Stationary states for a two-dimensional singular Schrödinger equation

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Abstract

In questo articolo studiamo problemi di Dirichlet singolari, lineari e semilineari, della forma $-|x|^2\Delta u = f(u)$ in Ω, $u = 0$ su $\partial\Omega$, dove Ω è un dominio in $\mathbf{R}^2$ e $f(u) = \lambda u$ o $f(u) = \lambda u + |u|^{p-2}u$ con $p > 2$ (o nonlinearità più generali). In tali problemi bidimensionali emergono alcune difficoltà a causa della non validità della disuguaglianza di Hardy in $\mathbf{R}^2$ e a causa delle invarianze dell'equazione $-|x|^2\Delta u = f(u)$. Pertanto opportune condizioni su λ e Ω sono necessarie al fine di garantire l'esistenza di una soluzione positiva. Per esempio, se Γ_0 è una curva non costante passante per l'origine e Γ_∞ è una curva non limitata, allora la disuguaglianza di Hardy vale su qualunque dominio Ω contenuto in $\mathbf{R}^2 \setminus (\Gamma_0 \cup \Gamma_\infty)$ e si possono ottenere alcuni risultati di esistenza.

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Introduction

In this paper we deal with the stationary singular Schrödinger equation in the limit case of "strong force" (see [9])

$$-\Delta u = \lambda \frac{u}{|x|^2} \quad \text{in } \Omega \tag{0.1}$$

where Ω is a domain in $\mathbf{R}^2$ and $\lambda \in \mathbf{R}$. We will also consider the nonlinear version

$$-\Delta u = \frac{\lambda u + g(u)}{|x|^2} \quad \text{in } \Omega \tag{0.2}$$

where $g \in C(\mathbf{R})$ is a superlinear function, e.g., $g(u) = |u|^{p-2}u$ with $p > 2$ (see Section 3 for the precise assumptions on g).

We focus our attention on two-dimensional domains since, as discussed in [8], this case exhibits some special features that are not shared with any other dimension. Let us note that some results concerning equation (0.2) and the corresponding Dirichlet problem were already proved in [8], while for variational, singular elliptic problems in dimensions different from two we refer, e.g., to the papers [10], [2], [3], [12], [13], [5] and [7].

The most striking phenomena related to equations (0.1) and (0.2), that appear *only in dimension two*, are:

- Failure of the Hardy-Sobolev inequality in $\mathbf{R}^2$ (see, e.g., [4]), in any domain containing the origin and in any exterior domain, that is, a domain with compact complement. This means that if Ω is a domain containing 0 or if Ω is an exterior domain, then the value

$$S_p(\Omega) = \inf \Big\{ \int_\Omega |\nabla u|^2 : u \in C_c^\infty(\Omega \setminus \{0\}) \ , \ \int_\Omega \frac{|u|^p}{|x|^2} = 1 \Big\} \tag{0.3}$$

  turns out to be zero for every $p \geq 2$.

- Invariance of the equation $-\Delta u = \frac{f(u)}{|x|^2}$ with respect to dilations $x \mapsto rx$ ($r > 0$) and with respect to the Kelvin transform $x \mapsto \frac{x}{|x|^2}$, whatever $f \in C(\mathbf{R})$ is.

A consequence of these facts is that when Ω is a domain containing the origin or when Ω is an exterior domain, for $\lambda > 0$ equation (0.1) admits no positive (super)solution, even in a very weak sense (precisely, in the sense of distributions in $\Omega \setminus \{0\}$, see Proposition 2.1).

These remarks lead us to start by examining the class of those two-dimensional domains on which the Hardy inequality holds, i.e., such that

$S_2(\Omega) > 0$, (that we call *Hardy-admissible domains*), and then by studying the corresponding linear problem.

For instance, as already noted in [8], any proper cone in $\mathbf{R}^2$ turns out to be a Hardy admissible domain. In fact, the property of Hardy-admissibility depends just on the shape of the domain at the origin and at infinity. More precisely, we can show that the Hardy inequality holds true in any domain Ω contained in $\mathbf{R}^2 \setminus (\Gamma_0 \cup \Gamma_\infty)$ where Γ_0 is a (non constant) curve such that $0 \in \Gamma_0$ and Γ_∞ is an unbounded curve. Actually, more general situations can be considered, see condition (H) stated in Theorem 1.4.

Whenever the domain Ω is Hardy-admissible, as it happens in the above described situations, then one can study the minimization problem corresponding to the definition of $S_2(\Omega)$. However, the invariance properties noted at the beginning reflect on phenomena of concentration at 0 or vanishing, and then on a possible lack of compactness of the minimizing sequences for $S_2(\Omega)$. This lack of compactness depends again on the shape of Ω near 0 and at infinity.

In Theorem 2.3 we state a sufficient condition for compactness and hence for existence of a positive eigenfunction for the problem

$$\begin{cases} -\Delta u = \lambda \frac{u}{|x|^2} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{0.4}$$

on a Hardy-admissible domain. This condition fits into the spirit of the concentration-compactness principle [11].

As a particular case we can prove that the eigenvalue problem (0.4) admits a positive solution when $\Omega = \mathbf{R}^2 \setminus \Gamma_{a,b}$ and $\Gamma_{a,b} = \{x \in \mathbf{R}^2 : 0 \leq x_1 \leq a,\ x_1 \geq b\}$, provided that $\frac{a}{b} > 0$ is small enough.

In the second part of the paper we deal with the semilinear Dirichlet problem

$$\begin{cases} -\Delta u = \frac{\lambda u + g(u)}{|x|^2} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\ . \end{cases} \tag{0.5}$$

As in the linear case, some non existence results are known when the domain Ω contains the origin or it is an exterior domain. For example, as proved in [3] and [8], because of the quadratic behaviour of the singularity at the origin, for $p > 2$ and $0 \in \Omega$ equation

$$-\Delta u = \frac{|u|^{p-2}u}{|x|^2} \quad \text{in } \Omega$$

has no positive solution, even in a very weak sense and also in higher dimensions. In addition, by the Kelvin invariance, when Ω is an exterior domain in $\mathbf{R}^2$ the same non existence result holds (this occurs only in dimension two). In these cases there is no variational setting suited to study the corresponding Dirichlet's problem, because there is an intrinsic lack of topology.

Therefore we restrict ourselves to study problem (0.5) assuming that the domain Ω is Hardy-admissible. In this case, $S_p(\Omega) > 0$ for every $p > 2$ and, for a nonlinearity $g \in C^1(\mathbf{R})$ of the form $g(u) = |u|^{p-2}u$ with $p > 2$, we can introduce a nice variational setting suited to study problem (0.5). Moreover, for $\lambda < S_2(\Omega)$ the variational functional associated to (0.5) turns out to have a mountain pass structure.

As for the eigenvalue problem, a lack of compactness may occur, because of the dilation and Kelvin invariances. Nevertheless, we can state a criterion ensuring that problem (0.5) admits a positive solution (Theorem 3.1). Let us note that a similar argument to recover some compactness was already used in [6] for a critical degenerate elliptic problem (see also [5]).

Finally, in Section 4 we illustrate some existence examples for problem (0.5).

1 On the Hardy-Sobolev inequality in two dimensions

Aim of this Section is to introduce a variational setting suited to study problems (0.4) and (0.5). As already observed in the Introduction, some difficulties arise because of the dimension two of the domain. Indeed, let us remark that in the N-dimensional case with $N \geq 3$ the Sobolev inequality holds true and one can take advantage from that in order to define the standard Sobolev space $D^1_0(\Omega)$ as the completion of $C^\infty_c(\Omega)$ with respect to the Dirichlet norm. It is known that $D^1_0(\Omega)$ turns out to be a Hilbert space endowed with the norm $\|\nabla u\|_{L^2}$. Moreover, for every $\alpha \in [0, 2]$ the space $D^1_0(\Omega)$ is continuously embedded into the weighted Lebesgue space $L^{p_\alpha}(\Omega, \frac{dx}{|x|^\alpha})$, where $p_\alpha = \frac{2(N-\alpha)}{N-2}$ (see [4]). In particular, for $\alpha = 0$ one recovers the Sobolev embedding ($p_0 = \frac{2N}{N-2}$), while for $\alpha = 2$ one gets the Hardy inequality ($p_2 = 2$).

All the above statements cannot be taken for granted at all when Ω is an arbitrary domain in $\mathbf{R}^2$. More precisely the lack of a Sobolev embedding in dimension two is an obstruction in order to define in a similar way the space $D^1_0(\Omega)$. Indeed, in general if Ω is an unbounded domain in $\mathbf{R}^2$, the completion of $C^\infty_c(\Omega)$ with respect to the Dirichlet norm is not contained in any space $L^p(\Omega)$ for $p \in [1, +\infty]$. In addition, the Hardy inequality fails in dimension two and the values $S_p(\Omega)$ defined in (0.3) turn out to be 0 for every $p \geq 2$ if $0 \in \Omega$ or if Ω is an exterior domain.

Our goal is to introduce a class of domains in $\mathbf{R}^2$ for which $S_p(\Omega) > 0$ for every $p \geq 2$. This will allow us to state a well-posed definition of the space $D_0^1(\Omega)$ as in the higher dimensional case.

Definition 1.1 We say that a domain Ω in $\mathbf{R}^2$ is *Hardy-admissible* if the Hardy inequality holds in Ω, that is, there exists $C > 0$ such that for every $u \in C_c^\infty(\Omega)$

$$\int_\Omega \frac{u^2}{|x|^2} \leq C \int_\Omega |\nabla u|^2 .$$

In this case we set

$$\lambda_1^H(\Omega) = \inf \Big\{ \int_\Omega |\nabla u|^2 : u \in C_c^\infty(\Omega) \ , \ \int_\Omega \frac{u^2}{|x|^2} = 1 \Big\} .$$

Remark 1.2 Note that if $0 \in \Omega$ then Ω is not Hardy-admissible. Moreover, by the invariance with respect to the Kelvin transform, also in case Ω is an exterior domain, Ω cannot be Hardy-admissible.

