Existence and multiplicity of homoclinic orbits

for potentials on unbounded domains

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Synopsis. We study the system $\ddot{q} = -V'(q)$ in $\mathbb{R}^N$ where V is a potential with a strict local maximum at 0 and possibly with a singularity. First, using a minimizing argument, we can prove existence of a homoclinic orbit when the component Ω of $\{x \in \mathbb{R}^N : V(x) < V(0)\}$ containing 0 is an arbitrary open set; in the case Ω unbounded we allow $V(x)$ to go to 0 at infinity, although at a slow enough rate. Then, we show that the presence of a singularity in Ω implies that a homoclinic solution can be found also via a min–max procedure and, comparing the critical levels of the functional associated to the system, we see that the two solutions are distinct whenever the singularity is "not too far" from 0.

Key words: Hamiltonian systems, homoclinic orbits, singular potentials, critical point, minimax argument.

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Introduction

In this work we study the conservative second order Hamiltonian system

$$\ddot{q} + V'(q) = 0 \tag{HS}$$

where $q \in \mathbf{R}^N$ and V is a C^1 real function defined on some open subset of $\mathbf{R}^N$. In particular we look for homoclinic orbits, i.e. solutions of (HS) defined on $\mathbf{R}$ and doubly asymptotic, with their derivatives, to some periodic solution of (HS). Actually, we consider the special case in which this periodic orbit is an equilibrium point p for V (i.e. a point where $V'(p) = 0$); in this case the conditions for a solution q of (HS) to be a homoclinic orbit to p reduce to $q(\pm\infty) = p$ and $\dot{q}(\pm\infty) = 0$ and obviously $q \not\equiv p$.

Although these kind of orbits were first shown by Poincaré [12], it is only recently that they have been tackled with a variational approach, which seems quite natural for the structure of the problem. In fact, through this approach, several questions have been successfully investigated, also for more general hamiltonian systems as

$$\dot{x} = JH'(x) \tag{H}$$

where $x \in \mathbf{R}^{2N}$, $J = \left(\begin{smallmatrix} 0_N & -I_N \\ I_N & 0_N \end{smallmatrix}\right)$ and a Hamiltonian of the following form:

$$H(x) = \frac{1}{2} x \cdot Ax + R(x)$$

being A a symmetric constant matrix such that JA is hyperbolic (i.e. $\mathrm{sp}(JA) \cap i\mathbf{R} = \emptyset$) and $R(x) = o(|x|^2)$ as $x \to 0$, so that $x = 0$ is a hyperbolic point for H.

A first existence result of homoclinic solutions of (H) was established by Coti Zelati, Ekeland and Séré [5] under the hypotheses R positive, convex and globally superquadratic (i.e. satisfying $R'(x) \cdot x \geq \alpha R(x)$ for all $x \in \mathbf{R}^{2N}$, with $\alpha > 2$); in this work the solution is obtained as critical point of the dual action functional, using the mountain–pass lemma. The lack of compactness due to the unboundedness of the domain is overcome by the concentration–compactness principle of P. L. Lions [11]. Then Hofer and Wysocki [10] dropped the convexity assumption and found a homoclinic orbit applying a linking theorem to the action functional defined on $H^{1/2}(\mathbf{R}; \mathbf{R}^{2N})$. The same result was achieved by Tanaka [20] who obtained the homoclinic orbit as limit in the C^1_{loc} topology of T–periodic solutions of (H) as $T \to \infty$. This method was introduced by Rabinowitz [14] to study the second order system (HS) with V of the form:

$$V(q) = -\frac{1}{2} q \cdot Lq + W(q) \tag{1}$$

being L a positive definite symmetric matrix and W globally superquadratic and such that $W'(q) = o(|q|)$ as $q \to 0$.

Some of the above–mentioned results are also valid if H —for (H)— or V —for (HS)— depend explicitly on time in a periodic way. Indeed, Séré in [16] first showed that this time dependence is the key to obtain multiplicity results; this fact was afterwards employed in other works as [6] and [18], concerning, like [16], the existence of infinitely many homoclinics.

In the autonomous case, recently, Ambrosetti and Bertotti [1] and Rabinowitz and Tanaka [15], with different techniques, were able to get the existence of a homoclinic solution for (HS) without the superquadraticity condition, but assuming that the component Ω of $\{x : V(x) < 0\} \cup \{0\}$ containing 0 is bounded and $V'(x) \neq 0$ for any $x \in \partial\Omega$. In some sense, the analogous thing was done by Séré [17] for the first order system as (H), supposing that $\Sigma \setminus \{0\}$ is compact and of restricted contact type, where $\Sigma = \{x : H(x) = 0\}$ is the zero–energy surface.

The main results here are theorems 2.2 and 3.1.

In theorem 2.2 we prove the existence of one homoclinic orbit when Ω is an open set in $\mathbf{R}^N$ (not necessarily bounded) and V is eventually singular in a point $e \in \Omega$ in the sense that $V(x) \to -\infty$ as $x \to e$.

This result generalizes [1], where Ω is assumed to be bounded and V to satisfy a strong force condition near the singularity, [15], which also assume Ω to be bounded and take V to be regular, and [19], which allows V to be singular (requiring the strong force condition) but take $\limsup_{|x|\to\infty} V(x) < 0$.

Our solution is founded as limit of minima of problems similar to that of [15]. The difficulties here are: (a) to show that such sequence converges to a solution bounded in L^∞, (b) to prove that the limit is a non–collision solution. We remark that our assumptions allow $V(x)$ to go to 0 as $|x| \to \infty$, even if not too fast (see assumptions *(V3)* and *(V3′)*).

In theorem 3.1 we prove the existence of a second solution under suitable assumptions. More precisely, we show that the presence of a singularity implies that a solution can be found also via a min–max procedure. We then prove, comparing the critical levels, that such a solution is different from the one found via theorem 2.2 whenever the singularity is "not too far" from 0, thus obtaining theorem 3.1.

Let us remark that the interest of this result lies in the fact that, contrary to the non–autonomous case, very little is known about the multiplicity of homoclinic solutions for conservative systems; we mention here a recent work of Ambrosetti and Coti Zelati [2], where the authors, with a variational approach and by means of the Ljusternik–Schnirelmann theory, prove the existence of two homoclinic orbits of (HS) for V of the form (1), with W superquadratic, satisfying a "pinching" condition:

$$a|q|^\alpha \le W(q) \le b|q|^\alpha$$

provided that $b/a < 2^{\frac{\alpha-2}{2}}$.

1. Homoclinics in unbounded domains

We study the system (HS) in $\mathbf{R}^N$ ruled by a potential $V \in C^1(\mathbf{R}^N;\mathbf{R})$ with a local maximum at some point p. Without loss of generality we can suppose $p = 0$ and $V(0) = 0$; so, the origin is an equilibrium for (HS). We look for homoclinic orbits to 0. Let Ω be the component of $\{x \in \mathbf{R}^N : V(x) < 0\} \cup \{0\}$ containing 0. Notice that $V(x) < 0$ if $x \in \Omega \setminus \{0\}$ and $V(x) = 0$ if $x \in \partial\Omega \cup \{0\}$. We will find a solution of (HS) on $\mathbf{R}_+ = [0, \infty)$ as critical point of the usual Lagrangian functional

$$I(u) = \int_0^\infty \left(\frac{1}{2}|\dot u|^2 - V(u)\right) dt$$

associated to (HS). To be precise, we consider the Hilbert space

$$E_+ = \left\{ u \in W^{1,2}_{\mathrm{loc}}(\mathbf{R}_+;\mathbf{R}^N) \ : \ \int_0^\infty |\dot u|^2 dt < \infty \right\}$$

equipped with the norm

$$\|u\|_+^2 = |u(0)|^2 + \int_0^\infty |\dot u|^2 dt.$$

Fixed an open set $\omega \subseteq \Omega$ containing 0, we define

$$\Gamma(\omega) = \{u \in E_+ \ : \ u(0) \in \partial\omega \ \ u(\infty) = 0 \ \ u(t) \in \overline{\omega} \ \forall t \in \mathbf{R}_+\}$$

and look for a solution of (HS) on $\mathbf{R}_+$ as minimum of I on $\Gamma(\omega)$. In particular we write $\Gamma = \Gamma(\Omega)$.
Rabinowitz and Tanaka [15] proved the following result:

Theorem 1.1. *Let $V \in C^1(\mathbf{R}^N; \mathbf{R})$ with $V(0) = 0$ satisfying*
(V1) Ω is open, i.e. the point 0 is a strict local maximum for V;
(V2) $V'(x) \neq 0$ for any $x \in \partial\Omega$.
Then, given an open bounded set $\omega \subseteq \Omega$ containing 0, there is a function $q \in C^2(\mathbf{R}_+; \mathbf{R}^N)$ solution of (HS) *with energy 0 and such that $q \in \Gamma(\omega)$, $q(t) \in \omega$ for $t > 0$ and $I(q) = \inf_{u \in \Gamma(\omega)} I(u)$.*
If in addition Ω is bounded, then (HS) *admits a homoclinic orbit described by a even function $x \in C^2(\mathbf{R}; \mathbf{R}^N)$, such that $x_+ = x|_{\mathbf{R}_+} \in \Gamma$ and $I(x_+) = \inf I(\Gamma)$.*

Now we want to discuss the case Ω unbounded to give a similar result using the previous theorem. For any $R > 0$ set $B_R = \{x \in \mathbf{R}^N : |x| < R\}$ and $\Gamma_R = \Gamma(\Omega \cap B_R)$. By theorem 1.1, for any $R > 0$ we can find a solution q_R of (HS) on $\mathbf{R}_+$, with energy zero and such that $q_R \in \Gamma_R$, $q_R(t) \in \Omega$ for $t > 0$ and $I(q_R) = \inf I(\Gamma_R)$. Observe that if $q_{\bar{R}}(0) \in \partial\Omega$ for some $\bar{R} > 0$, we can reflect $q_{\bar{R}}$, by defining $x(t) = q_{\bar{R}}(|t|)$ for $t \in \mathbf{R}$, to obtain a homoclinic orbit for (HS) with the desired properties. So, the problem is to show that it cannot happen that $|q_R(0)| = R$ for any $R > 0$.
Firstly we notice that for all $R > 0$

$$0 < I(q_R) \leq c \tag{2}$$

where $c = \inf I(\Gamma) < \infty$. Indeed, fixed $R > 0$, for any $u \in \Gamma$ there is a time $t_o \geq 0$ such that $u(\cdot + t_o) =: v \in \Gamma_R$ and so $I(v) \leq I(u)$; then $I(q_R) \leq I(u)$ for all $u \in \Gamma$, and (2) follows.
At this point, it is useful to distinguish the cases $\partial\Omega$ bounded, i.e. $\mathbf{R}^N \setminus \Omega$ bounded, and $\partial\Omega$ unbounded, even if the result obtained in the second case (theorem 1.4) is meaningful for any Ω with boundary.

