Unbounded solutions and periodic solutions of perturbed isochronous Hamiltonian systems at resonance

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Abstract

In this paper we deal with the existence of unbounded orbits of the map

$$\begin{cases} \theta_1\theta + \dfrac{1}{\rho}[\mu(\theta) - l_1(\rho)] + h_1(\rho, \theta), \\ \rho_1 = \rho - \mu'(\theta) + l_2(\rho) + h_2(\rho, \theta), \end{cases}$$

where μ is continuous and 2π-periodic, l_1, l_2 are continuous and bounded, $h_1(\rho, \theta) = o(\rho^{-1})$, $h_2(\rho, \theta) = o(1)$, for $\rho \to +\infty$. We prove that every orbit of the map tends to infinity in the future or in the past for ρ large enough provided that

$$[\liminf_{\rho\to+\infty} l_1(\rho), \limsup_{\rho\to+\infty} l_1(\rho)] \cap Range(\mu) = \emptyset$$

and other conditions hold. On the basis of this conclusion, we prove that the system $Jz' = \nabla H(z) + f(z) + p(t)$ has unbounded solutions when H is positively homogeneous of degree 2 and positive. Meanwhile, we also obtain the existence of 2π-periodic solutions of this system.

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1 Introduction

We are concerned with the coexistence of unbounded and periodic solutions of the system

$$Jz' = \nabla H(z) + f(z) + p(t), \tag{1.1}$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the standard symplectic matrix, the function $H : \mathbb{R}^2 \to \mathbb{R}$ is a C^1-function, with locally Lipschitz continuous gradient, $f : \mathbb{R}^2 \to \mathbb{R}^2$ is locally Lipschitz continuous and $p : \mathbb{R} \to \mathbb{R}^2$ is continuous and 2π-periodic.
We assume that the Hamiltonian function H is positively homogeneous of degree 2 and positive; in this situation the origin is an isochronous center for the autonomous Hamiltonian system

$$Jz' = \nabla H(z). \tag{1.2}$$

This means that all the solutions of (1.2) are periodic with the same minimal period T; we suppose that 2π is an integer multiple of T.
A classical example of (1.1) is the first order system equivalent to the well-known equation

$$x'' + \alpha x^+ - \beta x^- + g(x) = p(t), \tag{1.3}$$

where $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$, α and β are two positive constants satisfying

$$\frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}} = \frac{2}{n},$$

for some $n \in \mathbb{N}$. We recall that in this situation all solutions of the homogeneous equation $x'' + \alpha x^+ - \beta x^- = 0$ can be written in the form $x(t) = A\phi(t+\theta)$ for some $A \geq 0$ and $\theta \in [0, 2\pi/n)$, where ϕ is the $2\pi/n$-periodic function defined by

$$\phi(t) = \begin{cases} \dfrac{1}{\sqrt{\alpha}} \sin(\sqrt{\alpha} t), & t \in \left[0, \dfrac{\pi}{\sqrt{\alpha}}\right), \\ -\dfrac{1}{\sqrt{\beta}} \sin\left[\sqrt{\beta}(t - \dfrac{\pi}{\sqrt{\alpha}})\right], & t \in \left[\dfrac{\pi}{\sqrt{\alpha}}, \dfrac{2\pi}{n}\right]. \end{cases}$$

In order to deal with the existence of periodic solutions of Eq. (1.3), Dancer [6] first introduced the function

$$\Phi(\theta) = 2n\left[\frac{g(+\infty)}{\alpha} - \frac{g(-\infty)}{\beta}\right] - \int_0^{2\pi} p(t)\phi(t+\theta)dt,$$

where the limits $g(\pm\infty) = \lim_{x\to\pm\infty} g(x)$ exist and are finite. Later, Fabry and Fonda [7] proved that Eq. (1.3) has at least one 2π-periodic solution provided that Φ has a constant sign or has $2k$ simple zeros in $[0, 2\pi/n)$, with $k \geq 2$. More recently, Fabry and Mawhin [9] generalized in various directions the results in [7]; in particular, they replaced in the definition of Φ the constants $g(\pm\infty)$ with

$$G(\pm\infty) = \lim_{x\to\pm\infty} \frac{G(x)}{x},$$

where $G(x) = \int_0^x g(s)ds$. Moreover, they also proved the coexistence of periodic solutions and unbounded solutions of Eq. (1.3).
Later, these results have been improved or extended to various classes of forced Liénard and Rayleigh equations with asymmetric nonlinearities (cf. [2, 3, 4, 5, 14, 15, 17]). In particular, Fonda [10] investigated the dynamics of the solutions of a planar isochronous Hamiltonian system of the form

$$Jz' = \nabla H(z) + p(t). \tag{1.4}$$

It was proved in [10] that most of the known results for Eq. (1.3) still hold for system (1.4). Subsequently, Fonda and Mawhin [11] explored the coexistence of periodic solutions and unbounded solutions of the more general system (1.1) (see also [8]). To do this, it is assumed in [11] that $f : \mathbb{R}^2 \to \mathbb{R}^2$ can be written in the form

$$f(z) = \sum_{k=1}^{m} f_k(< z, e^{i\vartheta_k} >), \tag{1.5}$$

where

$$0 \leq \vartheta_1 < \vartheta_2 < \cdots < \vartheta_m < 2\pi$$

are $m \geq 1$ fixed directions and $f_k : \mathbb{R} \to \mathbb{R}^2$; here, $< \cdot, \cdot >$ denotes the Euclidean scalar product in $\mathbb{R}^2$. When f takes the form (1.5), system (1.1) can cover many equations such as forced Liénard equations, Rayleigh equations with asymmetric nonlinearities. Moreover, it is supposed in [11] that the limits

$$F_k^{\pm} = \lim_{s \to \pm\infty} \frac{F_k(s)}{s} \tag{1.6}$$

exist in $\mathbb{R}^2$, where $F_k(x) = \int_0^x f_k(s)ds$. Conditions (1.6) are always satisfied if the limits $f_k^{\pm} = \lim_{s \to \pm\infty} f_k(s)$ exist in $\mathbb{R}^2$. The results in [11] are based on a detailed analysis of the Poincaré map of system (1.1) via some suitable change of variables. More precisely, when (1.6) holds the asymptotic expression of the Poincaré map is

$$\begin{cases} \theta_1 = \theta + \dfrac{1}{\rho}[\mu(\theta) - c_1] + o(\rho^{-1}), \\ \rho_1 = \rho - \mu'(\theta) + c_2 + o(1), \quad \rho \to +\infty, \end{cases}$$

where c_1, c_2 are two constants depending on $F_k^{\pm}$ and

$$\mu(\theta) = \int_0^{2\pi} < p(t), \varphi(t + \theta) > dt,$$

being φ is a solution of (1.2) satisfying

$$H(\varphi(t)) = \frac{1}{2}, \quad t \in \mathbb{R}.$$

In [11] it is proved the existence of periodic solutions when $\mu - c_1$ or $\mu' - c_2$ has constant sign, and also when $\mu - c_1$ has zeros and $\mu' - c_2$ has constant sign or changes sign more than twice on

the zeros of $\mu - c_1$. On the other hand, when $\mu - c_1$ has constant sign or has only simple zeros, it is also proved that all solutions of (1.1) with sufficiently large amplitude are unbounded in the future or in the past.
In our paper we take [11] as a starting point; hence, we study systems of the form (1.1) with f having the form above described (see also [13]). The main difference with [11] will be in the fact that we shall only assume the boundedness of the functions f_k. In this case, the Poincaré map of system (1.1) can be expressed in the form:

$$\begin{cases} \theta_1 = \theta + \dfrac{1}{\rho}[\mu(\theta) - l_1(\rho)] + o(\rho^{-1}), \\ \\ \rho_1 = \rho - \mu'(\theta) + l_2(\rho) + o(1), \quad \rho \to +\infty, \end{cases} \tag{1.7}$$

where $l_1(\rho)$, $l_2(\rho)$ are two continuous bounded functions (which are constants if (1.6) holds). In our general situation, we cannot directly use the transformation $\rho = (\delta r)^{-1}$ as in [1] to get a difference equation which can be regarded as a numerical approximation of a differential equation. To overcome this difficulty, a new approach for the investigation of the iterates of the planar map (1.7) is necessary; more precisely, we carefully explore the dynamics of the family of maps

$$\begin{cases} \theta_1 = \theta + \dfrac{1}{\rho}[\mu(\theta) - l_1(s)] + o(\rho^{-1}), \\ \\ \rho_1 = \rho - \mu'(\theta) + l_2(s) + o(1), \quad \rho \to +\infty, \end{cases} \tag{1.8}$$

where $s > 0$ is a parameter. As a consequence of our result on planar maps, we are able to prove (cf. Theorem 3.4) the coexistence of periodic and unbounded solutions to (1.1).
Concerning the notations o, O, throughout this paper the involved limits are always intended *uniformly* w.r.t. all the other variables; for example, in (1.8) by writing $o(1), \rho \to +\infty$ we mean that the term tends to zero *uniformly* w.r.t. $\theta \in [0, 2\pi]$.

