How skewness influences optimal allocation in a risky asset

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Abstract: This paper extends the classic Samuelson (1970) and Merton (1973) model of optimal portfolio allocation with one risky asset and a riskless one to include the effect of the skewness. Using an extended version of Stein's Lemma, we provide the explicit solution for optimal demand that holds for *all* expected utility maximizing investors when the risky asset is skew-normally and normally distributed. A closed expression is achieved for investors with constant absolute risk aversion.

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Introduction

A well-known result in portfolio selection is that optimal asset weights are multiplicatively separable into investor risk aversion and market price of risk (see, e.g., Tobin, 1958; Samuelson, 1970; Merton, 1973). The correctness of this statement can be proved for *all* rational investors who invest in one risk-free asset and one normally distributed risky asset (see Rubinstein, 1973). The motivation for this paper is the growing body of empirical literature that documents the inconsistency of the normality assumption, especially with respect to the significant skewness observed in asset returns (see, e.g., Peiró, 1999; Su and Hung, 2011; Xu et al., 2011). We model the risky return with a skew-normal distribution (see Azzalini, 1985; Adcock and Shutes, 2001) that has many attractive features for modeling real asset returns (see Adcock, 2007; Harvey *et al*., 2010). A skew-normal variable is defined as a Gaussian perturbed via the addition of a skewness shock given by a truncated Gaussian.

The contribution of this paper is to derive the optimal allocation for a skew-normal portfolio which holds for *all* expected utility maximizing investors. The solution is an analog of the classical Samuelson-Merton optimal portfolio solution with the addition of a term dependent on the skewness shock and the agent utility function. In the special case, the investor has constant absolute risk aversion (CARA) a closed expression can be set up. We show that under feasible conditions a CARA agent invests more in a skew-normal asset than in a normal one if the skewness shock is positive, and vice versa if it is negative.

2.   The Model Set-Up

We consider an investor with initial unitary wealth who can invest in a risk-free asset and in a risky asset with returns R_f and R, respectively, at the end of the period. The final wealth is given by $W = (1-\omega) R_f + \omega R = (R - R_f) \omega + R_f$, where $\omega \in [0,1]$

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indicates the portion of wealth invested in the risky asset. As is typical in portfolio selection, we assume that the investor's utility function u is two times differentiable, increasing and strictly concave. The optimal allocation problem is the choice of ω that maximizes the expected utility of the final wealth. The first-order condition leads to:

$$\frac{\partial E\left[u(W)\right]}{\partial \omega} = E\left[u'(W)\left(R - R_f\right)\right] = 0$$

or, equivalently:

$$E\left[u'(W) \cdot R\right] - E\left[u'(W)\right] \cdot R_f = 0 \tag{1}$$

3.   The Normal Return Case

Let $R \sim N(\mu, \sigma^2)$, with $\mu > R_f$ and denote by ω_N the portion of wealth invested in the normal asset. Using Stein's Lemma for normal variables (see Rubinstein, 1973; Stein, 1973, 1981) we obtain:

$$E\left[u'(W) \cdot (R - \mu)\right] = E\left[\frac{\partial u'(W)}{\partial R}\right] \mathrm{var}(R) = E\left[u''(W)\right]\omega_N \sigma^2$$

Rearranging this formula achieves:

$$E\left[u'(W) \cdot R\right] = E\left[u'(W)\right]\mu + E\left[u''(W)\right]\omega_N \sigma^2$$

and substituting this in Equation (1) leads to:

$$E\left[u''(W)\right]\omega_N \sigma^2 + E\left[u'(W)\right]\left(\mu - R_f\right) = 0$$

Solving the last formula with respect to ω, the optimal allocation becomes:

$$\omega_N^* = \frac{1}{\gamma_N^*}\frac{\left(\mu - R_f\right)}{\sigma^2} \tag{2.1}$$

where $\gamma_N^* = -E\left[u''(W^*)\right]\big/E\left[u'(W^*)\right]$ in correspondence of the optimal wealth $W^* = W\left(\omega_N^*\right)$ can be interpreted as the analogue of the Arrow-Pratt coefficient of

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risk aversion. Equation (2.1) highlights the separability in optimal allocation between individual risk aversion expressed by γ_N^*, the market premium $\left(\mu - R_f\right)$ and the variance σ^2. Since the separability holds independently of the utility function u, the above statement holds for *all* expected utility investors.

