

# ECONOMETRICA

JOURNAL OF THE ECONOMETRIC SOCIETY

*An International Society for the Advancement of Economic  
Theory in its Relation to Statistics and Mathematics*

<http://www.econometricsociety.org/>

*Econometrica*, Vol. 80, No. 6 (November, 2012), 2827–2847

## AMBIGUITY IN THE SMALL AND IN THE LARGE

PAOLO GHIRARDATO

*Collegio Carlo Alberto, 10024 Moncalieri (TO), Italy*

MARCIANO SINISCALCHI

*Northwestern University, Evanston, IL 60208, U.S.A.*

---

The copyright to this Article is held by the Econometric Society. It may be downloaded, printed and reproduced only for educational or research purposes, including use in course packs. No downloading or copying may be done for any commercial purpose without the explicit permission of the Econometric Society. For such commercial purposes contact the Office of the Econometric Society (contact information may be found at the website <http://www.econometricsociety.org> or in the back cover of *Econometrica*). This statement must be included on all copies of this Article that are made available electronically or in any other format.

---

## AMBIGUITY IN THE SMALL AND IN THE LARGE

BY PAOLO GHIRARDATO AND MARCIANO SINISCALCHI<sup>1</sup>

This paper considers local and global multiple-prior representations of ambiguity for preferences that are (i) monotonic, (ii) Bernoullian, that is, admit an affine utility representation when restricted to constant acts, and (iii) locally Lipschitz continuous. We do not require either certainty independence or uncertainty aversion. We show that the set of priors identified by Ghirardato, Maccheroni, and Marinacci's (2004) "unambiguous preference" relation can be characterized as a union of Clarke differentials. We then introduce a behavioral notion of "locally better deviation" at an act and show that it characterizes the Clarke differential of the preference representation at that act. These results suggest that the priors identified by these preference statements are directly related to (local) optimizing behavior.

KEYWORDS: Ambiguity, Clarke differential, optimization, unambiguous preference.

### 1. INTRODUCTION

SEVERAL POPULAR MODELS of choice under ambiguity represent preferences over uncertain prospects (acts) via some function of their expected utilities, computed with respect to a distinguished set of probabilities. For instance, the maxmin-expected utility (MEU) model of Gilboa and Schmeidler (1989) ranks acts according to  $V(h) = \min_{Q \in D} E_Q[u \circ h]$ , where  $D$  is a set of priors over the state space  $S$ . For multiplier preferences (Hansen and Sargent (2001)),  $V(h) = \min_{Q \in \Delta(S)} E_Q[u \circ h] + \theta \cdot R(Q \| P)$ , where  $R(Q \| P)$  is the relative entropy of  $Q$  with respect to an approximating model  $P$ . In the smooth ambiguity model of Klibanoff, Marinacci, and Mukerji (2005),  $V(h) = \int_{\Delta(S)} \phi(E_Q[u \circ h]) d\mu(Q)$ , where  $\mu$  is a second-order belief over all priors.

The same preference may admit multiple representations that employ different sets of priors (see Siniscalchi (2006) for examples). Despite this fact, Ghirardato, Maccheroni, and Marinacci (2004; GMM henceforth) showed that a preference can be associated with a canonical set of priors that is independent of its functional representation. Their identification strategy is as follows. Let  $\succsim$  be the individual's preference; say that act  $f$  is *unambiguously preferred* to act  $g$ , written  $f \succsim^* g$ , if  $f \succsim g$  and this ranking is preserved across mixtures:

$$(1) \quad \lambda f + (1 - \lambda)h \succsim \lambda g + (1 - \lambda)h \quad \text{for all } \lambda \in (0, 1] \text{ and all acts } h.$$

<sup>1</sup>We thank a co-editor and two anonymous referees, as well as Simone Cerreia-Vioglio, Theo Diasakos, Fabio Maccheroni, Mark Machina, Pietro Ortoleva, Daniele Pennesi, Tomasz Strzalecki, and audiences at the 2009 RUD and SAET conferences, the Workshop in honor of Daniel Ellsberg (Vienna, May 2010), and at seminars at Rice, Northwestern, Penn, Princeton, and Montreal for helpful comments. The usual disclaimer applies. Ghirardato is also grateful to the Italian MIUR for financial support. A version of Theorem 2 in this paper first appeared in the working paper "A More Robust Definition of Multiple Priors" (Ghirardato and Siniscalchi (2010)).

GMM showed that, under suitable assumptions, there exist a utility function  $u$  and a unique set  $C$  of priors such that, for all acts  $f$  and  $g$ ,  $f \succ^* g$  if and only if  $E_P[u \circ f] \geq E_P[u \circ g]$  for all  $P \in C$  (a representation introduced by [Bewley \(2002\)](#)). Furthermore, under the assumptions in GMM, the representation  $V$  of the individual's preferences  $\succ$  can be written as  $V(h) = I(u \circ h)$  for a suitable real functional  $I$ ; GMM then showed that  $C$  is the [Clarke \(1983\)](#) differential of  $I$ , evaluated at the constant function 0. This characterization makes it practical to compute the set  $C$  for many decision models, including MEU and Choquet-expected utility (CEU) ([Schmeidler \(1989\)](#)). Notably, these results are *not* restricted to preferences that satisfy uncertainty aversion in the sense of [Schmeidler's \(1989\) Uncertainty Aversion axiom](#).<sup>2</sup>

However, the analysis in GMM has two limitations. First, GMM's differential characterization of the set  $C$  depends crucially on the assumption that preferences satisfy the Certainty Independence axiom of [Gilboa and Schmeidler \(1989\)](#).<sup>3</sup> This axiom restricts ambiguity attitudes and rules out several recent models of choice under ambiguity, including multiplier and smooth ambiguity preferences.<sup>4</sup> Second, the results in GMM do not fully reveal the usefulness and economic significance of the set of priors  $C$ , beyond the fact that it characterizes the unambiguous preference  $\succ^*$ .

The objective of this paper is to address both limitations. To begin, we do not assume certainty independence; as a result, our analysis does not impose any restriction on ambiguity attitudes, and accommodates virtually all classical and recent decision models under ambiguity, including those discussed above or referenced in footnote 4. Our first main result generalizes GMM's differential characterization of the set  $C$ : writing the representation of preferences as  $V(h) = I(u \circ h)$ , we show that, up to convex closure,  $C$  is the union of all (suitably normalized) Clarke differentials of the functional  $I$ , computed at all interior points rather than just at zero.

The Clarke differential of nonsmooth functions plays a similar role in optimization problems as the gradient of smooth functions. In particular, a function attains a local extremum at a point only if its Clarke differential at that point contains the zero vector—an analog of the familiar first-order conditions. Our result then implies that the probabilities in the set  $C$  are (up to convex closure) those that identify candidate solutions to optimization problems. Example 3 below illustrates this in a canonical portfolio choice application.

<sup>2</sup>Uncertainty aversion has been questioned both theoretically and experimentally: compare [Epstein \(1999\)](#), [Ghirardato and Marinacci \(2002\)](#), and [Baillon, L'Haridon, and Placido \(2012\)](#).

<sup>3</sup>On the other hand, the existence of a set  $C$  of priors that characterizes the unambiguous preference  $\succ^*$  follows under minimal regularity assumptions: see Section 3.

<sup>4</sup>Other models that do not assume certainty independence include variational preferences ([Maccheroni, Marinacci, and Rustichini \(2006\)](#)), confidence-function preferences ([Chateauneuf and Faro \(2009\)](#)), uncertainty-averse preferences ([Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio \(2011\)](#)), vector expected utility ([Siniscalchi \(2009\)](#)), and mean-dispersion preferences ([Grant and Polak \(2011\)](#), [Chambers, Grant, Polak, and Quiggin \(2011\)](#)).

Our second main result has no counterpart in GMM and sheds further light on the role of priors in  $C$  in the individual's choices. To illustrate, think of acts  $f, g$  as representing the state-contingent consequences of two actions the individual may choose, and think of act  $h$  as the status quo. Then Eq. (1) states that choosing the  $f$  action with some probability  $\lambda$ , thereby moving from  $h$  toward  $f$  in utility terms, is always at least as good as moving toward  $g$ , no matter what is the initial status quo point  $h$  and how far one moves away from  $h$ . That is,  $f$  is a uniformly better deviation than  $g$ . However, one is typically interested in optimality conditions at a specific status quo point  $h$ . With this in mind, we localize Eq. (1); that is, we apply it to a small neighborhood around a single status quo point  $h$ , and only consider small (but discrete) movements away from the status quo. Say that  $f$  is a better deviation than  $g$  near  $h$ , written  $f \succ_h^* g$ , if

$$(2) \quad \lambda f + (1 - \lambda)h' \succ \lambda g + (1 - \lambda)h' \quad \text{for all } \lambda \text{ small and all acts } h' \text{ near } h$$

(see Sec. 4.3 for details). This definition is naturally related to optimizing behavior at a point  $h$ : it identifies small profitable and unprofitable deviations away from the status quo. Our second result shows that the relation  $\succ_h^*$  characterizes the normalized Clarke differential  $C(h)$  of the functional  $I$  at the point  $h$ .<sup>5</sup> In our view, this result illustrates the connection between priors and optimizing behavior in a clearer and more direct way than GMM's global result (and our generalization thereof). That said, the local and global results are closely related:  $f \succ^* g$  if and only if  $f \succ_h^* g$  for all acts  $h$ , and  $C$  is the union of all sets  $C(h)$ .

Finally, a caveat. GMM suggested that the set  $C$  may represent ambiguous beliefs or perceived ambiguity. However, they also discussed (GMM, p. 137) potential difficulties with this interpretation; in particular,  $C$  may incorporate aspects of ambiguity attitude. We prefer to emphasize the connection between the priors in the set  $C$  and optimizing behavior, and do not take a stand as to whether such priors reflect beliefs, ambiguity attitudes, or both.

### 1.1. Intuition for the Results and Examples

For simplicity, let the state space be  $S = \{s_1, s_2\}$  and assume linear utility. To make the intuition as sharp as possible, we assume that  $I$  is continuously differentiable, so its Clarke differential at a point  $h$  is the gradient  $\nabla I(h)$ , and, importantly, the map  $h \mapsto \nabla I(h)$  is continuous.

