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SMALL BIALGEBRAS WITH A PROJECTION

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Abstract. Let A be a bialgebra with an H-bilinear coalgebra projection over an **arbitrary** subbialgebra H with antipode. In characteristic zero, we completely describe the bialgebra structure of A whenever H is either f.d. or cosemisimple and the H-coinvariant part R of A is connected with one dimensional space of primitive elements.

Contents

Introduction

Let A be a bialgebra and assume that the coradical H of A is a subbialgebra of A with antipode i.e. that A has the so-called dual Chevalley property.
The lifting method by N. Andruskiewitsch and H.-J. Schneider for the Hopf algebra A consists in analyzing the H-coinvariant part of the graded bialgebra $\mathrm{gr}\,(A)$, in transferring the information to $\mathrm{gr}\,(A)$ by usual bosonization, and finally in lifting it from $\mathrm{gr}\,(A)$ to A via the coradical filtration (see [AS]). In fact in [Rad] (and in [Maj] with categorical terms) it was proved that any Hopf algebra B having a projection, which is a bialgebra homomorphism, onto a Hopf algebra H can be reconstructed as a biproduct (called bosonization by Majid) of the H-coinvariant part of B and H itself. This applies in the above contest to $B = \mathrm{gr}\,(A)$ and to the usual projection of B onto $B_0 = H$.

Now, by using the Hochschild cohomology in monoidal categories, it was proved in [AMS, Theorem 2.35] that the canonical injection of H in A has a retraction $\pi : A \to H$ which is an H-bilinear coalgebra map. This led to the investigation of the structures of bialgebras A with an H-bilinear coalgebra projection onto an **arbitrary** subbialgebra H with antipode. There is a full description of these structures in terms of pre-bialgebras in ${}^H_H\mathcal{YD}$ with a cocycle (called dual Yetter-Drinfeld quadruples in [AMS, Definition 3.59]) and a bosonization type procedure. Namely (see [AMS, Theorem 3.64]) to such an A one associates a 5-tuple $(R, m, u, \delta, \varepsilon)$ (called pre-bialgebra), where (R, δ, ε) is a coalgebra in the category $({}^H_H\mathcal{YD}, \otimes, K)$, $u : K \to R$, $m : R \otimes R \to R$ are K-linear maps satisfying five equalities (see Definition 2.3) which make R a sort of unital bialgebra in ${}^H_H\mathcal{YD}$ with the following differences: the multiplication is non-associative and it is not a morphism of H-comodules. This particular pre-bialgebra is also endowed with a K-linear map $\xi : R \otimes R \to H$ (called associated cocycle) which fulfills six equalities (see Definition 3.1). Then A can be reconstructed by these data. In fact the bialgebra A is isomorphic to $R\#_\xi H$

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which is $R \otimes H$ endowed with a suitable bialgebra structure that depends on pre-bialgebra and its associated cocycle: this structure on $R \otimes H$ can be somehow regarded as a deformation of the usual bosonization structure recalled above via ξ. Our main goal is to describe the (co)algebra structure of $R \#_\xi H$. In this paper we do a first step: we consider the case when the coalgebra R is *thin* i.e. it is connected and the space of its primitive elements is one dimensional. We read the properties of R inside its associated graded ring and use these properties to show that this graded ring is in fact always a quantum line. Then we lift these type of information directly back to R (and not to $\mathrm{gr}(A)$ as in [AS]). It turns out that R, which usually carries a non-associative multiplication, is in fact **an associative K-algebra** but not a braided bialgebra in ${}_H^H\mathcal{YD}$. By means of this achievement, we can prove our main results. Explicitly in Theorem 3.30, we completely describe the bialgebra structure of A whenever H is either f.d. or cosemisimple. This new description allows us to construct in Theorem 4.2 another projection of A onto H which is normalized in the sense that it gives rise to a new pre-bialgebra $(R, m, u, \delta, \varepsilon)$ which is now a **braided bialgebra** in the category $({}_H^H\mathcal{YD}, \otimes, K)$ and in fact a quantum line.
In Theorem 4.5, we show how the obtained results apply to the special case when H is finite dimensional and it is the coradical of A. In this case the projection π is already normalized.
In a subsequent paper [AMSt] we will investigate the properties of ξ for a generic projection. We will construct for a given compatible datum (see Definition 3.27) a Hopf algebra with the required properties. This will enable us to construct some meaningful examples. In particular an example of a Hopf algebra of dimension 72 with a non normalized projection will be given.
The paper is organized as follows. Section 1 deals with general facts on thin coalgebras and divided power sequences of elements therein that will be used in the sequel. In Section 2 thin pre-bialgebras in ${}_H^H\mathcal{YD}$ are introduced and characterized by means of the associated graded coalgebra (see Theorem 2.14). Section 3 is devoted to the proof of the main results that is Theorem 3.29 and Theorem 3.30. Section 4 contains Theorem 4.2 and Theorem 4.5 that concern the normalization of the projection.

For the reader's sake we include here the following result that will be used in the sequel. In the finite dimensional case, a different proof can be found in [Ge, Lemma 0.2].

Theorem 0.1. *Let K be any field. Let A be a Hopf algebra over K. Let $z \in A$ such that*

$$\Delta_A(z) = g \otimes z + z \otimes 1_A, \qquad \text{and} \qquad gz = zg,$$

for some $g \in G(A)$. Suppose there exists a cosemisimple Hopf subalgebra B of A such that $z, g \in B$. Then there exists $\lambda(z) \in K$ such that

$$z = \lambda(z)(1_A - g).$$

Furthermore $\lambda(z) = 0$ whenever $g = 1_A$.
This holds whenever A is cosemisimple or A is f.d. and $\mathrm{char}(K) \nmid \dim(A)$.

Proof. Since B is cosemisimple, then B has a total integral $\lambda : B \to K$. By applying $B \otimes \lambda$ to both sides of $\Delta_A(z) = g \otimes z + z \otimes 1_A$, we get

$$1_A \lambda(z) = \sum z_{(1)} \lambda\left[z_{(2)}\right] = g\lambda(z) + z\lambda(1_A) = g\lambda(z) + z$$

so that $z = \lambda(z)(1_A - g)$.
If A is cosemisimple, then $B = A$ fulfills the initial assumption.
In the case when A is f.d., let B be the Hopf subalgebra of A generated by g and z.
Then B is a commutative Hopf subalgebra of A. In particular the antipode of B is involutive so that, since $\mathrm{char}(K) \nmid \dim(B)$, we obtain that B is cosemisimple. □

We assume for simplicity of the exposition that our ground field K **has characteristic** 0. Anyway we point out that many results below are valid under weaker hypotheses.

1. Thin Coalgebras

Recall that a *unital coalgebra* $((C,\Delta,\varepsilon),1_C)$ consists of a K-coalgebra (C,Δ,ε) and of a group like element, say $1_C \in C$. This means that there is a coalgebra homomorphism $u : K \to C, 1_C = u(1_K)$. Then, one can consider the set of primitive elements of the unital coalgebra $(C, 1_C)$ defined by

$$P(C) = \{c \in C \mid \Delta(c) = c \otimes 1_C + 1_C \otimes c\}.$$

For any coalgebra C we denote by

$$C_0 \leq C_1 \leq \cdots \leq C_n \leq \cdots$$

the coradical filtration of C. Set $C_{-1} = 0$. Let

$$\mathrm{gr}(C) = \bigoplus_{n\in\mathbb{N}} \frac{C_n}{C_{n-1}}$$

be the graded coalgebra associated to the coradical filtration of C. Recall that the coalgebra structure of $\mathrm{gr}(C)$ is defined as follows. For any $a, b \in \mathbb{N}$ such that $a + b \geq 1$, we define

$$\varphi_{a,b} : C_{a+b} \to \frac{C_a}{C_{a-1}} \otimes \frac{C_b}{C_{b-1}}$$

by setting $\varphi_{a,b}(c) = \sum (c_1 + C_{a-1}) \otimes (c_2 + C_{b-1})$. Note that this makes sense since

$$\Delta(c) \in \sum\nolimits_{0\leq i\leq a+b} C_i \otimes C_{a+b-i} \subseteq C_{a-1} \otimes C_{a+b} + C_{a+b} \otimes C_{b-1} + C_a \otimes C_b,$$

for every $c \in C_{a+b}$. Moreover $\ker(\varphi_{a,b}) = C_{a+b-1}$. Thus $\varphi_{a,b}$ factorizes through an injective morphism of K-vector spaces

$$\Delta_{a,b} : \frac{C_{a+b}}{C_{a+b-1}} \to \frac{C_a}{C_{a-1}} \otimes \frac{C_b}{C_{b-1}}.$$

For every $n \in \mathbb{N}$, let us define

$$\Delta_n : \mathrm{gr}(C)_n = \frac{C_n}{C_{n-1}} \to (\mathrm{gr}(C) \otimes \mathrm{gr}(C))_n = \bigoplus_{a+b=n} \mathrm{gr}(C)_a \otimes \mathrm{gr}(C)_b$$

to be the diagonal morphism of the family $(\Delta_{a,b})_{a+b=n}$. In this way one gets a graded K-linear map $\Delta : \mathrm{gr}(C) \to \mathrm{gr}(C) \otimes \mathrm{gr}(C)$. Define $\varepsilon_n : \mathrm{gr}(C)_n \to K$ by setting

$$\varepsilon_n = \varepsilon_{C|C_0}\delta_{0,n}.$$

In this way one obtains a graded K-linear map $\varepsilon : \mathrm{gr}(C) \to K$. Moreover

$$(\mathrm{gr}(C), \Delta, \varepsilon)$$

is a graded coalgebra. Recall that the coradical filtration of the associated graded coalgebra $\mathrm{gr}(C) = \oplus_{n\geq 0} \frac{C_n}{C_{n-1}}$ is given by

$$(\mathrm{gr}(C))_n = \oplus_{0\leq i\leq n} \frac{C_i}{C_{i-1}}.$$

Let C be a K-coalgebra, let $s \in \mathbb{N}$ and let $d_0, d_1, \ldots, d_s \in C$. Recall that $(d_i)_{0\leq i\leq s}$ is called a *divided power sequence* of elements in C whenever

$$\Delta(d_n) = \sum\nolimits_{t=0}^{n} d_t \otimes d_{n-t}$$

for any $0 \leq n \leq s$.

Definition 1.1. We will say that a K-coalgebra C is *a thin coalgebra* whenever

$$\dim_K C_0 = 1 \qquad \text{and} \qquad \dim_K P(C) = 1.$$

For every thin coalgebra C there is a unique coalgebra homomorphism $u : K \to C$ and $C_0 = Ku(1_K)$. In particular $(C, u(1_K))$ is a unital coalgebra.

Proposition 1.2. *Let C be a unital K-coalgebra. Then C is connected (i.e. $C_0 = K1_C$) if and only if* $\mathrm{gr}(C)$ *is connected. In this case*

$$P(\mathrm{gr}(C)) = \frac{C_1}{C_0} \qquad \text{and} \qquad \dim[P(\mathrm{gr}(C))] = \dim[P(C)].$$

In particular, if $\mathrm{gr}(C)$ *is a thin coalgebra, then C is thin too.*

Proof. The coradical of $\mathrm{gr}(C)$ coincides with the coradical of C. Hence the first assertion is trivial. If $\mathrm{gr}(C)$ is connected, then $P(\mathrm{gr}(C)) = \frac{C_1}{C_0}$. If C is connected then $C_1 = C_0 \oplus P(C)$ and hence $\dim[P(\mathrm{gr}(C))] = \dim[P(C)]$. □

Lemma 1.3. *Let C be an N-dimensional thin K-coalgebra. Then $\dim_K\left(\frac{C_n}{C_{n-1}}\right) = 1$ for any $0 \leq n \leq N-1$ and $C_n = C$ for any $n \geq N-1$.*

Proof. For any $n \geq 1$ consider the injective morphism of K-vector spaces

$$\Delta_{n,1} : \frac{C_{n+1}}{C_n} \to \frac{C_n}{C_{n-1}} \otimes \frac{C_1}{C_0}.$$

Since $C_1 = K1_C + P(R)$, then $\dim_K(C_1/C_0) = 1$ so that

$$\dim_K \frac{C_{n+1}}{C_n} \leq \dim_K \frac{C_n}{C_{n-1}} \text{ for any } n \geq 1.$$

Let $t = \min\{n \in \mathbb{N} \mid C_n = C_{n+1}\}$. Since

$$\dim_K \frac{C_t}{C_{t-1}} \leq \dim_K \frac{C_{t-1}}{C_{t-2}} \leq \cdots \leq \dim_K \frac{C_1}{C_0} = 1$$

and since, for $1 \leq n < t$ one has $C_n \neq C_{n+1}$, we deduce that

$$\dim_K \frac{C_t}{C_{t-1}} = \dim_K \frac{C_{t-1}}{C_{t-2}} = \cdots = \dim_K \frac{C_1}{C_0} = 1$$

Therefore $C = C_t$ has dimension $t+1$, so that $t = N-1$. □

Lemma 1.4. *Let C be an N-dimensional thin K-coalgebra. Let $t \in \mathbb{N}, 1 \leq t \leq N$ and let*

$$d_0, d_1, \ldots, d_{t-1}$$

be a divided power sequence of non-zero elements in C (e.g. $t = 1$).
Then $(d_i)_{0 \leq i \leq t-1}$ are linearly independent and can be completed to a basis

$$d_0, d_1, \ldots, d_{t-1}, d_t, \ldots, d_{N-1}$$

for C which is a divided power sequence of non-zero elements in C.
Moreover we have $d_0 = 1_C$, $P(C) = Kd_1$,

$$C_n = Kd_n + C_{n-1}$$

for any $0 \leq n \leq N-1$ and $C_{N-1} = C$.

Proof. The main idea comes from the proof of [AS, Theorem 3.2]. Let $A = C^*$ and let J be the Jacobson radical of A. Then, for any $n \in \mathbb{N}$, we have

$$J^n \simeq Hom_K\left(\frac{C}{C_{n-1}}, K\right) \qquad \text{and} \qquad \frac{J^n}{J^{n+1}} \simeq Hom_K\left(\frac{C_n}{C_{n-1}}, K\right),$$

where $C_{-1} = 0$ by definition.
By Lemma 1.3, we know that $\dim_K\left(\frac{C_n}{C_{n-1}}\right) = 1$ for any $0 \leq n \leq N-1$ and $C_n = C$ for any $n \geq N-1$.
Therefore $\dim_K\left(\frac{J^n}{J^{n+1}}\right) = 1$ for any $0 \leq n \leq N-1$ and $J^n = 0$ for any $n \geq N$.
Let $\alpha \in J \backslash J^2$. Then it is easy to show that $J^n = K\alpha^n + J^{n+1}$. In particular we have $K\alpha^N = J^N = 0$. Therefore we get that $1_A = \varepsilon_C, \alpha, \alpha^2, \ldots, \alpha^{N-1}$ is a system of generators of A (regarded as a vector space over K) and hence a basis since $\dim(A) = N$. We have also $\alpha^N = 0$.

Note that $d_0 \in G(C) = \{1_C\}$ so that $d_0 = 1_C$. Moreover

$$\Delta(d_1) = d_0 \otimes d_1 + d_1 \otimes d_0 = 1_C \otimes d_1 + d_1 \otimes 1_C$$

so that $d_1 \in P(C)$. Since $d_1 \neq 0$, we deduce that $P(C) = Kd_1$.
Let $0 \leq s \leq t-1$ be defined by

$$s = \max\{n \in \mathbb{N} \mid d_0, d_1, \cdots, d_n \text{ are linearly independent}\}.$$

Note that $s \geq 1$. Furthermore $d_0, d_1, \cdots, d_s$ are linearly independent and can so be completed to a basis of C.
Let $(e_i^*)_{0 \leq i \leq N-1}$ be the associated dual basis and set $\alpha = e_1^*$.
Note that $\alpha \in J \backslash J^2$. In fact $\alpha \in Hom_K\left(\frac{C}{C_0}, K\right) = J$ and $\alpha \notin Hom_K\left(\frac{C}{C_1}, K\right) = J^2$.
Thus we get that $1_A = \varepsilon_C, \alpha, \alpha^2, \ldots, \alpha^{N-1}$ is a basis of A regarded as a vector space over K and $\alpha^N = 0$. Let $(u_i)_{0 \leq i \leq N-1}$ be the dual basis associated to $(\alpha^i)_{0 \leq i \leq N-1}$ in C. The u_j's are uniquely determined by the relations $\alpha^i(u_j) = \delta_{i,j}$. Then

$$\left(\alpha^i \otimes \alpha^j\right)\Delta(u_n) = \alpha^{i+j}(u_n) = \delta_{i+j,n} = \left(\sum\nolimits_{t=0}^{n} \delta_{i,t}\delta_{j,n-t}\right) = \left(\alpha^i \otimes \alpha^j\right)\left(\sum\nolimits_{t=0}^{n} u_t \otimes u_{n-t}\right)$$

and hence

$$\Delta(u_n) = \sum\nolimits_{t=0}^{n} u_t \otimes u_{n-t}.$$

Thus the u_i's are a linearly independent divided power sequence of non-zero elements in C. Note that by duality it is clear that $C_n = Ku_n + C_{n-1}$ for any $0 \leq n \leq N-1$ and $C_n = C$ for any $n \geq N$.
Let us prove that $d_j = u_j$ for any $0 \leq j \leq t-1$. It is enough to check that $\alpha^i(d_j) = \delta_{i,j}$ for every $0 \leq i \leq N-1$ and $0 \leq j \leq t-1$.
First of all, let us prove that $d_j = u_j$ for any $0 \leq j \leq s$. Since $d_0, d_1, \cdots, d_s$ are linearly independent and by definition of α, we have that $\alpha(d_j) = \delta_{1,j}$ for every $0 \leq j \leq s$. Let $2 \leq n \leq N-1$ and assume $\alpha^i(d_j) = \delta_{i,j}$ for any $0 \leq i \leq n-1$ and for every $0 \leq j \leq s$. We have

$$\alpha^n(d_j) = \left(\alpha^{n-1} \otimes \alpha\right)\Delta(d_j) = \sum\nolimits_{a=0}^{j} \alpha^{n-1}(d_a)\,\alpha(d_{j-a}) = \sum\nolimits_{a=0}^{j} \delta_{n-1,a}\delta_{1,j-a} = \delta_{n,j}.$$

Therefore $d_j = u_j$ for any $0 \leq j \leq s$.
Assume $s \leq t-2$ and compute

$$\begin{aligned}
\Delta(d_{s+1} - u_{s+1}) &= \sum\nolimits_{a=0}^{s+1} d_a \otimes d_{s+1-a} - \sum\nolimits_{a=0}^{s+1} u_a \otimes u_{s+1-a} \\
&= 1_C \otimes (d_{s+1} - u_{s+1}) + (d_{s+1} - u_{s+1}) \otimes 1_C + \sum\nolimits_{a=1}^{s} d_a \otimes d_{s+1-a} - \sum\nolimits_{a=1}^{s} u_a \otimes u_{s+1-a} \\
&= 1_C \otimes (d_{s+1} - u_{s+1}) + (d_{s+1} - u_{s+1}) \otimes 1_C.
\end{aligned}$$

Then $d_{s+1} - u_{s+1} \in P(R) = Kd_1$ so that there exists $k \in K$ such that $u_{s+1} = d_{s+1} + kd_1$. Since $d_0, d_1, \cdots, d_{s+1}$ are linearly dependent and $d_0, d_1, \cdots, d_s$ are linearly independent, it follows that $d_{s+1} \in \sum_{i=0}^{s} Kd_i = \sum_{i=0}^{s} Ku_i$ and hence $u_{s+1} = d_{s+1} + kd_1 \in \sum_{i=0}^{s} Ku_i$. This contradicts the linear independence of u_j's. Thus $s = t-1$. □

Lemma 1.5. *Let C be an N-dimensional thin K-coalgebra. Let $d_0, d_1, \ldots, d_{N-1}$ be a divided power sequence of non-zero elements in C. Then*

$$\varepsilon(d_n) = \delta_{0,n}$$

for every $0 \leq n \leq N-1$.

