On the existence of homoclinic orbits for the asymptotically periodic Duffing equation

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Abstract. Using variational methods, we show the existence of a homoclinic orbit for the Duffing equation $-\ddot{u}+u=a(t)|u|^{p-1}u$, where $p>1$ and $a\in L^\infty(\mathbb{R})$ is a positive function of the form $a=a_0+a_\infty$ with a_∞ periodic, and $a_0(t)\to 0$ as $t\to\pm\infty$ satisfying suitable conditions. Under the same assumptions on a, we also prove that the perturbed equation $-\ddot{u}+u=a(t)|u|^{p-1}u+\alpha(t)g(u)$ admits a homoclinic orbit whenever $g\in C(\mathbb{R})$ satisfies $g(u)=O(u)$ as $u\to 0$ and $\alpha\in L^\infty(\mathbb{R})$, $\alpha(t)\to 0$ as $t\to\pm\infty$ and $\|\alpha\|_{L^\infty}$ is sufficiently small.

Key Words. Duffing equation, homoclinic orbits, critical points, locally compact case, minimax arguments.

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Introduction

In this article we are concerned with the asymptotically periodic Duffing equation in $\mathbb{R}$, that is

$$-\ddot{u} + u = a(t)|u|^{p-1}u \tag{D}$$

where $p > 1$ and $a\colon \mathbb{R} \to \mathbb{R}$ satisfies:

$(a1)$ $a \in L^\infty(\mathbb{R})$, $\inf a > 0$,

$(a2)$ $a = a_\infty + a_0$, with a_∞ T-periodic and $a_0(t) \to 0$ as $t \to \pm\infty$.

Noting that 0 is a hyperbolic rest point for (D), we look for homoclinic orbits to 0, namely non trivial solutions to (D) such that $u(t) \to 0$ and $\dot{u}(t) \to 0$ as $t \to \pm\infty$.

The homoclinic problem for equation (D), possibly with a more general nonlinearity, as well as the analogous subcritical elliptic problem on $\mathbb{R}^n$, has been successfully studied with variational methods by several authors, for different kinds of behavior of the coefficient a.

The main feature of the problem is a lack of global compactness, due to the unboundedness of the domain, and to the failure of the compact embedding of $H^1(\mathbb{R})$ into $L^p(\mathbb{R})$.

The existence of homoclinic solutions for (D) strongly depends on the behaviour of a. For instance, if a is a positive constant or is periodic, the invariance under translation permits to recover some compactness and to obtain existence results (see, e.g., [8], [15]). Instead, if a is monotone and non constant, one can easily see that that (D) has no homoclinic orbit.

On the other hand, adopting a different viewpoint like in [4], and considering the whole class of equations like (D) (or more general equations) with coefficients a satisfying $(a1)$, the existence of infinitely many homoclinics turns out to be a "generic" property (see also [3]). In particular, in [4] it is shown how, starting from a given function a satisfying $(a1)$ it is possible to construct a suitable L^∞ small perturbation α, in order that the perturbed equation $-\ddot{u}+u = (a+\alpha)|u|^{p-1}u$ admits infinitely many solutions in $H^1(\mathbb{R})$. Clearly, this kind of approach is not always useful if we want to handle with a specific equation (D) without modifying the coefficient a.

In the asymptotically periodic case, namely when a satisfies $(a1)$–$(a2)$, the problem can be studied by using concentration-compactness arguments and a comparison with the problem at infinity

$$-\ddot{u} + u = a_\infty(t)|u|^{p-1}u \tag{D_∞}$$

can be useful to prove existence of homoclinic solutions for (D).

In fact if the ground state level m of (D) is strictly lower that the ground state level m_∞ of (D_∞), then the Palais Smale condition holds at level m and (D) admits a homoclinic orbit characterized as ground state solution (see [15], [21], [12]).

However, if $m = m_\infty$, a different variational procedure has to be set up. This has been developed in [6], [7], [9], when a_∞ is a positive constant. In fact, the argument followed in these papers requires a precise knowledge of the critical set of the problem at infinity, that is possible because it admits a unique positive solution (up to translations). This fact is guaranteed when (D_∞) is autonomous (in the elliptic case this uniqueness result is proved in [14]), while in the non constant periodic case this kind of argument may fail.

We point out that the homoclinic problem for an asymptotically autonomous Duffing-like equation has been tackled also with perturbative methods (see [5], [18], [21]), or also using a geometrical approach, as in [13]. In all these works the fact that the problem at infinity is autonomous is fundamental in the argument followed there.

When a_∞ is a periodic, non constant, positive function, a deeper analysis of the local compactness properties of the variational problem associated to (D_∞) can be based on the study of the structure of the set of the homoclinics of (D_∞).

This argument involves some techniques developed in recent years, starting from [10] and [19] (see also [11] for the PDE case), to study certain aspects of the dynamics of (D_∞) and, more precisely, to detect a possible chaotic behavior due to the presence of so-called "multibump" solutions (see [19]). This rich structure of the set of solutions of (D_∞) appears as soon as a suitable non degeneracy condition on the set of the homoclinics is fulfilled.

This non degeneracy condition, stated in a precise way in Section 1, is a weaker version of the classical transversal intersection property between the stable and unstable manifolds, see [19], [17], [22]. Moreover it is suited to a variational approach to the problem, and, differently from the standard geometrical approach, permits to study Duffing-like equations with a more general time dependence, including the asymptotically periodic one, as done in [2], [16], [1].

The use of this information was already employed in [20] to treat the asymptotically periodic case (in a more general setting) when a_∞ is non constant and $\|a_0\|_{L^\infty}$ is small.

In the present paper, using some of the above mentioned arguments, we prove the following result.

Theorem 0.1 *Let* $a: \mathbb{R} \to \mathbb{R}$ *satisfy* $(a1)$–$(a2)$ *and let* $p > 1$. *If, in addition,*

$(a3)$ *there exists* $\theta > 2$ *such that* $\liminf_{|t|\to\infty} a_0(t)e^{\theta|t|} > -\infty$, *or*

$(a4)$ $a \geq 2^{-\frac{p-1}{2}} a_\infty$ *on* $\mathbb{R}$,

then (D) *admits a homoclinic orbit.*

We note that this result has a global character, i.e., no perturbation parameter appears, and is free from any non degeneracy condition.

Indeed, we are able to prove that the failure of the non degeneracy condition on (D_∞), which is responsible of the multibump dynamics both for (D_∞), and for (D), implies, and actually is equivalent to the uniqueness of the non zero critical level for the functional associated to the homoclinic problem for (D_∞). Then the procedure developed in [6] or [9] can be applied again, using one of the additional assumptions $(a3)$ or $(a4)$, to obtain the existence of a homoclinic for (D).

Finally, we point out that the existence result stated in Theorem 0.1 is stable with respect to small L^∞ perturbations that vanish at infinity. Precisely we can show:

Theorem 0.2 *Let* $a: \mathbb{R} \to \mathbb{R}$ *satisfy* $(a1)$–$(a2)$ *and either* $(a3)$ *or* $(a4)$. *Let* $g \in C(\mathbb{R})$ *be such that* $g(u) = O(u)$ *as* $u \to 0$. *Then there exists* $\bar{\varepsilon} = \bar{\varepsilon}(a, g) > 0$ *such that for any* $\alpha \in L^\infty(\mathbb{R})$ *with* $\|\alpha\|_{L^\infty} \leq \bar{\varepsilon}$ *and* $\alpha(t) \to 0$ *as* $|t| \to \pm\infty$, *the equation*

$$-\ddot{u} + u = a(t)|u|^{p-1}u + \alpha(t)g(u)$$

admits a homoclinic orbit.

