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Virtual encounters: the murky and furtive world of mathematical inventiveness

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Abstract Based on Châtelet's insights into the nature of mathematical inventiveness, drawn from historical analyses, we propose a new way of framing creativity in the mathematics classroom. The approach we develop emphasizes the social and material nature of creative acts. Our analysis of creative acts in two case studies involving primary school classrooms also reveals the characteristic ways in which digital technologies can occasion such acts.

1 Introduction

It is notoriously difficult to faithfully recall or relate those moments in mathematical thinking that constitute new, original or unusual ideas. In his historical study of mathematical practice, the philosopher Gilles Châtelet (1993) identified these 'inventive' ideas with the actual diagrams produced in those fateful moments. For Châtelet, the making of a mathematical diagram is a material process that precedes formalism and acts as a kind of mid-wife for implicit, intuitive and even irrational thought, and—in André Weil's (1992) words—for the obscure analogies, murky reflections, furtive caresses and inexplicable tiffs that animate mathematics knowledge. These diagrams are borne out of the mathematician's gesture as she "cuts out a form of articulation" (Châtelet 2000, p. 8). Diagrams are thus conceived as inherently gestural and grounded in the movement of hands. His interest is less in the fixed, representational diagrams that eventually get codified in textbooks, but in the sketches through which mathematicians create new spaces (new dimensions, new kinds of planes) on the piece of paper, with and through their hands. Such sketches are more like physico-mathematical beings in that they are not intended to *represent* abstract objects. Mathematical inventiveness, according to this approach, exists in the dance between the gesturing and drawing hand, which expresses and captures the temporal and dynamic moment when the new or the original comes into (*in-venire*) the world at hand.

While Châtelet studies historical moments of mathematical inventiveness, such as Hamilton's quaternions and Cauchy's residue theorem, all of which introduced new ideas to the discipline, we will be interested in inventiveness at a more local level, focusing on new ideas in the mathematics classroom. In the context of Leikin's (2009, p. 151) distinction between "relative creativity" and "absolute creativity," this paper focuses on the former. We will extend Châtelet's ideas to the context of the contemporary mathematics classroom and show how certain kinds of digital technologies can yield inventive moments for learners by enhancing the interplay between gestures and diagrams.

In the next section, we describe Châtelet's notion of inventiveness and its sourcing in the gesture/diagram interplay. We then propose a way to identify instances of inventiveness in the mathematics classroom, exploring the way in which computer-based technologies might occasion the leaps into the virtual that Châtelet identifies within his case studies of mathematicians. Finally, we use two different examples of classroom interactions (both with primary school children, but using different software

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environments) to illustrate our proposed characteristics of inventiveness and to highlight the role of the technologies involved.

2 Rethinking creativity

In this section, we describe our approach to thinking about mathematical creativity. We first explain how the concept of *virtuality* functions in the embodied materialist philosophy underlying Châtelet's approach to inventive diagramming; we then list criteria for identifying creative activity.

2.1 Actualizing the virtual through gesture and diagram

Châtelet (2000) selects episodes in the history of mathematics and physics to show how particular diagrams have functioned as inventive "cutting out gestures" by which new mathematical practices have emerged. For instance, he shows how the fourteenth century "kinemathematician" Oresme revolutionized the study of "the motion of motion" by generating new diagramming techniques. Oresme referred to these diagrams as "configuration" by which he was able to study the spatial and diagrammatic rendering of various physical and mathematical concepts (Clagett 1968). The most historically significant of these configuration (Fig. 1), were those that used the geometry of similar figures and their ratios to show the equality of a right triangle, which represented uniform acceleration, with a rectangle, which represented uniform motion, constructed and superimposed at the velocity of the middle instant of acceleration.

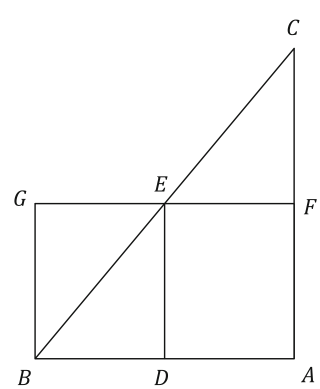

Fig. 1 Oresme's configuration for linear qualities unites extensive (time on the horizontal) and intensive (speed on the vertical) quantities so that distance can be calculated in terms of area. The area of triangle ABC gives the length travelled in time between B and A (equal to the area of BAFG)

Rather than conceptualizing diagrams as idealizations of mathematical relationships, however, Châtelet invites us to see diagramming as a dynamic process of excavation that conjures the virtual in sensible matter—in Oresme's case, the virtual acceleration of an object is conjured through an area diagram. In other words, the inventive diagram is an action that literally breaks down previously taken-for-granted determinations of what is sensible or intelligible, and actually carves up matter in new, unscripted ways. According to this approach the diagram is a physico-mathematical entity, with elasticity and mobility, that can "cut out" new dimensions in the plane—"the plane is made flesh, as it were" (Châtelet 2000, p. 34).

Châtelet's approach to mathematics is distinguished from both Platonic and Aristotelian traditions because of the way he leverages the two couplets: the virtual/actual and the possible/real. Mathematical activity, according to Châtelet, involves both *actualizing the virtual* and *realizing the possible*. Both realization and actualization bring forth something new into the situation (the possible and the virtual), but realization plays by the rules of logic, while actualization involves a different kind of determination, one that generates something ontologically new. The virtual marks that which is latent in an entity, while the possible is that which structures and limits the appearance of the entity according to current rules of inference and perceptual habits. The virtual (or potential) pertains to the indeterminacy at the source of all actions, whereas the possible pertains to the compliance of our actions with logical constraints. Thus novelty, genesis and creativity are fundamental concepts in a theory of actualization. Actualizing the virtual involves "an intrinsic genesis, not an extrinsic conditioning" (Deleuze 1994, p. 154). The virtual in sensible matter becomes intelligible, not by a reductionist abstraction or a "subtraction of determinations" (Aristotle's approach to abstraction), but by the actions (diagrams and gestures) that awaken the virtual or potential multiplicities that are implicit in any surface. Attending to processes of actualization demands that we reconceive the diagram less as a static figure and more in terms of the virtual motions generative of it. In other words, the virtuality of a diagram consists of all the gestures and future alterations that are in some fashion "contained" in it. *Inventive diagramming is an inherently gestural activity* that enlists the hands in all sorts of unscripted and unexpected ways. A triangle, for instance, does not exist as a rigid figure, or as a sign perched in space, but exists as a mobility or set of gestures. More generally, attending to the mobility (and potentiality) of a diagram allows one to grasp its inventiveness.