Theorem 1.2. *Let $V \in C^1(\mathbf{R}^N; \mathbf{R})$ with $V(0) = 0$ satisfying (V1) and (V2). Let Ω^c be bounded with $\partial\Omega$ non–empty and suppose:*
(V3) there are a constant $R_o > 0$ and a function $U \in C^1(\mathbf{R}^N; \mathbf{R})$ such that $|U(x)| \to \infty$ as $|x| \to \infty$ and $-V(x) \geq |U'(x)|^2$ for $|x| \geq R_o$.
Then (HS) *admits a homoclinic orbit described by a even function $x \in C^2(\mathbf{R}; \mathbf{R}^N)$, such that $x_+ = x|_{\mathbf{R}_+} \in \Gamma$ and $I(x_+) = \inf I(\Gamma)$.*

Remark 1.3. Notice that *(V3)* is formally the so–called strong force condition, introduced by Gordon in [9] to deal with singular potentials, i.e. functions φ such that $\varphi(x) \to -\infty$ as $x \to x_o$, where x_o is some point of $\mathbf{R}^N$, representing the singularity. Usually this condition governs the rate at which $\varphi(x) \to -\infty$ as $x \to x_o$; for example it is satisfied when $\varphi(x) = -|x - x_o|^{-\alpha}$ in a neighbourhood of x_o, with $\alpha \geq 2$. In our case the property *(V3)* tells us that, as $|x| \to \infty$, $V(x)$ can go to 0, but not too fast. For instance, if $V(x) = -|x|^{-\alpha}$ for $|x|$ large, *(V3)* is verified when $\alpha \leq 2$. As we will see in the proof, *(V3)* implies that if $u \in \Gamma$ is an unbounded function, then $I(u)$ must diverge.

Proof. Suppose, by contradiction, that $|q_R(0)| = R$ for any $R > 0$. Since $q_R \in \Gamma_R$, for any $R > R_o$ there is $T_R \in \mathbf{R}_+$ such that $|q_R(T_R)| = R_o$ and $|q_R(t)| > R_o$ for $t \in [0, T_R)$. Hence, using *(V3)* and (2), we get for any $R > R_o$:

$$
\begin{aligned}
|U(q_R(0))| &\leq |U(q_R(T_R)) - U(q_R(0))| + |U(q_R(T_R))| \\
&\leq \left| \int_0^{T_R} U'(q_R(t)) \cdot \dot{q}_R(t)\, dt \right| + |U(q_R(T_R))| \\
&\leq \left(\int_0^{T_R} |U'(q_R(t))|^2 dt \right)^{\frac{1}{2}} \left(\int_0^{T_R} |\dot{q}_R(t)|^2 dt \right)^{\frac{1}{2}} + |U(q_R(T_R))| \qquad (3) \\
&\leq \left(\int_0^{T_R} -V(q_R(t))\, dt \right)^{\frac{1}{2}} \left(\int_0^{\infty} |\dot{q}_R(t)|^2 dt \right)^{\frac{1}{2}} + |U(q_R(T_R))| \\
&\leq I(q_R) + |U(q_R(T_R))| \leq c' < \infty
\end{aligned}
$$

while $|U(q_R(0)| \to \infty$ as $R \to \infty$. Therefore for some $\bar{R} > 0$ it must be $q_{\bar{R}}(0) \in \partial\Omega$; hence $q_{\bar{R}} \in \Gamma$ and, from (2), $I(q_{\bar{R}}) = \inf I(\Gamma)$. Moreover $\dot{q}_{\bar{R}}(0) = 0$ and, setting $x(t) = q_{\bar{R}}(|t|)$ for $t \in \mathbf{R}$, we obtain the desired homoclinic orbit.

q.e.d.

Next we examine the case $\partial\Omega$ unbounded, and show how some conditions for V at infinity in Ω and along its boundary give the existence of a solution of (HS) characterized as minimum point of the Lagrangian functional I on the set Γ. We denote by $\rho(x)$ the distance of x from the boundary of Ω.

Theorem 1.4. *Let $V \in C^2(\mathbf{R}^N; \mathbf{R})$ with $V(0) = 0$ satisfying (V1) and (V2). Suppose $\partial\Omega \neq \emptyset$ and*

(V3′) there are constants $\delta, R_o > 0$ and $\alpha \in (0, 2]$ such that $V(x) \leq -|x|^{-\alpha} \min\{\delta, \rho(x)\}$ for any $x \in \Omega \setminus B_{R_o}$;

(V4) there are constants $a, b > 0$ and $\lambda \geq 1$ such that:

i. $V(x) \geq -\lambda \frac{\rho(x)}{|x|^\alpha}$ for any $x \in S$;

ii. $|\rho''(x)\xi \cdot \xi| \leq a\, |\xi|^2$ for any $x \in S$ and $\xi \in \mathbf{R}^N$;

iii. $\rho(x)\, \rho'(x) \cdot V'(x) \leq b\, V(x)$ for any $x \in S$;

where $S = \{x \in \Omega \setminus B_{R_o} :\, \rho(x) < \delta\}$.

Then (HS) *admits a homoclinic orbit described by a even function $x \in C^2(\mathbf{R}; \mathbf{R}^N)$, such that $x_+ = x|_{\mathbf{R}_+} \in \Gamma$ and $I(x_+) = \inf I(\Gamma)$.*

Remark 1.5. We point out that theorem 1.4 covers the cases Ω bounded and Ω^c bounded, too; indeed if Ω is bounded the hypotheses *(V3′)* and *(V4)* are trivially satisfied; if Ω^c is bounded, again the hypothesis *(V4)* is automatically true and *(V3′)* reduces to *(V3)* with $U(x) = \delta \log |x|$ if $\alpha = 2$ or $U(x) = \delta\, (1 - \frac{\alpha}{2})^{-1} |x|^{1-\frac{\alpha}{2}}$ if $\alpha \in (0, 2)$.

Remark 1.6. The condition *(V3′)* is the analogous of *(V3)* with the same meaning explained in remark 1.3; however the presence of $\partial\Omega$ is reflected by the term $\rho(x)$, by virtue of which $V(x)$ can go to 0 on $\partial\Omega$. Actually the assumption *(V4) i.* gives a boundedness from above for $|V|$ in the strip S and, with *(V3′)*, implies that for any $x \in S$ $\frac{\rho(x)}{|x|^\alpha} \leq |V(x)| \leq \lambda \frac{\rho(x)}{|x|^\alpha}$. We also notice that the condition *(V4) ii.* expresses a control about the curvature of the boundary of Ω that cannot become too much large. The *(V4) iii.* forces an orbit entering

S to have at most one point with least distance from $\partial\Omega$; more exactly, as we will see in the proof, the function $t \mapsto \rho(q(t))$ is convex when $q(t) \in S$, provided that q is a solution of (HS) with energy zero. This assumption looks like assumption *(V6)* of theorem 3.1.

Proof. The result follows from theorems 1.1 and 1.2 if $\partial\Omega$ is bounded (see remark 1.5). Then we will assume $\partial\Omega$ to be unbounded. As in the proof of theorem 1.2, we argue indirectly, supposing that $|q_R(0)| = R$ for any $R > 0$. First of all, we give some notations: since $q_R \in \Gamma_R$, for any $R \geq R_o$ there is $T_R \in \mathbf{R}_+$ such that $|q_R(T_R)| = R_o$ and $|q_R(t)| > R_o$ for $t \in [0, T_R)$. Call $\varepsilon_R = \inf_{t\in[0,T_R]} \rho(q_R(t))$; then, let $t_R \in [0, T_R]$ such that $\rho(q_R(t_R)) = \varepsilon_R$ and put $x_R = q_R(t_R)$. Moreover we will write $I_A(u) = \int_A (\frac{1}{2}|\dot{u}|^2 - V(u))dt$ for $A \subseteq \mathbf{R}_+$ and $u \in \Gamma$ and define for any $r > 0$ $\varphi(r) = \frac{2}{2-\alpha} r^{1-\frac{\alpha}{2}}$ in the case $0 < \alpha < 2$ and $\varphi(r) = \log r$ for $\alpha = 2$.

<u>1st step</u>: if $\rho(q_R(t)) \geq \varepsilon$ for any $t \in [t_1, t_2]$ and $t_2 \leq T_R$ then

$$\varphi(q_R(t_1)) \leq \varphi(q_R(t_2)) + \frac{1}{\sqrt{\min\{\varepsilon, \delta\}}} I_{[t_1,t_2]}(q_R) \tag{4}$$

In fact, we point out that $\varphi'(r)^2 = r^{-\alpha}$ and so, by *(V3′)*, we have:

$$\begin{aligned}\varphi(q_R(t_1)) &\leq \varphi(q_R(t_2)) + \int_{t_1}^{t_2} \varphi'(|q_R(t)|)\,|\dot{q}_R(t)|\,dt \\ &\leq \varphi(q_R(t_2)) + \left(\int_{t_1}^{t_2} \frac{dt}{|q_R(t)|^\alpha}\right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} |\dot{q}_R(t)|^2 dt\right)^{\frac{1}{2}}.\end{aligned}$$

Now, call $A = \{t \in [t_1, t_2] \,:\, \rho(q_R(t)) \geq \delta\}$ and $B = [t_1, t_2] \setminus A$ and observe that

$$\int_{t_1}^{t_2} \frac{dt}{|q_R(t)|^\alpha} \leq \frac{1}{\delta}\int_A -V(q_R(t))\,dt + \frac{1}{\varepsilon}\int_B -V(q_R(t))\,dt.$$

Thus we get

$$\begin{aligned}\varphi(q_R(t_1)) &\leq \varphi(q_R(t_2)) + \frac{1}{\sqrt{\min\{\varepsilon, \delta\}}} \left(\int_{t_1}^{t_2} -V(q_R(t))\,dt\right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} |\dot{q}_R(t)|^2 dt\right)^{\frac{1}{2}} \\ &\leq \varphi(q_R(t_2)) + \frac{1}{\sqrt{\min\{\varepsilon, \delta\}}} I_{[t_1,t_2]}(q_R).\end{aligned}$$

<u>2nd step</u>:

$$\lim_{R\to\infty} \varepsilon_R = 0 \tag{5}$$

We apply (4) to any q_R with $R \geq R_o$ in the interval $[0, T_R]$ and get:

$$\varphi(R) \leq \varphi(R_o) + \frac{1}{\sqrt{\min\{\varepsilon_R, \delta\}}} I(q_R) \leq c_o + \frac{c_1}{\sqrt{\min\{\varepsilon_R, \delta\}}}$$

so that, by the definition of φ, (5) follows immediately.

Before passing to the next step we point out that for any $R \geq R_o$ there exists a point $z_R \in \partial\Omega$ such that $|x_R - z_R| = \varepsilon_R$. By definition of ε_R, the segment $[x_R, z_R]$ is contained in $\overline{\Omega}$. If $|z_R| > R$ there is a unique $\bar{z}_R \in [x_R, z_R]$ such that $|\bar{z}_R| = R$; in this case we redefine $z_R = \bar{z}_R$. Hence $[x_R, z_R] \subset \overline{\Omega \cap B_R}$ and $z_R \in \partial(\Omega \cap B_R)$. Finally put $m_R = \max_{x\in[x_R,z_R]} |V(x)|$.

<u>3rd step</u>: for any $R \geq R_o$

$$\sqrt{2\varepsilon_R m_R} + \varphi(|x_R|) \geq \varphi(R) \tag{6}$$

If $t_R = 0$, (6) is obvious. If $t_R > 0$ we consider the function $Q_R : \mathbf{R}_+ \to \mathbf{R}^N$ defined in this way:

$$Q_R(t) = \begin{cases} z_R + v_R t & 0 \leq t \leq \tau_R \\ q_R(t - \tau_R + t_R) & t > \tau_R \end{cases}$$

where $v_R = \frac{x_R - z_R}{\tau_R}$ and $\tau_R > 0$ must be chosen in a suitable manner. We observe that $Q_R \in \Gamma_R$ and

$$I(Q_R) \leq \frac{\varepsilon_R^2}{2\tau_R} + \tau_R m_R + I(q_R) - I_{[0,t_R]}(q_R).$$

We minimize $\frac{\varepsilon_R^2}{2\tau_R} + \tau_R m_R$ taking $\tau_R = \frac{\varepsilon_R}{\sqrt{2m_R}}$. On the other hand, we can use (4) to estimate $I_{[0,t_R]}(q_R)$, so that

$$I(Q_R) \leq \varepsilon_R\sqrt{2m_R} + I(q_R) + \sqrt{\varepsilon_R}\varphi(|x_R|) - \sqrt{\varepsilon_R}\varphi(R).$$

Then, the fact that $I(q_R) = \inf I(\Gamma_R)$ implies readily (6).

