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Numerical integration based on trivariate C^2 quartic spline quasi-interpolants

Catterina Dagnino · **Paola Lamberti** · **Sara Remogna**

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Abstract In this paper we consider the space generated by the scaled translates of the trivariate C^2 quartic box spline B defined by a set X of seven directions, that forms a regular partition of the space into tetrahedra. Then, we construct new cubature rules for 3D integrals, based on spline quasi-interpolants expressed as linear combinations of scaled translates of B and local linear functionals.

We give weights and nodes of the above rules and we analyse their properties.

Finally, some numerical tests and comparisons with other known integration formulas are presented.

Keywords 3D Cubature · Trivariate box spline · Quasi-interpolation operator · Spline approximation

Mathematics Subject Classification (2000) MSC 65D07 · MSC 65D32 · MSC 41A15

1 Introduction

The numerical evaluation of integrals is one of the corner stones in Numerical Analysis and it is also an important tool in methods to solve integral and differential problems. In particular there is a wide literature concerning the numerical evaluation of integrals based on spline approximation. Indeed, splines have been used for numerical integration ever since they entered on the Numerical Analysis scene [19]. We recall, for instance, the papers [1,5–7,11,15,27,29], where quadrature formulas based on spline interpolants and quasi-interpolants (QIs) of different degrees are considered, also for singular integrals. Concerning the numerical evaluation of 2D integrals, we

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mention the cubatures proposed in [8–10, 22, 30, 32], based on tensor product of univariate splines, on C^1 quadratic and C^2 quartic quasi-interpolating splines, defined on criss-cross triangulations and on Powell-Sabin partitions. Furthermore, numerical integration over polygons using an eight-node quadrilateral spline finite element is presented and studied in [23–25].

Finally, we recall [13, 14], where cubature rules for a parallelepiped domain are defined by integrating tensor product of univariate C^1 quadratic spline QIs and blending sums of C^1 quadratic spline QIs in one and two variables.

In this paper, we propose new integration formulas for 3D integrals based on trivariate C^2 quartic spline quasi-interpolants on type-6 tetrahedral partitions with higher smoothness, useful, for example, in the numerical treatment of integral equations, where the unknown function can be reconstructed with C^2 smoothness.

In particular, we construct formulas based on four QIs, where the basic functions are the scaled translates of the trivariate C^2 quartic box spline B, defined on a type-6 tetrahedral partition, and the coefficient functionals are linear combinations of values of f at specific points in the support of the scaled translates of B. The first operator is based on the well-known Schoenberg-Marsden one and it is exact on the space of trilinear polynomials. The second one is exact on the space of polynomials contained in the spline space spanned by the scaled translates of B. The third one is exact on the space $\mathbb{P}_3$ of trivariate polynomials of total degree at most three, and it is of near-best type, i.e. it is constructed by minimizing an upper bound of its infinity norm. Finally, the fourth one is exact on $\mathbb{P}_3$ and shows some superconvergence properties at specific points of the domain (the vertices and the centers of each cube of the partition).

The paper is organized as follows: in Section 2, we recall the definitions and main properties of the trivariate C^2 quartic box spline B, the space spanned by its scaled translates and the four QIs. New cubature formulas, based on such QIs, are generated in Section 3 and their convergence and stability properties are studied. Finally, in Section 4, numerical results are presented, illustrating the performances of the proposed cubatures.

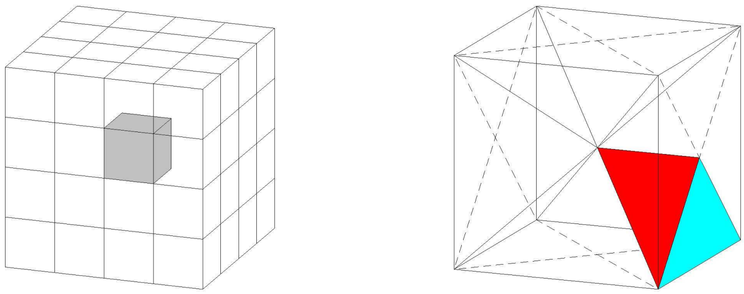
2 Trivariate C^2 quartic spline quasi-interpolants

2.1 The spline space $S_4^2(\Omega, \mathscr{T}_{\mathbf{m}})$

In order to define a box spline, it is necessary to specify a set of directions that determine the shape of its support and also its continuity properties. Following [26], we consider the set $X = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ of seven directions of $\mathbb{Z}^3$, spanning $\mathbb{R}^3$, where

$$e_1 = (1,0,0), \quad e_2 = (0,1,0), \quad e_3 = (0,0,1), \quad e_4 = (1,1,1),$$
$$e_5 = (-1,1,1), \quad e_6 = (1,-1,1), \quad e_7 = (-1,-1,1).$$

Therefore, the space is cut into a symmetric regular arrangement of tetrahedra called type-6 tetrahedral partition. The type-6 tetrahedral partitions are uniform partitions of $\mathbb{R}^3$ obtained from a given cube partition (see Fig. 2.1(a)) of the space by subdividing each cube into 24 tetrahedra with six planes (see Fig. 2.1(b)).



(a) (b)

Fig. 2.1 The uniform type-6 tetrahedral partition

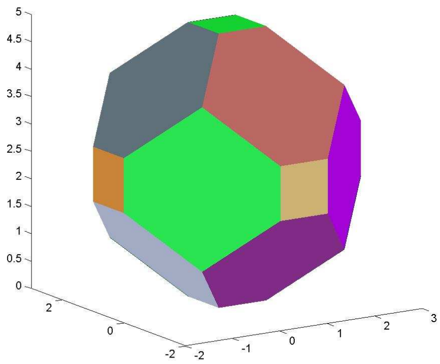


Fig. 2.2 The support of the seven directional box spline

According to [2, Chap. 11] and [4, Chap. 1], since the set X has seven elements and the domain is $\mathbb{R}^3$, the box spline $B(\cdot) = B(\cdot|X)$ is of degree four. Its continuity depends on the determination of the number d, such that $d+1$ is the minimal number of directions to be removed from X to obtain a reduced set that does not span $\mathbb{R}^3$. Then, the B continuity class is C^{d-1}. With the notation given in [2, Chap. 11],

$$d = \min\{|Y| : Y \in \mathscr{Y}\} - 1, \tag{2.1}$$

where

$$\mathscr{Y} = \mathscr{Y}(X) = \{Y \subset X : \langle X \backslash Y \rangle \neq \mathbb{R}^3\}.$$

In our case $d = 3$, thus the polynomial pieces defined over each tetrahedron are of degree four and they are joined with C^2 smoothness.

The support Ξ of the C^2 quartic box spline B is the truncated rhombic dodecahedron centered at the point $(\frac{1}{2}, \frac{1}{2}, \frac{5}{2})$ and contained in the cube $[-2,3] \times [-2,3] \times [0,5]$ (see Fig. 2.2).

Now, let $m_1, m_2, m_3 \geq 9$ be integers, let $\Omega = [0, m_1 h] \times [0, m_2 h] \times [0, m_3 h]$, $h > 0$, be a parallelepiped divided into $m_1 m_2 m_3$ equal cubes and endowed with the type-6 tetrahedral partition $\mathscr{T}_{\mathbf{m}}$, $\mathbf{m} = (m_1, m_2, m_3)$ (see Fig. 2.1). We set

$$\mathscr{A} = \left\{\alpha = (\alpha_1, \alpha_2, \alpha_3), -1 \le \alpha_i \le m_i + 2, 1 \le i \le 3, \alpha \notin \mathscr{A}'\right\},$$

with $\mathscr{A}'$ the set of indices defined by

$$\mathscr{A}' = \left\{ \begin{array}{ll} (\alpha_1, \alpha_2, -1),\ (\alpha_1, \alpha_2, m_3+2), & \text{for } -1 \le \alpha_1 \le m_1+2,\ \alpha_2 = -1, m_2+2, \\ (\alpha_1, \alpha_2, -1),\ (\alpha_1, \alpha_2, m_3+2), & \text{for } \alpha_1 = -1, m_1+2,\ 0 \le \alpha_2 \le m_2+1, \\ (\alpha_1, -1, \alpha_3),\ (\alpha_1, m_2+2, \alpha_3), & \text{for } \alpha_1 = -1, m_1+2,\ 0 \le \alpha_3 \le m_3+1 \end{array} \right\}.$$

Since B is centered at the point $\left(\frac{1}{2}, \frac{1}{2}, \frac{5}{2}\right)$, we define the scaled translates of B, $\{B_\alpha, \alpha \in \mathscr{A}\}$, in the following way:

$$B_\alpha(x,y,z) = B\left(\frac{x}{h} - \alpha_1 + 1, \frac{y}{h} - \alpha_2 + 1, \frac{z}{h} - \alpha_3 + 3\right), \tag{2.2}$$

whose supports Ξ_α are centered at the points $\left((\alpha_1 - \frac{1}{2})h, (\alpha_2 - \frac{1}{2})h, (\alpha_3 - \frac{1}{2})h\right)$ and overlap with Ω.

Then, we define the space generated by the functions $\{B_\alpha, \alpha \in \mathscr{A}\}$

$$S_4^2(\Omega, \mathscr{T}_{\mathbf{m}}) = \left\{ s = \sum_{\alpha \in \mathscr{A}} c_\alpha B_\alpha,\ c_\alpha \in \mathbb{R} \right\}.$$

This space is a subspace of the whole space of all trivariate C^2 quartic splines defined on $\mathscr{T}_{\mathbf{m}}$.

We also recall that the approximation power of $S_4^2(\Omega, \mathscr{T}_{\mathbf{m}})$ is the largest integer r for which

$$\operatorname{dist}(f, S_4^2(\Omega, \mathscr{T}_{\mathbf{m}})) = O(h^r)$$

for all sufficiently smooth f, with the distance measured in the $L_p(\Omega)$-norm $(1 \le p \le \infty)$ [4, Chap. 3]. From results given in [4, Chap. 3], we know that the approximation power of $S_4^2(\Omega, \mathscr{T}_{\mathbf{m}})$ does not exceed $d+1$, with d defined by (2.1). Therefore, in our case we have $r \le 4$ and in the following we show that $r = 4$ (Theorem 2.1).

From [28], we also know that the maximal space of polynomials included in $S_4^2(\Omega, \mathscr{T}_{\mathbf{m}})$ is $\mathscr{D}(X) = \mathbb{P}_3 \oplus$ span$\{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9\}$, with

$$\begin{array}{lll} p_1 = x^4, & p_2 = x^3y + 3xyz^2, & p_3 = xy^3 + 3xyz^2, \\ p_4 = y^4, & p_5 = x^3z + 3xy^2z, & p_6 = y^3z + 3x^2yz, \\ p_7 = xz^3 + 3xy^2z, & p_8 = yz^3 + 3x^2yz, & p_9 = z^4. \end{array}$$

2.2 Quasi-interpolants in $S_4^2(\Omega, \mathscr{T}_{\mathbf{m}})$

In the space $S_4^2(\Omega, \mathscr{T}_{\mathbf{m}})$, we consider several quasi-interpolation operators [28].