Proof. By Lemma 1.4, $d_0 = 1_C$.
If $n = 0$ then $\varepsilon(d_n) = \varepsilon(d_0) = \varepsilon(1_C) = 1_K$.
Let $1 \leq n \leq N-1$ and assume $\varepsilon(d_i) = \delta_{0,i}$ for any $0 \leq i \leq n-1$. We have

$$\varepsilon(d_n) = (\varepsilon \otimes \varepsilon)\Delta(d_n) = \sum\nolimits_{t=0}^{n} \varepsilon(d_t)\,\varepsilon(d_{n-t}) = \varepsilon(d_0)\,\varepsilon(d_n) + \varepsilon(d_n)\,\varepsilon(d_0) = 2\varepsilon(d_n)$$

so that $\varepsilon(d_n) = 0 = \delta_{0,n}$. □

LEMMA 1.6. *Let C be a unital K-coalgebra of finite dimension N over K. Let $t \in \mathbb{N}, 0 \leq t \leq N$ and let*

$$d_0 = 1_C, d_1, \ldots, d_{t-2}, d_{t-1}$$

be a divided power sequences of non-zero elements in C.
The following assertions are equivalent for an element $e_{t-1} \in C$:
(a) *$d_0, d_1, \ldots, d_{t-2}, e_{t-1}$ is a divided power sequences of non-zero elements in C;*
(b) *$e_{t-1} - d_{t-1} \in P(C)$.*

Proof. First note that $d_0, d_1, \ldots, d_{t-2}, e_{t-1}$ is a divided power sequences of non-zero elements in C if and only if

$$\Delta(e_{t-1}) = d_0 \otimes e_{t-1} + e_{t-1} \otimes d_0 + \sum_{i=1}^{t-2} d_i \otimes d_{t-1-i}.$$

$(a) \Rightarrow (b)$. By the above observation, we have

$$\begin{aligned}\Delta(e_{t-1} - d_{t-1}) &= d_0 \otimes e_{t-1} + e_{t-1} \otimes d_0 + \sum_{i=1}^{t-2} d_i \otimes d_{t-1-i} - \sum_{i=0}^{t-1} d_i \otimes d_{t-1-i} \\ &= 1_C \otimes (e_{t-1} - d_{t-1}) + (e_{t-1} - d_{t-1}) \otimes 1_C.\end{aligned}$$

Thus $e_{t-1} - d_{t-1} \in P(C)$.
$(b) \Rightarrow (a)$. Let $u := e_{t-1} - d_{t-1}$. Then, by hypothesis $u \in P(C)$ and hence

$$\begin{aligned}\Delta(e_{t-1}) &= \Delta(d_{t-1} + u) \\ &= \sum_{i=0}^{t-1} d_i \otimes d_{t-1-i} + d_0 \otimes u + u \otimes d_0 = \sum_{i=1}^{t-2} d_i \otimes d_{t-1-i} + d_0 \otimes e_{t-1} + e_{t-1} \otimes d_0\end{aligned}$$

so that $d_0, d_1, \ldots, d_{t-2}, e_{t-1}$is a divided power sequences of non-zero elements in C. □

2. PRE-BIALGEBRAS

Let H be a Hopf algebra over the field K. Recall that an object V in ${}^H_H\mathcal{YD}$ is a left H-module and a left H-comodule satisfying, for any $h \in H, v \in V$, the compatibility condition:

$$\sum (h_{(1)}v)_{<-1>} h_{(2)} \otimes (h_{(1)}v)_{<0>} = \sum h_{(1)} v_{<-1>} \otimes h_{(2)} v_{<0>}$$

or, equivalently,

$$\rho(hv) = \sum h_{(1)} v_{<-1>} S(h_{(3)}) \otimes h_{(2)} v_{<0>},$$

where $\rho : V \to H \otimes V$ is the coaction of H on V and for the action of H on V we used the notation hv, for every $h \in H, v \in V$. If there is danger of confusion we write ${}^h v$ instead of hv.
The tensor product $V \otimes W$ of two Yetter-Drinfeld modules is an object in ${}^H_H\mathcal{YD}$ via the diagonal action and the codiagonal coaction; the unit in ${}^H_H\mathcal{YD}$ is K regarded as a left H-comodule via the map $x \mapsto 1_H \otimes x$ and as a left H-module via ε_H. Recall that, for every $V, W \in {}^H_H\mathcal{YD}$ the braiding is given by:

$$(1) \qquad c_{V,W} : V \otimes W \to W \otimes V, \qquad c_{V,W}(v \otimes w) = \sum v_{\langle -1 \rangle} w \otimes v_{\langle 0 \rangle}.$$

If H has bijective antipode, then $\left({}^H_H\mathcal{YD}, c\right)$ is a braided category.

2.1. Let R and S be two algebras in the braided category ${}^H_H\mathcal{YD}$. We can define a new algebra structure on $R \otimes S$, by using the braiding (1), and not the usual flip morphism. The multiplication in this case is defined by the formula:

$$(2) \qquad (r \otimes s)(t \otimes v) = \sum r(s_{\langle -1 \rangle} t) \otimes s_{\langle 0 \rangle} v.$$

Let us remark that, for any algebra R in${}^H_H\mathcal{YD}$, the smash product $R \# H$ is a particular case of this construction. Just take $S = H$ with the left adjoint action (i.e. ${}^h x = \sum h_{(1)} x S h_{(2)}$, for every $h, x \in H$) and usual left H-comodule structure.

2.2. Let R and S be two coalgebras in the braided category ${}_H^H\mathcal{YD}$. We can define a new coalgebra structure on $R\otimes S$, by using the braiding (1), and not the usual flip morphism. The comultiplication in this case is defined by the formula:

$$\delta_{R\otimes S}\left(r\otimes s\right)=\sum r^{(1)}\otimes r^{(2)}_{\langle -1\rangle}s^{(1)}\otimes r^{(2)}_{\langle 0\rangle}\otimes s^{(2)}.$$

Let us remark that, for any coalgebra R in ${}_H^H\mathcal{YD}$, the smash coproduct $R\#H$ is a particular case of this construction. Just take $S=H$ with the left adjoint coaction (i.e. $\rho\left(h\right)=\sum h_{(1)}Sh_{(3)}\otimes h_{(2)}$, for every $h\in H$) and usual left H-module structure.

Definition 2.3. Let H be a Hopf algebra. A *pre-bialgebra* $(R,m,u,\delta,\varepsilon)$ in ${}_H^H\mathcal{YD}$ consists of

- a coalgebra (R,δ,ε) in the category $({}_H^H\mathcal{YD},\otimes,K)$.
- two K-linear maps

$$m:R\otimes R\to R \qquad \text{and} \qquad u:K\to R$$

such that, for all $r,s,t\in R$ and $h\in H$, the following relations are satisfied:

$$h\cdot u(1)=\varepsilon_H(h)u(1) \qquad \text{and} \qquad \rho_R u(1)=1_H\otimes u(1) \tag{3}$$

$$\delta u(1)=u(1)\otimes u(1) \qquad \text{and} \qquad \varepsilon u(1)=1_K; \tag{4}$$

$$hm_R(r\otimes s)=\sum m_R(h_{(1)}r\otimes h_{(2)}s); \tag{5}$$

$$\delta m_R=(m_R\otimes m_R)\delta_{R\otimes R} \qquad \text{and} \qquad \varepsilon m_R=m_K(\varepsilon\otimes\varepsilon); \tag{6}$$

$$m_R(R\otimes u)=R=m_R(u\otimes R); \tag{7}$$

Note that (3) and (4) mean that u is a coalgebra homomorphism in ${}_H^H\mathcal{YD}$, (5) and (6) mean that m_R is left H-linear coalgebra homomorphism while (7) means that u is a unit for m_R. We fix the following notation

$$\delta(r)=\sum r^{(1)}\otimes r^{(2)},\ \text{for every } r\in R.$$

Remark 2.4. To explain the meaning of the concept of pre-bialgebra in ${}_H^H\mathcal{YD}$, it is useful to compare it with the concept of a bialgebra in ${}_H^H\mathcal{YD}$. A pre-bialgebra is just a unital bialgebra in ${}_H^H\mathcal{YD}$ with the following differences:

a) the multiplication is non-associative;
b) the multiplication is not a morphism of H-comodules.

Let H be a Hopf algebra, let (R,δ,ε) be a coalgebra in the category $({}_H^H\mathcal{YD},\otimes,K)$
Let us consider the graded coalgebra

$$\mathrm{gr}\left(R\right)=\oplus_{n\geq 0}\frac{R_n}{R_{n-1}}$$

where, by definition, we set $R_{-1}=0$ and $(R_i)_{i\in\mathbb{N}}$ are the components of the coradical filtration of R. Now $\mathrm{gr}\left(R\right)$ is an ordinary coalgebra which becomes a coalgebra in the monoidal category ${}_H^H\mathcal{YD}$ whenever R_0 is a subcoalgebra of R in ${}_H^H\mathcal{YD}$. In fact, in this case, since, for any $n\geq 1$, we have $R_n=R_{n-1}\wedge_R R_0$ then inductively one has that R_n is a subcoalgebra of R in ${}_H^H\mathcal{YD}$.
Let $(R,m,u,\delta,\varepsilon)$ be a pre-bialgebra in ${}_H^H\mathcal{YD}$. In this case we also have a non necessarily associative multiplication on R. The following result explains how $\mathrm{gr}\left(R\right)$ inherits the multiplication of R.

Proposition 2.5. *Let H be a Hopf algebra and let $(R,m,u,\delta,\varepsilon)$ be a pre-bialgebra in ${}_H^H\mathcal{YD}$. Assume that R_0 is a subcoalgebra of R in ${}_H^H\mathcal{YD}$ such that $R_0\cdot R_0\subseteq R_0$.*
Then $\mathrm{gr}\left(R\right)$ inherits the pre-bialgebra structure in ${}_H^H\mathcal{YD}$ of R.

Proof. The coalgebra structure of R induces a coalgebra structure on $\mathrm{gr}\left(R\right)$. Let us prove that $\mathrm{gr}\left(R\right)$ inherits also the (eventually non associative) algebra structure of R. Let us check that $R_a\cdot R_b\subseteq R_{a+b}$ for any $a,b\in\mathbb{N}$.

We prove this by induction on $n=a+b$.
If $n=0$ there is nothing to prove. Let $n\geq 1$ and assume that $R_i\cdot R_j\subseteq R_{i+j}$ for any $i,j\in\mathbb{N}$ such that $0\leq i+j\leq n-1$. Let $a,b\in\mathbb{N}$. such that $n=a+b$. and let $r\in R_a$ and $s\in R_b$. Since

$\delta(R_a) \subseteq \sum_{i=0}^{a} R_i \otimes R_{a-i}$ and $c(r \otimes s) = r_{\langle -1\rangle} s \otimes r_{\langle 0\rangle} \in R_b \otimes R_a$, for every $r \in R_a, s \in R_b$, by (6), we have

$$\begin{aligned}\delta(R_a \cdot R_b) &= \delta(R_a)\,\delta(R_b) \subseteq \left(\sum\nolimits_{i=0}^{a} R_i \otimes R_{a-i}\right)\left(\sum\nolimits_{j=0}^{b} R_j \otimes R_{b-j}\right) \\ &\subseteq \sum\nolimits_{i=0}^{a}\sum\nolimits_{j=0}^{b} R_iR_j \otimes R_{a-i}R_{b-j} \subseteq R_{a+b-1} \otimes R + R \otimes R_0.\end{aligned}$$

Therefore $R_a \cdot R_b \subseteq R_{a+b}$. In this way $\mathrm{gr}(R)$ inherits the algebra structure of R; see the proof of [Mo, Lemma 5.2.8]. The last assertion is straightforward. □

Definitions 2.6. Let q be a primitive N-th root of unity. Let H be a Hopf algebra, $g \in H$ and $\chi \in H^*$.
Following [CDMM, Definition 2.1], we say that (H, g, χ) is a *Yetter-Drinfeld datum for q* whenever

- $g \in G(H)$,
- $\chi \in H^*$ is a character of H,
- $\chi(g) = q$,
- the following relation holds true

$$g \sum \chi(h_{(1)})h_{(2)} = \sum h_{(1)}\chi(h_{(2)})g \tag{8}$$

If (H, g, χ) is a Yetter-Drinfeld datum for q, we denote by R_q the graded algebra $K[X]/\left(X^N\right)$. Let $y = X + \left(X^N\right)$. Then R_q can be endowed with a unique braided bialgebra structure in $({}^H_H\mathcal{YD}, \otimes, K)$, where the Yetter-Drinfeld module structure is given by

$$hy = \chi(h)\,y \qquad \text{and} \qquad \rho(y) = g \otimes y$$

and the coalgebra structure is defined by setting

$$\delta(y) = y \otimes 1 + 1 \otimes y.$$

In this way R_q becomes a braided Hopf algebra that will be denoted by $R_q(H, g, \chi)$ and called a *quantum line* (see [AS]).

The very technical part of the following lemma is devoted to show that the order θ of the involved root of unity fulfills $2 \leq \theta \leq \dim_K(R)$. This relation will play a fundamental role in proving that θ is in fact equal to $\dim_K(R)$ (see Theorem 2.13).

Lemma 2.7. *Let H be a Hopf algebra and let $(R, m, u, \delta, \varepsilon)$ be a finite dimensional pre-bialgebra in ${}^H_H\mathcal{YD}$. Assume that R is a thin coalgebra where $P(R) = Ky$. Then there is a primitive θ-th root of unity $q \in K$, where $2 \leq \theta \leq \dim_K(R)$, and $g \in H, \chi \in H^*$ such that*
1) (H, g, χ) is a Yetter-Drinfeld datum for q,
2) ${}^H\rho_R(y) = g \otimes y$,
3) $hy = \chi(h)y$ for every $h \in H$.

Proof. Note that by (4) $u : K \to R$ is a coalgebra morphism, so that $(R, 1_R = u(1_K))$ is a unital coalgebra. Since R is thin, $C_0 = K1_R$. By (3), $u : K \to R$ is a morphism in ${}^H_H\mathcal{YD}$. Hence

$$P(R) = \{x \in R \mid \delta(x) = 1_R \otimes x + x \otimes 1_R\} = \mathrm{Ker}\,[\delta - (u \otimes R + R \otimes u)]$$

is a Yetter-Drinfeld submodule of R so that ${}^H\rho_R(y) \in H \otimes P(R)$ and $hy \in P(R) = Ky$ for every $h \in H$. Then there exists a $g \in G(H)$ such that ${}^H\rho_R(y) = g \otimes y$ and there exists a character $\chi \in H^*$ such that $hy = \chi(h)y$, for every $h \in H$.
Then, the Yetter-Drinfeld compatibility for $P(R) \in {}^H_H\mathcal{YD}$ writes as follows

$$\sum \chi(h_{(1)})gh_{(2)} \otimes y = \sum h_{(1)}g\chi(h_{(2)}) \otimes y$$

so that (H, g, χ) is a Yetter-Drinfeld datum for $q := \chi(g) \in K$. Let us prove that q has finite order. Set

$$w_0 = 1_R \qquad \text{and} \qquad w_n = m(y \otimes w_{n-1}), \text{ for every } n \geq 1.$$

Since $1_R \in G(R)$, inductively one can prove, by means of (6), that

$$\delta(w_n) = \sum\nolimits_{0\leq i\leq n} \binom{n}{i}_q (w_i \otimes w_{n-i}). \tag{9}$$

Since R is finite dimensional over K, there exists

$$N = \min \{n \in \mathbb{N} \mid w_0, \ldots, w_n \text{ are linearly dependent}\}.$$

From $y \in P(R)$, we deduce that $\{1_R, y\} \overset{(7)}{=} \{w_0, w_1\}$ is linearly independent and hence $N > 1$.
Let us prove that $o(q) \leq N$. By definition of N, there is a $\overline{k} := (k_0, k_1, ...k_N) \in K^{N+1} \backslash \{0\}$ such that $k_0 w_0 + k_1 w_1 + ... + k_{N-1} w_{N-1} + k_N w_N = 0$. Moreover, since $w_0, \ldots, w_{N-1}$ are linearly independent over K and $\overline{k} \neq 0$, the case $k_N = 0$ can not occur. Therefore, we can assume $k_N = -1_K$: $w_N = k_0 w_0 + k_1 w_1 + ... + k_{N-1} w_{N-1}$. If $k_i = 0$, for all $0 \leq i \leq N-1$, then $w_N = 0$ so that

$$0 = \delta(w_N) = \sum\nolimits_{0\leq i\leq N} \binom{N}{i}_q (w_i \otimes w_{N-i}) = \sum\nolimits_{1\leq i\leq N-1} \binom{N}{i}_q (w_i \otimes w_{N-i}).$$

Since $w_0, \ldots, w_{N-1}$ are linearly independent, so are $w_i \otimes w_{N-i}$, with $1 \leq i \leq N-1$.
Hence q is a solution of the system of equations $\binom{N}{i}_X = 0$ for every $i, 1 \leq i \leq N-1$. Therefore, since $\mathrm{char}(K) = 0$, we get $q^N = 1$ and $o(q) = N > 1$.
Assume now $k_i \neq 0$, for some $0 \leq i \leq N-1$. Clearly, by (5), one has that $gw_n = q^n w_n$, for every $n \in \mathbb{N}$. We get

$$\begin{aligned} gw_N &= q^N w_N = q^N k_0 + q^N k_1 w_1 + ... + q^N k_{N-1} w_{N-1} \text{ and} \\ gw_N &= k_0 g w_0 + k_1 g w_1 + ... + k_{N-1} g w_{N-1} = k_0 + k_1 q w_1 + ... + k_{N-1} q^{N-1} w_{N-1}. \end{aligned}$$

Since $w_0, \ldots, w_{N-1}$ are linearly independent over K, we have that $q^N k_i = k_i q^i$. From $k_i \neq 0$ one has $q^N = q^i$ so that $q^{N-i} = 1_K$ and hence $o(q) \leq N$.
It remains to prove that $o(q) \geq 2$. Suppose $q = 1$. In this case $\binom{n}{i}_q = \binom{n}{i}$ that is the usual binomial coefficient. Thus, by (9), we have

$$\delta\left(\sum\nolimits_{0\leq n\leq N-1} k_n w_n\right) = \sum\nolimits_{0\leq n\leq N-1} k_n \sum\nolimits_{1\leq i\leq N-1} \binom{n}{i} (w_i \otimes w_{n-i}) + w_0 \otimes w_N + w_N \otimes w_0$$

and

$$\delta(w_N) = \sum\nolimits_{0\leq i\leq N} \binom{N}{i} (w_i \otimes w_{N-i}) = \sum\nolimits_{1\leq i\leq N-1} \binom{N}{i} (w_i \otimes w_{N-i}) + w_0 \otimes w_N + w_N \otimes w_0.$$

Since $w_N = k_0 w_0 + k_1 w_1 + ... + k_{N-1} w_{N-1}$, we obtain

$$\sum\nolimits_{0\leq n\leq N-1} k_n \sum\nolimits_{1\leq i\leq N-1} \binom{n}{i} (w_i \otimes w_{n-i}) = \sum\nolimits_{1\leq i\leq N-1} \binom{N}{i} (w_i \otimes w_{N-i}).$$

As $w_0, \ldots, w_{N-1}$ are linearly independent, we get $\binom{N}{j} 1_K = 0$ for every $1 \leq j \leq N-1$. In particular, since $\mathrm{char}(K) = 0$, we obtain $0 = \binom{N}{1} = N \geq 2$. Contradiction. We conclude that $q \neq 1$. □

Definition 2.8. Let H be a Hopf algebra and let $(R, m, u, \delta, \varepsilon)$ be a finite dimensional pre-bialgebra in ${}^H_H\mathcal{YD}$. Assume that R is a thin coalgebra and let $P(R) = Ky$. Consider q and the Yetter-Drinfeld datum (H, g, χ) for q as in Lemma 2.7. Then (H, g, χ) will be called *the Yetter-Drinfeld datum associated to the pre-bialgebra* $(R, m, u, \delta, \varepsilon)$ *in* ${}^H_H\mathcal{YD}$ *relative to* y or simply *the Yetter-Drinfeld datum associated to* y whenever there is no risk of confusion.

Lemma 2.9. *Let H be a Hopf algebra and let $(R, m, u, \delta, \varepsilon)$ be a N-dimensional pre-bialgebra in ${}^H_H\mathcal{YD}$. Assume that R is a thin coalgebra where $P(R) = Ky$. Let $g \in H$ and $\chi \in H^*$ be such that (H, g, χ) is the Yetter-Drinfeld datum associated to y and let $q = \chi(g)$.*
Then there exists a divided power sequence of non-zero elements in R

$$d_0 = 1_R, d_1 = y, \ldots, d_{N-1}$$

such that

1) $gd_n = q^n d_n$, *for any* $0 \leq n \leq N-1$ *and also*
2) $d_1 d_{n-1} = (n)_q d_n$, *for any* $1 \leq n \leq N-1$.

Proof. By Lemma 1.4, there exists a divided power sequence of non-zero elements in R $d_0 = 1_R, d_1 = y, \ldots, d_{N-1}$ that completes $\{d_0, y\}$ to a basis of R. We have

$$gd_0 \overset{(3)}{=} 1_R \qquad \text{and} \qquad gy = qy.$$

Assume that $2 \leq n \leq N-2$ and that $d_0 = 1_R, d_1 = y, \ldots, d_{n-1}$ is a divided power sequence satisfying 1) and 2). Let $e_n = \frac{gd_n}{q^n}$. Then $d_0 = 1_C, d_1, \ldots, d_{n-1}, e_n$ is a divided power sequences of non-zero elements in R, as, by left H-linearity of δ, we have

$$\delta(e_n) = \frac{1}{q^n} \sum\nolimits_{j=0}^{n} gd_j \otimes gd_{n-j} = \sum\nolimits_{j=0}^{n} \frac{gd_j}{q^j} \otimes \frac{gd_{n-j}}{q^{n-j}} = d_0 \otimes e_n + e_n \otimes d_0 + \sum\nolimits_{i=1}^{n-1} d_i \otimes d_{t-1-i}.$$

Then, by Lemma 1.6, $e_n - d_n \in P(R) = Kd_1$.
Thus there is a $k \in K$ such that $gd_n = q^n d_n + kd_1$.
Now, if $k = 0$ then d_n satisfies 1).
Assume $k \neq 0$. We seek for an element $b \in K$ such that $g(d_n + bd_1) = q^n(d_n + bd_1)$ that is

$$gd_n + bgd_1 = q^n d_n + bq^n d_1, \text{ i.e. } q^n d_n + kd_1 + bqd_1 = q^n d_n + bq^n d_1, \text{ i.e. } k + bq = bq^n.$$

Since $k \neq 0$ we get $q^n \neq q$, so that $b = \frac{k}{q^n - q}$. Now, let $d_n' = d_n + bd_1$. Since $d_n' - d_n \in P(R)$, by Lemma 1.6, we have that $d_0, d_1, \ldots, d_{n-1}, d_n'$ is still a divided power sequences of non-zero elements in R so that we can substitute d_n with d_n' which satisfies 1).

