}^tL(n,K)$ is irreducible. But $S_p^{(1)}$ contains a K-basis of $\mathrm{M}_2(K)$ and it is well-known that symmetric powers of the standard representation of $\mathrm{SL}_2(K)$ are irreducible. ∎

Let $t \geq s$; for each Φ_s-module M there is the inflaction-restriction exact sequence

$$\begin{array}{ccccc} 0 & \longrightarrow & H^1(\Phi_s/\Phi_t, M^{\Phi_t}) & \xrightarrow{\text{infl}} & H^1(\Phi_s, M) \xrightarrow{\text{res}} H^1(\Phi_t, M)^{\Phi_s/\Phi_t} \longrightarrow \\ & \longrightarrow & H^2(\Phi_s/\Phi_t, M^{\Phi_t}) & \longrightarrow & H^2(\Phi_s, M) \end{array}$$

where $g \in \Phi_s/\Phi_t$ acts on $H^1(\Phi_t, M)$ by $(g\cdot\xi)(\gamma) = g\xi(g^{-1}\gamma g)$, for every 1-cocycle $\xi : \Phi_t \to M$. If $M =^t L(n,\mathcal{O})$ and $n \neq 0$ then by Lemma 6.2 $M^{\Phi_t} = 0$ and thus, if $n \neq 0$

$$\mathrm{Im}(\mathrm{res} : H^1(\Phi_s,{}^t L(n,\mathcal{O})) \to H^1(\Phi_t,{}^t L(n,\mathcal{O}))) = H^1(\Phi_t,{}^t L(n,\mathcal{O}))^{\Phi_s/\Phi_t}.$$

Therefore the restriction map from Φ_s to Φ_t sends $\Lambda_s^n(\mathcal{O})$ in $\Lambda_t^n(\mathcal{O})$ and the image is $\Lambda_t^n(\mathcal{O})^{\Phi_s/\Phi_t}$. In particular

$$\Lambda_t^n(\mathcal{O})/\mathrm{res}(\Lambda_s^n(\mathcal{O})) \text{ is } \mathcal{O}\text{-torsion-free if } n \neq 0. \tag{15}$$

Then we obtain the following result:

PROPOSITION 6.3. *If $s \leq t$ and $n \neq 0$ then the restriction map $V_s^n \to V_t^n$ (and so the restriction map $\mathcal{V}_s^n \to \mathcal{V}_t^n$) is injective.*

Proof. By definition $V_s^n \simeq H^1(\Phi_s,{}^t L(n,K))/\Lambda_s^n(\mathcal{O})$. Suppose that $x \in H^1(\Phi_s,{}^t L(n,K))$ and $\mathrm{res}(x) \in \Lambda_t^n(\mathcal{O})$. By (14) there exists a positive integer k such that $p^k x \in \Lambda_s^n(\mathcal{O})$, so that $\mathrm{res}(x) \in \mathrm{res}(\Lambda_s^n(\mathcal{O}))$ by (15). Notice that $\mathrm{res} : H^1(\Phi_s,{}^t L(n,K)) \to H^1(\Phi_t,{}^t L(n,K))$ is injective (this is evident, for example, from the fact that $\mathrm{cor} \circ \mathrm{res} = [\Phi_s : \Phi_t]$). Therefore $x \in \Lambda_s^n(\mathcal{O})$. ∎

If $n = 0$ then the Φ_s-action is trivial and the above argument does not work. However, we have the following result, which will be proved in Section 10.

THEOREM 6.4. *For every $n \in \mathbf{N}$, the inductive limit of the restriction maps*

$$I_s : H^1(\Phi_s, {}^t L(n, K/\mathcal{O})) \longrightarrow \varinjlim_t H^1(\Phi_t, {}^t L(n, K/\mathcal{O}))$$

has finite kernel.

We put $\mathcal{V}_\infty^n = \varinjlim_s \mathcal{V}_s^n$.

COROLLARY 6.5. a) *For every n, the Hecke algebra $\mathcal{H}_k(p^\infty, N; \mathcal{O})$ acts faithfully on $\mathcal{V}_\infty^n$.*

b) *If we give $\mathcal{V}_\infty^n$ the discrete topology, then the topology given by the projective limit and the topology induced by the compact-open topology in $\mathrm{End}_{\mathcal{O}}(\mathcal{V}_\infty^n)$ coincide over $\mathcal{H}_k(p^\infty, N; \mathcal{O})$.*

Proof. a) We saw that $\mathcal{H}_k(U_s, \mathcal{O})$ faithfully acts on $\mathcal{V}_s^n$, then taking the limit $\mathcal{H}_k(p^\infty, N; \mathcal{O})$ acts on $\mathcal{V}_\infty^n$. The action is faithful by Theorem 6.4, because $\mathcal{H}_k(U_s, \mathcal{O})$ has no $\mathcal{O}$-torsion.
b) Let K, A be subsets of $\mathcal{V}_\infty^n$, and suppose K finite (that is compact in the discrete topology); let $\Omega_{K,A}$ be the set of endomorphisms F of $\mathcal{V}_\infty^n$ such that $F(K) \subseteq A$. Then the subsets of the form $\Omega_{K,A}$ provide a basis for the compact-open topology in $\mathrm{End}_{\mathcal{O}}(\mathcal{V}_\infty^n)$. For each K, we can choose s and r such that $K \subseteq \mathrm{Im}(\mathcal{V}_s^n \to \mathcal{V}_\infty^n)$ and $K \subseteq \mathcal{V}_\infty^n[p^r]$. Let π_s be the projection $\mathcal{H}_k(p^\infty, N; \mathcal{O}) \to \mathcal{H}_k(U_s, \mathcal{O})$. If $T \in \mathcal{H}_k(p^\infty, N; \mathcal{O}) \cap \Omega_{K,A}$ then $T + \pi_s^{-1}(p^r \mathcal{H}_k(U_s, \mathcal{O})) \subseteq \Omega_{K,A}$. This shows that the sets $\Omega_{K,A}$ are open in $\mathcal{H}_k(p^\infty, N; \mathcal{O})$. Since the latter is compact and the compact-open topology is Hausdorff, claim b) is proved. ∎