Lemma 1.3 *If a domain Ω in $\mathbf{R}^2$ is Hardy-admissible, then the completion of $C_c^\infty(\Omega)$ with respect to the Dirichlet norm, denoted $D_0^1(\Omega)$, is a Hilbert space endowed with the norm $\|u\|_{D_0^1} = \|\nabla u\|_{L^2}$. Moreover $D_0^1(\Omega)$ is continuously embedded into $L^p(\Omega; \frac{dx}{|x|^2})$ for every $p \in [2, +\infty)$.*

Proof. The fact that $D_0^1(\Omega)$ is a Hilbert space endowed with the Dirichlet norm immediately follows by the assumption that the Hardy inequality holds in Ω. In order to prove that $D_0^1(\Omega)$ is continuously embedded into $L^p(\Omega; \frac{dx}{|x|^2})$ for every $p \in [2, +\infty)$, first of all, we point out that $\Omega \subset \mathbf{R}^2 \setminus \{0\}$ (Remark 1.2). Setting $\phi(s, \theta) = (e^s \cos\theta, e^s \sin\theta)$, we have that ϕ is a diffeomorphism between $\mathbf{R} \times S^1$ and $\mathbf{R}^2 \setminus \{0\}$. Moreover, if $u \in D_0^1(\Omega)$ and $v = u \circ \phi = \Phi(u)$, then $v \in H^1_{loc}(\mathbf{R} \times S^1)$, $|\nabla u|^2 = e^{-2s}|\nabla v|^2$ a.e., and

$$\int_\Omega |\nabla u|^2 = \int_\Sigma |\nabla v|^2 \ , \quad \int_\Omega \frac{u^2}{|x|^2} = \int_\Sigma v^2 \ , \tag{1.1}$$

where $\Sigma = \phi^{-1}(\Omega) \cap (\mathbf{R} \times [0, 2\pi))$. Hence $D_\phi^1(\Sigma) =: \{(u \circ \phi)|_\Sigma : u \in D_0^1(\Omega)\}$ is a Hilbert space endowed with the Dirichlet norm and is isomorphic to $D_0^1(\Omega)$, through the mapping Φ. In addition, since Ω is Hardy-admissible and (1.1) holds true, the Poincaré inequality holds in $D_\phi^1(\Sigma)$ and then $D_\phi^1(\Sigma)$ is a subspace of $H^1(\Sigma)$. In particular, by the classical Sobolev embedding

Theorem, $D^1_\phi(\Sigma)$ turns out to be continuously embedded into the spaces $L^p(\Sigma)$ for all $p \in [2, +\infty)$. Since for every $u \in C^\infty_c(\Omega)$

$$\int_\Sigma |u \circ \phi|^p = \int_\Omega \frac{|u|^p}{|x|^2}$$

using the isomorphism Φ, we conclude that $D^1_0(\Omega)$ is continuously embedded into $L^p(\Omega; \frac{dx}{|x|^2})$ for every $p \in [2, +\infty)$. $\square$

As already noted, in order that a domain in $\mathbf{R}^2$ is Hardy-admissible, it cannot contain the origin and it cannot be an exterior domain. In the next Theorem we introduce a quite general condition on the domain ensuring that it is Hardy-admissible.

Theorem 1.4 *If Ω is a domain in $\mathbf{R}^2$ satisfying the condition:*

(H) *there exists a finite or countable family of connected sets $\Gamma_n \subset \mathbf{R}^2$ of positive capacity such that:*

- *(i)* $\mathbf{R}^2 \setminus \Omega \supset \Gamma$, *where* $\Gamma = \bigcup_n \Gamma_n$,
- *(ii)* Γ *is unbounded and the origin is an accumulation point for* Γ,
- *(iii)* $\sup_n \rho(\Gamma_n, \Gamma_{n+1}) < +\infty$ *where* $\rho(\Gamma_n, \Gamma_{n+1}) = \inf\{\left|\log \frac{|x|}{|y|}\right| : x \in \Gamma_n \setminus \{0\},\ y \in \Gamma_{n+1} \setminus \{0\}\}$,

then Ω is Hardy-admissible.

Remark 1.5 According to Theorem 1.4, the Hardy inequality, that fails in $\mathbf{R}^2$, in fact holds in any domain Ω contained in $\mathbf{R}^2 \setminus (\Gamma_0 \cup \Gamma_\infty)$ where Γ_0 is a non constant curve passing through the origin and Γ_∞ is an unbounded curve. Clearly this includes every proper cone in $\mathbf{R}^2$ with vertex at the origin.

Proof. Let ϕ be the diffeomorphism between $\mathbf{R} \times S^1$ and $\mathbf{R}^2 \setminus \{0\}$ introduced in the proof of Lemma 1.3, and let $\Sigma = \phi^{-1}(\Omega)$. By (refE:change-of-variable), Ω is Hardy-admissible if and only if the Poincaré inequality is satified in Σ with respect to the class of functions $C^\infty_c(\Sigma) = \{v \in C^\infty(\mathbf{R} \times S^1) : \text{supp } v \subset \Sigma\}$. Under the diffeomorphism ϕ, condition (H) is equivalent to:

(H)$'$ there exists a finite or countable family of connected sets $F_n \subset \mathbf{R}^2 \times S^1$ of positive capacity such that:

- $(i)'$ $(\mathbf{R} \times S^1) \setminus \Sigma \supset F$, where $F =: \bigcup_n F_n$,

$(ii)'$ $\inf p_1(F) = -\infty$, $\sup p_1(F) = +\infty$, where $p_1 \colon \mathbf{R} \times S^1 \to \mathbf{R}$ is the projection with respect to the first component,

$(iii)'$ $\sup_n d_1(F_n, F_{n+1}) < +\infty$, where $d_1(F_n, F_{n+1}) = \inf\{|s - s'| : (s,\theta) \in F_n,\ (s',\theta') \in F_{n+1}\}$.

Hence, if $(\mathrm{H})'$ is fulfilled, taking $\bar{s} > \sup_n d_1(F_n, F_{n+1})$ and $\Sigma_j = ((j\bar{s}, (j+1)\bar{s}) \times S^1) \cap \Sigma$, then for every $j \in \mathbf{Z}$ there exists n_j such that $F_{n_j} \cap \Sigma_j$ is a set of positive capacity. Therefore, we have that

$$\int_{\Sigma_j} v^2 \le C \int_{\Sigma_j} |\nabla v|^2 \quad \text{for any } v \in C_c^\infty(\Sigma)\ ,$$

with $C > 0$ independent of v and j (C depends just on the diameter of Σ_j). Adding on j we find

$$\int_{\Sigma} v^2 \le C \int_{\Sigma} |\nabla v|^2 \quad \text{for any } v \in C_c^\infty(\Sigma)\ ,$$

that is, the Poincaré inequality on Σ holds. □

2 The linear Dirichlet problem

In this Section we study the existence of positive solutions for the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda \frac{u}{|x|^2} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

with $\lambda > 0$.

First of all, we state a non existence result, concerning the case of domains containing the origin and exterior domains.

Proposition 2.1 *If Ω is a domain in $\mathbf{R}^2$ containing 0, or if Ω is an exterior domain in $\mathbf{R}^2$, then for every $\lambda > 0$ the equation*

$$-\Delta u = \lambda \frac{u}{|x|^2}$$

admits no non negative supersolution in the sense of distributions in $\Omega \setminus \{0\}$, namely there is no function $u \in L^1_{loc}(\Omega \setminus \{0\})$, $u \ge 0$, $u \not\equiv 0$ such that $-\int_\Omega u\Delta\varphi \ge \lambda \int_\Omega \frac{u\varphi}{|x|^2}$ for every $\varphi \in C_c^\infty(\Omega \setminus \{0\})$, $\varphi \ge 0$.

Remark 2.2 The non existence result stated in Proposition 2.1 holds only in dimension two. Indeed, in the N-dimensional case for every $\lambda \le (\frac{N}{2} - 1)^2$ the equation $-|x|^2\Delta u = \lambda u$ admits (very weak) positive solutions on $\mathbf{R}^N \setminus \{0\}$ of the form $u(x) = |x|^{-\beta}$ for a suitable choice of $\beta = \beta(N, \lambda)$.

Proof. By the invariances of (0.1) with respect to dilations and to the Kelvin transform, we may assume that $\overline{B} \subset \Omega$ where $B = \{x \in \mathbf{R}^2 : |x| < 1\}$. Suppose that there exists a positive supersolution to $-\Delta u = \lambda \frac{u}{|x|^2}$ in the sense of distributions in $\Omega \backslash \{0\}$. Then, by standard regularization arguments, $-\int_\Omega u\Delta\varphi \geq \lambda \int_\Omega \frac{u\varphi}{|x|^2}$ for every $\varphi \in C^2_c(\Omega \setminus \{0\})$, $\varphi \geq 0$. For every $n \in \mathbf{N}$ let $\psi_n(r) = \pi(\frac{\log r}{\log n} + 1)$ $(r > 0)$ and

$$\varphi_n(x) = \begin{cases} (\sin \psi_n(|x|))^3 & \text{as } \frac{1}{n} < |x| < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Then $\varphi_n \in C^2_c(\Omega \setminus \{0\})$, $\varphi_n \geq 0$ and, after calculations,

$$\lambda \int_\Omega \frac{u\varphi_n}{|x|^2} \leq -\int_\Omega u\Delta\varphi_n \leq \frac{3\pi}{\log n} \int_\Omega \frac{u\varphi_n}{|x|^2} \ .$$