<u>4th step</u>:

$$\lim_{R\to\infty} \frac{|x_R|}{R} = 1 \tag{7}$$

By (5) and by the hypothesis *(V4) i.*, for any R large enough $m_R \leq m < \infty$ and so, for $0 < \alpha < 2$, (6) gives

$$c_2\sqrt{\varepsilon_R} + |x_R|^{1-\frac{\alpha}{2}} \geq R^{1-\frac{\alpha}{2}}$$

and then

$$1 - \frac{c_2\sqrt{\varepsilon_R}}{R^{1-\frac{\alpha}{2}}} \leq \left(\frac{|x_R|}{R}\right)^{1-\frac{\alpha}{2}} \leq 1$$

that implies (7). In the case $\alpha = 2$, we obtain

$$c_2\sqrt{\varepsilon_R} + \log|x_R| \geq \log R$$

that is

$$e^{c_2\sqrt{\varepsilon_R}} \geq \frac{R}{|x_R|} \geq 1$$

and (7) follows again.

For R sufficiently large, say $R \geq R_1$, by (5), we know that $x_R \in S$. But $q_R \in \Gamma_R$ and so, there is $s_R \in (t_R, T_R]$ such that $q_R(s_R) \in \partial S$ and $q_R(t) \in S$ if $t \in [t_R, s_R)$. Call $y_R = q_R(s_R)$.

<u>5th step</u>:

$$\lim_{R\to\infty} \frac{|y_R|}{R} = 1 \tag{8}$$

Since each q_R is solution of (HS) with energy zero, using the hypotheses *(V4) ii.* and *iii.* it holds that for any $t \in [t_R, s_R]$

$$\begin{aligned}\frac{d^2}{dt^2}\rho(q_R(t)) &= \rho''(q_R(t))\dot{q}_R(t)\cdot\dot{q}_R(t) + \rho'(q_R(t))\cdot\ddot{q}_R(t)\\ &\geq -a\,|\dot{q}_R(t)|^2 - \rho'(q_R(t))\cdot V'(q_R(t))\\ &\geq 2a\,V(q_R(t)) - b\,\frac{V(q_R(t))}{\rho(q_R(t))}\\ &\geq \frac{b-2a\delta}{|q_R(t)|^\alpha}\end{aligned}.$$

We can always suppose $\delta < \frac{b}{4a}$ so that for any $t \in [t_R, s_R]$

$$\frac{d^2}{dt^2}\rho(q_R(t)) \geq \frac{b}{2R^\alpha}.$$

Hence, taking into account that $\frac{d}{dt}\,\rho(q_R(t_R)) \geq 0$, we deduce that

$$\delta \geq \rho(q_R(s_R)) - \rho(q_R(t_R)) \geq \int_{t_R}^{s_R} dt \int_{t_R}^{t} ds\,\frac{d^2}{ds^2}\rho(q_R(s)) \geq \frac{b}{4R^\alpha}(s_R - t_R)^2$$

and so

$$\begin{aligned}|y_R| &\geq |x_R| - |y_R - x_R| \geq |x_R| - \int_{t_R}^{s_R} |\dot{q}_R|\,dt \geq\\ &\geq |x_R| - (s_R - t_R)^{\frac12}\left(\int_{t_R}^{s_R}|\dot{q}_R|^2 dt\right)^{\frac12} \geq |x_R| - c_3 R^{\frac{\alpha}{4}}\end{aligned}$$

where c_3 is a positive constant independent on R. Therefore

$$\frac{|x_R|}{R} - \frac{c_3}{R^{1-\frac{\alpha}{4}}} \leq \frac{|y_R|}{R} \leq 1$$

that gives (8).

We need again other quantities: for $R \geq R_1$ set $\varepsilon'_R = \inf_{t\in[s_R,T_R]} \rho(q_R(t))$, let $t'_R \geq s_R$ such that $\rho(q_R(t'_R)) = \varepsilon'_R$ and put $x'_R = q_R(t'_R)$.

<u>6th step</u>:

$$\lim_{R\to\infty} \varepsilon'_R = 0 \tag{9}$$

One can prove (9) exactly as (5), applying (4) to each q_R with $R \geq R_1$ in the interval $[s_R, T_R]$ and using (8).

<u>7th step</u>:

$$\lim_{R\to\infty} \frac{|x'_R|}{R} = 1 \tag{10}$$

Arguing as for the third step, we can show that for any R large enough

$$\sqrt{2\varepsilon'_R m'_R} + \varphi(|x'_R|) \geq \varphi(|y_R|)$$

where $m'_R = \max_{x \in [x'_R, z'_R]} |V(x)|$ and z'_R is a point of $\partial(\Omega \cap B_R)$ such that $[x'_R, z'_R] \subset \overline{\Omega \cap B_R}$ and $|x'_R - z'_R| \leq \varepsilon'_R$. Hence, following the line of the fourth step, we obtain (10).

From the results found up to now, we deduce that for any R large enough there exists an instant $s'_R \in [s_R, t'_R)$ such that $\rho(q_R(s'_R)) = \delta$ and $q_R(t) \in S$ for $t \in (s'_R, t'_R]$.

8th step: conclusion.
For any R sufficiently large we consider the function $Q'_R : \mathbf{R}_+ \to \mathbf{R}^N$ defined in this way:

$$Q'_R(t) = \begin{cases} z'_R + v'_R t & 0 \leq t \leq \tau'_R \\ q_R(t - \tau'_R + t'_R) & t > \tau'_R \end{cases}$$

where $v'_R = \frac{x'_R - z'_R}{\tau'_R}$ and $\tau'_R = \frac{\varepsilon'_R}{\sqrt{2m'_R}}$. We notice that $Q'_R \in \Gamma_R$ and

$$I(Q_R) \leq \varepsilon'_R \sqrt{2m'_R} + I(q_R) - I_{[0,t'_R]}(q_R).$$

Now we give a different estimate for $I_{[0,t'_R]}(q_R)$, arguing in this way: in general, for any $u \in \Gamma$, it can be easily shown that

$$I_{[t_1,t_2]}(u) \geq |u(t_2) - u(t_1)| \sqrt{2 \min_{t \in [t_1,t_2]} |V(u(t))|} \tag{11}$$

(see lemma 3.6 in [13]). Using this formula, we obtain that

$$I_{[0,t'_R]}(q_R) \geq I_{[s'_R,t'_R]}(q_R) \geq (\delta - \varepsilon'_R) \sqrt{\frac{2\varepsilon'_R}{R^\alpha}}$$

and consequently, for R large enough:

$$I(Q_R) \leq I(q_R) + \varepsilon'_R \sqrt{2m'_R} - \frac{\delta}{2} \sqrt{\frac{2\varepsilon'_R}{R^\alpha}}.$$

Since $I(q_R) = \inf I(\Gamma_R)$, it must be

$$\sqrt{\varepsilon'_R m'_R} \geq \frac{\delta}{2R^{\frac{\alpha}{2}}}. \tag{12}$$

But the hypothesis *(V4) iii.* implies that

$$m'_R \leq \frac{\lambda \varepsilon'_R}{(|x'_R| - \varepsilon'_R)^\alpha}$$

which, for (12), gives

$$\varepsilon'_R \geq \sqrt{\frac{\delta}{\lambda} \left(\frac{|x'_R|}{R} - \frac{\varepsilon'_R}{R} \right)^\alpha}$$

and this one is in contrast with (9) and (10). Now the conclusion of the proof is the same as for theorem 1.2.

q.e.d.

Remark 1.7. It is clear from the proof that it is possible to replace the hypotheses *(V4) ii.* and *iii.* with a weaker assumption, that is

$$2\,\|\rho''(x)\|\,V(x)+\rho'(x)\cdot V'(x)\leq -\frac{a}{|x|^{\beta}}$$

for any $x\in S$, being $a>0$ and $\beta\in(0,4)$.

2. Singular potentials and non–collision solutions

In this section we will deal with a potential V with a strict local maximum at 0 and a singularity at a point $e\in\mathbf{R}^N\setminus\{0\}$. We use the same notations of section 1 about Ω, E_+, Γ, etc. The problem (HS) in this setting was already studied by Tanaka [19] in the case $\Omega=\mathbf{R}^N$ and $\limsup_{|x|\to\infty}V(x)<0$ and by Ambrosetti and Bertotti [1] for Ω bounded; both of them assume the strong force condition near the singularity and use minimax methods, different in the two cases. Here we are interested in the case in which Ω is an arbitrary open with boundary and use the same minimizing argument of section 1. In this kind of approach it is not necessary to make any assumption about the behaviour of V near the singularity e. In this way we find a generalized solution (see [3] and definition 2.1 below) which could cross the singularity. But since this orbit is a minimum point of I on the set Γ one can show that the solution has actually no collisions whenever the potential satisfies some local property near e, weaker than the strong force condition.

We recall here the definition of generalized solution as given by Bahri and Rabinowitz in [3].

Definition 2.1. A function $q\in C(\mathrm{I};\mathbf{R}^N)$, where I is an interval of $\mathbf{R}$, is called *generalized solution of (HS) in* I*, with energy h,* if
(i) the set $q^{-1}(e)$ has measure zero;
(ii) $q\in C^2(\mathrm{I}\setminus q^{-1}(e);\mathbf{R}^N)$ satisfies (HS) on $\mathrm{I}\setminus q^{-1}(e)$;
(iii) $\frac{1}{2}|\dot q(t)|^2+V(q(t))=h$ for any $t\in\mathrm{I}\setminus q^{-1}(e)$.
The cardinality of $q^{-1}(e)$ defines the number of collisions of q. If $\mathrm{I}=\mathbf{R}$, $q\not\equiv 0$ and $q(\pm\infty)=\dot q(\pm\infty)=0$, q is said *generalized homoclinic orbit of (HS).* Clearly such a solution has energy zero.