A quasi-interpolant is a linear operator defined on a functional space $\mathscr{F}$, in the following way

$$Q : \mathscr{F} \to S_4^2(\Omega, \mathscr{T}_{\mathbf{m}})$$

$$Qf = \sum_{\alpha \in \mathscr{A}} \lambda_\alpha(f) B_\alpha$$

where the B_α's are the scaled translates of the box spline B, defined by (2.2) with support Ξ_α and the $\lambda_\alpha(f)$'s are linear functionals expressed as linear combination of values of f at specific points in Ξ_α.

We define four different QIs

$$Q^\nu f = \sum_{\alpha\in\mathscr{A}} \lambda_\alpha^\nu(f) B_\alpha, \qquad \nu = 1,2,3,4, \tag{2.3}$$

with

- $\lambda_\alpha^1(f) = f_\alpha$ (see Fig. 2.3(a));
- $\lambda_\alpha^2(f) = \frac{191}{64} f_\alpha - \frac{107}{288}(f_{\alpha\pm e_1} + f_{\alpha\pm e_2} + f_{\alpha\pm e_3}) + \frac{47}{1152}(f_{\alpha\pm 2e_1} + f_{\alpha\pm 2e_2} + f_{\alpha\pm 2e_3})$ (see Fig. 2.3(b));
- $\lambda_\alpha^3(f) = \frac{21}{16} f_\alpha - \frac{5}{96}(f_{\alpha\pm 2e_1} + f_{\alpha\pm 2e_2} + f_{\alpha\pm 2e_3})$ (see Fig. 2.4(a));
- $\lambda_\alpha^4(f) = \frac{16871}{4416} f_\alpha - \frac{507}{736}(f_{\alpha\pm e_1} + f_{\alpha\pm e_2} + f_{\alpha\pm e_3}) + \frac{47}{1152}(f_{\alpha\pm 2e_1} + f_{\alpha\pm 2e_2} + f_{\alpha\pm 2e_3}) + \frac{1435}{13248}(f_{\alpha\pm(e_1+e_2)} + f_{\alpha\pm(e_1-e_2)} + f_{\alpha\pm(e_1+e_3)} + f_{\alpha\pm(e_1-e_3)} + f_{\alpha\pm(e_2+e_3)} + f_{\alpha\pm(e_2-e_3)}) - \frac{2}{69}(f_{\alpha\pm e_4} + f_{\alpha\pm e_5} + f_{\alpha\pm e_6} + f_{\alpha\pm e_7})$ (see Fig. 2.4(b));

where $f_\beta = f(M_\beta)$ and

$$M_\beta = \left(\left(\beta_1 - \frac{1}{2}\right)h, \left(\beta_2 - \frac{1}{2}\right)h, \left(\beta_3 - \frac{1}{2}\right)h\right) \tag{2.4}$$

are the data points. They are the centers of each subcube of the partition and some of them lie outside Ω. Therefore, their corresponding set of indices is

$$\mathscr{A}^M = \begin{cases} \mathscr{A}, & \nu = 1 \\ \left\{\alpha = (\alpha_1,\alpha_2,\alpha_3), -3 \le \alpha_i \le m_i + 4, 1 \le i \le 3, \alpha \notin \mathscr{A}^{\bar{M}}\right\}, & \nu = 2,3,4 \end{cases}$$

with

$$\mathscr{A}^{\widetilde{M}} = \begin{cases} \mathscr{A}^{M^*} \cup \left\{ \begin{array}{ll} (\alpha_1,\alpha_2,\alpha_3) & \text{for } \alpha_1 = -1, m_1+2,\ \alpha_2 = -1, m_2+2, \\ & \alpha_3 = -2, m_3+3, \\ (\alpha_1,\alpha_2,\alpha_3) & \text{for } \alpha_1 = -2, m_1+3,\ \alpha_2 = -1, m_2+2, \\ & \alpha_1 = -1, m_1+2,\ \alpha_2 = -2,-1, m_2+2, m_2+3, \\ & \alpha_3 = -1, m_3+2 \end{array} \right\}, & \nu = 2,3 \\ \mathscr{A}^{M^*}, & \nu = 4 \end{cases}$$

and

$$\mathscr{A}^{M^*} = \left\{ \begin{array}{ll} (\alpha_1,\alpha_2,\alpha_3) & \text{for } \alpha_1 = -3,-2,-1, m_1+2, m_1+3, m_1+4,\ \alpha_2 = -3, m_2+4, \\ & \alpha_1 = -3,-2, m_1+3, m_1+4,\ \alpha_2 = -2, m_2+3, \\ & \alpha_1 = -3, m_1+4,\ \alpha_2 = -1, m_2+2, \\ & 0 \le \alpha_3 \le m_3+1; \\ (\alpha_1,\alpha_2,\alpha_3) & \text{for } \alpha_1 = -3,-2,-1, m_1+2, m_1+3, m_1+4,\ -3 \le \alpha_2 \le m_2+4, \\ & 0 \le \alpha_1 \le m_1+1,\ \alpha_2 = -3,-2,-1, m_2+2, m_2+3, m_2+4, \\ & \alpha_3 = -3, m_3+4; \\ (\alpha_1,\alpha_2,\alpha_3) & \text{for } \alpha_1 = -3,-2, m_1+3, m_1+4,\ -3 \le \alpha_2 \le m_2+4, \\ & -1 \le \alpha_1 \le m_1+2,\ \alpha_2 = -3,-2, m_2+3, m_2+4, \\ & \alpha_3 = -2, m_3+3; \\ (\alpha_1,\alpha_2,\alpha_3) & \text{for } \alpha_1 = -3, m_1+4,\ -3 \le \alpha_2 \le m_2+4, \\ & \alpha_1 = -2, m_1+3,\ \alpha_2 = -3,-2, m_2+3, m_2+4, \\ & -1 \le \alpha_1 \le m_1+2,\ \alpha_2 = -3, m_2+4, \\ & \alpha_3 = -1, m_3+2. \end{array} \right\}.$$

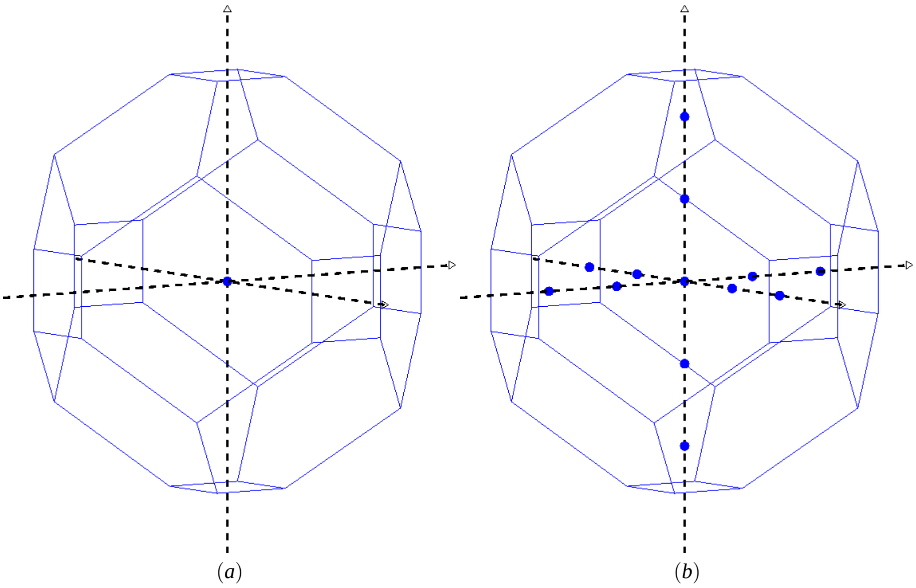


Fig. 2.3 Data points involved in (a) λ_α^1 and (b) λ_α^2

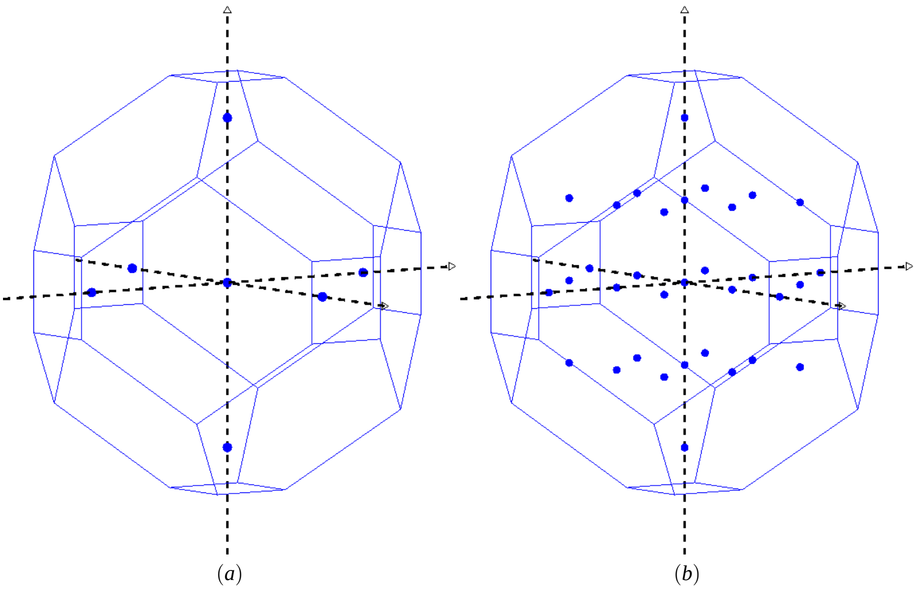


Fig. 2.4 Data points involved in (a) λ_α^3 and (b) λ_α^4

The first operator Q^1 is exact on the space of trilinear polynomials, the second one Q^2 is exact on the space $\mathscr{D}(X)$, the third one Q^3 is exact on the space $\mathbb{P}_3$ and it is of near-best type, i.e. it is constructed by minimizing an upper bound of its infinity norm. The fourth one Q^4 is exact on $\mathbb{P}_3$ and shows some superconvergence properties at specific points of the domain (the vertices and the centers of each cube of the partition, see [28] for more details).