Since $u \geq 0$ and $u \not\equiv 0$ on any neighborhood of 0 (otherwise $u \equiv 0$ on Ω), $\int_\Omega \frac{u\varphi_n}{|x|^2} > 0$ for every $n \in \mathbf{N}$. Then $\lambda \leq \frac{3\pi}{\log n}$ for every $n \in \mathbf{N}$, namely $\lambda \leq 0$. This concludes the proof. $\square$

The above non existence result leads us to study the eigenvalue problem (2.1) when the domain Ω is Hardy-admissible. In this case the space $D^1_0(\Omega)$ is well defined (see Lemma 1.3) and we can look for positive solutions to (2.1) as extremal functions corresponding to the continuous embedding of $D^1_0(\Omega)$ into $L^2(\Omega; \frac{dx}{|x|^2})$, namely as minimizers for the problem

$$\lambda^H_1(\Omega) = \inf\Big\{ \int_\Omega |\nabla u|^2 : u \in D^1_0(\Omega) \ , \ \int_\Omega \frac{u^2}{|x|^2} = 1 \Big\} \ . \tag{2.2}$$

Our first goal is to state a criterion ensuring that $\lambda^H_1(\Omega)$ is attained in $D^1_0(\Omega)$. To this aim, let us introduce the values:

$$\lambda^H_{1,0}(\Omega) = \sup_{r>0} \lambda^H_1(\Omega \cap B_r) \ , \quad \lambda^H_{1,\infty}(\Omega) = \sup_{r>0} \lambda^H_1(\Omega \setminus \overline{B}_r) \ ,$$

with the agreement that $\lambda^H_1(U) = \infty$ if $U = \emptyset$. Note that, since Ω is Hardy-admissible, also $\Omega \cap B_r$ and $\Omega \setminus \overline{B}_r$ are (when they are non empty), and $\lambda^H_1(\Omega \cap B_r) > 0$ and $\lambda^H_1(\Omega \setminus \overline{B}_r) > 0$ for every $r > 0$. Moreover the mappings $r \mapsto \lambda^H_1(\Omega \cap B_r)$ and $r \mapsto \lambda^H_1(\Omega \setminus \overline{B}_r)$ are respectively non increasing and non decreasing, and $\lambda^H_1(\Omega) \leq \min\{\lambda^H_{1,0}(\Omega), \lambda^H_{1,\infty}(\Omega)\}$. The following criterion holds.

Theorem 2.3 *Let Ω be a Hardy-admissible domain in $\mathbf{R}^2$. If*

$$\lambda^H_1(\Omega) < \min\{\lambda^H_{1,0}(\Omega), \lambda^H_{1,\infty}(\Omega)\} \tag{2.3}$$

then $\lambda^H_1(\Omega)$ is attained in $D^1_0(\Omega)$.

Proof. For every $n \in \mathbf{N}$ let $\Omega_n = \{x \in \Omega : \frac{1}{n} < |x| < n\}$. For $n \in \mathbf{N}$ large enough Ω_n is non empty and by standard arguments there exists $u_n \in H_0^1(\Omega_n)$ such that

$$\int_{\Omega_n} \frac{u_n^2}{|x|^2} = 1 \ , \quad \int_{\Omega_n} |\nabla u_n|^2 = \lambda_1^H(\Omega_n) \ ,$$

and u_n is a positive solution to

$$\begin{cases} -\Delta u = \lambda \frac{u}{|x|^2} & \text{in } \Omega_n \\ u = 0 & \text{on } \partial\Omega_n \end{cases} \tag{2.4}$$

with $\lambda = \lambda_1^H(\Omega_n)$. Moreover one can easily verify that $\lambda_1^H(\Omega_n) \geq \lambda_1^H(\Omega_{n+1})$ for all $n \in \mathbf{N}$ and $\lambda_1^H(\Omega_n) \to \lambda_1^H(\Omega)$ as $n \to \infty$. Hence the sequence (u_n) is bounded in $D_0^1(\Omega)$ and, for a subsequence, converges to some $u \in D_0^1(\Omega)$ weakly in $D_0^1(\Omega)$ and in $L^2(\Omega; \frac{dx}{|x|^2})$, and pointwise a.e. in Ω. Then $u \geq 0$ and, by weakly lower semicontinuity, $\int_\Omega \frac{u^2}{|x|^2} \leq 1$ and $\int_\Omega |\nabla u|^2 \leq \lambda_1^H(\Omega)$. Clearly, if $\int_\Omega \frac{u^2}{|x|^2} = 1$, then by (2.2), one has $\int_\Omega |\nabla u|^2 = \lambda_1^H(\Omega)$, namely $\lambda_1^H(\Omega)$ is attained. If $0 < \int_\Omega \frac{u^2}{|x|^2} < 1$, since

$$\begin{aligned} \int_\Omega \frac{(u_n - u)^2}{|x|^2} &= \int_\Omega \frac{u_n^2}{|x|^2} - \int_\Omega \frac{u^2}{|x|^2} + o(1) = 1 - \int_\Omega \frac{u^2}{|x|^2} + o(1) \\ \int_\Omega |\nabla(u_n - u)|^2 &= \int_\Omega |\nabla u_n|^2 - \int_\Omega |\nabla u|^2 + o(1) = \lambda_1^H(\Omega) - \int_\Omega |\nabla u|^2 + o(1) \end{aligned}$$

we infer that

$$\lambda_1^H(\Omega) \leq \frac{\int_\Omega |\nabla(u_n - u)|^2}{\int_\Omega \frac{(u_n-u)^2}{|x|^2}} = \frac{\lambda_1^H(\Omega) - \int_\Omega |\nabla u|^2}{1 - \int_\Omega \frac{u^2}{|x|^2}} + o(1)$$

and then

$$\lambda_1^H(\Omega) > \frac{\int_\Omega |\nabla u|^2}{\int_\Omega \frac{u^2}{|x|^2}}$$

in contradiction with (2.2). Hence the case $0 < \int_\Omega \frac{u^2}{|x|^2} < 1$ cannot occur. Finally, to exclude also the case $\int_\Omega \frac{u^2}{|x|^2} = 0$, namely $u = 0$, we use the hypothesis (2.3). For every $\epsilon > 0$ there exists $r_0 > 0$ such that $\lambda_1^H(\Omega \cap B_{r_0}) \geq \lambda_{1,0}^H(\Omega) - \epsilon$, and $r_\infty > r_0$ such that $\lambda_1^H(\Omega \setminus \overline{B}_{r_\infty}) \geq \lambda_{1,\infty}^H(\Omega) - \epsilon$. We claim that

$$\int_U |\nabla u_n|^2 \to 0 \ , \quad \int_U \frac{u_n^2}{|x|^2} \to 0 \ , \tag{2.5}$$

for every open, bounded set $U \subset \Omega$ such that $0 \notin \overline{U}$. Assuming for a moment that (2.5) is proved, we conclude that

$$
\begin{aligned}
\lambda_1^H(\Omega) &= \int_\Omega |\nabla u_n|^2 + o(1) \\
&= \int_{\Omega \cap B_{r_0}} |\nabla u_n|^2 + \int_{\Omega \setminus \overline{B}_{r_\infty}} |\nabla u_n|^2 + o(1) \\
&\geq \lambda_1^H(\Omega \cap B_{r_0}) \int_{\Omega \cap B_{r_0}} \frac{u_n^2}{|x|^2} + \lambda_1^H(\Omega \setminus \overline{B}_{r_\infty}) \int_{\Omega \setminus \overline{B}_{r_\infty}} \frac{u_n^2}{|x|^2} + o(1) \\
&\geq (\lambda_{1,0}^H(\Omega) - \epsilon) \int_{\Omega \cap B_{r_0}} \frac{u_n^2}{|x|^2} + (\lambda_{1,\infty}^H(\Omega) - \epsilon) \int_{\Omega \setminus \overline{B}_{r_\infty}} \frac{u_n^2}{|x|^2} + o(1) \\
&\geq \min\{\lambda_{1,0}^H(\Omega), \lambda_{1,\infty}^H(\Omega)\} - \epsilon + o(1) \ .
\end{aligned}
$$

Thus, for the arbitrariness of ϵ, we obtain $\lambda_1^H(\Omega) \geq \min\{\lambda_{1,0}^H(\Omega), \lambda_{1,\infty}^H(\Omega)\}$, contrary to the assumption (2.3). Hence it remains to prove the claim (2.5). Let U be an open, bounded subset of Ω with $0 \notin \overline{U}$. Firstly, we have that $\int_U \frac{u_n^2}{|x|^2} \to 0$ because $(u_n) \subset H^1(U)$, $u_n \to 0$ weakly in $H^1(U)$ and then strongly in $L^2(U)$. Secondly, let $\chi \in C_c^\infty(\mathbf{R}^2 \setminus \{0\}, [0,1])$ be such that $\chi = 1$ on U. By (2.4) we have that

$$
\int_\Omega \nabla u_n \cdot \nabla(\chi u_n) = \lambda_1^H(\Omega_n) \int_\Omega \chi \frac{u_n}{|x|^2} \ .
$$

By the previous part $\int_\Omega \chi \frac{u_n}{|x|^2} \to 0$. Moreover the sequence $(\lambda_1^H(\Omega_n))$ is bounded and then

$$
\int_\Omega \nabla u_n \cdot \nabla(\chi u_n) \to 0 \ .
$$

Hence

$$
\begin{aligned}
\int_U |\nabla u_n|^2 &\leq \int_\Omega \chi |\nabla u_n|^2 \\
&= \int_\Omega \nabla u_n \cdot \nabla(\chi u_n) - \int_\Omega u_n \nabla \chi \cdot \nabla u_n \\
&\leq \left(\int_\Omega u_n^2 |\nabla \chi|^2 \right)^{\frac{1}{2}} \left(\int_\Omega |\nabla u_n|^2 \right)^{\frac{1}{2}} + o(1) \\
&= o(1) \ ,
\end{aligned}
$$

because (u_n) is bounded and converges to 0 strongly in $L^2(\mathrm{supp}\ \chi)$. Therefore (2.5) holds true and the proof is completed. $\square$

In the second part of this Section we focus on the case $\Omega = \mathbf{R}^2 \setminus \Gamma_{a,b}$, where $\Gamma_{a,b} = \{x \in \mathbf{R}^2 : 0 \le x_1 \le a,\ x_1 \ge b\}$, being $b \ge a > 0$. As noted in Remark 1.5, Ω satisfies the condition (H) and then, by Theorem 1.4, it is Hardy-admissible. Moreover, by the dilation invariance, the minimization problem (2.2) as well as the eigenvalue problem (2.1) are equivalent to the corresponding problems for the domain $\mathbf{R}^2 \setminus \Gamma_{ta,tb}$ for every $t > 0$. Hence, without loss of generality, we may reduce ourselves to consider the domains

$$\Omega_\delta = \mathbf{R}^2 \setminus \Gamma_\delta \ ,$$

where $\Gamma_\delta = \{x \in \mathbf{R}^2 : 0 \le x_1 \le \delta,\ x_1 \ge \frac{1}{\delta}\}$, and $\delta \in (0,1]$. Our goal is to show the following result.