The following theorem will be proved in section 10

THEOREM 6.6. *For every $n \geq 0$*

$$\mathcal{V}_\infty^n = \varinjlim_s H^1(\mathbf{X}_s, \mathcal{L}(n, K/\mathcal{O})).$$

7. THE EVALUATION MAP

Consider the evaluation map

$$\begin{array}{rccc} \iota : & L(n, K/\mathcal{O}) & \longrightarrow & K/\mathcal{O} \\ & P(X,Y) & \longmapsto & P(1,0) \end{array}$$

If $s \geq t$ then the action of U_s over $L(n, p^{-t}\mathcal{O}/\mathcal{O})$ is trivial. Thus, the restriction of ι gives rise to a homomorphism of U_s-modules $L(n, p^{-t}\mathcal{O}/\mathcal{O}) \to p^{-t}\mathcal{O}/\mathcal{O}$. This induces an homomorphism of sheaves $\mathcal{L}(n, p^{-t}\mathcal{O}/\mathcal{O})_{/\mathbf{X}_s} \longrightarrow \mathcal{L}(0, p^{-t}\mathcal{O}/\mathcal{O})_{/\mathbf{X}_s}$ and a map on the cohomology groups

$$\iota_* : H^1(\mathbf{X}_s, \mathcal{L}(n, p^{-t}\mathcal{O}/\mathcal{O})) \longrightarrow H^1(\mathbf{X}_s, \mathcal{L}(0, p^{-t}\mathcal{O}/\mathcal{O})).$$

Since $\mathbf{X}_s$ is compact for every s there is a canonical identification

$$H^1(\mathbf{X}_s, \mathcal{L}(n, K/\mathcal{O})) \simeq \varinjlim_t H^1(\mathbf{X}_s, \mathcal{L}(n, p^{-t}\mathcal{O}/\mathcal{O})),$$

(see [1, II, Corollary 14.5]). We fix s and consider the map

$$\bar{\iota}_s : H^1(\mathbf{X}_s, \mathcal{L}(n, K/\mathcal{O})) \longrightarrow \varinjlim_t H^1(\mathbf{X}_t, \mathcal{L}(0, K/\mathcal{O})) \tag{16}$$

obtained by composition in the following way:

$$\begin{aligned} H^1(\mathbf{X}_s, \mathcal{L}(n, K/\mathcal{O})) &\xrightarrow{\sim} \varinjlim_t H^1(\mathbf{X}_s, \mathcal{L}(n, p^{-t}\mathcal{O}/\mathcal{O})) \\ &\xrightarrow{\text{res}} \varinjlim_{t\geq s} H^1(\mathbf{X}_t, \mathcal{L}(n, p^{-t}\mathcal{O}/\mathcal{O})) \\ &\xrightarrow{\iota_*} \varinjlim_t H^1(\mathbf{X}_t, \mathcal{L}(0, p^{-t}\mathcal{O}/\mathcal{O})) \\ &\xrightarrow{\sim} \varinjlim_t H^1(\mathbf{X}_t, \mathcal{L}(0, K/\mathcal{O})). \end{aligned}$$

We shall prove the following result; it is the analogous of [7, Theorem 8.7] in our situation, and our proof will closely follow Hida's one.

THEOREM 7.1. *The kernel of $\bar{\iota}_s$ is finite.*

REMARK 7.2. a) By point b) of Remark 6.1, Theorem 7.1 will be proved if we show the finiteness of the kernel of the map in cohomology of groups

$$\bar{j}_s : H^1(\Phi_s, L(n, K/\mathcal{O})) \longrightarrow \varinjlim_t H^1(\Phi_t, K/\mathcal{O})$$

obtained by composition as follows

$$\begin{aligned} H^1(\Phi_s, {}^tL(n, K/\mathcal{O})) &\xrightarrow{\sim} \varinjlim_t H^1(\Phi_s, {}^tL(n, p^{-t}\mathcal{O}/\mathcal{O})) \\ &\xrightarrow{\text{res}} \varinjlim_t H^1(\Phi_t, {}^tL(n, p^{-t}\mathcal{O}/\mathcal{O})) \\ &\xrightarrow{\iota_*} \varinjlim_t H^1(\Phi_t, p^{-t}\mathcal{O}/\mathcal{O}) \\ &\xrightarrow{\sim} \varinjlim_t H^1(\Phi_t, K/\mathcal{O}) \end{aligned}$$

where this time ι_* is the map induced by ι in cohomology of groups.

b) Since the definition of $\bar{\iota}_s$ only involves restriction and functoriality of cohomology the map $\bar{\iota}_s$ is equivariant with respect to the action of Hecke operators $T_q, T_{q,q}$.

LEMMA 7.3. *There exists a positive integer $\bar{n} = \bar{n}(s)$ such that $p^{\bar{n}}V = 0$ for every $\mathcal{O}$-submodule V of $\ker(\iota)$ (contained in $L(n, K/\mathcal{O})$) which is stable for the Φ_s-action.*

Proof. Let M be the subalgebra of $\mathrm{End}_K(L(n, K))$ generated over $\mathcal{O}$ by the action of Φ_s. We show that for $j = 1, ..., n$ there exists $\phi_j \in M \otimes K$ such that for each $P = \sum_i \lambda_i X^i Y^{n-i} \in L(n, K)$ it holds $\phi_j(P) = \lambda_j X^n + ...$