Under these assumptions,  $C$  is the convex closure of the set of all the probabilities  $\frac{\nabla I(k)}{\nabla I(k) \cdot [1,1]}$  for all  $k \in \mathbb{R}^2$ . To see that, for every  $f, g \in \mathbb{R}^2$ ,  $P \cdot f \geq P \cdot g$  for all  $P \in C$  implies Eq. (1), fix  $h \in \mathbb{R}^2$  and  $\lambda \in (0, 1]$ . By assumption,

<sup>5</sup>  $\succ_h^*$  is not a Bewley preference, and its connection with  $C(h)$  is more subtle than the relationship between  $\succ^*$  and  $C$ ; see Section 4.3 for details.

$\nabla I(k) \cdot (f - g) \geq 0$  for all  $k \in \mathbb{R}^2$ : then, by the mean value theorem, there is a point  $k^*$  in the segment joining  $\lambda f + (1 - \lambda)h$  and  $\lambda g + (1 - \lambda)h$  such that  $I(\lambda f + (1 - \lambda)h) - I(\lambda g + (1 - \lambda)h) = \nabla I(k^*) \cdot [\lambda f + (1 - \lambda)h - \lambda g + (1 - \lambda)h] = \nabla I(k^*) \cdot \lambda(f - g) \geq 0$ , so Eq. (1) holds. This argument generalizes to the non-smooth case.

For the converse implication, suppose that  $P^* \cdot f < P^* \cdot g$  for some  $P^* \in C$ , and hence  $\nabla I(k^*) \cdot (f - g) < 0$  for some  $k^*$ . Then, since  $\nabla I(k)$  is continuous in  $k$ ,  $\nabla I(k) \cdot (f - g) < 0$  for all  $k$  in some neighborhood  $N$  of  $k^*$ . But then we can let  $h = k^*$  and choose  $\lambda$  sufficiently small so that the segment joining  $\lambda f + (1 - \lambda)k^*$  and  $\lambda g + (1 - \lambda)k^*$  lies entirely in  $N$ ; thus, the mean value theorem implies that  $I(\lambda f + (1 - \lambda)k^*) - I(\lambda g + (1 - \lambda)k^*) < 0$ , so Eq. (1) does not hold. This argument relies crucially on the fact that there is a unique gradient at every point  $k$  and that the gradient is continuous in  $k$ . Both properties fail in the nonsmooth case, so our proof of Theorem 2 takes a different approach.

Turn now to our local characterization result:  $C(h) = \{P(h)\}$ , where  $P(h) = \frac{\nabla I(h)}{\nabla I(h) \cdot [1, 1]}$ . Assume that Eq. (2) holds and, in particular, consider the sequence  $h^n = \frac{1}{1 - \lambda^n} h$ . Then it is easy to see that, for all  $n$  large,  $\frac{I(\lambda f + h) - I(h)}{\lambda^n} \geq \frac{I(\lambda g + h) - I(h)}{\lambda^n}$ ; since  $I$  is differentiable, this implies that  $\nabla I(h) \cdot f \geq \nabla I(h) \cdot g$ . Thus,  $f \succ_h^* g$  implies that  $P(h) \cdot f \geq P(h) \cdot g$ . Here, differentiability allows us to focus on a specific sequence  $(h^n)$  and directly to link Eq. (2) to a property of the unique differential of  $I$  at  $h$ . The nonsmooth case again requires a different approach.

The converse implication is more delicate, even in the smooth case. By differentiability, if  $\nabla I(h) \cdot f > \nabla I(h) \cdot g$ , then  $\frac{I(\lambda f + h) - I(h)}{\lambda^n} > \frac{I(\lambda g + h) - I(h)}{\lambda^n}$  for  $n$  large, so Eq. (2) holds for the sequence  $h^n = \frac{1}{1 - \lambda^n} h$  considered above. To extend this conclusion to other sequences, one needs to invoke the fact that a continuously differentiable function is strictly differentiable (Clarke (1983, Proposition 2.2.1)). But if  $\nabla I(h) \cdot f = \nabla I(h) \cdot g$ , this argument clearly does not apply. Example 4 in Section 4.3 illustrates further subtleties. Theorems 6 and 7 circumvent these issues.

**EXAMPLE 1—Nonsmooth Preferences:** Example 17 in GMM characterizes the set  $C$  for Choquet preferences on a finite state space  $S = \{s_1, \dots, s_n\}$ . We briefly discuss the characterization of the local priors  $C(h)$ . For any permutation  $\sigma$  of  $\{1, \dots, n\}$ , a Choquet preference with capacity  $\nu$  admits an expected utility (EU) representation on the set  $\mathcal{F}_\sigma$  of acts  $h$  such that  $h(s_{\sigma(1)}) \succcurlyeq h(s_{\sigma(2)}) \succcurlyeq \dots \succcurlyeq h(s_{\sigma(n)})$ , with prior  $P_\sigma$  given by  $P_\sigma(s_{\sigma(i)}) = \nu(\{s_{\sigma(1)}, \dots, s_{\sigma(i)}\}) - \nu(\{s_{\sigma(1)}, \dots, s_{\sigma(i-1)}\})$ . Fix an act  $h$  that belongs only to  $\mathcal{F}_\sigma$ ; preferences are effectively EU in a neighborhood of  $h$ , so  $C(h) = \{P_\sigma\}$ . If instead  $h$  belongs to  $\mathcal{F}_{\sigma_1}, \dots, \mathcal{F}_{\sigma_k}$ , then, by Theorem 2.5.1 in Clarke (1983),  $C(h)$  is the convex hull of  $\{P_{\sigma_1}, \dots, P_{\sigma_k}\}$ . This result extends to piecewise linear preferences (defined in GMM, Section 5.2).

**EXAMPLE 2—Local versus Global Priors:** Let  $S = \{s_1, s_2\}$ ,  $X = \mathbb{R}_+$ , and the risk-neutral preference be represented by  $I(h) = \max(\frac{1}{2}h(s_1) + \frac{1}{2}h(s_2), \varepsilon +$

$\min(h(s_1), h(s_2))$  for some small  $\varepsilon > 0$ .<sup>6</sup> For acts  $h$  such that  $|h(s_1) - h(s_2)| \geq 2\varepsilon$ , preferences are consistent with EU, with a uniform prior  $P$  on  $S$ ; if  $\varepsilon$  is small, this is the case for most acts. However, for acts close to the diagonal, this preference behaves like MEU, with set of priors  $\Delta(S)$ .

Our generalization of GMM's result implies that the preference  $\succsim^*$  defined in Eq. (1) is represented by  $C = \Delta(S)$ , despite the fact that  $I$  is consistent with EU for most acts; we view this as a stark demonstration of the global nature of GMM's approach. By way of contrast,  $C(h) = \{P\}$  if  $|h(s_1) - h(s_2)| > 2\varepsilon$  and  $C(h) = \Delta(S)$  if  $|h(s_1) - h(s_2)| < 2\varepsilon$ , and our second main result implies that Eq. (2) correctly reflects the local behavior of this preference.

**EXAMPLE 3**—Based on [Dow and da Costa Werlang \(1992\)](#): An investor with wealth  $W$  and preferences characterized by the functional  $I$  and the utility  $u: X \rightarrow \mathbb{R}$  (where  $X \subset \mathbb{R}$ ) considers buying or selling an asset with uncertain returns  $R: \Omega \rightarrow \mathbb{R}$  on the finite state space  $S$ , at a price  $p$ . Thus, the agent's utility if she buys  $t \in \mathbb{R}$  units of the asset is  $I(u(W + t[R - p]))$ . [Dow and da Costa Werlang \(1992\)](#) assumed that  $I$  is an uncertainty-averse (i.e., concave) Choquet functional ([Schmeidler \(1989\)](#)) and that  $u$  is strictly increasing and continuously differentiable; they showed that the agent will optimally choose  $t = 0$  (i.e., no trade) if and only if (iff)  $I(R) \leq p \leq -I(-R)$ .

We now generalize this result. Assume that  $I$  is locally Lipschitz continuous; then (see [Clarke \(1983, Proposition 2.3.2\)](#)), a *necessary* condition for no trade to be optimal is that 0 be an element of the Clarke differential of the real function  $t \mapsto I(u(W + t[R - p]))$  at  $t = 0$ . By the chain rule for nonsmooth functions (see [Clarke \(1983, Proposition 2.3.9\)](#)), this translates to  $E_Q[u'(W)(R - p)] = 0$  for some  $Q \in \partial I(u(W))$ , the Clarke differential of  $I$  at  $u(W)$ . This generalizes the familiar first-order condition with EU preferences. Moreover, since  $W$  is constant and  $u'(W) > 0$ , we obtain

$$\min_{P \in C(1_S W)} E_P[R] \leq p \leq \max_{P \in C(1_S W)} E_P[R],$$

where  $C(1_S W)$  is the normalized Clarke differential characterized by Eq. (2), at  $h = 1_S W$ .

If, furthermore, the functional  $I$  and the function  $u$  are concave, this condition is also *sufficient*. This generalizes the result of [Dow and Werlang](#) to a broad class of uncertainty-averse preferences. Indeed, the above condition is also sufficient as long as the composite map  $t \mapsto I(u(W + t[R - p]))$  is concave, even though  $I$  is not. For instance (cf. [Heath and Tversky \(1991\)](#)), the investor may be uncertainty-averse with respect to  $R$ , yet feel "competent" enough to evaluate other prospects in a manner consistent with uncertainty appeal.

Finally, if  $I$  is an uncertainty-averse Choquet functional, by [Corollary 5](#),  $C(1_S W) = C(0_S) = C$ , the GMM set of priors. But since uncertainty-averse

<sup>6</sup>We thank an anonymous referee for suggesting this example.

Choquet preferences are MEU,  $I(u \circ h) = \min_{p \in C} E_p[u \circ h]$ . This yields Dow and Werlang's original result as a special case.

### 1.2. Related Literature

As noted above, GMM is the starting point of our work. The discussion of Corollaries 3–5 in Section 4.2 explains how our result specializes to GMM's under certainty independence. Nehring (2002) also identified the set  $C$  from behavior; our paper thus also extends his results.