Theorem 2.2. *Let $V\in C^1(\mathbf{R}^N\setminus\{e\};\mathbf{R})$ with $V(0)=0$ and $\lim\limits_{x\to e}V(x)=-\infty$ satisfying (V1) and (V2).*
Then, given an open bounded set $\omega\subseteq\Omega$ containing 0 and e, (HS) *possesses a generalized solution q on $\mathbf{R}_+$, with energy 0, at most one collision and such that $q\in\Gamma(\omega)$ and $I(q)=\inf\limits_{u\in\Gamma(\omega)}I(u)$.*
Moreover, suppose $\partial\Omega\neq\emptyset$ and, for Ω unbounded, assume that (V3′) and (V4) hold. Then (HS) *admits a generalized homoclinic orbit described by a even function $x\in C(\mathbf{R};\mathbf{R}^N)$, such that $x_+=x|_{\mathbf{R}_+}\in\Gamma$ and $I(x_+)=\inf I(\Gamma)$.*

Proof. Fix an open bounded set $\omega\subseteq\Omega$ containing 0 and e. With the same argument used to prove theorem 1.1, one can easily see that there is a function $q\in\Gamma(\omega)$ with the property *(ii)* of definition 2.1 and such that $I(q)=\inf\limits_{u\in\Gamma(\omega)}I(u)$. Thus, we have to show that q

satisfies *(i)* and *(iii)*. Actually we notice that q is an injective function; otherwise, if it were $q(t_1) = q(t_2)$ for some $t_2 > t_1 \geq 0$, we could consider a new function Q defined by

$$Q(t) = \begin{cases} q(t) & 0 \leq t \leq t_1 \\ q(t + t_2 - t_1) & t > t_1 \end{cases}$$

and observe that $Q \in \Gamma(\omega)$ and $I(Q) < \min\limits_{u \in \Gamma(\omega)} I(u)$. Then, in particular, we deduce that $q^{-1}(e)$ is at most a singleton. Finally we claim that q has always energy zero. In fact, first of all, since V does not depend on t, the energy is constant, i.e.

$$\frac{1}{2}|\dot{q}|^2 + V(q(t)) = h_o \quad \text{for } t \in (0, \tau) \tag{13.a}$$

$$\frac{1}{2}|\dot{q}|^2 + V(q(t)) = h_\infty \quad \text{for } t > \tau \tag{13.b}$$

where $\tau = \max\{0, q^{-1}(e)\}$. Knowing that $q \in E_+$ and $I(q) < \infty$ and using (13.b), we obtain $h_\infty = 0$ and $\dot{q}(\infty) = 0$. If $q^{-1}(e) = \tau > 0$ take the following function:

$$Q(t) = \begin{cases} q(\lambda t) & 0 \leq t \leq \frac{\tau}{\lambda} \\ q\big(t + \tau - \frac{\tau}{\lambda}\big) & t > \frac{\tau}{\lambda} \end{cases}$$

with a suitable $\lambda > 0$. It's clear that $Q \in \Gamma(\omega)$ and

$$\int_{\frac{\tau}{\lambda}}^{\infty} \big(\frac{1}{2}|\dot{Q}|^2 - V(Q)\big)\, dt = \int_{\tau}^{\infty} \big(\frac{1}{2}|\dot{q}|^2 - V(q)\big)\, dt$$

$$\int_0^{\frac{\tau}{\lambda}} \big(\frac{1}{2}|\dot{Q}|^2 - V(Q)\big)\, dt = \lambda \int_0^{\tau} \frac{1}{2}|\dot{q}|^2 dt + \frac{1}{\lambda}\int_0^{\tau} -V(q)\, dt$$

To minimize the last expression we choose $\lambda = \sqrt{\frac{P}{K}}$ where $P = \int_0^\tau -V(q)\, dt$ and $K = \int_0^\tau \frac{1}{2}|\dot{q}|^2 dt$. In this way we find that $\int_0^{\frac{\tau}{\lambda}} \big(\frac{1}{2}|\dot{Q}|^2 - V(Q)\big)\, dt = 2\sqrt{KP}$. Since $I(q) = \inf\limits_{u \in \Gamma(\omega)} I(u)$, it must be $K + P \leq 2\sqrt{KP}$, that is $K = P$. But (13.a) implies that $K - P = h_o \tau$; then $h_o = 0$ because $\tau > 0$.

q.e.d.

Now we are interested in finding conditions for V near the singularity assuring the existence of non collision solutions. This will be possible in the case of a strong force potential or when V is a radial function near the singularity, with a suitable behaviour, including the Keplerian case.

Proposition 2.3. *Under the hypotheses of theorem 2.2, if in addition V verifies the strong force condition, that is:*

(SF) there is a neighborhood N_e of e and a function $U \in C^1(N_e \setminus \{e\}; \mathbf{R})$ such that $|U(x)| \to \infty$ as $x \to e$ and $-V(x) \geq |U'(x)|^2$ for any $x \in N_e \setminus \{e\}$,

then the function q such that $I(q) = \inf I(\Gamma)$ (see theorem 2.2) describes a non–collision orbit, i.e. $q^{-1}(e) = \emptyset$; therefore q is a classical solution of (HS) *on $\mathbf{R}_+$.*

Proof. It's enough to notice that, in general, if $u \in \Gamma(\omega)$ and $I(u) < \infty$ then $u(t) \neq e$ for all $t \in \mathbf{R}_+$. In fact, by contradiction, if there were some $t > 0$ with $u(t) = e$ then there would exist $t_o \in (0,t)$ such that $u(t_o) \in \partial N_e$ and $u(s) \in N_e \setminus \{e\}$ for any $s \in (t_o,t)$. Therefore, with passages similar to (3), we infer that

$$|U(u(s))| \leq \sqrt{2}\, I(u) + |U(u(t_o))| < \infty$$

whereas $|U(u(s))| \to \infty$ as $s \to t_-$.

q.e.d.

Proposition 2.4. *Under the hypotheses of theorem 2.2, if in addition it holds that:*

(V5) there exist a constant $r_\circ > 0$ and a function $\phi \in C^1((0,r_\circ);\mathbf{R})$ such that $V(x) = \phi(|x-e|)$ for all $x \in B_{r_\circ}(e) \setminus \{e\}$ and $r\,\phi'(r) \to \infty$ as $r \to 0_+$,

then the function q such that $I(q) = \inf I(\Gamma)$ (see theorem 2.2) describes a non–collision orbit.

Remark 2.5. If *(V5)* holds, then $\phi(r)/\log r \to \infty$ as $r \to 0_+$ (but in general the viceversa is not true).
In the case $V(x) = -\big|\log|x-e|\big|^\beta$ for $x \in B_{r_\circ}(e) \setminus \{e\}$, the condition *(V5)* is satisfied if and only if $\beta > 1$. Moreover we note that *(V5)* is verified by potentials with the following behaviour:

$$V(x) = -\frac{1}{|x-e|^\alpha} + \varphi(|x-e|) \qquad \text{for} \quad x \in B_{r_\circ}(e) \setminus \{e\}$$

with $\alpha > 0$ and $\varphi \in C^1((0,r_\circ);\mathbf{R})$ such that $\lim_{r\to 0_+} r^{1+\alpha}\varphi'(r) \in (-\alpha,+\infty]$. It is also clear that this characterization does not exhaust all the cases described by *(V5)*.

Before proving the previous proposition, we recall the definition and the fundamental properties of the convexified of a real function.
Let I be an interval of $\mathbf{R}$ and let f be a function from I into $\mathbf{R}$, bounded from below. We call *convexified of f on* I the function $\hat{f} : \mathrm{I} \to \mathbf{R}$ defined by:

$$\hat{f}(x) = \sup_{(a,b)\in\mathcal{A}(f)} (ax+b) \qquad \text{for all } x \in \mathrm{I}$$

where $\mathcal{A}(f) = \{(a,b) \in \mathbf{R}^2 : ax+b \leq f(x)\ \forall x \in \mathrm{I}\}$. It holds that $\hat{f} \leq f$ on I and $\hat{f} = f$ if and only if f is convex and lower semicontinuous on I.
Moreover, given two applications $f,g : \mathrm{I} \to \mathbf{R}$ bounded from below, if $f \leq g$ on I, then $\hat{f} \leq \hat{g}$ on I.
In the sequel, we will make use of the following result.

Lemma 2.6. *Let $f \in C((0,a);\mathbf{R})$ bounded from below and such that $x\,f(x) \to \infty$ as $x \to 0$. Then $\hat{f} \in C((0,a);\mathbf{R})$ and $x\,\hat{f}(x) \to \infty$ as $x \to 0$.*

Proof. The continuity of $\hat{f}$ at an arbitrary point $x \in (0,a)$ follows by the convexity of $\hat{f}$ and by the fact that $\hat{f}$ is bounded from below. In addition, it holds that for any $M > 0$ there is some $x_M \in (0,a)$ for which $f(x) \geq \frac{M}{x}$ if $x \in (0,x_M)$. Therefore, noticing that the function $x \mapsto \frac{M}{x}$ is convex, we obtain that $\hat{f}(x) \geq \frac{M}{x}$ for any $x \in (0,x'_M)$ with $x'_M \in (0,x_M]$. Hence $x\,\hat{f}(x) \to \infty$ as $x \to 0$.

q.e.d.

Proof of proposition 2.4. The argument will be by contradiction, supposing $q^{-1}(e) = \{\tau\}$. For any $r \in (0,r_o)$, $t_1,t_2 > 0$ are uniquely determinated by these conditions: $t_1 < \tau < t_2$,

$q(t_i) \in \partial B_r(e)$ for $i=1,2$ and $q(t) \in B_r(e)$ for $t \in (t_1,t_2)$. We point out that, since V is radial on $B_{r_o}(e)$, as long as $q(t) \in B_{r_o}(e)$, the angular momentum does not change throughout the motion, and its value is zero, because the orbit goes through the singularity. Thus q follows a straight line inside $B_{r_o}(e)$, even if along possibly different directions, before and after the collision. To be precise:

$$\begin{aligned} q(t) &= e + \rho(t)\, e_1 \quad \text{if } t \in (t_1,\tau) \\ q(t) &= e + \rho(t)\, e_2 \quad \text{if } t \in (\tau,t_2) \end{aligned}$$

where $\rho(t) = |q(t)-e|$, $\quad e_i = \frac{q(t_i)-e}{|q(t_i)-e|}$ $(i=1,2)$. The contradiction will be reached constructing a function $Q \in \Gamma(\omega)$ such that $I(Q) < I(q)$.

To begin, we show that the singularity is crossed without change of direction. In fact, if not, consider the function Q different from q only for $t \in (t_1,t_2)$, where is defined as projection of the motion q along the segment joining $q(t_1)$ to $q(t_2)$. Explicitly:

$$Q(t) = \begin{cases} q(t) & t \in [0,t_1] \cup [t_2,+\infty) \\ \frac{1}{2}\big(q(t_1)+q(t_2)\big) + \hat{e}\cdot(q(t)-e)\,\hat{e} & t \in (t_1,t_2) \end{cases}$$

where $\hat{e} = \frac{e_2-e_1}{|e_2-e_1|}$ (notice that $e_1 \neq e_2$ since q is injective). It can easily checked that $Q \in \Gamma(\omega)$, $\int_{t_1}^{t_2} |\dot{Q}|^2 dt = \frac{1-e_1\cdot e_2}{2}\int_{t_1}^{t_2} |\dot{q}|^2 dt$ and $\int_{t_1}^{t_2} -V(Q)dt \le \int_{t_1}^{t_2} -V(q)dt$. Then, if $e_1 \neq -e_2$, $I(Q) < I(q)$.