If $\gamma \in \Phi_s$, let $\phi_\gamma \in \mathrm{End}_K(L(n, K))$ be the endomorphism corresponding to γ^{-1} and A_γ be the matrix of ϕ_γ with respect to the basis $X^n, X^{n-1}Y, ..., Y^n$ of $L(n, K)$. Since ${}^tL(n, K)$ is simple, the first rows of matrices A_γ generate K^{n+1}; otherwise there would be a non zero element $P(X, Y) \in L(n, K)$ such that $\Phi_s P(X, Y) \subseteq KX^{n-1}Y \oplus ... \oplus KY^n$. Therefore for $j = 0, ..., n$ the matrices A_γ generate over K a matrix A_j having on its first row $(0, ..., 1, ..., 0)$

with 1 in the j^{th} place; A_j is the matrix of the endomorphism ϕ_j we were looking for.
Let $\overline{n}$ be such that $p^{\overline{n}}\phi_j \in M$, for every j. Let $P = \sum_i a_i X^i Y^{n-i}$ be an element of V; then $p^{\overline{n}}\phi_j(P) = p^{\overline{n}}(a_j X^n + ...)$ belongs to V, for $j = 0, ..., n$, because V is Φ_s-stable. But $V \subseteq \ker(\iota)$, thus $p^{\overline{n}} a_j = 0$ for $j = 0, ..., n$; therefore $p^{\overline{n}} V = 0$. ∎

Proof of Theorem 7.1: Since the action of Φ_t over $L(n, p^{-t}\mathcal{O}/\mathcal{O})$ is trivial, the image of the restriction map I_s defined in Theorem 6.4 is contained in $\varinjlim_t \mathrm{Hom}_{\Phi_s}(\Phi_t, L(n, p^{-t}\mathcal{O}/\mathcal{O}))$. If $\xi \in \mathrm{Hom}_{\Phi_s}(\Phi_t, L(n, p^{-t}\mathcal{O}/\mathcal{O}))$, then $\xi(u\delta u^{-1}) = u\xi(\delta)$, for every $u \in \Phi_s$, $\delta \in \Phi_t$. Thus if $\iota \circ \xi = 0$ then $\iota(u\xi) = 0$. Therefore the image of ξ is contained in a $(\Phi_s, \mathcal{O})$-submodule of $\ker(\iota)$ in $L(n, K/\mathcal{O})$. By Lemma 7.3 we deduce that $p^{\bar{n}}\xi = 0$. In particular, if ξ is a cocycle in $Z^1(\Phi_s, L(n, K/\mathcal{O}))$ and the class of ξ in $H^1(\Phi_s, L(n, K/\mathcal{O}))$ belongs to $\ker(\bar{\iota}_s)$ then $\iota \circ I_s(\xi) = 0$, thus $p^{\bar{n}} I_s(\xi) = 0$. We deduce that $\ker(\bar{\iota}_s) \subseteq p^{-\bar{n}} \ker(I_s)$, which is finite by Theorem 6.4. ∎

8. Consequences for the Hecke algebra

Theorem 8.1. a) *For every positive integer s the homomorphism $\bar{\iota}_s$ defined in (16) induces a continuous surjective homomorphism of $\mathcal{O}[[\mathbf{Z}_p^\times]]$-algebras*

$$\bar{\iota}_s^* : \mathcal{H}_2(p^\infty, N; \mathcal{O}) \longrightarrow \mathcal{H}_k(U_s, \mathcal{O})$$

such that, for every prime $q \neq p$, $\bar{\iota}_s^*(T_q) = T_q$. The homomorphism $\bar{\iota}_s^*$ is compatible with the restriction $\mathcal{H}_k(U_t, \mathcal{O}) \to \mathcal{H}_k(U_s, \mathcal{O})$, if $t \geq s$.

b) *The projective limit of the homomorphisms $\bar{\iota}_s^*$:*

$$\bar{\iota}^* : \mathcal{H}_2(p^\infty, N; \mathcal{O}) \longrightarrow \mathcal{H}_k(p^\infty, N; \mathcal{O})$$

is an isomorphism.

Proof. a) We put $\mathcal{W}_s^n = \bar{\iota}_s(\mathcal{V}_s^n)$. The homomorphism $\bar{\iota}_s$ is a $\mathcal{R}(N)$-equivariant isogeny between $\mathcal{W}_s^n$ and $\mathcal{V}_s^n$. Since the algebra $\mathcal{H}_k(U_s, \mathcal{O})$ has no torsion, it faithfully acts over $\mathcal{W}_s^n$, and it has the compact-open topology with respect to the discrete module $\mathcal{W}_s^n$. Let A be the dense subalgebra of $\mathcal{H}_2(p^\infty, N; \mathcal{O})$ generated over $\mathcal{O}$ by $\mathcal{R}(N)$; then the restriction of operators in A to the module $\mathcal{W}_s^n$ gives a surjective homomorphism $A \to \mathcal{H}_k(U_s, \mathcal{O})$, which is continuous by Corollary 6.5; therefore it extends to a continuous surjective homomorphism of the algebra $\mathcal{H}_2(p^\infty, N; \mathcal{O})$ onto $\mathcal{H}_k(U_s, \mathcal{O})$.
b) The homomorphism $\bar{\iota}_s^*$ is surjective for every s and $\mathcal{H}_2(p^\infty, N; \mathcal{O})$ is compact; therefore the image of $\bar{\iota}^*$ is a dense and closed subset of $\mathcal{H}_k(p^\infty, N; \mathcal{O})$; thus $\bar{\iota}^*$ is surjective. We prove now that it is injective: define

$$\begin{array}{rcl} \kappa : p^{-t}\mathcal{O}/\mathcal{O} & \longrightarrow & L(n, p^{-t}\mathcal{O}/\mathcal{O}) \\ a & \longmapsto & aX^n; \end{array}$$

Then $\iota \circ \kappa = id$ and κ is a homomorphism of Φ_s-modules if $t \leq s$. Therefore $\iota_* : H^1(\mathbf{X}_s, \mathcal{L}(n, p^{-s}\mathcal{O}/\mathcal{O})) \to H^1(\mathbf{X}_s, \mathcal{L}(0, p^{-s}\mathcal{O}/\mathcal{O}))$ is surjective for every s. By Theorem 6.6 and Corollary 6.5 $\mathcal{H}_k(p^\infty, N; \mathcal{O})$ faithfully acts over $\varinjlim_s H^1(\mathbf{X}_s, \mathcal{L}(n, K/\mathcal{O}))$. Then the surjectivity of ι_* implies that $\bar{\iota}^*$ is injective. ∎

9. Proof of Theorem 6.4

If $n \neq 0$ then by Proposition 6.3 $V_s^n \cap \ker(I_s) = \{0\}$. Therefore $\ker(I_s)$ is injectively sent in $H^2(\Phi_s, {}^t L(n, \mathcal{O}))$ in the long exact sequence associated to sequence (12). By [17, Propositions 8.1 and 8.2], since Φ_s has no torsion, there is an isomorphism

$$H^2(\Phi_s, {}^t L(n, \mathcal{O})) \simeq H_0(\Phi_s, {}^t L(n, \mathcal{O})), \tag{17}$$

and the latter is the group of coinvariants of ${}^t L(n, \mathcal{O})$, which is finite by Lemma 6.2.