Consider, for instance, Archimedes Spiral, a curve generated by tracing a point as it moves away from a fixed point at a constant velocity along a straight line, which itself rotates around the fixed point at a constant velocity. Figure 2a shows the static, textbook version of the diagram. In Fig. 2b, the path travelled by the point can be seen in the faded traces, giving the spiral a more temporal, dynamic feel.

Figure 2a contains all the motion and gesture that was entailed in its construction, and yet we tend to perceive only the static image. The virtual is still *there* and can break out of the static diagram if inventive gesturing brings it forth. Through such gestures the boundary between the virtual and the actual is constantly shifted and re-made. In this sense, creative acts can be seen as ontological acts by which the new comes into being, forever changing the relationship between the virtual and the actual.

We find Châtelet's notion of the virtual powerful in large part because he has developed it specifically in the context of mathematics, where questions of the 'concrete' and the 'abstract' are so slippery. That said, the concept of the virtual has also been taken up in media studies. Burbules (2006), for instance, describes the virtual as that which creates the "feeling of immersion," which involves an extension or elaboration of what is present in experience. There is a sense that the virtual pertains to what is *potentially* present, but isn't actually present: "Actively going beyond the given is part of what engages us deeply in it" (Burbules 2006, p. 41). Burbules argues that digital technologies have particular characteristics that make them uniquely capable of engendering them. Burbules' construct of the virtual, however, has the disadvantage of imposing psychological states on the individual and thereby losing sight of the complex material interaction involved in such experiences. In the next section we draw on the Châteletian approach to inventiveness (and the actualizing of the virtual) and its focus on material acts (diagramming, gesturing) to explain our approach to creativity in the classroom.

2.2 Creative acts and material agency

Our approach treats creativity as *an action taken* that emerges in context, without being exhausted by it. In other words, our approach is in relation to existing theories that emphasize creativity as a property of a given individual. For example, Leikin et al. (2009) write "we view creativity as a personal creativity that can be developed in schoolchildren" (p. 151). In contrast, we propose a conceptualization of creativity that is not bound to the individual's choice or discernment between alternative possible paths. Creativity is not a property or competency of a child, as in the approaches that seek to measure the flexibility or fluency of the child's thinking—see Torrance (1974). Creativity does not exist independent of its exercise. It is not that individuals are creative or not creative, but that their actions, in concert with other material actions, may express creativity. Our approach thus focuses more on the processes of creation, rather than on the product, as proposed by Davis and Rimm (2004). Also, our sense of creativity focuses on novelty, which Plucker and Beghetto (2004) argue is one of the two key elements of creativity, the other one being usefulness. In some approaches, novelty (original, new, unique) qualifies the thing created, the product, but sometimes it also (eventually) qualifies the individual creator (and thus, as in Leikin et al. (2009), the schoolchild *is* creative when she is fluent, flexible and original). We believe that Châtelet's approach to actualization allows us to shift our attention away from the doer and focus on the doing—and resist the temptation to read these actions as reflections of a mental state—thus enabling us to study creativity in the classroom in new ways.

From this point of view, we propose to conceptualize mathematical inventiveness in terms of four essential characteristics. A creative act:

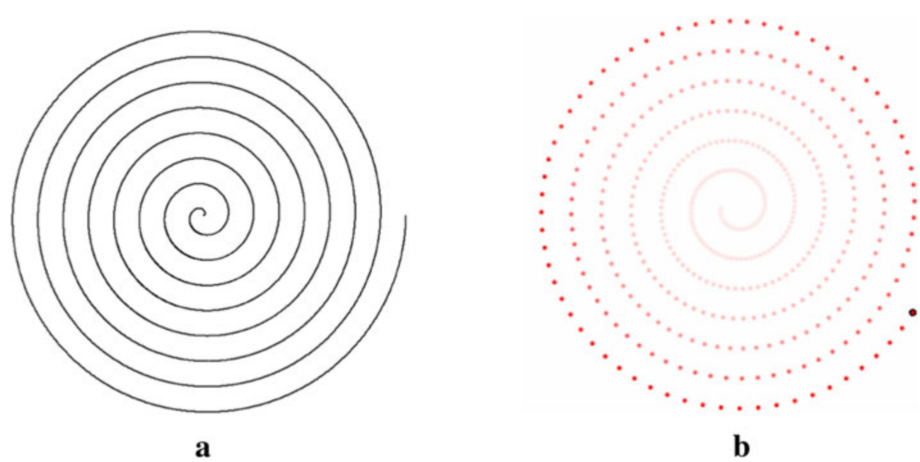

Fig. 2 Archimedes' spiral: **a** the static form and **b** a dynamic trace

1. introduces or catalyzes the new—quite literally, it brings forth or makes visible what was not present before,
2. is unusual in the sense that it must not align with current habits and norms of behavior,
3. is unexpected or unscripted, in other words, without prior determination or direct cause,
4. is without given content in that its meaning cannot be exhausted by existent meanings.

The first characteristic pertains to Châtelet's process of actualizing the virtual. This is an ontological claim about what constitutes the new. In actualizing the virtual, a creative act brings forth—literally makes manifest—an object which did not exist prior to the act. The second characteristic attends more carefully to the specific social context where the act occurs, and thereby frames the act as creative in relation to particular practices that are taken as norms. Thus creative acts are deemed such in relation to governing norms; the extent to which they are recognized as creative is conditioned by the context in which they occur. The third characteristic points to the collective emergent nature of creative acts whereby the new arises without being directly and formally determined by the intentions of the individuals involved. And finally, the fourth characteristic underscores the ways in which creative acts change the way language and other signs are used, and alter the meanings that circulate in a situation. Indeed, creative acts bring forth new uses of language and often break with the rules of common sign use, so that the new can be distinguished from that which is already familiar. These four qualities point to the centrality of the body and its movement (actions)—rather than internal mental disposition—in creative acts.