By introducing the fundamental splines

$$L^1_\alpha = B_\alpha, \tag{2.5}$$

$$\begin{aligned} L^2_\alpha = {} & \frac{191}{64} B_\alpha - \frac{107}{288}(B_{\alpha\pm e_1} + B_{\alpha\pm e_2} + B_{\alpha\pm e_3}) \\ & + \frac{47}{1152}(B_{\alpha\pm 2e_1} + B_{\alpha\pm 2e_2} + B_{\alpha\pm 2e_3}), \end{aligned} \tag{2.6}$$

$$L^3_\alpha = \frac{21}{16} B_\alpha - \frac{5}{96}(B_{\alpha\pm 2e_1} + B_{\alpha\pm 2e_2} + B_{\alpha\pm 2e_3}) \tag{2.7}$$

$$\begin{aligned} L^4_\alpha = {} & \frac{16871}{4416} B_\alpha - \frac{507}{736}(B_{\alpha\pm e_1} + B_{\alpha\pm e_2} + B_{\alpha\pm e_3}) + \frac{47}{1152}(B_{\alpha\pm 2e_1} + B_{\alpha\pm 2e_2} + B_{\alpha\pm 2e_3}) \\ & + \frac{1435}{13248}(B_{\alpha\pm(e_1+e_2)} + B_{\alpha\pm(e_1-e_2)} + B_{\alpha\pm(e_1+e_3)} + B_{\alpha\pm(e_1-e_3)} \\ & + B_{\alpha\pm(e_2+e_3)} + B_{\alpha\pm(e_2-e_3)}) - \frac{2}{69}(B_{\alpha\pm e_4} + B_{\alpha\pm e_5} + B_{\alpha\pm e_6} + B_{\alpha\pm e_7}), \end{aligned} \tag{2.8}$$

assuming $B_\alpha \equiv 0$ in case of $\alpha \notin \mathscr{A}$, the QIs defined in (2.3) can be written in the "quasi-Lagrange" form:

$$Q^\nu f = \sum_{\alpha\in\mathscr{A}^M} f(M_\alpha) L^\nu_\alpha, \qquad \nu = 1,2,3,4.$$

The infinity norms of the proposed quasi-interpolants have the following bounds:

$$\|Q^1\|_\infty = 1, \qquad \|Q^2\|_\infty \leq \tfrac{4674}{2323} \approx 2.01,$$

$$\|Q^3\|_\infty \leq \tfrac{47}{32} \approx 1.47, \; \|Q^4\|_\infty \leq \tfrac{1167}{493} \approx 2.37.$$

Standard results in approximation theory [4] allow us to deduce Theorem 2.1, for which we need the following notations:

- for any function $f \in C(H)$, with H a compact set, we denote the infinity norm of f by $\|f\|_H = \sup\{|f(x,y,z)| : (x,y,z) \in H\}$;
- $\omega(\varphi,t) = \max\{|\varphi(x_1,y_1,z_1) - \varphi(x_2,y_2,z_2)|;\ (x_1,y_1,z_1),(x_2,y_2,z_2) \in H,$ $\|(x_1,y_1,z_1) - (x_2,y_2,z_2)\| \leq t\}$ is the usual modulus of continuity of $\varphi \in C(H)$, where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^3$;
- $D^\beta = D^{\beta_1\beta_2\beta_3} = \frac{\partial^{|\beta|}}{\partial x^{\beta_1}\partial y^{\beta_2}\partial z^{\beta_3}}$, with $|\beta| = \beta_1 + \beta_2 + \beta_3$;
- $\omega(D^r\varphi,t) = \max\{\omega(D^\beta\varphi,t),\ |\beta| = r\}$;
- $\Omega_h = [-2h,(m_1+2)h] \times [-2h,(m_2+2)h] \times [-2h,(m_3+2)h]$ for Q^1 and $\Omega_h = [-4h,(m_1+4)h] \times [-4h,(m_2+4)h] \times [-4h,(m_3+4)h]$ for Q^ν, $\nu = 2,3,4$.

Theorem 2.1 *For each operator Q^ν, $\nu = 1,2,3,4$, there exist positive constants $\mathscr{C}_{r,\nu}$, such that*

i) if $f \in C^r(\Omega_h)$, $r = 0,1$, then $\left\| f - Q^1 f \right\|_\infty \le \mathscr{C}_{r,1} h^r \omega(D^r f, h)$;
if, in addition, $f \in C^2(\Omega_h)$, then $\left\| f - Q^1 f \right\|_\infty \le \mathscr{C}_{2,1} h^2 \max_{|\beta|=2} \left\| D^\beta f \right\|_\infty$;

ii) if $f \in C^r(\Omega_h)$, $r = 0,1,2,3$, then $\| f - Q^\nu f \|_\infty \le \mathscr{C}_{r,\nu} h^r \omega(D^r f, h)$, $\nu = 2,3,4$;
if, in addition, $f \in C^4(\Omega_h)$, then $\| f - Q^\nu f \|_\infty \le \mathscr{C}_{4,\nu} h^4 \max_{|\beta|=4} \left\| D^\beta f \right\|_\infty$, $\nu = 2,3,4$.

3 Cubature rules based on Q^ν, $\nu = 1,2,3,4$

For any function $f \in C(\Omega_h)$, we consider the evaluation of the integral

$$I(f) = I(f;\Omega) := \int_\Omega f(x,y,z) \; \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

by cubature rules defined by

$$I_{Q^\nu}(f) = I(Q^\nu f;\Omega) := \sum_{\alpha \in \mathscr{A}^M} w_\alpha^\nu f(M_\alpha), \quad \nu = 1,2,3,4, \tag{3.1}$$

where the M_α's are the evaluation points defined by (2.4) and the cubature weights are

$$w_\alpha^\nu = \int_{\mathrm{supp} L_\alpha^\nu \cap \Omega} L_\alpha^\nu(x,y,z) \; \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z. \tag{3.2}$$

In the following theorems we show some features of the rules (3.1). First we define the chains of equalities

$$\begin{aligned} E_1^\nu : w^\nu_{\alpha_1,\alpha_2,\alpha_3} &= w^\nu_{m_1-\alpha_1+1,\alpha_2,\alpha_3} = w^\nu_{\alpha_1,m_2-\alpha_2+1,\alpha_3} = w^\nu_{\alpha_1,\alpha_2,m_3-\alpha_3+1} \\ &= w^\nu_{m_1-\alpha_1+1,m_2-\alpha_2+1,m_3-\alpha_3+1} = w^\nu_{\alpha_1,m_2-\alpha_2+1,m_3-\alpha_3+1} \\ &= w^\nu_{m_1-\alpha_1+1,\alpha_2,m_3-\alpha_3+1} = w^\nu_{m_1-\alpha_1+1,m_2-\alpha_2+1,\alpha_3}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} E_2^\nu : w^\nu_{\alpha_1,s,t} &= w^\nu_{\alpha_1,m_2-s+1,t} = w^\nu_{\alpha_1,s,m_3-t+1} = w^\nu_{\alpha_1,m_2-s+1,m_3-t+1} \\ &= w^\nu_{\alpha_1,t,s} = w^\nu_{\alpha_1,m_2-t+1,s} = w^\nu_{\alpha_1,t,m_3-s+1} = w^\nu_{\alpha_1,m_2-t+1,m_3-s+1} \\ &= w^\nu_{s,\alpha_2,t} = w^\nu_{m_1-s+1,\alpha_2,t} = w^\nu_{s,\alpha_2,m_3-t+1} = w^\nu_{m_1-s+1,\alpha_2,m_3-t+1} \\ &= w^\nu_{t,\alpha_2,s} = w^\nu_{m_1-t+1,\alpha_2,s} = w^\nu_{t,\alpha_2,m_3-s+1} = w^\nu_{m_1-t+1,\alpha_2,m_3-s+1} \\ &= w^\nu_{s,t,\alpha_3} = w^\nu_{m_1-s+1,t,\alpha_3} = w^\nu_{s,m_2-t+1,\alpha_3} = w^\nu_{m_1-s+1,m_2-t+1,\alpha_3} \\ &= w^\nu_{t,s,\alpha_3} = w^\nu_{m_1-t+1,s,\alpha_3} = w^\nu_{t,m_2-s+1,\alpha_3} = w^\nu_{m_1-t+1,m_2-s+1,\alpha_3}, \end{aligned} \tag{3.4}$$

$$\begin{aligned} E_3^\nu : w^\nu_{\alpha_1,\alpha_2,t} &= w^\nu_{\alpha_1,\alpha_2,m_3-t+1} \\ &= w^\nu_{\alpha_1,t,\alpha_3} = w^\nu_{\alpha_1,m_2-t+1,\alpha_3} \\ &= w^\nu_{t,\alpha_2,\alpha_3} = w^\nu_{m_1-t+1,\alpha_2,\alpha_3}, \end{aligned} \tag{3.5}$$

for $\nu = 1,2,3,4$, where the involved indices will be specified in the subsequent Theorem 3.1 and Theorem 3.3.

Theorem 3.1 *The cubature weights in (3.1), in case $\nu = 1$, satisfy the following symmetry properties:*

Table 3.1 The 26 values $\overline{w}^1_\alpha$, with $w^1_\alpha = \frac{h^3}{840}\overline{w}^1_\alpha$

$\overline{w}^1_{0,0,-1} = 1/8$	$\overline{w}^1_{0,0,0} = 69/8$	$\overline{w}^1_{1,1,1} = 3081/8$	$\overline{w}^1_{2,2,2} = 819$	$\overline{w}^1_{3,3,3} = 840$
$\overline{w}^1_{1,0,-1} = 3/4$	$\overline{w}^1_{1,0,0} = 253/8$	$\overline{w}^1_{2,1,1} = 3975/8$	$\overline{w}^1_{3,2,2} = 826$	
$\overline{w}^1_{1,1,-1} = 43/8$	$\overline{w}^1_{1,1,0} = 937/8$	$\overline{w}^1_{2,2,1} = 2555/4$	$\overline{w}^1_{3,3,2} = 833$	
$\overline{w}^1_{2,0,-1} = 7/8$	$\overline{w}^1_{2,0,0} = 321/8$	$\overline{w}^1_{3,1,1} = 2009/4$		
$\overline{w}^1_{2,1,-1} = 49/8$	$\overline{w}^1_{2,1,0} = 148$	$\overline{w}^1_{3,2,1} = 5159/8$		
$\overline{w}^1_{2,2,-1} = 7$	$\overline{w}^1_{2,2,0} = 749/4$	$\overline{w}^1_{3,3,1} = 651$		
	$\overline{w}^1_{3,0,0} = 161/4$			
	$\overline{w}^1_{3,1,0} = 595/4$			
	$\overline{w}^1_{3,2,0} = 1505/8$			
	$\overline{w}^1_{3,3,0} = 189$			

- *for* $\alpha_1 = 0,1,\ \ \alpha_2 = 0,\dots,\alpha_1,\ \ \alpha_3 = -1$*;* $\alpha_3 = 0,1,2,\ \ \alpha_1 = \alpha_3,\dots,2,\ \ \alpha_2 = \alpha_3,\dots,\alpha_1$ *and for any permutation of the indices* α_1, α_2, α_3*, the equalities* E^1_1 *in (3.3) hold;*

- *for* $s = 0,1,\ \ t = -1,\ \ \alpha_r = 2,\dots,m_r-1$ $(r = 1,2,3)$*, the weights involved in the equalities* E^1_2 *of (3.4) and* E^1_3 *of (3.5) are all equal to* $w^1_{2,s,t}$ *and* $w^1_{2,2,t}$*, respectively;*

- *for* $t = 0,1,2,\ \ s = t,\dots,2,\ \ \alpha_r = 3,\dots,m_r-2$ $(r = 1,2,3)$*, the weights involved in the equalities* E^1_2 *of (3.4) and* E^1_3 *of (3.5) are all equal to* $w^1_{3,s,t}$ *and* $w^1_{3,3,t}$*, respectively;*

- *for* $\alpha_r = 3,\dots,m_r-2$ $(r = 1,2,3)$

$$w^1_{\alpha_1,\alpha_2,\alpha_3} = w^1_{3,3,3}.$$

The values of the twenty-six different weights for $I_{Q^1}(f)$ *are reported in Table 3.1.*

Proof From (2.5) and (3.2)

$$w^1_\alpha = \int_{\Xi_\alpha \cap \Omega} B_\alpha. \tag{3.6}$$

In order to compute (3.6), we recall that a trivariate polynomial $p \in \mathbb{P}_4$ on a tetrahedron T of the partition $\mathscr{T}_{\mathbf{m}}$ can be represented in the Bernstein basis [3] as

$$p(\tau) = \sum_{|\gamma|=4} c(\gamma) b_\gamma(\tau),$$

where $\{c(\gamma),\ \gamma = (\gamma_1,\gamma_2,\gamma_3,\gamma_4),\ |\gamma| = 4\}$ are the BB-coefficients of p, $b_\gamma(\tau) = \frac{4!}{\gamma!}\tau^\gamma$ are the Bernstein polynomials, $\tau = (\tau_1,\tau_2,\tau_3,\tau_4)$ are the barycentric coordinates with respect to T and $\gamma! = \gamma_1!\gamma_2!\gamma_3!\gamma_4!$, $\tau^\gamma = \tau_1^{\gamma_1}\tau_2^{\gamma_2}\tau_3^{\gamma_3}\tau_4^{\gamma_4}$.