2 Unbounded orbits of planar maps

Given $\sigma > 0$, let B_σ be the open ball centered at the origin and with radius σ. Set $E_\sigma = R^2 \backslash B_\sigma$. Assume that the map $P : E_\sigma \to R^2$ is a one-to-one and continuous map, whose lift (also denoted by P) can be expressed in the form:

$$P : \begin{cases} \theta_1 \theta + \frac{1}{\rho}[\mu(\theta) - l_1(\rho)] + h_1(\rho, \theta), \\ \rho_1 = \rho - \mu'(\theta) + l_2(\rho) + h_2(\rho, \theta), \end{cases} \tag{2.1}$$

where $\mu \in C^1(S^1)$ with $S^1 = \mathbb{R}^1/2\pi\mathbb{Z}$, $l_1, l_2 \in C[\sigma, +\infty)$ and $h_1, h_2 \in C([\sigma, +\infty) \times S^1)$ satisfy

$$h_1(\rho, \theta) = o\left(\frac{1}{\rho}\right), \quad h_2(\rho, \theta) = o(1), \quad \rho \to +\infty. \tag{2.2}$$

Given a point $(\rho_0, \theta_0) \in E_\sigma$, we denote by $\{(\rho_j, \theta_j)\}$ the orbit of the map P through the point (ρ_0, θ_0), i.e.

$$P(\rho_j, \theta_j) = (\rho_{j+1}, \theta_{j+1}).$$

For two continuous bounded functions l_1, l_2, we introduce the following notation:

$$a = \liminf_{\rho \to +\infty} l_1(\rho), \quad b = \limsup_{\rho \to +\infty} l_1(\rho);$$

$$c = \liminf_{\rho \to +\infty} l_2(\rho), \quad d = \limsup_{\rho \to +\infty} l_2(\rho).$$

We can prove the following result.

Proposition 2.1 *Assume $a = b$ and $\mu(\theta) - b \neq 0$, for every $\theta \in [0, 2\pi]$. Then the following conclusions hold:*

(1) if $c > 0$ then there exists $R_0 > 0$ such that, for $\rho_0 \geq R_0$, the orbit $\{(\rho_j, \theta_j)\}$ exists in the future and satisfies $\lim_{j \to +\infty} \rho_j = +\infty$.

(2) if $d < 0$ then there exists $R_0 > 0$ such that, for $\rho_0 \geq R_0$, the orbit $\{(\rho_j, \theta_j)\}$ exists in the past and satisfies $\lim_{j \to -\infty} \rho_j = +\infty$.

Remark 2.2 *In [16] the unboundedness of the orbits of the map P was studied in case when $l_1(\rho) \equiv constant$ and $l_2(\rho) \equiv constant$. Thus, the result in [16] can be regarded as a special case of Proposition 2.1.*

An analogous result is valid (under an additional condition) in case $a \neq b$ as well. It is stated at the end of this Section and its proof is similar to the one of Proposition 2.1.
In what follows we give a Lemma which is valid whenever $a = b$ or $a \neq b$ holds. For brevity, we only deal with the case $\mu(\theta) - b > 0$, for all θ. The other cases can be handled similarly.
Let us observe that, since l_1, l_2 depend on ρ, the methods in [1] cannot be applied. To overcome this difficulty we consider the family of planar maps $P_s : E_\sigma \to \mathbb{R}^2$ defined by

$$P_s : \begin{cases} \theta_1 \theta + \dfrac{1}{\rho}[\mu(\theta) - l_1(s)] + h_1(\rho, \theta), \\ \rho_1 = \rho - \mu'(\theta) + l_2(s) + h_2(\rho, \theta), \end{cases} \tag{2.3}$$

where $s \geq \sigma$ is a parameter.
Now we introduce the transformation (see [1])

$$\frac{1}{\rho} = \delta r,$$

where $\delta > 0$ is a parameter to be determined later. Under this transformation, (2.3) becomes

$$\tilde{P}_s : \begin{cases} \theta_1 = \theta + \delta r[\mu(\theta) - l_1(s)] + h_{11}(r, \theta, s, \delta), \\ r_1 = r + \delta r^2[\mu'(\theta) - l_2(s)] + \delta r^2 h_{21}(r, \theta, s, \delta), \end{cases} \tag{2.4}$$

where

$$h_{11}(r, \theta, s, \delta) = h_1(\delta^{-1} r^{-1}, \theta),$$

$$h_{21}(r, \theta, s, \delta) = -h_2(\delta^{-1} r^{-1}, \theta) + \frac{[l_2(s) - \mu'(\theta) + h_2(\delta^{-1} r^{-1}, \theta)]^2}{\delta^{-1} r^{-1} + l_2(s) - \mu'(\theta) + h_2(\delta^{-1} r^{-1}, \theta)}.$$

It follows from (2.2) that

$$\lim_{\delta\to 0^+} \delta^{-1} r^{-1} h_{11}(r,\theta,s,\delta) = 0, \quad \lim_{\delta\to 0^+} h_{21}(r,\theta,s,\delta) = 0 \tag{2.5}$$

uniformly in $\theta, s \geq \sigma$ and sufficiently small r.

We observe that in general the term $\mu'(\theta) - l_2(s)$ in (2.4) does not have constant sign; the next transformation leads to a planar mapping where the corresponding term has definite sign.
To this aim, we consider the system

$$\theta' = r\nu(\theta), \quad r' = r^2\nu'(\theta), \quad (r > 0), \tag{2.6}$$

where $\nu(\theta) = \mu(\theta) - b > 0$, for all θ. The first integral of (2.6) is

$$I(r,\theta) = \frac{\nu(\theta)}{r}. \tag{2.7}$$

Therefore, the orbits of (2.6) can be expressed in the form

$$\Gamma_h : I(r,\theta) = \frac{\nu(\theta)}{r} = h,$$

where $h > 0$ is an arbitrary constant. Let $(r(t), \theta(t))$ be the solution of (2.6) lying on the curve Γ_h. Obviously, $(r(t), \theta(t))$ is a periodic solution; we denote by $T(h)$ its minimal period. From the first equation in (2.6) and (2.7) we get that

$$T(h) = h\int_0^{2\pi} \frac{d\vartheta}{\nu^2(\vartheta)} = \lambda h,$$

where

$$\lambda = \int_0^{2\pi} \frac{d\vartheta}{\nu^2(\vartheta)} > 0.$$

We now introduce the functions

$$\omega(h) = \frac{2\pi}{T(h)} = \frac{2\pi}{\lambda h}, \quad K(r,\theta) = \frac{\nu(\theta)}{r}\int_0^{\theta} \frac{d\vartheta}{\nu^2(\vartheta)}.$$

Let us define

$$\tau(\theta) = \omega(I(r,\theta))K(r,\theta) = \frac{2\pi}{\lambda}\int_0^{\theta} \frac{d\vartheta}{\nu^2(\vartheta)}$$

and $\Psi : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+ \times \mathbb{R}$ by

$$\Psi : (r,\theta) \to (I,\tau) = (I(r,\theta), \tau(\theta)).$$

It is easy to check that the map Ψ is bijective; its inverse Ψ^{-1} satisfies the relations

$$\Psi^{-1}(I,\tau) = (r,\theta),$$

$$r(I,\tau) = \frac{\nu(\theta(\tau))}{I}, \quad \frac{2\pi}{\lambda}\int_0^{\theta(\tau)} \frac{d\vartheta}{\nu^2(\vartheta)} = \tau.$$

Moreover, the functions τ and θ fulfill

$$\tau(0) = 0, \quad \tau(\theta + 2\pi) = \tau(\theta) + 2\pi, \quad \forall\, \theta \in \mathbb{R},$$

$$\theta(0) = 0, \quad \theta(\tau + 2\pi) = \theta(\tau) + 2\pi, \quad \forall\, \tau \in \mathbb{R}.$$

Finally, we consider the map $\hat{P}_s$:

$$\hat{P}_s = \Psi \circ \tilde{P}_s \circ \Psi^{-1} : (I, \tau) \to (I_1, \tau_1) = \hat{P}_s(I, \tau).$$

Remark 2.3 *When $\mu(\theta) < a$, for every θ, we can proceed in a similar way. More precisely, we can consider the system*

$$\theta' = r\hat{\nu}(\theta), \quad r' = r^2\hat{\nu}'(\theta), \quad (r > 0),$$

where $\hat{\nu}(\theta) = \mu(\theta) - a < 0$, for every $\theta \in [0, 2\pi]$. Set

$$I(r, \theta) = \frac{\hat{\nu}(\theta)}{r}, \quad \tau(\theta) = \frac{2\pi}{\hat{\lambda}} \int_0^\theta \frac{d\theta}{\hat{\nu}^2(\theta)}.$$

with $\hat{\lambda} = -\int_{-2\pi}^{0} \frac{d\theta}{\hat{\nu}^2(\theta)} < 0$. Define $\hat{\Psi} : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^- \times \mathbb{R}, (r, \theta) \to (I, \tau) = (I(r, \theta), \tau(\theta))$ as follows:

$$I(r, \theta) = \frac{\hat{\nu}(\theta)}{r}, \quad \tau(\theta) = \frac{2\pi}{\hat{\lambda}} \int_0^\theta \frac{d\theta}{\hat{\nu}^2(\theta)}.$$

We can thus consider the map $\bar{P}_s$:

$$\bar{P}_s = \hat{\Psi} \circ \tilde{P}_s \circ \hat{\Psi}^{-1} : (I, \tau) \to (I_1, \tau_1) = \bar{P}_s(I, \tau).$$

Lemma 2.4 *Assume that l_1, l_2 are continuous and bounded, $\mu \in C^2[0, 2\pi]$ and $\mu(\theta) > b$, for every θ. Then the map $\hat{P}_s$ can be expressed in the form:*

$$\hat{P}_s : \begin{cases} \tau_1 = \tau + \delta\omega(I)\left[1 + \frac{b - l_1(s)}{\nu(\theta(\tau))}\right] + \delta h_{12}(I, \tau, s, \delta), \\ I_1 = I + \delta\left[\nu(\theta(\tau))l_2(s) + \nu'(\theta(\tau))(b - l_1(s))\right] + \delta h_{22}(I, \tau, s, \delta), \end{cases}$$

where h_{11} and h_{22} satisfy

$$\lim_{\delta \to 0^+} I h_{12}(I, \tau, s, \delta) = 0, \quad \lim_{\delta \to 0^+} h_{22}(I, \tau, s, \delta) = 0,$$

uniformly in $\tau \in \mathbb{R}$, $s \geq \sigma$ and sufficiently large I.