The paper is organized as follows. In Section 1 we introduce the variational setting useful to study the homoclinic problem for (D) and we recall some known facts. At the end of this Section we also state the non degeneracy condition $(*)$ on the problem at infinity (D_∞), that will discriminate the argument, according that it does hold or not. Then, in Section 2, we consider the case in which $(*)$ holds, while in Section 3 we study the case in which $(*)$ does not hold. In both the alternative cases we conclude that the equation (D) admits a homoclinic solution, under the assumptions of Theorem 0.1. Finally, in Section 4 we discuss further perturbative results, proving Theorem 0.2.

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1 Preliminaries

In this section we introduce the variational setting useful to study the homoclinic problem associated to (D).

Let $X = H^1(\mathbb{R})$ be the standard Sobolev space endowed with the inner product $\langle u, v\rangle = \int_{\mathbb{R}}(\dot u\dot v + uv)$ and norm $\|u\| = \langle u, u\rangle^{\frac{1}{2}}$. For every $u \in X$ let

$$\varphi(u) = \tfrac{1}{2}\|u\|^2 - \tfrac{1}{p+1}\int_{\mathbb{R}} a|u|^{p+1}.$$

It is well known, by the Sobolev embeddings, that $\varphi \in C^2(X, \mathbb{R})$ and the non zero critical points of φ are exactly the homoclinic orbits of (D).

Remark 1.1 The functional φ has a mountain pass geometry, since $\varphi(u) = \frac{1}{2}\|u\|^2 + o(\|u\|^2)$ as $\|u\| \to 0$, and for every $u \neq 0$, $\varphi(\lambda u) \to -\infty$ as $\lambda \to +\infty$. In particular, the mountain pass level of φ is given by

$$c = \inf_{\gamma\in\Gamma}\sup_{s\in[0,1]} \varphi(\gamma(s))$$

where $\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = 0,\ \varphi(\gamma(1)) < 0\}$. We note that for every $u \in X \setminus \{0\}$ there exist $s_0(u) > 0$ such that $\varphi(s_0(u)u) < 0$ and a unique $s(u) \in (0, s_0(u))$ such that $\frac{d}{ds}\varphi(su)|_{s=s(u)} = 0$ and hence $\varphi(s(u)u) = \max_{s\geq 0}\varphi(su)$. Then

$$c = \inf_{u\in X\setminus\{0\}}\sup_{s\geq 0}\varphi(su).$$

Remark 1.2 Setting $\mathcal{K} = \{u \in X : \varphi'(u) = 0,\ u \neq 0\}$ we observe that:
(i) If $\mathcal{K} \neq \emptyset$ then $c = \inf_{\mathcal{K}}\varphi = (\frac{1}{2} - \frac{1}{p+1})\inf_{\mathcal{K}}\|u\|^2$ and $\inf_{\mathcal{K}}\|u\|_{L^\infty} > 0$.
(ii) If $u \in \mathcal{K}$ and $\varphi(u) = c$ then $\pm u > 0$.
(iii) If $u \in \mathcal{K}$ and $u > 0$ then $\lim_{t\to\pm\infty} e^{\pm t}u(t) \in (0, +\infty)$ (see, e.g., [7]).

Remark 1.3 There exists $\bar\delta > 0$ such that for any interval $I \subset \mathbb{R}$ with length $|I| \geq 1$ we have

$$\text{if } \|u\|_{L^\infty(I)} \leq 2\bar\delta \text{ then } \varphi_I(u) \geq \tfrac{1}{4}\|u\|_I^2 \text{ and } \varphi_I'(u)v \geq \tfrac{1}{2}\|u\|_I\|v\|_I$$

where $\|u\|_I^2 = \int_I(\dot u^2 + u^2)$ and $\varphi_I(u) = \frac{1}{2}\|u\|_I^2 - \frac{1}{p+1}\int_I a|u|^{p+1}$. By the Sobolev imbedding Theorem, let $\bar\rho > 0$ be such that $\|u\|_I \leq \bar\rho$ implies $\|u\|_{L^\infty(I)} \leq \bar\delta$ for every interval I with $|I| \geq 1$.

Remark 1.4 Fixed any $\tau \in \mathbb{R}$, we set $I_\tau^+ = [\tau, +\infty)$, $I_\tau^- = (-\infty, \tau]$ and for all $x \in \mathbb{R}$ with $|x| \leq \bar{\delta}$, $\mathcal{U}_{\tau^\pm,x} = \{u \in H^1(I_\tau^\pm) \mid u(\tau) = x,\ \|u\|_{L^\infty(I_\tau^\pm)} \leq \bar{\delta}\}$. Then, the minimum problem

$$\min\{\varphi_{I_\tau^\pm}(u)\ :\ u \in \mathcal{U}_{\tau^\pm,x}\}$$

admits a unique solution $u_{\tau^\pm,x}$ for any $\tau \in \mathbb{R}$ and $|x| \leq \bar{\delta}$, depending continuously on x. Indeed, by the choice of $\bar{\delta}$, we have that $\varphi_{I_\tau^\pm}$ is strictly convex on the closed, convex set $\mathcal{U}_{\tau^\pm,x}$. Note that $u_{\tau^\pm,x}$ is the unique solution of (D) on $I_\tau^\pm$ which verifies the conditions $u_{\tau^\pm,x}(\tau) = x$ and $\|u_{\tau^\pm,x}\|_{L^\infty(I_\tau^\pm)} \leq \bar{\delta}$. Then, we infer that for any $\tau \in \mathbb{R}$ and $|x| \leq \bar{\delta}$ there results

$$|u_{\tau^\pm,x}(t)| \leq \bar{\delta} e^{-\frac{|t-\tau|}{4}}, \quad \forall t \in I_\tau^\pm. \tag{1.1}$$

Now we list some properties of Palais Smale sequences (briefly PS sequences) for φ, i.e., sequences $(u_n) \subset X$ such that $(\varphi(u_n))$ is bounded and $\|\varphi'(u_n)\| \to 0$. All the following results were stated, e.g., in [1] and [17], to which we refer for the proofs.

Remark 1.5 Any PS sequence (u_n) for φ is bounded and $\liminf \varphi(u_n) \geq 0$. Moreover, if $\limsup \varphi(u_n) < c$ then $u_n \to 0$. Furthermore, if (u_n) is a PS sequence for φ with $\limsup \|u_n\|_{L^\infty} \leq 2\bar{\delta}$ then $u_n \to 0$.

Lemma 1.6 *Let (u_n) be a PS sequence for φ weakly converging to $u \in X$. Then:*

(i) $\varphi(u) \leq \liminf \varphi(u_n)$ *and* $\varphi'(u) = 0$*,*

(ii) $(u_n - u)$ *is a PS sequence for φ with* $\limsup \varphi(u_n - u) \leq \liminf \varphi(u_n) - \varphi(u)$*.*

By Lemma 1.6 we are lead to study PS sequences that converge to 0 weakly in X and we have the following result.

Lemma 1.7 *If $u_n \to 0$ weakly in X and $\varphi'(u_n) \to 0$ then $u_n \to 0$ strongly in $H^1_{loc}(\mathbb{R})$ and the following alternative holds: either*

(i) $u_n \to 0$ strongly in X, or

(ii) there exists $(t_n) \subset \mathbb{R}$ such that $|t_n| \to \infty$ and $\liminf |u_n(t_n)| \geq 2\bar{\delta}$*.*

Hence, according to Lemma 1.7 we lose compactness of those PS sequences (u_n) which "carry mass" at infinity, in the sense explained in the case (ii). In order to obtain compactness results it is therefore useful to introduce the function $T\colon X \to \mathbb{R} \cup \{-\infty\}$ defined in the following way:

$$T(u) = \begin{cases} \sup\{t \in \mathbb{R} : |u(t)| = \bar{\delta}\} & \text{if } \|u\|_{L^\infty} \geq \bar{\delta} \\ -\infty & \text{otherwise .} \end{cases}$$

Then, arguing as in [17], we obtain:

Lemma 1.8 *Let (u_n) be a PS sequence for φ weakly converging to $u \in X$. If $(T(u_n))$ is bounded then $u \neq 0$ and $T(u_n) \to T(u)$. If in addition $\lim \varphi(u_n) \in [c, 2c)$ then $u_n \to u$ strongly in X.*

On the other hand, if (u_n) is a PS sequence for φ weakly converging to 0, with $(T(u_n))$ unbounded, we can follow the sequence $(u_n(\cdot + T(u_n)))$ that is a PS sequence for the functional corresponding to the problem "at infinity" (D_∞).