Therefore we can assume that we have found d_n which satisfies 1) such that $d_1, \ldots, d_{n-1}, d_n$ is a divided power sequence of non-zero elements in R.
By (6), we have

$$\begin{aligned}
& \delta(yd_{n-1}) \\
= \; & (y \otimes 1_R + 1_R \otimes y)\left(\sum\nolimits_{t=0}^{n-1} d_t \otimes d_{n-1-t}\right) = \sum\nolimits_{t=0}^{n-1} yd_t \otimes d_{n-1-t} + \sum\nolimits_{t=0}^{n-1} gd_t \otimes yd_{n-1-t} \\
= \; & yd_{n-1} \otimes 1_R + 1_R \otimes yd_{n-1} + \sum\nolimits_{t=0}^{n-2} (t+1)_q d_{t+1} \otimes d_{n-1-t} + \sum\nolimits_{t=1}^{n-1} q^t d_t \otimes (n-t)_q d_{n-t} \\
= \; & yd_{n-1} \otimes 1_R + 1_R \otimes yd_{n-1} + \sum\nolimits_{t=1}^{n-1} \left[(t)_q + q^t (n-t)_q\right] d_t \otimes d_{n-t}
\end{aligned}$$

Since $(t)_q + q^t(n-t)_q = (n)_q$, summing up, we get

$$\delta(yd_{n-1}) = yd_{n-1} \otimes 1_R + 1_R \otimes yd_{n-1} + \sum\nolimits_{t=1}^{n-1} (n)_q d_t \otimes d_{n-t}. \tag{10}$$

Assume that $o(q) \mid n$. Since, by Lemma 2.7, we have that $q \neq 1$, then $(n)_q = 0$ and hence

$$\delta(yd_{n-1}) \overset{(10)}{=} yd_{n-1} \otimes 1_R + 1_R \otimes yd_{n-1}$$

so that $yd_{n-1} \in P(R) = Kd_1$. Thus there is $k \in K$ such that $yd_{n-1} = kd_1$. Hence, since $o(q) \mid n$, from

$$g(yd_{n-1}) \overset{(5)}{=} (gy)(gd_{n-1}) = q^n yd_{n-1} = yd_{n-1} = kd_1 \quad \text{and} \quad g(kd_1) = kgd_1 = qkd_1$$

we deduce $kd_1 = g(yd_{n-1}) = g(kd_1) = qkd_1$ and so $k = 0$ (in fact $q \neq 1$). Therefore in this case $yd_{n-1} = 0$ and hence $yd_{n-1} = (n)_q d_n$.
Assume now $o(q) \nmid n$. Then

$$\begin{aligned}
& \delta\left[yd_{n-1} - (n)_q d_n\right] \\
\overset{(10)}{=} \; & yd_{n-1} \otimes 1_R + 1_R \otimes yd_{n-1} + \sum\nolimits_{t=1}^{n-1} (n)_q d_t \otimes d_{n-t} - (n)_q \sum\nolimits_{t=0}^{n} d_t \otimes d_{n-t} \\
= \; & \left[yd_{n-1} - (n)_q d_n\right] \otimes 1_R + 1_R \otimes \left[yd_{n-1} - (n)_q d_n\right].
\end{aligned}$$

Thus $yd_{n-1}-(n)_q\, d_n \in P(R)$ so that, by Lemma 1.6, we have that $d_0, d_1, \ldots, d_{n-1}, \frac{yd_{n-1}}{(n)_q}$ is still a divided power sequence of non-zero elements in R (we are in the case $(n)_q \neq 0$). This tells we can assume $d_n = yd_{n-1}/(n)_q$. Note that $gd_n = g\frac{yd_{n-1}}{(n)_q} = \frac{(gy)(gd_{n-1})}{(n)_q} = q^n\frac{yd_{n-1}}{(n)_q} = q^n d_n$. $\square$

2.10. Let $r \in R = \cup_{n\in\mathbb{N}} R_n$ and let $\nu_r = \min\{n \in \mathbb{N} \mid r \in R_n\}$. From now on, we will denote by $\overline{r} = r + R_{\nu_r - 1}$ the element of $\frac{R_{\nu_r}}{R_{\nu_r-1}}$ corresponding to r. We point out that $\overline{r}\cdot\overline{s} = rs + R_{\nu_r+\nu_s-1} \neq rs + R_{\nu_{r+s}-1} = \overline{rs}$ a priori.

THEOREM 2.11. *Let H be a Hopf algebra and let $(R, m, u, \delta, \varepsilon)$ be a pre-bialgebra in ${}^H_H\mathcal{YD}$. Assume that R is an N-dimensional thin coalgebra where $P(R) = Ky$. Let $g \in H$ and $\chi \in H^*$ be such that (H, g, χ) is the Yetter-Drinfeld datum associated to y and let $q = \chi(g)$. Consider a divided power sequence*

$$d_0 = 1_R, d_1 = y, \ldots, d_{N-1}$$

of non-zero elements in R such that $gd_n = q^n d_n$, for any $0 \leq n \leq N-1$ as in Lemma 2.9. Then $\left(\overline{d_n}\right)_{0\leq n\leq N-1}$ forms a divided power sequence of non-zero elements in $\mathrm{gr}(R)$ *such that*

1) $\rho\left(\overline{d_n}\right) = g^n \otimes \overline{d_n}$, *for any* $0 \leq n \leq N-1$.
2) $h\cdot\overline{d_n} = \chi^n(h)\,\overline{d_n}$, *for any* $0 \leq n \leq N-1$.
3) $\overline{d_a}\cdot\overline{d_b} = \binom{a+b}{a}_q \overline{d_{a+b}}$, *for* $0 \leq a+b \leq N-1$, *and* $\overline{d_a}\cdot\overline{d_b} = 0$, *for* $a+b \geq N$.

Moreover the pre-bialgebra $\mathrm{gr}(R)$ *is indeed a* ***braided bialgebra*** *in the monoidal category* $({}^H_H\mathcal{YD}, \otimes, K)$ *which is* ***commutative*** *as an algebra in the category of vector spaces.*

Proof. 1) For any $0 \leq n \leq N-1$, $\rho\left(\overline{d_n}\right) \in H \otimes \frac{R_n}{R_{n-1}}$ so that, since, by Lemma 1.4, $\frac{R_n}{R_{n-1}} = K\overline{d_n}$, there is a unique $h_n \in H$ such that $\rho\left(\overline{d_n}\right) = h_n \otimes \overline{d_n}$. Since the comultiplication on $\mathrm{gr}(R)$ is left H-colinear, we have $\sum_{i=0}^n h_i h_{n-i} \otimes \overline{d_i} \otimes \overline{d_{n-i}} = h_n \otimes \delta\left(\overline{d_n}\right) = \sum_{i=0}^n h_n \otimes \overline{d_i} \otimes \overline{d_{n-i}}$. Thus we deduce that $h_n = h_i h_{n-i}$ for any $0 \leq i \leq n$. Since $h_0 = 1_H$ and $h_1 = g$, by applying the above formula to the case $i = 1$, by induction on $n \geq 1$, it is easy to prove that $h_n = g^n$.
2) If $n = 0, 1$ there is nothing to prove.
Let $2 \leq n \leq N-1$ and assume $h\overline{d_i} = \chi^i(h)\,\overline{d_i}$ for any $0 \leq i \leq n-1$. We have

$$\begin{aligned}
\delta\left(h\overline{d_n}\right) &= \sum_{i=0}^{n}\left(h_1\overline{d_i} \otimes h_2\overline{d_{n-i}}\right) \\
&= \left(\varepsilon_H(h_1)\,\overline{d_0} \otimes h_2\overline{d_n}\right) + \left(h_1\overline{d_n} \otimes \varepsilon_H(h_2)\,\overline{d_0}\right) + \sum_{i=1}^{n-1}\left(\chi^i(h_1)\,\overline{d_i} \otimes \chi^{n-i}(h_2)\,\overline{d_{n-i}}\right) \\
&= \left(1_R \otimes h\overline{d_n}\right) + \left(h\overline{d_n} \otimes 1_R\right) + \chi^n(h)\sum_{i=1}^{n-1}\overline{d_i} \otimes \overline{d_{n-i}}.
\end{aligned}$$

From this, since $\delta\left[\chi^n(h)\,\overline{d_n}\right] = \chi^n(h)\,\delta\left[\overline{d_n}\right] = \chi^n(h)\sum_{i=0}^n \overline{d_i} \otimes \overline{d_{n-i}}$, and by Proposition 1.2, we infer that $h\overline{d_n} - \chi^n(h)\,\overline{d_n} \in P(\mathrm{gr}(R)) = \frac{R_1}{R_0}$. Since a priori $h\overline{d_n} - \chi^n(h)\,\overline{d_n} \in \frac{R_n}{R_{n-1}}$ and $n \geq 2$, we conclude that $h\overline{d_n} = \chi^n(h)\,\overline{d_n}$.
3) Observe that

$$\begin{aligned}
&\delta\left(\overline{d_a}\cdot\overline{d_b}\right) = \delta\left(\overline{d_a}\right)\delta\left(\overline{d_b}\right) = \sum_{i=0}^{a}\left(\overline{d_i} \otimes \overline{d_{a-i}}\right)\sum_{j=0}^{b}\left(\overline{d_j} \otimes \overline{d_{b-j}}\right) \\
=\ & \sum_{i=0}^{a}\sum_{j=0}^{b}\left[\overline{d_i}\cdot\left(\overline{d_{a-i}}\right)_{\langle -1\rangle}\overline{d_j}\right] \otimes \left[\left(\overline{d_{a-i}}\right)_{\langle 0\rangle}\cdot\overline{d_{b-j}}\right] = \sum_{i=0}^{a}\sum_{j=0}^{b}\overline{d_i}\cdot g^{a-i}\overline{d_j} \otimes \overline{d_{a-i}}\cdot\overline{d_{b-j}} \\
=\ & \sum_{i=0}^{a}\sum_{j=0}^{b}\overline{d_i}\cdot\chi^j\left(g^{a-i}\right)\overline{d_j} \otimes \overline{d_{a-i}}\cdot\overline{d_{b-j}} = \sum_{i=0}^{a}\sum_{j=0}^{b} q^{(a-i)j}\,\overline{d_i}\cdot\overline{d_j} \otimes \overline{d_{a-i}}\cdot\overline{d_{b-j}}
\end{aligned}$$

so that

$$\delta\left(\overline{d_a}\cdot\overline{d_b}\right) = 1_R \otimes \overline{d_a}\cdot\overline{d_b} + \overline{d_a}\cdot\overline{d_b} \otimes 1_R + \sum_{\substack{0\leq i\leq a, 0\leq j\leq b\\ 1\leq i+j\leq a+b-1}} q^{(a-i)j}\,\overline{d_i}\cdot\overline{d_j} \otimes \overline{d_{a-i}}\cdot\overline{d_{b-j}}. \tag{11}$$

Let us prove that $\overline{d_a}\cdot\overline{d_b} = \binom{a+b}{a}_q \overline{d_{a+b}}$ for any $a, b \in \mathbb{N}$ such that $1 \leq a+b \leq N-1$, by induction on $n = a+b$.

If $n=1$, then $\overline{d_a}\cdot\overline{d_b}=\overline{d_1}=\binom{a+b}{a}_q\overline{d_{a+b}}$. Let $2\leq n\leq N-1$ and assume $\overline{d_i}\cdot\overline{d_j}=\binom{i+j}{i}_q\overline{d_{i+j}}$ for any $i,j\in\mathbb{N}$ such that $0\leq i+j\leq n-1$. By (11), we have

$$\begin{aligned}
&\delta\left(\overline{d_a}\cdot\overline{d_b}\right)-\left[1_R\otimes\overline{d_a}\cdot\overline{d_b}+\overline{d_a}\cdot\overline{d_b}\otimes 1_R\right]\\
=&\sum\nolimits_{\substack{0\leq i\leq a,0\leq j\leq b\\1\leq i+j\leq a+b-1}} q^{(a-i)j}\binom{i+j}{i}_q\overline{d_{i+j}}\otimes\binom{a+b-(i+j)}{a-i}_q\overline{d_{a+b-(i+j)}}\\
=&\sum\nolimits_{t=1}^{a+b-1}\left[\sum\nolimits_{\substack{0\leq i\leq a,0\leq j\leq b\\i+j=t}} q^{(a-i)j}\binom{t}{i}_q\binom{a+b-t}{a-i}_q\right]\overline{d_t}\otimes\overline{d_{a+b-t}}.
\end{aligned}$$

From the X-analogue of Chu-Vandermonde formula [Ka, Proposition IV 2.3, page 75], we have

$$\binom{m+n}{i}_X=\sum\nolimits_{\substack{0\leq j\leq m,0\leq u\leq n\\j+u=i}} X^{(m-j)u}\binom{m}{j}_X\binom{n}{u}_X,$$

for any $1\leq t\leq a+b-1$ so that

$$\begin{aligned}
\binom{t+a+b-t}{a}_q &= \sum\nolimits_{\substack{0\leq i\leq t,0\leq(a-i)\leq a+b-t\\i+(a-i)=a}} q^{(t-i)(a-i)}\binom{t}{i}_q\binom{a+b-t}{a-i}_q,\text{ i.e.}\\
\binom{a+b}{a}_q &= \sum\nolimits_{\substack{0\leq i\leq a\\0\leq t-i\leq b}} q^{(t-i)(a-i)}\binom{t}{i}_q\binom{a+b-t}{a-i}_q=\sum\nolimits_{\substack{0\leq i\leq a\\0\leq j\leq b\\i+j=t}} q^{(a-i)j}\binom{t}{i}_q\binom{a+b-t}{a-i}_q
\end{aligned}$$

Finally, we get

$$\delta\left(\overline{d_a}\cdot\overline{d_b}\right)=\overline{1_R}\otimes\overline{d_a}\cdot\overline{d_b}+\overline{d_a}\cdot\overline{d_b}\otimes\overline{1_R}+\binom{a+b}{a}_q\sum\nolimits_{t=1}^{a+b-1}\overline{d_t}\otimes\overline{d_{a+b-t}}. \tag{12}$$

From this, the fact that

$$\delta\left[\binom{a+b}{a}_q\overline{d_{a+b}}\right]=\binom{a+b}{a}_q\delta\left[\overline{d_{a+b}}\right]=\binom{a+b}{a}_q\sum\nolimits_{t=0}^{a+b}\overline{d_t}\otimes\overline{d_{a+b-t}}$$

and by Proposition 1.2, we infer that $\overline{d_a}\cdot\overline{d_b}-\binom{a+b}{a}_q\overline{d_{a+b}}\in P\left(\mathrm{gr}\left(R\right)\right)=\frac{R_1}{R_0}$. But $\overline{d_a}\cdot\overline{d_b}-\binom{a+b}{a}_q\overline{d_{a+b}}\in\frac{R_{a+b}}{R_{a+b-1}}$ and $a+b\geq 2$ so that $\overline{d_a}\cdot\overline{d_b}=\binom{a+b}{a}_q\overline{d_{a+b}}$. Observe that, for any $0\leq a,b\leq N$ such that $a+b\geq N$ we have $R_{a+b}=R_{a+b-1}$ and hence $\overline{d_a}\cdot\overline{d_b}\in\frac{R_{a+b}}{R_{a+b-1}}=0$. If $0\leq a,b,c\leq N$ and $a+b+c\leq N-1$, we obtain

$$\left(\overline{d_a}\cdot\overline{d_b}\right)\cdot\overline{d_c}=\frac{(a+b+c)_q!}{(a)_q!\,(b)_q!\,(c)_q!}\cdot\overline{d_{a+b+c}}=\overline{d_a}\cdot\left(\overline{d_b}\cdot\overline{d_c}\right)$$

If $0\leq a,b,c\leq N$ and $a+b+c\geq N$, we get $\left(\overline{d_a}\cdot\overline{d_b}\right)\cdot\overline{d_c}\in\frac{R_{a+b+c}}{R_{a+b+c-1}}=0$ and $\overline{d_a}\cdot\left(\overline{d_b}\cdot\overline{d_c}\right)\in\frac{R_{a+b+c}}{R_{a+b+c-1}}=0$ so that $\left(\overline{d_a}\cdot\overline{d_b}\right)\cdot\overline{d_c}=\overline{d_a}\cdot\left(\overline{d_b}\cdot\overline{d_c}\right)$.

Hence we have proved that $\mathrm{gr}\left(R\right)$ **is an associative algebra**. Note also that, if $0\leq a,b\leq N$ and $a+b\leq N-1$, $\overline{d_a}\cdot\overline{d_b}=\binom{a+b}{a}_q\overline{d_{a+b}}=\overline{d_b}\cdot\overline{d_a}$ and that, for $a+b\geq N$, $\overline{d_a}\cdot\overline{d_b}=0=\overline{d_b}\cdot\overline{d_a}$ so that $\mathrm{gr}\left(R\right)$ is also **commutative.**

To see that $\mathrm{gr}\left(R\right)$ is a braided bialgebra in ${}_H^H\mathcal{YD}$ it remains to prove that the multiplication in $\mathrm{gr}\left(R\right)$ is left H-colinear. If $0\leq a,b\leq N$ and $a+b\leq N-1$, we have

$$\rho\left(\overline{d_a}\cdot\overline{d_b}\right)=\rho\left[\binom{a+b}{a}_q\overline{d_{a+b}}\right]=g^{a+b}\otimes\binom{a+b}{a}_q\overline{d_{a+b}}=g^ag^b\otimes\overline{d_a}\cdot\overline{d_b}.$$

If $a+b\geq N$ we have $\rho\left(\overline{d_a}\cdot\overline{d_b}\right)=0=g^ag^b\otimes\overline{d_a}\cdot\overline{d_b}$. □

Theorem 2.12. *Take the hypothesis and notations of Theorem 2.11. Then* $I=\overline{d_1}\cdot\mathrm{gr}\left(R\right)$ *is a two sided ideal and also a coideal of* $\mathrm{gr}\left(R\right)$ *regarded as a braided bialgebra in* ${}_H^H\mathcal{YD}$.

Proof. By Theorem 2.11, $\mathrm{gr}(R)$ is a commutative algebra. Then I is a two-sided ideal of $\mathrm{gr}(R)$. Moreover, for any $h \in H$ and for any $b \in \mathbb{N}$, we have

$$h\left(\overline{d_1} \cdot \overline{d_b}\right) = \sum h_{(1)}\overline{d_1} \cdot h_{(2)}\overline{d_b} = \sum \chi\left(h_{(1)}\right)\overline{d_1} \cdot \chi\left(h_{(2)}\right)\overline{d_b} \in I.$$

and $\rho\left(\overline{d_1} \cdot \overline{d_b}\right) = g^{1+b} \otimes \overline{d_1} \cdot \overline{d_b}$ for left H-colinearity of the multiplication. Hence I is an ideal of $\mathrm{gr}(R)$ regarded as an algebra in ${}^H_H\mathcal{YD}$. Furthermore, by (11), we have

$$\delta\left(\overline{d_1} \cdot \overline{d_b}\right) = 1_R \otimes \overline{d_1} \cdot \overline{d_b} + \overline{d_1} \cdot \overline{d_b} \otimes 1_R + \sum\nolimits_{\substack{0 \leq i \leq 1, 0 \leq j \leq b \\ 1 \leq i+j \leq 1+b-1}} q^{(1-i)j}\overline{d_i} \cdot \overline{d_j} \otimes \overline{d_{1-i}} \cdot \overline{d_{b-j}} \in R \otimes I + I \otimes R.$$

Finally $\varepsilon\left(\overline{d_1} \cdot \overline{d_b}\right) = \varepsilon\left(\overline{d_1}\right)\varepsilon\left(\overline{d_b}\right) = 0$ so that $\varepsilon(I) = 0$ and hence I is also a coideal of $\mathrm{gr}(R)$. □

The following result is known (see [AS, Theorem 3.2] and [CDMM, Proposition 3.4]) when R is a braided Hopf algebra in ${}^H_H\mathcal{YD}$.

Theorem 2.13. *Let H be a Hopf algebra and let $(R, m, u, \delta, \varepsilon)$ be a pre-bialgebra in ${}^H_H\mathcal{YD}$. Assume that R is an N-dimensional thin coalgebra where $P(R) = Ky$. Let $g \in H$ and $\chi \in H^*$ be such that (H, g, χ) is the Yetter-Drinfeld datum associated to y and let $q = \chi(g)$. Consider a divided power sequence*

$$d_0 = 1_R, d_1 = y, \ldots, d_{N-1}$$

of non-zero elements in R such that $gd_n = q^n d_n$, for any $0 \leq n \leq N-1$ as in Lemma 2.9. Then $N = o(q)$ and

$$\overline{d_n} = \frac{\left(\overline{d_1}\right)^n}{(n)_q!}$$

for any $0 \leq n \leq N-1$.
In particular, $\mathrm{gr}(R) = R_q(H, g, \chi)$ is a quantum line generated as an algebra by $\overline{d_1}$.

Proof. Let $\theta = o(q)$. By Lemma 2.7, we have that $2 \leq \theta \leq N$.
Since, by Theorem 2.11, $\mathrm{gr}(R)$ is an associative algebra, it makes sense to consider $\left(\overline{d_1}\right)^N$. Note that $\left(\overline{d_1}\right)^N \in \frac{R_N}{R_{N-1}} = 0$. Set $z = \overline{d_1}$ and let

$$t = \min\{n \in \mathbb{N}\backslash\{0\} \mid z^n = 0\}.$$

Then, we have $z^n \in \frac{R_n}{R_{n-1}}\backslash\{0\}$ for any $0 \leq n \leq t-1$ and hence $\overline{d_n}$ exists and is not zero for any $0 \leq n \leq t-1$. In particular we obtain that $t-1 \leq N-1$ that is $t \leq N$. Let us prove that $z^n = (n)_q!\overline{d_n}$ for any $0 \leq n \leq t-1$. For $n = 0, 1$ there is nothing to prove. Let $2 \leq n \leq t-1$ and assume $z^{n-1} = (n-1)_q!\overline{d_{n-1}}$. We have

$$z^n = z \cdot z^{n-1} = (n-1)_q!\, z \cdot \overline{d_{n-1}} = (n-1)_q!\, \overline{d_1} \cdot \overline{d_{n-1}} = (n-1)_q! \binom{n}{1}_q \overline{d_n} = (n)_q!\overline{d_n}.$$

Observe that, since $z^{t-1} \neq 0$, we have $(t-1)_q! \neq 0$ which means, being $q \neq 1$, that $q^n \neq 1$ for any $0 \leq n \leq t-1$ and hence $t \leq \theta$. By the quantum binomial formula we have

$$0 = \delta\left(z^t\right) = \sum\nolimits_{i=0}^{t} \binom{t}{i}_q z^i \otimes z^{t-i} = \sum\nolimits_{i=1}^{t-1} \binom{t}{i}_q z^i \otimes z^{t-i}.$$

Note that, since $z^n = (n)_q!\overline{d_n}$, then $(z^n)_{0 \leq n \leq t-1}$ are linearly independent so that $\binom{t}{i}_q = 0$ for any $1 \leq i \leq t-1$. In particular, for $i = 1$ we get $(t)_q = 0$ and hence $q^t = 1$. We deduce that $t = \theta$.