Proof. Let us first consider the expression of $\tilde{P} \circ \Psi^{-1}$. Set $\tilde{P} \circ \Psi^{-1}(I, \tau) = (\theta_1, r_1)$. Since $\Psi^{-1}(I, \tau) = (r(I, \tau), \theta(\tau))$, from (2.4) we get that

$$\begin{cases} \theta_1 &= \theta(\tau) + \delta r(I, \tau)\nu(\theta(\tau)) + \delta r(I, \tau)\left[b - l_1(s)\right] + h_{11}\left(r(I, \tau), \theta(\tau), s, \delta\right), \\ r_1 &= r(I, \tau) + \delta r^2(I, \tau)[\nu'(\theta(\tau)) - l_2(s)] + \delta r^2(I, \tau) h_{21}(r(I, \tau), \theta(\tau), s, \delta). \end{cases}$$

Using the relation $r(I,\tau)=\nu(\theta(\tau))/I$, we can infer that

$$\begin{cases} \theta_1 &= \theta(\tau)+\dfrac{\delta\nu^2(\theta(\tau))}{I}+\dfrac{\delta\nu(\theta(\tau))}{I}[b-l_1(s)]+h_{11}\left(\dfrac{\nu(\theta(\tau))}{I},\theta(\tau),s,\delta\right), \\ r_1 &= \dfrac{\nu(\theta(\tau))}{I}+\dfrac{\delta\nu^2(\theta(\tau))[\nu'(\theta(\tau)-l_2(s)]}{I^2} \\ & \quad +\dfrac{\delta\nu^2(\theta(\tau))}{I^2}h_{21}\left(\dfrac{\nu(\theta(\tau))}{I},\theta(\tau),s,\delta\right). \end{cases}$$

In what follows, we shall give an asymptotic expression of (I_1,τ_1). Let us recall that

$$I_1=\frac{\nu(\theta_1)}{r_1},\quad \tau_1=\frac{2\pi}{\lambda}\int_0^{\theta_1}\frac{d\vartheta}{\nu^2(\vartheta)}.$$

Expanding $\nu(\theta_1)$, we have that

$$\nu(\theta_1)=\nu(\theta(\tau))+\frac{\delta\nu'(\theta(\tau))\nu^2(\theta(\tau))}{I}+\frac{\delta\nu'(\theta(\tau))\nu(\theta(\tau))}{I}[b-l_1(s)]+\tilde{h}_{11}, \tag{2.8}$$

where $\tilde{h}_{11}=\tilde{h}_{11}(I,\tau,s,\delta)$ is defined by

$$\begin{aligned} \tilde{h}_{11}= \;& \nu'(\theta(\tau))h_{11}\left(\frac{\nu(\theta(\tau))}{I},\theta(\tau),s,\delta\right) \\ &+\int_0^1(1-\zeta)\nu''\left[\theta(\tau)+\zeta\frac{\delta\nu^2(\theta(\tau))}{I}+\zeta\frac{\delta\nu(\theta(\tau))}{I}(b-l_1(s))+\zeta h_{11}\left(\frac{\nu(\theta(\tau))}{I},\theta(\tau),s,\delta\right)\right] \\ &\times\left[\frac{\delta\nu^2(\theta(\tau))}{I}+\frac{\delta\nu(\theta(\tau))}{I}(b-l_1(s))+h_{11}\left(\frac{\nu(\theta(\tau))}{I},\theta(\tau),s,\delta\right)\right]^2 d\zeta. \end{aligned}$$

On the other hand, we have that

$$\frac{1}{r_1}=\frac{I}{\nu(\theta(\tau))[1+\delta\nu(\theta(\tau))(\nu'(\theta(\tau))-l_2(s))/I+\delta\nu(\theta(\tau))\bar{h}_{21}(I,\tau,s,\delta)/I]}$$

with $\bar{h}_{21}(I,\tau,s,\delta)=h_{21}\left(\frac{\nu(\theta(\tau)}{I},\theta(\tau),s,\delta\right)$. Therefore, we get that

$$\frac{1}{r_1}=\frac{I}{\nu(\theta(\tau))}+\delta[l_2(s)-\nu'(\theta(\tau))]+\delta\tilde{h}_{21}, \tag{2.9}$$

with $\tilde{h}_{21}=\tilde{h}_{21}(I,\tau,s,\delta)$ defined by

$$\tilde{h}_{21}=-\bar{h}_{21}(I,\tau,s,\delta)+\frac{\delta\nu(\theta(\tau))}{I}\frac{[\nu'(\theta(\tau))-l_2(s)+\bar{h}_{21}(I,\tau,s,\delta)]^2}{1+\delta\nu(\theta(\tau))[\nu'(\theta(\tau))-l_2(s)+\bar{h}_{21}(I,\tau,s,\delta)]/I}.$$

From (2.8) and (2.9) we obtain that

$$\begin{aligned} I_1=I\; &+\delta[l_2(s)\nu(\theta(\tau))+\nu'(\theta(\tau))(b-l_1(s))]+\delta\nu(\theta(\tau))\tilde{h}_{21}(I,\tau,s,\delta) \\ &+\frac{\delta^2[l_2(s)-\nu'(\theta(\tau))]}{I}\nu'(\theta(\tau))\nu^2(\theta(\tau))+\frac{\delta^2\nu'(\theta(\tau))\nu^2(\theta(\tau))}{I}\tilde{h}_{21}(I,\tau,s,\delta) \\ &+\frac{\delta^2[l_2(s)-\nu'(\theta(\tau))][b-l_1(s)]}{I}\nu'(\theta(\tau))\nu(\theta(\tau))+\frac{\delta^2\nu'(\theta(\tau))\nu(\theta(\tau))}{I}[b-l_1(s)]\tilde{h}_{21}(I,\tau,s,\delta) \\ &+\frac{I}{\nu(\theta(\tau))}\tilde{h}_{11}(I,\tau,s,\delta)+\delta[l_2(s)-\nu'(\theta(\tau))]\tilde{h}_{11}(I,\tau,s,\delta)+\delta\tilde{h}_{11}(I,\tau,s,\delta)\tilde{h}_{21}(I,\tau,s,\delta). \end{aligned}$$

Consequently, I_1 can be expressed in the form

$$I_1 = I + \delta[l_2(s)\nu(\theta(\tau)) + \nu'(\theta(\tau))(b - l_1(s))] + \delta h_{22}(I, \tau, s, \delta),$$

where

$$\begin{aligned} h_{22}(I,\tau,s,\delta) = \;& \nu(\theta(\tau))\tilde{h}_{21}(I,\tau,s,\delta) + \frac{\delta[l_2(s) - \nu'(\theta(\tau))]}{I}\nu'(\theta(\tau))\nu^2(\theta(\tau)) \\ & + \frac{\delta\nu'(\theta(\tau))\nu^2(\theta(\tau))}{I}\tilde{h}_{21}(I,\tau,s,\delta) + \frac{\delta[l_2(s) - \nu'(\theta(\tau))][b - l_1(s)]}{I}\nu'(\theta(\tau))\nu(\theta(\tau)) \\ & + \frac{\delta\nu'(\theta(\tau))\nu(\theta(\tau))}{I}[b - l_1(s)]\tilde{h}_{21}(I,\tau,s,\delta) + \frac{I}{\delta\nu(\theta(\tau))}\tilde{h}_{11}(I,\tau,s,\delta) \\ & + [l_2(s) - \nu'(\theta(\tau))]\tilde{h}_{11}(I,\tau,s,\delta) + \tilde{h}_{11}(I,\tau,s,\delta)\tilde{h}_{21}(I,\tau,s,\delta). \end{aligned}$$

Next, we shall prove that

$$\lim_{\delta\to 0^+} h_{22}(I,\tau,s,\delta) = 0 \tag{2.10}$$

uniformly in $\tau \in [0, 2\pi]$, $s \geq \sigma$ and sufficiently large I.
In fact, since $\nu(\theta) = \mu(\theta) - b > 0$ for all θ, from (2.5) we can infer that

$$\lim_{\delta\to 0^+} \delta^{-1} I h_{11}\left(\frac{\nu(\theta(\tau))}{I}, \theta(\tau), s, \delta\right) = 0, \quad \lim_{\delta\to 0^+} h_{21}\left(\frac{\nu(\theta(\tau))}{I}, \theta(\tau), s, \delta\right) = 0 \tag{2.11}$$

uniformly in $\tau \in [0, 2\pi]$, $s \geq \sigma$ and sufficiently large I. Therefore, we have that

$$\lim_{\delta\to 0^+} \delta^{-1} I \tilde{h}_{11}(I,\tau,s,\delta) = 0, \tag{2.12}$$

and

$$\lim_{\delta\to 0^+} \tilde{h}_{21}(I,\tau,s,\delta) = 0 \tag{2.13}$$

uniformly in $\tau \in [0, 2\pi]$, $s \geq \sigma$ and sufficiently large I. From (2.12) we have obtained that

$$\lim_{\delta\to 0^+} \tilde{h}_{11}(I,\tau,s,\delta) = 0, \quad \lim_{\delta\to 0^+} \frac{I}{\delta\nu(\theta(\tau))}\tilde{h}_{11}(I,\tau,s,\delta) = 0 \tag{2.14}$$

uniformly in $\tau \in [0, 2\pi]$, $s \geq \sigma$ and sufficiently large I. From (2.13), (2.14) and the boundedness of ν, ν', l_1, l_2 we can deduce that (2.10) holds.
We are now in position to give the estimate on τ_1. From the definition of τ_1 we have that

$$\tau_1 = \frac{2\pi}{\lambda}\int_0^{\theta(\tau) + \frac{\delta\nu^2(\theta(\tau))}{I} + \frac{\delta\nu(\theta(\tau))}{I}[b - l_1(s)] + h_{11}\left(\frac{\nu(\theta(\tau))}{I}, \theta(\tau), s, \delta\right)} \frac{d\vartheta}{\nu^2(\vartheta)}.$$