Gilboa, Maccheroni, Marinacci, and Schmeidler (2010) considered a decision maker (DM) who is endowed with a possibly incomplete preference over acts reflecting objective information, and a complete preference reflecting her actual behavior. The objective preference has a Bewley-style representation via a set  $C$  of priors. Thus, while there are natural similarities, our objectives are clearly different. We do not posit the existence of objective information. Moreover, our main contribution is the operational characterization of the sets  $C$  and  $C(h)$ .

Siniscalchi (2006) proposed a related notion of "plausible priors." The main difference with the present paper, and with the GMM approach is the fact that plausible priors are identified individually, rather than as elements of a set. This requires restrictions on preferences that we do not need (in addition to certainty independence).

Klibanoff, Mukerji, and Seo (2011; KMS henceforth) considered infinite repetitions of an experiment with outcomes in some set  $S$  and imposed a "symmetry" requirement on preferences. They showed that, in this setting,  $C = \{\int \ell^\infty dm(\ell) : m \in M\}$ , where  $\ell^\infty$  denotes the independent and identically distributed (i.i.d.) product of  $\ell \in \Delta(S)$  and  $M \subset \Delta(\Delta(S))$ . KMS proposed a "relevance" condition that identifies measures in the support of some  $m \in M$ . This approach differs substantially from GMM's identification strategy. For instance, consider an EU preference with a prior  $P$  that, by the symmetry requirement, satisfies  $P = \int \ell^\infty dm$  for some  $m \in \Delta(\Delta(S))$ . KMS's definition deems relevant all measures in the support of  $m$ , whereas GMM (and we) find that  $C = \{P\}$ .

None of the above papers provides a counterpart to our local characterization result.

Rigotti, Shannon, and Strzalecki (2008) proposed different, equivalent notions of belief at an act  $h$  in a setting with monetary outcomes and preferences represented by a quasiconcave function  $V$ , and used them to analyze efficiency and trade in a competitive environment. When  $V(h) = I(u(h))$  and  $I$  and  $u$  are suitably regular,<sup>7</sup> their beliefs at  $h$  can be computed from the set  $C(h)$  that

<sup>7</sup>In particular, if  $I$  is locally Lipschitz and nice in the sense of Section 4.1, and  $u$  is differentiable.



we characterize and the derivative of  $u$  via an appropriate chain rule (cf. Example 3). Thus, up to marginal utilities, our Eq. (2) provides a complementary behavioral interpretation of Rigotti, Shannon, and Strzalecki’s (2008) beliefs at  $h$  and relates these to the GMM set of priors  $C$ . On the other hand, our results do not require quasiconcavity.

Finally, Machina (2005) defined “event derivatives” of a representation  $V(\cdot)$ , a subjective counterpart to derivatives with respect to lotteries in Machina (1982). A representation is *event-smooth* if it admits suitably regular event derivatives. Machina showed how to generalize EU-based characterizations of, for example, likelihood rankings or comparative risk aversion to event-smooth representations of preferences; however, his paper does not provide a preference foundation for event smoothness. Instead, our paper focuses on the behavioral properties that characterize the normalized Clarke differential  $C(h)$  at an act  $h$ . At a formal level, we consider Clarke derivatives with respect to outcomes, rather than events, and do not assume smoothness.

2. NOTATION AND PRELIMINARIES

We consider a state space  $S$ , endowed with a sigma-algebra  $\Sigma$ . The notation  $B_0(\Sigma, \Gamma)$  indicates the set of simple  $\Sigma$ -measurable real functions on  $S$  with values in the interval<sup>8</sup>  $\Gamma \subset \mathbb{R}$ , endowed with the topology induced by the supremum norm; for simplicity, write  $B_0(\Sigma, \mathbb{R})$  as  $B_0(\Sigma)$ . Recall that since  $\Sigma$  is a sigma-algebra,  $B(\Sigma)$  is the closure of  $B_0(\Sigma)$ , and it is a Banach space.

The set of finitely additive probabilities on  $\Sigma$  is denoted  $ba_1(\Sigma)$ .  $ba_1(\Sigma)$  is endowed with the (relative) weak\* topology; that is,  $\sigma(ba(\Sigma), B_0(\Sigma))$  (equivalently,  $\sigma(ba(\Sigma), B(\Sigma))$ ). We identify elements of  $ba(\Sigma)$  and the linear functionals they identify; if  $a \in B(\Sigma)$  and  $Q \in ba(\Sigma)$ , then  $Q(a) = \int a dQ$ .

If  $B$  is one of  $B_0(\Sigma, \Gamma)$  for some interval  $\Gamma$  or  $B(\Sigma)$ , a functional  $I : B \rightarrow \mathbb{R}$  is *monotonic* if  $I(a) \geq I(b)$  for all  $a \geq b$ , *continuous* if it is sup-norm continuous, *isotone* if, for all  $\alpha, \beta \in \Gamma$ ,  $I(\alpha 1_S) \geq I(\beta 1_S)$  if and only if  $\alpha \geq \beta$ , *normalized* if  $I(\alpha 1_S) = \alpha$  for all  $\alpha \in \Gamma$ , *constant-additive* if  $I(a + \alpha 1_S) = I(a) + \alpha$  for all  $a \in B$  and  $\alpha \in \mathbb{R}$  such that  $a + \alpha 1_S \in B$ , *positively homogeneous* if  $I(\alpha a) = \alpha I(a)$  for all  $a \in B$  and  $\alpha \in \mathbb{R}_+$  such that  $\alpha a \in B$ , and *constant-linear* if it is constant-additive and positively homogeneous.

Finally, fix a convex subset  $X$  of a vector space. (Simple) acts are  $\Sigma$ -measurable functions  $f : S \rightarrow X$  such that  $f(S) = \{f(s) : s \in S\}$  is finite; the set of all (simple) acts is denoted by  $\mathcal{F}$ . We define mixtures of acts pointwise: for any  $\alpha \in [0, 1]$ ,  $\alpha f + (1 - \alpha)g$  is the act that delivers the prize  $\alpha f(s) + (1 - \alpha)g(s)$  in state  $s$ . Given a preference  $\succsim$  on  $\mathcal{F}$ , we say that an act  $h \in \mathcal{F}$  is *interior* if there exist prizes  $x, y \in X$  such that  $x \succ h(s) \succ y$  for all  $s \in S$ , and we denote the set of interior acts by  $\mathcal{F}^{int}$ . (The dependence of  $\mathcal{F}^{int}$  on  $\succsim$ , while not made explicit, should be kept in mind.)

<sup>8</sup>That is,  $\Gamma \subset \mathbb{R}$  is one of  $[\alpha, \beta]$ ,  $[\alpha, \beta)$ ,  $(\alpha, \beta]$ , or  $(\alpha, \beta)$ , where  $\alpha = -\infty$  and  $\beta = \infty$  are allowed where applicable.



## 3. PREFERENCES

The main object of study is a binary relation  $\succsim$  on  $\mathcal{F}$ . As usual,  $\succ$  (resp.  $\sim$ ) denotes the asymmetric (resp. symmetric) component of  $\succsim$ . With a small abuse of notation, we denote with the same symbol the prize  $x$  and the constant act that delivers  $x$  for all  $s$ . We assume throughout that the preference  $\succsim$  admits a numerical representation that satisfies a regularity property:

DEFINITION 1: A preference relation  $\succsim$  is (nontrivial) *monotonic*, *Bernoullian*, and *locally Lipschitzian* (henceforth MBL) if there exists a nonconstant, affine function  $u: X \rightarrow \mathbb{R}$  and a monotonic, isotone functional  $I: B_0(\Sigma, u(X)) \rightarrow \mathbb{R}$  that is locally Lipschitz in the interior of its domain, and such that, for all  $f, g \in \mathcal{F}$ ,

$$(3) \quad f \succsim g \iff I(u \circ f) \geq I(u \circ g).$$

MBL preferences admit *certainty equivalents*: for any  $f \in \mathcal{F}$ , there is  $x_f \in C$  such that  $x_f \sim f$ .

Most preference models considered in the classic and recent literature on ambiguity belong to this class. Virtually all have monotonic, isotone, Bernoullian, and continuous representations; [Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci, and Siniscalchi \(2011\)](#) provided an axiomatization of preferences satisfying these assumptions. Furthermore, if the representing functional is also *constant-additive*, it is globally Lipschitz; this applies to the preferences considered by GMM (including MEU,  $\alpha$ -MEU, and CEU), as well as multiplier, variational, vector expected utility (VEU) and mean-dispersion preferences ([Grant and Polak \(2011\)](#)). Alternatively, if the representing functional is *concave or convex*, then it is locally Lipschitz on the interior of its domain by a classic result of [Roberts and Varberg \(1974\)](#); this includes smooth uncertainty-averse preferences and the confidence-function preferences studied by [Chateauneuf and Faro \(2009\)](#). Also, if  $I$  is *continuously Frechet differentiable*, then again it is locally Lipschitz (see [Clarke \(1983, Proposition 2.2.1 and Corollary\)](#)).

In addition, Appendix S.A in the Supplemental Material ([Ghirardato and Siniscalchi \(2012\)](#)) introduces a novel axiom that is equivalent to the existence of a locally Lipschitz, normalized representation for monotonic, Bernoullian, and continuous preferences. This enables us to apply our results below even to preferences that do not fall into any of the above categories; for instance, uncertainty-averse preferences that are not “concavifiable” or the generalized mean-dispersion preferences of [Chambers et al. \(2011\)](#).

Though MBL preferences are more general than those considered in GMM, these authors’ notion of unambiguous preference still identifies a unique set of priors via a Bewley-like representation. The proof is a straightforward adaptation of GMM’s and hence is omitted. As we argued in the [Introduction](#), it is

useful to interpret GMM's definition as stating that an act  $f$  is a better deviation than another act  $g$  regardless of what the status quo  $h$  is and regardless of how far one moves away from  $h$  (i.e., how much weight one places on  $f$  or, respectively,  $g$ ).

DEFINITION 2: Let  $f, g \in \mathcal{F}$ . We say that  $f$  is a *uniformly (weakly) better deviation than*  $g$ , denoted by  $f \succ^* g$ , if and only if, for each  $h \in \mathcal{F}$  and each  $\lambda \in (0, 1]$ ,  $\lambda f + (1 - \lambda)h \succ \lambda g + (1 - \lambda)h$ .