Recall that $\theta \leq N$. Assume $N \geq \theta + 1$. Then we know that $\overline{d_\theta}$ exists and it is not zero. Let $I = \overline{d_1} \cdot \mathrm{gr}(R)$ be the ideal of Theorem 2.12. Assume that $\overline{d_\theta} \in I$. In this case there exists $r \in \mathrm{gr}(R)$ such that $\overline{d_\theta} = \overline{d_1}r$. Since $r \in \mathrm{gr}(R)$ we have $r = \sum_{i=0}^{N-1} k_i\overline{d_i}, k_i \in K$. Hence, by Theorem 2.11, we have

$$\overline{d_\theta} = \overline{d_1}r = \sum\nolimits_{i=0}^{N-1} k_i\overline{d_1} \cdot \overline{d_i} = \sum\nolimits_{i=0}^{N-2} k_i \binom{1+i}{1}_q \overline{d_{1+i}} + k_{N-1}\overline{d_1} \cdot \overline{d_{N-1}} = \sum\nolimits_{i=0}^{N-2} k_i (1+i)_q \overline{d_{1+i}}$$

Since $0\leq\theta\leq N-1$, we get $\overline{d_\theta}=k_{\theta-1}\left(1+\theta-1\right)_q\overline{d_{1+\theta-1}}=k_{\theta-1}\left(\theta\right)_q\overline{d_\theta}=0$, a contradiction. Hence we always have that $\overline{d_\theta}\notin I$.

Consider the braided bialgebra $Q=\frac{\mathrm{gr}(R)}{I}$. As observed above, for every $1\leq n\leq\theta-1$ we have that $\overline{d_n}=\frac{z^n}{(n)_q!}\in I$. Set $w=\overline{d_\theta}+I$. Then

$$\delta_Q\left(w\right)=\sum\nolimits_{t=0}^{\theta}\binom{\theta}{t}_q\left(\overline{d_t}+I\otimes\overline{d_{\theta-t}}+I\right)=w\otimes 1_Q+1_Q\otimes w$$

i.e. $w\in P\left(Q\right)$. Moreover we have $h\cdot w=\chi^\theta\left(h\right)w$ and $\rho\left(w\right)=g^\theta\otimes w$. Then

$$\begin{aligned}&\delta_Q\left(w\right)\delta_Q\left(w\right)\\ =\;&\left(w\otimes 1_Q+1_Q\otimes w\right)\left(w\otimes 1_Q+1_Q\otimes w\right)=w^2\otimes 1_Q+1_Q\otimes w^2+w\otimes w+w_{\langle-1\rangle}w\otimes w_{\langle-1\rangle}\\ =\;&w^2\otimes 1_Q+1_Q\otimes w^2+w\otimes w+g^\theta w\otimes w=w^2\otimes 1_Q+1_Q\otimes w^2+w\otimes w+q^{\theta^2}w\otimes w,\end{aligned}$$

so that

$$\delta_Q\left(w\right)\delta_Q\left(w\right)=w^2\otimes 1_Q+1_Q\otimes w^2+2w\otimes w. \tag{13}$$

Let us write $N-1=a\theta+r$, where $a\geq 1$ ($\theta\leq N-1$) and $0\leq r\leq\theta-1$.

If $a=1$ then $N-1=\theta+r\leq 2\theta-1$ so that $\overline{d_\theta}\cdot\overline{d_\theta}=0$ and hence $w^2=0$. By (13), we deduce $2w\otimes w=0$ so that, since $char\left(K\right)\neq 2$, we infer $w=0$. This contradicts $\overline{d_\theta}\notin I$. Hence we have $a\geq 2$, so that $\overline{d_{2\theta}}$ exists. Therefore

$$ww=\left(\overline{d_\theta}+I\right)\left(\overline{d_\theta}+I\right)=\left(\overline{d_\theta d_\theta}+I\right)=\binom{\theta+\theta}{\theta}_q\left(\overline{d_{2\theta}}+I\right)=\binom{\theta}{0}_q\left(\overline{d_{2\theta}}+I\right)=\overline{d_{2\theta}}+I.$$

Thus

$$\begin{aligned}&\delta_Q\left(w\right)\delta_Q\left(w\right)=\delta_Q\left(ww\right)=\delta_Q\left(\overline{d_{2\theta}}+I\right)=\sum\nolimits_{i=0}^{2\theta}\binom{2\theta}{i}_q\left(\overline{d_i}+I\right)\otimes\left(\overline{d_{2\theta-i}}+I\right)\\ =\;&\left(\overline{d_0}+I\right)\otimes\left(\overline{d_{2\theta-0}}+I\right)+\binom{2\theta}{\theta}_q\left(\overline{d_\theta}+I\right)\otimes\left(\overline{d_{2\theta-\theta}}+I\right)+\left(\overline{d_{2\theta}}+I\right)\otimes\left(\overline{d_{2\theta-2\theta}}+I\right)\\ =\;&1_Q\otimes w^2+w\otimes w+w^2\otimes 1_Q\end{aligned}$$

Comparing with (13), we get $w\otimes w=0$ and hence $w=0$, a contradiction. In conclusion $\theta=N$. □

Theorem 2.14. *Let H be a Hopf algebra and let $(R,m,u,\delta,\varepsilon)$ be a finite dimensional pre-bialgebra in ${}_H^H\mathcal{YD}$. Then the following assertions are equivalent:*

(1) $\mathrm{gr}\left(R\right)$ *is a thin coalgebra.*
(2) R *is a thin coalgebra.*
(3) $R_0R_0\subseteq R_0$ *and* $\mathrm{gr}\left(R\right)$ *is a quantum line with respect to the structures inherited from* R.

Proof. (1) ⇒ (2) It follows by Proposition 1.2.

(2) ⇒ (3) Since $R_0=K1_R$, by 7, we get $R_0R_0\subseteq R_0$ so that, by Theorem 2.13, $\mathrm{gr}\left(R\right)$ is a quantum line with respect to the structures inherited from R.

(3) ⇒ (1) Quantum lines are thin coalgebras. □

Lemma 2.15. *Let H be a Hopf algebra and let $(R,m,u,\delta,\varepsilon)$ be a pre-bialgebra in ${}_H^H\mathcal{YD}$. Assume that R is an N-dimensional thin coalgebra where $P(R)=Ky$. Let $g\in H$ and $\chi\in H^*$ be such that (H,g,χ) is the Yetter-Drinfeld datum associated to y and let $q=\chi(g)$. Consider a divided power sequence*

$$d_0=1_R,d_1=y,\ldots,d_{N-1}$$

of non-zero elements in R such that $gd_n=q^nd_n$, for any $0\leq n\leq N-1$ and $yd_{n-1}=\left(n\right)_q d_n$, for any $1\leq n\leq N-1$ as in Lemma 2.9.
Then

$$hd_n=\chi^n\left(h\right)d_n$$

for any $0\leq n\leq N-1$.

Proof. By Theorem 2.13, we have $N = o(q)$ so that $(n)_q \neq 0$ for any $0 \leq n \leq N-1$. The statement is clear for $n = 0, 1$. Let $2 \leq n \leq N-1$ and assume $hd_{n-1} = \chi^{n-1}(h)\, d_{n-1}$. Then

$$h(yd_{n-1}) = \sum h_{(1)} y \cdot h_{(2)} d_{n-1} = \sum \chi\left(h_{(1)}\right) y \cdot \chi^{n-1}\left(h_{(2)}\right) d_{n-1} = \chi^n(h)\, yd_{n-1}$$

so that $hd_n = \frac{1}{(n)_q} h(yd_{n-1}) = \frac{1}{(n)_q}\chi^n(h)\, yd_{n-1} = \chi^n(h)\, d_n$. □

3. Pre-bialgebras with a Cocycle

Definitions 3.1. Let H be a Hopf algebra. A *cocycle* for a pre-bialgebra $(R, m, u, \delta, \varepsilon)$ in ${}^H_H\mathcal{YD}$ is a K-linear map

$$\xi : R \otimes R \to H$$

such that, for all $r, s, t \in R$ and $h \in H$, the following relations are satisfied:

$$\sum \xi(h_{(1)} r \otimes h_{(2)} s) = \sum h_{(1)} \xi(r \otimes s) S h_{(2)}; \tag{14}$$

$$\Delta_H \xi = (m_H \otimes \xi)(\xi \otimes \rho_{R\otimes R})\delta_{R\otimes R} \quad \text{and} \quad \varepsilon_H \xi = m_K(\varepsilon \otimes \varepsilon); \tag{15}$$

$$c_{R,H}(m \otimes \xi)\delta_{R\otimes R} = (m_H \otimes m_R)(\xi \otimes \rho_{R\otimes R})\delta_{R\otimes R}; \tag{16}$$

$$m_R(R \otimes m_R) = m_R(R \otimes \mu_R)[(m_R \otimes \xi)\delta_{R\otimes R} \otimes R]; \tag{17}$$

$$m_H(\xi \otimes H)[R \otimes (m_R \otimes \xi)\delta_{R\otimes R}] = m_H(\xi \otimes H)(R \otimes c_{H,R})[(m_R \otimes \xi)\delta_{R\otimes R} \otimes R]; \tag{18}$$

$$\xi(R \otimes u) = \xi(u \otimes R) = \varepsilon 1_H. \tag{19}$$

We will also say that $(R, m, u, \delta, \varepsilon)$ is a pre-bialgebra in ${}^H_H\mathcal{YD}$ with cocycle ξ.
For a pre-bialgebra $(R, m, u, \delta, \varepsilon)$ in ${}^H_H\mathcal{YD}$ with cocycle ξ, we have that (R, u, m, ξ) is a dual Yetter-Drinfeld quadruple in the sense of [AMS, Definition 3.59]

To any pre-bialgebra $(R, m, u, \delta, \varepsilon)$ in ${}^H_H\mathcal{YD}$ with cocycle ξ we associate (see [AMS, Theorem 3.62]) a bialgebra $B = R\#_\xi H$ as follows. As a vector space it is $R \otimes H$.

The coalgebra structures are:

$$\begin{aligned} \Delta_B(r\#h) &= \sum r^{(1)} \# r^{(2)}_{\langle -1\rangle} h_{(1)} \otimes r^{(2)}_{\langle 0\rangle} \# h_{(2)}, \qquad \text{where } \delta(r) = \sum r^{(1)} \otimes r^{(2)}, \\ \varepsilon_B(r\#h) &= \varepsilon(r)\,\varepsilon_H(h). \end{aligned}$$

The algebra structures are:

$$\begin{aligned} m_B[(r\#h) \otimes (s\#k)] &= \sum \widetilde{m}^0(r \otimes h_{(1)} s) \otimes \widetilde{m}^1(r \otimes h_{(1)} s)\ h_{(2)} k. \\ u_B(1) &= u(1)\#1_H \end{aligned}$$

where we use the notation

$$(m \otimes \xi)\delta_{R\otimes R}(r \otimes s) = \widetilde{m}(r \otimes s) = \sum \widetilde{m}^0(r \otimes s) \otimes \widetilde{m}^1(r \otimes s) \tag{20}$$

The canonical injection $\sigma : H \hookrightarrow R\#_\xi H$ is a bialgebra homomorphism. Furthermore the map

$$\pi : R\#_\xi H \to H : r\#h \longmapsto \varepsilon(r)\, h$$

is an H-bilinear coalgebra retraction of σ.

Definitions and Notations 3.2. *Let H be a Hopf algebra, let A be a bialgebra and let $\sigma : H \to A$ be an injective morphism of bialgebras having a retraction $\pi : A \to H$ (i.e. $\pi\sigma = H$) that is an H-bilinear coalgebra map. Set*

$$R = A^{Co(H)} = \left\{ a \in A \mid \sum a_{(1)} \otimes \pi\left(a_{(2)}\right) = a \otimes 1_H \right\}$$

Let $\tau : A \to R, \tau(a) = \sum a_{(1)} \sigma S \pi\left(a_{(2)}\right)$ (see Proposition 3.4). The map

$$\omega : R \otimes H \to A,\ \omega(r \otimes h) = r\sigma(h)$$

is an isomorphism of K-vector spaces, the inverse being defined by

$$\omega^{-1} : A \to R \otimes H,\ \omega^{-1}(a) = \sum a_{(1)} \sigma S_H \pi\left(a_{(2)}\right) \otimes \pi\left(a_{(3)}\right) = \sum \tau\left(a_{(1)}\right) \otimes \pi\left(a_{(2)}\right).$$

Clearly A defines, via ω, a bialgebra structure on $R\otimes H$ that will depend on the chosen σ and π. To describe this structure, we need the following data. Set

$$\delta(r)=\sum r_{(1)}\sigma S\pi(r_{(2)})\otimes r_{(3)}=\sum \tau\left(r_{(1)}\right)\otimes r_{(2)},\qquad \varepsilon=\varepsilon_{A|R}.$$

By [Scha, 6.1] *and* [AMS, Theorem 3.64], *(R,δ,ε) is a coalgebra in ${}_H^H\mathcal{YD}$ where the Yetter-Drinfeld module structure of R is given by*

$$^{h}r=\sum \sigma\left(h_{(1)}\right)r\sigma S_H\left(h_{(1)}\right),\qquad \rho\left(r\right)=\sum \pi\left(r_{(1)}\right)\otimes r_{(2)}$$

and the maps $u:K\to R$ and $m:R\otimes R\to R$, given by

$$u=u_A^{|R},\qquad m(r\otimes s)=\sum r_{(1)}s_{(1)}\sigma S\pi(r_{(2)}s_{(2)})=\tau\left(r\cdot_A s\right).$$

*define on R a unital algebra structure (which **might be non associative**).*
Let $\xi:R\otimes R\to H$ be the map defined by setting

$$\xi(r\otimes s)=\pi(r\cdot_A s).$$

Remark 3.3. As proved in [AMS, Theorem 3.64], the datum $(R,m,u,\delta,\varepsilon)$ constructed from (A,π,σ) is a pre-bialgebra in ${}_H^H\mathcal{YD}$ with cocycle ξ. This will be called *the pre-bialgebra in ${}_H^H\mathcal{YD}$ associated to (A,π,σ)*. Moreover ξ will be called *the cocycle corresponding to $(R,m,u,\delta,\varepsilon)$*. Then (cf. [Scha, 6.1]) $\omega:R\#_\xi H\to A$ is a bialgebra isomorphism.

Conversely, note that, starting from a pre-bialgebra $(R,m,u,\delta,\varepsilon)$ in ${}_H^H\mathcal{YD}$ with cocycle ξ, if we consider the maps

$$\sigma:H\hookrightarrow R\#_\xi H\qquad\text{and}\qquad \pi:R\#_\xi H\to H$$

as in Definitions 3.1, then the pre-bialgebra in ${}_H^H\mathcal{YD}$ associated to $(R\#_\xi H,\pi,\sigma)$ is exactly $(R,m,u,\delta,\varepsilon)$ and the corresponding cocycle is exactly ξ.

Proposition 3.4. *Let H be a Hopf algebra with antipode S, let A be a bialgebra and let $\sigma:H\to A$ be an injective morphism of bialgebras having a retraction $\pi:A\to H$ (i.e. $\pi\sigma=H$) that is an H-bilinear coalgebra map. Let $(R,m,u,\delta,\varepsilon)$ be the pre-bialgebra in ${}_H^H\mathcal{YD}$ associated to (A,π,σ).*

Then the map τ of 3.2 is a surjective coalgebra homomorphism. Moreover

$$\begin{aligned}\tau\left[a\sigma\left(h\right)\right] &= \tau\left(a\right)\varepsilon_H\left(h\right),\qquad \tau\left[\sigma\left(h\right)a\right]={}^{h}\tau\left(a\right),\\ r\cdot_R s &= \tau\left(r\cdot_A s\right),\qquad \tau\left(a\right)\cdot_R\tau\left(b\right)=\tau\left[\tau\left(a\right)\cdot_A b\right],\end{aligned}$$

where $a\in A,h\in H$ and $r,s\in R$.

Proof. First of all, let us prove that $\tau\left(a\right)\in A$ is in fact an element of R. Note that

$$\pi\tau\left(a\right)=\sum\pi\left[a_{(1)}\sigma S\pi\left(a_{(2)}\right)\right]=\sum\pi\left(a_{(1)}\right)S\pi\left(a_{(2)}\right)=\varepsilon\left(a\right)1_H.\tag{21}$$

Since $\Delta_A\tau\left(a\right)=a_{(1)}\sigma S\pi\left(a_{(3)}\right)\otimes\tau\left(a_{(2)}\right)$ we get

$$\sum\tau\left(a\right)_{(1)}\otimes\pi\left[\tau\left(a\right)_{(2)}\right]=\sum a_{(1)}\sigma S\pi\left(a_{(3)}\right)\otimes\pi\tau\left(a_{(2)}\right)\overset{(21)}{=}\sum a_{(1)}\sigma S\pi\left(a_{(3)}\right)\otimes 1_H=\tau\left(a\right)\otimes 1_H$$

so that $\tau\left(a\right)\in R$. We have

$$\tau\left[a\sigma\left(h\right)\right]=\sum a_{(1)}\sigma\left(h_{(1)}\right)\sigma S_H\left\{\pi\left[a_{(2)}\right]h_{(2)}\right\}=\tau\left(a\right)\varepsilon_H\left(h\right).\tag{22}$$

Since π is left H-linear we have

$$\tau\left[\sigma\left(h\right)a\right]=\sum\sigma\left(h_{(1)}\right)a_{(1)}\sigma S_H\pi\left[\sigma\left(h_{(2)}\right)a_{(2)}\right]=\sum\sigma\left(h_{(1)}\right)\tau\left(a\right)\sigma S_H\left(h_{(2)}\right)={}^{h}\tau\left(a\right).$$

Let us prove that τ is a coalgebra homomorphism. Since $\delta\left(r\right)=\sum r_{(1)}\sigma S\pi\left(r_{(2)}\right)\otimes r_{(3)}=\sum\tau\left(r_{(1)}\right)\otimes r_{(2)}$ for every $r\in R$, we get

$$\begin{aligned}\delta\tau\left(a\right) &= \sum\tau\left(\tau\left(a\right)_{(1)}\right)\otimes\tau\left(a\right)_{(2)}=\sum\tau\left[a_{(1)}\sigma S\pi\left(a_{(3)}\right)\right]\otimes\tau\left(a_{(2)}\right)\overset{(22)}{=}\left(\tau\otimes\tau\right)\Delta_A a,\\ \varepsilon\tau\left(a\right) &= \varepsilon_{A|R}\left[\sum a_{(1)}S_H\pi\left(a_{(2)}\right)\right]=\varepsilon_A\left(a\right).\end{aligned}$$

Note that for every $r\in R$ we have $\tau(r)=\sum r_{(1)}\sigma S_H\pi\left[r_{(2)}\right]=r\sigma S_H(1_H)=r$ so that τ is surjective. Since $r\cdot_R s=m(r\otimes s)=\sum r_{(1)}s_{(1)}\sigma S\pi\left[r_{(2)}\cdot_A s_{(2)}\right]=\tau(r\cdot_A s)$, for every $r,s\in R$, we obtain $\tau(a)\cdot_R\tau(b)=\tau[\tau(a)\cdot_A\tau(b)]\overset{(22)}{=}\tau\left[\tau(a)\cdot_A b_{(1)}\right]\varepsilon_H S\pi\left(b_{(2)}\right)=\tau[\tau(a)\cdot_A b]$. □

3.5. Let H be a Hopf algebra and let $\chi\in H^*$ be a character. Let (M,ρ_M) be a left H-comodule and (N,ρ_N) be a right H-comodule. In the sequel we will use the well known K-linear automorphisms $\varphi_M:M\to M$ and $\psi_N:N\to N$ defined by

$$\varphi_M(m)=(m\leftharpoonup\chi)=\sum\chi\left(m_{\langle-1\rangle}\right)m_{\langle0\rangle}\quad\text{and}\quad\psi_N(n)=(\chi\rightharpoonup n)=\sum n_{\langle0\rangle}\chi\left(n_{\langle1\rangle}\right)$$

Recall that φ_M and ψ_N are (co)algebra automorphisms whenever M and N are H-comodule (co)algebras.

Proposition 3.6. *Let H be a Hopf algebra and let $(R,m,u,\delta,\varepsilon)$ be a N-dimensional pre-bialgebra in ${}_H^H\mathcal{YD}$. Let ξ be a cocycle for the pre-bialgebra $(R,m,u,\delta,\varepsilon)$. Let $\chi\in H^*$ be a character of H such that*

$$\chi[\xi(r\otimes s)]=\varepsilon(r)\varepsilon(s),\ \textit{for every } r,s\in R. \tag{23}$$

Then the map

$$\varphi_R:R\to R,\varphi_R(r)=\sum\chi\left(r_{\langle-1\rangle}\right)r_{\langle0\rangle}$$

defines an isomorphism of coalgebras which is also an algebra homomorphism. Moreover

$$\varphi_H[\xi(r\otimes s)]=\xi[\varphi_R(r)\otimes\varphi_R(s)],\qquad\psi_H[\xi(r\otimes s)]=\xi(r\otimes s).$$

Proof. Since R is a left H-comodule coalgebra, by 3.5, we have that φ_R is a coalgebra automorphism. We outline that, since the multiplication of R is, in general, not colinear, R need not to be an H-comodule algebra, so that we cannot apply 3.5 to get that φ_R is an algebra homomorphism. By (16), we get

$$\begin{aligned}&\sum\left[r^{(1)}\left(r^{(2)}_{\langle-1\rangle}s^{(1)}\right)\right]_{\langle-1\rangle}\xi(r^{(2)}_{\langle0\rangle}\otimes s^{(2)})\otimes\left[r^{(1)}\left(r^{(2)}_{\langle-1\rangle}s^{(1)}\right)\right]_{\langle0\rangle}\\ =&\sum\xi\left(r^{(1)}\otimes r^{(2)}_{\langle-2\rangle}s^{(1)}\right)r^{(2)}_{\langle-1\rangle}s^{(2)}_{\langle-1\rangle}\otimes r^{(2)}_{\langle0\rangle}s^{(2)}_{\langle0\rangle}.\end{aligned}$$

If we apply $l_R(\chi\otimes R)$ to both sides, we obtain

$$\sum\chi\left[\xi(r^{(2)}_{\langle0\rangle}\otimes s^{(2)})\right]\varphi_R\left[r^{(1)}\left(r^{(2)}_{\langle-1\rangle}s^{(1)}\right)\right]=\sum\chi\left[\xi\left(r^{(1)}\otimes r^{(2)}_{\langle-1\rangle}s^{(1)}\right)\right]\varphi_R\left(r^{(2)}_{\langle0\rangle}\right)\varphi_R\left(s^{(2)}\right).$$

By (23) we get $\varphi_R(r\cdot s)=\varphi_R(r)\cdot\varphi_R(s)$. Moreover $\varphi_R(1_R)=1_R$. We have

$$\begin{aligned}&\Delta_H\xi(r\otimes s)\overset{(15)}{=}(m_H\otimes\xi)(\xi\otimes\rho_{R\otimes R})\delta_{R\otimes R}(r\otimes s)\\ =&\sum\left[\xi\left(r^{(1)}\otimes r^{(2)}_{\langle-2\rangle}s^{(1)}\right)r^{(2)}_{\langle-1\rangle}s^{(2)}_{\langle-1\rangle}\otimes\xi\left(r^{(2)}_{\langle0\rangle}\otimes s^{(2)}_{\langle0\rangle}\right)\right].\end{aligned}$$

so that

$$\begin{aligned}&\varphi_H[\xi(r\otimes s)]=\sum\chi\left[\xi(r\otimes s)_{(1)}\right]\xi(r\otimes s)_{(2)}=l_H(\chi\otimes H)\Delta_H\xi(r\otimes s)\\ =&\sum l_H(\chi\otimes H)\left[\xi\left(r^{(1)}\otimes r^{(2)}_{\langle-2\rangle}s^{(1)}\right)r^{(2)}_{\langle-1\rangle}s^{(2)}_{\langle-1\rangle}\otimes\xi\left(r^{(2)}_{\langle0\rangle}\otimes s^{(2)}_{\langle0\rangle}\right)\right]\\ =&\sum\chi\left[\xi\left(r^{(1)}\otimes r^{(2)}_{\langle-2\rangle}s^{(1)}\right)r^{(2)}_{\langle-1\rangle}s^{(2)}_{\langle-1\rangle}\right]\xi\left(r^{(2)}_{\langle0\rangle}\otimes s^{(2)}_{\langle0\rangle}\right)\\ =&\sum\chi\left[\xi\left(r^{(1)}\otimes r^{(2)}_{\langle-1\rangle}s^{(1)}\right)\right]\xi\left[\varphi_R\left(r^{(2)}_{\langle0\rangle}\right)\otimes\varphi_R\left(s^{(2)}\right)\right]\overset{(23)}{=}\xi[\varphi_R(r)\otimes\varphi_R(s)].\end{aligned}$$

In a similar way one can prove $\psi_H[\xi(r\otimes s)]=\xi(r\otimes s)$. □

Definition 3.7. Let H be a Hopf algebra and let R be a braided bialgebra in the category ${}_H^H\mathcal{YD}$. The tensor product $R\otimes H$ endowed with the smash product and the smash coproduct is a bialgebra that will be denoted by $R\#H$ and called the *Radford-Majid bosonization of* R (see [Rad] and [Maj]).