More precisely, let

$$\varphi_\infty(u) = \tfrac{1}{2}\|u\|^2 - \tfrac{1}{p+1}\int_{\mathbb{R}} a_\infty |u|^{p+1}$$

and $\mathcal{K}_\infty = \{u \in X : \varphi_\infty'(u) = 0,\ u \neq 0\}$. Note that all the above results, stated for φ clearly hold even for φ_∞. In particular, setting $\Gamma_\infty = \{\gamma \in C([0,1], X) : \gamma(0) = 0,\ \varphi_\infty(\gamma(1)) < 0\}$, the mountain pass level for φ_∞ is given by $c_\infty = \inf_{\gamma\in\Gamma_\infty} \sup_{s\in[0,1]} \varphi_\infty(\gamma(s)) = \inf_{u\in X\setminus\{0\}} \sup_{s\geq 0} \varphi_\infty(su)$.

Remark 1.9 Thanks to the invariance under translation in the problem (D_∞), one can easily show that there exists $u_\infty \in \mathcal{K}_\infty$ with $\varphi_\infty(u_\infty) = c_\infty$. Moreover, arguing as in Remark 1.1, there exists $s_0 > 0$ such that, setting $\gamma_\infty(s) = s_0 s u_\infty$ for every $s \in [0,1]$, we have $\gamma_\infty \in \Gamma_\infty$ and

(i) $\gamma_\infty([0,1]) \subset \{\varphi_\infty \leq c_\infty\}$,

(ii) for any $r \in (0, \bar{\rho})$ there exists $h_r > 0$ such that if $\gamma_\infty(s) \notin B_r(u_\infty)$ then $\varphi_\infty(\gamma_\infty(s)) \leq c_\infty - h_r$.

Since $\gamma_\infty([0,1])$ is compact, for all $\delta > 0$ there exists R_δ such that

(iii) $\max_{s\in[0,1]} \|\gamma_\infty(s)\|_{L^\infty(\mathbb{R}\setminus[-R_\delta,R_\delta])} \leq \delta$.

Then, for any $h > 0$ there exists $j_h \in \mathbb{N}$ such that

(iv) $\max_{s\in[0,1]} |\varphi(\gamma_\infty(s)(\cdot - jT)) - \varphi_\infty(\gamma_\infty(s)(\cdot - jT))| \leq h$ for all $|j| \geq j_h$.

In particular (iv) implies that $c \leq c_\infty$.

Finally, the study of PS sequences for φ can be completed by the following results (see, e.g., [1] for the proofs).

Lemma 1.10 *If $(u_n) \subset X$ is a PS sequence for φ weakly converging to 0, then (u_n) is a PS sequence for φ_∞ and $\limsup \varphi(u_n) = \limsup \varphi_\infty(u_n)$.*

Lemma 1.11 *If $(u_n) \subset X$ is a PS sequence for φ at level b then there exist $u \in \mathcal{K} \cup \{0\}$, $v_1, \dots, v_j \in \mathcal{K}_\infty$, with $j \in \mathbb{N} \cup \{0\}$, and sequences $(t_n^1), \dots, (t_n^j) \subset \mathbb{R}$ with $\lim_{n\to\infty} |t_n^i| = \infty$ $(1 \leq i \leq j)$ and $\lim_{n\to\infty}(t_n^{i+1} - t_n^i) = +\infty$ $(1 \leq i \leq j-1)$, such that, up to a subsequence, $\|u_n - u - \sum_{i=1}^j v_i(\cdot - t_n^i)\| \to 0$ as $n \to \infty$. Moreover $b = \varphi(u) + \sum_{i=1}^j \varphi_\infty(v_i)$.*

By Lemma 1.11, the set of PS sequences for φ can be described in terms of the critical set $\mathcal{K}_\infty$ of the functional at infinity. Hence, as we will see, the topological structure of $\mathcal{K}_\infty$ reflect possibly on compactness properties for φ. In particular the non connectedness of $\mathcal{K}_\infty$, expressed by the following condition

$$(*) \qquad \text{there exists } \bar{h} \in (0, c_\infty) \text{ such that } T(\mathcal{K}_\infty \cap \{\varphi_\infty \leq c_\infty + \bar{h}\}) \neq \mathbb{R},$$

will allow us to recover compactness for φ_∞. On the other hand, the failure of $(*)$ can be used to get a precise knowledge of the set of critical levels of φ_∞, that, together with Lemma 1.11, gives information on the values at which the functional φ satisfies the PS condition.

Therefore, we will adopt a different strategy, according that the condition $(*)$ holds or does not, and in both cases we will prove that (D) admits at least a homoclinic solution. Precisely, in Section 2 we show that, under the assumption $(*)$, the equation (D) admits infinitely many homoclinics, while in Section 3 we prove the existence of a non trivial critical point for φ, by following a minimax procedure already introduced in [6].

2 If $(*)$ holds

In this Section we show that if the assumption $(*)$ is fulfilled then the equation (D) admits a homoclinic. The procedure developed here shows in fact that if $(*)$ holds then (D) actually admits infinitely many homoclinics.

Suppose that condition $(*)$ holds. Then, by periodicity and Lemma 1.8, there exists $\tau \in [0, T)$, $\eta \in (0, \frac{T}{2})$ and $\nu > 0$ such that

$$\text{if } \varphi_\infty(u) \leq c_\infty + \bar{h} \text{ and } T(u) \in [\tau - \eta, \tau + \eta] \text{ then } \|\varphi'_\infty(u)\| \geq \nu. \qquad (2.1)$$

For all $j \in \mathbb{N}$, let us denote $\tau_j^- = \tau + \eta + jT$ and $\tau_j^+ = \tau - \eta + (j+1)T$ and

$$K_j = \{u \in \mathcal{K}_\infty \cap \{\varphi_\infty \leq c_\infty + \bar{h}\} \,:\, T(u) \in [\tau_j^-, \tau_j^+]\}.$$

Note that by (2.1) we have $\mathcal{K}_\infty \cap \{\varphi_\infty \leq c_\infty + \bar{h}\} = \cup_{j\in\mathbb{Z}} K_j$ and, by Lemma 1.8, each K_j is compact. Moreover, arguing e.g. as in [17], one can prove that there exists $\bar{r} \in (0, \frac{\bar{\rho}}{4})$ such that

$$\mathrm{dist}(K_j, \mathcal{K}_\infty \cap \{\varphi_\infty \leq c_\infty + \bar{h}\} \setminus K_j) \geq 2\bar{r}, \quad \forall j \in \mathbb{Z}.$$

In other words the assumption $(*)$ together with the recurrence properties of the function a_∞ give information about the critical set under the level $c_\infty + \bar{h}$. This set turns out to be the union of the uniformly disjoint compact sets K_j defined above.