Since T is included in a cube with edge of length h, its volume is equal to $\frac{h^3}{24}$ and, since [3]

$$\int_T b_\gamma = \frac{1}{35}\frac{h^3}{24}$$

for all b_γ, then

$$\int_T p = \frac{1}{840} h^3 \sum_{|\gamma|=4} c(\gamma),$$

where the $c(\gamma)$ can be obtained by the procedure given in [20]. Therefore, by considering all the tetrahedra involved in $\Xi_\alpha \cap \Omega$, the cubature weights are given by

$$w_\alpha^1 = \sum_{T/T \in \Xi_\alpha \cap \Omega} \frac{h^3}{840} \sum_{|\gamma|=4} c(\gamma). \tag{3.7}$$

By considering the symmetry of the domain Ω and of the support Ξ_α of B_α, by tedious computations we can deduce the symmetry properties of the weights. Now, by the above features and (3.7), we get the twenty-six different weights that we report in Table 3.1. □

From Theorem 3.1, after some easy algebraic computations, we obtain the following result.

Corollary 3.1 *For any $f \in C(\Omega_h)$, the rule (3.1), with $\nu = 1$, can be written in the following more compact form:*

$$I_{Q^1}(f) = \sum_{\alpha_1=0}^{2} \sum_{\alpha_2=0}^{\alpha_1} w^1_{\alpha_1,\alpha_2,-1} z_{\alpha_1,\alpha_2,-1}(f) + \sum_{\alpha_3=0}^{3} \sum_{\alpha_1=\alpha_3}^{3} \sum_{\alpha_2=\alpha_3}^{\alpha_1} w^1_{\alpha_1\alpha_2\alpha_3} z_{\alpha_1\alpha_2\alpha_3}(f),$$

defined only by the twenty-six weights, given in Table 3.1, with

- $z_{333}(f) = \sum_{\alpha_1=3}^{m_1-2} \sum_{\alpha_2=3}^{m_2-2} \sum_{\alpha_3=3}^{m_3-2} f_{\alpha_1\alpha_2\alpha_3}$;
- *for* $s = 0, 1, 2$

$$\begin{aligned} z_{3ss}(f) =& \sum_{\alpha_1=3}^{m_1-2} (f_{\alpha_1 ss} + f_{\alpha_1,m_2-s+1,s} + f_{\alpha_1,s,m_3-s+1} + f_{\alpha_1,m_2-s+1,m_3-s+1}) \\ &+ \sum_{\alpha_2=3}^{m_2-2} (f_{s\alpha_2 s} + f_{m_1-s+1,\alpha_2,s} + f_{s,\alpha_2,m_3-s+1} + f_{m_1-s+1,\alpha_2,m_3-s+1}) \\ &+ \sum_{\alpha_3=3}^{m_3-2} (f_{ss\alpha_3} + f_{m_1-s+1,s,\alpha_3} + f_{s,m_2-s+1,\alpha_3} + f_{m_1-s+1,m_2-s+1,\alpha_3}); \end{aligned}$$

- *for* $s = 0, 1, 2$

$$\begin{aligned} z_{sss}(f) =& f_{sss} + f_{m_1-s+1,s,s} + f_{s,m_2-s+1,s} + f_{m_1-s+1,m_2-s+1,s} \\ &+ f_{s,s,m_3-s+1} + f_{m_1-s+1,s,m_3-s+1} \\ &+ f_{s,m_2-s+1,m_3-s+1} + f_{m_1-s+1,m_2-s+1,m_3-s+1}; \end{aligned}$$

- *for* $s = 0, 1$

$$z_{1+s,s,-1+s}(f) = \sum_{(\alpha_1,\alpha_2,\alpha_3)\in\Pi} \big(f_{\alpha_1\alpha_2\alpha_3} + f_{m_1-\alpha_1+1,\alpha_2,\alpha_3} + f_{\alpha_1,m_2-\alpha_2+1,\alpha_3} + f_{m_1-\alpha_1+1,m_2-\alpha_2+1,\alpha_3} + f_{\alpha_1,\alpha_2,m_3-\alpha_3+1} + f_{m_1-\alpha_1+1,\alpha_2,m_3-\alpha_3+1} + f_{\alpha_1,m_2-\alpha_2+1,m_3-\alpha_3+1} + f_{m_1-\alpha_1+1,m_2-\alpha_2+1,m_3-\alpha_3+1}\big),$$

where $\Pi = \{(-1+s,s,1+s),\ (-1+s,1+s,s),\ (s,-1+s,1+s),\ (s,1+s,-1+s),\ (1+s,s,-1+s),\ (1+s,-1+s,s)\}$ *is the permutation set of* $(1+s,s,-1+s)$*;*

- *for* $(s,t) = (2,-1), (3,\ell)$, $\ell = 0,1,2$

$$z_{sst}(f) = \sum_{\alpha_1=s}^{m_1-s+1}\sum_{\alpha_2=s}^{m_2-s+1}\big(f_{\alpha_1\alpha_2 t} + f_{\alpha_1,\alpha_2,m_3-t+1}\big) + \sum_{\alpha_1=s}^{m_1-s+1}\sum_{\alpha_3=s}^{m_3-s+1}\big(f_{\alpha_1 t\alpha_3} + f_{\alpha_1,m_2-t+1,\alpha_3}\big) + \sum_{\alpha_2=s}^{m_2-s+1}\sum_{\alpha_3=s}^{m_3-s+1}\big(f_{t\alpha_2\alpha_3} + f_{m_1-t+1,\alpha_2,\alpha_3}\big);$$

- *for* $(r,s,t) = (2,s,-1), (3,s+1,t)$, $s = 0,1$, $t = 0,\ldots,s$

$$\begin{aligned} z_{rst}(f) = &\sum_{\alpha_1=r}^{m_1-r+1}\big(f_{\alpha_1 st} + f_{\alpha_1,m_2-s+1,t} + f_{\alpha_1,s,m_3-t+1} + f_{\alpha_1,m_2-s+1,m_3-t+1} \\ &+ f_{\alpha_1 ts} + f_{\alpha_1,m_2-t+1,s} + f_{\alpha_1,t,m_3-s+1} + f_{\alpha_1,m_2-t+1,m_3-s+1}\big) \\ &+ \sum_{\alpha_2=r}^{m_2-r+1}\big(f_{s\alpha_2 t} + f_{m_1-s+1,\alpha_2,t} + f_{s,\alpha_2,m_3-t+1} + f_{m_1-s+1,\alpha_2,m_3-t+1} \\ &+ f_{t\alpha_2 s} + f_{m_1-t+1,\alpha_2,s} + f_{t,\alpha_2,m_3-s+1} + f_{m_1-t+1,\alpha_2,m_3-s+1}\big) \\ &+ \sum_{\alpha_3=r}^{m_3-r+1}\big(f_{st\alpha_3} + f_{m_1-s+1,t,\alpha_3} + f_{s,m_2-t+1,\alpha_3} + f_{m_1-s+1,m_2-t+1,\alpha_3} \\ &+ f_{ts\alpha_3} + f_{m_1-t+1,s,\alpha_3} + f_{t,m_2-s+1,\alpha_3} + f_{m_1-t+1,m_2-s+1,\alpha_3}\big); \end{aligned}$$

- *for* $s = 0,1,2$, $t = -1,\ldots,s-1$ *and* $(s,t) \neq (2,-1)$

$$\begin{aligned} z_{sst}(f) = &\ f_{sst} + f_{m_1-s+1,s,t} + f_{s,m_2-s+1,t} + f_{m_1-s+1,m_2-s+1,t} \\ &+ f_{sts} + f_{m_1-s+1,t,s} + f_{s,m_2-t+1,s} + f_{m_1-s+1,m_2-t+1,s} \\ &+ f_{tss} + f_{m_1-t+1,s,s} + f_{t,m_2-s+1,s} + f_{m_1-t+1,m_2-s+1,s}, \\ &+ f_{s,s,m_3-t+1} + f_{m_1-s+1,s,m_3-t+1} \\ &+ f_{s,m_2-s+1,m_3-t+1} + f_{m_1-s+1,m_2-s+1,m_3-t+1} \\ &+ f_{s,t,m_3-s+1} + f_{m_1-s+1,t,m_3-s+1} \\ &+ f_{s,m_2-t+1,m_3-s+1} + f_{m_1-s+1,m_2-t+1,m_3-s+1} \\ &+ f_{t,s,m_3-s+1} + f_{m_1-t+1,s,m_3-s+1} \\ &+ f_{t,m_2-s+1,m_3-s+1} + f_{m_1-t+1,m_2-s+1,m_3-s+1}; \end{aligned}$$

Table 3.2 The number F_α of function evaluations related to the weights w^1_α's of Table 3.1

$F_{0,0,-1} = 24$
$F_{1,0,-1} = 48$
$F_{1,1,-1} = 24$
$F_{2,0,-1} = 8(m_1+m_2+m_3-6)$
$F_{2,1,-1} = 8(m_1+m_2+m_3-6)$
$F_{2,2,-1} = 2[(m_1-2)(m_2-2)+(m_1-2)(m_3-2)+(m_2-2)(m_3-2)]$
$F_{0,0,0} = 8$
$F_{1,0,0} = 24$
$F_{1,1,0} = 24$
$F_{2,0,0} = 24$
$F_{2,1,0} = 48$
$F_{2,2,0} = 24$
$F_{3,0,0} = 4(m_1+m_2+m_3-12)$
$F_{3,1,0} = 8(m_1+m_2+m_3-12)$
$F_{3,2,0} = 8(m_1+m_2+m_3-12)$
$F_{3,3,0} = 2[(m_1-4)(m_2-4)+(m_1-4)(m_3-4)+(m_2-4)(m_3-4)]$
$F_{1,1,1} = 8$
$F_{2,1,1} = 24$
$F_{2,2,1} = 24$
$F_{3,1,1} = 4(m_1+m_2+m_3-12)$
$F_{3,2,1} = 8(m_1+m_2+m_3-12)$
$F_{3,3,1} = 2[(m_1-4)(m_2-4)+(m_1-4)(m_3-4)+(m_2-4)(m_3-4)]$
$F_{2,2,2} = 8$
$F_{3,2,2} = 4(m_1+m_2+m_3-12)$
$F_{3,3,2} = 2[(m_1-4)(m_2-4)+(m_1-4)(m_3-4)+(m_2-4)(m_3-4)]$
$F_{3,3,3} = (m_1-4)(m_2-4)(m_3-4)$

- *for* $s = 1,2,\ t = 0,\dots,s-1$

$$\begin{aligned} z_{stt}(f) = {} & f_{stt} + f_{m_1-s+1,t,t} + f_{s,m_2-t+1,t} + f_{m_1-s+1,m_2-t+1,t} \\ & + f_{tst} + f_{m_1-t+1,s,t} + f_{t,m_2-s+1,t} + f_{m_1-t+1,m_2-s+1,t} \\ & + f_{tts} + f_{m_1-t+1,t,s} + f_{t,m_2-t+1,s} + f_{m_1-t+1,m_2-t+1,s} \\ & + f_{s,t,m_3-t+1} + f_{m_1-s+1,t,m_3-t+1} \\ & + f_{s,m_2-t+1,m_3-t+1} + f_{m_1-s+1,m_2-t+1,m_3-t+1} \\ & + f_{t,s,m_3-t+1} + f_{m_1-t+1,s,m_3-t+1} \\ & + f_{t,m_2-s+1,m_3-t+1} + f_{m_1-t+1,m_2-s+1,m_3-t+1} \\ & + f_{t,t,m_3-s+1} + f_{m_1-t+1,t,m_3-s+1} \\ & + f_{t,m_2-t+1,m_3-s+1} + f_{m_1-t+1,m_2-t+1,m_3-s+1}. \end{aligned}$$

Remark 3.1 From Corollary 3.1, each of the twenty-six weights is associated with a certain number of function evaluations, as shown in Table 3.2. Therefore the total number of function evaluations is $m_1m_2m_3 + 4(m_1m_2 + m_1m_3 + m_2m_3) + 12(m_1 + m_2 + m_3) + 32$.