Proposition 2.4 *There exists $\delta_0 \in (0,1]$ such that for every $\delta \in (0,\delta_0)$ the value $\lambda_1^H(\Omega_\delta)$ is attained in $D_0^1(\Omega_\delta)$. Moreover $\lambda_1^H(\Omega_\delta) \to \frac{1}{4}$ as $\delta \to \delta_0$.*

The proof of Proposition 2.4, is based on the following auxiliary results. The first one concerns the limit case $\delta = 1$.

Lemma 2.5 *$\lambda_1^H(\Omega_1) = \frac{1}{4}$ and it is not attained in $D_0^1(\Omega_1)$. Moreover $\lambda_1^H(\Omega_1 \cap B_r) = \lambda_1^H(\Omega_1 \setminus \overline{B}_r) = \frac{1}{4}$ for every $r > 0$.*

Proof. With the change of variable $\phi(s,\theta) = (e^s \cos\theta, e^s \sin\theta)$ we get that

$$\lambda_1^H(\Omega_1) = \inf\Big\{ \int_\Sigma |\nabla v|^2 : u \in H_0^1(\Sigma) \ , \ \int_\Sigma v^2 = 1 \Big\} \ . \tag{2.6}$$

It is known that the infimum in (2.6) is $\lambda_1(I)$, with $I = (0,2\pi)$, and it is not attained in $H_0^1(\Sigma)$. In addition $\lambda_1(I) = \frac{1}{4}$. Let us prove that $\lambda_1^H(\Omega_1 \cap B_r) = \lambda_1^H(\Omega_1)$. Clearly $\lambda_1^H(\Omega_1 \cap B_r) \ge \lambda_1^H(\Omega_1)$. Given $\epsilon > 0$ there exists $u \in C_c^\infty(\Omega_1)$ such that $\int_{\Omega_1} \frac{u^2}{|x|^2} = 1$ and $\int_{\Omega_1} |\nabla u|^2 \le \lambda_1^H(\Omega_1) + \epsilon$. Let $u_\delta(x) = u(\frac{x}{\delta})$. Then for $\delta > 0$ small enough $u_\delta \in C_c^\infty(\Omega_1 \cap B_r)$ and, by the dilation invariance $\int_{\Omega_1} \frac{u_\delta^2}{|x|^2} = \int_{\Omega_1} \frac{u^2}{|x|^2}$ and $\int_{\Omega_1} |\nabla u_\delta|^2 = \int_{\Omega_1} |\nabla u|^2$. Hence $\lambda_1^H(\Omega_1 \cap B_r) \le \lambda_1^H(\Omega_1) + \epsilon$. By the arbitrariness of ϵ we conclude that $\lambda_1^H(\Omega_1 \cap B_r) \le \lambda_1^H(\Omega_1)$. Therefore we get the thesis. In a similar way one can prove that $\lambda_1^H(\Omega_1) = \lambda_1^H(\Omega_1 \setminus \overline{B}_r)$. $\square$

The next result concerns some properties of the mapping $\delta \mapsto \lambda_1^H(\Omega_\delta)$.

Lemma 2.6 *The mapping $\delta \mapsto \lambda_1^H(\Omega_\delta)$ is continuous and non decreasing on $(0,1]$. Moreover $\lambda_1^H(\Omega_\delta) \to 0$ as $\delta \to 0$.*

Proof. For $0 < \delta < \delta' \leq 1$ we have $\Omega_\delta \subset \Omega_{\delta'}$. This immediately implies that the mapping $\delta \mapsto \lambda_1^H(\Omega_\delta)$ is non decreasing on $(0,1]$. To prove that $\lambda_1^H(\Omega_\delta) \to 0$ as $\delta \to 0$, one uses the fact that, setting $\Omega_0 = \mathbf{R}^2 \setminus \{0\}$,

$$\inf\Big\{ \int_{\mathbf{R}^2} |\nabla u|^2 : u \in C_c^\infty(\Omega_0)\ ,\ \int_{\mathbf{R}^2} \frac{u^2}{|x|^2} = 1 \Big\} = 0\ .$$

Also to prove the continuity from the right it is enough to apply the definition of $\lambda_1^H(\Omega_\delta)$. Now let us show that the mapping $\delta \mapsto \lambda_1^H(\Omega_\delta)$ is continuous from the left at any $\delta \in (0,1]$. By contradiction, let $\epsilon > 0$ and $0 < \delta_0 < \delta \leq 1$ be such that

$$\lambda_1^H(\Omega_{\delta'}) \leq \lambda_1^H(\Omega_\delta) - \epsilon \ \text{ for every } \delta' \in (\delta_0, \delta)\ . \tag{2.7}$$

Let $(\delta_n) \subset (\delta_0, \delta)$ be such that $\delta_n \to \delta$. By (2.7), $\lambda_1^H(\Omega_{\delta_n}) < \lambda_1^H(\Omega_1)$ for every $n \in \mathbf{N}$, and then, by Lemma 2.5 and by Theorem 2.3, $\lambda_1^H(\Omega_{\delta_n})$ is attained in $D_0^1(\Omega_{\delta_n})$ by some u_n which satisfies

$$\int_{\Omega_{\delta_n}} \frac{u_n^2}{|x|^2} = 1\ ,\quad \int_{\Omega_{\delta_n}} |\nabla u_n|^2 = \lambda_1^H(\Omega_{\delta_n})\ , \tag{2.8}$$

and is a positive solution to

$$\begin{cases} -\Delta u = \lambda_1^H(\Omega_{\delta_n}) \frac{u}{|x|^2} & \text{in } \Omega_{\delta_n} \\ u = 0 & \text{on } \partial\Omega_{\delta_n}\ . \end{cases} \tag{2.9}$$

Since, by (2.8), the sequence (u_n) is bounded in $D_0^1(\Omega_{\delta_0})$, there exists $u \in D_0^1(\Omega_{\delta_0})$ and a subsequence of (u_n), still denoted (u_n), such that $u_n \to u$ weakly in $D_0^1(\Omega_{\delta_0})$ and pointwise a.e. in Ω_{δ_0}. Hence $u \geq 0$, $u \in D_0^1(\Omega_\delta)$, $\int_{\Omega_\delta} \frac{u^2}{|x|^2} \leq 1$ and $\int_{\Omega_\delta} |\nabla u|^2 \leq \lambda_1^H(\Omega_\delta) - \epsilon$, because of (2.7). If $\int_{\Omega_\delta} \frac{u^2}{|x|^2} = 1$, we obtain a contradiction with (2.2). Also the case $0 < \int_{\Omega_\delta} \frac{u^2}{|x|^2} < 1$ can be excluded, arguing as in the proof of Theorem 2.3. Let us consider the last case $\int_{\Omega_\delta} \frac{u^2}{|x|^2} = 0$, namely $u = 0$. Arguing again as in the proof of Theorem 2.3, using (2.9), we can show that

$$\int_U |\nabla u_n|^2 \to 0\ ,\quad \int_U \frac{u_n^2}{|x|^2} \to 0\ , \tag{2.10}$$

for every open, bounded set $U \subset \Omega$ such that $0 \notin \overline{U}$. Then, taking $r_0 > 0$ such that $\lambda_1^H(\Omega_\delta \cap B_{r_0}) \geq \lambda_{1,0}^H(\Omega_\delta) - \frac{\epsilon}{2}$, and $r_\infty > r_0$ such that $\lambda_1^H(\Omega_\delta \setminus \overline{B}_{r_\infty}) \geq \lambda_{1,\infty}^H(\Omega_\delta) - \frac{\epsilon}{2}$, we have that, by (2.10),

$$\begin{aligned} \lambda_1^H(\Omega_{\delta_n}) &= \int_{\Omega_\delta \cap B_{r_0}} |\nabla u_n|^2 + \int_{\Omega_\delta \setminus \overline{B}_{r_\infty}} |\nabla u_n|^2 + o(1) \\ &\geq \min\{\lambda_{1,0}^H(\Omega_\delta), \lambda_{1,\infty}^H(\Omega_\delta)\} - \frac{\epsilon}{2} + o(1)\ . \end{aligned}$$

Thus, using (2.7) and Lemma 2.5, we obtain that $\lambda_1^H(\Omega_\delta) \geq \lambda_1^H(\Omega_1) + \frac{\epsilon}{2}$, a contradiction. Therefore also the case $u = 0$ cannot occur, and thus the continuity from the left is proved. $\square$

Hence Proposition 2.4 immediately follows by Lemmata 2.5 and 2.6 and by Theorem 2.3. Moreover, as a consequence of the above statements, we plainly obtain the following result.