The proof for weight 2 (corresponding to $n = 0$) is the following adaptation of a theorem of Shimura [15, 3.2.1]. We can suppose that K is the p-adic closure of K_0 in $\overline{\mathbf{Q}}_p$ (according to the fixed immersion $\bar{\mathbf{Q}} \to \bar{\mathbf{Q}}_p$.) Indeed, let K' be a finite extension of K and $\mathcal{O}'$ be its integer ring; then $H^1(\Phi_s, K'/\mathcal{O}') = H^1(\Phi_s, K/\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}'$ by the universal coefficients theorem. The kernel of the restriction map $H^1(\Phi_s, K'/\mathcal{O}') \to H^1(\Phi_t, K'/\mathcal{O}')$ is the image of

$$\ker(\mathrm{res} : H^1(\Phi_s, K/\mathcal{O}) \longrightarrow H^1(\Phi_t, K/\mathcal{O})) \otimes_{\mathcal{O}} \mathcal{O}';$$

if the claim is proved for K, then its order is then bounded by a constant which does not depend on t.

Let L be a finite Galois extension of $\mathbf{Q}$ satisfying

- L is totally real and linearly disjoint from K_0 over $\mathbf{Q}$;
- every prime factor of $N\Delta\mathrm{Disc}(K_0)$ totally split in L;
- $d = [L : \mathbf{Q}] \geq 2$.

We put $B' = B \otimes_{\mathbf{Q}} L$; it is a division algebra over L, of discriminant Δ. Let $K_0' = L \otimes_{\mathbf{Q}} K_0$; then K_0' is a field, because we assumed that $L \cap K_0 = \mathbf{Q}$; let $\mathcal{O}_L$ be the ring of integers of L and $\mathcal{O}_0' = \mathcal{O}_L \otimes_{\mathbf{Z}} \mathcal{O}_0$ be the ring of integers of K_0' (because $\mathrm{Disc}(K_0)$ is prime to $\mathrm{Disc}(L)$, see for example [12, III,§3, Prop. 17]). We put $R' = R \otimes_{\mathbf{Z}} \mathcal{O}_L$. The p-adic closure of K_0' in $\mathbf{Q}_p$ coincide with K. The embedding $K_0' \hookrightarrow K$ corresponds to a prime ideal $\mathfrak{p}|p$ of $\mathcal{O}_0'$. We put $\mathcal{O}_0'' = \mathcal{O} \cap K_0' = \mathcal{O}'_{0,(\mathfrak{p})}$.

Let $N = \prod_{j=1}^m q_j^{e_j}$ be the decomposition of N in a product of powers of primes; we choose for every j $(1 \leq j \leq m)$ a prime ideal $\mathfrak{q}_j$ of L over q_j. Since the primes dividing N totally split over L, the isomorphisms i_{q_j} introduced in Section 2 induce isomorphisms

$$i_j' : R' \otimes_{\mathcal{O}_L} \mathcal{O}_{L,\mathfrak{q}_j} \longmapsto \mathrm{M}_2(\mathbf{Z}_{q_j}).$$

We define

$$R'(N)_{\mathfrak{q}_j} = (i_j')^{-1} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbf{Z}_{q_j}) \mid q_j^{e_j} \text{ divides } c \right\}.$$

Since p totally splits in L

$$B_{\mathfrak{p}}' = B' \otimes_{\mathcal{O}_L} \mathbf{Q}_p = B_p.$$

We define the compact open subgroup of $(B'_{\mathbf{A}_L})^{\times,\infty} = (B' \otimes_L \mathbf{A}_L^\infty)^\times$

$$U'_s = \prod_{\mathfrak{q}\neq\mathfrak{p},\mathfrak{q}_i} (R'_\mathfrak{q})^\times \times \prod_{i=1}^m R'(N)_{\mathfrak{q}_i}^\times \times (1+p^s R_p)$$

and its subgroup

$$\Phi'_s = (\mathrm{GL}_2(\mathbf{R})^d \times U'_s) \cap B'^{(1)}.$$

There is a natural immersion $B \hookrightarrow B' = B \otimes_\mathbf{Q} L$. We can easily see that $\Phi'_s \cap B = \Phi_s$, for every s. Therefore, if $t \geq s$, there is a natural injection

$$\Psi : \Phi_s/\Phi_t \longrightarrow \Phi'_s/\Phi'_t.$$

Proposition 9.1. *The map Ψ is surjective.*

Proof. By Proposition 2.2 we know that Φ_s is dense in $(1+p^sR_p)^{(1)}$. Moreover, the groups $(1+p^tR_p)^{(1)}$, for $t \geq s$, are a fondamental system of neightbours of unity in $(1+p^sR_p)^{(1)}$. If $x \in (1+p^sR_p)^{(1)}$, then there exists an element $y \in \Phi_s$ such that $x^{-1}y \in (1+p^tR_p)^{(1)}$, so that the natural homomorphism $\Phi_s/\Phi_t \to (1+p^sR_p)^{(1)}/(1+p^tR_p)^{(1)}$ is an isomorphism. Proposition 2.2 holds also for a quaternion algebra over the number field L; since p totally splits in L we have that Φ'_s is dense in $(1+p^sR_p)^{(1)}$, so that $\Phi'_s/\Phi'_t \simeq (1+p^sR_p)^{(1)}/(1+p^tR_p)^{(1)}$. ■

There is an isomorphism $K'_0/\mathcal{O}''_0 \simeq K/\mathcal{O}$; therefore

$$H^1(\Phi_s/\Phi_t, K/\mathcal{O}) \simeq H^1(\Phi'_s/\Phi'_t, K'_0/\mathcal{O}''_0). \tag{18}$$