Remark 2.1 Since K_0 is compact and $u \in K_j$ if and only if $u(\cdot + jT) \in K_0$, for all $j \in \mathbb{Z}$, there exists $\bar{R} \geq 1$ such that

$$\sup_{u\in K_j} \|u\|_{L^\infty(\mathbb{R}\setminus[\tau_j^- - \bar{R}, \tau_j^+ + \bar{R}])} \leq \frac{\bar{\delta}}{4}, \quad \forall j \in \mathbb{Z}.$$

Moreover, since $\bar{r} < \frac{\bar{\rho}}{4}$, we have that

$$\sup_{u\in B_{\bar{r}}(K_j)} \|u\|_{L^\infty(\mathbb{R}\setminus[\tau_j^- - \bar{R}, \tau_j^+ + \bar{R}])} \leq \frac{\bar{\delta}}{2}, \quad \forall j \in \mathbb{Z}$$

from which it follows that φ_∞ satisfies the PS condition on every $B_{\bar{r}}(K_j)$.

The structure of the critical set of the functional φ_∞ reflects on the PS sequences of the functional φ as we see in the next Lemma.

Lemma 2.2 *For any $r \in (0, \bar{r})$, there exist $j_r \in \mathbb{N}$ and $\nu_r > 0$ such that $\|\varphi'(u)\| \geq \nu_r$ for any $u \in B_{\bar{r}}(K_j) \cap \{\varphi \leq c_\infty + \bar{h}\} \setminus B_r(K_j)$ with $|j| \geq j_r$.*

Proof. Arguing by contradiction, there exists a sequence $u_n \in B_{\bar{r}}(K_{j_n}) \cap \{\varphi \leq c_\infty + \bar{h}\} \setminus B_r(K_{j_n})$ with $j_n \to \infty$ with $\varphi'(u_n) \to 0$. Then, since by Remark 2.1, we have $\|u_n\|_{L^\infty(\mathbb{R}\setminus[\tau_{j_n}^- - \bar{R}, \tau_{j_n}^+ + \bar{R}])} \leq \frac{\bar{\delta}}{2}$, we obtain $u_n(\cdot + j_n T) \to u$ strongly in X. Moreover, by Lemma 1.10, $u \in \mathcal{K}_\infty$ and $\varphi_\infty(u) \leq c_\infty + \bar{h}$. Therefore $u \in \cup_{j\in\mathbb{Z}} K_j$, a contradiction. $\square$

Then for $|j| \geq j_r$ the gradient of φ is uniformly bounded from below by the positive constant μ_r on anyone of the annulus type regions $B_{\bar{r}}(K_j) \cap \{\varphi \leq c_\infty + \bar{h}\} \setminus B_r(K_j)$. Moreover, by Remark 2.1 and Lemma 1.7, the PS condition holds in anyone of the sets $B_{\bar{r}}(K_j)$.

Next Remark says that inside $B_{\bar{r}}(K_j)$ a well characterized local mountain pass structure for the functional φ is defined. These three properties will be the key to prove Theorem 2.4 below.

Remark 2.3 By Remark 1.9, we have that there exists $u_\infty \in \mathcal{K}_\infty$ with $\varphi_\infty(u_\infty) = c_\infty$. By the previous properties, we can assume that $u_\infty \in K_0$. Then, setting $\gamma_j(s) = \gamma_\infty(s)(\cdot - jT)$, we obtain

(i) $\gamma_j([0,1]) \subset \{\varphi_\infty \leq c_\infty\}$,

(ii) for any $r \in (0, \bar{r})$ there exists $h_r > 0$ such that $\varphi_\infty(\gamma_j(s)) \leq c_\infty - h_r$ for any $\gamma_j(s) \notin B_r(K_j)$.

Moreover, we can assume that $\bar{R}$ in Remark 2.1 is so large that
(iii) $\max_{s\in[0,1]} \|\gamma_j(s)\|_{L^\infty(\mathbb{R}\setminus[\tau_j^- - \bar{R}, \tau_j^+ + \bar{R}])} \leq \frac{\bar{\delta}}{4}$ for all $j \in \mathbb{Z}$.
Finally, for any $r \in (0, \bar{r})$, setting $\bar{h}_r = \frac{1}{4}\min\{\bar{h}, h_r, r\nu_r, -2\varphi_\infty(\gamma_\infty(1))\}$, there exists $\bar{\jmath}_r \geq j_r$ such that
(iv) $\max_{s\in[0,1]} |\varphi(\gamma_j(s)) - \varphi_\infty(\gamma_j(s))| \leq \bar{h}_r$ for all $|j| \geq \bar{\jmath}_r$.

Theorem 2.4 *If $(*)$ holds, then (D) admits infinitely many solutions. Precisely, for any $r \in (0, \frac{\bar{r}}{6})$ there exists $\tilde{\jmath}_r \geq \bar{\jmath}_r$, such that $\mathcal{K} \cap B_r(K_j) \neq \emptyset$ for any $|j| \geq \tilde{\jmath}_r$.*

Proof. Assume by contradiction that for all $\tilde{\jmath}_r \geq \bar{\jmath}_r$ there exists $j \in \mathbb{Z}$ with $|j| \geq \tilde{\jmath}_r$ such that $\mathcal{K} \cap B_{\bar{r}}(K_j) = \emptyset$. Then, since φ satisfies the PS condition in $B_{\bar{r}}(K_j)$, there exists $\mu_j > 0$ such that $\|\varphi'(u)\| \geq \mu_j$ for all $u \in B_{\bar{r}}(K_j)$. Let $\eta_j : [0,1] \times X \to X$ be the flow associated to the Cauchy problem

$$\begin{cases} \frac{d}{dt}\eta_j(t,u) = -\psi(\eta_j(t,u))\frac{\varphi'(\eta_j(t,u))}{\|\varphi'(\eta_j(t,u))\|}, \\ \eta_j(0,u) = u, \quad \forall u \in X, \end{cases}$$

where $\psi : X \to [0,1]$ is a locally Lipschitz continuous function such that $\psi(u) = 1$ for all $u \in B_{2r}(K_j)$ and $\psi(u) = 0$ for all $u \in X \setminus B_{3r}(K_j)$.
It is standard to check that φ decreases along the flow lines and that $X \setminus B_{3r}(K_j)$ is invariant under η_j. Moreover, since η_j sends bounded sets in

bounded sets, there exists $\bar{t} > 0$ such that for all $u \in B_r(K_j) \cap \{\varphi \le c_\infty + \frac{\bar{h}}{2}\}$ there exists $t \in (0, \bar{t}]$ such that $\eta_j(t, u) \notin B_{2r}(K_j)$. Hence, by Lemma 2.2, for any $u \in B_r(K_j) \cap \{\varphi \le c_\infty + \frac{\bar{h}}{2}\}$ we get $\varphi(\eta_j(\bar{t}, u)) \le \varphi(u) - r\nu_r$. Consider the path γ_j given by Remark 2.3. Then, setting $\bar{\gamma}_j(s) = \eta_j(\bar{t}, \gamma_j(s))$ for any $s \in [0,1]$, we obtain

$$\max_{s \in [0,1]} \varphi(\bar{\gamma}_j(s)) \le c_\infty - 3\bar{h}_r. \tag{2.2}$$

Indeed, if $\gamma_j(s) \notin B_r(K_j)$ then, by Remark 2.3 (ii) and (iv),

$$\varphi(\bar{\gamma}_j(s)) \le \varphi(\gamma_j(s)) \le \varphi_\infty(\gamma_j(s)) + \bar{h}_r \le c_\infty - h_r + \bar{h}_r \le c_\infty - 3\bar{h}_r.$$

Otherwise, if $\gamma_j(s) \in B_r(K_j)$, since by Remark 2.3 (i) and (iv) we have $\gamma_j([0,1]) \subset \{\varphi \le c_\infty + \bar{h}_r\} \subset \{\varphi \le c_\infty + \frac{\bar{h}}{2}\}$, we obtain

$$\varphi(\bar{\gamma}_j(s)) = \varphi(\eta_j(\bar{t}, \gamma_j(s))) \le \varphi(\gamma_j(s)) - r\nu_r \le c_\infty + \bar{h}_r - r\bar{\nu}_r \le c_\infty - 3\bar{h}_r.$$