In the case $e_1 = -e_2$ take $Q = q + g e_o$ where e_o is a fixed vector of $\mathbf{R}^N$ with norm one, orthogonal to e_1 and g is a scalar function defined in the following way:

$$g(t) = \begin{cases} \frac{t-t_1}{\delta}\mu & t \in (t_1,t_1+\delta) \\ \mu & t \in (t_1+\delta,t_2-\delta) \\ \frac{t_2-t}{\delta}\mu & t \in (t_2-\delta,t_2) \\ 0 & t \in [0,t_1] \cup [t_2,+\infty) \end{cases}$$

with appropriate $\mu, \delta > 0$. Observe that $Q \in \Gamma(\omega)$ and $|Q(t)-e| \le r$ for $t \in [t_1,t_2]$ if μ and δ are small enough. Moreover, called $h = g e_o$, it holds that

$$\int_{t_1}^{t_2} |\dot{Q}|^2 dt = \int_{t_1}^{t_2} |\dot{q}|^2 dt + 2\frac{\mu^2}{\delta} \tag{14}$$

$$\int_{t_1}^{t_2} \big(V(q)-V(Q)\big)\,dt = -\int_{t_1}^{t_2} \left(\int_0^1 V'(q+\lambda h)\cdot h\; d\lambda\right) dt.$$

Let f be the convexified of ϕ' on $(0,r_o)$. Since $r\phi'(r) \to +\infty$ as $r \to 0_+$, for lemma 2.6, $rf(r) \to +\infty$ and, changing r_o if necessary, we can say that $f(r) > 0$ for $r \in (0,r_o)$. Being $q(t)+\lambda h(t) \in B_r(e)$ for $t \in (t_1,t_2)$ and $\lambda \in (0,1)$, it holds that

$$\begin{aligned} V'(q+\lambda h)\cdot h &= \phi'(|q+\lambda h-e|)\,\frac{(q+\lambda h-e)\cdot h}{|q+\lambda h-e|} \ge \\ &\ge \lambda\,|h|^2\,\frac{f(|q+\lambda h-e|)}{|q+\lambda h-e|} \ge \lambda\,|h|^2\,\frac{f(|q-e|+|h|)}{|q-e|+|h|} \end{aligned}$$

and thus

$$\int_{t_1}^{t_2} \big(V(q)-V(Q)\big)\,dt \le -\frac{1}{2}\int_{t_1}^{t_2} |h|^2\,\frac{f(|q-e|+|h|)}{|q-e|+|h|}\,dt.$$

Since $|q(t)-e| \le \|\dot{q}\|_2 |t-\tau|^{\frac{1}{2}} = C|t-\tau|^{\frac{1}{2}}$, then $|q(t)-e| \le \mu$ if $|t-\tau| \le \sigma_\mu = \frac{\mu^2}{C^2}$. Therefore

$$\int_{t_1}^{t_2} \big(V(q)-V(Q)\big)\,dt \le -\frac{1}{2}\int_{\tau-\sigma_\mu}^{\tau+\sigma_\mu} |h|^2\frac{f(|q-e|+|h|)}{|q-e|+|h|}\,dt \le -\frac{1}{2}\,\frac{f(2\mu)}{2\mu}\int_{\tau-\sigma_\mu}^{\tau+\sigma_\mu} |h|^2 dt.$$

Taking μ so that $[\tau - \sigma_\mu, \tau + \sigma_\mu] \subseteq [t_1 + \delta, t_2 - \delta]$ we conclude that

$$\int_{t_1}^{t_2} \big(V(q) - V(Q)\big)dt \le -\frac{\mu^3}{2C^2}\, f(2\mu)$$

and finally, using (14):

$$I(Q) - I(q) \le -\mu^2 \left(-\frac{1}{\delta} + \frac{\mu}{2C^2} f(2\mu)\right)$$

But we know that $r\, f(r) \to +\infty$ as $r \to 0_+$; so, choosing μ sufficiently small, we find $I(Q) < I(q)$.

q.e.d.

3. Existence of a second homoclinic orbit

As in the previous section, we consider a potential V with a strict local maximum at 0 and a singularity at a point $e \in \mathbf{R}^N \setminus \{0\}$ and, at the beginning, assume that Ω is bounded and the strong force condition holds near the singularity. Under the hypotheses *(V1)* and *(V2)* we know that (HS) admits a homoclinic classic solution given by a even function $x \in C^2(\mathbf{R}; \mathbf{R}^N)$, such that $x(0) \in \partial\Omega$.

Now, our purpose is to find an other homoclinic orbit, geometrically different from x; to do this, we exploit the fact that V is singular at the point e and, roughly speaking, we look for an orbit describing a loop inside Ω around e. This new solution will be obtained with an argument similar to that used by Benci and Giannoni in [4] and by Tanaka in [19]. Actually, in [19] the potential V is assumed strictly negative at any $x \neq 0$, e so that the Lagrangian functional is automatically bounded from below. In our case we overcome this difficulty by modifying V in a suitable way. Moreover, as in [19], we assume a condition of quasi–concavity for V near 0 —hypothesis *(V6)*— verified, for instance, for V like (1). Finally we require a geometrical property, which will allow us to compare the critical levels of the two solutions.

Before stating our result, we introduce some notations: for any $\delta > 0$ let $V_\delta = \{x \in \Omega : V(x) \le -\delta\}$, let V_δ^0 and V_δ^1 be the components of $\overline{\Omega} \setminus V_\delta$ containing 0 and $\partial\Omega$ respectively and call $r_\delta =$dist(V_δ^0, V_δ^1). Then set $\Sigma = \{x \in \mathbf{R}^N : |x - e| = |e|\}$ and $v = \sup\limits_{x \in \Sigma \cap \Omega} |V(x)|$.

Theorem 3.1. *Let $V \in C^1(\mathbf{R}^N \setminus \{e\}; \mathbf{R})$ with $V(0) = 0$, Ω bounded, $e \in \Omega \setminus \{0\}$ and $\lim\limits_{x \to e} V(x) = -\infty$. Moreover suppose V satisfies (V1), (V2), (SF) and :*

(V6) (only for $N \ge 3$) there is some $\delta > 0$ such that $V(x) + \frac{1}{2}V'(x) \cdot x \le 0$ for any $x \in B_\delta$ and $V \in C^{1,1}(B_\delta; \mathbf{R})$;

(V7) there is some $\delta' > 0$ such that $2\pi|e|\sqrt{2v} < r_{\delta'}\sqrt{\delta'}$.

Then (HS) admits two geometrically distinct homoclinic orbits.

The proof consists of four main steps: firstly, fixed $T > 0$, we set up an approximating Dirichlet problem on $[0, T]$ using a potential V_T obtained cutting V out of Ω to a level T^{-1}; with a minimax method we get a solution q_T describing a sort of loop with initial and final point at 0, around the singularity. In the second part we give some uniform estimates for the approximating solutions that allow to pass to the limit $T \to \infty$ (third step) and, after some remarks, find a function $y \in C^1(\mathbf{R}; \mathbf{R}^N)$ satisfying the energy equation on $\mathbf{R}$ and solution of (HS) on the set $\{t : y(t) \in \Omega\}$. At the end, using the geometrical hypothesis *(V7)* we compare the values of the Lagrangian functional at x and y and deduce that the second orbit y cannot touch the boundary of Ω. This implies that y is a homoclinic solution different from x and describes a loop inside Ω.

I – APPROXIMATING PROBLEM

We start by modifying the potential V out of Ω, as explained before.

Lemma 3.2. *Fixed $\lambda > 0$, for any $T > 0$ there is a function $V_T \in C^1(\mathbf{R}^N \setminus \{e\}; \mathbf{R})$ with the following properties:*
(i) $V_T(x) = V(x)$ for $x \in \Omega \setminus \{e\}$;
(ii) $V_T(x) = 0$ if $dist(x, \Omega) \geq \frac{c_o}{T}$;
(iii) $V_T(x) \leq \frac{\lambda}{T}$ for any $x \in \mathbf{R}^N \setminus \{e\}$;
(iv) $|V_T'(x)| \leq c_1$ for any $x \notin \Omega$,
where c_o and c_1 are positive constants independent on x and T.

Proof. To begin, we prove that given $\varepsilon > 0$ there is a function $f_\varepsilon \in C^\infty(\mathbf{R}^N; \mathbf{R})$ such that $0 \leq f_\varepsilon \leq 1$, $f_\varepsilon(x) = 0$ for $x \in \overline{\Omega}$, $f_\varepsilon(x) = 1$ if $\mathrm{dist}(x, \Omega) \geq \varepsilon$ and $|f_\varepsilon'(x)| \leq \frac{A_N}{\varepsilon}$ for any $x \in \mathbf{R}^N$ with A_N constant with respect to x and ε but depending on the dimension N. In fact, pick $\varphi \in C^\infty(\mathbf{R}^N; \mathbf{R})$ such that $0 \leq \varphi \leq 1$, $\varphi(x) = 0$ if $|x|_\infty \leq \frac{1}{2}$ and $\varphi(x) = 1$ if $|x|_\infty \geq 1$; for any $x \in \mathbf{R}^N$ let $\varphi_\varepsilon(x) = \varphi\left(\frac{x}{\varepsilon}\right)$ and

$$g_\varepsilon(x) = \prod_{y \in Y_\varepsilon} \varphi_\varepsilon(x - y)$$

where

$$Y_\varepsilon = \{y = (y^1, \ldots, y^N) \in \mathbf{R}^N : y^1, \ldots, y^N \in \varepsilon\mathbf{Z},\ \mathrm{dist}_\infty(y, \Omega) \leq \tfrac{1}{2}\varepsilon\}.$$

One can readily see that $g_\varepsilon(x) = 0$ for $x \in \overline{\Omega}$, $g_\varepsilon(x) = 1$ if $\mathrm{dist}_\infty(x, \Omega) \geq \frac{3}{2}\varepsilon$ and clearly $0 \leq g_\varepsilon \leq 1$ and $g_\varepsilon \in C^\infty(\mathbf{R}^N; \mathbf{R})$. In addition, taken $x \in \Omega^c$ with $\mathrm{dist}_\infty(x, \Omega) < \frac{3}{2}\varepsilon$ the derivative of g_ε at x involves at most 2^N terms corresponding to some points $y_1, \ldots, y_{2^N} \in Y_\varepsilon$ so that

$$\left|\frac{\partial g_\varepsilon}{\partial x_i}(x)\right| = \left|g_\varepsilon(x) \sum_{j=1}^{2^N} \frac{1}{\varphi_\varepsilon(x - y_j)} \frac{1}{\varepsilon} \frac{\partial \varphi}{\partial x_i}\left(\frac{x - y_j}{\varepsilon}\right)\right| \leq \frac{A_N}{\varepsilon}.$$

Since $|x|_\infty \leq \sqrt{N}|x|$, taking $f_\varepsilon = g_\eta$ with $\eta = \frac{2}{3\sqrt{N}}\varepsilon$ we obtain the desired function. Now, fixed $\lambda > 0$, let $a = (2\max_{x \in \partial\Omega} |V'(x)|)^{-1}$. If $\mathrm{dist}(x, \partial\Omega) < \frac{\lambda a}{T}$ then $|V(x)| \leq \frac{\lambda}{T}$ because there is a point $y_x \in \partial\Omega$ such that $|x - y_x| < \frac{\lambda a}{T}$ and so $|V(x)| \leq |V'(y_x)|\,|x - y_x| + \frac{1}{2a}|x - y_x| \leq \frac{1}{a}|x - y_x| \leq \frac{\lambda}{T}$. Finally, for any $x \in \mathbf{R}^N \setminus \{e\}$ define:

$$V_T(x) = (1 - f_{\frac{\lambda a}{T}}(x))V(x).$$

The properties of $f_{\frac{\lambda a}{T}}$ imply the thesis.

q.e.d.

For $T > 0$ let $E_T = W_0^{1,2}([0, T]; \mathbf{R}^N)$ be the Hilbert space with the usual norm $\|u\|_T^2 = \int_0^T |\dot{u}|^2 dt$; consider its open subset $\Lambda_T = \{u \in E_T : u(t) \neq e\ \forall t \in [0, T]\}$ and the functional $I_T : \Lambda_T \to \mathbf{R}$ given by $I_T(u) = \int_0^T \left(\frac{1}{2}|\dot{u}|^2 - V_T(u)\right)dt$ for $u \in \Lambda_T$. Notice that, for lemma 3.2 *(iii)*, the functional I_T is bounded from below: $I_T(u) \geq \frac{1}{2}\|u\|_T^2 - \lambda$ for any $u \in \Lambda_T$.

It can be shown in a standard way that $I_T \in C^1(\Lambda_T; \mathbf{R})$ and $q \in \Lambda_T$ is a critical point of I_T if and only if is classical solution of the following Dirichlet problem:

$$(D_T) \qquad \begin{cases} \ddot{q} + V_T'(q) = 0 \quad \text{in } (0,T) \\ q(0) = q(T) = 0 \end{cases}$$

We approach to the problem of the existence of critical points for I_T in a different way according to $N = 2$ or $N > 2$.