In the contexts we discuss in this paper, mathematical inventiveness is considered as a relation between the learner and the material world. This allows us to resist the tendency to locate learning in an individual body and, instead, to consider the ways in which learning is distributed over a collective social/material set of bonds. Here we follow Rotman (2008), who insists that the concept of body—and embodiment—has to be reconceived in terms of distributed agency across a network of interactions, the properties of which are constantly changing. Rotman's refrain of "becoming beside ourselves" captures this new acentered sense of agency, emerging this century, in part, because of new digital technologies that herald and hail a network "I" which thinks of itself as permeated by other collectives and assemblages. "Such an 'I' is plural and distributed, 'spilling out of itself' while forming new assemblages and new folds within its tissue" (de Freitas and Sinclair 2012, p. 7).

The virtual or potential multiplicities implicit in any of these assemblages can be awakened by material actions—gestures and diagrams—that constitute inventive moments (processes of actualization). Identifying that which affords such an environment a creative impulse then becomes the challenge. In Sect. 2 we examine two episodes from classroom interactions to illustrate our characteristics of inventiveness. But first, we elaborate on the role that digital technologies might play in our approach to creative activity.

2.3 Machines, mathematics and impulse

Although the word 'virtual' is often associated with the computer, we seek to remain faithful to Châtelet's use of the term, which has no digital requirements. That said, we think there are features of certain digital technologies that make them particularly conducive to creative acts.

Since its exciting and eye-opening beginnings with Papert's (1980) Turtle Geometry, the field of ICT has touted new digital technologies as being capable of radically changing the way students think and learn. Of particular interest to us are the so-called *expressive* technologies that provide tools that enable learners to construct mathematical objects and explore the relationships between them. The body syntonicity of Logo carved out a new subjectivity for the learner: she was now at the center of the mathematics, she *was* the mathematics. The square became ways of moving, within a vocabulary of walks and turns, that was in stark contrast to the square as a particular visual configuration or a particular property-based definition. We see such square-making experiences as potential inventive moments in which the human–technology interaction gives rise to new ways of thinking and moving. Similarly, data collection and physical output devices (e.g. motion probes and detectors) have introduced into education significant ways to connect simulations and real phenomena. They eliminate the algebraic channel as the sole channel into mathematical modeling and entail for learners a challenging *kinesthetic active* engagement with the technology (see Ferrara et al. 2006).

In this paper, we focus on two particular digital technologies that both attempt to mobilize mathematics: Dynamic Geometry Environments (DGEs) and motion detectors. Unlike Logo's more static drawn surface, these digital technologies temporalize mathematical behavior. In a DGE, for example, a triangle is not a representation of the abstract triangle, nor an example of a particular triangle, but *all* and *any* possible triangles, which the user can make by dragging the vertices that define it. The triangle has been inscribed in a new space, a stretchy space of continuous transformation. In describing the shift from declarative geometry (written proofs and even command-driven constructions) to dragging geometry on one's screen with the mouse, Jackiw (2006) writes: "one's actions are inquisitive and usually tentative: one is seeking, rather than stating" (p. 156). The seeking hand, and—with some motion-controller technologies, the seeking body—can not only move freely across the screen, but can also put into motion an "improvisational choreography" of mathematical objects, with the trajectory of one object mathematically dependent on others. The result is "a single possible performance drawn from the limitless configuration space of the mathematics spread across the stage" (p. 156). Jackiw's words are profoundly important in understanding the possibilities of the gestural/diagrammatic interplay, because of the way in which the seeking hand—tentative and awkward at first—learns to move.

In the case of motion detection devices, the real-time feedback of the tool is what makes the graphs on the screen dynamic and responsive to *all* and *any* possible motion, which the user can perform with her body or an object. New ways of thinking are offered through the experience of this sensory-motor feedback (for example, you will move your hand faster when you anticipate steeper graphs; you will imagine and draw and gesture new diagrams as generated by particular motions you have not performed before). In discussing the change provoked by data capture technologies, Ferrara et al. (2006) emphasize the important interplay between the physical actions of the student and the real-time appearance of the graph on the screen—not only does the graph capture the mobility of the student, but as students see what happens on the screen, they can change the way they move. There's a double sense of controlling and being controlled in this simulation. Nemirovsky and Ferrara (2009) illustrate gestural/diagrammatic interplay in their description of one girl's gestures tracing the motion of two laser lights in order to discover a defined triangle shape that gives the trajectory of the composed motion.

The virtual is actualized in large part by the fact that, in these environments driven by the hand or body, the human is constantly reinscribing herself into the idealized, abstract mathematics. Speaking specifically of dynamic geometry, Jackiw (2006) writes that it is a milieu in which "the individual 'touches' raw mathematical ideas, where personal volition and physical exertion can make seismic impact on disembodied abstractions" (p. 155). Of particular interest here is the materiality of the mathematical objects being "touched" and creating something that is more than the sum of its parts.

3 Creative acts in the mathematics classroom

Our aim in this paper is to exemplify our proposed characterization of creative acts and to reflect back on the way in which digital technologies facilitate actualizations of the virtual. Drawing on two different contexts involving young children engaged in computer-based mathematical explorations, we use the fourfold characterization described in Sect. 2.2 to exemplify our notion of creative acts.

3.1 When do two lines intersect?

The episode described in this section was part of a larger research project aimed at studying the potential for using DGE in grades K-3. The particular lesson was conducted in a grade 1 classroom at a University Lab pre-K-6 school in an urban middle SES district. The children came from diverse ethnic backgrounds and with a wide range of academic abilities, with 25 % being special needs learners. The lesson lasted approximately 30 min and was conducted with a small group of 11 children (half the class) with the children seated on a carpet in front of a large screen. Two researchers (one being the first author), and the classroom teacher, were present for each lesson. The lesson presented in this paper focused on conceptualizing intersecting and parallel lines, which the students had never formally encountered. The students had already had two previous lessons involving *Sketchpad*.