In general, γ_N^* depends on $W^* = W\left(\omega_N^*\right)$. Therefore the value of ω_N^* can be computed only if the analytical expression of the utility function is given. If the investor has a CARA utility function, i.e., $u(W) = -e^{-\gamma W}$, where γ is the absolute constant Arrow-Pratt coefficient of risk aversion, $\gamma_N^* = \gamma$ and Equation (2.1) gives a closed formula

$$\omega_N^* = \frac{1}{\gamma} \frac{\left(\mu - R_f\right)}{\sigma^2} \tag{2.2}$$

Due to the strictly concavity of u, the optimum ω_N^* is unique (see Samuelson, 1970); Merton, 1973); the seminal proof can be traced back to Rubinstein, 1973).

4.   The Skew-Normal Return Case

We now model the risky asset with the extended version of a skew-normal distribution proposed by Adcock and Shutes (2001). We denote the *skew-normal* return by $R = Y + \lambda|U|$ with $Y \sim N(\mu, \sigma^2)$ and independently $U \sim N(\tau, 1)$; $|\;|$ means truncation from below at zero. The four parameters μ, σ, τ, and λ are unrestricted and we use the notation $R \sim SN\left(\mu, \sigma^2, \lambda, \tau\right)$. If $\lambda = 0$, R collapses in $Y \sim N(\mu, \sigma^2)$. We call $\lambda|U|$ *skewness shock*.

Denoted by Φ and ϕ the cumulative distribution and the density function of a standard normal variable, the mean and the variance of R are, respectively

$$E(R) = \mu + \lambda\left(\tau + \frac{\phi(\tau)}{\Phi(\tau)}\right); \quad var(R) = \sigma^2 + \lambda^2\left[1 + \frac{\phi'(\tau)}{\Phi(\tau)} - \frac{\phi(\tau)^2}{\Phi(\tau)^2}\right]; \text{ see Adcock and}$$

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Shutes (2001). Since R is the sum of two independent variables, the skewness is just the sum of the skewness of addenda, so $Sk(R) = \lambda^3 Sk(|U|)$. Since $Sk(|U|)$ is positive (the proof is available from the authors upon request), it follows that $Sk(R)$ has the same sign of λ.

We compute the optimal allocation for $R \sim SN(\mu, \sigma^2, \lambda, \tau)$. As in the normal case, we assume $E(R) > R_f$. Using Stein's Lemma for the extended multivariate skew-normal (see Adcock, 2007, Corollary 4.1 (b)) we prove the following.

Proposition 1. *Let $R \sim SN(\mu, \sigma^2, \lambda, \tau)$. The optimal allocation is given*

$$\omega_{SN}^* = \frac{E(R) - R_f}{\gamma_{SN}^* (\sigma^2 + \lambda^2)} + \lambda \frac{Eu'(W_{(N)}^*) - Eu'(W^*)}{\gamma_{SN}^* (\sigma^2 + \lambda^2) Eu'(W^*)} \frac{\phi(\tau)}{\Phi(\tau)} \tag{3}$$

where $\gamma_{SN}^* = E\left[-u''(W^*)\right] \big/ E\left[u'(W^*)\right]$ and $W_{(N)}^* = W^* - \lambda |U| \omega_{SN}^*$.

Proof. See the Appendix. Since Stein's Lemma holds independently on the utility function u, the optimal allocation (3) holds for *all* expected utility investors. That is given by a term multiplicatively separable into investor risk aversion γ_{SN}^* and the market premium $(E(R) - R_f)$ that is analog to the classic Samuelson-Merton result; and, an addendum that is a function of the skewness shock $\lambda |U|$ and the agent utility function u. If $\lambda = 0$, solution (3) reduces to (2.1). As in the normal case, ω_{SN}^* appears in the both hand-sides of (3), so its closed expression is achievable only if the utility function is specified.