Corollary 2.7 *For every $\lambda \in (0, \frac{1}{4})$ there exists $\delta \in (0,1)$ such that the eigenvalue problem*

$$\begin{cases} -\Delta u = \lambda \frac{u}{|x|^2} & \text{in } \Omega_\delta \\ u = 0 & \text{on } \partial\Omega_\delta \end{cases}$$

admits a positive solution in $D_0^1(\Omega_\delta)$.

Remark 2.8 One can easily check that $\delta_0 > e^{-\pi}$. Indeed, consider the function $u\colon \mathbf{R}^2 \to \mathbf{R}$ defined as follows:

$$u(x) = \begin{cases} \cos(\frac{1}{2}\log|x|) & \text{as } e^{-\pi} \leq |x| \leq e^{\pi} \\ 0 & \text{elsewhere.} \end{cases}$$

One can see that $u \in D_0^1(\Omega_{\bar\delta})$ with $\bar\delta = e^{-\pi}$, and $\int_{\mathbf{R}^2} |\nabla u|^2 / \int_{\mathbf{R}^2} \frac{u^2}{|x|^2} = \frac{1}{4}$. Hence, if it were $\lambda_1^H(\Omega_{\bar\delta}) = \frac{1}{4}$, then u should be a minimizer and consequently should be a positive solution to (2.1) on $\Omega_{\bar\delta}$. In particular, by the maximum principle, $u(x) > 0$ for every $x \in \Omega_{\bar\delta}$, contrary to the definition of u. Therefore we conclude that $\delta_0 > \bar\delta$.

3 The semilinear Dirichlet problem

In this section we study the Dirichlet's problem (0.5) assuming that Ω is a Hardy-admissible domain in $\mathbf{R}^2$, $\lambda < \lambda_1^H(\Omega)$ and $g \in C^1(\mathbf{R})$ satisfies:

$(g1)$ there exists $p > 2$ such that $|g'(u)| = O(u^{p-2})$ as $u \to +\infty$,

$(g2)$ $g(u) = o(u)$ as $u \to 0$,

$(g3)$ there exists $q > 2$ such that $0 < qG(u) \leq g(u)u$ for any $u > 0$,

$(g4)$ $\frac{g(u)}{u} < g'(u)$ for every $u > 0$,

being $G(u) = \int_0^u g(t)dt$. Note that the function $u \mapsto |u|^{p-2}u$ verifies $(g1)$–$(g4)$ for $p > 2$.

Since we look for positive solutions to (0.5), following a standard procedure, we may modify g on $(-\infty, 0)$ setting $g(u) = 0$ for $u < 0$.

In Theorem 3.1 we will state a criterion in order that problem (0.5) admits a (weak) positive solution.

Since the domain Ω is Hardy-admissible, by Lemma 1.3 the space $D_0^1(\Omega)$ is well defined and $\lambda_1^H(\Omega) > 0$. Furthermore, since $\lambda < \lambda_1^H(\Omega)$, we can take

$$\|u\| = \left(\int_\Omega \left(|\nabla u|^2 - \lambda \frac{u^2}{|x|^2}\right)\right)^{1/2}$$

as a norm in $D_0^1(\Omega)$.

Thanks to $(g1)$ and $(g2)$, by Lemma 1.3, the functional $I\colon D_0^1(\Omega) \to \mathbf{R}$ defined by

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega \frac{u^2}{|x|^2} - \int_\Omega \frac{G(u)}{|x|^2}$$

is of class C^2 on $D_0^1(\Omega)$ and its critical points are weak solutions to (0.5).

Moreover, by $(g2)$, one has that $\int_\Omega \frac{G(u)}{|x|^2} = o(\|u\|^2)$, as $\|u\| \to 0$. By $(g3)$, $G(u) \geq G(1)u^q$ for $u \geq 1$ and then $I(su) \to -\infty$ as $s \to +\infty$, for every $u \in D_0^1(\Omega)$, $u \geq 0$, $u \neq 0$.

Hence the functional I has a mountain pass geometry. More precisely, setting $\mathcal{G} = \{\gamma \in C([0,1], D_0^1(\Omega)) : \gamma(0) = 0,\ I(\gamma(1)) < 0\}$ the class of the mountain pass paths, and

$$c(\Omega) = \inf_{\gamma\in\mathcal{G}} \max_{s\in[0,1]} I(\gamma(s)) \tag{3.1}$$

the corresponding minimax level of I, we have that $c(\Omega) > 0$.

To check the Palais Smale (briefly PS) condition for I at level $c(\Omega)$, we use a comparison estimate with the "problems at infinity", that keeps into account of possible concentration phenomena at 0, or vanishing at infinity. This is accomplished similarly to what done for the eigenvalue problem (2.1). Precisely, using the agreement that $c(\emptyset) = +\infty$, let us define

$$c_0(\Omega) = \sup_{r>0}\ c(\Omega \cap B_r)\ ,\ c_\infty(\Omega) = \sup_{r>0}\ c(\Omega \setminus \overline{B}_r)\ . \tag{3.2}$$

Notice that $c(\Omega\cap B_r)$ and $c(\Omega\setminus\overline{B}_r)$ are well defined for every $r > 0$. Indeed if $\Omega\cap B_r$ (or $\Omega\setminus\overline{B}_r$) is non empty, then it is Hardy-admissible, $\lambda < \lambda_1^H(\Omega\cap B_r)$, because $\lambda_1^H(\Omega) \leq \lambda_1^H(\Omega\cap B_r)$, and then also the restriction of I to $D_0^1(\Omega\cap B_r)$ has a mountain pass geometry at a level $c(\Omega \cap B_r) \geq c(\Omega)$. Hence we have that $c(\Omega) \leq \min\{c_0(\Omega), c_\infty(\Omega)\}$. The following existence criterion holds.

Theorem 3.1 *Let Ω be a Hardy-admissible domain in $\mathbf{R}^2$, $\lambda < \lambda_1^H(\Omega)$, and $g \in C^1(\mathbf{R})$ satisfy $(g1)$–$(g4)$. If*

$$c(\Omega) < \min\{c_0(\Omega), c_\infty(\Omega)\} \ , \tag{3.3}$$

then problem (0.5) *admits a weak positive solution.*

According to what stated before, in order to prove Theorem 3.1, we have just to check that if $(u_n) \subset D_0^1(\Omega)$ satisfies $I(u_n) \to c(\Omega)$ and $I'(u_n) \to 0$, then (u_n) is precompact in $D_0^1(\Omega)$. This will be carried out in the next Lemmata.

Let us first remark that, by $(g2)$, for every $u \in D_0^1(\Omega)$ we have

$$(\frac{1}{2} - \frac{1}{q})\|u\|^2 \le I(u) + \frac{1}{q}\|I'(u)\| \ \|u\| \ . \tag{3.4}$$

In particular, (3.4) implies that any PS sequence for I is bounded and hence it has a weakly convergent subsequence. Moreover, by (3.4) there is no PS sequence for I at a level $b < 0$.

The next two Lemmata are standard and we omit the proof that can be plainly obtained by following a procedure already used (see, e.g., [11]).

Lemma 3.2 *Let $(u_n) \subset D_0^1(\Omega)$ be a PS at level b such that $u_n \to u$ weakly in $D_0^1(\Omega)$ for some $u \in D_0^1(\Omega)$. Then $I'(u) = 0$ and $(u_n - u)$ is a PS sequence at level $b - I(u)$.*

By Lemma 3.2 we may restrict ourselves to study PS sequences weakly converging to 0. We have the following result.

Lemma 3.3 *Let $(u_n) \subset D_0^1(\Omega)$ be a PS sequence weakly converging to 0. Then:*

(i) $\int_U |\nabla u_n|^2 \to 0$ *for every open bounded set $U \subset \Omega$ with $0 \notin \overline{U}$,*

(ii) given a cut-off function $\chi \in C_c^1(\mathbf{R}^2, [0,1])$ with $\chi = 1$ in a neighborhood of 0, then (χu_n) and $((1-\chi)u_n)$ are PS sequences.

Now, let us state an auxiliary result concerning bounded sequences in $D_0^1(\Omega)$.

Lemma 3.4 *Let $(u_n) \subset D_0^1(\Omega)$ be a bounded sequence. Let $A_j = \{x \in \mathbf{R}^2 : \rho^j < |x| < \rho^{j+1}\}$ where $j \in \mathbf{Z}$ and $\rho > 1$ is fixed. For every $p > 2$, if $\sup_{j \in \mathbf{Z}} \int_{\Omega \cap A_j} \frac{|u_n|^p}{|x|^2} \to 0$ as $n \to \infty$, then $\int_\Omega \frac{|u_n|^p}{|x|^2} \to 0$.*

Proof. For every $n \in \mathbf{N}$ there exists $r_n > 0$ such that

$$\int_{\Omega\cap B_{r_n}} \frac{|u_n|^p}{|x|^2} = \int_{\Omega\setminus B_{r_n}} \frac{|u_n|^p}{|x|^2} = \frac{1}{2}\int_\Omega \frac{|u_n|^p}{|x|^2}.$$