Theorem 3.2 *If in (3.1) we assume*

- $\nu = 2$, *then*

$$w_\alpha^2 = \int_{suppL_\alpha^2 \cap \Omega} L_\alpha^2(x,y,z)\ dx\,dy\,dz$$
$$= \frac{191}{64} w_\alpha^1 - \frac{107}{288}(w^1_{\alpha\pm e_1} + w^1_{\alpha\pm e_2} + w^1_{\alpha\pm e_3}) + \frac{47}{1152}(w^1_{\alpha\pm 2e_1} + w^1_{\alpha\pm 2e_2} + w^1_{\alpha\pm 2e_3}),$$

- $\nu = 3$, *then*

$$w_\alpha^3 = \int_{suppL_\alpha^3 \cap \Omega} L_\alpha^3(x,y,z)\ dx\,dy\,dz$$
$$= \frac{21}{16} w_\alpha^1 - \frac{5}{96}(w^1_{\alpha\pm 2e_1} + w^1_{\alpha\pm 2e_2} + w^1_{\alpha\pm 2e_3}),$$

- $\nu = 4$, *then*

$$w_\alpha^4 = \int_{suppL_\alpha^4 \cap \Omega} L_\alpha^4(x,y,z)\ dx\,dy\,dz$$
$$= \frac{16871}{4416} w_\alpha^1 - \frac{507}{736}(w^1_{\alpha\pm e_1} + w^1_{\alpha\pm e_2} + w^1_{\alpha\pm e_3}) + \frac{47}{1152}(w^1_{\alpha\pm 2e_1} + w^1_{\alpha\pm 2e_2} + w^1_{\alpha\pm 2e_3})$$
$$+ \frac{1435}{13248}(w^1_{\alpha\pm(e_1+e_2)} + w^1_{\alpha\pm(e_1-e_2)} + w^1_{\alpha\pm(e_1+e_3)} + w^1_{\alpha\pm(e_1-e_3)}$$
$$+ w^1_{\alpha\pm(e_2+e_3)} + w^1_{\alpha\pm(e_2-e_3)}) - \frac{2}{69}(w^1_{\alpha\pm e_4} + w^1_{\alpha\pm e_5} + w^1_{\alpha\pm e_6} + w^1_{\alpha\pm e_7}),$$

with the convention $w_\alpha^1 \equiv 0$, $\alpha \notin \mathscr{A}$.

Proof The proof immediately follows from (3.2) and (2.6)-(2.7)-(2.8). □

Theorem 3.3 *The cubature weights of (3.1), in case*

i) $\nu = 2,3$, *satisfy the following symmetry properties:*

- *for* $\alpha_1 = 0,1,\ \ \alpha_2 = 0,\dots,\alpha_1,\ \ \alpha_3 = -3$*;* $\alpha_3 = -2,-1,\ \ \alpha_1 = 0,\dots,\alpha_3+4,$ $\alpha_2 = 0,\dots,\alpha_1$*;* $\alpha_1 = 0,1,\ \ \alpha_2 = -1,\ \ \alpha_3 = -2,-1$*;* $\alpha_3 = 0,1,2,3,4,$ $\alpha_1 = \alpha_3,\dots,4,\ \ \alpha_2 = \alpha_3,\dots,\alpha_1$ *and for any permutation of the indices* α_1, α_2, α_3, *the equalities* E_1^ν *in (3.3) hold.*

- *for* $s = 0,1,\ \ t = -3,\ \ \alpha_r = 2,\dots,m_r-1$ *(*$r = 1,2,3$*), the weights involved in the equalities* E_2^ν *of (3.4) and* E_3^ν *of (3.5) are all equal to* $w^\nu_{2,s,t}$ *and* $w^\nu_{2,2,t}$*, respectively;*

- *for* $s = -1,\ \ t = -2\ \ \alpha_r = 2,\dots,m_r-1$ *(*$r = 1,2,3$*), the weights involved in the equalities* E_2^ν *of (3.4) are all equal to* $w^\nu_{2,0,-3}$*;*

- *for* $s = 0,1,2,\ \ t = -2\ \ \alpha_r = 3,\dots,m_r-2$ *(*$r = 1,2,3$*), the weights involved in the equalities* E_2^ν *of (3.4) and* E_3^ν *of (3.5) are all equal to* $w^\nu_{3,s,t}$ *and* $w^\nu_{3,3,t}$*, respectively;*

- *for* $s=t=-1 \quad \alpha_r=2,\dots,m_r-1$ *(*$r=1,2,3$*), the weights involved in the equalities* E_2^ν *of (3.4) are all equal to* $w_{2,s,t}^\nu$*;*

- *for* $s=0,1,2,3,\ \ t=-1 \quad \alpha_r=4,\dots,m_r-3$ *(*$r=1,2,3$*), the weights involved in the equalities* E_2^ν *of (3.4) and* E_3^ν *of (3.5) are all equal to* $w_{4,s,t}^\nu$ *and* $w_{4,4,t}^\nu$*, respectively;*

- *for* $t=0,1,2,3,4,\ \ s=t,\dots,4,\ \ \alpha_r=5,\dots,m_r-4$ *(*$r=1,2,3$*), the weights involved in the equalities* E_2^ν *of (3.4) and* E_3^ν *of (3.5) are all equal to* $w_{5,s,t}^\nu$ *and* $w_{5,5,t}^\nu$*, respectively;*

- *for* $\alpha_r=5,\dots,m_r-4$ *(*$r=1,2,3$*)*

$$w_{\alpha_1,\alpha_2,\alpha_3}^\nu = w_{5,5,5}^\nu.$$

The values of the ninety different weights for $I_{Q^2}(f)$ *and* $I_{Q^3}(f)$ *are reported in Table 3.3 and Table 3.4, respectively;*

ii) $\nu=4$*, satisfy the following symmetry properties:*

- *for* $\alpha_1=0,1,\ \ \alpha_2=0,\dots,\alpha_1,\ \ \alpha_3=-3$*;* $\alpha_3=-2,-1,\ \ \alpha_1=-1,\dots,\alpha_3+4,\ \ \alpha_2=-1,\dots,\alpha_1$*;* $\alpha_3=0,1,2,3,4,\ \ \alpha_1=\alpha_3,\dots,4,\ \ \alpha_2=\alpha_3,\dots,\alpha_1$ *and for any permutation of the indices* α_1*,* α_2*,* α_3*, the equalities* E_1^4 *in (3.3) hold;*

- *for* $s=0,1,\ \ t=-3,\ \ \alpha_r=2,\dots,m_r-1$ *(*$r=1,2,3$*), the weights involved in the equalities* E_2^4 *of (3.4) and* E_3^4 *of (3.5) are all equal to* $w_{2,s,t}^4$ *and* $w_{2,2,t}^4$*, respectively;*

- *for* $s=-1,0,1,2,\ \ t=-2 \quad \alpha_r=3,\dots,m_r-2$ *(*$r=1,2,3$*), the weights involved in the equalities* E_2^4 *of (3.4) and* E_3^4 *of (3.5) are all equal to* $w_{3,s,t}^4$ *and* $w_{3,3,t}^4$*, respectively;*

- *for* $s=-1,0,1,2,3,\ \ t=-1 \quad \alpha_r=4,\dots,m_r-3$ *(*$r=1,2,3$*), the weights involved in the equalities* E_2^4 *of (3.4) and* E_3^4 *of (3.5) are all equal to* $w_{4,s,t}^4$ *and* $w_{4,4,t}^4$*, respectively;*

- *for* $t=0,1,2,3,4,\ \ s=t,\dots,4,\ \ \alpha_r=5,\dots,m_r-4$ *(*$r=1,2,3$*), the weights involved in the equalities* E_2^4 *of (3.4) and* E_3^4 *of (3.5) are all equal to* $w_{5,s,t}^4$ *and* $w_{5,5,t}^4$*, respectively;*

- *for* $\alpha_r=5,\dots,m_r-4$ *(*$r=1,2,3$*)*

$$w_{\alpha_1,\alpha_2,\alpha_3}^4 = w_{5,5,5}^4.$$

The values of the ninety-eight different weights for $I_{Q^4}(f)$ *are reported in Table 3.5.*

Table 3.3 The 90 values $\bar{w}^2_\alpha$, with $w^2_\alpha = \frac{h^3}{840}\bar{w}^2_\alpha$