5. Normal versus Skew-Normal for CARA utility

We now investigate how the optimal risky allocation varies as the risky asset moves from normal to skew-normal. To compare Equations (2.1) and (3) we assume that the agent is endowed with constant risk aversion $\gamma = \gamma_{SN}^* = \gamma_N^*$.

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Proposition 2. *Let* $R \sim SN\left(\mu, \sigma^2, \lambda, \tau\right)$ *and CARA utility function* $u\left(W\right) = -e^{-\gamma W}$ *with* $\gamma > 0$. *It follows that*

$$\omega_{SN}^* = \omega_N^* \frac{\sigma^2}{\sigma^2 + \lambda^2} + \frac{\lambda}{\gamma\left(\sigma^2 + \lambda^2\right)}\left[\tau + \frac{\phi(\tau)\Phi\left(\gamma\omega_{SN}^*\lambda + \tau\right)}{\Phi^2(\tau)}e^{\left(\gamma\omega_{SN}^*\lambda\right)\tau + \frac{1}{2}\left(\gamma\omega_{SN}^*\lambda\right)^2}\right] \tag{4}$$

where ω_N^* is defined in (2.2).

Proof. See the Appendix. Formula (4) highlights the fact that the direction of change is not only driven by λ, but also by the location parameter τ and risk aversion γ. At first let's intuitively tackle the problem. The skewness shock $\lambda|U|$ induces a shift of the probability mass on the right part[d] of the support if $\lambda > 0$, and on the left one if $\lambda < 0$. That implies that the all first three central moments are perturbed. Specifically, if $\lambda > 0$ the mean, the variance and the skewness increase with respect to the normal case. If $\lambda < 0$, the mean decreases, the variance increases and the skewness turns to negative.

We now conjecture how the optimal allocation changes as a skewness shock occurs. We assume that the agent exhibits preference for odd order moments (as mean and skewness) and dislike even order moments (variance, kurtosis); see for example Scott and Horvath (1980). If $\lambda > 0$ the first three moments move to desirable directions, however due to the favourable probability mass shift it seems reasonable to expect an increase in the risky asset allocation under possible restrictions on the

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[d] It is worthwhile noting that $E\left[|U|\right] = \tau + \frac{\phi(\tau)}{\Phi(\tau)}$ is positive. If $\tau \geq 0$ the proof is trivial. If $\tau < 0$ that follows from the Theorem of the Mean $\Phi(\tau) < \frac{\phi(\tau)}{-\tau}$. The fact that $E\left(|U|\right) > 0$ is not surprising, since the support of $|U|$ is non-negative.

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tradeoffs among the higher central moments and risk aversion. Vice versa if $\lambda < 0$, the mean, the variance and the skewness move to undesirable directions and we expect a decrease in the risky allocation under possible restrictions. Above is confirmed in the following.

Proposition 3. *Let* $u(W) = -e^{-\gamma W}$ *with* $\gamma > 0$ *and* $R \sim SN(\mu, \sigma^2, \lambda, \tau)$. *Denote by* ω_{SN}^* *and* ω_N^* *the optimal solutions in (4) and (2.2), respectively. Then, if* $\lambda \neq 0$ *there exists a threshold*

$$T = T(\lambda, \tau, \gamma, \omega_{SN}^*) = \frac{1}{\lambda\gamma}\left[\tau + \frac{\phi(\tau)}{\Phi(\tau)} \cdot \frac{\Phi(\gamma\omega_{SN}^*\lambda + \tau)}{\Phi(\tau)} \cdot e^{(\gamma\omega_{SN}^*\lambda)\tau + \frac{1}{2}(\gamma\omega_{SN}^*\lambda)^2}\right] \text{ such that}$$

If $\lambda > 0$: *the investor increases the investment in the risky asset, if and only if*

$$\omega_N^* < T; \tag{5.1}$$

If $\lambda < 0$: *the investor decreases the investment in the risky asset, if and only if*

$$\omega_N^* > T. \tag{5.2}$$

If $\omega_N^* = T$ *no change occurs.*

Proof. See the Appendix.