By the inflaction-restriction sequence, the kernel of the restriction map $H^1(\Phi_s, K/\mathcal{O}) \to H^1(\Phi_t, K/\mathcal{O})$ is isomorphic to $H^1(\Phi_s/\Phi_t, K/\mathcal{O})$, because the Φ_t-action over $K/\mathcal{O}$ is trivial. Therefore, in order to prove the finiteness of the kernel of I_s it suffices to show that

$$\left|H^1(\Phi_s/\Phi_t, K/\mathcal{O})\right| \leq M \tag{19}$$

where M is a constant not depending on t. By the isomorphism (18) this is equivalent to show that

$$\left|H^1(\Phi'_s/\Phi'_t, K'_0/\mathcal{O}''_0)\right| \leq M \tag{20}$$

with M not depending on t. Since $H^1(\Phi'_s/\Phi'_t, K'_0/\mathcal{O}''_0)$ injects by inflaction in $H^1(\Phi'_s, K'_0/\mathcal{O}''_0)$, we are reduced to show that the latter is a finite group.

Consider the exact sequence

$$0 \longrightarrow \mathcal{O}''_0 \longrightarrow K'_0 \longrightarrow K'_0/\mathcal{O}''_0 \longrightarrow 0.$$

It induces a long exact sequence in cohomology

$$H^1(\Phi'_s, K'_0) \longrightarrow H^1(\Phi'_s, K'_0/\mathcal{O}''_0) \xrightarrow{\delta} H^2(\Phi'_s, \mathcal{O}''_0).$$

The central group is torsion, therefore its image by δ is contained in $H^2(\Phi'_s, \mathcal{O}''_0)_{\mathrm{tors}}$, which finite, because it is a torsion submodule of a finitely generated module over a principal ideal domain. It remains to show that $H^1(\Phi'_s, K'_0)$ is finite, i.e. that it is zero. Now, $H^1(\Phi'_s, K'_0)$ maps injectively to $H^1(\Phi'_s, K'_0 \otimes_{K_0} \mathbf{C})$. We have $(B' \otimes_\mathbf{Q} \mathbf{R})^\times \simeq \mathrm{GL}_2(\mathbf{R})^d$ and this isomorphism identifies Φ'_s to a discrete irreducible subgroup of $\mathrm{SL}_2(\mathbf{R})^d$ (see [20, IV, §1]). By [13, Théorème 7.1] we deduce that $H^1(\Phi'_s, K'_0 \otimes_{K_0} \mathbf{C}) = 0$. ■

REMARK 9.2. This proof works for every n.

10. Proof of theorem 6.6

By (13) the theorem is trivially true if $n = 0$.
By appying the inductive limit to the exact sequences

$$0 \to \mathcal{V}_s^n \to H^1(\mathbf{X}_s, \mathcal{L}(n, K/\mathcal{O})) \to H^2(\mathbf{X}_s, \mathcal{L}(n, \mathcal{O})) \to 0$$

we see that the theorem will be proved if we show that $\varinjlim_s H^2(\Phi_s, {}^t L(n, \mathcal{O}))$ is zero if $n \neq 0$.

10.1. **The index of Φ_{s+1} in Φ_s.** By definition, the ring $\mathcal{O}$ contains $\mathbf{Z}_{p^2}$. Let

$$S_p = \left\{ \begin{pmatrix} a & b \\ pb^\sigma & a^\sigma \end{pmatrix} \mid a, b \in \mathbf{Z}_{p^2} \right\} \subseteq \mathrm{M}_2(\mathcal{O}).$$

As we have seen in the proof of Lemma 6.2, the two rings R_p and S_p are conjugated by a matrix δ_0 of $\mathrm{GL}_2(K)$: therefore we have

$$S_p = \delta_0 R_p {\delta_0}^{-1}, \quad S_p^\times = \delta_0 R_p^\times {\delta_0}^{-1}, \quad 1 + p^s S_p = \delta_0 (1 + p^s R_p) {\delta_0}^{-1},$$

for every positive integer s. We shall denote by $[a, b]$ the element $\begin{pmatrix} a & b \\ pb^\sigma & a^\sigma \end{pmatrix}$ of $\mathrm{M}_2(\mathbf{Z}_{p^2})$. Let u be the matrix $[0, 1] \in S_p$; then $u^2 = p$. The map

$$\begin{array}{rcl} S_p & \longrightarrow & \mathbf{Z}_{p^2} \\ [a, b] & \longmapsto & a \end{array}$$

induce a isomorphism of rings

$$S_p / u S_p \xrightarrow{\sim} \mathbf{Z}_{p^2} / p \mathbf{Z}_{p^2} = \mathbf{F}_{p^2}.$$

There is also an isomorphism of groups

$$\begin{array}{rcl} 1 + u^s S_p / 1 + u^{s+1} S_p & \xrightarrow{\sim} & S_p / u S_p \\ 1 + u^s \alpha & \longmapsto & \alpha, \end{array}$$

Therefore $1 + u^s S_p / 1 + u^{s+1} S_p$ is isomorphic to $\mathbf{F}_{p^2}$.