Case $N = 2$

Let $\Gamma_T^* = \{u \in \Lambda_T : \operatorname{ind}(u) = 1\}$ where $\operatorname{ind}(u)$ denotes the number of winding of u around the point e in some direction. To be precise $\operatorname{ind}(u) = \int_\gamma \frac{dz}{z-e}$ with $z \in \mathbf{R}^2$ and γ a closed curve in $\mathbf{R}^2$ parametrized by u. Clearly Γ_T^* is not empty, so that we can consider $\inf I(\Gamma_T^*)$.

Lemma 3.3. *For any $T > 0$ there exists $q_T \in \Gamma_T^*$ such that $I_T(q_T) = \inf I_T(\Gamma_T^*)$. Moreover q_T is a critical point of I_T.*

Proof. Let $(u_n) \subset \Gamma_T^*$ be a sequence such that $I_T(u_n) \to \inf I_T(\Gamma_T^*)$ as $n \to \infty$. Then for any $n \in \mathbf{N}$ it holds that $\frac{1}{2}\|u_n\|_T^2 = I_T(u_n) + \int_0^T V_T(u_n)dt \le C + \lambda$ with C independent on n. Hence, possibly for a subsequence, (u_n) converges to some $q \in E_T$ weakly in E_T and uniformly on $[0,T]$. The strong force condition implies that $q \in \Lambda_T$; otherwise, if $q(t) = e$ for some $t \in [0,T]$, then for any $n \in \mathbf{N}$ there is $t_n \in [0,t)$ such that $u_n(t_n) \in \partial N_e$ and $u_n(s) \in N_e$ for $s \in (t_n, t)$; hence, arguing as in proposition 2.3, we obtain for any $n \in \mathbf{N}$

$$|U(u_n(t))| \le \sqrt{2}\, I_T(u_n) + |U(u_n(t_n))| \le C' + \max_{x \in \partial N_e} |U(x)| < \infty$$

while $|U(u_n(t))| \to \infty$ as $n \to \infty$. In addition $\operatorname{ind}(q) = \lim \operatorname{ind}(u_n) = 1$, so that $q \in \Gamma_T^*$. For the dominated convergence theorem we infer that $\int_0^T -V_T(q)dt = \lim \int_0^T -V_T(u_n)dt$ and, on the other hand, using the weak convergence, $\liminf \int_0^T |\dot{u}_n|^2 dt \ge \int_0^T |\dot{q}|^2 dt$; thus $I_T(q) \le \liminf I_T(u_n) = \inf I_T(\Gamma_T^*)$ and so, for $q \in \Gamma_T^*$, $I_T(q) = \min I_T(\Gamma_T^*)$.
To verify the second statement it suffices to notice that if $u \in \Gamma_T^*$, $\varphi \in C_c((0,T); \mathbf{R}^2)$ and $s \in \mathbf{R}$ then $u + s\varphi \in \Gamma_T^*$ for $|s|$ small enough. Therefore, since $q \in \Gamma_T^*$ and $I_T(q) = \min I_T(\Gamma_T^*)$, for any $\varphi \in C_c((0,T); \mathbf{R}^2)$ it holds that

$$\lim_{s \to 0} \frac{I_T(q + s\varphi) - I_T(q)}{s} = \frac{\partial I_T(q)}{\partial \varphi} = 0$$

that is $I_T'(q) = 0$.

q.e.d.

Case $N > 2$

Here, we follow Bahri and Rabinowitz [3] to define and solve a minimax problem giving the existence of a non–zero solution for (D_T).

Let $\Gamma_T = \{\gamma \in C(D^{N-2}; \Lambda_T) : \gamma(x) = 0\ \forall x \in \partial D^{N-2}\}$ where $D^{N-2} = \{x \in \mathbf{R}^{N-2} : |x| \le 1\}$. Given $\gamma \in \Gamma_T$, the function $(x,t) \mapsto \tilde{\gamma}(x,t) = \frac{\gamma(x)(t) - e}{|\gamma(x)(t) - e|}$ is well defined on

$D^{N-2}\times[0,T]$. Being $\gamma(x)(t)=0$ for $(x,t)\in\partial(D^{N-2}\times[0,T])$, we can consider as domain of $\tilde\gamma$ the quotient $D^{N-2}\times[0,T]/\partial(D^{N-2}\times[0,T])\simeq S^{N-1}$ so that $\tilde\gamma$ is a continuous function from S^{N-1} into S^{N-1} (we use the notation $S^M=\{x\in\mathbf{R}^{M+1}:|x|=1\}$). In this way the set $\Gamma_T^*=\{\gamma\in\Gamma_T:\deg(\tilde\gamma)\neq 0\}$ is well defined and non–empty (see lemma 1.2 of [3]) and the number

$$c(T)=\inf_{\gamma\in\Gamma_T^*}\max_{x\in D^{N-2}} I_T\big(\gamma(x)\big)$$

is meaningful. Notice that, adopting the agreement $D^0=\{0\}$ for $N=2$, then Γ_T^* corresponds to that one defined in the case $N=2$ and $c(T)=\inf I_T(\Gamma_T^*)$.

Lemma 3.4. *I_T satisfies the Palais–Smale condition.*

Proof. Let $(u_n)\in\Lambda_T$ be a Palais–Smale sequence. With the same argument of the previous proof, we obtain that, possibly passing to a subsequence, u_n converges to some $q\in\overline{\Lambda}_T$ weakly in E_T and uniformly on $[0,T]$ and, by the strong force condition, $q\in\Lambda_T$. Now, using the fact that $I_T'(u_n)\to 0$, in a standard way one can verify that $u_n\to q$ strongly in E_T.

q.e.d.

Lemma 3.5. *For any $T>0$ $c(T)>0$ and $c(T)$ is a critical level for I_T.*

Proof. First, we show that $c(T)>0$. In fact, fix $\delta\in(0,\frac{|e|}{2})$ such that $B_{2\delta}\subseteq\Omega$; taken a generic $\gamma\in\Gamma_T^*$, since $\deg(\tilde\gamma)\neq 0$ there exists $(x_o,t_o)\in D^{N-2}\times[0,T]$ such that $|\gamma(x_o)(t_o)|\geq 2\delta$. Call $u=\gamma(x_o)$. For $u(0)=0$, there are $t_1,t_2\in(0,t_o]$ such that $t_1<t_2$, $u(t_1)\in\partial B_\delta$, $u(t_2)\in\partial B_{2\delta}$ and $u(t)\in B_{2\delta}\setminus B_\delta$ for $t\in(t_1,t_2)$. Let $A=\{t\in[0,T]\ :\ u(t)\in\overline{\Omega}\}$ and $B=[0,T]\setminus A$. Then, called $\mu_\delta=\inf\{-V(x)\ :\ x\in B_{2\delta}\setminus B_\delta\}$ it holds that

$$\begin{aligned} I_T(u) &\geq \frac{1}{2}\int_{t_1}^{t_2}|\dot u|^2dt+\int_A -V(u)dt+\int_B -V_T(u)dt\\ &\geq \frac{1}{2}\frac{\delta^2}{t_2-t_1}+\mu_\delta(t_2-t_1)-\lambda\geq\sqrt{2\mu_\delta}\delta-\lambda=\mu. \end{aligned}$$

Fixing $\lambda\in(0,\sqrt{2\mu_\delta}\delta)$, we deduce that $\max_{x\in D^{N-2}} I_T(\gamma(x))\geq\mu>0$ and so, by the arbitrariness of γ, $c(T)\geq\mu>0$ follows for any $T>0$.
To prove the second statement we use a standard deformation lemma according to that, since I_T satisfies the Palais–Smale condition, if c is not a critical value of I_T, then for any $\bar\varepsilon>0$ there are an $\varepsilon\in(0,\bar\varepsilon)$ and a deformation η of Λ_T in E_T that sends the sublevel $I_T^{c+\varepsilon}$ into the sublevel $I_T^{c-\varepsilon}$, lowers the values of the functional I_T and acts identically out of $I_T^{c+\bar\varepsilon}\setminus I_T^{c-\bar\varepsilon}$ (here $I_T^a=\{u\in\Lambda_T:I_T(u)\leq a\}$). By contradiction suppose that $c(T)$ is not a critical level of I_T. For the first part, we can apply the previous result choosing $c=c(T)$ and $\bar\varepsilon=\frac{1}{2}c(T)$ and find a deformation η of Λ_T and an $\varepsilon\in(0,\bar\varepsilon)$ with the above–mentioned properties. Given an arbitrary $\gamma\in\Gamma_T^*$ we show that $\eta\circ\gamma\in\Gamma_T^*$. Clearly $\eta\circ\gamma\in C(D^{N-2};E_T)$ and for any $x\in D^{N-2}$ $I_T((\eta\circ\gamma)(x))\leq I_T(\gamma(x))<\infty$ so that, by (SF), $\eta\circ\gamma(x)\in\Lambda_T$. Since $0\notin I_T^{c+\bar\varepsilon}\setminus I_T^{c-\bar\varepsilon}$, if $x\in\partial D^{N-2}$ then $\eta\circ\gamma(x)=\eta(0)=0$. Hence $\eta\circ\gamma\in\Gamma_T$. Moreover the homotopy invariance of the degree implies that $\deg(\widetilde{\eta\circ\gamma})=\deg(\tilde\gamma)\neq 0$ and so $\eta\circ\gamma\in\Gamma_T^*$. Now take $\gamma_o\in\Gamma_T^*$ such that $\max_{x\in D^{N-2}} I_T(\gamma_o(x))\leq c+\varepsilon$ and call $\gamma_1=\eta\circ\gamma_o$. Then $\gamma_1\in\Gamma_T^*$ and being $\eta(I_T^{c+\varepsilon})\subseteq I_T^{c-\varepsilon}$, we get $\max_{x\in D^{N-2}} I_T(\gamma_1(x))\leq c-\varepsilon$ in contrast with the definition of c.

q.e.d.

II – UNIFORM ESTIMATES

For any $T > 0$ let $q_T \in \Lambda_T$ be the solution of (D_T), founded as minimum point of I_T on Γ_T^* for $N = 2$, or as minimax point of I_T with respect to Γ_T^* in the case $N > 2$.

Lemma 3.6. *There is $\delta_o > 0$ such that $|q_T|_\infty \geq \delta_o$ for each $T > 0$.*

Proof. If $N = 2$ it suffices to notice that $|q_T|_\infty \geq |e|$ for any $T > 0$ because $\mathrm{ind}(q_T) = 1$. In the case $N > 2$ we use the hypothesis *(V6)* in the following way: let h_T be the total energy of q_T; it holds that $h_T = \frac{1}{2}|\dot{q}_T(0)|^2 \geq 0$. Now, if $h_T = 0$, then q_T would be solution of the Cauchy problem:

$$\begin{cases} \ddot{q} + V_T'(q) = 0 \quad \text{in } (0,T) \\ q(0) = \dot{q}(0) = 0 \end{cases}$$

and, by the uniqueness of the solution, for $V_T = V$ is $C^{1,1}$ in a neighborhood of 0, it follows that $q_T(t) = 0$ for any $t \in [0,T]$, in contrast with the fact that $c(T) > 0$. So, $h_T > 0$ for any $T > 0$ and consequently:

$$\frac{1}{2}\frac{d^2}{dt^2}|q_T(t)|^2 = -2V_T(q_T(t)) - V_T'(q_T(t)) \cdot q_T(t) + 2h_T > -2V_T(q_T(t)) - V_T'(q_T(t)) \cdot q_T(t).$$

For *(V6)* $\frac{d^2}{dt^2}|q_T(t)|^2 > 0$ if $q_T(t) \in B_{\delta_o}$ (we can always suppose $B_{\delta_o} \not\ni e$ and $B_{\delta_o} \subset \Omega$ so that $V_T = V$ in B_{δ_o}). Let $t_T^* \in [0,T]$ such that $|q_T(t_T^*)| = |q_T|_\infty$. Being $\frac{d^2}{dt^2}|q_T(t_T^*)|^2 \leq 0$, we deduce that $q_T(t_T^*) \notin B_{\delta_o}$, i.e. $|q_T|_\infty \geq \delta_o$.

q.e.d.