$\bar{w}^2_{0,0,-3} = 47/9216$	$\bar{w}^2_{0,-1,-1} = -73/2304$	$\bar{w}^2_{0,0,0} = -14551/3072$	$\bar{w}^2_{1,1,1} = 1619357/3072$	$\bar{w}^2_{2,2,2} = 479045/512$
$\bar{w}^2_{1,0,-3} = 47/1536$	$\bar{w}^2_{0,0,-1} = -6227/3072$	$\bar{w}^2_{1,0,0} = 22163/9216$	$\bar{w}^2_{2,1,1} = 5962451/9216$	$\bar{w}^2_{3,2,2} = 129437/144$
$\bar{w}^2_{1,1,-3} = 2021/9216$	$\bar{w}^2_{1,-1,-1} = -547/4608$	$\bar{w}^2_{1,1,0} = 317837/3072$	$\bar{w}^2_{2,2,1} = 3606763/4608$	$\bar{w}^2_{3,3,2} = 1323847/1536$
$\bar{w}^2_{2,0,-3} = 329/9216$	$\bar{w}^2_{1,0,-1} = -62813/9216$	$\bar{w}^2_{2,0,0} = -2809/9216$	$\bar{w}^2_{3,1,1} = 642883/1024$	$\bar{w}^2_{4,2,2} = 4163887/4608$
$\bar{w}^2_{2,1,-3} = 2303/9216$	$\bar{w}^2_{1,1,-1} = -150869/9216$	$\bar{w}^2_{2,1,0} = 1038487/9216$	$\bar{w}^2_{3,2,1} = 3483131/4608$	$\bar{w}^2_{4,3,2} = 7987217/9216$
$\bar{w}^2_{2,2,-3} = 329/1152$	$\bar{w}^2_{2,-1,-1} = -77/512$	$\bar{w}^2_{2,2,0} = 69805/576$	$\bar{w}^2_{3,3,1} = 1679335/2304$	$\bar{w}^2_{4,4,2} = 1003919/1152$
	$\bar{w}^2_{2,0,-1} = -40601/4608$	$\bar{w}^2_{3,0,0} = -24305/9216$	$\bar{w}^2_{4,1,1} = 5809561/9216$	$\bar{w}^2_{5,2,2} = 1388401/1536$
$\bar{w}^2_{0,0,-2} = 2815/9216$	$\bar{w}^2_{2,1,-1} = -53147/2304$	$\bar{w}^2_{3,1,0} = 322117/3072$	$\bar{w}^2_{4,2,1} = 6998635/9216$	$\bar{w}^2_{5,3,2} = 887761/1024$
$\bar{w}^2_{1,0,-2} = 9323/9216$	$\bar{w}^2_{2,2,-1} = -24367/768$	$\bar{w}^2_{3,2,0} = 1023587/9216$	$\bar{w}^2_{4,3,1} = 6749995/9216$	$\bar{w}^2_{5,4,2} = 125531/144$
$\bar{w}^2_{1,1,-2} = 8545/3072$	$\bar{w}^2_{3,0,-1} = -81541/9216$	$\bar{w}^2_{3,3,0} = 232477/2304$	$\bar{w}^2_{4,4,1} = 3391325/4608$	$\bar{w}^2_{5,5,2} = 334859/384$
$\bar{w}^2_{2,0,-2} = 12091/9216$	$\bar{w}^2_{3,1,-1} = -53491/2304$	$\bar{w}^2_{4,0,0} = -2393/1024$	$\bar{w}^2_{5,1,1} = 968597/1536$	
$\bar{w}^2_{2,1,-2} = 8669/2304$	$\bar{w}^2_{3,2,-1} = -294119/9216$	$\bar{w}^2_{4,1,0} = 30481/288$	$\bar{w}^2_{5,2,1} = 388941/512$	$\bar{w}^2_{3,3,3} = 316673/384$
$\bar{w}^2_{2,2,-2} = 23219/4608$	$\bar{w}^2_{3,3,-1} = -147917/4608$	$\bar{w}^2_{4,2,0} = 1035349/9216$	$\bar{w}^2_{5,3,1} = 1125383/1536$	$\bar{w}^2_{4,3,3} = 955577/1152$
$\bar{w}^2_{3,0,-2} = 2023/1536$	$\bar{w}^2_{4,0,-1} = -40747/4608$	$\bar{w}^2_{4,3,0} = 941717/9216$	$\bar{w}^2_{5,4,1} = 2261651/3072$	$\bar{w}^2_{4,4,3} = 961135/1152$
$\bar{w}^2_{3,1,-2} = 17479/4608$	$\bar{w}^2_{4,1,-1} = -106841/4608$	$\bar{w}^2_{4,4,0} = 158921/1536$	$\bar{w}^2_{5,5,1} = 848407/1152$	$\bar{w}^2_{5,3,3} = 477953/576$
$\bar{w}^2_{3,2,-2} = 15589/3072$	$\bar{w}^2_{4,2,-1} = -48965/1536$	$\bar{w}^2_{5,0,0} = -10745/4608$		$\bar{w}^2_{5,4,3} = 40061/48$
$\bar{w}^2_{3,3,-2} = 5887/1152$	$\bar{w}^2_{4,3,-1} = -295505/9216$	$\bar{w}^2_{5,1,0} = 487837/4608$		$\bar{w}^2_{5,5,3} = 961793/1152$
	$\bar{w}^2_{4,4,-1} = -12299/384$	$\bar{w}^2_{5,2,0} = 172613/1536$		
		$\bar{w}^2_{5,3,0} = 471023/4608$		$\bar{w}^2_{4,4,4} = 322231/384$
		$\bar{w}^2_{5,4,0} = 953855/9216$		$\bar{w}^2_{5,4,4} = 483511/576$
		$\bar{w}^2_{5,5,0} = 119273/1152$		$\bar{w}^2_{5,5,4} = 967351/1152$
				$\bar{w}^2_{5,5,5} = 840$

Proof Taking into account Theorem 3.2, the symmetry of the domain Ω and of the support Ξ_α of B_α, as in Theorem 3.1, we can deduce the symmetry properties of the weights and the values of the different ones, reported in Tables 3.3, 3.4 and 3.5. □

Remark 3.2 For $\nu = 2,3$, the values $\bar{w}^\nu_{0,-1,-2}$ and $\bar{w}^\nu_{1,-1,-2}$, appearing in Theorem 3.3, are equal to $\bar{w}^\nu_{0,0,-3}$ and $\bar{w}^\nu_{1,0,-3}$, respectively.

Remark 3.3 With the request $m_1, m_2, m_3 \geq 9$, in the construction of the weights w^ν_α, $\nu = 2,3,4$, as in (3.2), we have at least one fundamental spline L^ν_α, $\nu = 2,3,4$, with support completely included in the domain Ω. Therefore, we have at least one weight assuming the value of $w^\nu_{5,5,5} = \frac{h^3}{840}\bar{w}^\nu_{5,5,5}$, $\nu = 2,3,4$, with $\bar{w}^\nu_{5,5,5}$ given in Tables 3.3, 3.4 and 3.5.

Remark 3.4 The total number of function evaluations for $I_{Q^\nu}(f)$ is $m_1m_2m_3 + 8(m_1m_2 + m_1m_3 + m_2m_3) + 40(m_1 + m_2 + m_3) + 152$ in case $\nu = 2,3$ and it is $m_1m_2m_3 + 8(m_1m_2 + m_1m_3 + m_2m_3) + 40(m_1 + m_2 + m_3) + 184$ in case $\nu = 4$.

Remark 3.5 From Theorems 3.1 and 3.2, the sum of the absolute values of the weights, for each cubature $I_{Q^\nu}(f)$, $\nu = 1,2,3,4$, is bounded as follows

$$\sum_{\alpha \in \mathscr{A}^M} |w^\nu_\alpha| \leq \tilde{\mathscr{C}}_\nu \, |\Omega| \,, \tag{3.8}$$

Table 3.4 The 90 values $\bar{w}^3_\alpha$, with $w^3_\alpha = \frac{h^3}{840}\bar{w}^3_\alpha$

$\bar{w}^3_{0,0,-3} = -5/768$	$\bar{w}^3_{0,-1,-1} = -5/64$	$\bar{w}^3_{0,0,0} = 1293/256$	$\bar{w}^3_{1,1,1} = 109097/256$	$\bar{w}^3_{2,2,2} = 117327/128$
$\bar{w}^3_{1,0,-3} = -5/128$	$\bar{w}^3_{0,0,-1} = -403/256$	$\bar{w}^3_{1,0,0} = 6141/256$	$\bar{w}^3_{2,1,1} = 423995/768$	$\bar{w}^3_{3,2,2} = 28847/32$
$\bar{w}^3_{1,1,-3} = -215/768$	$\bar{w}^3_{1,-1,-1} = -215/384$	$\bar{w}^3_{1,1,0} = 86227/768$	$\bar{w}^3_{2,2,1} = 91185/128$	$\bar{w}^3_{3,3,2} = 113379/128$
$\bar{w}^3_{2,0,-3} = -35/768$	$\bar{w}^3_{1,0,-1} = -1403/256$	$\bar{w}^3_{2,0,0} = 7837/256$	$\bar{w}^3_{3,1,1} = 139401/256$	$\bar{w}^3_{4,2,2} = 342559/384$
$\bar{w}^3_{2,1,-3} = -245/768$	$\bar{w}^3_{1,1,-1} = -10477/768$	$\bar{w}^3_{2,1,0} = 108859/768$	$\bar{w}^3_{3,2,1} = 269387/384$	$\bar{w}^3_{4,3,2} = 224343/256$
$\bar{w}^3_{2,2,-3} = -35/96$	$\bar{w}^3_{2,-1,-1} = -245/384$	$\bar{w}^3_{2,2,0} = 538/3$	$\bar{w}^3_{3,3,1} = 132517/192$	$\bar{w}^3_{4,4,2} = 27741/32$
	$\bar{w}^3_{2,0,-1} = -893/128$	$\bar{w}^3_{3,0,0} = 7549/256$	$\bar{w}^3_{4,1,1} = 137911/256$	$\bar{w}^3_{5,2,2} = 342419/384$
$\bar{w}^3_{0,0,-2} = -115/256$	$\bar{w}^3_{2,1,-1} = -297/16$	$\bar{w}^3_{3,1,0} = 35305/256$	$\bar{w}^3_{4,2,1} = 533099/768$	$\bar{w}^3_{5,3,2} = 672749/768$
$\bar{w}^3_{1,0,-2} = -1265/768$	$\bar{w}^3_{2,2,-1} = -4781/192$	$\bar{w}^3_{3,2,0} = 133975/768$	$\bar{w}^3_{4,3,1} = 524363/768$	$\bar{w}^3_{5,4,2} = 20797/24$
$\bar{w}^3_{1,1,-2} = -4685/768$	$\bar{w}^3_{3,0,-1} = -5413/768$	$\bar{w}^3_{3,3,0} = 32543/192$	$\bar{w}^3_{4,4,1} = 86443/128$	$\bar{w}^3_{5,5,2} = 83153/96$
$\bar{w}^3_{2,0,-2} = -535/256$	$\bar{w}^3_{3,1,-1} = -229/12$	$\bar{w}^3_{4,0,0} = 22307/768$	$\bar{w}^3_{5,1,1} = 206759/384$	
$\bar{w}^3_{2,1,-2} = -185/24$	$\bar{w}^3_{3,2,-1} = -19579/768$	$\bar{w}^3_{4,1,0} = 13085/96$	$\bar{w}^3_{5,2,1} = 88809/128$	$\bar{w}^3_{3,3,3} = 27825/32$
$\bar{w}^3_{2,2,-2} = -3745/384$	$\bar{w}^3_{3,3,-1} = -3339/128$	$\bar{w}^3_{4,2,0} = 44135/256$	$\bar{w}^3_{5,3,1} = 87353/128$	$\bar{w}^3_{4,3,3} = 82565/96$
$\bar{w}^3_{3,0,-2} = -805/384$	$\bar{w}^3_{4,0,-1} = -903/128$	$\bar{w}^3_{4,3,0} = 128597/768$	$\bar{w}^3_{5,4,1} = 518413/768$	$\bar{w}^3_{4,4,3} = 81655/96$
$\bar{w}^3_{3,1,-2} = -2975/384$	$\bar{w}^3_{4,1,-1} = -7343/384$	$\bar{w}^3_{4,4,0} = 63511/384$	$\bar{w}^3_{5,5,1} = 64771/96$	$\bar{w}^3_{5,3,3} = 13755/16$
$\bar{w}^3_{3,2,-2} = -7525/768$	$\bar{w}^3_{4,2,-1} = -3269/128$	$\bar{w}^3_{5,0,0} = 3717/128$		$\bar{w}^3_{5,4,3} = 20405/24$
$\bar{w}^3_{3,3,-2} = -315/32$	$\bar{w}^3_{4,3,-1} = -20069/768$	$\bar{w}^3_{5,1,0} = 52325/384$		$\bar{w}^3_{5,5,3} = 27195/32$
	$\bar{w}^3_{4,4,-1} = -2513/96$	$\bar{w}^3_{5,2,0} = 66185/384$		
		$\bar{w}^3_{5,3,0} = 21427/128$		$\bar{w}^3_{4,4,4} = 26915/32$
		$\bar{w}^3_{5,4,0} = 42329/256$		$\bar{w}^3_{5,4,4} = 40355/48$
		$\bar{w}^3_{5,5,0} = 15869/96$		$\bar{w}^3_{5,5,4} = 80675/96$
				$\bar{w}^3_{5,5,5} = 840$