Now, note that, by Remark 2.1, $\|u\|_{L^\infty(\mathbb{R} \setminus [\tau_j^- - \bar{R}, \tau_j^+ + \bar{R}])} \le \frac{\bar{\delta}}{2}$ for all $u \in B_{\bar{r}}(K_j)$. Then, since $X \setminus B_{\bar{r}}(K_j)$ is invariant under η_j, by Remark 2.3 (iii) we infer that $\|\bar{\gamma}_j(s)\|_{L^\infty(\mathbb{R} \setminus [\tau_j^- - \bar{R}, \tau_j^+ + \bar{R}])} \le \frac{\bar{\delta}}{2}$. Therefore, if we denote $\tau^\pm = \tau_j^\pm \pm \bar{R}$ and $x^\pm(s) = \bar{\gamma}_j(s)(\tau^\pm)$, $s \in [0,1]$, we can consider the function $u^\pm(s)(\cdot) = u_{\tau^\pm, x^\pm(s)}$ defined in Remark 1.4. Therefore it is well defined and continuous the path $\tilde{\gamma}_j : [0,1] \to X$ given by

$$\tilde{\gamma}_j(s)(t) = \begin{cases} u^-(s)(t), & \text{if } t \le \tau^-, \\ \bar{\gamma}_j(s)(t), & \text{if } \tau^- \le t \le \tau^+, \\ u^+(s)(t), & \text{if } \tau^+ \le t \end{cases} \quad \forall s \in [0,1].$$

By construction $\varphi(\tilde{\gamma}_j(s)) \le \varphi(\bar{\gamma}_j(s))$ for any $s \in [0,1]$. Moreover, by (1.1), taking $\bar{j}_r$ large enough, we obtain that

$$\max_{s \in [0,1]} |\varphi(\tilde{\gamma}_j(s)) - \varphi_\infty(\tilde{\gamma}_j(s))| \le \bar{h}_r.$$

Therefore, by (2.2), we conclude that $\max_{s \in [0,1]} \varphi_\infty(\tilde{\gamma}_j(s)) \le c_\infty - 2\bar{h}_r < c_\infty$ which is a contradiction since $\tilde{\gamma}_j \in \Gamma_\infty$. Indeed, $0 \notin B_{3r}(K_j)$ and moreover $\varphi_\infty(\tilde{\gamma}_j(1)) \le \varphi_\infty(\gamma_j(1)) + 2\bar{h}_r < 0$. □

Remark 2.5 Note that Theorem 2.4 holds true only under the assumptions $(a1)$ and $(a2)$ on a and moreover it can be proved for more general nonlinearities than the power $|u|^{p-1}u$, see e.g. [3].

3 If $(*)$ does not hold

In this section we discuss the existence of a non trivial critical point for the functional φ under the assumption that *condition* $(*)$ *is not satisfied*.

The most relevant consequence of the failure of $(*)$ is the fact that c_∞ is the only non zero critical level of φ_∞. More precisely, the following facts hold.

Lemma 3.1 *For every $\tau \in \mathbb{R}$ there exists a unique pair $\pm u_\tau \in \mathcal{K}_\infty$ such that $T(\pm u_\tau) = \tau$. Moreover, if $u \in \mathcal{K}_\infty$ then $\varphi_\infty(u) = c_\infty$ and $\pm u > 0$.*

Proof. Since $(*)$ does not hold, $T(\mathcal{K}_\infty \cap \{\varphi_\infty < c_\infty + \bar{h}\}) = \mathbb{R}$ for every $\bar{h} \in (0, c_\infty)$. Then, given $\tau \in \mathbb{R}$, there exists $(u_n) \subset \mathcal{K}_\infty$ such that $T(u_n) = \tau$ and $\varphi_\infty(u_n) < c_\infty + \frac{1}{n}$ for every $n \in \mathbb{N}$. By Remark 1.2 (i) and Lemma 1.8, the sequence (u_n) is precompact and thus there exists $u \in \mathcal{K}_\infty$ such that $T(u) = \tau$ and $\varphi_\infty(u) = c_\infty$. In particular $u(\tau) = \pm\bar{\delta}$, $\|u\|_{L^\infty([\tau,+\infty))} \le \bar{\delta}$. Supposing $u(\tau) = \bar{\delta}$, by Remark 1.4, we obtain $u = u_{\tau^+,\bar{\delta}}$ on $[\tau, +\infty)$ from which we conclude that there exists a unique $u_\tau \in \mathcal{K}_\infty \cap \mathcal{U}_{\tau^+,\bar{\delta}}$. Moreover, by Remark 1.2 (ii), $u_\tau > 0$. An analogous argument holds if $u(\tau) = -\delta$. Finally, since the equation (D) is odd, if $u \in \mathcal{K}_\infty$, $T(u) = \tau$ and $u(\tau) = -\delta$, then $u = -u_\tau$. $\square$

Remark 3.2 (i) According to Lemma 3.1, by the uniqueness (up to a sign) of the critical point u_τ, we infer that $u_\tau(\cdot - jT) = u_{\tau+jT}$ for every $\tau \in \mathbb{R}$ and $j \in \mathbb{Z}$.

(ii) The mapping $\tau \mapsto u_\tau$ is continuous from $\mathbb{R}$ into X. Indeed, if $\tau_n \to \tau$, there exists $u \in X$ such that, for a subsequence, $u_{\tau_{n_j}} \to u$ weakly in X. Moreover, by Lemma 1.8, $u \in \mathcal{K}_\infty$, $T(u) = \tau$, and $u_{\tau_{n_j}} \to u$ strongly. Hence, by uniqueness, $u = u_\tau$, and $u_{\tau_n} \to u_\tau$.

(iii) By Lemma 3.1, since $\varphi_\infty(u_\tau) = c_\infty$ and $\varphi'_\infty(u_\tau)u_\tau = 0$, it follows that $\|u_\tau\|^2 = \int_{\mathbb{R}} a_\infty u_\tau^{p+1} = 2c_\infty \frac{p+1}{p-1}$ for every $\tau \in \mathbb{R}$.

As important consequence of Lemma 3.1, the following compactness result holds.

Corollary 3.3 *If $(u_n) \subset X$ is a PS sequence for φ at a level $b \in (c_\infty, c+c_\infty)$ then (u_n) is precompact.*

Proof. By Remark 1.5, we may assume that $u_n \to u$ weakly in X. If $u = 0$, by Lemma 1.10, (u_n) is a PS sequence also for φ_∞ at level b. By Corollary 1.11 and Lemma 3.1, $b = jc_\infty$ for some integer $j \geq 0$, in contradiction with the assumption $b \in (c_\infty, c + c_\infty)$, since, by Remark 1.9 (iv), $c \leq c_\infty$. Hence $u \neq 0$ and, by Lemma 1.6, $u \in \mathcal{K}$ with $\varphi(u) \geq c$ (see Remark 1.2 (i)), and $(u_n - u)$ is a PS sequence for φ weakly converging to 0. Then, by Lemma 1.10, $(u_n - u)$ is a PS sequence for φ_∞ and, since $\limsup \varphi_\infty(u_n - u) = \limsup \varphi(u_n - u) \leq b - c < c_\infty$, by Remark 1.5, $u_n \to u$ strongly in X. $\square$

We point out that for every $u \neq 0$

$$\max_{s\geq 0} \varphi(su) = (\tfrac{1}{2} - \tfrac{1}{p+1}) J(u)^{\frac{p+1}{p-1}} \tag{3.1}$$

where

$$J(u) = \frac{\|u\|^2}{\left(\int_{\mathbb{R}} a|u|^{p+1}\right)^{2/(p+1)}} \ .$$

We note that $J \in C^2(X \setminus \{0\}, \mathbb{R})$ and, by Remark 1.1,

$$\inf_{u\neq 0} J(u) = m = (\tfrac{1}{2} - \tfrac{1}{p+1})^{-1} c^{\frac{p-1}{p+1}}. \tag{3.2}$$

Moreover, by Remark 1.9 (iv), we have

$$\lim_{|\tau|\to\infty} \max_{s\geq 0} \varphi(su_\tau) = c_\infty \ . \tag{3.3}$$

According to what stated in Section 1, the value c is a candidate to be a critical value for φ, since by Remark 1.1 there exists a PS sequence for φ at level c. Indeed, we observe that, by Lemma 1.11 and Remark 1.2, if $c < c_\infty$ then there exists $u \in \mathcal{K}$ such that $\varphi(u) = c$ (see [15]).