Therefore,

$$\begin{aligned} \tau_1 = \;& \frac{2\pi}{\lambda}\int_0^{\theta(\tau)} \frac{d\vartheta}{\nu^2(\vartheta))} + \frac{2\pi}{\lambda}\int_{\theta(\tau)}^{\theta(\tau) + \frac{\delta\nu^2(\theta(\tau))}{I} + \frac{\delta\nu(\theta(\tau))}{I}[b - l_1(s)]} \frac{d\vartheta}{\nu^2(\vartheta)} \\ & + \frac{2\pi}{\lambda}\int_{\theta(\tau) + \frac{\delta\nu^2(\theta(\tau))}{I} + \frac{\delta\nu(\theta(\tau))}{I}[b - l_1(s)]}^{\theta(\tau) + \frac{\delta\nu^2(\theta(\tau))}{I} + \frac{\delta\nu(\theta(\tau))}{I}[b - l_1(s)] + h_{11}\left(\frac{\nu(\theta(\tau))}{I}, \theta(\tau), s, \delta\right)} \frac{d\vartheta}{\nu^2(\vartheta)}. \end{aligned}$$

From the definition of $\theta(\tau)$ we get

$$\frac{2\pi}{\lambda}\int_0^{\theta(\tau)}\frac{d\vartheta}{\nu^2(\vartheta)}=\tau.$$

On the other hand, we have

$$\begin{aligned}
&\frac{2\pi}{\lambda}\int_{\theta(\tau)}^{\theta(\tau)+\frac{\delta\nu^2(\theta(\tau))}{I}+\frac{\delta\nu(\theta(\tau))}{I}[b-l_1(s)]}\frac{d\vartheta}{\nu^2(\vartheta)}\\
=\ &\frac{2\pi}{\lambda}\int_{\theta(\tau)}^{\theta(\tau)+\frac{\delta\nu^2(\theta(\tau))}{I}+\frac{\delta\nu(\theta(\tau))}{I}[b-l_1(s)]}\frac{d\vartheta}{\nu^2(\theta(\tau))}\\
&+\frac{2\pi}{\lambda}\int_{\theta(\tau)}^{\theta(\tau)+\frac{\delta\nu^2(\theta(\tau))}{I}+\frac{\delta\nu(\theta(\tau))}{I}[b-l_1(s)]}\frac{[\nu^2(\theta(\tau))-\nu^2(\vartheta)]d\vartheta}{\nu^2(\vartheta)\nu^2(\theta(\tau))}\\
=\ &\delta\omega(I)\left[1+\tfrac{b-l_1(s)}{\nu(\theta(\tau))}\right]+\delta\tilde{h}_{12}(I,\tau,s,\delta),
\end{aligned}$$

with

$$\tilde{h}_{12}(I,\tau,s,\delta)=\frac{2\pi}{\delta\lambda}\int_{\theta(\tau)}^{\theta(\tau)+\frac{\delta\nu^2(\theta(\tau))}{I}+\frac{\delta\nu(\theta(\tau))}{I}[b-l_1(s)]}\frac{[\nu^2(\theta(\tau))-\nu^2(\vartheta)]d\vartheta}{\nu^2(\vartheta)\nu^2(\theta(\tau))}.$$

Since

$$\nu^2(\theta(\tau))-\nu^2(\vartheta)=[\nu(\theta(\tau))+\nu(\vartheta)][\nu(\theta(\tau))-\nu(\vartheta)],$$

using the Lagrange mean-value theorem and the fact $\nu(\theta)>0$ for all θ, we infer that there exists a constant $c'>0$ such that

$$\left|\int_{\theta(\tau)}^{\theta(\tau)+\frac{\delta\nu^2(\theta(\tau))}{I}+\frac{\delta\nu(\theta(\tau))}{I}[b-l_1(s)]}\frac{[\nu^2(\theta(\tau))-\nu^2(\vartheta)]d\vartheta}{\nu^2(\vartheta)\nu^2(\theta(\tau))}\right|\le\frac{c'\delta^2}{I^2}.$$

As a consequence, we obtain

$$\lim_{\delta\to0^+}I\tilde{h}_{12}(I,\tau,s,\delta)=0 \tag{2.15}$$

uniformly in $\tau\in[0,2\pi]$, $s\ge\sigma$ and sufficiently large I. From the fact that $\nu(\theta>0$ for all θ, we deduce that there exists $c''>0$ such that

$$\left|\int_{\theta(\tau)+\frac{\delta\nu^2(\theta(\tau))}{I}+\frac{\delta\nu(\theta(\tau))}{I}[b-l_1(s)]}^{\theta(\tau)+\frac{\delta\nu^2(\theta(\tau))}{I}+\frac{\delta\nu(\theta(\tau))}{I}[b-l_1(s)]+h_{11}\left(\frac{\nu(\theta(\tau))}{I},\theta(\tau),s,\delta\right)}\frac{d\vartheta}{\nu^2(\vartheta)}\right|\le c''\left|h_{11}\left(\frac{\nu(\theta(\tau))}{I},\theta(\tau),s,\delta\right)\right|,$$

which, together with the first limit in (2.11), implies that

$$\lim_{\delta\to0^+}I\bar{h}_{12}(I,\tau,s,\delta)=0, \tag{2.16}$$

where

$$\bar{h}_{12}(I,\tau,s,\delta)=\frac{2\pi}{\lambda\delta}\int_{\theta(\tau)+\frac{\delta\nu^2(\theta(\tau))}{I}+\frac{\delta\nu(\theta(\tau))}{I}[b-l_1(s)]}^{\theta(\tau)+\frac{\delta\nu^2(\theta(\tau))}{I}+\frac{\delta\nu(\theta(\tau))}{I}[b-l_1(s)]+h_{11}\left(\frac{\nu(\theta(\tau))}{I},\theta(\tau),s,\delta\right)}\frac{d\vartheta}{\nu^2(\vartheta)}.$$

Therefore, we obtain

$$\tau_1=\tau+\delta\omega(I)\left[1+\frac{b-l_1(s)}{\nu(\theta(\tau))}\right]+\delta h_{12}(I,\tau,s,\delta),$$

where $h_{12}(I,\tau,s,\delta)=\tilde{h}_{12}(I,\tau,s,\delta)+\bar{h}_{12}(I,\tau,s,\delta)$. Combining (2.15) and (2.16), we deduce that $\lim_{\delta\to0^+}h_{12}(I,\tau,s,\delta)=0$ uniformly in $\tau\in[0,2\pi]$, $s\geq\sigma$ and sufficiently large I. ■

Remark 2.5 *In case when $\mu\in C^2[0,2\pi]$ and $\mu(\theta)-a<0$, for every θ, we can prove that the map $\bar{P}_s$ can be expressed in the form:*

$$\bar{P}_s:\begin{cases}\tau_1=\tau+\delta\omega(I)\left[1+\frac{a-l_1(s)}{\nu(\theta(\tau))}\right]+\delta\hat{h}_{12}(I,\tau,s,\delta),\\ I_1=I+\delta\left[\nu(\theta(\tau))l_2(s)+\nu'(\theta(\tau))(a-l_1(s))\right]+\delta\hat{h}_{22}(I,\tau,s,\delta),\end{cases}$$

where $\hat{h}_{11}$ and $\hat{h}_{22}$ satisfy

$$\lim_{\delta\to0^+}I\hat{h}_{12}(I,\tau,s,\delta)=0,\quad \lim_{\delta\to0^+}\hat{h}_{22}(I,\tau,s,\delta)=0,$$

uniformly in $\tau\in\mathbb{R}$, $s\geq\sigma$ and sufficiently large $|I|$.

To finish the proof of Proposition 2.1, we still the following lemma.

Lemma 2.6 *Assume that $\mu\in C^1(S^1)$. Then, for any sufficiently small $\varepsilon>0$, there exists a function $\tilde{\mu}\in C^2(S^1)$ such that the following inequalities hold:*

$$|\mu(\theta)-\tilde{\mu}(\theta)|<\varepsilon,\quad |\mu'(\theta)-\tilde{\mu}'(\theta)|<\varepsilon,\quad \forall\theta\in[0,2\pi].$$

Proof. Since $\mu\in C^1(S^1)$, there exists a constant $\varrho>0$ such that, if $|\tau|<\varrho$, then

$$|\mu(\theta+\tau)-\mu(\theta)|<\varepsilon,\quad |\mu'(\theta+\tau)-\mu'(\theta)|<\varepsilon,\quad \forall\theta\in[0,2\pi].$$

Let us define

$$\Psi_\varrho(\theta)=\begin{cases}A_\varrho\left(1-\frac{\theta^2}{\varrho^2}\right)^3, & |\theta|\leq\varrho,\\ 0, & |\theta|>\varrho,\end{cases}$$

where the positive constant A_ϱ is defined by

$$\int_{-\infty}^{+\infty}\Psi_\varrho(\theta)d\theta=1. \tag{2.17}$$

It is straightforward to check that (2.17) is equivalent to

$$A\varrho \int_0^1 (1-\tau^2)^3 d\tau = \frac{1}{2}.$$

Define

$$\tilde{\mu}(\theta) = \int_{-\infty}^{+\infty} \Psi_\varrho(\theta-\tau)\mu(\tau)d\tau.$$

It is easy to check that $\tilde{\mu} \in C^2(\mathbb{R})$. Moreover, we have that

$$\tilde{\mu}(\theta) = A_\varrho \varrho \int_{-1}^1 (1-\tau^2)^3 \mu(\theta+\varrho\tau)d\tau.$$