3.1.1 Exploring intersecting lines

The lesson began with the children being shown several examples of pairs of points tracing out thickly-colored linear paths, with some pairs intersecting and others not (see Fig. 3). In talking about these pairs of lines, the children described the former as "touching." After students successfully identified pairs of lines that "touch" or not, the instructor offered the more technical word "intersection" to describe the former, which the children immediately connected to road crossings and car crashes.

The teacher opened a new sketch and used the line tool to construct two lines, coloring one red and the other blue. The lines were positioned so as to be non-parallel, but so that the intersection was not visible (see Fig. 4). When asked "Do you think these two lines meet?" the students all said "No" in chorus. Then one girl said, "But they can if you tilt it all the way down." The teacher began dragging

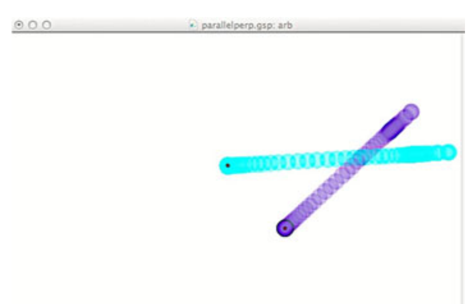

Fig. 3 Two points tracing intersecting paths in *Sketchpad*

the top line toward the bottom one and as the intersection became visible, one student said, "Now they have an intersection" and another added, "a very small one." The teacher dragged the top line up again to its original position (as in Fig. 4) and asked, "And here do they make an intersection?" The students chorused "No." After a few seconds, one boy said, "Oh yes they do, they do." Several students began talking at once, and one said, "Because they go out of the screen." So the teacher adjusted the screen (dragging the right corner of the window to enlarge it) so that the intersection was made visible.

The teacher then dragged the lines even further apart, so that their intersection was not visible, and asked the students to "use your imaginations" to decide whether they intersect. This time most children said "Yes." Then a few said that they wouldn't, with one girl explaining "because they are very far apart." Other children hedged, "I think it might."

T: Can we make some theories about why it might intersect?

Natasha: Because it's tilting (*referring to the red [top] line*).

The teacher invites other children to explain their reasoning.

Robert: The lines, um, can't meet at the edge of the screen because they are too far apart (*left hand raised with index finger and thumb forming a 'C' shape*) and they can't just like suddenly just have a straight line going down and meet (*index finger and thumb coming together,* Fig. 5a).

Jamie: Cause they are going like this (*two arms moving along a linear path,* Fig. 5b).

T: But do you think they would ever meet?

Robert: Yes, because they are both slanting and the red one is slanting toward the blue one.

The teacher repeated Robert's reasoning and then invited more contributions.

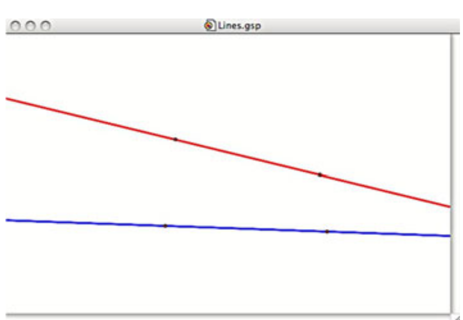

Fig. 4 A non-visible intersection in *Sketchpad*

Natasha: It's going to always connect somewhere because the red one is slanting (*tracing index finger along a linear path,* Fig. 5c) so it's going to connect somewhere over here (*having moved hand to end of screen, turned it into a vertical position and moving it up and down,* Fig. 5d).

T: Even if we can't see it, it's going to connect, it's going to intersect somewhere over here?

Jamie: I think it's never going to intersect.

T: Why?

Jamie: Because I just do.

T: What do you think about the theory though that this (*pointing to the red line*) is slanting more and more toward the blue?

Jamie: (*Standing up*) But the blue is also going like this (*using hands and arms to show that both lines are slanting,* Fig. 6a).

T: Oh I see. Interesting, so the blue is slanting as well.

Jamie: As long as both, the red's going down the blue's going down beside it so the line can't just go like that (*bringing his hands together, curving the top one down to touch the bottom one,* Fig. 6b) and then intersect.

T: That's interesting. Let's look at a situation where we can definitely see an intersection (*dragging the two lines so that their intersection is visible on the screen*). So now they're both slanting just like Jamie said before.

Natasha: But it's always going to slant because right there (*pointing to the left on the screen*) that's how thick it was so it's always going to slant.

T: It's always going to slant.

Saskia: It's going to intersect.

Robert: It's going to intersect at one point but it might, it might intersect somewhere far, far away.

T: We need to figure out how we're going to know when the lines are going to intersect even when we can't see it. So Jamie, no Natasha, said they're going to intersect because the red one is slanting toward the blue one.

Natasha: No, because that right there (*hand positioned so that index and thumb at a certain distance away,* Fig. 6c) isn't the same thickness and it's going to always intersect because it always gets smaller.

When asked what gets smaller, Natasha came to the screen and put her index finger on the red line and her thumb on the blue and moved toward the intersection while decreasing the gap between her index finger and thumb.

The teacher then announced they would look at another situation in which the intersection is not visible. After

Fig. 5 Children's gesturing with the lines

dragging the red line, Robert asserts that the lines will intersect "because it's slanting enough." When the teacher proposed to look at another one, Jamie asked, "Can we see if it is going to intersect or not?" No one expressed any surprise when the window was enlarged in order to make the intersection visible. Jamie then got up and traced his fingers along the intersection.

Finally, the teacher dragged the red line so that the two lines were parallel to each other and asked the students whether they would intersect. All students said "Nooo." Camille used Natasha's gesture of measuring the thickness. Jamie used both arms and said, "because they are going away from each other." The teacher invited a student who hadn't spoken yet to contribute:

Charlotte: Because they are both going the same way. One of them, they're not slanted, so, they're kind of slanted but they're not going to meet since one of them is not really slanted because they're just going like (*gesturing with one straight arm the direction of a line*) they're both going (*now bringing the other arm to move parallel with the first*) like that so they're never going to meet (*using her right hand to curve down towards the left one,* Fig. 6d).