PROPOSITION 1—GMM, Propositions 4 and 5: *Let  $\succ$  be an MBL preference. Then there exists a nonempty, unique, convex, and closed set  $C \subset \text{ba}_1(\Sigma)$  such that for each  $f, g \in \mathcal{F}$ ,*

$$(4) \quad f \succ^* g \iff \int u \circ f \, dP \geq \int u \circ g \, dP \quad \text{for all } P \in C,$$

where  $u$  is the function in Definition 1 and the set  $C$  is independent of the choice of normalization of  $u$ .

The set  $C$  in Proposition 1 is the set of *relevant priors* for the preference  $\succ$ . The following terminology is convenient: a binary relation  $\widehat{\succ}$  on  $\mathcal{F}$  admits a *Bewley representation* (and hence is a *Bewley preference*) if there are an affine function  $v: X \rightarrow \mathbb{R}$  and a set  $D \subset \text{ba}_1(\Sigma)$  such that  $f \widehat{\succ} g$  if and only if  $P(v \circ f) \geq P(v \circ g)$  for all  $P \in D$ .<sup>9</sup> Thus, Proposition 1 states that  $\succ^*$  is a Bewley preference represented by  $u$  and  $C$ .

#### 4. RELEVANT PRIORS: CHARACTERIZATIONS

##### 4.1. Clarke Differentials

DEFINITION 3—Clarke (1983, Sec. 2.1); Lebourg (1979, Sec. 1): Let  $B$  denote either  $B_0(\Sigma)$  or  $B(\Sigma)$ . Consider a locally Lipschitz functional  $I: U \rightarrow \mathbb{R}$ , where  $U \subset B$  is open. For every  $c \in U$  and  $a \in B(\Sigma)$ , the *Clarke (upper) derivative* of  $I$  in  $c$  in the direction  $a$  is

$$I^\circ(c; a) = \limsup_{b \rightarrow c, t \downarrow 0} \frac{I(b + ta) - I(b)}{t}.$$

The *Clarke (sub)differential* of  $I$  at  $c$  is the set

$$\partial I(c) = \{Q \in \text{ba}(\Sigma) : Q(a) \leq I^\circ(c; a), \forall a \in B\}.$$

<sup>9</sup>Clearly, a set  $D \subset \text{ba}_1(\Sigma)$  and its convex closure  $\overline{\text{co}}D$  induce the same Bewley preference. Proposition A.2 in GMM characterizes Bewley preferences.

It is important to point out that, like the usual notion of gradient, the definition of Clarke differential is seldom used directly (although we do so in proving the results in this section). It is useful chiefly because of its convenient calculus properties (see, e.g., Clarke (1983)).

Consider an MBL preference with representation  $(I, u)$ . Given an interior act  $h$ , the functionals in  $\partial I(u \circ h)$  are linear, but in general not normalized. For consistency with the GMM approach, we normalize the elements of  $\partial I(u \circ h)$  to obtain

$$(5) \quad C(h) = \left\{ \frac{Q}{Q(S)} : Q \in \partial I(u \circ h), Q(S) > 0 \right\}.$$

Given  $C(h) \neq \emptyset$  and  $u$ , we can define a Bewley preference  $\succ_{C(h)}$  on  $\mathcal{F}$  as

$$(6) \quad f \succ_{C(h)} g \iff P(u \circ f) \geq P(u \circ h) \quad \forall P \in C(h).$$

Say that the functional  $I$  is *nice at*  $c \in \text{int} B_0(\Sigma, \Gamma)$  if the zero measure  $Q_0 \in \text{ba}(\Sigma)$  is not an element of  $\partial I(c)$ . This condition strengthens monotonicity and loosely speaking, requires that preferences remain nontrivial in arbitrarily small neighborhoods of an act. It plays a role in our local results (Propositions 6 and 7), though not in our global result (Theorem 2). All preferences considered by GMM and, more generally, all MBL preferences represented by a constant-additive functional  $I$ , are everywhere nice; the same is true for concave preferences; see Supplemental Material Appendix S.B. To cover the remaining cases, Appendix S.B also provides an axiom for arbitrary MBL preferences that ensures the existence of a nice representation.

#### 4.2. Global Characterization

We are ready to state our first main result.

**THEOREM 2:** *For any MBL preference  $\succ$  with representation  $(I, u)$  and relevant priors  $C$ ,*

$$C = \overline{\text{co}} \left( \bigcup_{h \in \mathcal{F}^{\text{int}}} C(h) \right).$$

For the proof, see Appendix A.2.

Thus, up to convex closure, the set  $C$  can be computed by considering the normalized Clarke differentials  $C(h)$  for all interior acts  $h$  and then taking the union of such objects. Equivalently,  $f \succ^* g$  if and only if  $P(u \circ f) \geq P(u \circ g)$  for all  $P \in C(h)$  and  $h \in \mathcal{F}^{\text{int}}$ .

We now review specific independence properties of the preference  $\succ$  that have been analyzed in the literature. This will also clarify the relationship between Theorem 2 and its counterpart in GMM. First, if preferences satisfy the

weak certainty independence axiom of **Maccheroni, Marinacci, and Rustichini (2006)**, the functional  $I$  is constant-additive, and all elements  $Q \in \partial I(e)$  satisfy  $Q(S) = 1$  (see part 2 of Proposition A.3 in GMM). Thus,  $C(h) = \partial I(u \circ h)$ , and we obtain the following corollary.

**COROLLARY 3:** *If  $I$  is normalized and constant-additive, then*

$$C = \overline{\text{co}} \left( \bigcup_{h \in \mathcal{F}^{\text{int}}} \partial I(u \circ h) \right).$$

If instead an MBL preference satisfies the homotheticity axiom of **Cerreia-Vioglio et al. (2011)**,  $I$  is positively homogeneous. If  $I$  is normalized and there is a prize  $z \in X$  with  $u(z) = 0 \in \text{int } u(X)$ , then  $\partial I(u \circ h) \subset \partial I(0)$  for all  $h \in \mathcal{F}^{\text{int}}$  (cf. part 1 of Proposition A.3 in GMM). We obtain another corollary.

**COROLLARY 4:** *If  $I$  is normalized and positively homogeneous, and  $z$  is as above, then  $C = \overline{\text{co}} C(z)$ .*

Finally, GMM considered preferences that satisfy certainty independence, and hence admit a representation with  $I$  normalized and constant-linear. With  $z$  as above, we obtain an additional corollary.

**COROLLARY 5—GMM, Theorem 14:** *If  $I$  is constant-linear, then*

$$C = \partial I(0) = C(z).$$

Notice that, if  $I$  is constant-linear, the Clarke upper derivative at 0 in the direction  $a \in B$  takes a particularly simple form (cf. GMM, Proposition A.3), which GMM exploited in their proofs:

$$I^\circ(0; a) = \sup_{b \in B} I(b + a) - I(b).$$

### 4.3. Local Characterization

We turn to the behavioral characterization of locally relevant priors. Recalling the discussion in the Introduction, the definition of locally better deviation concerns the behavior of  $\succsim$  near an interior act  $h$ .<sup>10</sup> Thus, its formal statement requires a notion of convergence for acts. We say that a sequence  $(f^n) \subset \mathcal{F}$

<sup>10</sup>Clearly, given a representation  $(I, u)$  of  $\succsim$ ,  $h \in \mathcal{F}^{\text{int}}$  if and only if the function  $u \circ h$  is in the interior of  $B_0(\Sigma, u(X))$ .

converges to an act  $f \in \mathcal{F}$ , written  $f^n \rightarrow f$ , iff, for all prizes  $x, y \in X$  with  $x \succ y$ , there exists  $K$  such that  $k \geq K$  implies

$$\forall s \in S, \quad \frac{1}{2}f(s) + \frac{1}{2}y \prec \frac{1}{2}f^k(s) + \frac{1}{2}x \quad \text{and} \quad \frac{1}{2}f^k(s) + \frac{1}{2}y \prec \frac{1}{2}f(s) + \frac{1}{2}x.$$

This corresponds to uniform convergence in  $B_0(\Sigma, u(X))$ .

Intuitively, we then apply Definition 2 to a neighborhood of an interior act  $h$ : we consider mixtures of the acts  $f$  and  $g$  with an act near  $h$ , assigning most of the weight to the latter.

DEFINITION 4: For any triple of acts  $f, g, h \in \mathcal{F}$ , say that  $f$  is a (weakly) better deviation than  $g$  near  $h$ , written  $f \succ_h^* g$ , if, for every  $(\lambda^n)_{n \geq 0} \subset [0, 1]$  and  $(h^n)_{n \geq 0} \subset \mathcal{F}$  such that  $\lambda^n \downarrow 0$  and  $h^n \rightarrow h$ ,

$$(7) \quad \lambda^n f + (1 - \lambda^n)h^n \succ \lambda^n g + (1 - \lambda^n)h^n \quad \text{eventually.}$$

Unlike  $\succ^*$ , the relation  $\succ_h^*$  is not always a Bewley preference, because it may fail continuity (Supplemental Material Appendix S.E indicates the properties it does satisfy). Our first main result in this section shows that it nonetheless uniquely identifies the set  $C(h)$ .

THEOREM 6: For any MBL preference  $\succ$  with representation  $(I, u)$ , and any interior act  $h \in \mathcal{F}$ , the following statements hold:

- (i) For all  $f, g \in \mathcal{F}$ ,  $f \succ_h^* g$  implies that  $P(u \circ f) \geq P(u \circ g)$  for all  $P \in C(h)$ .
- (ii) If  $I$  is nice at  $u \circ h$ , then the preference  $\succ_{C(h)}$  is the unique minimal Bewley preference that extends  $\succ_h^*$  (i.e., the intersection of all Bewley preferences that contain  $\succ_h^*$ ).

See Appendix A.1 for the proof.

Thus,  $f \succ_h^* g$  always implies  $f \succ_{C(h)} g$ ; moreover, if  $I$  is nice at  $u \circ h$ , then  $\succ_h^*$  fully identifies the set  $C(h)$ . There is, however, a different, more direct, way to identify  $C(h)$  from  $\succ_h^*$ . The following example illustrates this idea, as well as the role of the niceness assumption.