However, in general it is not always true that c is a critical value for φ. For instance in the case $a_0 < 0$ one can check that $c = c_\infty$ and every PS sequence for φ at level c converges to 0 strongly in $H^1_{\mathrm{loc}}(\mathbb{R})$.

On the other hand, if $c = c_\infty$, we can set up a new minimax at a possibly larger level, following the same procedure developed in [6], [9], and [7].

Taken $\tau > 0$, let

$$G_\tau = \{g \in C([0,1], X^+) : g(0) = u_{-\tau},\ g(1) = u_\tau\}$$

where $X^+ = \{u \in X : u \neq 0, u \geq 0\}$. Now, let

$$c_\tau = \inf_{g\in G_\tau} \max_{\substack{r\in[0,1]\\ s\geq 0}} \varphi(sg(r)).$$

Clearly $c_\tau \geq c$. In particular we can conclude about the existence of a non trivial critical point for φ in the following case.

Lemma 3.4 *If* $\liminf_{\tau\to+\infty} c_\tau = c_\infty$, *then* $\mathcal{K} \neq \emptyset$.

Proof. Let $\tau_n \to +\infty$ and $g_n \in G_{\tau_n}$ be such that $\max_{r\in[0,1],s\geq 0} \varphi(sg_n(r)) \to c_\infty$. For every $u \in X \setminus \{0\}$ let $\beta(u) = \int_{\mathbb{R}} \frac{t}{|t|}(\dot{u}^2 + u^2)dt\ \|u\|^{-2}$. We note that $\beta(u_{\pm\tau_n}) \to \pm 1$ as $n \to \infty$. Hence, by the continuity of β and g_n there exists a sequence $(r_n) \subset [0,1]$ such that $\beta(g_n(r_n)) = 0$ for every n sufficiently large. Using the notation of Remark 1.1, let $v_n = s(g_n(r_n))g_n(r_n)$. By the Ekeland principle, there exists a PS sequence (u_n) for φ such that $\|v_n - u_n\| \to 0$ and $\varphi(u_n) \to c_\infty$. Moreover, since $\beta(v_n) = 0$ and $\liminf \|v_n\| > 0$, we have $\beta(u_n) \to 0$. By Remark 1.5 and Lemma 1.6, up to a subsequence, $u_n \to u$ weakly in X, with $u \in \mathcal{K} \cup \{0\}$. If $u = 0$ then $|T(u_n)| \to \infty$ and, using Lemma 1.10, there exists $v \in \mathcal{K}_\infty$ such that $u_n(\cdot + T(u_n)) \to v$ strongly in X. Hence $\liminf |\beta(u_n)| > 0$, a contradiction. $\square$

By Lemma 3.4 we are reduced to consider the case $\liminf_{\tau\to+\infty} c_\tau > c_\infty$. The following result holds.

Lemma 3.5 *If* $\liminf_{\tau\to+\infty} c_\tau > c_\infty$, *then there exists* $\bar{\tau} > 0$ *such that for* $\tau > \bar{\tau}$, *there exists a PS sequence for* φ *at level* c_τ. *In addition, if* $c_\tau < 2c_\infty$ *then there exists* $u \in \mathcal{K}$ *with* $\varphi(u) = c_\tau$.

Proof. By (3.3) and Lemma 3.4, there exist $\epsilon > 0$ and $\bar{\tau} > 0$ such that if $\tau > \bar{\tau}$ then $\max\{\varphi(u_\tau), \varphi(u_{-\tau})\} \leq c_\infty + \epsilon < c_\tau$. Now, the first part of the Lemma follows by a standard deformation argument (see [6]). The second part is a consequence of the first one and of Corollary 3.3. $\square$

Hence, to conclude, we only have to construct for some $\tau > \bar{\tau}$ a particular $g \in G_\tau$ such that $\max_{r\in[0,1],s\geq 0} \varphi(sg(r)) < 2c_\infty$. This can be achieved arguing as in [6] and [7], or in [9], with an additional assumption on the behavior of a_0.

We remark that only at this point, the hypothesis (a_3) (or (a_4)) and the fact that we deal with a homogeneous potential play a crucial role in the argument.

Lemma 3.6 *If* a_0 *satisfies* (a3), *then* $\max_{\substack{r\in[0,1]\\ s\geq 0}} \varphi(s(ru_\tau + (1-r)u_{-\tau})) < 2c_\infty$ *for* $\tau \in T\mathbb{N}$ *large.*

Proof. For $\tau > \bar{\tau}$ let $g_\tau(r) = ru_\tau + (1-r)u_{-\tau}$ for every $r \in [0,1]$. By (3.1), since $c = c_\infty$, the lemma is proved if we show that there exist $\tau > \bar{\tau}$ such that for every $r \in [0,1]$

$$J(g_\tau(r)) < 2^{\frac{p-1}{p+1}} m \ . \tag{3.4}$$

For every $r \in [0,1]$ we have

$$\|g_\tau(r)\|^2 = r^2\|u_\tau\|^2 + (1-r)^2\|u_{-\tau}\|^2 + 2r(1-r)\langle u_\tau, u_{-\tau}\rangle \tag{3.5}$$

$$\int_{\mathbb{R}} a g_\tau(r)^{p+1} \geq \int_{\mathbb{R}} a_\infty g_\tau(r)^{p+1} - C_0 \int_{\mathbb{R}} e^{-\theta|t|} g_\tau(r)^{p+1} \tag{3.6}$$

where in (3.6) we have used the assumption $(a3)$. Let us recall now the following inequality (see [6], Lemma 2.1): there exists $C_p \geq 0$ such that for every $x, y \geq 0$

$$(x+y)^{p+1} \geq x^{p+1} + y^{p+1} + (p+1)(x^p y + x y^p) - C_p x^{\frac{p+1}{2}} y^{\frac{p+1}{2}}. \tag{3.7}$$

By (3.7), we obtain:

$$\begin{aligned}\int_{\mathbb{R}} a_\infty g_\tau(r)^{p+1} \ \geq \ & r^{p+1}\int_{\mathbb{R}} a_\infty u_\tau^{p+1} + (1-r)^{p+1}\int_{\mathbb{R}} a_\infty u_{-\tau}^{p+1} \\ + \ & (p+1)\left(r^p(1-r)\int_{\mathbb{R}} a_\infty u_\tau^p u_{-\tau} + r(1-r)^p \int_{\mathbb{R}} a_\infty u_\tau u_{-\tau}^p\right) \\ - \ & C_p r^{\frac{p+1}{2}}(1-r)^{\frac{p+1}{2}} \int_{\mathbb{R}} a_\infty u_\tau^{\frac{p+1}{2}} u_{-\tau}^{\frac{p+1}{2}} \,.\end{aligned} \tag{3.8}$$

Moreover, for every $r \in [0,1]$

$$\int_{\mathbb{R}} e^{-\theta|t|} g_\tau(r)^{p+1} \leq 2^{p+1}\left(\int_{\mathbb{R}} e^{-\theta|t|} u_{-\tau}^{p+1} + \int_{\mathbb{R}} e^{-\theta|t|} u_\tau^{p+1}\right). \tag{3.9}$$