Lemma 3.8. *let H be a Hopf algebra, let A be a bialgebra and let $\sigma : H \to A$ be an injective morphism of bialgebras having a retraction $\pi : A \to H$ (i.e. $\pi\sigma = H$) that is an H-bilinear coalgebra map. Let $(R, m, u, \delta, \varepsilon)$ be the pre-bialgebra in ${}_H^H\mathcal{YD}$ associated to (A, π, σ) with corresponding cocycle ξ.*
Then for every $r \in R$ and $h \in H$ we have

$$\pi\left(r\sigma\left(h\right)\right) = \varepsilon\left(r\right)h.$$

Moreover the following assertions are equivalent:
(1) $\xi = \varepsilon \otimes \varepsilon$.
(2) $\pi : A \to H$ *is a bialgebra homomorphism.*
(3) R *is a braided bialgebra in* ${}_H^H\mathcal{YD}$ *and* $R\#_\xi H = R\#H$ *is the Radford-Majid bosonization of* R.

Proof. For every $r \in R$ and $h \in H$ we have $\pi\left(r\sigma\left(h\right)\right) = \pi\left(r\right)h = \sum \varepsilon\left(r_{(1)}\right)\pi\left(r_{(2)}\right)h = \varepsilon\left(r\right)h$.
(1) $\Rightarrow$ (2). Clearly $\{r\sigma\left(h\right) \mid r \in R, h \in H\}$ generates A. We have

$$\begin{aligned}
\pi\left[r\sigma\left(h\right)s\sigma\left(k\right)\right] &= \sum \pi\left[r\sigma\left(h_{(1)}\right)s\sigma S\left(h_{(2)}\right)\sigma\left(h_{(3)}k\right)\right] = \sum \pi\left(r \cdot_A {}^{h_{(1)}}s\right)h_{(2)}k \\
&= \sum \xi\left(r \otimes {}^{h_{(1)}}s\right)h_{(2)}k = \sum \varepsilon\left(r\right)\varepsilon\left({}^{h_{(1)}}s\right)h_{(2)}k \\
&= \sum \varepsilon\left(r\right)\varepsilon_H\left(h_{(1)}\right)\varepsilon\left(s\right)h_{(2)}k = \varepsilon\left(r\right)h\varepsilon\left(s\right)k = \pi\left(r\sigma\left(h\right)\right) \cdot_H \pi\left(s\sigma\left(k\right)\right).
\end{aligned}$$

(2) $\Rightarrow$ (1) follows easily by the definition of $\xi : \xi\left(r \otimes s\right) = \pi\left(r \cdot_A s\right) = \pi\left(r\right) \cdot_H \pi\left(s\right) = \varepsilon\left(r\right)\varepsilon\left(s\right)1_H$.
(1) $\Rightarrow$ (3) can be easily proved by direct computation.
(3) $\Rightarrow$ (2) Observe that $\pi = \pi' \circ \omega^{-1}$ where the map $\pi' : R\#_\xi H \to H : r\#h \longmapsto \varepsilon\left(r\right)h$. One easily check that π' is an algebra homomorphism so that π is an algebra homomorphism too. □

Theorem 3.9. *Let H be a Hopf algebra and let $(R, m, u, \delta, \varepsilon)$ be a pre-bialgebra in ${}_H^H\mathcal{YD}$ with cocycle ξ. The following assertions are equivalent:*
(*a*) $Corad\left(R\right) = K1_R$ *i.e.* R *is connected.*
(*b*) $Corad\left(R\#_\xi H\right) \subseteq K \otimes H$.
(*c*) $Corad\left(R\#_\xi H\right) = K \otimes Corad\left(H\right)$.
Moreover, in this case $R\#_\xi H$ is a Hopf algebra.

Proof. Set $B := R\#_\xi H$. Recall that the coalgebra structures of B are:

$$\Delta_B\left(r\#h\right) = \sum r^{(1)}\#r^{(2)}{}_{\langle -1\rangle}h_{(1)} \otimes r^{(2)}{}_{\langle 0\rangle}\#h_{(2)}, \qquad \varepsilon_B\left(r\#h\right) = \varepsilon\left(r\right)\varepsilon_H\left(h\right),$$

(a) $\Rightarrow$ (b). Assume that $R_0 = Corad\left(R\right) = K1_R$ i.e. R is connected.
Let $R_0 \leq R_1 \leq \cdots \leq R_{n-1} \leq R_n \leq \cdots \leq R$ be the coradical filtration of R. Let $B_i = R_i \otimes H$. Let $r \in R_n$, $n \in \mathbb{N}$. Then $\delta\left(r\right) = \sum_{i=0}^{n} r_i \otimes s_{n-i}$, where $r_i, s_i \in R_i$. Thus

$$\begin{aligned}
\Delta_{R\#H}\left(r\#h\right) &= \sum r^{(1)}\#r^{(2)}{}_{\langle -1\rangle}h_{(1)} \otimes r^{(2)}{}_{\langle 0\rangle}\#h_{(2)} \\
&= \sum_{i=0}^{n} r_i\#\left(s_{n-i}\right)_{\langle -1\rangle}h_{(1)} \otimes \left(s_{n-i}\right)_{\langle 0\rangle}\#h_{(2)} \in \sum_{i=0}^{n} B_i \otimes B_{n-i}.
\end{aligned}$$

Therefore $\Delta_{R\#H}\left(B_n\right) \subseteq \sum_{i=0}^{n} B_i \otimes B_{n-i}$ and hence

$$H \simeq K \otimes H = B_0 \leq B_1 \leq \cdots \leq B_{n-1} \leq B_n \leq \cdots \leq B$$

defines a coalgebra filtration for B. This entails that $Corad\left(B\right) \subseteq H$ (see [Sw, page 226]).
(b) $\Rightarrow$ (a). Assume that $Corad\left(B\right) \subseteq K \otimes H$. Apply Proposition 3.4 to the case when $\sigma : H \to B$ is the canonical injection and $\pi : B \to H$ is defined by $\pi\left(r\#h\right) = \varepsilon\left(r\right)h$ (as observed in Definitions 3.1 π is a left H-bilinear coalgebra retraction of σ). Then $\tau : B \to R, r\#h \mapsto r\varepsilon_H\left(h\right)$ is a surjective coalgebra homomorphism. By [Mo, Corollary 5.3.5, page 66], we have that

$$Corad\left(R\right) \subseteq \tau\left(Corad\left(B\right)\right) \subseteq \tau\left(K \otimes H\right) = K.$$

$(c)\Rightarrow(b)$ is trivial.
$(b)\Rightarrow(c)$. We get $Corad\,(R\#_{\xi}H)=Corad\,(K\otimes H)=K\otimes Corad\,(H)$ as $Corad\,(R\#_{\xi}H)\subseteq K\otimes H$. Let us prove that $R\#_{\xi}H$ has an antipode, whenever (b) holds. Since H is a Hopf algebra and $Corad\,(R\#_{\xi}H)\subseteq K\otimes H=H$, the antipode of H gives an inverse of the canonical inclusion $Corad\,(R\#_{\xi}H)\subseteq R\#_{\xi}H$ in $\mathrm{Hom}_{K}\left(Corad\,(R\#_{\xi}H),R\#_{\xi}H\right)$ so that, in view of a famous Takeuchi's result [Mo, Lemma 5.2.10], $R\#_{\xi}H$ has an antipode. □

Our aim is to characterize those pre-bialgebras $(R,m,u,\delta,\varepsilon)$ in ${}_{H}^{H}\mathcal{YD}$ with cocycle ξ such that $\mathrm{gr}\,(R)$ is a quantum line. The first step is to lift the properties of $\mathrm{gr}\,(R)$ to R which a priori is a non-associative algebra.

Let $B=R\#_{\xi}H$. We point out that our procedure differs from the classical lifting method by N. Andruskiewitsch and H.-J. Schneider. Namely, since B_0 does not need to be a Hopf subalgebra of B, its associated graded coalgebra $\mathrm{gr}\,(B)$ is not a (graded) Hopf algebra in general.

Lemma 3.10. *Keep the assumptions and notations of Lemma 2.15. Let ξ be a cocycle for the pre-bialgebra $(R,m,u,\delta,\varepsilon)$. Let $0\leq a,b\leq N-1$. Then,*

$$\chi^{a+b}\,(h)\,\xi\,(d_a\otimes d_b)=\sum h_{(1)}\xi\,(d_a\otimes d_b)\,Sh_{(2)},\ \text{for every } h\in H. \tag{24}$$

In particular, for any $c\in\mathbb{N}$, we have

$$\left[\chi^{a+b}\,(h)-\varepsilon_H\,(h)\right]\chi^{c}\left[\xi\,(d_a\otimes d_b)\right]=0,\ \text{for every } h\in H. \tag{25}$$

Moreover if $a+b\neq 0$ and

$$\chi^{c+1}\left[\xi\,(d_a\otimes d_b)\right]=\chi^{c}\left[\xi\,(d_a\otimes d_b)\right]+\chi\left[\xi\,(d_a\otimes d_b)\right] \tag{26}$$

for any $c\in\mathbb{N}$, then we have $\chi^{c}\left[\xi\,(d_a\otimes d_b)\right]=0$ for every $c\in\mathbb{N}$.

Proof. By (14), we have: $\sum\xi(h_{(1)}r\otimes h_{(2)}s)=\sum h_{(1)}\xi(r\otimes s)Sh_{(2)}$ for any $r,s\in R$ and $h\in H$. We apply this in the case $r=d_a$ and $s=d_b$,where $0\leq a,b\leq N-1$ to obtain $\sum\xi(h_{(1)}d_a\otimes h_{(2)}d_b)=\sum h_{(1)}\xi\,(d_a\otimes d_b)\,Sh_{(2)}$. Now, by Lemma 2.15, we have

$$\sum\xi(h_{(1)}d_a\otimes h_{(2)}d_b)=\sum\xi(\chi^{a}\left(h_{(1)}\right)d_a\otimes\chi^{b}\left(h_{(2)}\right)d_b)=\chi^{a+b}\,(h)\,\xi\,(d_a\otimes d_b)$$

and hence we obtain (24). Then, by applying χ^{c}, $c\geq 0$, to both sides of this formula, we get $\chi^{a+b}\,(h)\,\chi^{c}\left[\xi\,(d_a\otimes d_b)\right]=\varepsilon_H\,(h)\,\chi^{c}\left[\xi\,(d_a\otimes d_b)\right]$, so that we obtain (25).

Assume that $a+b\neq 0$ and that (26) holds for any $c\in\mathbb{N}$. By induction on $c\geq 0$, one can prove that

$$\chi^{c}\left[\xi\,(d_a\otimes d_b)\right]=c\cdot\chi\left[\xi\,(d_a\otimes d_b)\right],\ \text{for any } c\in\mathbb{N}. \tag{27}$$

Now, by (15), and since $\varepsilon\,(d_n)=\delta_{n,0}=0$ for any $n\geq 1$ (see Lemma 1.5), we obtain $\varepsilon_H\left[\xi\,(d_a\otimes d_b)\right]=\varepsilon\,(d_a)\,\varepsilon\,(d_b)=0$. Thus, by (25), applied to the case $c=1$ and $h=\xi\,(d_a\otimes d_b)$, by (27) and since $a+b\neq 0$, we obtain $\chi\left[\xi\,(d_a\otimes d_b)\right]=0$ so that $\chi^{c}\left[\xi\,(d_a\otimes d_b)\right]=c\chi\left[\xi\,(d_a\otimes d_b)\right]=0$, for any $c\geq 0$. □

Theorem 3.11. *Keep the assumptions and notations of Lemma 2.15. Let ξ be a cocycle for the pre-bialgebra $(R,m,u,\delta,\varepsilon)$.*
Let $0\leq a,b\leq N-1$. We have

$$\sum\chi^{c}\left[(d_a)_{\langle -1\rangle}\right](d_a)_{\langle 0\rangle}=q^{ca}d_a\ \text{for any } c\in\mathbb{N}, \tag{28}$$

$$\delta_{R\otimes R}\,(d_a\otimes d_b)=\sum\nolimits_{0\leq i\leq a,0\leq j\leq b}q^{j(a-i)}d_i\otimes d_j\otimes d_{a-i}\otimes d_{b-j}. \tag{29}$$

$$\begin{aligned}&\Delta_H\xi\,(d_a\otimes d_b)\\ &=\sum\nolimits_{0\leq i\leq a,0\leq j\leq b}q^{j(a-i)}\xi\,(d_i\otimes d_j)\,(d_{a-i})_{\langle -1\rangle}\,(d_{b-j})_{\langle -1\rangle}\otimes\xi\left[(d_{a-i})_{\langle 0\rangle}\otimes(d_{b-j})_{\langle 0\rangle}\right].\end{aligned} \tag{30}$$

$$\chi^{c}\left[\xi\,(d_1\otimes d_a)\right]=0,\ \text{for any } c\in\mathbb{N}. \tag{31}$$

If $b \leq N-a$, we have

$$\rho(d_a d_b) = \sum (d_a)_{\langle -1\rangle}(d_b)_{\langle -1\rangle} \otimes (d_a)_{\langle 0\rangle}(d_b)_{\langle 0\rangle} + \tag{32}$$
$$+\sum_{\substack{0\leq i\leq a, 0\leq j\leq b\\ 0<i+j<a+b}} \left[\begin{array}{c} q^{(b-j)i}\xi(d_{a-i}\otimes d_{b-j})(d_i)_{\langle -1\rangle}(d_j)_{\langle -1\rangle}\otimes (d_i)_{\langle 0\rangle}(d_j)_{\langle 0\rangle} + \\ -q^{j(a-i)}(d_i d_j)_{\langle -1\rangle}\xi(d_{a-i}\otimes d_{b-j})\otimes (d_i d_j)_{\langle 0\rangle} \end{array}\right]$$

$$\sum \chi^c\left[(d_1 d_a)_{\langle -1\rangle}\right](d_1 d_a)_{\langle 0\rangle} = q^{c(1+a)} d_1 d_a \tag{33}$$

Proof. Recall that, by Lemma 2.7, there are $g\in G(H)$ such that $\rho(y) = g\otimes y$ and $\chi\in H^*$, a character such that $hy=\chi(h)y$ for every $h\in H$.

Let us proceed by induction on $0\leq a\leq N-1$. The case $a=0$ is straightforward. Let $a\geq 1$ and assume that the statements hold for every $0\leq i\leq a-1$. By assumption, for any $c\in\mathbb{N}$, we have $\sum\chi^c\left[(d_1 d_{a-1})_{\langle -1\rangle}\right](d_1 d_{a-1})_{\langle 0\rangle} = q^{ca} d_1 d_{a-1}$. By Lemma 2.9 we get $d_1 d_{a-1} = (a)_q d_a$ so that $(a)_q\sum\chi^c\left[(d_a)_{\langle -1\rangle}\right](d_a)_{\langle 0\rangle} = (a)_q q^{ca} d_a$. By Theorem 2.13, we get $N = o(q)$. Since $a\leq N-1$ then $(a)_q\neq 0$ so that we get (28). By means of Lemma 2.15, we have

$$\begin{aligned}
\delta_{R\otimes R}(d_a\otimes d_b) &= (R\otimes c_{R,R}\otimes R)(\delta\otimes\delta)(d_a\otimes d_b)\\
&= \sum_{0\leq i\leq a, 0\leq j\leq b} d_i\otimes (d_{a-i})_{\langle -1\rangle} d_j\otimes (d_{a-i})_{\langle 0\rangle}\otimes d_{b-j}\\
&= \sum_{0\leq i\leq a, 0\leq j\leq b} d_i\otimes \chi^j\left[(d_{a-i})_{\langle -1\rangle}\right] d_j\otimes (d_{a-i})_{\langle 0\rangle}\otimes d_{b-j}\\
&\overset{(28)}{=} \sum_{0\leq i\leq a, 0\leq j\leq b} q^{j(a-i)} d_i\otimes d_j\otimes d_{a-i}\otimes d_{b-j}
\end{aligned}$$

so that we get (29) and

$$\begin{aligned}
&(\xi\otimes\rho_{R\otimes R})\delta_{R\otimes R}(d_a\otimes d_b)\\
\overset{(29)}{=}\ & \sum_{0\leq i\leq a, 0\leq j\leq b} q^{j(a-i)}\xi(d_i\otimes d_j)\otimes\rho_{R\otimes R}(d_{a-i}\otimes d_{b-j})\\
=\ & \sum_{0\leq i\leq a, 0\leq j\leq b} q^{j(a-i)}\xi(d_i\otimes d_j)\otimes (d_{a-i})_{\langle -1\rangle}(d_{b-j})_{\langle -1\rangle}\otimes (d_{a-i})_{\langle 0\rangle}\otimes (d_{b-j})_{\langle 0\rangle}
\end{aligned}$$

that is

(34)

$$(\xi\otimes\rho_{R\otimes R})\delta_{R\otimes R}(d_a\otimes d_b) = \sum_{0\leq i\leq a, 0\leq j\leq b} q^{j(a-i)}\xi(d_i\otimes d_j)\otimes (d_{a-i})_{\langle -1\rangle}(d_{b-j})_{\langle -1\rangle}\otimes (d_{a-i})_{\langle 0\rangle}\otimes (d_{b-j})_{\langle 0\rangle}.$$

By means of (15), we obtain

$$\begin{aligned}
&\Delta_H\xi(d_a\otimes d_b) = (m_H\otimes\xi)(\xi\otimes\rho_{R\otimes R})\delta_{R\otimes R}(d_a\otimes d_b)\\
\overset{(34)}{=}\ & \sum_{0\leq i\leq a, 0\leq j\leq b} q^{j(a-i)}\xi(d_i\otimes d_j)(d_{a-i})_{\langle -1\rangle}(d_{b-j})_{\langle -1\rangle}\otimes\xi\left[(d_{a-i})_{\langle 0\rangle}\otimes (d_{b-j})_{\langle 0\rangle}\right].
\end{aligned}$$

so that we get (30). Let us prove (31).

We have

$$\begin{aligned}
&\chi^{c+1}\left[\xi(d_1\otimes d_a)\right] = m_K(\chi\otimes\chi^c)\Delta_H\xi(d_1\otimes d_a)\\
\overset{(30)}{=}\ & \sum_{0\leq i\leq 1, 0\leq j\leq a} q^{j(1-i)}\chi\left[\xi(d_i\otimes d_j)(d_{1-i})_{\langle -1\rangle}(d_{a-j})_{\langle -1\rangle}\right]\cdot\chi^c\left\{\xi\left[(d_{1-i})_{\langle 0\rangle}\otimes (d_{a-j})_{\langle 0\rangle}\right]\right\}\\
=\ & \sum_{0\leq i\leq 1, 0\leq j\leq a} q^{j(1-i)}\chi\left[\xi(d_i\otimes d_j)\right]\cdot\chi^c\left\{\xi\left[\chi\left[(d_{1-i})_{\langle -1\rangle}\right](d_{1-i})_{\langle 0\rangle}\otimes\chi\left[(d_{a-j})_{\langle -1\rangle}\right](d_{a-j})_{\langle 0\rangle}\right]\right\}\\
\overset{(28)}{=}\ & \sum_{0\leq i\leq 1, 0\leq j\leq a} q^{j(1-i)} q^{1+a-(i+j)}\chi\left[\xi(d_i\otimes d_j)\right]\cdot\chi^c\left[\xi(d_{1-i}\otimes d_{a-j})\right]\\
=\ & q^{1+a}\chi^c\left[\xi(d_1\otimes d_a)\right] + \chi\left[\xi(d_1\otimes d_a)\right]
\end{aligned}$$

as $\xi(d_0\otimes d_u) = \delta_{u,0}1_H$, for every $0\leq u\leq N-1$. Therefore we obtain

$$\chi^{c+1}\left[\xi(d_1\otimes d_a)\right] = q^{1+a}\chi^c\left[\xi(d_1\otimes d_a)\right] + \chi\left[\xi(d_1\otimes d_a)\right],\ \text{for every } c\in\mathbb{N}.$$

If $q^{1+a}=1$, by Lemma 3.10, we obtain $\chi^{c}\left[\xi\left(d_{1}\otimes d_{a}\right)\right]=0$, for every $c\in\mathbb{N}$.
If $q^{1+a}\neq 1$, by (25) applied in the case $h=g$, for any $c\in\mathbb{N}$, we have

$$0=\left[\chi^{1+a}\left(h\right)-\varepsilon_{H}\left(h\right)\right]\chi^{c}\left[\xi\left(d_{1}\otimes d_{a}\right)\right]=\left[\chi^{1+a}\left(g\right)-\varepsilon_{H}\left(g\right)\right]\chi^{c}\left[\xi\left(d_{1}\otimes d_{a}\right)\right]=\left[q^{1+a}-1\right]\chi^{c}\left[\xi\left(d_{1}\otimes d_{a}\right)\right]$$

whence $\chi^{c}\left[\xi\left(d_{1}\otimes d_{a}\right)\right]=0$. In both cases we got (31).