EXAMPLE 4: Let  $S = \{s_1, s_2\}$ , let  $X = \mathbb{R}$ , and let  $u$  be the identity. Thus,  $\mathcal{F} = B_0(2^S, u(X)) = \mathbb{R}^2$ , and we identify acts  $h$  with vectors  $[h_1, h_2] \in \mathbb{R}^2$ . Fix  $p \in (\frac{1}{2}, 1)$ , let  $P^1 = [p, 1 - p]$  and  $P^2 = [1 - p, p]$ , and consider the smooth ambiguity preference represented by  $u$  and by the strictly increasing and continuously differentiable (hence, locally Lipschitz) function  $I: \mathbb{R}^2 \rightarrow \mathbb{R}$  defined by  $I(h) = \sum_{i=1,2} (P^i \cdot h)^3$ . For any  $h$ , the Clarke differential  $\partial I(h)$  coincides with the gradient  $\nabla I(h) = 3(P^1 \cdot h)^2 P^1 + 3(P^2 \cdot h)^2 P^2$ . Therefore, for  $h \neq 0$ ,  $C(h) = \{\frac{1}{3} \nabla I(h)\}$ , while  $C(0) = \emptyset$  since  $\nabla I(0) = 0$ .

As expected,  $f \succ_h^* g$  implies  $f \succ_{C(h)} g$ . As to the opposite implication, we argue in Supplemental Material Appendix S.C that if  $f \succ_{C(h)} g$  (i.e.,  $\nabla I(h) \cdot f > \nabla I(h) \cdot g$ ), then Eq. (7) eventually holds for all sequences  $\lambda^k \downarrow 0$  and  $h^n \rightarrow h$ . We now show that if  $f \sim_{C(h)} g$  or  $C(h) = \emptyset$ , Eq. (7) does not necessarily hold. Again, details and proofs of all claims below are found in Supplemental Material Appendix S.C.

Let  $f = [1, 0]$  and  $g = [0, 1]$ . Fix  $h \in \mathbb{R}^2$  arbitrarily and define sequences  $(\lambda^n), (h^n), (k^n)$  by

$$\lambda^n = \frac{1}{n}, \quad h^n = \frac{1}{1 - \lambda^n}(\lambda^n[2, 1] + h), \quad k^n = \frac{1}{1 - \lambda^n}(\lambda^n[1, 2] + h).$$

Notice that as  $n \rightarrow \infty$ ,  $\lambda^n \downarrow 0$ ,  $h^n \rightarrow h$  and  $k^n \rightarrow h$ . Then  $f \succ_h^* g$  requires that, for  $n$  large,

$$(8) \quad \begin{aligned} \lambda^n f + (1 - \lambda^n)h^n &\succ \lambda^n g + (1 - \lambda^n)h^n \quad \text{and} \\ \lambda^n f + (1 - \lambda^n)k^n &\succ \lambda^n g + (1 - \lambda^n)k^n. \end{aligned}$$

We focus on three cases (we omit the other cases for brevity).

*Case 1*— $h_1 > h_2 \geq 0$ . In this case,  $f \succ_{C(h)} g$ . As noted above, this implies  $f \succ_h^* g$ .

*Case 2*— $h_1 = h_2 = \gamma > 0$ . In this case,  $\nabla I(h) \cdot f = \nabla I(h) \cdot g$ , that is,  $f \sim_{C(h)} g$ . Then, for  $n$  large, the second preference in Eq. (8) is violated, so it is not the case that  $f \succ_h^* g$ . However, clearly  $\nabla I(h) \cdot (f + \varepsilon) > \nabla I(h) \cdot (g - \varepsilon)$  for any  $\varepsilon > 0$ . As noted above, this implies that  $f + \varepsilon \succ_h^* g - \varepsilon$ .

*Case 3*— $h_1 = h_2 = 0$ . Since  $\nabla I(0) = 0$ ,  $I$  is not nice at  $h = 0$ . Then Eq. (8) does not hold, and continues to be violated if  $f$  and  $g$  are replaced with  $f + \varepsilon$  and  $g - \varepsilon$  for  $\varepsilon > 0$  sufficiently small. Thus, neither  $f \succ_h^* g$  nor  $f + \varepsilon \succ_h^* g - \varepsilon$  holds.

Example 4 suggests that while  $f \succ_{C(h)} g$  may not imply  $f \succ_h^* g$ , it may still imply that  $f' \succ_h^* g'$  for all  $f', g'$  with  $f'(s) > f(s)$  and  $g(s) > g'(s)$  for all  $s \in S$ ; we call such a pair of acts  $(f', g')$  a *spread* of  $(f, g)$ . The following result confirms that this is indeed the case and provides a direct characterization of  $C(h)$  in terms of  $\succ_h^*$ .

**THEOREM 7:** *Consider an MBL preference  $\succ$  and a representation  $(I, u)$ . Fix an interior act  $h \in \mathcal{F}$  and assume that  $I$  is nice at  $u \circ h$ . Then  $C(h)$  is the only weak\*-closed, convex set  $D \subset \text{ba}_1(\Sigma)$  for which the following statements are equivalent for every pair  $(f, g)$  of interior acts:*

- (i)  $f' \succ_h^* g'$  for all spreads  $(f', g')$  of  $(f, g)$ .
- (ii)  $P(u \circ f) \geq P(u \circ g)$  for all  $P \in D$

See Appendix A.1 for the proof.

Case 3 in the Example 4 shows that niceness is required in Theorems 6 and 7.<sup>11</sup>

Finally, as noted in the Introduction, there is a tight connection between the global preference  $\succ^*$  and the local preferences  $\succ_h^*$ :

**COROLLARY 8:** *For all  $f, g \in \mathcal{F}$ ,  $f \succ^* g$  if and only if  $f \succ_h^* g$  for all interior  $h \in \mathcal{F}$ .*

See Appendix A.2 for the proof.

## 5. EXTENSIONS

All the results in this paper apply verbatim if preferences are defined on the set of bounded (rather than simple) acts, as defined, for example, in Gilboa and Schmeidler (1989).

Theorem 2 can be generalized to preferences that are continuous but possibly not locally Lipschitz. For details, see Ghirardato and Siniscalchi (2010).

Supplemental Material Appendix S.D shows that, given an interior act  $h$ , whether a given probability  $P \in \text{ba}_1(\Sigma)$  belongs to the set  $C(h)$  can be directly ascertained using the DM's preferences without invoking Theorems 6 or 7.

Finally, for preferences that satisfy the weak certainty independence axiom of Maccheroni, Marinacci, and Rustichini (2006) (e.g., multiplier, variational, or vector expected utility preferences) and under additional regularity conditions (in particular, concavity or continuous differentiability of  $I$  suffice), the sets  $C(h)$  pin down the preference  $\succ$  uniquely. This follows from non-smooth analogs of the fundamental theorem of calculus (cf. Ngai, Luc, and Théra (2000)). We leave a fuller investigation of this fact to future research.

## APPENDIX: PROOFS OF THE MAIN RESULTS

### A.1. Proof of Theorems 6 and 7, and Corollary 4

Throughout,  $\succ$  is an MBL preference with representation  $(I, u)$  and relevant priors  $C$ . For any  $D \subset \text{ba}_1(\Sigma)$ , we also use the notation  $f \succ_D g$  to mean that  $P(u \circ f) \geq P(u \circ g)$  for all  $P \in D$ .

We use freely the following facts. (i) Since  $I$  is monotonic,  $\partial I(u \circ h)$  consists of positive linear functionals (Rockafellar (1980, Theorem 6, Corollary 3)), and consequently  $a \mapsto I^\circ(c; a)$  is monotonic. (ii)  $a \mapsto I^\circ(u \circ h; a)$  is continuous by Rockafellar (1980, Corollary 1, p. 268).

**LEMMA 9:**  *$C$  is the smallest weak\*-compact, convex set  $D \subset \text{ba}_1(\Sigma)$  such that, for all  $f, g \in \mathcal{F}$ ,  $f \succ_D g$  implies  $f \succ g$ .*

<sup>11</sup> In Example 4,  $\partial I(0)$  contains *only* the zero vector. However, we show in Supplemental Material Appendix S.C how to modify preferences so that  $\partial I(0)$  contains vectors other than 0 without changing the conclusions of the example.



PROOF: That  $C$  satisfies this property is clear, because  $f \succ_C g$  implies  $f \succ^*_h g$  by Proposition 1, and hence  $f \succ g$ . Now suppose another set  $D \subset \text{ba}_1(\Sigma)$  also satisfies this property. If  $f \succ_D g$ , then, for all  $\lambda \in (0, 1]$  and  $h \in \mathcal{F}$ , also  $\lambda f + (1 - \lambda)h \succ_D \lambda g + (1 - \lambda)h$ . Then, by assumption,  $\lambda f + (1 - \lambda)h \succ \lambda g + (1 - \lambda)h$  for all  $\lambda \in (0, 1]$  and  $h \in \mathcal{F}$ ; that is,  $f \succ^*_h g$ . But by Proposition 1, this implies that  $f \succ_C g$ . By Proposition A.1 in GMM, this implies that  $C \subset \overline{c \circ D}$ . *Q.E.D.*

LEMMA 10:  $f \succ^*_h g$  implies that, for all  $\mu \in (0, 1]$  and  $c \in B_0(\Sigma)$ ,  $I^\circ(u \circ h; \mu u \circ f + (1 - \mu)c) \geq I^\circ(u \circ h; \mu u \circ g + (1 - \mu)c)$ .<sup>12</sup>

PROOF: *Step 1.* Fix  $(\lambda_n)$  and  $(h_n)$  as in Definition 4. Functionally, Eq. (7) is equivalent to  $I(\lambda_n u \circ f + (1 - \lambda_n)u \circ h_n) \geq I(\lambda_n u \circ g + (1 - \lambda_n)u \circ h_n)$  eventually; in other words,

$$I(\lambda_n u \circ f + c_n) \geq I(\lambda_n u \circ g + c_n) \quad \text{eventually}$$

for all sequences  $(\lambda_n) \downarrow 0$  and  $(c_n)$  such that  $c_n = (1 - \lambda_n)u \circ h_n$  for some sequence  $(h_n) \subset \mathcal{F}$  with  $h_n \rightarrow h$ .