Hence, $\tilde{\mu} \in C^2(S^1)$ and we have that, for any $\theta \in [0, 2\pi]$,

$$|\tilde{\mu}(\theta) - \mu(\theta)| \leq A_\varrho \varrho \int_{-1}^1 (1-\tau^2)^3 |\mu(\theta+\varrho\tau) - \mu(\theta)| d\tau < \varepsilon.$$

Similarly, we can prove that $|\mu'(\theta) - \tilde{\mu}'(\theta)| < \varepsilon$, $\forall \theta \in [0, 2\pi]$. ■

Proof of Proposition 2.1. As already announced, we only deal with the case $\nu(\theta) = \mu(\theta) - b > 0$, for every θ. Since $a = b$, we have $l_1(\rho) = b + o(1)$, $\rho \to +\infty$. Therefore, we know from Lemma 2.4 that $\hat{P}_s$ can be expressed in the form:

$$\hat{P}_s : \begin{cases} \tau_1 = \tau + \delta\omega(I) + \delta h_{12}(I, \tau, s, \delta), \\ I_1 = I + \delta\nu(\theta(\tau))l_2(s) + \delta h_{22}(I, \tau, s, \delta), \end{cases} \tag{2.18}$$

where h_{11} and h_{22} satisfy

$$\lim_{\delta \to 0^+} I h_{12}(I, \tau, s, \delta) = 0, \quad \lim_{\delta \to 0^+} h_{22}(I, \tau, s, \delta) = 0, \tag{2.19}$$

uniformly in $\tau \in \mathbb{R}$, $s \geq \sigma$ and sufficiently large I.

Let us consider the orbit $\{(\rho_j, \theta_j)\}$ of the map P through the point (ρ_0, θ_0) with $\rho_0 > \sigma$. Setting $s_j = \rho_j$, we have

$$P_{s_j}(\rho_j, \theta_j) = (\rho_{j+1}, \theta_{j+1})$$

and

$$P_{s_j} \circ \cdots \circ P_{s_1} \circ P_{s_0}(\rho_0, \theta_0) = (\rho_{j+1}, \theta_{j+1}).$$

Letting $r_j = 1/(\delta\rho_j)$, we get

$$\tilde{P}_{s_j}(r_j, \theta_j) = (r_{j+1}, \theta_{j+1}),$$

and

$$\tilde{P}_{s_j} \circ \cdots \circ \tilde{P}_{s_1} \circ \tilde{P}_{s_0}(r_0, \theta_0) = (r_{j+1}, \theta_{j+1}).$$

Set $\Psi(r_j, \theta_j) = (I_j, \tau_j)$. From the definition of Ψ we have

$$I_0 = \frac{\nu(\theta_0)}{r_0}, \quad \tau_0 = \frac{2\pi}{\lambda} \int_0^{\theta_0} \frac{d\vartheta}{\nu^2(\vartheta)}$$

and

$$I_j = \frac{\nu(\theta_j)}{r_j}, \quad \tau_j = \frac{2\pi}{\lambda} \int_0^{\theta_j} \frac{d\vartheta}{\nu^2(\vartheta)}.$$

Obviously,

$$\hat{P}_{s_j}(I_j, \tau_j) = (I_{j+1}, \tau_{j+1}),$$

and

$$\hat{P}_{s_j} \circ \cdots \circ \hat{P}_{s_1} \circ \hat{P}_{s_0}(I_0, \tau_0) = (I_{j+1}, \tau_{j+1}).$$

Next, we first prove the conclusion under the additional condition $\mu \in C^2(S^1)$. In order to obtain the result we will distinguish two cases.

(1) $c > 0$. Since $\nu(\theta) > 0$, $\theta \in [0, 2\pi]$ and $\liminf_{s \to +\infty} l_2(s) = c > 0$, we know that there exist positive constants $\varsigma(> \sigma)$ and γ such that, for $s \geq \varsigma$,

$$\nu(\theta) l_2(s) \geq \gamma, \quad \forall \theta \in [0, 2\pi].$$

Moreover, it follows from (2.19) that there exist positive constants δ_0 and ϱ_0 satisfying $\varrho_0 \geq \varsigma\delta_0\nu_0, \nu_0 = \max\{\nu(\theta) : \theta \in [0, 2\pi]\}$, such that, for $I \geq \varrho_0$ and $s \geq \varsigma$,

$$|h_{22}(I, \tau, s, \delta_0)| \leq \frac{\gamma}{2}, \quad \forall \tau \in [0, 2\pi].$$

If $I_0 \geq \varrho_0$, then $s_0 = \rho_0 = \frac{I_0}{\delta_0 \nu(\theta_0)} \geq \frac{\varsigma\nu_0}{\nu(\theta_0)} \geq \varsigma$. Therefore, for $I_0 \geq \varrho_0$, we have

$$I_1 = I_0 + \delta_0 \nu(\theta(\tau_0)) l_2(s_0) + \delta_0 h_{22}(I_0, \tau_0, s_0, \delta_0) \geq I_0 + \frac{\delta_0 \gamma}{2},$$

which implies that $I_1 \geq I_0 \geq \varrho_0$. Inductively, we get that, for $j = 1, 2, \cdots$,

$$I_{j+1} = I_j + \delta_0 \nu(\theta(\tau_j)) l_2(s_j) + \delta_0 h_{22}(I_j, \tau_j, s_j, \delta_0) \geq I_j + \frac{\gamma}{2} \geq \cdots \geq I_0 + \frac{(j+1)\delta_0\gamma}{2}.$$

Hence,

$$\lim_{j \to +\infty} I_j = +\infty. \tag{2.20}$$

Since $\nu(\theta) > 0$ for every $\theta \in [0, 2\pi]$ and $r_j = \nu(\theta_j)/I_j$, it follows from (2.20) that

$$\lim_{j \to +\infty} r_j = 0,$$

which, together with the transformation $\rho_j = \frac{1}{\delta_0 r_j}$, implies that $\lim_{j \to +\infty} \rho_j = +\infty$.

(2) $d < 0$. Since $\nu(\theta) > 0$ for all θ and $\limsup_{s \to +\infty} l_2(s) = d < 0$, we know that there exist positive constants $\varsigma'(> \sigma)$ and γ' such that, for $s \geq \varsigma'$,

$$\nu(\theta) l_2(s) \leq -\gamma', \quad \forall \theta \in [0, 2\pi].$$

From (2.19) we know that there exist $\delta_0' > 0$ and $\rho_0' > 0$ such that, for $I \geq \varrho_0'$ and $s \geq \varsigma'$,

$$|h_{22}(I, \tau, s, \delta_0')| \leq \frac{\gamma'}{2}, \quad \forall \tau \in [0, 2\pi].$$

Set $\varrho_0'' = \max\{\varrho_0', \delta_0'\varsigma'\nu_0)\}$ with $\nu_0 = \max\{\nu(\theta) : \theta \in \mathbb{R}\}$. Let us define

$$\Omega = \{(I, \tau) : I \geq \varrho_0'', \tau \in \mathbb{R}\}.$$

From (2.18) and (2.19) we infer that, for any $s \geq \varsigma'$, $\hat{P}_s(\Omega)$ contains a neighborhood of infinity; therefore, there exists a positive constant $\hat{\varrho}_0$ independent of s such that if $I_0 \geq \hat{\varrho}_0$, $s \geq \varsigma'$ and $\hat{P}_s^{-1}(I_0, \tau_0) = (I_{-1}, \tau_{-1})$, then $I_{-1} \geq \varrho_0''$.
If $I_0 \geq \hat{\varrho}_0$, then we get

$$s_{-1} = \rho_{-1} = \frac{I_{-1}}{\delta_0'\nu(\theta_{-1})} \geq \frac{\varrho_0''}{\delta_0'\nu(\theta_{-1})} \geq \frac{\varsigma'\nu_0}{\nu(\theta_{-1})} \geq \varsigma'.$$

Since

$$\begin{cases} \tau_0 = \tau_{-1} + \delta_0'\omega(I_{-1}) + \delta_0' h_{12}(I_{-1}, \tau_{-1}, s_{-1}, \delta_0'), \\ I_0 = I_{-1} + \delta_0'\nu(\theta(\tau_{-1}))l_2(s_{-1}) + \delta_0' h_{22}(I_{-1}, \tau_{-1}, s_{-1}, \delta_0'), \end{cases}$$

we have

$$\begin{cases} \tau_{-1} = \tau_0 - \delta_0'\omega(I_{-1}) - \delta_0' h_{12}(I_{-1}, \tau_{-1}, s_{-1}, \delta_0'), \\ I_{-1} = I_0 - \delta_0'\nu(\theta(\tau_{-1}))l_2(s_{-1}) - \delta_0' h_{22}(I_{-1}, \tau_{-1}, s_{-1}, \delta_0'). \end{cases} \tag{2.21}$$

From the second equation of (2.21) we obtain that, for $I_0 \geq \hat{\varrho}_0$,

$$\begin{aligned} I_{-1} = & I_0 - \delta_0'\nu(\theta(\tau_{-1}))l_2(s_{-1}) - \delta_0' h_{22}(I_{-1}, \tau_{-1}, s_{-1}, \delta_0') \\ \geq & I_0 + \delta_0'|\nu(\theta(\tau_{-1}))l_2(s_{-1}) - \delta_0'|h_{22}(I_{-1}, \tau_{-1}, s_{-1}, \delta_0')| \\ \geq & I_0 + \frac{1}{2}\delta_0\gamma'. \end{aligned}$$