For $i = 1, 2$ let $\phi_i \in C^1([0, +\infty), [0, 1])$ satisfying: $\phi_1(r) = 1$ for $0 \le r \le \frac{1}{\rho}$ and $\phi_1(r) = 0$ for $r \ge 1$, $\phi_2(r) = 0$ for $0 \le r \le 1$ and $\phi_2(r) = 1$ for $r \ge \rho$. Now, for $i = 1, 2$ and $n \in \mathbf{N}$ set $\chi_{i,n}(x) = \varphi_i(\frac{|x|}{r_n})$, for every $x \in \mathbf{R}^N$. Then, define $u_{i,n} = \chi_{i,n} u_n$. Firstly, let us prove that for $i = 1, 2$

$$\frac{1}{2}\int_\Omega \frac{|u_n|^p}{|x|^2} - \int_\Omega \frac{|u_{i,n}|^p}{|x|^2} \to 0 \tag{3.5}$$

as $n \to \infty$. Indeed, for every $n \in \mathbf{N}$ there exists $j_n \in \mathbf{Z}$ such that $r_n \in (\rho^{j_n}, \rho^{j_n+1}]$. Then $\frac{r_n}{\rho} \in (\rho^{j_n-1}, \rho^{j_n}]$ and thus

$$\begin{aligned} 0 \ &\le\ \frac{1}{2}\int_\Omega \frac{|u_n|^p}{|x|^2} - \int_\Omega \frac{|u_{1,n}|^p}{|x|^2} = \int_{\Omega\cap B_{r_n}} (1-\chi_{1,n}^p)\frac{|u_n|^p}{|x|^2} \\ &\le\ \int_{\Omega\cap A_{j_n-1}} \frac{|u_n|^p}{|x|^2} + \int_{\Omega\cap A_{j_n}} \frac{|u_n|^p}{|x|^2} \le 2\sup_{j\in\mathbf{Z}} \int_{\Omega\cap A_j} \frac{|u_n|^p}{|x|^2} \end{aligned}$$

and then, by the assumption, (3.5) holds true for $i = 1$. A similar argument holds for $i = 2$. Secondly, we claim that for $i = 1, 2$

$$\int_\Omega |\nabla\chi_{i,n}|^2 u_n^2 \to 0 \tag{3.6}$$

as $n \to \infty$. Indeed

$$\begin{aligned} \int_\Omega |\nabla\chi_{1,n}|^2 u_n^2 \ &\le\ \frac{C}{r_n^2}\int_{\Omega\cap\{\frac{r_n}{\rho}<|x|<r_n\}} u_n^2 \\ &\le\ \frac{C}{r_n^2}\left(\int_{\Omega\cap\{\frac{r_n}{\rho}<|x|<r_n\}} |x|^{\frac{4}{p-2}}\right)^{\frac{p-2}{p}} \left(\int_{\Omega\cap\{\frac{r_n}{\rho}<|x|<r_n\}} \frac{|u_n|^p}{|x|^2}\right)^{\frac{2}{p}} \\ &\le\ C\left(\sup_{j\in\mathbf{Z}}\int_{\Omega\cap A_j} \frac{|u_n|^p}{|x|^2}\right)^{\frac{2}{p}}. \end{aligned}$$

Hence, using again the assumption, (3.6) is proved for $i = 1$. Similarly it holds also for $i = 2$. Now we show that

$$\int_\Omega |\nabla u_n|^2 \ge 2S_p(\Omega)\left(\frac{1}{2}\int_\Omega \frac{|u_n|^p}{|x|^2}\right)^{\frac{2}{p}} + o(1) \tag{3.7}$$

as $n \to \infty$, where $S_p(\Omega)$ is defined by (0.3). Indeed, setting $\chi_n = \chi_{1,n} + \chi_{2,n}$, we have

$$
\begin{aligned}
\int_\Omega |\nabla(\chi_n u_n)|^2 &\leq \int_\Omega |\nabla \chi_n|^2 u_n^2 + \int_\Omega \chi_n^2 |\nabla u_n|^2 \\
&\quad +2 \left(\int_\Omega |\nabla \chi_n|^2 u_n^2 \right)^{\frac{1}{2}} \left(\int_\Omega \chi_n^2 |\nabla u_n|^2 \right)^{\frac{1}{2}} \\
&\leq o(1) + \int_\Omega |\nabla u_n|^2 \qquad (3.8)
\end{aligned}
$$

because (u_n) is bounded in $D_0^1(\Omega)$ and by (3.6). Then, using (0.3), we infer that

$$
\begin{aligned}
\int_\Omega |\nabla u_n|^2 &\geq \int_\Omega |\nabla u_{1,n}|^2 + \int_\Omega |\nabla u_{2,n}|^2 + o(1) \\
&\geq S_p(\Omega) \left(\int_\Omega \frac{|u_{1,n}|^p}{|x|^2} \right)^{\frac{2}{p}} + S_p(\Omega) \left(\int_\Omega \frac{|u_{2,n}|^p}{|x|^2} \right)^{\frac{2}{p}} + o(1) \\
&\geq 2S_p(\Omega) \left(\frac{1}{2} \int_\Omega \frac{|u_n|^p}{|x|^2} \right)^{\frac{2}{p}} + o(1)
\end{aligned}
$$

because of (3.5). Now we observe that for $i = 1, 2$ the sequence $(u_{i,n})$ satisfies the same assumptions of (u_n), namely $(u_{i,n})$ is bounded, by (3.8), and clearly $\sup_{j \in \mathbf{Z}} \int_{\Omega \cap A_j} \frac{|u_{i,n}|^p}{|x|^2} \to 0$ as $n \to \infty$. Hence we also have

$$
\int_\Omega |\nabla u_{i,n}|^2 \geq 2S_p(\Omega) \left(\frac{1}{2} \int_\Omega \frac{|u_{i,n}|^p}{|x|^2} \right)^{\frac{2}{p}} + o(1)
$$

as $n \to \infty$. Then, using (3.5) and (3.8), we deduce that

$$
\int_\Omega |\nabla u_n|^2 \geq 2^2 S_p(\Omega) \left(\frac{1}{2^2} \int_\Omega \frac{|u_n|^p}{|x|^2} \right)^{\frac{2}{p}} + o(1) \, .
$$

By recurrence, for every $k \in \mathbf{N}$ we have that

$$
\int_\Omega |\nabla u_n|^2 \geq 2^k S_p(\Omega) \left(\frac{1}{2^k} \int_\Omega \frac{|u_n|^p}{|x|^2} \right)^{\frac{2}{p}} + o(1)
$$

as $n \to \infty$. Since $p > 2$ and (u_n) is bounded in $D_0^1(\Omega)$, and since, by Lemma 1.3, $S_p(\Omega) > 0$, we conclude that $\int_\Omega \frac{|u_n|^p}{|x|^2} \to 0$. $\square$

As a consequence of Lemma 3.4 we have the following result.

Lemma 3.5 *Let $(u_n) \subset D_0^1(\Omega)$ be a PS sequence weakly converging to 0. Then the following alternative holds: either*

(i) $\liminf \|v_n\| = 0$, *or*

(ii) *there exists a sequence $(j_n) \subset \mathbf{Z}$ such that $|j_n| \to \infty$ and* $\liminf \int_{\Omega \cap A_{j_n}} \frac{|u_n|^p}{|x|^2} > 0$, *where A_j is defined as in Lemma 3.4.*

Proof. Suppose that (i) does not hold. By $(g1)$–$(g3)$, for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that $0 \le g(t)t \le \epsilon t^2 + C_\epsilon |t|^p$ for any $t \in \mathbf{R}$. Then, taking $\epsilon < \lambda_1^H(\Omega) - \lambda$ we have

$$\|u_n\|^2 = I'(u_n)u_n + \int_\Omega \frac{g(u_n)u_n}{|x|^2} \le I'(u_n)u_n + \epsilon\|u_n\|^2 + C_\epsilon \int_\Omega \frac{|u_n|^p}{|x|^2} \ . \tag{3.9}$$

Since $\liminf \|u_n\| > 0$ and $I'(u_n) \to 0$, fixing $\epsilon \in (0, \lambda_1^H(\Omega) - \lambda)$ small enough, we obtain that $\liminf \int_\Omega \frac{|u_n|^p}{|x|^2} > 0$, and then, by Lemma 3.4,

$$\liminf_{n\to\infty} \sup_{j \in \mathbf{Z}} \int_{\Omega \cap A_j} \frac{|u_n|^p}{|x|^2} > 0 \ .$$

In particular there exists a sequence $(j_n) \subset \mathbf{Z}$ such that

$$\liminf \int_{\Omega \cap A_{j_n}} \frac{|u_n|^p}{|x|^2} > 0 \ .$$

Since $\int_U \frac{|u_n|^p}{|x|^2} \to 0$ for every bounded open set $U \subset \Omega$ with $0 \notin \overline{U}$, we have that $|j_n| \to \infty$. Therefore (ii) follows. Conversely, suppose that (ii) does not hold. Let $(j_n) \subset \mathbf{Z}$ be such that $\sup_{j\in\mathbf{Z}} \int_{\Omega\cap A_j} \frac{|u_n|^p}{|x|^2} = \int_{\Omega \cap A_{j_n}} \frac{|u_n|^p}{|x|^2}$. If (j_n) is bounded, since $u_n \to 0$ weakly in $D_0^1(\Omega)$, we have that $\int_{\Omega\cap A_{j_n}} \frac{|u_n|^p}{|x|^2} \to 0$ and then, using Lemma 3.4, $\liminf \int_\Omega \frac{|u_n|^p}{|x|^2} = 0$. The same holds if (j_n) is unbounded, because we are assuming that (ii) does not hold. Hence, by (3.9), $\liminf \|u_n\| = 0$, namely (i).

□

In the next Lemma we show that, by the assumption $(g4)$, there is no PS sequence at levels $b \in (0, c(\Omega))$.