The teacher then offered the word "parallel" to describe two lines that are never going to intersect.

3.1.2 Creating a new space for potential intersection

Two strategies are collectively generated for solving the problem of deciding when two lines will intersect: (1) the idea of the lines intersecting because one is slanted more than the other (or is slanted enough); and, (2) the idea that the lines intersect because the thickness between them is changing. Gestures are used throughout as the children make arguments about what will happen to the lines. The first gesture by Robert shows the lines "far apart" and the fact that they cannot suddenly "meet" at the edge of the screen. Interestingly, Natasha's gesturing of "thickness" also relates to the distance between the lines, albeit hers is one that she will describe as being able to change over time. However, before Natasha talks about thickness, Jamie and Natasha use their hands and fingers to invoke the current and future paths of the lines. Jamie's hands *are* the lines, moving steadily from left to right, whereas Natasha seems to point to the path of the line on the screen. Jamie's use of arms-as-lines is later used by Charlotte to explain why the lines will never intersect. We see these gestures as being evoked by the dynamic tracing out of lines they saw previously: the lines are all drawn out temporally and not just represented by, say, static arms placed at an angle to each other. This evocation of the lines is precisely what enables the movement past the limits of the screen and enables the children to create the possibility of an intersection that is not visible, beyond the objects visible on the screen. If, before, an intersection was something concrete and visible, it later becomes something that can be

Fig. 6 Children's gestures evoking new objects

imagined, potentially existing by virtue of reasoning about relative slanting of changing distance.

If the gesturing of extending lines brings to life the invisible intersection, the "thickness" gesture invokes a new relationship between the two lines, that of distance. First used by Robert to explain why the lines couldn't meet at the end of the screen, Natasha uses it after seeing the screen scroll in order to make the intersection between the lines visible. In scrolling the screen, the lines themselves remain static, but the "thickness" changes. It is this changing quantity that Natasha becomes aware of. Again, this gesture is later used by Camille to describe the invariance of the distance between two lines that will never intersect.

In summary, we see this episode as involving a series of gestural and verbal thought experiments that eventually unleash the potential point of intersection. The creative act involves the slow expansion of the plane circumscribed by the screen, extending it beyond what was previously visible to a plane that can welcome the crossing of lines not drawn. The potential point emerges both in the interactions amongst the students and the interactions with the screen/software (which sometimes shows the intersection, sometimes not, but always maintains the line as perfectly straight yet infinitely variable in that straightness). This reading of the episode focuses less on the creativity of any given child and more on the unexpected interactions between the material and human players in the classroom. In terms of our fourfold characterization of creativity, we claim the following:

1. There are several creative acts in this example. The first is the extension of the surface of the plane, which literally brings forth or makes visible what was not present before and unleashes the potential point of intersection. The technology plays a central role in affording this material act of creation. In addition, the students perform creative acts in gestures that literally make manifest the convergence and intersection of the lines. This catalyzes two ways of explaining when two lines will intersect when the point of intersection is not visible.
2. Given that the norms of behavior in the classroom in relation to lines and planes involved working with the concrete and visible, the collective actions (both movement and discourse) by which the plane was extended and the point of intersection created can be considered unusual, since such actions involved the non-visible and the potential. One could also argue that the particular gestures deployed by the children were unusual and broke with gesture norms, although we are unable to say definitively without more data. It is evident, however, from facial and other expressions, as well as from the teacher's invitation to repeat the gesture, that Jamie was creating and using gestures in ways that were entirely new to him.
3. The creative acts were genuinely unexpected as well as unscripted in the sense that the teacher was experimenting with a new technology as well as with ideas that are not usually part of the grade 1 curriculum. More importantly, the creative acts were also unexpected for the children. This is important because the teacher needs to be able to occasion similar creative acts with other groups of children. But even more importantly, the creative acts were unexpected in the sense that they were not *directly caused* either by the software or by the teacher, or by any individual student.
4. The existent meanings for "line" and "intersection" were in terms of their concrete and visible nature. The unfolding path of the lines on the screen as well as the 'uncovering' of a hidden intersection provoked gestures amongst the children that actualized infinitely extending lines and their invisible points of intersection. The new meanings of "line" and "intersection" were by no means exhausted by the old ones in the sense that the shift from the possible objects on the screen to the potential ones travelling off and on the screen fundamentally changed their nature.

We have purposely refrained from ascribing creativity to any one individual. Instead, in addition to the chorus of words and gestures circulating in the classroom, we highlight the agency of the projected dynamic image, as well as the computer and the teacher in the collective and creative activity that furnished the virtual space for the invisible intersection point and catalyzed new gestures and meanings for the students.

3.2 What kind of motion makes a vertical line?

The episode described in this section was part of a larger research project aimed at studying the potential for a graphical approach to functions through the aid of motion detectors in grades 2 through 5. A researcher (the third author) and the classroom teacher were present for each lesson. The particular lesson was conducted in a regular grade 4 classroom in Northern Italy. The children came from a peripheral area in the countryside and with a wide range of academic abilities, with 15 % having learning disabilities. The whole lesson lasted approximately 3 h and was conducted with a group of 16 children (the whole class). The episode presented is focused on conceptualizing straight lines as models of motion and begins by prompting the children to recollect the previous grade 3 explorations with the software *Motion Visualizer DV* (MV). The software—installed on a computer—works through the aid of a web camera linked to the computer. Based on live input, the MV captures and tracks, in real time, the motion of a colored object in a plane (an orange glove was used to track hand movements). As the object is moved in front of the web camera, the software displays on the right side two graphs decomposing the motion into the two dimensions of the plane and, on the left side, the trajectory in the 3D space of the room and the live video of the student moving the object on the plane (Fig. 7).