Hence, taking $\tau \in T\mathbb{N}$, and setting $\omega(\tau) = \langle u_\tau, u_{-\tau}\rangle = \int_{\mathbb{R}} a_\infty u_{\pm\tau}^p u_{\mp\tau}$ and $A = 2c_\infty \frac{p+1}{p-1} = \|u_{\pm\tau}\|^2 = \int_{\mathbb{R}} a_\infty u_{\pm\tau}^{p+1}$ (see Remark 3.2 (iii)), by (3.8)–(3.9), the estimates (3.5) and (3.6) become

$$\|g_\tau(r)\|^2 = r^2 A + (1-r)^2 A + 2r(1-r)\omega(\tau) \tag{3.10}$$

$$\begin{aligned}\int_{\mathbb{R}} a g_\tau(r)^{p+1} \ \geq \ & r^{p+1} A + (1-r)^{p+1} A \\ & + (p+1)(r^p(1-r) + r(1-r)^p)\omega(\tau) - R(\tau)\end{aligned} \tag{3.11}$$

where

$$R(\tau) = C\left(\int_{\mathbb{R}} a_\infty u_\tau^{\frac{p+1}{2}} u_{-\tau}^{\frac{p+1}{2}} + \int_{\mathbb{R}} e^{-\theta|t|} u_\tau^{p+1} + \int_{\mathbb{R}} e^{-\theta|t|} u_{-\tau}^{p+1}\right). \tag{3.12}$$

We claim that

$$\frac{R(\tau)}{\omega(\tau)} \to 0 \text{ as } \tau \to +\infty. \tag{3.13}$$

Hence, the lemma follows. Indeed, by (3.10)–(3.11) and (3.13), setting

$$f_\tau(r) = \frac{r^2 A + (1-r)^2 A + 2r(1-r)\omega(\tau)}{\left(r^{p+1}A + (1-r)^{p+1}A + (p+1)\left(r^p(1-r) + r(1-r)^p\right)\omega(\tau)\right)^{\frac{2}{p+1}}} \ ,$$

we have that (3.4) is true if

$$f_\tau(r) < 2^{\frac{p-1}{p+1}} m \quad \text{for every } r \in [0,1] \ , \tag{3.14}$$

whenever $\tau \in T\mathbb{N}$ is large enough. One can check that, since $\omega(\tau) \to 0$ as $\tau \to +\infty$,

$$\max_{r\in[0,1]} f_\tau(r) = f_\tau(\tfrac{1}{2}) = 2^{\frac{p-1}{p+1}} \frac{A + \omega(\tau)}{\left(A + (p+1)\omega(\tau)\right)^{2/(p+1)}}$$

for $\tau \in T\mathbb{N}$ sufficiently large. Moreover

$$(A + (p+1)\omega(\tau))^{\frac{2}{p+1}} = A^{\frac{2}{p+1}} + 2A^{\frac{2}{p+1}-1}\omega(\tau) + \omega(\tau)o(1) \text{ as } \tau \to +\infty \ .$$

Hence, noting that $m = A^{\frac{p-1}{p+1}}$, we infer that (3.14) holds for $\tau \in T\mathbb{N}$ large enough.
To conclude, we have to prove the claim (3.13). To this aim, as in [6], we use the following

Lemma 3.7 ([6]) *Let $f \in C(\mathbb{R})$ and $\alpha > 0$ be such that $e^{\alpha|t|}f(t) \to \ell_\pm \in \mathbb{R}$ as $t \to \pm\infty$, and let $g \in C(\mathbb{R})$ be such that $e^{\alpha|t|}g \in L^1(\mathbb{R})$. Then $e^{\pm\alpha t_n}\int_{\mathbb{R}} f(t \pm t_n)g(t)\, dt \to L_\pm$ if $t_n \to \pm\infty$, being $L_\pm = \ell_\pm \int_{\mathbb{R}} e^{\mp\alpha t}g(t)\, dt$.*

By Lemma 3.7 and by Remarks 1.2 and 3.2 (i), one can check that

$$\lim_{\tau\to+\infty} e^{2\tau}\omega(\tau) \in (0,+\infty) \ . \tag{3.15}$$

Moreover, for every $\alpha \in (0,1)$ there exists $C_\alpha > 0$ such that $u_0 \le C_\alpha u_0^\alpha$ on $\mathbb{R}$. Then $\int_{\mathbb{R}} a_\infty u_\tau^{\frac{p+1}{2}} u_{-\tau}^{\frac{p+1}{2}} \le C_\alpha \int_{\mathbb{R}} a_\infty u_0^{\frac{p+1}{2}} u_{2\tau}^{\alpha\frac{p+1}{2}}$, and, using again Lemma 3.7 and Remark 1.2 (iii), there exists

$$\lim_{\tau\to+\infty} e^{\alpha\tau(p+1)} \int_{\mathbb{R}} a_\infty u_0^{\frac{p+1}{2}} u_{2\tau}^{\alpha\frac{p+1}{2}} \in (0,+\infty) \ . \tag{3.16}$$

Finally, noting that we can always assume that $\theta < p+1$ in $(a3)$, by Lemma 3.7 and Remark 1.2 (iii), we have that

$$\lim_{\tau\to\pm\infty} e^{\pm\theta\tau} \int_{\mathbb{R}} e^{-\theta|t|} u_{\pm\tau}^{p+1} \in (0,+\infty) \ . \tag{3.17}$$

Hence, taking $\alpha \in (\frac{2}{p+1}, 1)$ in (3.16), (3.15)-(3.17) yield (3.13). $\square$

Remark 3.8 If a_0 satisfies $(a3)$ then $\limsup_{\tau\to+\infty} c_\tau < 2c_\infty$. Indeed, by Lemma 3.6 and by (3.3) there exists $\hat{\tau} \in T\mathbb{N}$ such that $c_{\hat{\tau}} < 2c_\infty$ and $\max_{s\geq 0} \varphi(su_\tau) < \frac{3}{2}c_\infty$ for $|\tau| \geq \hat{\tau}$. Then, gluing together the paths $g_-(r) = u_{-r\hat{\tau}-(1-r)\tau}$, $g(r) = ru_{\hat{\tau}} + (1-r)u_{-\hat{\tau}}$ and $g_+(r) = u_{r\tau+(1-r)\hat{\tau}}$ (note that $g_\pm$ are continuous by Remark 3.2 (iii)), we conclude that $c_\tau \leq \max\{c_{\hat{\tau}}, \frac{2}{3}c_\infty\}$ for $|\tau| \geq \hat{\tau}$.

Alternatively to the condition $(a3)$, to have $c_\tau < 2c_\infty$, we can argue as in [9], assuming a global bound for the ratio $\frac{a}{a_\infty}$, without any convergence control for a_0 at infinity.

Lemma 3.9 *If a satisfies $(a4)$ then $\max_{r\in[0,1],s\geq 0} \varphi(su_{r\tau-(1-r)\tau}) < 2c_\infty$ for every $\tau > 0$, and the mapping $r \mapsto u_{r\tau-(1-r)\tau}$ belongs to G_τ.*

Proof. The first statement follows by the fact that $J(u_\tau) < 2^{\frac{p-1}{p+1}}m$ for every $\tau \in \mathbb{R}$. Indeed, this is equivalent to show that $\int_{\mathbb{R}} au_\tau^{p+1} > 2^{-\frac{p-1}{2}} \int_{\mathbb{R}} a_\infty u_\tau^{p+1}$, that follows by $(a4)$ (Note that $u_\tau > 0$ and $a(t) > 2^{-\frac{p-1}{2}}a_\infty(t)$ for $|t|$ large). The second part is a consequence of Remark 3.2 (ii). □

4 Further results

The techniques developed in the previous sections can be easily adapted to study also perturbative situations.