*Step 2.* For any sequence  $(\lambda_n) \downarrow 0$  and  $(c_n) \rightarrow u \circ h$ , and for any  $\mu \in (0, 1]$  and  $c \in B_0(\Sigma)$ ,

$$\begin{aligned} \lambda_n[\mu u \circ f + (1 - \mu)c] + c_n &= (\lambda_n \mu)u \circ f + [\lambda_n(1 - \mu)c + c_n] \\ &\equiv (\lambda_n \mu)u \circ f + d_n, \end{aligned}$$

and analogously  $\lambda_n[\mu u \circ g + (1 - \mu)c] + c_n = (\lambda_n \mu)u \circ g + d_n$ . Since  $c_n \rightarrow u \circ h$ , eventually  $(1 - \lambda_n \mu)^{-1}d_n \in \text{int } B_0(\Sigma, u(X))$  because  $h$  is interior,  $\lambda_n(1 - \mu)c \rightarrow 0$ , and  $1 - \lambda_n \mu \rightarrow 1$ ; also,  $d_n \rightarrow u \circ h$ . Therefore, there is a sequence  $(h_n) \subset \mathcal{F}$  such that  $(1 - \lambda_n \mu)u \circ h_n = d_n$ ; this sequence necessarily satisfies  $h_n \rightarrow h$ , and so, by Step 1, eventually

$$\begin{aligned} I(\lambda_n[\mu u \circ f + (1 - \mu)c] + c_n) & \\ &= I((\lambda_n \mu)u \circ f + d_n) \geq I((\lambda_n \mu)u \circ g + d_n) \\ &= I(\lambda_n[\mu u \circ g + (1 - \mu)c] + c_n). \end{aligned}$$

Subtracting  $I(c_n)$  from both sides and dividing by  $\lambda_n > 0$  yields

$$\begin{aligned} &\frac{I(\lambda_n[\mu u \circ f + (1 - \mu)c] + c_n) - I(c_n)}{\lambda_n} \\ &\geq \frac{I(\lambda_n[\mu u \circ g + (1 - \mu)c] + c_n) - I(c_n)}{\lambda_n} \quad \text{eventually} \end{aligned}$$

<sup>12</sup>By Lemma S5,  $\succ^*_h$  is independent. Thus, just showing that  $f \succ^*_h g$  implies  $I^\circ(u \circ h; u \circ f) \geq I^\circ(u \circ h; u \circ g)$  would be enough to establish the claim in this lemma for  $c \in B_0(\Sigma, u(X))$ . However, the proof of Theorem 6 requires that the claim hold for all  $c \in B_0(\Sigma)$ .

for all  $(\lambda_n) \downarrow 0$ ,  $\mu \in (0, 1]$ ,  $c \in B_0(\Sigma)$ , and  $(c_n) \rightarrow u \circ h$ .

Step 3. Finally, fix  $\mu$ ,  $c$ , and  $\varepsilon > 0$ . By the definition of  $I^\circ(u \circ h; \mu u \circ g + (1 - \mu)c)$ , there are sequences  $(\lambda_n) \subset [0, 1]$ ,  $(c_n) \subset B_0(\Sigma, u(X))$  such that  $\lambda_n \downarrow 0$ ,  $c_n \rightarrow u \circ h$ , and  $\lim_n \frac{I(\lambda_n[\mu u \circ g + (1 - \mu)c] + c_n) - I(c_n)}{\lambda_n} \geq I^\circ(u \circ h; \mu u \circ g + (1 - \mu)c) - \varepsilon$ . Taking a subsequence if necessary,<sup>13</sup> it follows from Step 2 that  $\lim_n \frac{I(\lambda_n[\mu u \circ f + (1 - \mu)c] + c_n) - I(c_n)}{\lambda_n} \geq I^\circ(u \circ h; \mu u \circ g + (1 - \mu)c) - \varepsilon$ . This implies that  $I^\circ(u \circ h; \mu u \circ f + (1 - \mu)c) \geq I^\circ(u \circ h; \mu u \circ g + (1 - \mu)c) - \varepsilon$ . Since  $\varepsilon > 0$  was arbitrary, the claim follows. Q.E.D.

COROLLARY 11: *If  $\partial I(u \circ h) \neq \{Q_0\}$  (the zero measure), then  $\succ_h^*$  agrees with  $\succ$  on  $X$ .*

PROOF: By monotonicity of  $\succ$ ,  $x \succ y$  implies  $x \succ_h^* y$ , so it is enough to prove that  $x \succ y$  implies that  $y \succ_h^* x$  does not hold. By contradiction, suppose that  $x \succ y$  (hence,  $x \succ_h^* y$ ) and  $y \succ_h^* x$ . Then, by Lemma 10, for every  $c \in B_0(\Sigma)$  and  $\mu \in (0, 1]$ ,  $y \succ_h^* x$  implies  $I^\circ(u \circ h; \mu u(y) + (1 - \mu)c) \geq I^\circ(u \circ h; \mu u(x) + (1 - \mu)c)$ . Now let  $c = 1_S$  and choose  $\mu > 0$  small enough so that  $\alpha \equiv \mu u(x) + (1 - \mu) > 0$  and  $\beta \equiv \mu u(y) + (1 - \mu) > 0$ . Then  $I^\circ(u \circ h; \alpha) = \max_{Q \in \partial I(u \circ h)} \alpha Q(S) = \alpha \max_{Q \in \partial I(u \circ h)} Q(S) \equiv \alpha M$  and similarly  $I^\circ(u \circ h; \beta) = \beta M$ , because  $\alpha, \beta > 0$ . By assumption,  $Q_0$  is not the only functional in  $\partial I(u \circ h)$  and, therefore, since  $I$  is monotonic,  $M > 0$ . But then  $\alpha \leq \beta$ , which contradicts the fact, so  $u(x) > u(y)$ . Q.E.D.

LEMMA 12: *Assume that  $I$  is nice at  $u \circ h$ . For any pair  $f, g \in \mathcal{F}$ ,  $f \succ_{C(h)} g$  implies that  $f' \succ_h^* g'$  for any spread  $(f', g')$  of  $(f, g)$ .*

PROOF: The claim is vacuously true if  $f$  or  $g$  are not interior acts, because in this case there is no spread of  $(f, g)$ . Thus, consider a spread  $(f', g')$  of an interior pair of acts  $(f, g)$ . Then there is  $\varepsilon > 0$  such that  $u \circ f' \geq u \circ f + \varepsilon$  and  $u \circ g' \leq u \circ g - \varepsilon$ . Thus,  $P(u \circ f') > P(u \circ g')$  for all  $P \in C(h)$ .

Suppose there are sequences  $(\lambda_n) \subset [0, 1]$  and  $(h^n) \subset \mathcal{F}$  such that  $\lambda_n \downarrow 0$ ,  $h^n \rightarrow h$  and, by taking subsequences if necessary,  $\lambda^n f' + (1 - \lambda^n)h^n < \lambda^n g' + (1 - \lambda^n)h^n$  for all  $n$ . Passing to the functional representation of  $\succ$ ,  $I(\lambda^n u \circ f' + (1 - \lambda^n)u \circ h^n) < I(\lambda^n u \circ g' + (1 - \lambda^n)u \circ h^n)$  for all  $n$ .

Let  $c^n = \lambda^n u \circ f' + (1 - \lambda^n)h^n$ , so  $\lambda^n u \circ g' + (1 - \lambda^n)u \circ h^n = c^n + \lambda^n[u \circ g' - u \circ f']$  and  $c^n \rightarrow u \circ h$ . Then  $I(c^n) < I(c^n + \lambda^n[u \circ g' - u \circ f'])$  for all  $n$ , so  $\frac{I(c^n + \lambda^n[u \circ g' - u \circ f']) - I(c^n)}{\lambda^n} > 0$  for all  $n$ .

It follows that  $\max_{Q \in \partial I^\circ(u \circ h)} Q(u \circ g' - u \circ f') = I^\circ(u \circ h; u \circ g' - u \circ f') \geq 0$ . Hence, since  $I$  is nice at  $u \circ h$ , there exists  $Q \neq Q_0$  in  $\partial I(u \circ h)$  such that

<sup>13</sup>The sequence  $\frac{I(\lambda_n[\mu u \circ f + (1 - \mu)c] + c_n) - I(c_n)}{\lambda_n}$  may fail to converge. However, since  $I^\circ(u \circ h; \mu u \circ f + (1 - \mu)c) < \infty$  as  $I$  is locally Lipschitz, this sequence must be bounded and hence contain a convergent subsequence.

$Q(u \circ g') \geq Q(u \circ f')$ ; then, for  $P = \frac{Q}{Q(S)} \in C(h)$ ,  $P(u \circ g') \geq P(u \circ f')$ —a contradiction. Q.E.D.

PROOF OF THEOREM 6: (i) If  $\partial I(u \circ h) = \{Q_0\}$ , then  $C(h) = \emptyset$ , so the assertion holds vacuously. Thus, assume henceforth that there is  $Q \neq Q_0$  such that  $Q \in \partial I(u \circ h)$ .

Define a relation  $\geq_h^*$  on  $B_0(\Sigma)$  by letting  $a \geq_h^* b$  iff  $I^\circ(u \circ h; \lambda a + (1 - \lambda)c) \geq I^\circ(u \circ h; \lambda b + (1 - \lambda)c)$  for all  $\lambda \in (0, 1]$  and  $c \in B_0(\Sigma)$ . Since the map  $a \mapsto I^\circ(u \circ h; a)$  is monotonic and continuous, by adapting the proof of Proposition 4 in GMM one can easily show that  $\geq_h^*$  is monotonic, reflexive, transitive, continuous (if  $a^n \rightarrow a$ ,  $b^n \rightarrow b$  and  $a^n \geq_h^* b^n$ , then  $a \geq_h^* b$ ), and conic (if  $a \geq_h^* b$ , then  $\lambda a + (1 - \lambda)c \geq_h^* \lambda b + (1 - \lambda)c$  for all  $\lambda \in (0, 1]$  and  $c \in B_0(\Sigma)$ ). Finally, let  $\alpha > \beta > 0$  and suppose that  $\geq_h^*$  is trivial. Then, since  $\alpha \geq_h^* \beta$  by monotonicity, we must have  $\beta \geq_h^* \alpha$ , so for all  $\lambda \in (0, 1]$  and  $c \in B_0(\Sigma)$ ,  $I^\circ(u \circ h; \lambda \beta + (1 - \lambda)c) \geq I^\circ(u \circ h; \lambda \alpha + (1 - \lambda)c)$ . Take  $c = 1_S$  and any  $\lambda \in (0, 1]$ ; then  $\lambda \alpha + 1 - \lambda, \lambda \beta + 1 - \lambda > 0$ , so  $I^\circ(u \circ h; \lambda \beta + (1 - \lambda)) = [\lambda \beta + (1 - \lambda)]M$  and  $I^\circ(u \circ h; \lambda \alpha + (1 - \lambda)) = [\lambda \alpha + (1 - \lambda)]M$ , where  $M = \max_{Q \in \partial I(u \circ h)} Q(S) > 0$  because  $\partial I(u \circ h)$  contains positive functionals other than  $Q_0$ . Hence,  $\beta \geq_h^* \alpha$  requires  $\lambda \beta + (1 - \lambda) \geq \lambda \alpha + (1 - \lambda)$ , a contradiction. Thus  $\geq_h^*$  is nontrivial.