Lemma 3.6 *Let $(u_n) \subset D_0^1(\Omega)$ be a PS sequence at level b, such that $u_n \to u$ weakly in $D_0^1(\Omega)$ for some $u \in D_0^1(\Omega)$. If $b < c(\Omega)$ then $u = 0$, and $u_n \to 0$ strongly in $D_0^1(\Omega)$. In particular $b = 0$.*

Proof. By $(g4)$, for every $u \in D_0^1(\Omega)$, $u \neq 0$, there exists a unique $t_u > 0$ such that $\frac{d}{dt} I(tu)|_{t=t_u} = 0$. Moreover $I(t_u u) = \max_{t \geq 0} I(tu) \geq c(\Omega)$. In particular, if $I'(u) = 0$ then $I(u) \geq c(\Omega)$. Let (u_n) be a PS sequence at level $b < c(\Omega)$ such that $u_n \to u$ weakly in $D_0^1(\Omega)$. By Lemma 3.2, $I'(u) = 0$ and $(u_n - u)$ is a PS sequence at level $b' = b - I(u)$. If $u \neq 0$, then $b' < 0$, which is impossible, because of (3.4). Hence $u_n \to 0$ weakly in $D_0^1(\Omega)$. Suppose, by contradiction, that for a subsequence, still denoted (u_n), $\lim \|u_n\| > 0$ holds. For any $n \in \mathbf{N}$ let $t_n > 0$ be such that $\max_{t \geq 0} I(tu_n) = I(t_n u_n)$. We claim that $I(t_n u_n) - I(u_n) \to 0$, and then $I(t_n u_n) \to b$, contradicting the fact that $I(t_n u_n) \geq c(\Omega) > b$. Therefore $u_n \to 0$ strongly in $D_0^1(\Omega)$. Hence, to complete the proof, we have to show that $I(t_n u_n) - I(u_n) \to 0$. To this aim, let us introduce for every $n \in \mathbf{N}$ the function $j_n(t) = I(tu_n)$. Note that $j_n \in C^2(\mathbf{R}^+)$, with $j_n'(t) = t\|u_n\|^2 - \int_\Omega \frac{g(tu_n)u_n}{|x|^2}$ and $j_n''(t) = \|u_n\|^2 - \int_\Omega \frac{g'(tu_n)u_n^2}{|x|^2}$. By $(g4)$ we have that for any $n \in \mathbf{N}$

$$j_n''(t) \leq \tfrac{1}{t} j_n'(t) \quad \text{for every } t > 0 \ . \tag{3.10}$$

Moreover $j_n'(t) > 0$ for $t \in (0, t_n)$, $j_n'(t_n) = 0$, and $j_n'(t) < 0$ for $t > t_n$. Hence, by (3.10), we obtain that

$$|j_n'(t)| \leq t|j_n'(1)| \text{ for every } t \in [\min\{t_n, 1\}, \max\{t_n, 1\}] \ . \tag{3.11}$$

Let us prove that the sequence (t_n) is bounded. By $(g2)$ and $(g3)$, fixing $\epsilon > 0$ there exists $C_\epsilon > 0$ such that $g(u)u \geq -\epsilon u^2 + C_\epsilon |u|^q$ for every $u \in \mathbf{R}$. Therefore $\int_\Omega \frac{g(tu_n)u_n}{|x|^2} \geq -\epsilon t_n \int_\Omega \frac{u_n^2}{|x|^2} + C_\epsilon t_n^{q-1} \int_\Omega \frac{|u_n|^q}{|x|^2}$ and then

$$(1+\epsilon)\|u_n\|^2 \geq C_\epsilon t_n^{q-2} \int_\Omega \frac{|u_n|^q}{|x|^2} \ . \tag{3.12}$$

By Lemma 3.5, there exists a sequence $(j_n) \subset \mathbf{Z}$ such that $\liminf \int_{\Omega \cap A_{j_n}} \frac{|u_n|^p}{|x|^2} > 0$, where $A_j = \{x \in \mathbf{R}^2 : \rho^j < |x| < \rho^{j+1}\}$. Then, $\liminf \int_\Omega \frac{|u_n|^q}{|x|^2} > 0$, that, together with (3.12), implies that $\sup t_n < +\infty$. Finally, we can show that $j_n(t_n) - j_n(1) \to 0$, that is the initial claim. Indeed, by (3.11) we have that

$$0 \le j_n(t_n) - j_n(1) \le \left| \int_{t_n}^1 |j_n'(t)| \, dt \right| \le |j_n'(1)| \left| \int_{t_n}^1 t \, dt \right| = o(1)$$

because $\sup t_n < +\infty$ and $j_n'(1) = I'(u_n)u_n \to 0$. This concludes the proof. □

Remark 3.7 If Ω is a proper cone in $\mathbf{R}^2$, then, under the assumptions $(g1)$–$(g4)$, for $\lambda < \lambda_1^H(\Omega)$, the mountain pass level $c(\Omega)$ is always a critical value for I. Indeed, given a PS sequence $(u_n) \subset D_0^1(\Omega)$ at level $c(\Omega)$, thanks to the invariance of the problem (0.5) with respect to dilation, we may suppose that, up to dilations, $\liminf \int_{\Omega \cap A_0} \frac{|u_n|^p}{|x|^2} > 0$, where $A_0 = B_\rho \setminus \overline{B}_1$. Hence there exists $u \in D_0^1(\Omega)$, $u \neq 0$, such that, for a subsequence, $u_n \to u$ weakly. By Lemmata 3.2 and 3.5, $I'(u) = 0$ and $I(u) = c(\Omega)$ (in fact $u_n \to u$ strongly in $D_0^1(\Omega)$, see below). Note that in this case $c(\Omega) = c_0(\Omega) = c_\infty(\Omega)$ (see Lemma 4.1).

Lemma 3.8 *Let $(u_n) \subset D_0^1(\Omega)$ be a PS sequence weakly converging to* 0. *Let $(j_n) \subset \mathbf{Z}$ be such that $|j_n| \to \infty$ and* $\liminf \int_{\Omega \cap A_{j_n}} \frac{|u_n|^p}{|x|^2} > 0$, *where A_j is defined as in Lemma 3.4. Then* $\liminf I(u_n) \ge \min\{c_0(\Omega), c_\infty(\Omega)\}$.

Proof. Let us suppose that, for a subsequence, $j_n \to -\infty$. Let $r > 0$ and $\chi \in C_c^\infty(\mathbf{R}^2, [0,1])$ be such that $\chi = 1$ on $B_{\frac{r}{2}}$ and $\chi = 0$ outside B_r. By Lemma 3.3, (χu_n) is a PS sequence for $I|_{D_0^1(\Omega \cap B_r)}$ Then, by Lemma 3.6, $\liminf I(\chi u_n) \ge c(\Omega \cap B_r)$, because $\liminf \|\chi u_n\| > 0$. One can also see that $\liminf I(u_n) \ge \liminf I(\chi u_n)$. Thus, by the arbitrariness of $r > 0$, we infer that $\liminf I(u_n) \ge c_0(\Omega)$, by (3.2). Instead, if $\inf j_n > -\infty$, then $j_n \to +\infty$, and, arguing as before, we conclude that $\liminf I(u_n) \ge c_\infty(\Omega)$. Hence, the Lemma is proved. □

Conclusion of the proof of Theorem 3.1. Let $(u_n) \subset D_0^1(\Omega)$ be a PS sequence at level $c(\Omega)$. Up to a subsequence, $u_n \to u$ weakly in $D_0^1(\Omega)$. If $u \neq 0$ we have finished, because, by Lemma 3.2, $(u_n - u)$ is again a PS sequence at a level $b \le c(\Omega) - I(u) < c(\Omega)$ and thus, $u_n \to u$ strongly in $D_0^1(\Omega)$, by Lemma 3.6. Hence, by contradiction, suppose that $u_n \to 0$ weakly in $D_0^1(\Omega)$. Since $\liminf \|u_n\| > 0$ (otherwise $I(u_n) \to 0$, while $c(\Omega) > 0$), by Lemma 3.5, there exists a sequence $(j_n) \subset \mathbf{Z}$ such that $|j_n| \to \infty$ and $\liminf \int_{\Omega \cap A_{j_n}} \frac{|u_n|^p}{|x|^2} > 0$, being $A_j = \{x \in \mathbf{R}^2 : \rho^j < |x| < \rho^{j+1}\}$. Then, by Lemma 3.8, $\liminf I(u_n) \ge \min\{c_0(\Omega), c_\infty(\Omega)\}$, contrary to the assumption (3.3). This concludes the proof. □

Remark 3.9 In Theorem 3.1 the assumption $(g1)$ on g can be slightly weakened, by requiring just
$(g1)'$ $\log(|g'(u)|) = o(u^2)$ as $u \to +\infty$.
Indeed, since the Trudinger-Moser inequality holds in $H^1(\Sigma)$, where $\Sigma = \mathbf{R} \times (0, 2\pi)$ (see [1]), the condition $(g1)'$ is sufficient to guarantee that the functional $\varphi(u) = \int_\Omega \frac{G(u)}{|x|^2}$ is well defined and of class C^2 on $D_0^1(\Omega)$, and φ' maps weakly convergent sequences into weakly convergent ones. Moreover, if U is an open bounded subset of Ω such that $0 \notin \overline{U}$, denoting by φ_U the restriction of φ to $D_0^1(U)$, then φ_U and φ'_U are relatively compact mappings.

We conclude this Section by discussing the role of the assumption of Hardy-admissibility of the domain. Consider the case of a domain Ω containing 0. Then it is not Hardy-admissible and, if $\lambda \geq 0$, it is known that equation (0.2) admits no positive solution, even in a very weak sense, as proved in [3] and in [8]. More precisely, the following non existence result holds (see [8]).

Theorem 3.10 *Let $f \in C(\mathbf{R})$ be such that $\liminf_{u \to +\infty} \frac{f(u)}{u^p} > 0$ for some $p > 1$, and $f(u) > 0$ for every $u > 0$. If Ω is a domain in $\mathbf{R}^2$ containing 0, or if Ω is an exterior domain in $\mathbf{R}^2$, then the equation*

$$-\Delta u = \frac{f(u)}{|x|^2} \quad \text{in } \Omega$$

admits no non negative supersolution in the sense of distributions in $\Omega \setminus \{0\}$, namely there is no function $u \in L^1_{loc}(\Omega \setminus \{0\})$ such that $f(u) \in L^1_{loc}(\Omega \setminus \{0\})$, $u \geq 0$, $u \neq 0$ and $-\int_\Omega u \Delta\varphi \geq \int_\Omega \frac{f(u)}{|x|^2}\varphi$ for every $\varphi \in C_c^\infty(\Omega \setminus \{0\})$, $\varphi \geq 0$.