Inductively, we deduce that

$$I_j \geq I_{j+1} + \frac{1}{2}\delta_0\gamma' \geq \cdots \geq I_0 + \frac{1}{2}\delta_0|j|\gamma',$$

for every $j = -1, -2, -3, \cdots$. Hence, we have that, if I_0 is large enough, then the orbit $\{(I_j, \tau_j)\}$ exists in the past and satisfies

$$\lim_{j \to -\infty} I_j = +\infty.$$

Arguing as in case 1, we can deduce that, if ρ_0 is large enough, then the orbit $\{(\rho_j, \theta_j)\}$ exists in the past and satisfies $\lim_{j \to -\infty} \rho_j = +\infty$.
In what follows, we shall prove the conclusion under the condition $\mu \in C^1(S^1)$. In this case, we know from Lemma 2.6 that there exist $\mu_k \in C^2(S^1)$ $(k = 1, 2, \cdots)$ such that

$$\mu_k(\theta) \to \mu(\theta), \quad \mu_k'(\theta) \to \mu'(\theta) \quad (k \to +\infty)$$

uniformly in $\theta \in [0, 2\pi]$. If $\mu(\theta) > b$ for every θ, then we have that, for k large enough,

$$\mu_k(\theta) > b, \quad \forall \theta \in [0, 2\pi]. \tag{2.22}$$

Let us consider the maps $P_k : E_\sigma \to \mathbb{R}^2$,

$$P_k : \begin{cases} \theta_1 \theta + \dfrac{1}{\rho}[\mu_k(\theta) - l_1(\rho)] + h_1(\rho, \theta), \\ \rho_1 = \rho - \mu_k'(\theta) + l_2(\rho) + h_2(\rho, \theta), \end{cases}$$

where $k \in \mathbb{N}$, l_1, l_2 and h_1, h_2 satisfy the same conditions as in (2.1). Obviously, we have that

$$\lim_{k \to +\infty} P_k(\rho, \theta) = P(\rho, \theta)$$

holds uniformly in $(\rho, \theta) \in E_\sigma$. Given a point $(\rho_0, \theta_0) \in E_\sigma$, we denote by $\{(\rho_j^{(k)}, \theta_j^{(k)})\}$ the orbit of the map P_k through the point (ρ_0, θ_0), i.e.

$$P_k \left(\rho_j^{(k)}, \theta_j^{(k)} \right) = \left(\rho_{j+1}^{(k)}, \theta_{j+1}^{(k)} \right).$$

From (2.22) and the result in case 1 we know that, if $c > 0$ and ρ_0 is large enough, then the orbit $\{(\rho_j^{(k)}, \theta_j^{(k)})\}$ exists in the future and

$$\lim_{j \to +\infty} \rho_j^{(k)} = +\infty, \tag{2.23}$$

for k sufficiently large. Moreover, since $\mu_k(\theta) \to \mu(\theta)$ and $\mu_k'(\theta) \to \mu'(\theta)$ $(k \to +\infty)$ uniformly in $\theta \in [0, 2\pi]$, we can prove, analogously to the proof in case 1, that (2.23) holds uniformly in k large enough. As a result, if ρ_0 is large enough, then for every $j \in \mathbb{N}$ and sufficiently large k,

$$\rho_j^{(k)} \geq \sigma;$$

this implies that $\rho_j \geq \sigma$, $j \in \mathbb{N}$. On the other hand, since

$$P_k^j(\rho_0, \theta_0) = (\rho_j^{(k)}, \theta_j^{(k)}),$$

we get that, for any fixed $j \in \mathbb{N}$,

$$\lim_{k \to +\infty} (\rho_j^{(k)}, \theta_j^{(k)}) = \lim_{k \to +\infty} P_k^j(\rho_0, \theta_0) = P^j(\rho_0, \theta_0) = (\rho_j, \theta_j).$$

This equality, together with (2.23), implies that, for ρ_0 large enough, the orbit $\{(\rho_j, \theta_j)\}$ satisfies $\lim_{j \to +\infty} \rho_j = +\infty$.
The case $d < 0$ can be treated similarly. ∎

Arguing as in the proof of Proposition 2.1 and using Lemma 2.4, Lemma 2.6, Remark 2.5 we can obtain the following more general result. For brevity, we omit the technical proof.

Proposition 2.7 *Assume that $a \neq b$. Then the following conclusions hold:*

(1) if $c > 0, \mu(\theta) > b$ and $c\mu(\theta) + (b - a)\mu'(\theta) > bc$ for every θ, then there exists $R_0 > 0$ such that, for $\rho_0 \geq R_0$, the orbit $\{(\rho_j, \theta_j)\}$ exists in the future and satisfies $\lim\limits_{j \to +\infty} \rho_j = +\infty$.

(2) if $d < 0, \mu(\theta) > b$ and $d\mu(\theta) + (b - a)\mu'(\theta) < bd$ for every θ, then there exists $R_0 > 0$ such that, for $\rho_0 \geq R_0$, the orbit $\{(\rho_j, \theta_j)\}$ exists in the past and satisfies $\lim\limits_{j \to -\infty} \rho_j = +\infty$.

(3) if $c > 0, \mu(\theta) < a$ and $c\mu(\theta) + (b - a)\mu'(\theta) < ac$ for every θ, then there exists $R_0 > 0$ such that, for $\rho_0 \geq R_0$, the orbit $\{(\rho_j, \theta_j)\}$ exists in the future and satisfies $\lim\limits_{j \to +\infty} \rho_j = +\infty$.

(4) if $d < 0, \mu(\theta) < a$ and $c\mu(\theta) + (b - a)\mu'(\theta) > bc$ for every θ, then there exists $R_0 > 0$ such that, for $\rho_0 \geq R_0$, the orbit $\{(\rho_j, \theta_j)\}$ exists in the past and satisfies $\lim\limits_{j \to -\infty} \rho_j = +\infty$.

3 Unbounded solutions and periodic solutions

In this section we consider the system

$$Jz' = \nabla H(z) + f(z) + p(t), \tag{3.1}$$

where the function $H : \mathbb{R}^2 \to \mathbb{R}$ is of class C^1 with locally Lipschitz continuous gradient, $f : \mathbb{R}^2 \to \mathbb{R}^2$ is locally Lipschitz continuous and $p : \mathbb{R} \to \mathbb{R}^2$ is continuous and 2π-periodic.
We assume that the Hamiltonian H is positively homogeneous of degree 2 and positive, i.e. for every $z \in \mathbb{R}^2 - \{0\}$ and $\lambda > 0$ we have

$$H(\lambda z) = \lambda^2 H(z) > 0.$$

Under this condition, all solutions of

$$Jz' = \nabla H(z) \tag{3.2}$$

are periodic with the same minimal period, which will be denoted by T. Assume that 2π is an integer multiple of T. Let $\varphi : \mathbb{R} \to \mathbb{R}^2$ be a solution of (3.2) satisfying

$$H(\varphi(t)) = \frac{1}{2}, \quad t \in \mathbb{R}.$$

Then we have

$$< J\varphi'(t), \varphi(t) > = < \nabla H(\varphi(t)), \varphi(t) > = 2H(\varphi(t)) = 1, \quad t \in \mathbb{R}.$$

Therefore, the orbit of φ is strictly star-shaped and any solution of (3.2) can be expressed in the form $z(t) = A\varphi(t + \theta)$, for some $A > 0$, $\theta \in [0, T)$.
Moreover, we suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2$ can be written in the form

$$f(z) = \sum_{k=1}^{m} f_k(< z, e^{i\vartheta_k} >),$$

where

$$0 \leq \vartheta_1 < \vartheta_2 < \cdots < \vartheta_m < 2\pi$$

are $m \geq 1$ fixed directions and $f_k : \mathbb{R} \to \mathbb{R}^2$.
Assuming that every function $f_k : \mathbb{R} \to \mathbb{R}^2$ is locally Lipschitz continuous and bounded, we obtain that f is locally Lipschitz continuous and bounded. For every $k = 1, \cdots, m$, we set

$$F_k(x) = \int_0^x f_k(s)ds, \quad x \in \mathbb{R}.$$

Finally, we suppose that the set

$$\{u \in \mathbb{R}^2 : ||u|| = 1 \text{ and } \frac{\nabla H(u)}{||\nabla H(u)||} = \pm e^{i\vartheta_k}\}$$

has only isolated points, for every $k = 1, 2, \cdots, m$; let us observe that this assumption is satisfied when H is a strictly convex function.

Now, let $z(t)$ be a solution of system (3.1) satisfying the initial condition $z(0) \neq 0$. Write

$$z(t) = \rho(t)\varphi(t + \theta(t)),$$

with $\rho(0) > 0$. If $\rho(t) > 0$, then the functions $\rho(t)$ and $\theta(t)$ are of class C^1 and satisfy

$$\begin{cases} \theta' = & \frac{1}{\rho} < f(\rho\varphi(t+\theta)), \varphi(t+\theta) > + \frac{1}{\rho} < p(t), \varphi(t+\theta) >, \\ \rho' = & - < f(\rho\varphi(t+\theta)), \varphi'(t+\theta) > - < p(t), \varphi'(t+\theta) > . \end{cases} \tag{3.3}$$

Denote by $(\theta(t), \rho(t)) = (\theta(t, \theta_0, \rho_0), \rho(t, \theta_0, \rho_0))$ the solution of (3.3) through the initial point

$$\theta(0, \theta_0, \rho_0) = \theta_0, \quad \rho(0, \theta_0, \rho_0) = \rho_0$$

and consider the Poincaré map

$$P : (\theta_0, \rho_0) \to (\theta_1, \rho_1) = (\theta(2\pi, \theta_0, \rho_0), \rho(2\pi, \theta_0, \rho_0)).$$

It is immediate to check that, for ρ_0 large enough, P can be written as

$$\begin{cases} \theta_1 = & \theta_0 + \int_0^{2\pi} \frac{1}{\rho(t)} < f(\rho(t)\varphi(t+\theta(t))), \varphi(t+\theta(t)) > dt \\ & + \int_0^{2\pi} \frac{1}{\rho(t)} < p(t), \varphi(t+\theta(t)) > dt, \\ \rho_1 = & \rho_0 - \int_0^{2\pi} < f(\rho(t)\varphi(t+\theta(t))), \varphi'(t+\theta(t)) > dt \\ & - \int_0^{2\pi} < p(t), \varphi'(t+\theta(t)) > dt. \end{cases} \tag{3.4}$$

Let us observe that the boundedness of f, p and φ imply that

$$\rho(t) = \rho_0 + O(1), t \in [0, 2\pi]. \tag{3.5}$$

Therefore, for $\rho_0 \to +\infty$, we obtain

$$\theta(t) = \theta_0 + o(1), t \in [0, 2\pi]. \tag{3.6}$$

We are now in position to prove the following result.