Let $b\leq N-a$. Let us compute (16):

$$c_{R,H}(m\otimes\xi)\delta_{R\otimes R}\left(d_{a}\otimes d_{b}\right)=(m_{H}\otimes m_{R})(\xi\otimes\rho_{R\otimes R})\delta_{R\otimes R}\left(d_{a}\otimes d_{b}\right).$$

The left side:

$$\begin{aligned}
&c_{R,H}(m\otimes\xi)\delta_{R\otimes R}\left(d_{a}\otimes d_{b}\right)\overset{(29)}{=}\sum\nolimits_{0\leq i\leq a,0\leq j\leq b}q^{j(a-i)}c_{R,H}(m\otimes\xi)\left(d_{i}\otimes d_{j}\otimes d_{a-i}\otimes d_{b-j}\right)\\
=\;&\sum\nolimits_{0\leq i\leq a,0\leq j\leq b}q^{j(a-i)}\left(d_{i}d_{j}\right)_{\langle -1\rangle}\xi\left(d_{a-i}\otimes d_{b-j}\right)\otimes\left(d_{i}d_{j}\right)_{\langle 0\rangle}\\
=\;&\left(d_{a}d_{b}\right)_{\langle -1\rangle}\otimes\left(d_{a}d_{b}\right)_{\langle 0\rangle}+\sum\nolimits_{\substack{0\leq i\leq a,0\leq j\leq b\\ i+j<a+b}}q^{j(a-i)}\left(d_{i}d_{j}\right)_{\langle -1\rangle}\xi\left(d_{a-i}\otimes d_{b-j}\right)\otimes\left(d_{i}d_{j}\right)_{\langle 0\rangle}
\end{aligned}$$

The right side

$$\begin{aligned}
&(m_{H}\otimes m_{R})(\xi\otimes\rho_{R\otimes R})\delta_{R\otimes R}\left(d_{a}\otimes d_{b}\right)\\
\overset{(34)}{=}\;&\sum\nolimits_{0\leq i\leq a,0\leq j\leq b}q^{j(a-i)}\xi\left(d_{i}\otimes d_{j}\right)\left(d_{a-i}\right)_{\langle -1\rangle}\left(d_{b-j}\right)_{\langle -1\rangle}\otimes\left(d_{a-i}\right)_{\langle 0\rangle}\left(d_{b-j}\right)_{\langle 0\rangle}\\
=\;&\sum\left(d_{a}\right)_{\langle -1\rangle}\left(d_{b}\right)_{\langle -1\rangle}\otimes\left(d_{a}\right)_{\langle 0\rangle}\left(d_{b}\right)_{\langle 0\rangle}+\\
&+\sum\nolimits_{\substack{0\leq i\leq a,0\leq j\leq b\\ i+j>0}}q^{j(a-i)}\xi\left(d_{i}\otimes d_{j}\right)\left(d_{a-i}\right)_{\langle -1\rangle}\left(d_{b-j}\right)_{\langle -1\rangle}\otimes\left(d_{a-i}\right)_{\langle 0\rangle}\left(d_{b-j}\right)_{\langle 0\rangle}\\
=\;&\sum\left(d_{a}\right)_{\langle -1\rangle}\left(d_{b}\right)_{\langle -1\rangle}\otimes\left(d_{a}\right)_{\langle 0\rangle}\left(d_{b}\right)_{\langle 0\rangle}+\\
&+\sum\nolimits_{\substack{0\leq i\leq a,0\leq j\leq b\\ i+j<a+b}}q^{(b-j)i}\xi\left(d_{a-i}\otimes d_{b-j}\right)\left(d_{i}\right)_{\langle -1\rangle}\left(d_{j}\right)_{\langle -1\rangle}\otimes\left(d_{i}\right)_{\langle 0\rangle}\left(d_{j}\right)_{\langle 0\rangle}
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&\rho\left(d_{a}d_{b}\right)=\sum\left(d_{a}\right)_{\langle -1\rangle}\left(d_{b}\right)_{\langle -1\rangle}\otimes\left(d_{a}\right)_{\langle 0\rangle}\left(d_{b}\right)_{\langle 0\rangle}+\\
&+\sum\nolimits_{\substack{0\leq i\leq a,0\leq j\leq b\\ i+j<a+b}}\left[\begin{array}{c} q^{(b-j)i}\xi\left(d_{a-i}\otimes d_{b-j}\right)\left(d_{i}\right)_{\langle -1\rangle}\left(d_{j}\right)_{\langle -1\rangle}\otimes\left(d_{i}\right)_{\langle 0\rangle}\left(d_{j}\right)_{\langle 0\rangle}+\\ -q^{j(a-i)}\left(d_{i}d_{j}\right)_{\langle -1\rangle}\xi\left(d_{a-i}\otimes d_{b-j}\right)\otimes\left(d_{i}d_{j}\right)_{\langle 0\rangle}\end{array}\right]\\
=\;&\sum\left(d_{a}\right)_{\langle -1\rangle}\left(d_{b}\right)_{\langle -1\rangle}\otimes\left(d_{a}\right)_{\langle 0\rangle}\left(d_{b}\right)_{\langle 0\rangle}+\\
&+\sum\nolimits_{\substack{0\leq i\leq a,0\leq j\leq b\\ 0<i+j<a+b}}\left[\begin{array}{c} q^{(b-j)i}\xi\left(d_{a-i}\otimes d_{b-j}\right)\left(d_{i}\right)_{\langle -1\rangle}\left(d_{j}\right)_{\langle -1\rangle}\otimes\left(d_{i}\right)_{\langle 0\rangle}\left(d_{j}\right)_{\langle 0\rangle}+\\ -q^{j(a-i)}\left(d_{i}d_{j}\right)_{\langle -1\rangle}\xi\left(d_{a-i}\otimes d_{b-j}\right)\otimes\left(d_{i}d_{j}\right)_{\langle 0\rangle}\end{array}\right]
\end{aligned}$$

so that we got (32). Let us apply this formula in the case $(a,b)=(1,a)$.

$$\begin{aligned}
&\sum\chi^{c}\left[\left(d_{1}d_{a}\right)_{\langle -1\rangle}\right]\left(d_{1}d_{a}\right)_{\langle 0\rangle}\\
\overset{(32)}{=}\;&\sum\chi^{c}\left[\left(d_{1}\right)_{\langle -1\rangle}\left(d_{a}\right)_{\langle -1\rangle}\right]\left(d_{1}\right)_{\langle 0\rangle}\left(d_{a}\right)_{\langle 0\rangle}+\\
&+\sum\nolimits_{\substack{0\leq i\leq 1,0\leq j\leq a\\ 0<i+j<1+a}}\left[\begin{array}{c} q^{(a-j)i}\chi^{c}\left[\xi\left(d_{1-i}\otimes d_{a-j}\right)\left(d_{i}\right)_{\langle -1\rangle}\left(d_{j}\right)_{\langle -1\rangle}\right]\left(d_{i}\right)_{\langle 0\rangle}\left(d_{j}\right)_{\langle 0\rangle}+\\ -q^{j(1-i)}\chi^{c}\left[\left(d_{i}d_{j}\right)_{\langle -1\rangle}\xi\left(d_{1-i}\otimes d_{a-j}\right)\right]\left(d_{i}d_{j}\right)_{\langle 0\rangle}\end{array}\right]\\
\overset{(28)}{=}\;&q^{c(1+a)}d_{1}d_{a}
\end{aligned}$$

where, the last equality follows as $\chi^{c}\left[\xi\left(d_{1-i}\otimes d_{a-j}\right)\right]=0$ for $0\leq i\leq 1,0\leq j\leq a$ and $0<i+j<1+a$: in fact, by (31) and (19), $\chi^{c}\left[\xi\left(d_{1-i}\otimes d_{a-j}\right)\right]=0$ unless $i=1$ and $j=a$. Hence we get (33). □

THEOREM 3.12. *Keep the assumptions and notations of Lemma 2.15. Let ξ be a cocycle for the pre-bialgebra $(R, m, u, \delta, \varepsilon)$. Then*

$$\chi^c\left[\xi\left(d_a \otimes d_b\right)\right] = 0 \text{ unless } a = 0 \text{ and } b = 0, \tag{35}$$

for any a, b such that $0 \leq a, b \leq N-1$ and for any $c \in \mathbb{N}$.

Proof. Let us prove, by induction on $t \geq 1$, that $\chi^c\left[\xi\left(d_a \otimes d_b\right)\right] = 0$,for any $c \in \mathbb{N}$ and for any $0 \leq a, b \leq N-1$ such that $t = a + b$. If $t = 1$, then $\xi\left(d_a \otimes d_b\right) = 0$ so that there is nothing to prove. Let $t \geq 2$ be such that $\chi^c\left[\xi\left(d_i \otimes d_j\right)\right] = 0$ for any $1 \leq i + j \leq t - 1$ and for any $c \in \mathbb{N}$. Now, for any $c \in \mathbb{N}$, by means of (30), (28) and the inductive hypothesis, in the style of the proof of (31), one gets $\chi^c\left[\xi\left(d_a \otimes d_b\right)\right] = 0$. □

NOTATION 3.13. *Let H be a Hopf algebra and let $(R, m, u, \delta, \varepsilon)$ be a N-dimensional pre-bialgebra in ${}^H_H\mathcal{YD}$. Assume that R is a thin coalgebra where $P(R) = Ky$. Let $g \in H$ and $\chi \in H^*$ be such that (H, g, χ) is the Yetter-Drinfeld datum associated to y and let $q = \chi(g)$.*
From now on, we fix a basis of R consisting of a divided power sequence of non-zero elements in R

$$d_0 = 1_R, d_1 = y, \ldots, d_{N-1}$$

such that

$$\begin{aligned} gd_n &= q^n d_n, \text{ for any } 0 \leq n \leq N-1, \\ yd_{n-1} &= (n)_q\, d_n, \text{ for any } 1 \leq n \leq N-1, \\ hd_n &= \chi^n(h)\, d_n, \text{ for any } 0 \leq n \leq N-1. \end{aligned}$$

Such a basis exists in view of Lemma 2.9 and of Lemma 2.15.

THEOREM 3.14. *Let H be a Hopf algebra and let $(R, m, u, \delta, \varepsilon)$ be a N-dimensional pre-bialgebra in ${}^H_H\mathcal{YD}$. Assume that R is a thin coalgebra where $P(R) = Ky$. Let $g \in H$ and $\chi \in H^*$ be such that (H, g, χ) is the Yetter-Drinfeld datum associated to y and let $q = \chi(g)$. Let ξ be a cocycle for the pre-bialgebra $(R, m, u, \delta, \varepsilon)$.*
Then:

1) *R is an **associative algebra** over K spanned by y.*
2) *$o(q) = N$.*
3) *$y^n = (n)_q! d_n$, for every $0 \leq n \leq N-1$ and $y^N = 0$.*
4) *$\left(y^i\right)_{0 \leq i \leq N-1}$ is a basis for R .*
5) *$R = R_q(H, g, \chi)$ is a quantum line, whenever m is left H-colinear.*

Proof. Recall that, by Lemma 2.7, there are $g \in G(H)$ such that $\rho(y) = g \otimes y$ and $\chi \in H^*$, a character such that $hy = \chi(h)y$ for every $h \in H$.

For any a, b integers such that $0 \leq a, b \leq N-1$, we have:

$$(m_R \otimes \xi)\delta_{R \otimes R}\left(d_a \otimes d_b\right) \overset{(29)}{=} \sum\nolimits_{0 \leq i \leq a, 0 \leq j \leq b} q^{j(a-i)}\left(d_i d_j\right) \otimes \xi\left(d_{a-i} \otimes d_{b-j}\right). \tag{36}$$

We obtain,

$$\begin{aligned} & d_a\left(d_b d_c\right) = m_R(R \otimes m_R)\left(d_a \otimes d_b \otimes d_c\right) \\ \overset{(17)}{=} \; & m_R(R \otimes \mu_R)[(m_R \otimes \xi)\delta_{R \otimes R} \otimes R]\left(d_a \otimes d_b \otimes d_c\right) \\ \overset{(36)}{=} \; & \sum\nolimits_{0 \leq i \leq a, 0 \leq j \leq b} q^{j(a-i)}\left(d_i d_j\right) \cdot \left[\xi\left(d_{a-i} \otimes d_{b-j}\right) d_c\right] \\ = \; & \sum\nolimits_{0 \leq i \leq a, 0 \leq j \leq b} q^{j(a-i)}\left(d_i d_j\right) \cdot \left[\chi^c\left[\xi\left(d_{a-i} \otimes d_{b-j}\right)\right] d_c\right] \\ \overset{(35)}{=} \; & q^{b(a-a)}\left(d_a d_b\right) \cdot \chi^c\left[\xi(d_{a-a} \otimes d_{b-b})\right] d_c = \left(d_a d_b\right) d_c. \end{aligned}$$

Therefore R is an associative algebra.
By Theorem 2.13, $N = o(q)$ so that $(n)_q \neq 0$, for any $1 \leq n \leq N-1$. Since $yd_{n-1} = (n)_q\, d_n$, for any $1 \leq n \leq N-1$, we infer that $d_n = \frac{y^n}{(n)_q!}$ which means that R, as an associative algebra,

is spanned by y and that $\left(y^i\right)_{0\leq i\leq N-1}$ is a basis for R. Assume $y^N\neq 0$. Since $y^N\in R$ we have $y^N=k_0 1_R+k_1y+...+k_{N-1}y^{N-1}$ for suitable $k_i\in K$ and there is a t with $0\leq t\leq N-1$ such that $k_t\neq 0$. Note that $0=\varepsilon(y)^N=\varepsilon\left(y^N\right)=k_0$ so that $y^N=k_1y+...+k_{N-1}y^{N-1}$ and $1\leq t\leq N-1$. We get

$$\begin{aligned} gy^N &= (gy)^N=q^Ny^N=y^N=k_1y+...+k_{N-1}y^{N-1} \text{ and}\\ gy^N &= k_1gy+...+k_{N-1}gy^{N-1}=k_1qy+...+k_{N-1}q^{N-1}y^{N-1}. \end{aligned}$$

Since $\{1_R,y,...y^{N-1}\}$ is a linearly independent set over K, we infer that $k_t=k_tq^t$. Since $k_t\neq 0$, one gets $1=q^t$. Since $1\leq t\leq N-1$ and $o(q)=N$ we have a contradiction. Assume that m, which is associative, is also left H-colinear. Then R is a braided bialgebra in $({}_H^H\mathcal{YD},\otimes,K)$ and, in this case, $R=R_q(H,g,\chi)$ is a quantum line. □

Proposition 3.15. *Take the hypothesis and notations of 3.13. Let ξ be a cocycle for the pre-bialgebra $(R,m,u,\delta,\varepsilon)$. We have*

$$\chi^c[\xi(r\otimes s)]=\varepsilon(r)\varepsilon(s),\ \textit{for every } r,s\in R,c\in\mathbb{N},$$

and the map $\varphi_R:R\to R,\varphi_R(r)=\sum\chi\left(r_{\langle -1\rangle}\right)r_{\langle 0\rangle}$ defines an isomorphism of coalgebras which is also an algebra homomorphism. Moreover

$$\varphi_H[\xi(r\otimes s)] = \xi[\varphi_R(r)\otimes\varphi_R(s)], \tag{37}$$

$$\psi_H[\xi(r\otimes s)] = \xi(r\otimes s), \tag{38}$$

$$\varphi_R^n(d_a) = q^{na}d_a,\ \textit{for every } a,n\in\mathbb{N}. \tag{39}$$

Furthermore, for every $0\leq a,b\leq N-1,c\geq 0$, we have:

$$\varphi_H^c[\xi(d_a\otimes d_b)]=\sum\chi^c(\xi(d_a\otimes d_b)_{(1)})\xi(d_a\otimes d_b)_{(2)}=q^{c(a+b)}\xi(d_a\otimes d_b) \tag{40}$$

$$\psi_H^c[\xi(d_a\otimes d_b)]=\sum\xi(d_a\otimes d_b)_{(1)}\chi^c(\xi(d_a\otimes d_b)_{(2)})=\xi(d_a\otimes d_b). \tag{41}$$

Proof. By (35) we have

$$\chi^c[\xi(d_a\otimes d_b)]=0=\varepsilon(d_a)\varepsilon(d_b)\ \text{ unless } a=0 \text{ and } b=0,$$

for every $0\leq a,b\leq N-1$ and for every $c\in\mathbb{N}$. Since

$$\chi^c[\xi(d_0\otimes d_0)]=\chi^c(1_H)=1_K=\varepsilon(d_0)\varepsilon(d_0)$$

and $(d_i)_{0\leq i\leq N-1}$ is a basis for R as a vector space, we infer that $\chi^c[\xi(r\otimes s)]=\varepsilon(r)\varepsilon(s)$, for every $r,s\in R$. Therefore we can apply Proposition 3.6 to obtain (37), (38) and the first statement involving φ_R. Moreover we get:

$$\begin{aligned} &\varphi_R^n(d_a)=\sum\chi^n\left[(d_a)_{\langle -1\rangle}\right](d_a)_{\langle 0\rangle}\overset{(28)}{=}q^{na}d_n,\\ &\varphi_H^c[\xi(d_a\otimes d_b)]\overset{(37)}{=}\xi[\varphi_R^c(d_a)\otimes\varphi_R^c(d_b)]\overset{(39)}{=}q^{c(a+b)}\xi(d_a\otimes d_b),\\ &\psi_H^c[\xi(d_a\otimes d_b)]\overset{(38)}{=}\xi(d_a\otimes d_b). \end{aligned}$$

□

Lemma 3.16. *Take the hypothesis and notations of 3.13. Let ξ be a cocycle for the pre-bialgebra $(R,m,u,\delta,\varepsilon)$. The following relations hold true*

$$q^{a+b}\xi(d_a\otimes d_b)g = g\xi(d_a\otimes d_b). \tag{42}$$

$$q^{a+b}g\xi(d_a\otimes d_b) = \xi(d_a\otimes d_b)g. \tag{43}$$

for any $a,b\in\mathbb{N}$ such that $0\leq a,b\leq N-1$. We have that

$$\xi(d_a\otimes d_b)=0\ \textit{unless } a+b=0,\frac{N}{2},N,\frac{3N}{2} \tag{44}$$

whenever they make sense.

Proof. By applying (24) to the case $h=g$, we obtain $q^{a+b}\xi\left(d_a\otimes d_b\right)=g\xi\left(d_a\otimes d_b\right)g^{-1}$ for any $a,b\in\mathbb{N}$ such that $0\leq a,b\leq N-1$, and hence we get (42). Moreover, by applying (8) to the case $h=\xi\left(d_a\otimes d_b\right)$ we obtain

$$g\sum\chi(\xi\left(d_a\otimes d_b\right)_{(1)})\xi\left(d_a\otimes d_b\right)_{(2)}=\sum\xi\left(d_a\otimes d_b\right)_{(1)}\chi(\xi\left(d_a\otimes d_b\right)_{(2)})g.$$

By (40) and (41), we infer $gq^{a+b}\xi\left(d_a\otimes d_b\right)=\xi\left(d_a\otimes d_b\right)g$ and hence we get (43). We have

$$g\xi\left(d_a\otimes d_b\right)\overset{(42)}{=}q^{a+b}\left[\xi\left(d_a\otimes d_b\right)g\right]\overset{(43)}{=}q^{a+b}\left[q^{a+b}g\xi\left(d_a\otimes d_b\right)\right]=gq^{2(a+b)}\xi\left(d_a\otimes d_b\right)$$

so that $\left[q^{2(a+b)}-1_K\right]\xi\left(d_a\otimes d_b\right)=0$. Therefore we obtain $\xi\left(d_a\otimes d_b\right)=0$ unless $2\left(a+b\right)=tN$, for some $t\in\mathbb{N}$. Since $0\leq a,b\leq N-1$, we have $a+b\leq(N-1)+(N-1)=2N-2$ and hence

$$tN=2\left(a+b\right)\leq 4N-4<4N\Longrightarrow t<4.$$

Thus we have only the cases $t=0,1,2,3$ that is $a+b=0,\frac{N}{2},N,\frac{3N}{2}$, whenever they make sense. $\square$

Lemma 3.17. *Take the hypothesis and notations of 3.13. Let ξ be a cocycle for the pre-bialgebra $(R,m,u,\delta,\varepsilon)$ and let $B=R\#_\xi H$ as in Definitions 3.1. We have that*

$$\begin{aligned}\widetilde{m}\left(1_R\otimes s\right)&=s\#1_H\quad\text{and}\quad\widetilde{m}\left(r\otimes 1_R\right)=r\#1_H,\\ \left(r\#h\right)\cdot_B\left(1_R\#k\right)&=r\#hk\quad\text{and}\quad\left(1_R\#h\right)\cdot_B\left(s\#k\right)=\sum h_{(1)}s\#h_{(2)}k,\\ \left(r\#1_H\right)\cdot_B\left(s\#1_H\right)&=\widetilde{m}\left(r\otimes s\right)\end{aligned}$$

for any $r,s\in R$ and for any $h,k\in H$, where $\widetilde{m}$ is the map defined in (20).
In particular, for any $0\leq a\leq N-1$ and any $h\in H$ we have

$$\left(y^a\#1_H\right)\left(1_R\#h\right)=y^a\#h,\qquad\left(1_R\#h\right)\left(y^a\#1_H\right)=y^a\#\varphi_H^a\left(h\right)\tag{45}$$

Proof. Using (3), (7), (19), we get $\widetilde{m}\left(1_R\otimes s\right)=s\#1_H$ and $\widetilde{m}\left(r\otimes 1_R\right)=r\#1_H$. Using these equalities one proves that $\left(r\#h\right)\cdot_B\left(1_R\#k\right)=r\#hk$ and $\left(1_R\#h\right)\cdot_B\left(s\#k\right)=\sum h_{(1)}s\#h_{(2)}k$. In particular, for any $0\leq a\leq N-1$ and any $h\in H$ we have $\left(y^a\#1_H\right)\left(1_R\#h\right)=y^a\#h$ and

$$\left(1_R\#h\right)\left(y^a\#1_H\right)=\sum h_{(1)}y^a\#h_{(2)}=\sum\chi^a\left(h_{(1)}\right)y^a\#h_{(2)}=y^a\#\varphi_H^a\left(h\right).$$

The equality $\left(r\#1_H\right)\cdot_B\left(s\#1_H\right)=\widetilde{m}\left(r\otimes s\right)$ is trivial. $\square$

Proposition 3.18. *Take the hypothesis and notations of 3.13. Let ξ be a cocycle for the pre-bialgebra $(R,m,u,\delta,\varepsilon)$. Let*

$$B=R\#_\xi H,Y=y\#1_H,\ \Gamma=\sigma\left(g\right).$$

Then, we have

$$\begin{aligned}\sigma\left(h\right)Y^a&=Y^a\sigma\left[\varphi_H^a\left(h\right)\right]\quad\text{for any }a\in\mathbb{N},\\ \sigma\left(h\right)\sigma\left(k\right)&=\sigma\left(hk\right),\\ \Gamma Y&=qY\Gamma.\end{aligned}$$

and

$$\begin{aligned}\Delta_B\left(Y^n\right)&=\sum\nolimits_{i=0}^{n}\binom{n}{i}_q Y^{n-i}\Gamma^i\otimes Y^i\quad\text{for any }n\in\mathbb{N}.\\ \Delta_B\left(\sigma\left(h\right)\right)&=\sum\sigma\left(h_{(1)}\right)\otimes\sigma\left(h_{(2)}\right),\end{aligned}$$

for any $h,k\in H$.