If $s = 2r$ is even, then the reduced norm $\nu : 1 + u^s S_p \to 1 + p^r \mathbf{Z}_p$ is surjective, since S_p contains all matrices of the form $[a, 0]$ with $a \in \mathbf{Z}_{p^2}$. If $s = 2r + 1$ is odd, then $\nu(1 + u^s S_p) = 1 + p^{r+1} \mathbf{Z}_p$, because $\mathrm{trace}(u S_p) \subseteq p \mathbf{Z}_p$. We study now the index $[(1 + u^s S_p)^{(1)} : (1 + u^{s+1} S_p)^{(1)}]$:

- If $s = 2r$ is even, then there is an exact sequence

  $$0 \to C \to 1 + u^s S_p / 1 + u^{s+1} S_p \xrightarrow{\nu} 1 + p^r \mathbf{Z}_p / 1 + p^{r+1} \mathbf{Z}_p \to 0$$

  where

  $$C = \{\alpha \in 1 + u^s S_p \mid \nu(\alpha) \in 1 + p^{r+1} \mathbf{Z}_p\} / 1 + u^{s+1} S_p.$$

  The natural inclusion $(1 + u^s S_p)^{(1)} / (1 + u^{s+1} S_p)^{(1)} \hookrightarrow C$ is surjective; therefore

  $$[(1 + u^s S_p)^{(1)} : (1 + u^{s+1} S_p)^{(1)}] = p \quad \text{if } s \text{ is even.}$$

  A system of representatives of the cosets of $(1 + u^{s+1} S_p)^{(1)}$ in $(1 + u^s S_p)^{(1)}$ is given by the elements of the form $\alpha_i \gamma_i$, where $\alpha_i = 1 +$

$u^s[a_i, 0]$, the a_i's vary in the set of trace 0 elements in $\mathbf{F}_{p^2}$ and the $\gamma_i \in 1 + u^{s+1}S_p$ are such that $\nu(\alpha_i\gamma_i) = 1$.

- If $s = 2r + 1$ is odd, by the same argument there is an isomorphism

  $$(1 + u^s S_p)^{(1)}/(1 + u^{s+1}S_p)^{(1)} \xrightarrow{\sim} 1 + u^s S_p/1 + u^{s+1}S_p$$

  therefore

  $$[(1 + u^s S_p)^{(1)} : (1 + u^{s+1}S_p)^{(1)}] = p^2 \quad \text{if } s \text{ is odd.}$$

  A system of representatives of the cosets of $(1 + u^{s+1}S_p)^{(1)}$ in $(1 + u^s S_p)^{(1)}$ is given by the elements of the form $\alpha_i\gamma_i$, where $\alpha_i = 1 + u^s[a_i, 0]$, the a_i's vary in $\mathbf{F}_{p^2}$ and the $\gamma_i \in 1 + u^{s+1}S_p$ are such that $\nu(\alpha_i\gamma_i) = 1$.

If Φ' is the p-adic closure of Φ in $R_p^\times$, then by Proposition 2.2 we have $\Phi_s/\Phi_{s+1} \simeq \Phi'_s/\Phi'_{s+1} \simeq (1 + p^s R_p)^{(1)}/(1 + p^{s+1}R_p)^{(1)}$, so that

$$[\Phi_s : \Phi_{s+1}] = p^3.$$

10.2. **The action of $S_p^\times$ over $L(n, \mathcal{O})$.** Let G be a group, M be a G-module and let $D(G, M)$ be the subgroup of M generated by the $\gamma m - m$, $m \in M$, $\gamma \in G$. Then $H_0(G, M) = M/D(G, M)$ is the biggest quotient of M over which G acts trivially. For every positive integer s, let $G_s = (1+p^s S_p)^{(1)}$. We let $S_p^\times$ act on $L(n, \mathcal{O})$ by the matrices $[a, b]$. We put $D'_s = D(G_s, L(n, \mathcal{O}))$; then $D'_s \subseteq p^s L(n, \mathcal{O})$. If $s \leq t$ we consider the transfer map:

$$\begin{array}{rcl} Tr_{G_s/G_t} : L(n, \mathcal{O})/D'_s & \longrightarrow & L(n, \mathcal{O})/D'_t \\ P & \longmapsto & \displaystyle\sum_\gamma \gamma^{-1}P \end{array}$$

for a decomposition $G_s = \coprod_\gamma \gamma G_t$.

Proposition 10.1. *If $n > 0$ then there exists a positive integer l_0, depending only on n, such that if $s > l_0$*

$$p^{s+l_0}L(n, \mathcal{O}) \subseteq D'_s.$$

Proof. We choose an element $x \in \mathbf{Z}_{p^2}^\times$ such that $N(1 + p^s x) = 1$, where N denotes the norm of $\mathbf{Q}_{p^2}$ over $\mathbf{Q}_p$. Such an element exists: in fact, let $m \in \mathbf{Z}_p$ such that $p \not| m$ and $-m$ is not a square modulo p; then the polynomial $X^2 + p^s mX + m$ has discriminant $m(p^{2s}m - 4)$, which is a unity and a non-square in $\mathbf{Q}_p$, because p is odd; therefore it is irreducible over $\mathbf{Q}_p$ and its roots are in $\mathbf{Z}_{p^2}$. If x is a root, then $p^s N(x) = p^s m = -\mathrm{tr}(x)$; so $1 + p^s x$ has norm $1 + p^s\mathrm{tr}(x) + p^{2s}N(x) = 1$. The matrix $\alpha = [1 + p^s x, 0]$ belongs to G_s. For $i = 0, ..., n$ we have

(21)
$$\begin{array}{rcl} \alpha^{-1}X^iY^{n-i} - X^iY^{n-i} & = & (1 + p^s x)^i(1 + p^s x^\sigma)^{n-i}X^iY^{n-i} - X^iY^{n-i} \\ & = & (p^s(2i - n)x + p^{2s}(...))X^iY^{n-i}, \end{array}$$

since $x^\sigma \equiv -x \bmod p^s$. Let l be such that $p^l > n$; since $|2i - n| \leq n$, if $2i-n \neq 0$, we have $v_p(2i-n) < l$; if $s > l$ then the order at p of the coefficient of X^iY^{n-i} in (21) is equal to $sv_p(2i - n)$, therefore $p^{s+l}X^iY^{n-i} \in D'_s$ if $2i - n \neq 0$. Let $L'(n, \mathcal{O})$ be the sub-$\mathcal{O}$-module of $L(n, \mathcal{O})$ generated by the X^iY^{n-i}, for $i \neq n/2$; the argument above shows that $p^{s+l}L'(n, \mathcal{O}) \subseteq D'_s$ if