Proposition A.2 in GMM yields a unique weak\*-compact, convex set  $C(h) \subset \text{ba}_1(\Sigma)$  such that  $a \geq_h^* b$  iff  $P(a) \geq P(b)$  for all  $P \in C(h)$ . We claim that  $C(h) = \overline{\text{co}}\{\frac{Q}{Q(S)} : Q \in \partial I(u \circ h), Q(S) > 0\} \equiv D(h)$ .

First, we show that  $C(h)$  is the smallest weak\*-compact, convex set  $D \subset \text{ba}_1(\Sigma)$  with the following property, henceforth denoted (P):  $P(a) \geq P(b)$  for all  $P \in D$  implies  $I^\circ(u \circ h; a) \geq I^\circ(u \circ h; b)$ . Clearly,  $C(h)$  satisfies (P), so consider another set  $D$  that also satisfies (P). If  $P(a) \geq P(b)$  for all  $P \in D$ , then, for all  $\lambda \in (0, 1]$  and  $c \in B_0(\Sigma)$ , also  $P(\lambda a + (1 - \lambda)c) \geq P(\lambda b + (1 - \lambda)c)$  for all  $P \in D$ , so by assumption,  $I^\circ(u \circ h; \lambda a + (1 - \lambda)c) \geq I^\circ(u \circ h; \lambda b + (1 - \lambda)c)$ . But this means that  $a \geq_h^* b$ . In other words, the relation  $\geq_D$ , defined by  $a \geq_D b$  iff  $P(a) \geq P(b)$  for all  $P \in D$ , is a subset of  $\geq_h^*$ . By Proposition A.1 in GMM,  $C(h) \subset \overline{\text{co}} D$ , as claimed.

We now show that  $D(h)$  is also the smallest weak\*-compact convex set that satisfies (P), which obviously implies the claim. First, suppose that  $P(a) \geq P(b)$  for all  $P \in D(h)$ , so  $P(b - a) \leq 0$  for all  $P \in D(h)$ . Then also  $Q(b - a) \leq 0$  for all  $Q \in \partial I(u \circ h)$  [this is trivially true for  $Q = Q_0$ , in case  $Q_0 \in \partial I(u \circ h)$ ]. Hence  $I^\circ(u \circ h; b) = I^\circ(u \circ h; a + (b - a)) \leq I^\circ(u \circ h; a) + I^\circ(u \circ h; b - a) \leq I^\circ(u \circ h; a)$ , because  $I^\circ(u \circ h; b - a) = \sup_{Q \in \partial I(u \circ h)} Q(b - a) \leq 0$ . Thus  $D(h)$  satisfies (P).

Let  $D \subset \text{ba}_1(\Sigma)$  be another weak\*-compact, convex set that satisfies (P). Suppose there is  $P \in D(h) \setminus D$ . By the separating hyperplane theorem,<sup>14</sup> there is

<sup>14</sup>For example, Aliprantis and Border (2007, Corollary 5.80 and Theorem 5.93). Note that since the topologies  $\sigma(\text{ba}(\Sigma), B(\Sigma))$  and  $\sigma(\text{ba}(\Sigma), B_0(\Sigma))$  coincide on  $\text{ba}_1(\Sigma)$  (Maccheroni, Marinacci, and Rustichini (2006, Appendix A)), we can restrict attention to  $\sigma(\text{ba}(\Sigma), B_0(\Sigma))$ -continuous linear functionals on  $\text{ba}(\Sigma)$ .

$a \in B_0(\Sigma)$  and  $\alpha \in \mathbb{R}$  such that  $P(a) > \alpha$  and  $P'(a) \leq \alpha$  for all  $P' \in D$ . Letting  $b = a - \alpha$ , we have  $P(b) > 0$  and  $P'(b) \leq 0$  for all  $P' \in D$ . By assumption,  $P'(b) \leq 0$  for all  $P' \in D$  implies  $I^\circ(u \circ h; 0) \geq I^\circ(u \circ h; b)$ ; however,  $P(b) > 0$  implies that there is<sup>15</sup>  $Q \in \partial I(u \circ h)$  with  $Q(S) \neq 0$  and  $Q(b) > 0$ , so  $I^\circ(u \circ h; b) = \sup_{Q' \in \partial I(u \circ h)} Q'(b) \geq Q(b) > 0 = \sup_{Q' \in \partial I(u \circ h)} 0 \cdot Q'(S) = I^\circ(u \circ h; 0)$ —a contradiction. Thus,  $D(h) \subset D$ .

To complete the proof, assume that  $f \succ_h^* g$ . Then, by Lemma 10,  $u \circ f \geq_h^* u \circ g$ ; hence  $P(u \circ f) \geq P(u \circ g)$  for all  $P \in C(h) = D(h)$ .

(ii) Suppose that  $\succ_D$  is another Bewley refinement of  $\succ_h^*$ ; recall that, by definition,  $\succ_D$  on  $X$  is represented by  $u$ , but by Corollary 11, this must be true for any Bewley refinement of  $\succ_h^*$ , because  $I$  is nice at  $u \circ h$  and so a fortiori  $\partial I(u \circ h) \neq \{Q_0\}$ . Fix an interior act  $f \in \mathcal{F}$  and let  $x$  be such that  $u(x) = \max_{P \in C(h)} P(u \circ f)$ . We will show that  $u(x) \geq \max_{P \in D} P(u \circ f)$ .

Clearly,  $P(u(x)) \geq P(u \circ f)$  for all  $P \in C(h)$ . Since  $f$  is interior and, by monotonicity, so is  $x$ , there are  $\varepsilon > 0$ ,  $y \in X$ , and  $g \in \mathcal{F}$  with  $u(y) = u(x) + \varepsilon$  and  $u \circ g = u \circ f - \varepsilon$ . Indeed, for all  $\delta \in (0, \varepsilon)$ , there are  $y_\delta$  and  $g_\delta$  such that  $u(y_\delta) = u(x) + \delta$  and  $u \circ g_\delta = u \circ f - \delta$ . For each such  $\delta \in (0, \varepsilon)$ ,  $(y_\delta, g_\delta)$  is a spread of  $(x, f)$ . By Lemma 12, it must then be the case that  $y_\delta \succ_h^* g_\delta$ .

Since  $\succ_D$  extends  $\succ_h^*$ , conclude that  $y_\delta \succ_D g_\delta$  for all  $\delta \in (0, \varepsilon)$ ; hence,  $u(y_\delta) = P(u(y_\delta)) \geq P(u \circ g_\delta)$  for all  $P \in D$ . Therefore,  $u(x) + \delta \geq P(u \circ f) - \delta$  for all  $P \in D$  and all  $\delta \in (0, \varepsilon)$ . It follows that  $u(x) \geq P(u \circ f)$  for all  $P \in D$ , that is,  $u(x) \geq \max_{P \in D} P(u \circ f)$ , as claimed.

To sum up, for all interior  $f \in \mathcal{F}$ ,  $\max_{P \in C(h)} P(u \circ f) \geq \max_{P \in D} P(u \circ f)$ . Since any  $a \in B_0(\Sigma)$  can be written as  $\alpha u \circ f + \beta$  for some  $f \in \mathcal{F}^{\text{int}}$  and  $\alpha, \beta \in \mathbb{R}$ , then  $\max_{P \in C(h)} P(a) \geq \max_{P \in D} P(a)$  for all  $a \in B_0(\Sigma)$ . By standard results (e.g., Aliprantis and Border (2007, Theorem 7.51)), this implies that  $D \subset C(h)$ , that is,  $\succ_D$  is a richer Bewley relation than  $\succ_h^*$ . Q.E.D.

**PROOF OF THEOREM 7:** Fix an interior pair  $(f, g)$ . Assume that (i) holds and fix  $\varepsilon > 0$  such that  $u \circ f + \varepsilon, u \circ g - \varepsilon \in B_0(\Sigma, u(X))$ . Then, for all  $\delta \in (0, \varepsilon)$ , there exist  $f_\delta, g_\delta \in \mathcal{F}$  with  $u \circ f_\delta = u \circ f + \delta$  and  $u \circ g_\delta = u \circ g - \delta$ ; note that  $(f_\delta, g_\delta)$  is a spread of  $(f, g)$ . Then  $f_\delta \succ_h^* g_\delta$ , so by (i) in Theorem 6,  $P(u \circ f) + \delta = P(u \circ f_\delta) \geq P(u \circ g_\delta) = P(u \circ g) - \delta$  for all  $P \in C(h)$  and all  $\delta \in (0, \varepsilon)$ . Therefore, (ii) with  $D = C(h)$  follows.

The converse, again with  $D = C(h)$ , is established in Lemma 12. Finally, suppose there is another set  $D$  for which (i) and (ii) are equivalent (again using utility  $u$  in view of Corollary 11). Consider a pair  $(f, g)$  of interior acts. Suppose that  $f \succ_{C(h)} g$ : then (ii) holds for set  $C(h)$ , hence (i) must hold. But by assumption, this implies that (ii) must hold for set  $D$  as well: therefore,  $f \succ_D g$ . Since

<sup>15</sup>This is immediate if  $P = \frac{Q}{Q(S)}$  for some  $Q \in \partial I(u \circ h)$ . If not, there is a net  $(P_i)$  in  $\text{co}\{\frac{Q}{Q(S)} : Q \in \partial I(u \circ h), Q(S) > 0\}$  that converges to  $P$  in the weak\* topology. Then  $P_i(b) \rightarrow P(b)$ , so there is  $\bar{i}$  such that  $P_i(b) > 0$  for all  $i$  following  $\bar{i}$ . Since any such  $P_i$  is a convex combination of elements of  $\{\frac{Q}{Q(S)} : Q \in \partial I(u \circ h), Q(S) > 0\}$ , the claim follows.