Considering again a domain Ω containing 0, one could ask if there exists a positive solution u to problem (0.5) for $\lambda < 0$, with $\int_\Omega (|\nabla u|^2 + \frac{u^2}{|x|^2}) < \infty$. In general also in this case we have non existence. Precisely, let us recall the following result, proved in [8].

Theorem 3.11 *Let $g \in C(\mathbf{R})$ satisfy $(g1)$–$(g2)$, $\lambda \in \mathbf{R}$, and let Ω be a smooth, bounded, star-shaped domain in $\mathbf{R}^2$ containing 0. If u is a solution to (0.5) such that $\int_\Omega (|\nabla u|^2 + \frac{u^2}{|x|^2}) < \infty$, then $u = 0$.*

In this case the reason of non existence is the invariance under dilation and a concentration phenomenon to the problem on the whole space $\mathbf{R}^2$, which admits a positive solution radial solution for $\lambda < 0$ and $g \in C(\mathbf{R})$ satisfying $(g1)$–$(g3)$ (see [8, Theorem 2.5]).

4 Examples

As an application of Theorem 3.1, in this section we consider the case of domains contained in $\mathbf{R}^2 \setminus \Gamma_\delta$, where $\Gamma_\delta = \{x \in \mathbf{R}^2 : 0 \le x_1 \le \delta,\ x_1 \ge \frac{1}{\delta}\}$, and $\delta \in (0, 1]$. Let us introduce the cones

$$\Omega_\theta = \{(\rho \cos \tau, \rho \sin \tau) : \rho > 0,\ 0 < \tau < \theta\}$$

where $0 < \theta \le 2\pi$. Note that, setting $I_\theta = (0, \theta)$ and $\Sigma_\theta = \mathbf{R} \times I_\theta$, we have

$$\lambda_1^H(\Omega_\theta) = \lambda_1(\Sigma_\theta) = \lambda_1(I_\theta) = \frac{\pi^2}{\theta^2}\ . \tag{4.1}$$

We first discuss the case of a domain of the form $\Omega = \Omega_\theta \cup U$, where U is a bounded open set such that $0 \notin \overline{U}$.

To evaluate $c_0(\Omega)$ and $c_\infty(\Omega)$, the following result is useful.

Lemma 4.1 *$c(\Omega_\theta) = c_0(\Omega_\theta) = c_\infty(\Omega_\theta)$.*

Proof. On one hand, the inequalities $c(\Omega_\theta) \le c_0(\Omega_\theta)$ and $c(\Omega_\theta) \le c_\infty(\Omega_\theta)$ are always true. To prove the opposite inequalities, let us consider the solution u to (0.5) on Ω_θ obtained as critical point of I in $D_0^1(\Omega_\theta)$ at the mountain pass level $c(\Omega_\theta)$ (see Remark 3.7). The sequence (u_n) defined by $u_n(x) = u(nx)$ satisfies the assumption of Lemma 3.8. Hence $c_0(\Omega_\theta) \le \liminf I(u_n) = c(\Omega_\theta)$. A similar argument holds to prove that $c_\infty(\Omega_\theta) \le c(\Omega_\theta)$. □

Theorem 4.2 *Let $g \in C^1(\mathbf{R})$ satisfy (g1)–(g4), and let Ω be a domain in $\mathbf{R}^2$ of the form $\Omega = \Omega_\theta \cup U$, where Ω_θ is a cone of angle $\theta \in (0, 2\pi]$ and U is a bounded open set such that $0 \notin \overline{U}$. Then, for every $\lambda < \lambda_1^H(\Omega)$, problem (0.5) admits a weak positive solution.*

Let us note that Theorem 4.2 applies in particular for $\Omega = \mathbf{R}^2 \setminus \Gamma_\delta$, $\delta \in (0, 1]$.

Proof. If $\Omega = \Omega_\theta$ then the result has been already discussed in Remark 3.7. Suppose that $U \setminus \Omega_\theta \neq \emptyset$. By the assumptions on the domain, there exist $r_0, r_\infty > 0$ such that $\Omega \cap B_r = \Omega_\theta \cap B_r$ for every $r \in (0, r_0)$, and $\Omega \setminus B_r = \Omega_\theta \setminus B_r$ for every $r > r_\infty$. Hence, by Lemma 4.1 and by (3.2), we obtain $c_0(\Omega) = c_\infty(\Omega) = c(\Omega_\theta)$. We claim that $c(\Omega) < c(\Omega_\theta)$, and therefore, by the previous equalities, the result follows as an application of Theorem 3.1. Suppose by contradiction, that $c(\Omega) = c(\Omega_\theta)$. Let $u_\theta \in D_0^1(\Omega_\theta)$ be the solution to (0.5) on Ω_θ obtained as critical point of I in $H_0^1(\Omega_\theta)$ at the mountain pass level $c(\Omega_\theta)$. Then, setting $M = \{u \in D_0^1(\Omega) : I'(u)u = 0,\ u \neq 0\}$ and $M_\theta = \{u \in D_0^1(\Omega_\theta) : I'(u)u = 0,\ u \neq 0\}$, we have that $I(u_\theta) = c(\Omega_\theta) = \inf_{M_\theta} I = \inf_M I$. Therefore, by $(g4)$, u_θ solves problem (0.5) on Ω. In particular $u = 0$ in $U \setminus \Omega_\theta$, contradicting the maximum principle. □

Let us consider now the case of a domain Ω such that $0 \in \partial\Omega$ is a cusp point, that is, there exists a unit versor $\nu \in \mathbf{R}^2$ such that for every $\theta > 0$ there exists $r > 0$ for which $\Omega \cap B_r \subset \Omega_{\theta,\nu}$, where $\Omega_{\theta,\nu} = \{x \in \mathbf{R}^2 : x \cdot \nu > |x| \cos \frac{\theta}{2}\}$. The estimate of $c_0(\Omega)$ will be obtained by using the following result.

Lemma 4.3 *$c(\Omega_\theta) \to +\infty$ as $\theta \to 0_+$.*

Proof. Firstly we note that, by (4.1) and by Remark 3.7, given $\lambda \in \mathbf{R}$, there exists $\theta_0 \in (0, 2\pi]$ such that for $\theta \in (0, \theta_0]$, the functional I on $D_0^1(\Omega_\theta)$ admits a mountain pass geometry and a critical point u_θ at the mountain pass level $c(\Omega_\theta)$. By the dilation invariance, we may suppose that $\sup_{j \in \mathbf{Z}} \int_{\Omega_\theta \cap A_j} \frac{|u_\theta|^p}{|x|^2} = \int_{\Omega_\theta \cap A_0} \frac{|u_\theta|^p}{|x|^2}$, where $A_j = \{x \in \mathbf{R}^2 : \rho^j < |x| < \rho^{j+1}\}$. Using (3.1), we have that for $0 < \theta' < \theta$, $c(\Omega_\theta) \le c(\Omega_{\theta'})$ and then $\lim_{\theta \to 0} c(\Omega_\theta) = \sup_{\theta > 0} c(\Omega_\theta)$. Suppose by contradiction that $\sup_{\theta > 0} c(\Omega_\theta) < \infty$. Then, by (3.4), we infer that $\sup_{\theta > 0} \|u_\theta\|_{D_0^1(\Omega_\theta)} < \infty$, and thus, up to a subsequence, $u_\theta \to u_0$ weakly in $D_0^1(\Omega_{\theta_0})$. Since $\operatorname{supp} u_\theta = \overline{\Omega}_\theta$, and $\theta \to 0$, we have that $u_0 = 0$. If $u_\theta \to 0$ strongly in $D_0^1(\Omega_{\theta_0})$, then $c(\Omega_\theta) \to 0$, a contradiction. Hence, $\liminf \|u_\theta\|_{D_0^1(\Omega_\theta)} > 0$, and then, arguing as in the proof of Lemma 3.5, $\liminf_{\theta \to 0} \int_{\Omega_\theta \cap A_0} \frac{|u_\theta|^p}{|x|^2} > 0$. This contradicts the fact that, since $u_\theta \to 0$ weakly, $\int_{\Omega_\theta \cap A_0} \frac{|u_\theta|^p}{|x|^2} \to 0$. This concludes the proof. □

Hence we have the following result.

Proposition 4.4 *If Ω is a domain in $\mathbf{R}^2$ with $0 \in \partial\Omega$ and such that 0 is a cusp point, then $c_0(\Omega) = +\infty$.*

Proof. By definition of cusp point, there exists a unit versor $\nu \in \mathbf{R}^2$ such that for every $\theta > 0$ one can find $r > 0$ for which $\Omega \cap B_r \subset \Omega_{\theta,\nu}$. Then $c(\Omega \cap B_r) \ge c(\Omega_{\theta,\nu}) = c(\Omega_\theta)$. Hence, by (3.2) and (4.1), $c_0(\Omega) = +\infty$. □

A result similar to Proposition 4.4 holds in the case of an unbounded domain Ω such that $\Omega \setminus B_r$ is contained in a strip, for some $r > 0$.

Proposition 4.5 *Let Ω be a domain in $\mathbf{R}^2$ satisfying the following property: there exist $r > 0$, unit versors $\nu_0, \nu_1 \in \mathbf{R}^2$ with $\nu_0 \cdot \nu_1 = 0$, and a bounded interval I such that $\Omega \setminus B_r \subset \{a\nu_0 + b\nu_1 : a \in \mathbf{R},\ b \in I\}$. Then $c_\infty(\Omega) = +\infty$.*

The proof goes on as for Proposition 4.4. Hence, if Ω is a domain in $\mathbf{R}^2$ satisfying the assumptions of Propositions 4.4 and 4.5, for $\lambda < \lambda_1^H(\Omega)$ and $g \in C^1(\mathbf{R})$ satisfying $(g1)$–$(g4)$, the existence of a weak positive solution to (0.5) is guaranteed by Theorem 3.1.

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