The class referred to the paper on which the movements were performed as "Movilandia" and the screen showing the graphs, which are generated by the movements of objects across the surface of the paper, as "Cartesiolandia." Each graph shows the movement of the object in relation to the particular dimension (in this case, horizontal and vertical dimensions). The students had moved the glove in Movilandia along straight trajectories—horizontal, vertical and oblique—and had watched the corresponding motion graphs that were generated on the screen, as well as observing the graphs generated when the glove was kept still, in each case investigating the associated relationships between position and time.

3.2.1 Recalling motion trajectories

The episode below occurred in a lesson aimed at recovering and sharing competencies with various kinds of motions already experienced in experiments with the MV. It began with the children being asked what they remembered about past activities with the MV, starting from 'shapes' in Movilandia, that is, motion trajectories. The discussion occurred at a time when no graphs were projected or movements performed. In talking about how many 'shapes' had been seen in Movilandia, several children recalled three main straight trajectories, and one child summarized them as "oblique, vertical and horizontal." At this point, the teacher perceived a latent confusion between the two worlds of Movilandia and Cartesiolandia, and invited the children to discuss the case of the oblique trajectory (in the following LH means left hand, RH right hand, LA left arm, RA right arm):

T: Oblique, how?

Arianna: It is made in a diagonal (*LH raised moving in an oblique line*) that, when you move in Movilandia, in Cartesiolandia, um, you move in a certain way in Movilandia and a line, vertical or horizontal or oblique, appears in Cartesiolandia.

Elisabetta: But it depends on the way you move.

The confusion became apparent, but Marco immediately pointed out the impossibility for a vertical line to appear in Cartesiolandia: "It cannot be vertical (*speaking in a whisper*)." The teacher repeated Arianna's words to give emphasis to the particular case of the vertical line as a possible graph: "You said that a line, vertical or horizontal or oblique, appears in Cartesiolandia."

Marco: No, a vertical line never appeared in Cartesiolandia.

T: Did a vertical line never appear?

Marco: In Movilandia we moved along vertical segments, but in Movilandia (*sic*: meaning Cartesiolandia) a vertical line never, it never appeared.

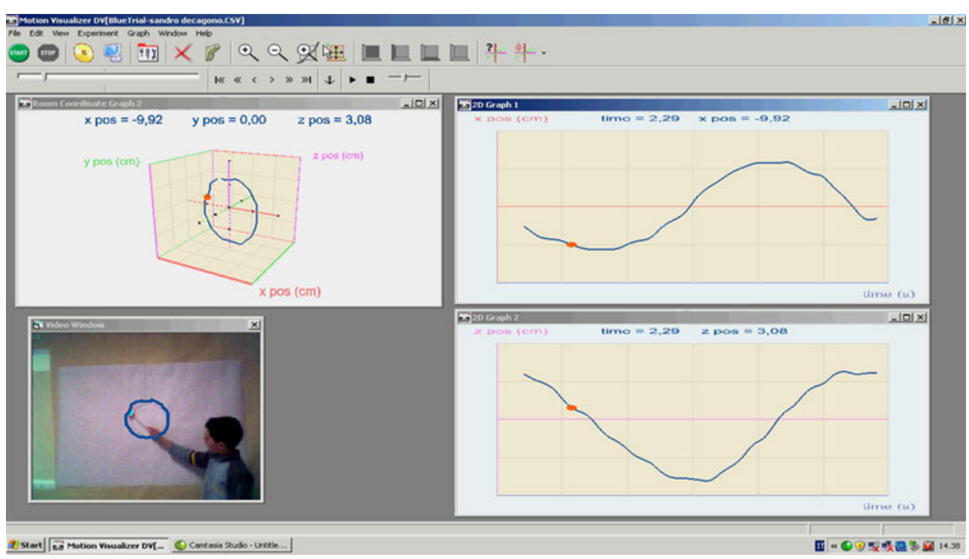

Fig. 7 Interface of the MV

Elisa: Or in a diagonal direction, um, or in a horizontal direction.

The children were strongly attached to the phenomenological side of the experience, recollecting past perception of the graphs encountered using the device. In Châtelet's terms, the students are focusing on the limits of the possible (of what can be realized) in the context of Cartesiolandia. The teacher invited the children to explain the fact that a vertical line "never appeared," stressing the impossibility of its occurrence with a conditional form in speech: "Why could not a vertical line appear?"

Gaia: Because the glove, um, when it moves, it moves (*RH raised in the air miming a short movement*) from bottom (*RH closed in a fist indicating a specific position*), since, um, when it appears in Cartesiolandia, the glove is always at the bottom (*indicating a specific position*) and then it makes the line in this way (*RH shifting horizontally from left to right:* Fig. 8a, b) as they moved, and it does not start in this way (*RH moving twice along a vertical direction, from top to bottom,* Fig. 8c, d) to make, um, the vertical line.

T: What do you want to add, Elisa?

Elisa: To me, because in the table (*LH kept still in the air, RH miming the axes*) that is in Cartesiolandia, it appears, um, to come vertical, it does not arrive at the end of the table (*open RH moving horizontally, from left to right:* Fig. 9a, b), but it should arrive at the end (*RH repeating previous gesture,* Fig. 9c, d).

T: What about you, Beniamino?

Beniamino: I wanted to say that, as Elisa said, there is the table (*LA raised vertically*), where here there is time (*RH moving twice horizontally direction, from left to right*) and here (*LA shifting twice vertically direction, from top to bottom*) there is the movement you make, um, but you cannot, for example, in little time, say, 10 s, in few seconds make, um, be able to have such a movement (*LH miming a vertical line*) on a platform, that is in a place making you understand that time passes (*LA raised vertically, and RH moving horizontally from left to right,* Fig. 10a, b), since it would be as if you stopped time (*LH pointing to a specific position,* Fig. 10c) and moved (*LH jumping twice in the air,* Fig. 10d).

The teacher then helped the children to share this "as if" movement in the class and to translate it in a straightforward 'as if' relationship between variables in the graph.

T: If you stop time, it is as if time didn't change, but what does it change?

Beniamino: Um, the movement.

T: The movement?

David: The position.

T: In what you call table, what does it appear vertically?

Ss: The position!

T: So, to have a vertical line (*RH miming it*) it should be, um, I stop time but?