$s > l$; it remains to find l_0 such that $p^{s+l_0}X^{n/2}Y^{n/2} \in D'_s$ if n is even and $s > l_0$. Consider the matrix $\beta = [1, p^s] \in 1 + p^s S_p$; then $\nu(\beta) \in 1 + p^{2s+1}\mathbf{Z}_p$; we choose $\delta \in 1 + p^{2s}S_p$ such that $\nu(\beta\delta) = 1$; so $\gamma = \beta\delta$ is in G_s. If $P \in L(n, \mathcal{O})$ then $\gamma^{-1}P \equiv \beta^{-1}P \bmod p^{2s}L(n, \mathcal{O})$. If n is even, put $i = \frac{n-2}{2}$; then

$$\begin{aligned}
&\gamma^{-1}X^iY^{n-i} - X^iY^{n-i} \equiv \\
&\equiv (X + p^sY)^i(p^{s+1}X + Y)^{n-i} - X^iY^{n-i} \\
&\equiv (X^i + p^s iX^{i-1}Y)(Y^{n-i} + p^{s+1}(n-i)XY^{n-i-1}) - X^iY^{n-i} \\
&\equiv p^{s+1}(n-i)X^{i+1}Y^{n-i-1} + p^s iX^{i-1}Y^{n-i+1} \bmod p^{2s}L(n, \mathcal{O}).
\end{aligned}$$

Since $i + 1 = n/2$ we find that

$$\begin{aligned}
&((n-i)p^{s+1} + p^{2s}z)X^{n/2}Y^{n/2} = \\
&= (\gamma^{-1}X^iY^{n-i} - X^iY^{n-i}) + p^{2s}Q - p^s iX^{i-1}Y^{n-i+1}
\end{aligned}$$

where $z \in \mathcal{O}$, $Q \in L'(n, \mathcal{O})$, therefore

$$((n-i)p^{s+1} + p^{2s}z)X^{n/2}Y^{n/2} \equiv -p^s iX^{i-1}Y^{n-i+1} \bmod D'_s \quad \text{if } s > l.$$

Since $n - 2i = 2$, $n - i$ and i have the same p-adic valuation, and $X^{i-1}Y^{n-i+1} \in L'(n, \mathcal{O})$, multiplying by $p^{l-v_p(i)}$ we find that, if $s > l$ then $p^{s+l+1}X^{n/2}Y^{n/2} \in D'_s$; therefore $l_0 = l + 1$ is the number we were looking for. ∎

We have seen that $[G_s : G_{s+1}] = p^3$; since G_s acts trivially over $L(n, \mathcal{O}/p^r\mathcal{O})$ if $r \leq s$ we have

$$Tr_{G_s/G_{s+1}}L(n, \mathcal{O}) \subseteq p^3L(n, \mathcal{O}) \quad \text{if } s \geq 3$$

therefore

$$Tr_{G_s/G_{s+k}}L(n, \mathcal{O}) \subseteq p^{3k}L(n, \mathcal{O})$$

for a suitable choice of representatives of the cosets of G_{s+k} in G_s. By Proposition 10.1 we obtain, if $s > l_0$, $s > 3$, $2k > s + l_0$

$$Tr_{G_s/G_{s+k}}L(n, \mathcal{O}) \subseteq p^{3k}L(n, \mathcal{O}) \subseteq D'_{s+k}.$$

Therefore we obtain the following result:

PROPOSITION 10.2.

$$\varinjlim_s H_0(G_s, {}^tL(n, \mathcal{O})) = 0$$

where the inductive limit is taken with respect to the maps $Tr_{G_s/G_{s+k}}$.

Let now M be a cotorsion $\mathcal{O}$-submodule of $L(n, \mathcal{O})$, stable for the action of $S_p^\times$. For every positive integer s, put $D_s^M = D(G_s, M)$. Let c be such that $p^cL(n, \mathcal{O}) \subseteq M$; then $p^cD'_s \subseteq D_s^M$, for every s; put $d_0 = l_0 + c$; then

$$p^{s+d_0}M \subseteq p^cD'_s \subseteq D_s^M \quad \text{if } s > l_0.$$

Since $Tr_{G_s/G_{s+1}}$ multiplies by p^3 we have also

$$\varinjlim_s H_0(G_s, M) = 0. \tag{22}$$

10.3. **Triviality of $\varinjlim_s H^2(\Phi_s,{}^t L(n,\mathcal{O}))$ for $n \neq 0$.** By isomorphism (17) the restriction map

$$\mathrm{res}_{\Phi_s/\Phi_t} : H^2(\Phi_s,{}^t L(n,\mathcal{O})) \longrightarrow H^2(\Phi_t,{}^t L(n,\mathcal{O}))$$

corresponds to the transfer

$$Tr_{\Phi_s/\Phi_t} : L(n,\mathcal{O})/D_s \longrightarrow L(n,\mathcal{O})/D_t,$$

where $D_s = D(\Phi_s,{}^t L(n,\mathcal{O}))$.

Let Φ'_s be the p-adic closure of Φ_s in $\mathrm{GL}_2(\mathcal{O})$ and $\tilde{D}_s = D(\Phi'_s,{}^t L(n,\mathcal{O}))$.

Lemma 10.3. *$D_s = \tilde{D}_s$.*

Proof. Of course $D_s \subseteq \tilde{D}_s$. Since $L(n,K)$ is a simple Φ_s-module, and D_s is Φ_s-invariant, $H^2(\Phi_s,{}^t L(n,\mathcal{O}))$ is finite; therefore D_s is open in $L(n,\mathcal{O})$ and there exists a positive integer k such that $p^k L(n,\mathcal{O}) \subseteq D_s$. If $\gamma \in \Phi'_s$ then by Proposition 2.2 there exists $\delta \in \Phi_s$ such that $\delta^{-1}\gamma \in \Phi'_k$. Therefore for every $P \in L(n,\mathcal{O})$

$$\gamma P - P = \delta(\delta^{-1}\gamma P - P) + \delta P - P \in D_s,$$

because $\delta P - P \in D_s$, $\delta^{-1}\gamma P - P \in p^k L(n,\mathcal{O})$ and D_s is Φ_s-invariant. ■