Lemma 3.7. *Chosen $T_o > 0$, for each $T \geq T_o$ $\|q_T\|_T \leq$ constant.*

Proof. Firstly we notice that $\frac{1}{2}\|q_T\|^2 \leq I_T(q_T) + \lambda$ for any $T > 0$. Then we have to show that $c(T) \leq$ constant independent on T. For $e \in \Omega$ and Ω is open, fixed $T_o > 0$, there is $\gamma \in \Gamma_{T_o}^*$ such that $\gamma(D^{N-2} \times [0,T_o]) \subset \Omega$. Let $T \geq T_o$. For any $x \in D^{N-2}$ define $\gamma_T(x) = \gamma(x)\chi_{[0,T_o]}$, where $\chi_{[0,T_o]}(t) = 1$ if $t \in [0,T_o]$ and 0 otherwise. We point out that $\gamma_T \in \Gamma_T^*$ and $I_T(\gamma_T(x)) = I_{T_o}(\gamma(x))$ for all $x \in D^{N-2}$, so that $c(T) \leq \max_{x \in D^{N-2}} I_{T_o}(\gamma(x))$ for any $T \geq T_o$. Notice also that T_o is arbitrary.

q.e.d.

Lemma 3.8. *There is $\rho > 0$ such that $|q_T(t) - e| \geq \rho$ for each $t \in [0,T]$ and $T > 0$.*

Proof. Suppose the lemma is false. Then there are two sequences (T_n) and (t_n), with $0 < t_n < T_n$, such that $q_{T_n}(t_n) \to e$ as $n \to \infty$; moreover for any $n \in \mathbf{N}$ there is $s_n \in (0,t_n)$ such that $q_{T_n}(s_n) \in \partial N_e$ and $q_{T_n}(t) \in N_e$ for $t \in (s_n,t_n)$. The usual inequality, obtained with (SF),

$$|U(q_{T_n}(t_n))| \leq \sqrt{2}\, I_{T_n}(q_{T_n}) + |U(q_{T_n}(s_n))| \leq C' + \max_{x \in \partial N_e} |U(x)| < \infty$$

gives the contradiction.

q.e.d.

III – LIMIT PROCESS

By lemma (3.6), for any $T > 0$ there is $\tau_T \in (0,T)$ such that $|q_T(\tau_T)| = \delta_o$ and $|q_T(t)| < \delta_o$ if $t \in (0, \tau_T)$. Now, define

$$y_T = \begin{cases} q_T(t+\tau_T) & \text{for } t \in [-\tau_T, T-\tau_T] \\ 0 & \text{for } t \in \mathbf{R} \setminus [-\tau_T, T-\tau_T] \end{cases}$$

All the functions y_T belong to the Hilbert space $E = \{u \in W^{1,2}_{\mathrm{loc}}(\mathbf{R}; \mathbf{R}^N) : \int_{-\infty}^{\infty} |\dot{u}|^2 dt < \infty\}$ endowed with the norm $\|u\|^2 = |u(0)|^2 + \int_{-\infty}^{\infty} |\dot{u}|^2 dt$; moreover, by lemma 3.7, $\|y_T\| \leq$ constant for any $T > T_o$. Then there are a sequence $T_n \to \infty$ and a function $y \in E$ to that (y_{T_n}) converges, weakly in E and uniformly on the compact subsets of $\mathbf{R}$. Clearly $y \not\equiv 0$ because $|y(0)| = \delta_o$. Other properties of this function y are listed in the following lemma.

Lemma 3.9.
(i) $y \in C^1(\mathbf{R}; \mathbf{R}^N)$ *and, possibly for a subsequence,* $y_{T_n} \to y$ *in* $C^1_{\mathrm{loc}}(\mathbf{R}; \mathbf{R}^N)$*;*
(ii) $y(t) \in \overline{\Omega} \setminus B_\rho(e)$ *for any* $t \in \mathbf{R}$*;*
(iii) $\frac{1}{2}|\dot{y}(t)|^2 + V(y(t)) = 0$ *for any* $t \in \mathbf{R}$*;*
(iv) $y \in C^2(\mathbf{R} \setminus \mathcal{T}; \mathbf{R}^N)$*;*
(v) $\ddot{y}(t) + V'(y(t)) = 0$ *for any* $t \in \mathbf{R} \setminus \mathcal{T}$*,*
where $\mathcal{T} = \{t \in \mathbf{R} : y(t) \in \partial\Omega\}$*.*

Proof. (i). Let $R_o = 2 \operatorname{diam} \Omega$, fix $T_o > 0$ and call $C_1 = \sup\{|V_T(x)| : T > T_o,\ x \in B_{R_o} \setminus B_\rho(e)\}$ and $C_2 = \sup\{|V_T'(x)| : T > T_o,\ x \in B_{R_o} \setminus B_\rho(e)\}$. Clearly C_1, $C_2 < \infty$ and, from lemmas 3.5 and 3.8 (or 3.3 and 3.8 for the case $N = 2$), we deduce that for any $n \in \mathbf{N}$:

$$|\dot{y}_{T_n}|_\infty + |\ddot{y}_{T_n}|_\infty \leq C_3$$

with C_3 positive constant independent on n. So, by Ascoli–Arzelà theorem, passing to a subsequence, if necessary, $\dot{y}_{T_n} \to z$ in $C_{\mathrm{loc}}(\mathbf{R}; \mathbf{R}^N)$; but $\dot{y}_{T_n} \to \dot{y}$ weakly in $L^2(\mathbf{R}; \mathbf{R}^N)$, so that $y_{T_n} \to y$ in $C^1_{\mathrm{loc}}(\mathbf{R}; \mathbf{R}^N)$.

(ii) and *(iii)*. For lemma 3.8, $y(t) \notin B_\rho(e)$ for all $t \in \mathbf{R}$. Then, notice that $h_{T_n} \to \infty$ as $n \to \infty$, because $0 \leq h_{T_n} T_n \leq I_{T_n}(q_{T_n}) + 2\int_0^{T_n} V_{T_n}(q_{T_n}) dt \leq C_4 + 2\lambda < \infty$. Therefore, passing to the limit $n \to \infty$ in the energy equation for y_{T_n}, we get

$$\begin{array}{cll} \frac{1}{2}|\dot{y}(t)|^2 + V(y(t)) = 0 & \text{if} & y(t) \in \Omega \\ \dot{y}(t) = 0 & \text{if} & y(t) \notin \overline{\Omega}^c \end{array}$$

and this implies $y(t) \in \overline{\Omega}$ for any $t \in \mathbf{R}$, so that the energy equation is satisfied on $\mathbf{R}$.

(iv) and *(v)*. Pick a compact interval $[t_1, t_2]$ in an arbitrary component of $\mathbf{R} \setminus \mathcal{T}$. For any $t \in [t_1, t_2]$ $y(t) \in \Omega$ and then, for n sufficiently large $y_{T_n}(t) \in \Omega$, so that $V'_{T_n}(y_{T_n}(t)) = V'(y_{T_n}(t)) \to V'(y(t))$. From the dominated convergence theorem and from the result *(i)* we deduce that

$$\int_{t_1}^{t_2} -V'(y(t)) dt = \lim_{n\to\infty} \int_{t_1}^{t_2} -V'_{T_n}(y_{T_n}(t)) dt = \lim_{n\to\infty} \int_{t_1}^{t_2} \ddot{y}_{T_n}(t) dt = \dot{y}(t_2) - \dot{y}(t_1).$$

Hence, for the continuity of $-V' \circ y$ on $[t_1, t_2]$ we can say that $\dot{y}$ admits derivative and solves (HS) on $[t_1, t_2]$. For the arbitrariness of $[t_1, t_2]$, the proof is complete. q.e.d.

In the next lemma we complete the description of the solution y.

Lemma 3.10.
(vi) $y(-\infty) = 0$ *and* $\dot{y}(\pm\infty) = 0$*;*
(vii) If $\mathcal{T}$ *is empty or bounded from above then* $y(\infty) = 0$*;*
(viii) If $\mathcal{T} \neq \emptyset$ *then* $t_o := \inf \mathcal{T} > -\infty$ *and the function* $t \mapsto y_+(t) = y(t_o - t)$*, for* $t \geq 0$*, belongs to* Γ *and solves* (HS) *on* $\mathbf{R}_+$ *with energy zero.*

Proof. Let L_- be the α–limit set of the function $(y, \dot{y})$, given by

$$L_- = \{\, (\xi, \dot{\xi}) \in \mathbf{R}^N \times \mathbf{R}^N : \textit{ there exist } t_n \to -\infty \textit{ with } (y(t_n), \dot{y}(t_n)) \to (\xi, \dot{\xi}) \,\}.$$

For the properties *(ii)* and *(iii)* of y, $L_- \subseteq \{(\xi, \dot{\xi}) \ : \ \xi \in \overline{\Omega},\ \frac{1}{2}|\dot{\xi}|^2 + V(\xi) = 0\}$. Since $y \in E$, there is $\xi \in \partial\Omega \cup \{0\}$ such that $(\xi, 0) \in L_-$. Now, if $\zeta \in \Omega \setminus \{0\}$ is an other limit point of $y(t)$ as $t \to -\infty$, then the trajectory of y would cross infinitely many often the corona $B_{2r}(\zeta) \setminus B_r(\zeta)$ where $r > 0$ is choosen small enough so that $\beta = \min\limits_{r \leq |x-\zeta| \leq 2r} |V(x)| > 0$. Hence, for the formula (11), it should be $\int_{\mathbf{R}} |\dot{y}|^2 dt \geq \sqrt{2\beta}\, r\, n$ for any $n = 1, 2, \ldots$, contradicting the fact that $y \in E$. Therefore $L_- \subseteq (\partial\Omega \cup \{0\}) \times \{0\}$. In the same manner we can prove that the ω–limit set of $(y, \dot{y})$, denoted by L_+, is contained in $(\partial\Omega \cup \{0\}) \times \{0\}$. So $\dot{y}(t) \to 0$ as $t \to \pm\infty$.
Since $|y(t)| < \delta_o$ for any $t < 0$ and δ_o can be choosen in such a way $B_{2\delta_o} \subset \Omega$, $\mathcal{T}$ is bounded from below and y is a solution of (HS) on $(-\infty, 0)$. Therefore L_- is invariant and consequently cannot contain the points $(x, 0)$ with $x \in \partial\Omega$ because of *(V2)*. Then $L_- = \{(0, 0)\}$ and in particular $y(t) \to 0$ as $t \to -\infty$.
The same argument holds in the case $t \to \infty$ if we assume $\mathcal{T}$ empty or bounded from above.
The statement *(viii)* follows immediately from the properties of y given in lemma 3.9.
q.e.d.