Proof. We have

$$\begin{aligned}\Delta_B\left(Y\right)&=\Delta_{R\#H}\left(y\otimes 1_H\right)=\sum y^{(1)}\otimes y^{(2)}{}_{\langle-1\rangle}\otimes y^{(2)}{}_{\langle0\rangle}\otimes 1_H\\ &=\sum y\otimes\left(1_R\right)_{\langle-1\rangle}\otimes\left(1_R\right)_{\langle0\rangle}\otimes 1_H+\sum 1_R\otimes y_{\langle-1\rangle}\otimes y_{\langle0\rangle}\otimes 1_H\\ &=y\otimes 1_H\otimes 1_R\otimes 1_H+1_R\otimes g\otimes y\otimes 1_H=Y\otimes 1_B+\Gamma\otimes Y.\end{aligned}$$

Let us prove that $\sigma(h)Y^a = Y^a\sigma[\varphi_H^a(h)]$, for every $a \in \mathbb{N}$, where Y^a denotes the a-th iterated power of Y in B. If $a = 0$, then $\varphi_H^a = H$ and there is nothing to prove. If $a = 1$, we have $Y\sigma(h) = (y \otimes 1_H)(1_R \otimes h) = y \otimes h$ so that we get

$$\sigma(h)Y = (1_R \otimes h)(y \otimes 1_H) \overset{(45)}{=} y \otimes \varphi_H(h) = Y\sigma[\varphi_H(h)].$$

Let $2 \leq a$ and assume $\sigma(h)Y^{a-1} = Y^{a-1}\sigma\left[\varphi_H^{a-1}(h)\right]$. Then we obtain

$$\sigma(h)Y^a = \sigma(h)Y^{a-1}Y = Y^{a-1}\sigma\left[\varphi_H^{a-1}(h)\right]Y = Y^{a-1}Y\sigma\left[\varphi_H\left(\varphi_H^{a-1}(h)\right)\right] = Y^a\sigma\left[\varphi_H^a(h)\right].$$

From this we deduce that $\Gamma Y = \sigma(g)Y = Y\sigma[\varphi_H(g)] = Y\sigma(qg) = qY\Gamma$ and hence that

$$\Delta_B(Y^n) = [Y \otimes 1_B + \Gamma \otimes Y]^n = \sum\nolimits_{i=0}^{n} \binom{n}{i}_q (Y \otimes 1_B)^{n-i}(\Gamma \otimes Y)^i = \sum\nolimits_{i=0}^{n} \binom{n}{i}_q Y^{n-i}\Gamma^i \otimes Y^i.$$

The remaining statements follows as, by 3.1, σ is a morphism of bialgebras. □

Proposition 3.19. *Take the hypothesis and notations of 3.13. Let ξ be a cocycle for the pre-bialgebra $(R, m, u, \delta, \varepsilon)$.*
If N is odd we have

$$\widetilde{m}(d_1 \otimes d_b) = \begin{cases} d_1 d_b \otimes 1_H & \text{for any } 0 \leq b \leq N-2. \\ 1_R \otimes \xi(d_1 \otimes d_{N-1}) & \text{for } b = N-1 \end{cases}$$

If N is even, we have

$$\widetilde{m}(d_1 \otimes d_b) = \begin{cases} d_1 d_b \otimes 1_H & \text{for any } 0 \leq b \leq N/2-2. \\ q^{1+b-N/2} d_{1+b-N/2} \otimes x + d_1 d_b \otimes 1_H & \text{for any } N/2-1 \leq b \leq N-2. \\ 1_R \otimes \xi(d_1 \otimes d_{N-1}) - d_{N/2} \otimes x & \text{for } b = N-1 \end{cases}$$

where $x = \xi\left(d_1 \otimes d_{N/2-1}\right)$.

Proof. We compute $\widetilde{m}(d_1 \otimes d_b)$ for any $0 \leq b \leq N-1$.
We have

$$\begin{aligned} \widetilde{m}(d_1 \otimes d_b) &= (m \otimes \xi)\delta_{R \otimes R}(d_1 \otimes d_b) \\ &\overset{(29)}{=} (m \otimes \xi)\left[\sum\nolimits_{0 \leq j \leq b} q^j 1_R \otimes d_j \otimes d_1 \otimes d_{b-j} + \sum\nolimits_{0 \leq j \leq b} d_1 \otimes d_j \otimes 1_R \otimes d_{b-j}\right] \\ &= \sum\nolimits_{0 \leq j \leq b} q^j d_j \otimes \xi(d_1 \otimes d_{b-j}) + \sum\nolimits_{0 \leq j \leq b} d_1 d_j \otimes \xi(1_R \otimes d_{b-j}) \\ &\overset{(19)}{=} \sum\nolimits_{0 \leq j \leq b} q^j d_j \otimes \xi(d_1 \otimes d_{b-j}) + d_1 d_b \otimes 1_H \end{aligned}$$

so that

$$\widetilde{m}(d_1 \otimes d_b) = \sum\nolimits_{0 \leq j \leq b} q^j d_j \otimes \xi(d_1 \otimes d_{b-j}) + d_1 d_b \otimes 1_H \text{ for any } 0 \leq b \leq N-1. \tag{46}$$

Now, if $0 \leq j \leq b \leq N-1$, then $1 \leq 1 + (b-j) \leq 1 + b \leq N$ so that, by (44), $\xi(d_1 \otimes d_{b-j}) = 0$ unless $1 + (b-j) = \frac{N}{2}, N$. If N is odd, then $\xi(d_1 \otimes d_{b-j}) = 0$ unless $1 + (b-j) = N$. Thus $\widetilde{m}(d_1 \otimes d_b) = d_1 d_b \otimes 1_H$ for any $0 \leq b \leq N-2$ and for $b = N-1$ we have $\xi(d_1 \otimes d_{b-j}) = 0$ unless $j = 0$ so that $\widetilde{m}(d_1 \otimes d_{N-1}) \overset{(46)}{=} 1_R \otimes \xi(d_1 \otimes d_{N-1})$. In fact $d_1 d_{N-1} = 0$ by Theorem 3.14.

Assume now that N is even. Thus we have the following cases.

$0 \leq b \leq N/2-2)$ In this case $1+(b-j) \leq 1+(N/2-2-j) \leq N/2-1$ so that $\xi(d_1 \otimes d_{b-j}) = 0$ always and hence $\widetilde{m}(d_1 \otimes d_b) = d_1 d_b \otimes 1_H$ for any $0 \leq b \leq N/2-2$.

$N/2-1 \leq b \leq N-2)$ In this case $1+(b-j) \leq 1+(N-2-j) \leq N-1$ so that $\xi(d_1 \otimes d_{b-j}) = 0$ unless $1 + (b-j) = N/2$ and hence

$$\widetilde{m}(d_1 \otimes d_b) \overset{(46)}{=} q^{1+b-N/2} d_{1+b-N/2} \otimes x + d_1 d_b \otimes 1_H.$$

$b = N-1)$ In this case $1 + (b-j) = 1 + (N-1-j) = N-j$ so that $\xi\left(y \otimes y^{b-j}\right) = 0$ unless $N - j = N/2, N$ which means $j = 0, N/2$ and hence

$$\widetilde{m}(d_1 \otimes d_{N-1}) \overset{(46)}{=} 1_R \otimes \xi(d_1 \otimes d_{N-1}) - d_{N/2} \otimes x.$$

□

NOTATION 3.20. *Take the hypothesis and notations of 3.13. Let ξ be a cocycle for the pre-bialgebra $(R,m,u,\delta,\varepsilon)$. From now on, we will use the following notation*

$$Y := y\otimes 1_H, \Gamma = \sigma(g).$$

Let $\mathcal{B}(H)$ be a basis for H. Next aim is to prove, under suitable hypothesis, that

$$\left\{Y^i\sigma(h) \mid 0\leq i\leq N-1, h\in\mathcal{B}(H)\right\},$$

defines a basis for $B=R\#_\xi H$ and that there exists $\lambda(N)\in K$ such that:

$$Y^N = \lambda(N)\left(1_A-\Gamma^N\right)$$

where $\lambda(N)=0$ whenever $g^N=1_H$. This will lead a complete description of the Hopf algebra structure of B.

PROPOSITION 3.21. *Take the hypothesis and notations of 3.20. If N is odd we have*

$$Y^a=\begin{cases} y^a\otimes 1_H & \text{for } 0\leq a\leq N-1\\ \sigma\left[\xi\left(y\otimes y^{N-1}\right)\right] & \text{for } a=N.\end{cases}$$

If N is even, we have

$$Y^a=\begin{cases} y^a\otimes 1_H & \text{for } 0\leq a\leq N/2-1\\ \binom{a}{N/2}_q Y^{a-N/2}\cdot_A X+y^a\otimes 1_H & \text{for } N/2\leq a\leq N-1\\ \sigma\left[\xi\left(y\otimes y^{N-1}\right)\right]+\binom{N-1}{N/2}_q X^2 & \text{for } a=N\end{cases}$$

where

$$X=1_R\otimes (N/2-1)_q!\xi\left(d_1\otimes d_{N/2-1}\right)=1_R\otimes(N/2-1)_q!x=(N/2-1)_q!\sigma(x).$$

Proof. Recall that, by Theorem 3.14, we have $y^n=(n)_q!d_n$ for every $0\leq n\leq N-1$ so that $\widetilde{m}(y\otimes y^n)=(n)_q!\widetilde{m}(d_1\otimes d_n)$. Assume now that N is odd and let us prove, by induction on $0\leq a\leq N-1$, that $Y^a=y^a\otimes 1_H$

For $a=0$ there is nothing to prove. Let $1\leq a\leq N-1$ and assume $Y^{a-1}=y^{a-1}\otimes 1_H$. Since $a-1\leq N-2$, by Proposition 3.19 we have

$$\widetilde{m}\left(y\otimes y^{a-1}\right)=(a-1)_q!\widetilde{m}(d_1\otimes d_{a-1})=(a-1)_q!d_1d_{a-1}\otimes 1_H=y^a\otimes 1_H$$

so that

$$Y^a=Y\cdot_B Y^{a-1}=m_{R\#H}[(y\#1_H)\otimes_H(y^{a-1}\#1_H)]=\widetilde{m}\left(y\otimes y^{a-1}\right)=y^a\otimes 1_H.$$

Moreover we have

$$\begin{aligned}\widetilde{m}\left(y\otimes y^{N-1}\right) &= (N-1)_q!\widetilde{m}(d_1\otimes d_{N-1})\\ &= 1_R\otimes\xi\left[d_1\otimes(N-1)_q!d_{N-1}\right]=1_R\otimes\xi\left(y\otimes y^{N-1}\right)=\sigma\left[\xi\left(y\otimes y^{N-1}\right)\right]\end{aligned}$$

so that

$$Y^N=Y\cdot_B Y^{N-1}=m_{R\#H}[(y\#1_H)\otimes_H(y^{N-1}\#1_H)]=\widetilde{m}\left(y\otimes y^{N-1}\right)=\sigma\left[\xi\left(y\otimes y^{N-1}\right)\right].$$

Assume N even. Since $(y\otimes 1_H)\cdot_B(y^n\otimes 1_H)=\widetilde{m}(y\otimes y^n)=(n)_q!\widetilde{m}(d_1\otimes d_n)$ for every $0\leq n\leq N-1$, in view of Proposition 3.19 and as $d_n=\frac{y^n}{(n)_q!}$, we obtain

$$\begin{aligned}&(y\otimes 1_H)\cdot_B(y^n\otimes 1_H)\\ =&\begin{cases} y^{n+1}\otimes 1_H & \text{if } 0\leq n\leq N/2-2\\ X+y^{N/2}\otimes 1_H & \text{if } n=N/2-1\\ \binom{n}{n+1-N/2}_q q^{n+1-N/2}Y^{n+1-N/2}\cdot_B X+y^{n+1}\otimes 1_H & \text{if } N/2\leq n\leq N-2\\ \sigma\left[\xi\left(y\otimes y^{N-1}\right)\right]-\binom{N-1}{N/2}_q y^{N/2}\cdot_B X+\binom{N-1}{N/2}_q X^2 & \text{if } n=N-1.\end{cases}\end{aligned}$$

Let us prove by induction on $0\leq a\leq N/2-1$ that $Y^a=y^a\otimes 1_H$.

For $a=0$ there is nothing to prove. Let $1\leq a\leq N/2-1$ and assume $Y^{a-1}=y^{a-1}\otimes 1_H$. We deduce, as above, that $Y^a=Y\cdot_B Y^{a-1}=y^a\otimes 1_H$. Since, for any $0\leq a\leq N/2-1$, we have $Y^a\sigma(h)=(y^a\otimes 1_H)(1_H\otimes h)=y^a\otimes h$, if we choose $h=x=\xi\left(d_1\otimes d_{N/2-1}\right)$, we get $Y^a\sigma(x)=y^a\otimes x$ for any $0\leq a\leq N/2-1$. Let us compute Y^a for $N/2\leq a\leq N-1$.

Let us prove that, for any $0\leq t\leq N/2-1$, we have $Y^{N/2+t}=\binom{N/2+t}{N/2}_q Y^t\cdot_A X+y^{t+N/2}\otimes 1_H$. We prove it by induction on t.
If $t=0$ we have $Y^{N/2}=Y\cdot_B Y^{N/2-1}=(y\otimes 1_H)\cdot_B(y^{N/2-1}\otimes 1_H)=X+y^{N/2}\otimes 1_H$.Let $1\leq t\leq N/2-1$ and assume that the formula holds for $t-1$. We have

$$\begin{aligned}
Y^{N/2+t} &= Y\cdot_B Y^{N/2+t-1}=Y\cdot_B\left\{\binom{N/2+t-1}{N/2}_q Y^{t-1}\cdot_B X+y^{t-1+N/2}\otimes 1_H\right\}\\
&= \binom{N/2+t-1}{N/2}_q Y\cdot_B Y^{t-1}\cdot_B X+Y\cdot_B\left(y^{t-1+N/2}\otimes 1_H\right)\\
&= \left[\binom{N/2+t-1}{N/2}_q+\binom{t-1+N/2}{t}_q q^t\right]Y^t\cdot_B X+y^{t+N/2}\otimes 1_H\\
&= \binom{N/2+t}{N/2}_q Y^t\cdot_B X+y^{t+N/2}\otimes 1_H.
\end{aligned}$$

In fact, by [Ka, Poposition IV.2.1, page 74], we have $\binom{N/2+t}{N/2}_q=q^t\binom{N/2+t-1}{N/2-1}_q+\binom{N/2+t-1}{N/2}_q$.

In particular, for $t=N/2-1$ we get

$$Y^{N-1}=\binom{N-1}{N/2}_q Y^{N-1-N/2}\cdot_B X+y^{N-1}\otimes 1_H=\binom{N-1}{N/2}_q Y^{N/2-1}\cdot_B X+y^{N-1}\otimes 1_H.$$

Moreover, we have

$$Y^{N/2}\cdot_B X=\left(X+y^{N/2}\otimes 1_H\right)\cdot_B X=X^2+\left(y^{N/2}\otimes 1_H\right)\cdot_B X=X^2+y^{N/2}\otimes(N/2-1)_q!x.$$

We have

$$\begin{aligned}
Y^N &= Y\cdot_B Y^{N-1}=Y\cdot_B\left[\binom{N-1}{N/2}_q Y^{N/2-1}\cdot_B X+y^{N-1}\otimes 1_H\right]\\
&= \binom{N-1}{N/2}_q Y^{N/2}\cdot_B X+Y\cdot\left(y^{N-1}\otimes 1_H\right)\\
&= \binom{N-1}{N/2}_q Y^{N/2}\cdot_B X+\sigma\left[\xi\left(y\otimes y^{N-1}\right)\right]-\binom{N-1}{N/2}_q y^{N/2}\cdot_B X+\binom{N-1}{N/2}_q X^2\\
&= \sigma\left[\xi\left(y\otimes y^{N-1}\right)\right]+\binom{N-1}{N/2}_q X^2.
\end{aligned}$$

□

Corollary 3.22. *Take the hypothesis and notations of 3.20. If H is f.d. or cosemisimple, then there exists $\lambda(N)\in K$ such that*

$$Y^N=\lambda(N)\left(1_B-\Gamma^N\right).$$

Furthermore $\lambda(N)=0$ whenever $g^N=1_H$.

Proof. By Proposition 3.21 we have that $Y^N\in K\otimes H\cong H$. Since $N=o(q)$, $\binom{N}{i}_q=0$, for every $1\leq i\leq N-1$, so that, by Proposition 3.18, we have that

$$\Delta_B\left(Y^N\right)=\sum\nolimits_{i=0}^{N}\binom{N}{i}_q Y^{N-i}\Gamma^i\otimes Y^i=Y^N\otimes 1_B+\Gamma^N\otimes Y^N$$

and that $\Gamma Y^N=q^N Y^N\Gamma=Y^N\Gamma$, we can apply Theorem 0.1. □

Lemma 3.23. *Take the hypothesis and notations of 3.20. Let $n \in \mathbb{N}$, $0 \leq n \leq N-1$. Assume that*

$$\rho\left(d_1 d_t\right) = \sum \left(d_1\right)_{\langle -1\rangle} \left(d_t\right)_{\langle -1\rangle} \otimes \left(d_1\right)_{\langle 0\rangle} \left(d_t\right)_{\langle 0\rangle},$$

for any $0 \leq t \leq n-1$. Then $\rho\left(d_a\right) = g^a \otimes d_a$, for any $0 \leq a \leq n$.

Proof. If $a = 0$ there is nothing to prove.
If $1 \leq a \leq n$ and $\rho\left(d_{a-1}\right) = g^{a-1} \otimes d_{a-1}$, since $d_a = \frac{1}{(a)_q} d_1 d_{a-1}$, we have

$$\begin{aligned} \rho\left(d_a\right) &= \frac{1}{(a)_q} \rho\left(d_1 d_{a-1}\right) = \frac{1}{(a)_q} \sum \left(d_1\right)_{\langle -1\rangle} \left(d_{a-1}\right)_{\langle -1\rangle} \otimes \left(d_1\right)_{\langle 0\rangle} \left(d_{a-1}\right)_{\langle 0\rangle} = \\ &= \frac{1}{(a)_q} g g^{a-1} \otimes d_1 d_{a-1} = g^a \otimes d_a. \end{aligned}$$

□

Lemma 3.24. *Take the hypothesis and notations of 3.20. If N is odd then $\rho\left(d_a\right) = g^a \otimes d_a$, for any $0 \leq a \leq N-1$.*
If N is even then $\rho\left(d_a\right) = g^a \otimes d_a$, for any $0 \leq a \leq N/2$.

Proof. By Lemma 3.23. it is enough to prove that

$$\rho\left(d_1 d_t\right) = \sum \left(d_1\right)_{\langle -1\rangle} \left(d_t\right)_{\langle -1\rangle} \otimes \left(d_1\right)_{\langle 0\rangle} \left(d_t\right)_{\langle 0\rangle} \tag{47}$$

for any $0 \leq t \leq n-1$ where $n = N-1$ if N is odd and $n = N/2$ otherwise.

Assume N odd. Let $0 \leq t \leq N-2$. Then, for any $0 \leq i \leq 1, 0 \leq j \leq t$ such that $0 < i+j < 1+t$, we have $(1-i)+(t-j) = (1+t)-(i+j)$ and $1 \leq (1+t)-(i+j) \leq 1+t-1 \leq N-2$ so that, by (44), we get $\xi\left(d_{1-i} \otimes d_{t-j}\right) = 0$.

Hence, by (32), for $a = 1, b = t$, we obtain (47).

Assume N even. Let $0 \leq t \leq N/2-1$. Then, for any $0 \leq i \leq 1, 0 \leq j \leq t$ such that $0 < i+j < 1+t$, we have $(1-i)+(t-j) = (1+t)-(i+j)$ and $1 \leq (1+t)-(i+j) \leq 1+t-1 \leq N/2-1$ so that $\xi\left(d_{1-i} \otimes d_{t-j}\right) = 0$.