By Proposition 2.2, $\Phi_s/\Phi_t = \Phi'_s/\Phi'_t$. Therefore $\mathrm{res}_{\Phi_s/\Phi_t} L(n,\mathcal{O}) \subseteq p^{3(t-s)} L(n,\mathcal{O})$, because Φ'_s acts trivially over $L(n,\mathcal{O}/p^r\mathcal{O})$ if $r \leq s$, and $[\Phi_s : \Phi_t] = p^{3(t-s)}$. We have seen in the proof of Lemma 6.2 that there exists a matrix $\delta_0 \in \mathrm{GL}_2(K)$ such that $\delta_0 \Phi'_s \delta_0^{-1} = G_s$ for every positive integer s. There is a commutative diagram

$$\begin{array}{ccc} L(n,\mathcal{O}) & \overset{\alpha \in R_p^\times}{\longrightarrow} & L(n,\mathcal{O}) \\ \delta_0 \downarrow & & \downarrow \delta_0 \\ \delta_0 L(n,\mathcal{O}) & \overset{\delta_0 \alpha \delta_0^{-1} \in S_p^\times}{\longrightarrow} & \delta_0 L(n,\mathcal{O}) \end{array} \tag{23}$$

Proposition 10.4. *$\varinjlim_s H^2(\Phi_s,{}^t L(n,\mathcal{O})) = 0$; therefore $\varinjlim_s H^2(\mathbf{X}_s,\mathcal{L}(n,\mathcal{O})) = 0$.*

Proof. By structure transfer using diagram (23) it suffices to prove that for every $s \gg 0$ there exists a $t \geq s$ such that

$$Tr_{G_s/G_t} : H_0(G_s, \delta_0^t L(n,\mathcal{O})) \longrightarrow H_0(G_t, \delta_0^t L(n,\mathcal{O}))$$

is zero. Let b such that $p^b \delta_0 L(n,\mathcal{O}) \subseteq L(n,\mathcal{O})$. Then $p^b \delta_0 L(n,\mathcal{O})$ is a sub-module of $L(n,\mathcal{O})$ stable for the $S_p^\times$-action, and it is isomorphic to the $R_p^\times$-module $L(n,\mathcal{O})$ by multiplication for $p^{-b}\delta_0^{-1}$. By 22 there exists t such that

$$Tr_{G_s/G_t} p^b \delta_0 L(n,\mathcal{O}) \subseteq D(G_t, p^b \delta_0^t L(n,\mathcal{O})).$$

Therefore $Tr_{\Phi_s/\Phi_t} L(n,\mathcal{O}) \subseteq \tilde{D}_t = D_t$. ■

Acknowledgments

This work was supported by the framework of PRIN 2010/11 "Geometria delle varietà algebriche", cofinanced by MIUR.
The author would like to thank the anonymous referee for his/her thorough review and very useful comments that helped improve the clarity and the relevance of this paper.

References

[1] G. E. Bredon, *Sheaf Theory*, second ed., GTM 170, (Springer, New York, 1997).
[2] F. Calegari and M. Emerton, Completed cohomology - a survey, in *Non-abelian fundamental groups and Iwasawa theory*, London Math. Soc. Lecture Note Ser., vol. 393, (Cambridge Univ. Press, 2012), pp. 239–257.
[3] M. Ciavarella, Congruences between modular forms and related modules, *Funct. Approx. Comment. Math.* **41**(1) (2009) 55–70.
[4] S. Gelbart, *Automorphic forms on adele groups*, Annals of Math. Studies, vol. 83, (Princeton Univ. Press, 1975).
[5] H. Hida, Galois representations into $\mathrm{GL}_2(\mathbf{Z_p}[[\mathbf{X}]])$ attached to ordinary cusp forms, *Invent. Math.* **85** (1986) 545–613.
[6] H. Hida, Iwasawa modules attached to congruences of cusp forms, *Ann. Sci. Éc. Norm. Sup. (*4^e *série)* **19**, (1986), 231–273.
[7] H. Hida, Modules of congruence of Hecke algebras and L-functions associated with cusp forms, *Amer. J. Math.* **110** (1988) 323–382.
[8] H. Hida, On p-adic Hecke algebras for GL_2 over totally real fields, *Ann. of Math.* **128** (1988) 295–384.
[9] H. Hida, On nearly ordinary Hecke algebras for GL(2) over totally real fields, *Adv. Studies Pure Math.* **17** (1989) 139–169.
[10] H. Hida, *Elementary theory of L-functions and Eisenstein series*, (Cambridge University Press, 1993).
[11] H. Jacquet and R. Langlands, *Automorphic forms on* GL_2, Lecture Notes Math. vol. 114 (Springer, 1970).
[12] S. Lang, *Algebraic Number Theory*, Graduate Texts in Mathematics vol. 110, (Springer, New York, 1986).
[13] Y. Matsushima and G. Shimura, On the cohomology groups attached to certain vector valued differential forms on the product of the upper half planes, *Ann. of Math.* **78** (1963) 417–449.
[14] A. Mori and L. Terracini, A canonical map between Hecke algebras. *Boll. Un. Mat. Ital. (8) 2-B* (1999) 429–452.
[15] M. Ohta, On ℓ-adic representations attached to automorphic forms. *Japan. J. Math* **8** (1982) 1–47.
[16] J.-P. Serre, Cohomologie des groupes discrets, in *Prospects in mathematics (Proc. Sympos., Princeton Univ., Princeton, N.J.)*, Ann. of Math. Studies vol. 70 , (Princeton Univ. Press, 1971), pp. 77–169.
[17] G. Shimura, *Introduction to the arithmetic theory of automorphic functions*, (Iwanami Shoten and Princeton Univ. Press, 1971).
[18] L. Terracini, Sur quelques propriétés des algèbres de Hecke quaternioniques. *Boll. Un. Mat. Ital. (8) 2-B* (2002) 677–700.
[19] L. Terracini, A Taylor-Wiles system for quaternionic Hecke algebras. *Compositio Mathematica* **137** (2003) 23–47.
[20] M.F. Vignéras, *Arithmétique des algèbres de quaternions*, Lecture Notes in Mathematics vol. 800, (Springer, 1980).

Dipartimento di Matematica, Università di Torino, Via Carlo Alberto 10, 10123 Torino,Italy, e-mail: lea.terracini@unito.it