Table 3.5 The 98 values $\bar{w}^4_\alpha$, with $w^4_\alpha = \frac{h^3}{840}\bar{w}^4_\alpha$

$\bar{w}^4_{0,0,-3} = 47/9216$	$\bar{w}^4_{-1,-1,-1} = -7397/35328$	$\bar{w}^4_{0,0,0} = -58939/70656$	$\bar{w}^4_{1,1,1} = 37791221/70656$	$\bar{w}^4_{2,2,2} = 259825/276$
$\bar{w}^4_{1,0,-3} = 47/1536$	$\bar{w}^4_{0,-1,-1} = 2311/35328$	$\bar{w}^4_{1,0,0} = 168097/70656$	$\bar{w}^4_{2,1,1} = 139053119/211968$	$\bar{w}^4_{3,2,2} = 23867863/26496$
$\bar{w}^4_{1,1,-3} = 2021/9216$	$\bar{w}^4_{0,0,-1} = -236021/211968$	$\bar{w}^4_{1,1,0} = 21114467/211968$	$\bar{w}^4_{2,2,1} = 41892839/52992$	$\bar{w}^4_{3,3,2} = 3971947/4608$
$\bar{w}^4_{2,0,-3} = 329/9216$	$\bar{w}^4_{1,-1,-1} = 28871/17664$	$\bar{w}^4_{2,0,0} = 563495/211968$	$\bar{w}^4_{3,1,1} = 1942013/3072$	$\bar{w}^4_{4,2,2} = 4172413/4608$
$\bar{w}^4_{2,1,-3} = 2303/9216$	$\bar{w}^4_{1,0,-1} = -994019/211968$	$\bar{w}^4_{2,1,0} = 22612777/211968$	$\bar{w}^4_{3,2,1} = 26821019/35328$	$\bar{w}^4_{4,3,2} = 2662541/3072$
$\bar{w}^4_{2,2,-3} = 329/1152$	$\bar{w}^4_{1,1,-1} = -507221/23552$	$\bar{w}^4_{2,2,0} = 12210089/105984$	$\bar{w}^4_{3,3,1} = 840175/1152$	$\bar{w}^4_{4,4,2} = 1003919/1152$
	$\bar{w}^4_{2,-1,-1} = 33725/17664$	$\bar{w}^4_{3,0,0} = 1303/1024$	$\bar{w}^4_{4,1,1} = 5845289/9216$	$\bar{w}^4_{5,2,2} = 1391243/1536$
$\bar{w}^4_{-1,-1,-2} = -1/276$	$\bar{w}^4_{2,0,-1} = -621953/105984$	$\bar{w}^4_{3,1,0} = 928289/9216$	$\bar{w}^4_{4,2,1} = 7026649/9216$	$\bar{w}^4_{5,3,2} = 7990255/9216$
$\bar{w}^4_{0,-1,-2} = -73/23552$	$\bar{w}^4_{2,1,-1} = -1476365/52992$	$\bar{w}^4_{3,2,0} = 22849751/211968$	$\bar{w}^4_{4,3,1} = 750225/1024$	$\bar{w}^4_{5,4,2} = 125531/144$
$\bar{w}^4_{0,0,-2} = 279/1024$	$\bar{w}^4_{2,2,-1} = -1259011/35328$	$\bar{w}^4_{3,3,0} = 12859/128$	$\bar{w}^4_{4,4,1} = 3391325/4608$	$\bar{w}^4_{5,5,2} = 334859/384$
$\bar{w}^4_{1,-1,-2} = 2927/35328$	$\bar{w}^4_{3,-1,-1} = 60181/35328$	$\bar{w}^4_{4,0,0} = 14191/9216$	$\bar{w}^4_{5,1,1} = 2923655/4608$	
$\bar{w}^4_{1,0,-2} = 11657/9216$	$\bar{w}^4_{3,0,-1} = -1229383/211968$	$\bar{w}^4_{4,1,0} = 58729/576$	$\bar{w}^4_{5,2,1} = 292873/384$	$\bar{w}^4_{3,3,3} = 316673/384$
$\bar{w}^4_{1,1,-2} = 21271/9216$	$\bar{w}^4_{3,1,-1} = -2788285/105984$	$\bar{w}^4_{4,2,0} = 1007335/9216$	$\bar{w}^4_{5,3,1} = 844291/1152$	$\bar{w}^4_{4,3,3} = 955577/1152$
$\bar{w}^4_{2,-1,-2} = 5891/70656$	$\bar{w}^4_{3,2,-1} = -2389013/70656$	$\bar{w}^4_{4,3,0} = 313229/3072$	$\bar{w}^4_{5,4,1} = 2261651/3072$	$\bar{w}^4_{4,4,3} = 961135/1152$
$\bar{w}^4_{2,0,-2} = 326521/211968$	$\bar{w}^4_{3,3,-1} = -49441/1536$	$\bar{w}^4_{4,4,0} = 158921/1536$	$\bar{w}^4_{5,5,1} = 848407/1152$	$\bar{w}^4_{5,3,3} = 477953/576$
$\bar{w}^4_{2,1,-2} = 41099/11776$	$\bar{w}^4_{4,-1,-1} = 2611/1536$	$\bar{w}^4_{5,0,0} = 791/512$		$\bar{w}^4_{5,4,3} = 40061/48$
$\bar{w}^4_{2,2,-2} = 524315/105984$	$\bar{w}^4_{4,0,-1} = -6685/1152$	$\bar{w}^4_{5,1,0} = 469973/4608$		$\bar{w}^4_{5,5,3} = 961793/1152$
$\bar{w}^4_{3,-1,-2} = 245/3072$	$\bar{w}^4_{4,1,-1} = -7553/288$	$\bar{w}^4_{5,2,0} = 20993/192$		
$\bar{w}^4_{3,0,-2} = 1771/1152$	$\bar{w}^4_{4,2,-1} = -17269/512$	$\bar{w}^4_{5,3,0} = 58751/576$		$\bar{w}^4_{4,4,4} = 322231/384$
$\bar{w}^4_{3,1,-2} = 343/96$	$\bar{w}^4_{4,3,-1} = -32879/1024$	$\bar{w}^4_{5,4,0} = 953855/9216$		$\bar{w}^4_{5,4,4} = 483511/576$
$\bar{w}^4_{3,2,-2} = 46361/9216$	$\bar{w}^4_{4,4,-1} = -12299/384$	$\bar{w}^4_{5,5,0} = 119273/1152$		$\bar{w}^4_{5,5,4} = 967351/1152$
$\bar{w}^4_{3,3,-2} = 5887/1152$				
				$\bar{w}^4_{5,5,5} = 840$

where $|\Omega|$ denotes the measure of Ω and

$$\tilde{\mathscr{C}}_\nu = \begin{cases} 1, & \nu = 1 \\ \frac{131}{24}, & \nu = 2 \\ \frac{13}{8}, & \nu = 3 \\ \frac{5371}{552}, & \nu = 4 \end{cases}.$$

Since the weights w^1_α are positive, then, in case $\nu = 1$, (3.8) is an equality.

Therefore, from (3.8) and the multivariate version of the Polya-Steklov theorem, the cubatures over Ω are stable [21,31].

Finally, from Theorem 2.1 we can immediately deduce some results on the convergence of sequences of cubatures $I_{Q^\nu}(f)$, $\nu = 1,2,3,4$ to $I(f)$.

Theorem 3.4 *Let $f \in C(\Omega_h)$, then we have*

$$I_{Q^\nu}(f) \to I(f) \quad \text{as } h \to 0, \qquad \nu = 1,2,3,4.$$

In particular

$$|\, I(f) - I_{Q^\nu}(f)\,| \le \overline{\mathscr{C}_\nu}\,\omega(f,h),$$

where $\overline{\mathscr{C}_\nu}$ is a positive constant independent on m_1, m_2 and m_3.

Moreover,

- *if $f \in C^k(\Omega_h)$, $k = 1,2$, then $|\, I(f) - I_{Q^1}(f)\,| = O(h^k)$;*
- *if $f \in C^k(\Omega_h)$, $k = 1,2,3,4$, then $|\, I(f) - I_{Q^\nu}(f)\,| = O(h^k)$, $\nu = 2,3,4$.*

Remark 3.6 Thanks to the symmetry properties of the rules (3.1), if $\Omega = [-m_1h, m_1h] \times [-m_2h, m_2h] \times [-m_3h, m_3h]$, then

$$I_{Q^\nu}(f) = I(f), \qquad f = x^{r_1} y^{r_2} z^{r_3},$$

when at least one of the r_j's, $j = 1,2,3$ is odd.

4 Numerical results

In this section we present some numerical results obtained by computational procedures developed in a Matlab environment.

We compare our cubatures with other known ones having the same approximation order $O(h^4)$ of the error, i.e. we consider

- $I_{Q^\nu}(f)$, $\nu = 2,3,4$, defined in (3.1);

- $I_P(f)$ cubatures based on tensor product of univariate C^1 quadratic spline QIs proposed in [13];
- $I_R(f)$ cubatures based on blending sums of univariate and bivariate C^1 quadratic spline QIs proposed in [14];
- $I_S(f)$ cubatures based on tensor product of univariate composite Simpson's rules;
- $I_{H^1}(f)$ composite non-product formulas for a cube exact on $\mathbb{P}_3$ proposed in [12, p. 367], [16, p. 204], [31, p. 230];
- $I_{H^2}(f)$ composite non-product formulas for a cube exact on $\mathbb{P}_3$ proposed in [12, p. 368], [16, p. 204], [31, p. 230] (different nodes and weights with respect to the rules $I_{H^1}(f)$).

We remark that the total number of function evaluations for each cubature rule is

- $m_1m_2m_3 + 2(m_1m_2 + m_1m_3 + m_2m_3) + 4(m_1 + m_2 + m_3) + 8$ for $I_P(f)$ and $I_R(f)$;
- $m_1m_2m_3 + (m_1m_2 + m_1m_3 + m_2m_3) + (m_1 + m_2 + m_3) + 1$ for $I_S(f)$;
- $3m_1m_2m_3 + (m_1m_2 + m_1m_3 + m_2m_3)$ for $I_{H^1}(f)$;
- $6m_1m_2m_3$ for $I_{H^2}(f)$.

For $I_{Q^\nu}(f)$, $\nu = 2,3,4$, the number of function evaluations is reported in Remark 3.4.

We assume as integration domain the standard cube $\Omega = [0,1]^3$, $m_1 = m_2 = m_3 = m$ and $h = 1/m$. We assume m an even number, since we consider the composite Simpson's rule on $m+1$ equally spaced points on $[0,1]$, for each direction x, y and z. The rules $I_{H^1}(f)$ and $I_{H^2}(f)$ are used in Ω by a composite technique, applying in each subcube of edge h the non-product rules proposed in [12,16,31].

We apply the above cubature rules to several smooth integrand functions. The first three ones come from the testing package of Genz [17,18], the fourth one from [13].