David: The position changes.

Ss: Yeah (*laughing*).

Elisa: So, it's impossible (*with emphasis*)!

3.2.2 Creation of timeless motion

In this episode the new idea of the vertical line as a model of motion emerged. Arguments about what would happen in this instance are driven by hand and arm gestures. Gaia referred to the movement of the glove as she thinks of the real time origin (movement) of the graphs in Cartesiolandia ("when it appears in Cartesiolandia", "it makes the line in

Fig. 8 Gaia's RH moving horizontally and actualizing the vertical line

Fig. 9 Elisa's RH moving twice horizontally

Fig. 10 Beniamino's RH miming the passage of time and LH actualizing a timeless motion

this way as they moved"). Gaia's RH gestures revealed a tension between the motion experiments and the logical necessity of thinking of the vertical line as a potential graph. Gaia did not detect the difference between the two worlds, and went on to use the subject "the glove" in talking about what she experienced with the MV. Although the vertical line was not present before, it could now be imagined and conjured through gesture as a potential graph of Cartesiolandia. The line was actualized in her RH moving vertically up and down, while in speech she specified the impossibility of actually seeing it in Cartesiolandia: "and it does not start in this way to make, um, the vertical line." This conflict between gesture and speech reveals the power of gesture to conjure (and in our terms, create) an entity that has no existence. Elisa referred to some of the potential graphs of Cartesiolandia introducing "the table." Like Gaia, she recalled the visual experience with the real time origin of the graphs, when she said "it appears" and "to come vertical, it does not arrive at the end of the table." Again, the RH gestures function centrally in allowing the students to make manifest what is impossible—that being a vertical line in Cartesiolandia. Here we see how the impossible, rather than the possible, comes to exist (through gesture) in ways that move the discussion forward. The gesture evokes that which is denied existence by the constraints of Cartesiolandia.

In effect, the actualization of the potential line ("to come vertical") entails another consequence, that is, a line not arriving "at the end of the table" in Movilandia. This was clear from words of Beniamino: "you cannot, um, be able to have such a movement on a platform, that is, in a place making you understand that time passes," together with his gestures. Beniamino has realized that the motion that would generate a vertical line in Cartesiolandia cannot be a *real* motion in Movilandia because one would be at different positions at the same time. The fact that the motion cannot be actualized in Movilandia does not prevent its actualization through the gestures, with Beniamino's LH pointing to a specific position in the air ("you stopped time") and jumping from left to right ("and moved"), specifying in speech that "it would be as if you stopped time and moved." The experience is so immersive that Beniamino uses the 'as if' form and the subject 'you' (a generic 'you', not necessarily me).

All the arguments expressed the logical necessity of thinking of the impossibility of the vertical line, by making present and admitting its negation instead, that is, its imaginary possibility. If before the vertical line was something not at all present in Cartesiolandia, it later becomes something that can be imagined and potentially exist as generated by some movement (although an absurd movement). Gaia, Elisa and Beniamino made a series of gestural and verbal thought experiments with the hypothesis of the vertical line as a potential graph. The creative act involves the actualization of this graph and of gestural conjuring of its characteristics: a motion that does not move, and the occupying of two distinct positions at the same time. The virtual vertical line emerges in the lived contraposition between the real experiments and their possible models, through a recollection of past experiences with the MV and the visible graphs.

In terms of our fourfold characterization of creativity:

1. A creative act is the actualization of the potential vertical line in the gestures used by the students, which literally brings forth or makes visible an impossible object. This shifts the boundary between the virtual and the actual, and the related but distinct boundary between the possible and the real. Here the new that comes into being is an impossible vertical line evoked by the gestures. These gestures highlight the difference between creative acts of actualizing versus logical inferences that realize the possible. They also point to the role of the absurd (or impossible) in inventive activity.
2. Given that the norms of behavior involved working with the concrete and visible, the act is unusual, since it involves deploying gestures that engender a previously non-existent entity, and in this case, an impossible one.
3. The creative act was genuinely unexpected as well as unscripted in the sense that the teacher was experimenting with a new technology as well as with ideas that are not usually part of the grade 4 curriculum. More importantly, the creative act was also unexpected for the children. This is important because the teacher needs to be able to occasion similar creative acts with other groups of children. But even more importantly, the creative act was unexpected in the sense that it was not directly caused either by the software nor by the teacher, nor by any individual student. The creation of an impossible vertical line through gesturing emerged collectively through interaction between the students and researcher as they recollected previous encounters with the MV. The potentiality of the vertical line emerged by the discussion itself and by the need to understand the contradiction between the thoughts of two children (Arianna and Marco).
4. The existent meanings for "line" were in terms of their concrete and visible nature. The unfolding path of the vertical line as a graph generated by some movement provoked attempts at an explanation of its impossibility. The attempts were driven by gestures that actualized consequences of the invisible vertical line, and its meaning as a model of motion. This changed the nature of the meaning associated with the graph of a vertical line in an unexpected way, and favored a shift from the possible graphs on the screen to the potential new graph corresponding to an imagined movement.

The children created a new space where they could reason about the graph of the vertical line: a gestural space not physically possible, but mathematically actualizable. While the two girls keep thinking of the mathematical impossibility of the vertical line (in the context of the visible motions and graphs), Benny's thought experiment shifts attention to its actualization through a movement that happens in "no" time. This creative act is the seed for the idea—shared in the classroom—that the instantaneous motion *cannot* happen but *it could* happen. The discussion shows how the conditional language maps onto the virtual space of potentiality.

4 Discussion

The two excerpts exemplify the conception of creativity we developed based on Châtelet's work. They were chosen for this reason, of course, so it is worth considering what kinds of conditions were present to occasion them and, in particular, what roles the digital technologies played.