Remark 3.11. Let us observe that, for V like (1), the above discussion holds also replacing our min–max class Γ^*_T by the mountain–pass one:

$$\tilde{\Gamma}_T = \{\gamma \in C([0,1]; \Lambda_T) \ : \ \gamma(0) = 0\,,\ I_T(\gamma(1)) < 0\}.$$

Using such a class one finds a sequence of critical points $\tilde{q}_T$ of I_T which satisfy the same uniform estimates of lemmas 3.6, 3.7 and 3.8 and so converge weakly to a non–zero solution of (HS) on $\mathbf{R}$. This is indeed the approach taken by Ambrosetti and Bertotti in [1]; but their modification of V makes it more difficult to obtain estimates for $\tilde{q}_T$. On the other hand we have not followed such a procedure since the estimates we need to find a second solution (see below) do not hold for the sequence $(\tilde{q}_T)$.

IV – EXISTENCE OF TWO HOMOCLINICS

At this point, to conclude the proof of theorem 3.1 we have to show the following last result.

Lemma 3.12. $\mathcal{T} = \emptyset$.

Proof. Fix $T_o > 0$ and let $\gamma_o \in \Gamma^*_{T_o}$ such that $\gamma_o(D^{N-2} \times [0, T_o]) = \Sigma$. Then, for any $x \in D^{N-2}$ define $\gamma_T(x) = \gamma_o(x)\chi_{[0,T_o]}$, with $\chi_{[0,T_o]}$ as in the proof of lemma 3.7. Clearly $\gamma_T \in \Gamma^*_T$,

$\gamma_T(D^{N-2}\times[0,T])=\Sigma$ and $I_T(\gamma_T(x))=K(x)+P_T(x)$ where $K(x)=\frac{1}{2}\int_0^{T_o}\left|\frac{d}{dt}\gamma_o(x)\right|^2 dt$ and $P_T(x)=\int_0^{T_o}-V_T(\gamma_o(x))dt$. Since $P_T(x)\le T_o v$ for any $x\in D^{N-2}$, it follows that for $T\ge T_o$

$$c(T)\le T_o v+\max_{x\in D^{N-2}}K(x). \tag{14}$$

We can always suppose that for any $x\in D^{N-2}$ $\gamma_o(x)$ describes a circular orbit passing for the origin, with radius $\rho_s\le|e|$ and angular speed $\omega_o=\frac{2\pi}{T_o}$ so that $\max_{x\in D^{N-2}}K(x)=\dfrac{2\pi^2|e|^2}{T_o}$. Hence, with an appropriate choice of T_o (14) becomes:

$$c(T)\le 2\pi|e|\sqrt{2v}$$

and, consequently

$$\|y_T\|_T^2\le 4\pi|e|\sqrt{2v}+2\lambda$$

where y_T is the approximating solution solving (D_T). Therefore, since y is weak limit of y_{T_n} as $n\to\infty$, the following inequality holds:

$$I(y_+)\le 4\pi|e|\sqrt{2v}+2\lambda. \tag{15}$$

Now, if $x\in E$ denotes the homoclinic orbit of (HS) given by theorem 2.2 and $x_+=x|_{\mathbf{R}_+}$, for $x(0)\in\partial\Omega$ and x has energy zero, we get that

$$I(x_+)=\int_0^\infty|\dot x_+|^2dt\ge\int_{\mathcal{T}_{\delta'}}|\dot x_+|^2dt\ge\frac{(2r_{\delta'})^2}{|\mathcal{T}_{\delta'}|}$$

where $\mathcal{T}_{\delta'}=\{t\in\mathbf{R}_+:\ x(t)\in V_{\delta'}\}$. On the other hand:

$$I(x_+)=\int_0^\infty -V(x_+)\,dt\ge\int_{\mathcal{T}_{\delta'}}-V(x_+)\,dt\ge\delta'|\mathcal{T}_{\delta'}|$$

and so

$$I(x_+)\ge 2r_{\delta'}\sqrt{\delta'}. \tag{16}$$

Then, by *(V7)*, fixing $\lambda\in(0,r_{\delta'}\sqrt{\delta'}-2\pi|e|\sqrt{2v})$, from (15) and (16) it follows that $I(y_+)<I(x_+)$. Now keeping into account that $I(x_+)=\inf I(\Gamma)$, from lemma 3.10 *(viii)* we infer that $\mathcal{T}=\emptyset$. q.e.d.

By eliminating the hypothesis (SF) we find a result similar to theorem 3.1 concerning generalized homoclinic orbits.

Theorem 3.13. *Let $V\in C^1(\mathbf{R}^N\setminus\{e\};\mathbf{R})$ with $V(0)=0$, Ω bounded, $e\in\Omega\setminus\{0\}$ and $\lim_{x\to e}V(x)=-\infty$. Suppose (V1), (V2), (V6) and (V7) hold.*
Then (HS) admits two geometrically distinct generalized homoclinic orbits.

Proof. For any $\varepsilon>0$ small enough, let $V_\varepsilon\in C^1(\mathbf{R}^N\setminus\{e\};\mathbf{R})$ satisfying the hypotheses of theorem 3.1 and such that $V_\varepsilon(x)=V(x)$ if $|x-e|\ge\varepsilon$ and $V_\varepsilon\le V$ on $\mathbf{R}^N\setminus\{e\}$. Let $(q_{T,\varepsilon})_T$ be the family of the approximating solutions corresponding to the problem for V_ε, ruled by the equation

$$\ddot q+V_\varepsilon'(q)=0. \tag{HS$_\varepsilon$}$$

Coming back to lemmas 3.6 and 3.7, it is easy to recognize that for any $\varepsilon \in (0, \varepsilon_o)$ and $T > T_o$ it holds that

$$|q_{T,\varepsilon}|_\infty \geq \delta_o \ , \quad \|q_{T,\varepsilon}\|_T \leq C$$

with δ_o and C positive constants independent on ε and T. We define $y_{T,\varepsilon}$ by translating $q_{T,\varepsilon}$ so that $|y_{T,\varepsilon}(0)| = \delta_o$ and $|y_{T,\varepsilon}(t)| < \delta_o$ for $t < 0$. Finally let y_ε be the limit of the approximating solutions $y_{T,\varepsilon}$ as $T \to \infty$. Such a y_ε is a homoclinic solution of $(\mathrm{HS})_\varepsilon$. Moreover, for any $\varepsilon \in (0, \varepsilon_o)$ we get that $|y_\varepsilon(0)| = \delta_o$ $|y_\varepsilon(t)| \leq \delta_o$ for $t \leq 0$ and $\int_{-\infty}^{\infty} |\dot{y}_\varepsilon|^2 dt \leq C'$. Hence there are a sequence $\varepsilon_n \to 0$ and a function $y \in E$ such that $y_{\varepsilon_n} \to y$ uniformly on the compact subsets of $\mathbf{R}$ and $\dot{y}_{\varepsilon_n} \to \dot{y}$ weakly in $L^2_{loc}(\mathbf{R}, \mathbf{R}^N)$. Then $|y(0)| = \delta_o$, $|y(t)| \leq \delta_o$ for $t \leq 0$, $y(t) \in \overline{\Omega}$ for all $t \in \mathbf{R}$ and $\int_{-\infty}^{\infty} -V(y)dt < \infty$; in fact $\int_{-\infty}^{\infty} -V(y)dt \leq \liminf \int_{-\infty}^{\infty} -V_{\varepsilon_n}(y_{\varepsilon_n})dt = \liminf \int_{-\infty}^{\infty} \frac{1}{2}|\dot{y}_{\varepsilon_n}|^2 dt \leq C'$. Therefore $y^{-1}(e)$ is a set of measure zero and $I(y) < \infty$. Moreover, keeping into account that for any $\varepsilon \in (0, \varepsilon_o)$ y_ε solves $(\mathrm{HS})_\varepsilon$ and its energy is zero, for any compact K contained in an arbitrary component of $\mathbf{R} \setminus y^{-1}(e)$ we get:

$$y_{\varepsilon_n} \to y \ \text{ in } C^2(K; \mathbf{R}^N)$$

$$\ddot{y}(t) + V'(y(t)) = 0 \ \text{ for any } t \in K$$

$$\frac{1}{2}|\dot{y}(t)|^2 + V(y(t)) = 0 \ \text{ for any } t \in K.$$

Therefore y is a non–zero generalized solution of (HS) on $\mathbf{R}$, with energy zero. Then, we argue exactly as in the proofs of lemmas 3.10 and 3.12 and obtain the thesis; more precisely, first we get $y(t) \to 0$ as $t \to -\infty$ and $\dot{y}(t) \to 0$ as $t \to \pm\infty$; then we can see that $y(t) \in \Omega$ for all $t \in \mathbf{R}$. Otherwise, if $t_o = \inf\{t \ : \ y(t) \in \partial\Omega\} > -\infty$, we can define the function $y_+(t) = y(t_o - t)$ for $t \geq 0$ and, noticing that $y_+ \in \Gamma$ and using *(V7)*, we have the contradiction $I(y_+) < I(x_+) = \inf I(\Gamma)$, where x_+ is the generalized solution of (HS) given by theorem 2.2. Finally we infer that $y(t) \to 0$ as $t \to \infty$. Thus y is a generalized homoclinic orbit of (HS) describing a loop inside Ω and so different from the solution found in theorem 2.2.

q.e.d.

Using proposition 1.3, we can state similar results when Ω is unbounded and its boundary is non–empty and bounded.

Theorem 3.14. *Let $V \in C^1(\mathbf{R}^N \setminus \{e\}; \mathbf{R})$ with $V(0) = 0$, Ω^c bounded, $\partial\Omega \neq \emptyset$, $e \in \Omega \setminus \{0\}$ and $\lim\limits_{x \to e} V(x) = -\infty$. Suppose (V1), (V2), (V3), (V6), (V7) and (SF) hold. Then* (HS) *admits two geometrically distinct homoclinic orbits.*

Proof. Theorem 2.2 and proposition 1.3 give a first solution x. To find the second one, we can repeat the argument used to prove theorem 3.1; we notice that all the passages hold again and the only new thing to show is a limitation for the approximating solutions q_T with respect to the sup norm. Arguing indirectly, if it were $|q_{T_n}|_\infty \to \infty$ for some sequence (T_n), then, for *(V3)*, with the same passages of (3), we would infer that for any $n \in \mathbf{N}$:

$$|U(q_{T_n}(t_n))| \leq \sqrt{2}\, I_{T_n}(q_{T_n}) + |U(q_{T_n}(s_n))|$$

where t_n is the time that achieves $|q_{T_n}(t_n)| = |q_{T_n}|_\infty$ and $s_n \in (0, t_n)$ is such that $|q_{T_n}(s_n)| = R_o$ while $|q_{T_n}(t)| > R_o$ if $t \in (s_n, t_n)$. But $I_T(q_T)$ is bounded independently on T and the same holds for $|U(q_{T_n}(s_n))|$; therefore $|U(q_{T_n}(t_n))| <$ constant, contradicting the fact that $|U(q_{T_n}(t_n))| \to \infty$, if $|q_{T_n}(t_n)| \to \infty$. Now, since $|q_T|_\infty$ and $\|q_T\|_T$ are bounded by constants independent on T, the weak limit of the approximating solutions, possibly translated, is a non–zero function $y \in C^2(\mathbf{R};\mathbf{R}^N)$ with the properties *(i)–(viii)* of lemmas 3.9 and 3.10. Finally we distinguish x from y using the hypothesis *(V7)* in the same way of the proof of lemma 3.12.

q.e.d.

Remark 3.15. As for theorem 3.13, if we omit the hypothesis *(SF)* in the previous theorem we get two generalized homoclinic solutions, geometrically distinct.

Remark 3.16. If $\Omega = \mathbf{R}^N$ we are in the situation studied in [19], where the author finds a homoclinic orbit assuming that $\limsup_{|x|\to\infty} V(x) < 0$. Arguing exactly as in the previous proof, we can improve this assumption allowing V to go to 0 at infinity.

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