The test functions are the following, for which we report the exact value of the integral:

- $f_1(x,y,z) = \cos\left(\frac{9\pi x}{2} + \frac{9\pi y}{2} + \frac{9\pi z}{2}\right)$ (Oscillatory function), $I(f_1) = -\frac{16}{729\pi^3}$;
- $f_2(x,y,z) = 1/\left(1+x+y+z\right)^4$ (Corner peak function), $I(f_2) = \frac{1}{24}$;
- $f_3(x,y,z) = e^{\left((x-0.5)^2+(y-0.5)^2+(z-0.5)^2\right)}$ (Gaussian function), $I(f_3) = 0.7852115962$;
- $f_4(x,y,z) = \frac{\pi y}{2(e-2)} e^{xy} \sin(\pi z)$, $I(f_4) = 1$.

We compute the absolute errors

$$E_{\mathscr{Q}}(f) = |I(f) - I_{\mathscr{Q}}(f)|, \text{ for } \mathscr{Q} = Q^2, Q^3, Q^4, P, R, S, H^1, H^2$$

for each test function and for increasing values of m, i.e. $m = 16$, 32, 64, 128, and we report the corresponding graphs in Figs. 4.1, 4.2, 4.3, 4.4.

The numerical results shown in Figs. 4.1, 4.2, 4.3, 4.4 confirm the convergence properties given in Section 3 for $I_{Q^\nu}(f)$, $\nu = 2,3,4$ and in the literature for the other rules.

Moreover, our cubature rules are comparable and the formula $I_{Q^4}(f)$ seems to be better than the other ones.

In Figs. 4.1, 4.2, 4.3 the error is smaller for the two formulas I_{H^1}, I_{H^2}, but we remark that such formulas use a greater number of functional evaluations, namely

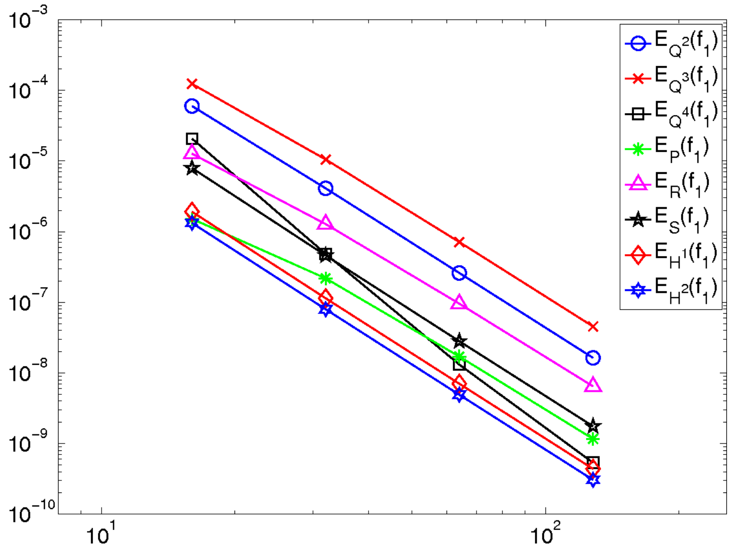


Fig. 4.1 Absolute errors for $I(f_1)$

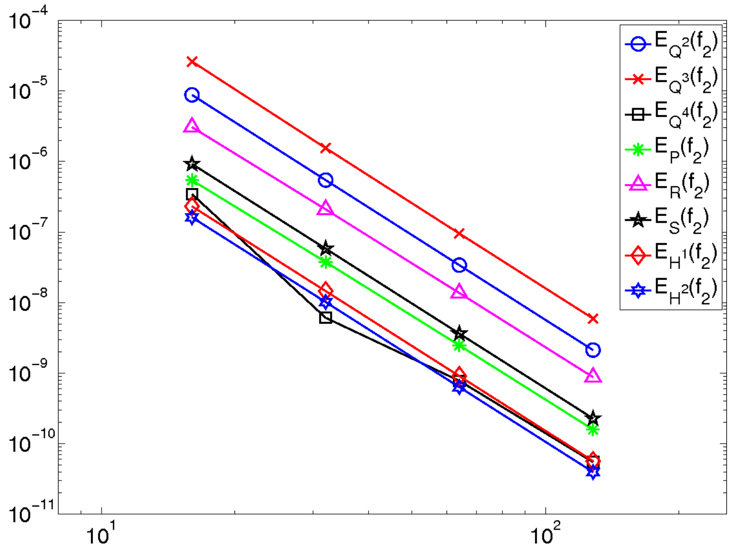


Fig. 4.2 Absolute errors for $I(f_2)$

$O(3m^3)$ and $O(6m^3)$, respectively, instead of $O(m^3)$ and this is evident especially for high values of m.

Furthermore, we recall that the cubature rules $I_P(f)$ are based on C^1 splines of degree six (tensor product of univariate C^1 quadratic spline QIs), the rules $I_R(f)$ are based on C^1 splines of degree four (blending sums of univariate and bivariate C^1 quadratic spline QIs) and here we have proposed new integration formulas based on trivariate spline quasi-interpolants on type-6 tetrahedral partitions of total degree four

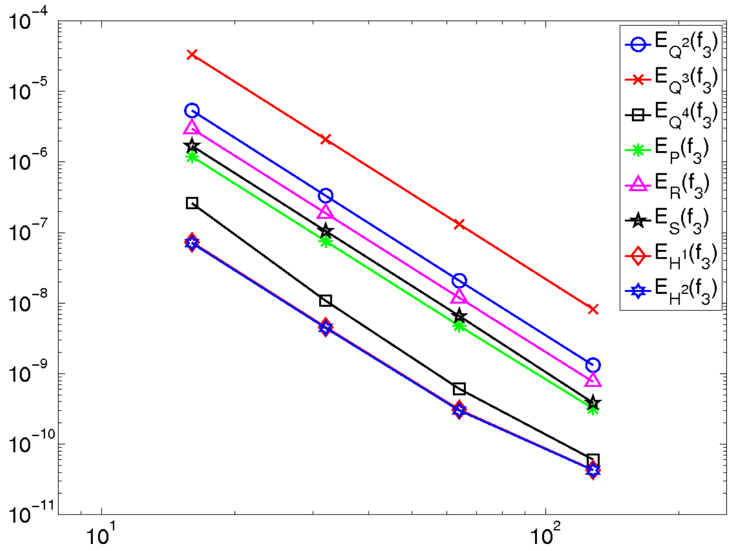


Fig. 4.3 Absolute errors for $I(f_3)$

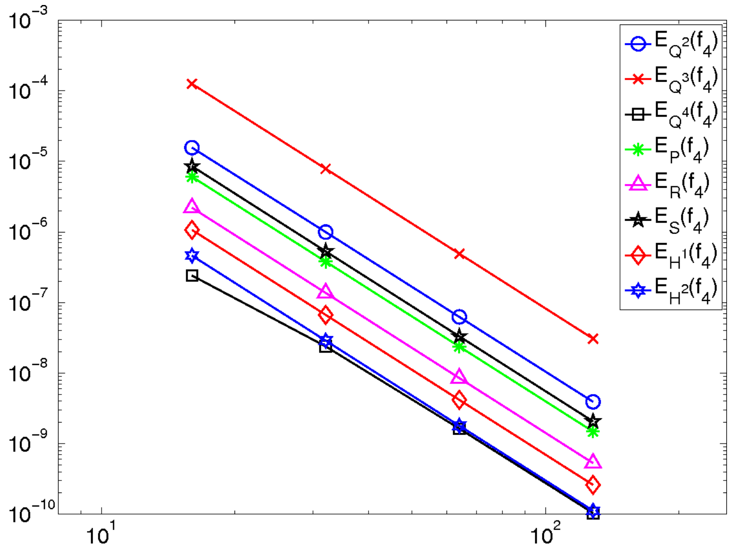


Fig. 4.4 Absolute errors for $I(f_4)$

with C^2 smoothness. Such a higher smoothness is useful, for example, in the numerical treatment of integral equations, where the unknown function can be reconstructed with C^2 smoothness.

Finally, we propose another example in case of integration domain different from the standard cube. We want to evaluate the integral

$$I(f) = \int_{\Omega'} \frac{x^2}{x^2+z^2}\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \tag{4.1}$$

where $\Omega' = \{(x,y,z) \in \mathbb{R}^3 : 1 < x^2+y^2+z^2 < 2,\ x^2-y^2+z^2 < 0,\ y > 0\}$. We know that $I(f) = \frac{\pi}{6}(5\sqrt{2}-6)$. By using the spherical coordinates and affine transformations, we get an integral on $[0,1]^3$. Then, we evaluate it by the three cubatures $I_{Q^\nu}(f)$, $\nu = 2,3,4$, for increasing values of m ($m = 16,\ 32,\ 64,\ 128$) and we compute the corresponding absolute errors, obtaining the results shown in Fig. 4.5, where we note again the better behaviour of $I_{Q^4}(f)$.

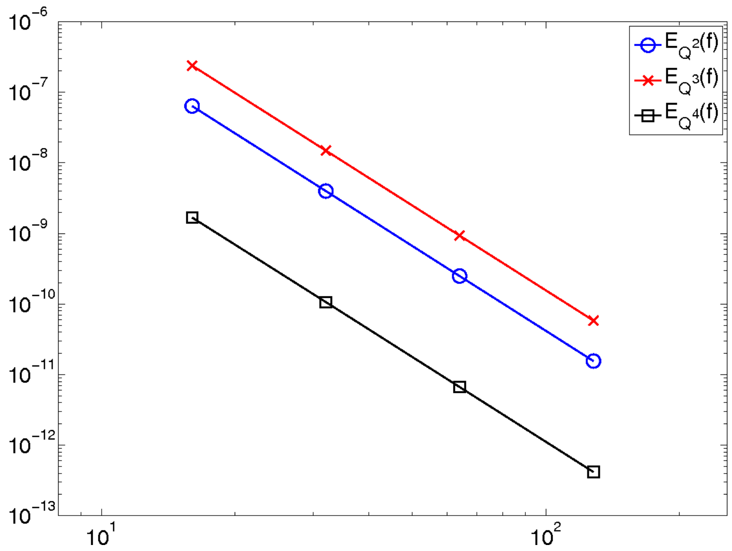


Fig. 4.5 Absolute error for (4.1)

5 Final remarks

In this paper we have considered the space $S_4^2(\Omega, \mathscr{T}_{\mathbf{m}})$ generated by the scaled translates of the trivariate C^2 quartic box spline B defined by a set X of seven directions, that forms a regular partition of the space into tetrahedra. Then, we have constructed new cubature rules for 3D integrals, based on spline quasi-interpolants expressed as linear combinations of scaled translates of B and local linear functionals.

We have provided weights and nodes of the above rules and we have analysed their properties.

Finally, some numerical tests and comparisons with other known integration formulas have been presented.

We remark that the points used in the integration formulas here proposed lie also outside the integration domain. Since the function to be integrated may not be defined outside the domain of integration, an interesting development of this paper could be the study and construction of spline cubature rules, based on linear combinations of the scaled translates of the box spline *B*, making use of evaluation points inside or on the boundary of the domain.

Moreover, in case of integrands with singularities in the first partial derivatives, it could be interesting the construction of 3D cubature formulas based on trivariate B-splines defined on non-uniform partitions, in order to simulate such singularities.

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