Lemma 3.1 *For $\rho_0 \to +\infty$ the following conclusions hold:*

$$\int_0^{2\pi} \frac{1}{\rho(t)} < f(\rho(t)\varphi(t+\theta(t))), \varphi(t+\theta(t)) > dt =$$
$$= \frac{1}{\rho_0} \int_0^{2\pi} < f(\rho_0\varphi(t+\theta_0)), \varphi(t+\theta_0) > dt + o(\frac{1}{\rho_0}) =$$
$$= \frac{1}{\rho_0} \int_0^{2\pi} < f(\rho_0\varphi(t), \varphi(t) > dt + o(\frac{1}{\rho_0});$$
$$\int_0^{2\pi} < f(\rho(t)\varphi(t+\theta(t))), \varphi'(t+\theta(t)) > dt =$$
$$= \int_0^{2\pi} < f(\rho_0\varphi(t+\theta_0)), \varphi'(t+\theta_0)) > dt + o(1) =$$
$$= \int_0^{2\pi} < f(\rho_0\varphi(t), \varphi'(t) > dt + o(1).$$

Proof. We follow an argument similar to the one developed in [11]. For every $k = 1, \cdots, m$, we have

$$\begin{aligned}&\frac{d}{dt} F_k(< \rho(t)\varphi(t+\theta(t)), e^{i\vartheta_k} >)\\ &\quad = f_k(< \rho(t)\varphi(t+\theta(t)), e^{i\vartheta_k} >)[< \rho'(t)\varphi(t+\theta(t)), e^{i\vartheta_k} >\\ &\qquad + < \rho(t)\varphi'(t+\theta(t)), e^{i\vartheta_k} > (1+\theta'(t))].\end{aligned}$$

Let $[x, y]$ be an arbitrary interval contained in the set

$$\{t \in \mathbb{R} :< \varphi'(t+\theta_0), e^{i\vartheta_k} > \neq 0\}.$$

It follows from (3.3)-(3.5) that

$$< \rho'(t)\varphi(t+\theta(t)), e^{i\vartheta_k} > + < \rho(t)\varphi'(t+\theta(t)), e^{i\vartheta_k} > (1+\theta'(t)) \neq 0,$$

for every $t \in [x, y]$ and for ρ_0 large enough. Then, for $t \in [x, y]$, it follows that

$$\begin{aligned}&f_k(< \rho(t)\varphi(t+\theta(t)), e^{i\vartheta_k} >)\\ &= \frac{\frac{d}{dt} F_k(< \rho(t)\varphi(t+\theta(t)), e^{i\vartheta_k} >)}{< \rho'(t)\varphi(t+\theta(t)), e^{i\vartheta_k} > + < \rho(t)\varphi'(t+\theta(t)), e^{i\vartheta_k} > (1+\theta'(t))}.\end{aligned}$$

Therefore, we have that, for $\rho_0 \to +\infty$,

$$\begin{aligned}&\int_x^y \frac{1}{\rho(t)} < f_k(< \rho(t)\varphi(t+\theta(t)), e^{i\vartheta_k} >), \varphi(t+\theta(t)) > dt\\ &= \int_x^y \frac{1}{\rho(t)} \frac{< \frac{d}{dt} F_k(< \rho(t)\varphi(t+\theta(t)), e^{i\vartheta_k} >), \varphi(t+\theta(t)) >}{< \rho'(t)\varphi(t+\theta(t)), e^{i\vartheta_k} > + < \rho(t)\varphi'(t+\theta(t)), e^{i\vartheta_k} > (1+\theta'(t))} dt\\ &= \frac{1}{\rho_0} \int_x^y \frac{< \frac{d}{dt} F_k(< \rho(t)\varphi(t+\theta(t)), e^{i\vartheta_k} >), \varphi(t+\theta_0) >}{\rho_0 < \varphi'(t+\theta_0), e^{i\vartheta_k} >} dt + o(\frac{1}{\rho_0}).\end{aligned}$$

Integrating by parts,

$$\begin{aligned}&\int_x^y \frac{< \frac{d}{dt} F_k(< \rho(t)\varphi(t+\theta(t)), e^{i\vartheta_k} >), \varphi(t+\theta_0) >}{\rho_0 < \varphi'(t+\theta_0), e^{i\vartheta_k} >} dt\\ &\quad = \langle F_k(< \rho(t)\varphi(t+\theta(t)), e^{i\vartheta_k} >), \frac{\varphi(t+\theta_0)}{\rho_0 < \varphi'(t+\theta_0), e^{i\vartheta_k} >} \rangle |_x^y\\ &\qquad - \int_x^y < F_k(< \rho(t)\varphi(t+\theta(t)), e^{i\vartheta_k} >), \frac{d}{dt}(\frac{\varphi(t+\theta_0)}{\rho_0 < \varphi'(t+\theta_0), e^{i\vartheta_k} >}) > dt.\end{aligned}$$

Since f_k is bounded, we get

$$F_k(< \rho(t)\varphi(t+\theta(t)), e^{i\vartheta_k} >) = F_k(< \rho_0\varphi(t+\theta_0), e^{i\vartheta_k} >) + O(1).$$

Therefore, we obtain

$$
\begin{aligned}
&< F_k(< \rho(t)\varphi(t+\theta(t)), e^{i\vartheta_k} >), \frac{\varphi(t+\theta_0)}{\rho_0 < \varphi'(t+\theta_0), e^{i\vartheta_k} >} > |_x^y \\
&=< F_k(< \rho_0\varphi(t+\theta_0), e^{i\vartheta_k} >), \frac{\varphi(t+\theta_0)}{\rho_0 < \varphi'(t+\theta_0), e^{i\vartheta_k} >} > |_x^y + o(1).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\int_x^y < F_k(< \rho(t)\varphi(t+\theta(t)), e^{i\vartheta_k} >), \frac{d}{dt}(\frac{\varphi(t+\theta_0)}{\rho_0 < \varphi'(t+\theta_0), e^{i\vartheta_k} >}) > dt \\
&= \int_x^y < F_k(< \rho_0\varphi(t+\theta_0), e^{i\vartheta_k} >), \frac{d}{dt}(\frac{\varphi(t+\theta_0)}{\rho_0 < \varphi'(t+\theta_0), e^{i\vartheta_k} >}) > dt + o(1).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\int_x^y \frac{< \frac{d}{dt}F_k(< \rho(t)\varphi(t+\theta(t)), e^{i\vartheta_k} >), \varphi(t+\theta_0) >}{\rho_0 < \varphi'(t+\theta_0), e^{i\vartheta_k} >} dt \\
&=< F_k(< \rho_0\varphi(t+\theta_0)), e^{i\vartheta_k} >), \frac{\varphi(t+\theta_0)}{\rho_0 < \varphi'(t+\theta_0), e^{i\vartheta_k} >} > |_x^y \\
&\quad - \int_x^y < F_k(< \rho_0\varphi(t+\theta_0)), e^{i\vartheta_k} >), \frac{d}{dt}(\frac{\varphi(t+\theta_0)}{\rho_0 < \varphi'(t+\theta_0), e^{i\vartheta_k} >}) > dt + o(1) \\
&= \int_x^y \frac{< \frac{d}{dt}F_k(< \rho_0\varphi(t+\theta_0)), e^{i\vartheta_k} >), \varphi(t+\theta_0) >}{\rho_0 < \varphi'(t+\theta_0), e^{i\vartheta_k} >} dt + o(1) \\
&= \int_x^y < f_k(< \rho_0\varphi(t+\theta_0)), e^{i\vartheta_k} >), \varphi(t+\theta_0) > dt + o(1).
\end{aligned}
$$

Consequently, we get that, for $\rho_0 \to +\infty$,

$$
\begin{aligned}
&\int_x^y \frac{1}{\rho(t)} < f_k(< \rho(t)\varphi(t+\theta(t)), e^{i\vartheta_k} >), \varphi(t+\theta(t)) > dt \\
&= \frac{1}{\rho_0} \int_x^y < f_k(< \rho_0\varphi(t+\theta_0)), e^{i\vartheta_k} >), \varphi(t+\theta_0) > dt + o(\frac{1}{\rho_0}).
\end{aligned}
$$

The assumption on H implies that the set $\{t \in \mathbb{R} :< \varphi'(t+\theta_0), e^{i\theta_k} >= 0\}$ has only isolated points. Therefore, for any sufficiently small constant $\eta > 0$, we can take a finite number of intervals $[x_i, y_i](i = 1, 2, \cdots, n)$ as above such that

$$
meas[0, 2\pi] \backslash \bigcup_{i=1}^{i=n} [x_i, y_i] \leq \eta.
$$

Since f_k is bounded, we have that

$$
\begin{aligned}
&\int_0^{2\pi} \frac{1}{\rho(t)} < f_k(< \rho(t)\varphi(t+\theta(t)), e^{i\vartheta_k} >), \varphi(t+\theta(t)) > dt \\
&= \frac{1}{\rho_0} \int_0^{2\pi} < f_k(< \rho_0\varphi(t+\theta_0)), e^{i\vartheta_k} >), \varphi(t+\theta_0) > dt + o(\frac{1}{\rho_0}).
\end{aligned}
$$

Summing up for $k = 1, 2, \cdots, m$, we obtain

$$\begin{aligned}&\int_0^{2\pi} \frac{1}{\rho(t)} < f(\rho(t)\varphi(t+\theta(t)), \varphi(t+\theta(t)) > dt \\ &= \frac{1}{\rho_0}\int_0^{2\pi} < f(\rho_0\varphi(t+\theta_0)), \varphi(t+\theta_0) > dt + o(\frac{1}{\rho_0}).\end{aligned} \tag{3.7}$$

Since φ is 2π-periodic, it follows that

$$\int_0^{2\pi} < f(\rho_0\varphi(t+\theta_0)), \varphi(t+\theta_0) > dt = \int_0^{2\pi} < f(\rho_0\varphi(t)), \varphi(t) > dt. \tag{3.8}$$

Therefore, from (3.7) and (3.8) we can write

$$\begin{aligned}&\int_0^{2\pi} \frac{1}{\rho(t)} < f(\rho(t)\varphi(t+\theta(t)), \varphi(t+\theta(t)) > dt \\ &= \frac{1}{\rho_0}\int_0^{2\pi} < f(\rho_0\varphi(t)), \varphi(t) > dt + o(\frac{1}{\rho_0}).\end{aligned}$$

The second conclusion can be proved similarly. ■

With a similar argument, based again on (3.5)-(3.6) and the periodicity of φ, it is possible to prove the following result.