Bewley preferences satisfy independence,  $\succ_{C(h)} \subset \succ_D$ . By the same argument,  $\succ_D \subset \succ_{C(h)}$ . It follows that  $D = C(h)$ . Q.E.D.

A.2. Proof of Theorem 2 and Corollary 8

We must show that  $C$  is the closed convex hull of all  $C(h)$  for  $h \in \mathcal{F}^{\text{int}}$ .

CLAIM: For all  $f, g \in \mathcal{F}$ ,  $f \succ_{C(h)} g$  for all  $h \in \mathcal{F}^{\text{int}}$  implies  $f \succ g$ .

PROOF: Assume first that  $f$  and  $g$  are interior. By Lebourg’s mean value theorem (Lebourg (1979, Theorem 1.7)), there are  $\mu \in (0, 1)$  and  $Q \in \partial I(\mu u \circ f + (1 - \mu)u \circ g)$  such that  $I(u \circ f) - I(u \circ g) = Q(u \circ f) - Q(u \circ g)$ . Since  $\mu f + (1 - \mu)g$  is interior, the assumption that  $P(u \circ f) \geq P(u \circ g)$  for all  $P \in C(\mu f + (1 - \mu)g)$  implies that  $Q(u \circ f) \geq Q(u \circ g)$  [if  $Q = Q_0$ , this is trivially true]. Hence,  $I(u \circ f) \geq I(u \circ g)$ , that is,  $f \succ g$ , as claimed. If now  $f$  and  $g$  are not interior, pick  $x$  interior and consider  $\lambda x + (1 - \lambda)f, \lambda x + (1 - \lambda)g$ . If  $P(u \circ f) \geq P(u \circ g)$  for all interior  $h$  and all  $P \in C(h)$ , then also  $P(\lambda u(x) + (1 - \lambda)u \circ f) \geq P(\lambda u(x) + (1 - \lambda)u \circ g)$  for all such  $h$  and  $P$ . As was just shown, this implies  $\lambda x + (1 - \lambda)f \succ \lambda x + (1 - \lambda)g$ . Since this holds for all  $\lambda$ , continuity yields  $f \succ g$ , as required. Q.E.D.

By Lemma 9, the Claim implies that  $C \subset \overline{\text{co}} \bigcup_{h \in \mathcal{F}^{\text{int}}} C(h)$ . Conversely, suppose  $f \succ_C g$ . Then  $f \succ^* g$ ; in particular, for every  $h \in \mathcal{F}^{\text{int}}$ ,  $f \succ_h^* g$ . But then, part (i) of Theorem 6 shows that  $f \succ_{C(h)} g$ . Applying Proposition A.1 in GMM to the Bewley preference  $\succ_{C(h)}$  now implies that  $C(h) \subset C$ .

Note that the above also shows that  $f \succ_{C(h)} g$  for all interior  $h$  if and only if  $f \succ^* g$ . Since  $f \succ^* g$  directly and trivially implies that  $f \succ_h^* g$ , and part (i) of Theorem 6 shows that  $f \succ_h^* g$  implies  $f \succ_{C(h)} g$ , we can also conclude that  $f \succ^* g$  if and only if  $f \succ_h^* g$  for all interior  $h$ .

REFERENCES

ALIPRANTIS, C., AND K. BORDER (2007): *Infinite Dimensional Analysis*. Berlin: Springer-Verlag. [2843,2844]  
 BAILLON, A., O. L’HARIDON, AND L. PLACIDO (2012): “Ambiguity Models and the Machina Paradoxes,” *American Economic Review*, 101 (4), 1547–1560. [2828]  
 BEWLEY, T. (2002): “Knightian Decision Theory: Part I,” *Decisions in Economics and Finance*, 25 (2), 79–110. [2828]  
 CERREIA-VIOGLIO, S., P. GHIRARDATO, F. MACCHERONI, M. MARINACCI, AND M. SINISCALCHI (2011): “Rational Preferences Under Ambiguity,” *Economic Theory*, 48, 341–375. [2834]  
 CERREIA-VIOGLIO, S., F. MACCHERONI, M. MARINACCI, AND L. MONTRUCCHIO (2011): “Uncertainty Averse Preferences,” *Journal of Economic Theory*, 146 (4), 1275–1330. [2828,2837]  
 CHAMBERS, R. G., S. GRANT, B. POLAK, AND J. QUIGGIN (2011): “A Two-Parameter Model of Dispersion Aversion,” RSGM Working Paper Series, School of Economics and Political Science, University of Queensland. [2828,2834]  
 CHATEAUNEUF, A., AND J. H. FARO (2009): “Ambiguity Through Confidence Functions,” *Journal of Mathematical Economics*, 45 (9–10), 535–558. [2828,2834]

- CLARKE, F. H. (1983): *Optimization and Nonsmooth Analysis*. New York: Wiley. [2828,2830,2831, 2834-2836]
- DOW, J., AND S. R. DA COSTA WERLANG (1992): "Uncertainty Aversion, Risk Aversion, and the Optimal Choice of Portfolio," *Econometrica*, 60 (1), 197–204. [2831]
- EPSTEIN, L. G. (1999): "A Definition of Uncertainty Aversion," *Review of Economic Studies*, 66, 579–608. [2828]
- GHIRARDATO, P., AND M. MARINACCI (2002): "Ambiguity Made Precise: A Comparative Foundation," *Journal of Economic Theory*, 102, 251–289. [2828]
- GHIRARDATO, P., AND M. SINISCALCHI (2010): "A More Robust Definition of Multiple Priors," Working Paper, Collegio Carlo Alberto and Economics Department, Northwestern University. [2827,2840]
- (2012): "Supplement to 'Ambiguity in the Small and in the Large'," *Econometrica Supplemental Material*, 80, [http://www.econometricsociety.org/ecta/Supmat/9367\\_proofs.pdf](http://www.econometricsociety.org/ecta/Supmat/9367_proofs.pdf). [2834]
- GHIRARDATO, P., F. MACCHERONI, AND M. MARINACCI (2004): "Differentiating Ambiguity and Ambiguity Attitude," *Journal of Economic Theory*, 118, 133–173. [2827]
- GILBOA, I., AND D. SCHMEIDLER (1989): "Maxmin Expected Utility With a Non-Unique Prior," *Journal of Mathematical Economics*, 18, 141–153. [2827,2828,2840]
- GILBOA, I., F. MACCHERONI, M. MARINACCI, AND D. SCHMEIDLER (2010): "Objective and Subjective Rationality in a Multiple Prior Model," *Econometrica*, 78, 755–770. [2832]
- GRANT, S., AND B. POLAK (2011): "Mean-Dispersion Preferences and Absolute Uncertainty Aversion," Technical Report 1805, Cowles Foundation. [2828,2834]
- HANSEN, L. P., AND T. J. SARGENT (2001): "Robust Control and Model Uncertainty," *American Economic Review*, 91, 60–66. [2827]
- HEATH, C., AND A. TVERSKY (1991): "Preference Belief: Ambiguity and Competence in Choice Under Uncertainty," *Journal of Risk and Uncertainty*, 4, 5–28. [2831]
- KLIBANOFF, P., M. MARINACCI, AND S. MUKERJI (2005): "A Smooth Model of Decision Making Under Ambiguity," *Econometrica*, 73, 1849–1892. [2827]
- KLIBANOFF, P., S. MUKERJI, AND K. SEO (2011): "Relevance and Symmetry," Discussion Paper 539, Oxford University. [2832]
- LEBOURG, G. (1979): "Generic Differentiability of Lipschitzian Functions," *Transactions of the American Mathematical Society*, 256, 125–144. [2835,2845]
- MACCHERONI, F., M. MARINACCI, AND A. RUSTICHINI (2006): "Ambiguity Aversion, Robustness, and the Variational Representation of Preferences," *Econometrica*, 74, 1447–1498. [2828,2837, 2840,2843]
- MACHINA, M. J. (1982): "'Expected Utility' Analysis Without the Independence Axiom," *Econometrica*, 50, 277–323. [2833]
- (2005): "'Expected Utility/Subjective Probability' Analysis Without the Sure-Thing Principle or Probabilistic Sophistication," *Economic Theory*, 26 (1), 1–62. [2833]
- NEHRING, K. (2002): "Imprecise Probabilistic Beliefs as a Context for Decision-Making Under Ambiguity," Report, Economics Department, UC Davis. [2832]
- NGAI, H. V., D. T. LUC, AND M. THÉRA (2000): "Approximate Convex Functions," *Journal of Nonlinear and Convex Analysis*, 1 (2), 155–176. [2840]
- RIGOTTI, L., C. SHANNON, AND T. STRZALECKI (2008): "Subjective Beliefs and ex ante Trade," *Econometrica*, 76 (5), 1167–1190. [2832,2833]
- ROBERTS, A. W., AND D. E. VARBERG (1974): "Another Proof That Convex Functions Are Locally Lipschitz," *The American Mathematical Monthly*, 81 (9), 1014–1016. [2834]
- ROCKAFELLAR, R. T. (1980): "Generalized Directional Derivatives and Subgradients of Nonconvex Functions," *Canadian Journal of Mathematics*, 32, 257–280. [2840]
- SCHMEIDLER, D. (1989): "Subjective Probability and Expected Utility Without Additivity," *Econometrica*, 57, 571–587. [2828,2831]
- SINISCALCHI, M. (2006): "A Behavioral Characterization of Plausible Priors," *Journal of Economic Theory*, 128, 91–135. [2827,2832]

——— (2009): “Vector Expected Utility and Attitudes Toward Variation,” *Econometrica*, 77, 801–855. [2828]

*Collegio Carlo Alberto, Via Real Collegio 30, 10024 Moncalieri (TO), Italy;*  
*paolo.ghirardato@carloalberto.org*

*and*

*Economics Dept., Northwestern University, 2001 Sheridan Rd., Evanston, IL*  
*60208, U.S.A.; marciano@northwestern.edu.*

*Manuscript received June, 2010; final revision received March, 2012.*