If $\lambda < 0$ and $\tau \geq 0$, condition (5.2) is always fulfilled. Note the key role played by the risk aversion γ. The higher the risk aversion γ, the lower the threshold T and the more the properness to reduce the risky allocation, no matter the sign of the skewness. That confirms the fact that for the given level of γ a CARA agent may choose risky assets with lower preferable moments (also see the counter-examples in Peel, 2012).

6.   **Conclusion**

A positive skewness shock thus induces CARA investors to allocate more in a skew-normal asset than in a normal one, and vice versa if the skewness shock is negative.

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Our work may suggest different avenues for future research. Using extended versions of Stein's Lemma further solutions for the classical Samuelson-Merton model can be set up. See the extended version in Söderlind (2009) for assets which are a mixture of bivariate normal distributions and that in Gron *et al.* (2012) for assets with stochastic volatility. Another interesting aspect would be to extend the classical farm manager's land allocation problem to incorporate the skewness of the crop yields (see Haley, 2012).

Appendix

Proof of Proposition 1.

Adcock (2007, Corollary 4.1 (b)) proved the Stein's Lemma for the extended multivariate skew-normal; in one dimension it reduces to

$$\text{cov}\{R, h(R)\} = \left(\sigma^2 + \lambda^2\right) E\left[h'(R)\right] + \lambda\left\{E\left[h(Y)\right] - E\left[h(R)\right]\right\}\frac{\phi(\tau)}{\Phi(\tau)}$$

where $Y \sim \text{N}\left(\mu, \sigma^2\right)$. Substituting $h(R) = u'\left((1 - \omega_{SN})R_f + \omega_{SN}R\right) = u'(W)$, so

$h'(R) = u''(W)\omega_{SN}$. It follows that

$$E\left(u'(W)\cdot R\right) = \text{cov}\left(u'(W), R\right) + Eu'(W)\cdot E(R)$$

$$= \omega_{SN}\left(\sigma^2 + \lambda^2\right)Eu''(W) + \lambda\left\{Eu'\left(W_{(N)}\right) - Eu'(W)\right\}\frac{\phi(\tau)}{\Phi(\tau)} + Eu'(W)\cdot E(R) \qquad \text{where}$$

$W_{(N)} = \left(Y - R_f\right)\omega_{SN} + R_f = W^* - \lambda|U|\omega_{SN}^*$. Then, substituting above in Equation (1), we obtain:

$$\omega_{SN}\left(\sigma^2 + \lambda^2\right)Eu''(W) + \lambda\left\{Eu'\left(W_{(N)}\right) - Eu'(W)\right\}\frac{\phi(\tau)}{\Phi(\tau)} + Eu'(W)\cdot\left(E(R) - R_f\right) = 0$$

Solving with respect to ω_{SN}, the optimal allocation becomes:

$$\omega_{SN}^* = \frac{\left(E(R) - R_f\right)}{\gamma_{SN}^*\left(\sigma^2 + \lambda^2\right)} + \lambda\frac{Eu'\left(W_{(N)}^*\right) - Eu'\left(W^*\right)}{\gamma_{SN}^*\left(\sigma^2 + \lambda^2\right)Eu'\left(W^*\right)}\cdot\frac{\phi(\tau)}{\Phi(\tau)}$$

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where $\gamma_{SN}^* = E\left[-u''\left(W^*\right)\right]\big/E\left[u'\left(W^*\right)\right]$.

Proof of Proposition 2.

We write (3) for the CARA utility function with $\gamma_{SN}^* = \gamma$. Then,

$$Eu'\left(W_{(N)}^*\right) = \gamma e^{-\gamma\left(1-\omega_{SN}^*\right)R_f} E\left(e^{-\gamma\omega_{SN}^* Y}\right)$$

$$Eu'\left(W^*\right) = \gamma e^{-\gamma\left(1-\omega_{SN}^*\right)R_f} E\left(e^{-\gamma\omega_{SN}^* Y}\right) E\left(e^{-\gamma\omega_{SN}^*\lambda|U|}\right)$$

then[e]

$$\frac{Eu'\left(W_{(N)}^*\right)}{Eu'\left(W^*\right)} = \frac{1}{E\left(e^{-\gamma\omega_{SN}^*\lambda|U|}\right)} = E\left(e^{\gamma\omega_{SN}^*\lambda|U|}\right)$$