With respect to the latter issue, we do not believe that creative acts in the mathematics classroom require the use of digital technologies, nor that the use of DGEs and MBLs are sufficient to occasion creative acts. Rather, remaining true to our commitment to distributed agency, we focus on the specific ways in which the technologies were used—with particular tasks, around particular mathematical situations and particular ways of interacting between teachers and students. Keeping this in mind, it is possible to investigate the features of the use of these technologies that enabled actualizations of the virtual. Might mobilizations of mathematics—as exemplified here in technologies that animate diagrams and evoke the vibrant dynamic potential (or virtuality) which couples the mathematical to the material—open up all sorts of opportunities for creative acts? As mentioned above, Burbules (2006) pursues a similar exercise in his attempt to identify the features of digital technologies that may produce the sense of immersion associated with his construct of virtuality. Despite differences with our approach, we find useful the five features of digital technologies associated with producing virtual experiences: mobility, inhabitation, action at a distance, haptic sensitivity, and performative identities. All of these features essentially involve the potential: mobility is about being able to really move things (lines, points, ourselves) in new spaces (not the ones that satisfy our normal physical laws); inhabitation is about the extension or transformation of space and time, and the bodily occupation of that space and time; action at a distance is about our ability to transform the temporal dimension of our participation; haptic sensitivity is about the way in which our bodies are firmly implicated in the virtual spaces we explore—enabling a rapprochement of body and machine—and how sight, touch and feel create "as if" experiences; and, finally, performative identities is about the extension and transformation of our identities in cyberspaces.

Both technologies are first and foremost about mobility. But in the context of mathematics, this mobility is even more poignant than Burbules lets on, in part because of the ongoing program of detemporalization that *is* formal mathematics and in part because of the status of mathematical objects as being more or less inaccessible to actually being moved.

For the intersecting lines example, the movement of the points and lines occurred in a frictionless, infinitely extendable two-dimensional space. The children soon joined this new world, using their bodies, arms and hands to conjure more lines, thus extending and transforming their own spaces beyond that of the visible and the concrete (e.g. "it's going to connect somewhere over here", "it's always going to slant because right there", "it might intersect somewhere far, far away"). And while they do not interact directly with the mouse, or even the points and lines (the teacher does the "dragging"), their bodily involvement is acute, as can be seen in the dynasties of gestures they produce. It was initially important for the children that there be the possibility of moving the screen in order to make visible and real the point of intersection—here the children used the language of what "might" happen. And perhaps the shift to the potential was aided by the fact that they did not have direct access to the mouse in the sense that it brought forth shared gestures.

In the graphing example, we see movement both in children walking in certain ways so as to create graphs, and in the child's avatar on the screen, answering to his movement. The movement is highly coupled with the sense of action at a distance, as the child brings into being shapes on the screen through the behavior of his body. As with the previous example, the bodily involvement (now in a space where the technological device is no longer physically present) is palpable as the children use their arms and hands to conjure lines they have seen as well as lines they can imagine (e.g. "the glove is always at the bottom", "it does not start in this way", "it does not arrive at the end of the table, but it should arrive at the end", "be able to have such a movement", "it would be as if you stopped time and moved"). While their previous work with the technology began in the real, the interplay of their mobility and the inscriptions on the screen first led to possibilities ("I could move this way", "I could produce that graph") and eventually to the potential of timeless motion (expressed in terms of "as if").

Burbules' notion of performative identities, which emerges from his consideration of technologies such as social networks and virtual realities, seems at first blush much less relevant in our examples. However, we follow Rotman (2008) in asserting the way in which mathematical activity co-involves the discipline, the person and the material world—and that this co-involvement means that mathematical activity does not just produce more mathematics (or more learning), but also produces a new person in a new material world. We are fascinated by the question of how the children in these episodes can be thought of as performing new identities as they move in new ways in the classroom.

While we accord an important role to the digital technologies used in these two case studies, we also want to underline the way in which the creative acts we identified involved not only material agency, but also the agencies of the people in the classroom and the agency of the mathematics discipline itself. The tasks were designed so to develop ways of thinking about mathematical objects that are usually introduced in more formal ways later in the school curriculum. Both tasks also explicitly engaged students in the question of whether or not—as well as when—something exists, a question that is arguably one of the motivating concerns of the discipline. Both teachers were also able to use the multimodal expressions of the children (talk and gestures) to help coordinate emerging understandings. It is in this sense that we see the creative acts as occurring in the confluence of these multiple agencies and not just in the hands of a given child or a given technology.

5 Conclusion

In both examples, the creation of the new came about from situations previously unimagined, impossible, unusual and unexpected: the creative acts collectively engendered a new space, which enabled new forms of arguments to emerge. As we showed in our analysis, the diagram/gesture interplay provided a gateway to virtuality. The children's gestures were not windows into deduced or induced inferences—rather, they brought into being new mathematical objects that could be shared, in full sensuous inventiveness, in the classroom. The embodied materialist philosophy of Châtelet provided an alternative and complementary perspective to current research on gestures and diagrams. This research has focused on their potential for prompting or communicating intuitions and other visual or kinesthetic understandings, but has often overlooked the ways in which gestures and diagrams intersect. Using Châtelet, we shift interest to the ways in which gesturing and diagramming can together occasion new ways of thinking, moving and imagining, and thereby give rise to inventive processes. By separating the processes of actualization from processes of realization, and distinguishing between the potential (virtual) and the possible, Châtelet allows us to study the ways that students bring forth mathematical entities as material inventions and not simply as logical deductions. In a sense, the concept of the virtual becomes the animating force of the mathematical, giving flesh and mobility to what might have been otherwise considered abstract, ideal and inert.

Our criteria for identifying creative acts—which we can summarize as acts that introduce the new in an unpredictable way that transgresses current habits of behavior and exceed existent meanings—are consistent with Châtelet's approach to studying inventiveness in mathematics while also sensible to the distributed and collective enterprise of the classroom. These criteria should open up new areas of research, at once suggesting that creative acts might be less the exception than the rule and pointing to curricular possibilities for achieving this more democratic access to mathematical creativity.

Our use of Burbules enabled us to show how mobility is relevant to virtual encounters in mathematics and plays a seminal role in the shifting of boundaries between the actual (real or possible) and the virtual. Although we have only had space here to examine two very brief episodes in which creative acts flow and animate the interaction, we offer this analysis as a starting point for further studies of creativity as a material process of mathematical invention.

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