First, we observe that if $a\colon \mathbb{R} \to \mathbb{R}$ satisfies $(a1)$ and $a = a_\infty + \varepsilon a_0$, with a_∞ periodic and $a_0(t) \to 0$ as $t \to \pm\infty$, then $(a2)$ holds and also the assumption $(a4)$ is satisfied if $|\varepsilon|$ is sufficiently small. Hence the corresponding equation $-\ddot{u} + u = (a_\infty + \varepsilon a_0)|u|^{p-1}u$ admits a homoclinic solution (see [20] for a more general setting).

Next theorem shows that if we perturb a function $a\colon \mathbb{R} \to \mathbb{R}$ satisfying $(a1)$–$(a2)$ with any $\alpha \in L^\infty(\mathbb{R})$ that vanishes at infinity and has L^∞ norm small enough, then the corresponding equation still has a homoclinic solution. More generally, we have:

Theorem 4.1 *Let $a\colon \mathbb{R} \to \mathbb{R}$ satisfy $(a1)$–$(a2)$ and either $(a3)$ or $(a4)$. Let $g \in C(\mathbb{R})$ be such that $g(u) = O(u)$ as $u \to 0$. Then there exists $\bar{\varepsilon} = \bar{\varepsilon}(a, g) > 0$ such that for any $\alpha \in L^\infty(\mathbb{R})$ with $\|\alpha\|_{L^\infty} \leq \bar{\varepsilon}$ and $\alpha(t) \to 0$ as $|t| \to \pm\infty$, the equation*

$$-\ddot{u} + u = a(t)|u|^{p-1}u + \alpha(t)g(u) \qquad (D_\alpha)$$

admits a homoclinic orbit.

Proof. Let $G(u) = \int_0^u g(y)\, dy$ and, given $\alpha \in L^\infty(\mathbb{R})$, let $\psi_\alpha(u) = \int_\mathbb{R} \alpha G(u)$ and $\varphi_\alpha = \varphi + \psi_\alpha$. Note that $\psi_\alpha \in C^1(X)$ and sends bounded sets into bounded sets. Moreover, since $\alpha(t) \to 0$ as $t \to \pm\infty$, $\psi_\alpha(u(\cdot - t)) \to 0$ as $t \to \pm\infty$, uniformly on compact sets of X. Furthermore the problem at infinity corresponding to (D_α) is (D_∞). We distinguish the cases in which $(*)$ holds or does not.
If $(*)$ holds the Theorem follows arguing exactly as in Section 2.
If $(*)$ does not hold we argue as in Section 3 using the assumption $(a3)$ or $(a4)$ to prove via the Lemmas 3.6 and 3.9 that $\limsup_{\tau\to\infty} c_\tau < 2c_\infty$. This can be concluded also for the functional φ_α if $\bar{\varepsilon} > 0$ is small, because of the previous remarks, and we can prove the existence of a non trivial critical point for φ_α following again the argument of Section 3. $\square$

We finally note that the argument developed in this paper can be used to study also the cases in which the function a_∞ is almost periodic or more generally the cases in which a_∞ is just Poisson stable, i.e., there exists a sequence $(t_n)_{n\in\mathbb{Z}}$ such that $t_n \to \pm\infty$ as $n \to \pm\infty$ and $a_\infty(t - t_n) \to a_\infty(t)$ for any $t \in \mathbb{R}$. With minor changes in the proofs, these cases can be treated following the same scheme used before to study the asymptotically periodic case.

References

[1] S. Abenda, P. Caldiroli and P. Montecchiari, *Multibump Solutions for Duffing-like Systems*, Rend. Ist. Mat. Univ. Trieste, **28** (1996), 115-143.

[2] S. Alama and Y.Y. Li, *On "multibump" bound states for certain semilinear elliptic equations*, J. Diff. Eq. **96** (1992), 89-115.

[3] F. Alessio, P. Caldiroli and P. Montecchiari, *Genericity of the multibump dynamics for almost periodic Duffing-like systems*, preprint (1997).

[4] F. Alessio, P. Caldiroli and P. Montecchiari, *Genericity of the existence of infinitely many solutions for a class of semilinear elliptic equations in* $\mathbb{R}^N$, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), to appear. See also: F. Alessio, P. Caldiroli and P. Montecchiari, *On the existence of infinitely many solutions for a class of semilinear elliptic equations in* $\mathbb{R}^N$, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8), to appear.

[5] A. Ambrosetti and M. Badiale, *Variational Perturbative Methods in Bifurcation of Bound States from the Essential Spectrum*, Proc. Roy. Soc. Edinburgh Sect. A, to appear.

[6] A. Bahri and Y.Y. Li, *On a Min-Max Procedure for the Existence of a Positive Solution for Certain Scalar Field Equation in* $\mathbb{R}^n$, Rev. Mat. Iberoamericana **6** (1990), 1-15.

[7] A. Bahri and P.L. Lions, *On the existence of a positive solution of semilinear elliptic equations in unbounded domains*, Ann. Inst. H. Poincaré, Anal. non linéaire **14** (1997), 365-413.

[8] H. Berestycki and P.L. Lions, *Nonlinear scalar filed equations, I, II,* Arch. Rat. Mech. Anal. **82** (1983), 313-376.

[9] D.M. Cao, *Positive solutions and bifurcation from the essential spectrum of a semilinear elliptic equation in* $\mathbb{R}^n$, Nonlinear Anal. T.M.A. **15** (1990), 1045-1052.

[10] V. Coti Zelati, I. Ekeland and E. Séré, *A variational approach to homoclinic orbits in Hamiltonian systems*, Math. Ann. **288** (1990), 133-160.

[11] V. Coti Zelati, P.H. Rabinowitz, *Homoclinic type solutions for a semilinear elliptic PDE on* $\mathbb{R}^n$, Comm. Pure Appl. Math. **45** (1992), 1217-1269.

[12] W.Y. Ding and W.M. Ni, *On the existence of a positive entire solution of a semilinear elliptic equation*, Arch. Rat. Mech. Anal. **91** (1986), 283-308.

[13] P.J. Holmes and C.A. Stuart, *Homoclinic orbits for eventually autonomous planat flows*, Z. Angew. Math. Phys. **43** (1992), 598-625.

[14] M.K. Kwong, *Uniqueness of positive solutions of* $\Delta u - u + u^p = 0$ *in* $\mathbb{R}^n$, Arch. Rat. Mech. Anal. **105** (1989), 243-266.

[15] P.-L. Lions, *The concentration–compactness principle in the calculus of variations: the locally compact case, Part I and II*, Ann. Inst. H. Poincaré, Anal. Non Linéaire **1** (1984), 109-145 and 223-283.

[16] P. Montecchiari, *Existence and multiplicity of homoclinic solutions for a class of asymptotically periodic second order Hamiltonian systems*, Ann. Mat. Pura Appl. (4) **168** (1995), 317-354.

[17] P. Montecchiari, M. Nolasco and S. Terracini, *A global condition for periodic Duffing-like equations*, Trans. Amer. Math. Soc., to appear.

[18] E. Montefusco, *Perturbation Results for Semilinear Elliptic Equations in* $\mathbb{R}^n$, Preprint SNS Pisa, 1998.

[19] E. Séré, *Looking for the Bernoulli shift*, Ann. Inst. H. Poincaré, Anal. Non Linéaire **10** (1993), 561-590.

[20] G.S. Spradlin, *A perturbation of a periodic Hamiltonian system*, Preprint, 1997.

[21] C.A. Stuart, *Bifurcation of homoclinic orbits and bifurcation from the essential spectrum*, SIAM J. Math. Anal. **20** (1989), 1145-1171.

[22] S. Wiggins, *Global bifurcations and chaos*, Applied Math. Sci., Springer, 1988.