That coincides with the moment generator functions of $\lambda|U|$ at $t = \gamma\omega_{SN}^*$. Therefore

$$\frac{Eu'\left(W_{(N)}^*\right)}{Eu'\left(W^*\right)} = e^{\lambda\gamma\omega_{SN}^*\tau+\frac{1}{2}\left(\lambda\gamma\omega_{SN}^*\right)^2} \cdot \frac{\Phi\left(\lambda\gamma\omega_{SN}^* +\tau\right)}{\Phi(\tau)} = g_U\left(\lambda t\right)\cdot\frac{\Phi\left(\lambda\gamma\omega_{SN}^* +\tau\right)}{\Phi(\tau)}$$

where g_U is the moment generator function of $U \sim N\left(\tau,1\right)$. Substituting above into

Equation (3), we obtain:

$$\omega_{SN}^* = \frac{1}{\gamma}\left\{\frac{\left(\mu-R_f\right)}{\sigma^2}\frac{\sigma^2}{\sigma^2+\lambda^2}+\right.$$

$$\left.\frac{\lambda}{\sigma^2+\lambda^2}\left[\tau+\frac{\phi(\tau)}{\Phi(\tau)}\cdot\frac{\Phi\left(\gamma\omega_{SN}^*\lambda+\tau\right)}{\Phi(\tau)}\cdot e^{\left(\gamma\omega_{SN}^*\lambda\right)\tau+\frac{1}{2}\left(\gamma\omega_{SN}^*\lambda\right)^2}\right]\right\}$$

Denoting $\omega_N^* = \frac{1}{\gamma}\frac{\left(\mu-R_f\right)}{\sigma^2}$, solution (4) follows. And that concludes the proof.

Proof of Proposition 3.

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[e] Note that $\dfrac{Eu'\left(W_{(N)}^*\right)}{Eu'\left(W^*\right)} \geq 1$ if $\lambda \geq 0$ and $\dfrac{Eu'\left(W_{(N)}^*\right)}{Eu'\left(W^*\right)} < 1$ if $\lambda < 0$.

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We rewrite ω_{SN}^* as a function of ω_N^*. From (4)

$$\omega_{SN}^* = \omega_N^* + \frac{1}{\gamma}\left\{ -\lambda^2 \frac{\left(\mu - R_f\right)}{\sigma^2} \frac{1}{\sigma^2 + \lambda^2} + \right.$$

$$\left. \frac{\lambda}{\sigma^2 + \lambda^2}\left[\tau + \frac{\phi(\tau)}{\Phi(\tau)} \cdot \frac{\Phi\left(\gamma\omega_{SN}^*\lambda + \tau\right)}{\Phi(\tau)} \cdot e^{\left(\gamma\omega_{SN}^*\lambda\right)\tau + \frac{1}{2}\left(\gamma\omega_{SN}^*\lambda\right)^2}\right]\right\} =$$

$$= \omega_N^* + \frac{\lambda}{\sigma^2 + \lambda^2}\left\{ -\lambda\omega_N^* + \right.$$

$$\left. \frac{1}{\gamma}\left[\tau + \frac{\phi(\tau)}{\Phi(\tau)} \cdot \frac{\Phi\left(\gamma\omega_{SN}^*\lambda + \tau\right)}{\Phi(\tau)} \cdot e^{\left(\gamma\omega_{SN}^*\lambda\right)\tau + \frac{1}{2}\left(\gamma\omega_{SN}^*\lambda\right)^2}\right]\right\}$$

If $\{\ \}$ is positive, the sign of change in optimal allocation is the same as that of λ.

Denoting with $T = T\left(\lambda, \tau, \gamma, \omega_{SN}^*\right) = \frac{1}{\lambda\gamma}\left[\tau + \frac{\phi(\tau)}{\Phi(\tau)} \cdot \frac{\Phi\left(\gamma\omega_{SN}^*\lambda + \tau\right)}{\Phi(\tau)} \cdot e^{\left(\gamma\omega_{SN}^*\lambda\right)\tau + \frac{1}{2}\left(\gamma\omega_{SN}^*\lambda\right)^2}\right]$

the results follow.

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