Lemma 3.2 *For $\rho_0 \to +\infty$ the following conclusions hold:*

$$\int_0^{2\pi} \frac{1}{\rho(t)} < p(t), \varphi(t+\theta(t)) > dt = \frac{1}{\rho_0}\int_0^{2\pi} < p(t), \varphi(t+\theta_0) > dt + o(\frac{1}{\rho_0}),$$

$$\int_0^{2\pi} < p(t), \varphi'(t+\theta(t)) > dt = \int_0^{2\pi} < p(t), \varphi'(t+\theta_0) > dt + o(1).$$

Now, let us define

$$l_1(\rho) = -\int_0^{2\pi} < f(\rho\varphi(t)), \varphi(t) > dt, \quad l_2(\rho) = -\int_0^{2\pi} < f(\rho\varphi(t)), \varphi'(t) > dt,$$

for $\rho > 0$, and

$$\mu(\theta) = \int_0^{2\pi} < p(t), \varphi(t+\theta) > dt,$$

for every $\theta \in [0, 2\pi]$. From the fact that ∇H is locally Lipschitz continuous and φ is the solution of system $Jz' = \nabla H(z)$ we deduce that $\varphi(t)$ is continuously differentiable on $[0, 2\pi]$. Hence, $\mu \in C^1(S^1)$; moreover, we have

$$\mu'(\theta) = \int_0^{2\pi} < p(t), \varphi'(t+\theta) > dt,$$

for every $\theta \in [0, 2\pi]$.
From (3.4), Lemma 3.1 and Lemma 3.2 we plainly deduce the following result.

Lemma 3.3 *The Poincaré map of (3.3) can be expressed in the form:*

$$\begin{cases} \theta_1 = & \theta_0 + \dfrac{1}{\rho_0}[\mu(\theta_0) - l_1(\rho_0)] + h_1(\rho_0, \theta_0), \\ \rho_1 = & \rho_0 - \mu'(\theta_0) + l_2(\rho_0) + h_2(\rho_0, \theta_0), \end{cases}$$

where h_1, h_2 satisfy

$$h_1(\rho_0, \theta_0) = o\left(\frac{1}{\rho_0}\right), \quad h_2(\rho_0, \theta_0) = o(1), \quad \rho_0 \to +\infty.$$

To state the main theorems of this section, we still use notations of a, b, c, d and v given in section 2, namely, $a = \liminf_{\rho\to+\infty} l_1(\rho), b = \limsup_{\rho\to+\infty} l_1(\rho); c = \liminf_{\rho\to+\infty} l_2(\rho), d = \limsup_{\rho\to+\infty} l_2(\rho)$.

Theorem 3.4 *Assume that $a = b$ and $\mu(\theta) \neq b$, for every $\theta \in [0, 2\pi]$. Then (3.1) has at least one 2π-periodic solution. Moreover, the following conclusions hold:*

(1) if $c > 0$, then there exists $R_0 > 0$ such that all solutions of (3.1) with $||z(0)|| \geq R_0$ satisfy

$$\lim_{t\to+\infty} ||z(t)|| = +\infty;$$

(2) if $d < 0$, then there exists $R_0 > 0$ such that all solutions of (3.1) with $||z(0)|| \geq R_0$ satisfy

$$\lim_{t\to-\infty} ||z(t)|| = +\infty.$$

Proof. We first prove that (3.1) has at least one 2π-periodic solution. Since

$$\mu(\theta) \neq b, \theta \in [0, 2\pi],$$

we have that, for ρ_0 large enough, the image (ρ_1, θ_1) of the point (ρ_0, θ_0) under the Poincaré map P cannot lie on the ray $\theta = \theta_0$. According to the Poincaré-Bohl theorem [12], the map P has at least one fixed point. Therefore, (3.1) has at least one 2π-periodic solution.

Now we prove the unboundedness of the solutions of (3.1) when ρ_0 is large enough; we will concentrate on the first case. The other cases can be treated similarly.

From Lemma 3.3 we deduce that we can apply Proposition 2.1 to the Poincaré map P; hence, there exists $R_0 > 0$ such that, if $\rho_0 \geq R_0$, then $\{(\rho_j, \theta_j)\}$ exists in the future and satisfies $\lim_{j\to+\infty} \rho_j = +\infty$.

On the other hand, since f is bounded, from the second equality of (3.3) we infer that there exists a constant $c_0 > 0$ such that $|\rho(t) - \rho(s)| \leq c_0$ for t and s satisfying $|t - s| \leq 2\pi$. Therefore, we obtain

$$\lim_{t\to+\infty} \rho(t) = +\infty.$$

Now, let us observe that the assumptions on the Hamiltonian H and the fact that $H(\varphi(t)) = 1/2$ ($t \in [0, 2\pi]$) imply that there exists a constant $d_0 > 0$ such

$$||\varphi(t)|| \geq d_0, t \in [0, 2\pi].$$

Hence, we obtain

$$\lim_{t\to+\infty} ||z(t)|| = \lim_{t\to+\infty} \rho(t)||\varphi(t)|| = +\infty.$$

■

Using Proposition 2.7 and the same method as in the proof of Theorem 3.4 we can prove the following

Theorem 3.5 *Assume that $a \neq b$ and*

$$[a, b] \cap Range(\mu) = \emptyset. \tag{3.9}$$

Then (3.1) has at least one 2π-periodic solution. Moreover, the following conclusions hold:

(1) if $c > 0$, $\mu(\theta) > b$ and $c\mu(\theta) + (b-a)\mu'(\theta) > bc$, for every $\theta \in [0, 2\pi]$, then there exists $R_0 > 0$ such that all solutions of (3.1) with $||z(0)|| \geq R_0$ satisfy

$$\lim_{t\to+\infty} ||z(t)|| = +\infty;$$

(2) if $d < 0$, $\mu(\theta) > b$ and $d\mu(\theta) + (b-a)\mu'(\theta) < bd$, for every $\theta \in [0, 2\pi]$, then there exists $R_0 > 0$ such that all solutions of (3.1) with $||z(0)|| \geq R_0$ satisfy

$$\lim_{t\to-\infty} ||z(t)|| = +\infty;$$

(3) if $c > 0$, $\mu(\theta) < a$ and $c\mu(\theta) - \upsilon\mu'(\theta) < ac$, for every $\theta \in [0, 2\pi]$, then there exists $R_0 > 0$ such that all solutions of (3.1) with $||z(0)|| \geq R_0$ satisfy

$$\lim_{t\to+\infty} ||z(t)|| = +\infty;$$

(4) if $d < 0$, $\mu(\theta) < a$ and $d\mu(\theta) - \upsilon\mu'(\theta) > ad$, for every $\theta \in [0, 2\pi]$, then there exists $R_0 > 0$ such that all solutions of (3.1) with $||z(0)|| \geq R_0$ satisfy

$$\lim_{t\to-\infty} ||z(t)|| = +\infty.$$

Remark 3.6 *Theorem 3.4 is a generalization of Corollary 2 (case 1) and of Theorem 2 in [11]. Indeed, in [11] it is assumed that f is bounded and (1.6) holds, while in our paper it is sufficient to suppose the boundedness of f. Hypothesis (1.6) which we avoid (and which was crucial in [11]) causes the cancellation, in the development of the Poincaré map obtained in Lemma 1 of [11], of the term arising from the presence of f. Similar unboundedness results can be found in [8], where a different class of (homogeneous) nonlinearities f is considered.*
We also observe that the coexistence of periodic solutions and unbounded solutions (on the lines of case 2 in Corollary 2 of [11]) can be obtained when (3.9) *may not hold and it is assumed that*

$$[\liminf_{\rho\to+\infty} l_2(\rho), \limsup_{\rho\to+\infty} l_2(\rho)] \cap Range(\mu') = \emptyset.$$

Remark 3.7 *Various applications of Theorems 3.4 and 3.5 to second order equations are possible. In particular, this is true for the classical Liénard equation $x'' + \psi(x)x' + \alpha x^+ - \beta x^- + g(x) = p(t)$ and the Rayleigh equation $x'' + \psi(x') + \alpha x^+ - \beta x^- + g(x) = p(t)$ when α, β satisfy $1/\sqrt{\alpha} + 1/\sqrt{\beta} = 2/n$, $n \in \mathbf{N}$. The coexistence of periodic and unbounded solutions follows from our result when we limit ourselves to assume that g and any primitive of ψ are bounded.*

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