Hence, by (32), for $a = 1, b = t$, as above we obtain (47). □

Lemma 3.25. *Take the hypothesis and notations of 3.20. If N is odd and $1 \leq b \leq N-1$ or if N is even and $1 \leq b \leq N/2$, we have*

$$\Delta_H \xi(d_1 \otimes d_b) = g^{1+b} \otimes \xi\left(d_1 \otimes d_b\right) + \xi\left(d_1 \otimes d_b\right) \otimes 1_H.$$

Proof. Let $1 \leq b$. Using (3) and (19) we get

$$\begin{aligned} & \Delta_H \xi\left(d_1 \otimes d_b\right) \\ \overset{(30)}{=} & \sum\nolimits_{0 \leq i \leq 1, 0 \leq j \leq b} q^{j(1-i)} \xi\left(d_i \otimes d_j\right) \left(d_{1-i}\right)_{\langle -1\rangle} \left(d_{b-j}\right)_{\langle -1\rangle} \otimes \xi\left[\left(d_{1-i}\right)_{\langle 0\rangle} \otimes \left(d_{b-j}\right)_{\langle 0\rangle}\right] \\ = & \left(d_1\right)_{\langle -1\rangle} \left(d_b\right)_{\langle -1\rangle} \otimes \xi\left[\left(d_1\right)_{\langle 0\rangle} \otimes \left(d_b\right)_{\langle 0\rangle}\right] + \xi\left(d_1 \otimes d_b\right) \otimes 1_H \\ = & g \left(d_b\right)_{\langle -1\rangle} \otimes \xi\left[d_1 \otimes \left(d_b\right)_{\langle 0\rangle}\right] + \xi\left(d_1 \otimes d_b\right) \otimes 1_H. \end{aligned}$$

If N is odd and $1 \leq b \leq N-1$ or if N is even and $1 \leq b \leq N/2$, by Lemma 3.24, we have $\rho\left(d_b\right) = g^b \otimes d_b$ and hence $\Delta_H \xi(d_1 \otimes d_b) = g^{1+b} \otimes \xi\left(d_1 \otimes d_b\right) + \xi\left(d_1 \otimes d_b\right) \otimes 1_H$. □

Lemma 3.26. *Take the hypothesis and notations of 3.20. Assume that N is even and let $x = \xi\left(d_1 \otimes d_{N/2-1}\right)$. Then $x = 0$ whenever H is cosemisimple.*

Proof. By Lemma 3.25, for any $1 \leq b \leq N/2$, we have

$$\Delta_H \xi(d_1 \otimes d_b) = g^{1+b} \otimes \xi\left(d_1 \otimes d_b\right) + \xi\left(d_1 \otimes d_b\right) \otimes 1_H.$$

In particular, if $N \geq 4$, then $1 \leq N/2-1 \leq N/2$ so that we can apply this formula for $b = N/2-1$ and obtain $\Delta_H(x) = g^{N/2} \otimes x + x \otimes 1_H$. This equality still holds whenever $N = 2$ as in this case $x = \xi\left(d_1 \otimes d_{N/2-1}\right) = \xi\left(d_1 \otimes d_0\right) = 0$. By applying (24), to the case $(a,b) = (1, N/2-1)$, we get

$$\chi^{N/2}(h)\, x = \sum h_{(1)} x S h_{(2)}, \text{ for any } h \in H.$$

If $h = g$, we have $q^{N/2}x = gxg^{-1}$ that is $xg + gx = 0$. Assume now that H is cosemisimple and let $\lambda \in H^*$ be a total integral. Then by applying $H \otimes \lambda$ to $\Delta_H(x) = g^{N/2} \otimes x + x \otimes 1_H$ we get $x = \lambda(x)\left(1_H - g^{N/2}\right)$ so that $xg = gx$. From $xg + gx = 0$ we obtain $xg = 0$ and hence $x = 0$. □

Definition 3.27. Let q be a primitive N-th root of unity. A *compatible datum* for q is a quadruple $(H, g, \chi, \lambda(N))$, where

- (H, g, χ) is a Yetter-Drinfeld datum for q,
- $\lambda(N) \in K$ and $\lambda(N) = 0$ if

$$g^N = 1_H, \qquad \text{or} \qquad \chi^N(h)\left(1_H - g^N\right) \neq \sum h_{(1)}\left(1_H - g^N\right) S h_{(2)}, \text{ for some } h \in H,$$

while $\lambda(N)$ is an arbitrary otherwise.

A compatible datum is called *trivial* whenever $\lambda(N) = 0$ and it is called *non-trivial* otherwise.

Remark 3.28. A compatible datum $(H, g, \chi, \lambda(N))$ is trivial if and only if $\lambda(N)(1_H - g^N) = 0$.

Theorem 3.29. *Let H be a Hopf algebra and let $(R, m, u, \delta, \varepsilon)$ be a N-dimensional pre-bialgebra in ${}^H_H\mathcal{YD}$. Assume that R is a thin coalgebra where $P(R) = Ky$. Let $g \in H$ and $\chi \in H^*$ be such that (H, g, χ) is the Yetter-Drinfeld datum associated to y and let $q = \chi(g)$. Let ξ be a cocycle for the pre-bialgebra $(R, m, u, \delta, \varepsilon)$.*

Assume that H is either f.d. or cosemisimple. Then

1) q is a primitive N-th root of unity.

2) R is an associative algebra over K spanned by y and the N-th power of y in R is zero.

3) The map $\varphi_H : H \to H, \varphi_H(h) = \sum \chi\left(h_{(1)}\right) h_{(2)}$ is an algebra automorphism of H.

4) The map $\sigma : H \to R\#H, \sigma(h) = 1_R \otimes h$ is a morphism of bialgebras.

5) There exists $\lambda(N) \in K$ such that a $(H, g, \chi, \lambda(N))$ is a compatible datum for q.

Let $Y := y \otimes 1_H, \Gamma = \sigma(g)$ and let $\mathcal{B}(H)$ be a basis for H.

Then $B = R\#_\xi H$ is the Hopf algebra with basis

$$\left\{Y^i \sigma(h) \mid 0 \leq i \leq N-1, h \in \mathcal{B}(H)\right\},$$

where Y^i denotes the i-th iterated power of Y in B, for every $i \in \mathbb{N}$, with algebra structure given by

$$\begin{aligned} Y^N &= \lambda(N)\left(1_B - \Gamma^N\right) \\ \sigma(h)\, Y^a &= Y^a \sigma\left[\varphi_H^a(h)\right] \text{ for any } a \in \mathbb{N}, \\ \sigma(h)\, \sigma(k) &= \sigma(hk), \end{aligned}$$

and coalgebra structure given by

$$\begin{aligned} \Delta_B(Y) &= Y \otimes 1_B + \Gamma \otimes Y \\ \Delta_B(\sigma(h)) &= \sum \sigma\left(h_{(1)}\right) \otimes \sigma\left(h_{(2)}\right), \end{aligned}$$

for any $h, k \in H$. Furthermore $Y^n = y^n \otimes 1_H$ for every $0 \leq n \leq N/2-1$ whenever N is even and $Y^n = y^n \otimes 1_H$ for every $0 \leq n \leq N-1$ whenever N is odd or $x = 0$.

Proof. 1) and 2) follow by Theorem 3.14. In view of 3.5, we get 3). Statement 4) follows by 3.1. By Corollary 3.22, we have $Y^N = \lambda(N)\left(1_B - \Gamma^N\right)$ so that, by Proposition 3.18, we get all the displayed equalities.

5) Set $z = \lambda(N)(1_H - g^N)$ so that $Y^N = \sigma(z)$. We have

$$\sigma(hz) = \sigma(h)\, \sigma(z) = \sigma(h)\, Y^N = Y^N \sigma\left[\varphi_H^N(h)\right] = \sigma(z)\, \sigma\left[\varphi_H^N(h)\right] = \sigma\left[z \varphi_H^N(h)\right]$$

so that $hz = z\varphi_H^N(h)$. From this equality we get

$$\sum h_{(1)} z S h_{(2)} = z\varphi_H^N\left(h_{(1)}\right) S h_{(2)} = z \sum \chi^N\left(h_{(1)}\right) h_{(2)} S h_{(3)} = \chi^N(h)\, z$$

and hence

$$\chi^N(h)\,\lambda(N)\left(1_H - g^N\right) = \sum h_{(1)}\lambda(N)\left(1_H - g^N\right) S h_{(2)}. \tag{48}$$

Now, by Corollary 3.22, if $g^N = 1_H$, then $\lambda(N) = 0$.
If there exists an element $h \in H$ such that

$$\chi^N(h)\left(1_H - g^N\right) \neq \sum h_{(1)}\left(1_H - g^N\right) S h_{(2)},$$

still, by (48), we get $\lambda(N) = 0$. Thus we have proved that $(H, g, \chi, \lambda(N))$ is a compatible datum for $q = \chi(g)$.

It remains to prove the statement concerning the basis of B.

If H is cosemisimple, by Lemma 3.26 one has $x = 0$ whenever N is even. By Proposition 3.21 we deduce that

$$Y^a = y^a \otimes 1_H \text{ for } 0 \leq a \leq N-1$$

regardless the parity of N. Since $\mathcal{B}(B) = \left\{y^i \otimes h \mid 0 \leq i \leq N-1, h \in \mathcal{B}(H)\right\}$ is a basis for B, we conclude by observing that, in view of Lemma 3.17, one has $y^i \otimes h = Y^i \sigma(h)$ for any $0 \leq i \leq N-1, h \in H$.
Assume now that H is finite dimensional.

If N is odd we have $Y^a = y^a \otimes 1_H$ for $0 \leq a \leq N-1$ and we conclude as in the cosemisimple case.

If N is even, by Proposition 3.21, we have

$$Y^a = \begin{cases} y^a \otimes 1_H & \text{for } 0 \leq a \leq N/2-1 \\ \binom{a}{N/2}_q Y^{a-N/2} \cdot_A X + y^a \otimes 1_H & \text{for } N/2 \leq a \leq N-1. \end{cases}$$

Then, from $\left(y^i \otimes 1_H\right)\sigma(h) = y^i \otimes h$ and by definition of X, for any $h \in H$, we get

$$Y^a \sigma(h) = \begin{cases} y^a \otimes h & \text{for } 0 \leq a \leq N/2-1 \\ \binom{a}{N/2}_q Y^{a-N/2}\sigma\left[(N/2-1)_q! xh\right] + y^a \otimes h & \text{for } N/2 \leq a \leq N-1. \end{cases}$$

Therefore we obtain

$$y^a \otimes h = \begin{cases} Y^a \sigma(h) & \text{for } 0 \leq a \leq N/2-1 \\ Y^a \sigma(h) - \binom{a}{N/2}_q (N/2-1)_q! Y^{a-N/2}\sigma(xh) & \text{for } N/2 \leq a \leq N-1. \end{cases}$$

Since $\mathcal{B}(B) = \left\{y^i \otimes h \mid 0 \leq i \leq N-1, h \in \mathcal{B}(H)\right\}$ is a basis for B, then

$$W = \left\{Y^i \sigma(h) \mid 0 \leq i \leq N-1, h \in \mathcal{B}(H)\right\}$$

generates B as a K-vector space. Since $|W| \leq N \cdot |\mathcal{B}(H)| = |\mathcal{B}(B)|$, we deduce that W is a basis for B. Finally we point out that, since $R_0 = K1_R$, by Theorem 3.9, B is in fact a Hopf algebra. □

Theorem 3.30. *Let H be a Hopf algebra over a field K. Let A be a bialgebra and let $\sigma : H \to A$ be an injective morphism of bialgebras having a retraction $\pi : A \to H$ (i.e. $\pi\sigma = H$) that is an H-bilinear coalgebra map. Let $(R, m, u, \delta, \varepsilon)$ be the pre-bialgebra in ${}_H^H\mathcal{YD}$ associated to (A, π, σ) with corresponding cocycle ξ.*
Assume that

- *H is either f.d. or cosemisimple;*
- *R is an N-dimensional thin coalgebra where $P(R) = Ky$.*

Let $g \in H$ and $\chi \in H^*$ be such that (H, g, χ) is the Yetter-Drinfeld datum associated to y and let $q = \chi(g)$. Then

1) There exists $\lambda(N) \in K$ such that a $(H, g, \chi, \lambda(N))$ is a compatible datum for q.

2) (R, m, u) is an associative algebra over K spanned by y (a priori not in ${}_H^H\mathcal{YD}$) and the N-th power of y in R is zero.

3) A is a Hopf algebra with basis

$$\left\{y^i \sigma(h) \mid 0 \leq i \leq N-1, h \in \mathcal{B}(H)\right\},$$

where y^i denotes the i-th iterated power of y in A, for every $i \in \mathbb{N}$, algebra structure given by

$$\begin{aligned} y^N &= \lambda(N)\left(1_A - \Gamma^N\right), \\ \sigma(h)\, y^a &= y^a \sigma\left[\varphi_H^a(h)\right] \text{ for any } a \in \mathbb{N}, \text{ and } h \in H \end{aligned}$$

and coalgebra structure given by

$$\Delta_A(y) = y \otimes 1_A + \Gamma \otimes y.$$

Here $\varphi_H : H \to H$ denotes the algebra automorphism of H defined by $\varphi_H(h) = \sum \chi\left(h_{(1)}\right) h_{(2)}$ and $\Gamma = \sigma(g)$.

4) The n-th iterated power of y in R and the n-th iterated power of y in A coincides for every $0 \leq n \leq N/2 - 1$ whenever N is even.

5) The n-th iterated power of y in R and the n-th iterated power of y in A coincides for every $0 \leq n \leq N-1$, whenever N is odd or $\xi\left(y \otimes y^{N/2-1}\right) = 0$.

Proof. By Theorem 3.29,

- q is a primitive N-th root of unity
- R is an associative algebra over K spanned by y and the N-th power of y in R is zero.
- The map $\varphi_H : H \to H, \varphi_H(h) = \sum \chi\left(h_{(1)}\right) h_{(2)}$ is an algebra automorphism of H.
- The map $\gamma : H \to R\#H, \gamma(h) = 1_R \otimes h$ is a bialgebra homomorphism.
- There exists $\lambda(N) \in K$ such that a $(H, g, \chi, \lambda(N))$ is a compatible datum for q.

Let $Y := y \otimes 1_H, \Theta = \gamma(g)$ and let $\mathcal{B}(H)$ be a basis for H. Then $B = R\#_\xi H$ is the Hopf algebra with basis $\left\{Y^i \gamma(h) \mid 0 \leq i \leq N-1, h \in \mathcal{B}(H)\right\}$, with algebra structure given by

$$\begin{aligned} Y^N &= \lambda(N)\left(1_B - \Theta^N\right), \\ \gamma(h)\, Y^a &= Y^a \gamma\left[\varphi_H^a(h)\right] \text{ for any } a \in \mathbb{N}, \\ \gamma(h)\,\gamma(k) &= \gamma(hk), \end{aligned}$$

and coalgebra structure given by

$$\Delta_B(Y) = Y \otimes 1_B + \Theta \otimes Y, \qquad \Delta_B(\gamma(h)) = \sum \gamma\left(h_{(1)}\right) \otimes \gamma\left(h_{(2)}\right),$$

for any $h, k \in H$. As explained in Remark 3.3, the map $\omega : R\#_\xi H \to A$, $\omega(r\#h) = r\sigma(h)$, is a bialgebra isomorphism. Let y^n denote the n-th iterated power of y in R. We have that

(1) $\omega(Y) = \omega(y\#1_H) = y$,
(2) $\omega(\gamma(h)) = \omega(1_R\#h) = \sigma(h)$, so that
(3) $\omega(\Theta) = \omega(\gamma(g)) = \sigma(g) = \Gamma$.
(4) $Y^n = y^n \otimes 1_H$ for every $0 \leq n \leq N/2 - 1$.
(5) If N is odd or $x = 0$, then $Y^n = y^n \otimes 1_H$ for every $0 \leq n \leq N-1$

If N is even, as $(N/2-1)_q!x = \xi(y \otimes y^{N/2-1})$, we have that $x = 0$ if and only if $\xi(y \otimes y^{N/2-1}) = 0$. Let $0 \leq n \leq N-1$ be such that $Y^n = y^n \otimes 1_H$. Then

$$\omega(Y^n) = \omega(y^n\#1_H) = y^n = n\text{-th iterated power of } y \text{ in } R.$$

On the other hand since ω is an algebra homomorphism, then

$$\omega(Y^n) = \omega(Y)^n = n\text{-th iterated power of } y \text{ in } A.$$

□

Remark 3.31. Note that in the statement of Theorem 3.30, the basis of A is

$$\left\{y^{i}\sigma\left(h\right)\mid 0\leq i\leq N-1,h\in\mathcal{B}\left(H\right)\right\},$$

where y^i is the i-th iterated power of y in A. In fact, since R is not a subalgebra of A, one should not mix up the powers of y in A with the powers of y in R. In [AMSt] we will provide an example showing that these may be different.

4. Normalization of the Projection

Theorem 4.1. *Let $N\in\mathbb{N}\backslash\left\{0\right\}$. Let H be a Hopf algebra, let A be a bialgebra, let $\sigma:H\rightarrow A$ be an injective bialgebra map and let $y\in A$ be an element such that*

$$\mathcal{B}\left(A\right)=\left\{y^{i}\sigma\left(h\right)\mid 0\leq i\leq N-1,h\in\mathcal{B}\left(H\right)\right\}$$

is a basis for A, where y^i denotes the i-th iterated power of y in A, for every $i\in\mathbb{N}$. Assume that the algebra structure of A is defined by

$$\begin{aligned} y^{N} &= \lambda\left(N\right)\left(1_{A}-\Gamma^{N}\right),\lambda\left(N\right)\in K,\Gamma=\sigma\left(g\right)\\ \sigma\left(h\right)y^{a} &= y^{a}\sigma\left[\varphi^{a}\left(h\right)\right]\ \text{for any } a\in\mathbb{N},\ \text{and } h\in H,\end{aligned}$$

where $g\in G\left(H\right)$, $\varphi:H\rightarrow H$ is an isomorphism of algebras, $\varphi\left(g\right)=qg$ where q is a primitive N-th root of unity. Assume also that the coalgebra structure is given by

$$\Delta_{A}\left(y\right)=y\otimes 1_{A}+\Gamma\otimes y.$$

Let

$$p:A\rightarrow H,\ p\left[y^{n}\sigma\left(h\right)\right]=\delta_{n,0}h,\ \text{for every } 0\leq n\leq N-1,h\in\mathcal{B}\left(H\right).$$

Then p is an H-bilinear coalgebra (not necessarily algebra) retraction ($p\sigma=H$) of σ. Moreover $(H,g,\varepsilon_{H}\varphi)$ is a Yetter-Drinfeld datum for q and the pre-bialgebra in ${}_{H}^{H}\mathcal{YD}$ associated to (A,p,σ) is $(R,m,u,\delta,\varepsilon)$ with corresponding cocycle ξ where

1) $R=R_{q}\left(H,g,\varepsilon_{H}\varphi\right)$ is a braided bialgebra in ${}_{H}^{H}\mathcal{YD}$, in fact a quantum line spanned by y of dimension N and the N-th power of y in R is zero.

2) for any $0\leq n\leq N-1$, the n-th power of y in R coincides with the n-th power of y in A, namely y^n.

3) for any $0\leq a,b\leq N-1$, we have

$$\xi(y^{a}\otimes y^{b})=\left\{\begin{array}{ll} 1 & \text{for } a+b=0\\ \lambda(N)(1_{H}-g^{N}) & \text{for } a+b=N\\ 0 & \text{otherwise.}\end{array}\right.$$

4) $\varphi\left(h\right)=\sum\varepsilon_{H}\varphi\left(h_{(1)}\right)h_{(2)}$, for every $h\in H$. Furthermore A is a Hopf algebra.

Proof. Clearly we have $p\sigma=H$. Since $\sigma\left(h\right)y^{a}=y^{a}\sigma\left[\varphi^{a}\left(h\right)\right]$ and by definition of p, it is straightforward to check that p is H-bilinear. Let us prove that p is a coalgebra homomorphism. Since p is H-bilinear, it is enough to check it on the powers of y. Since $\left(\Gamma\otimes y\right)\left(y\otimes 1_{A}\right)=q\left(y\otimes 1_{A}\right)\Gamma\otimes y$, by the quantum binomial formula, for any $0\leq n\leq N-1$, we deduce $\Delta_{A}\left(y^{n}\right)=\sum_{i=0}^{n}\binom{n}{i}_{q}y^{n-i}\Gamma^{i}\otimes y^{i}$so that $\left(p\otimes p\right)\Delta_{A}\left(y^{n}\right)=\Delta_{H}p\left(y^{n}\right)$ and $\varepsilon_{H}p\left(y^{n}\right)=\varepsilon_{A}\left(y^{n}\right)$. Thus p is an H-bilinear coalgebra retraction of σ.

Therefore we can consider the pre-bialgebra $(R,m,u,\delta,\varepsilon)$ in ${}_{H}^{H}\mathcal{YD}$ associated to (A,p,σ) with corresponding cocycle ξ. We want to compute

$$R=A^{co(H)}=\left\{a\in A\mid\sum a_{(1)}\otimes p\left(a_{(2)}\right)=a\otimes 1_{H}\right\}.$$

It is easy to check that $y^{n}\in R$, for any $0\leq n\leq N-1$. Let us prove that $\left(y^{n}\right)_{0\leq n\leq N-1}$ defines a basis for R. Clearly, since $\mathcal{B}\left(A\right)$ is a basis of A, they are linearly independent over K. Let us check that they also generate R as a vector space over K. Recall from Proposition 3.4 that the map $\tau:A\rightarrow R,\tau\left(a\right)=\sum a_{(1)}\sigma S_{H}p\left[a_{(2)}\right]$ defines a surjective coalgebra homomorphism such that $\tau\left(a\sigma\left(h\right)\right)=\tau\left(a\right)\varepsilon_{H}\left(h\right)$. Moreover, since $y^{n}\in R$, then $\tau\left(y^{n}\right)=y^{n}$, for every $0\leq n\leq N-1$, while

$$\tau\left(y^{n}\right)=\tau\left(y^{n-N}y^{N}\right)=\lambda\left(N\right)\tau\left[y^{n-N}\left(1_{A}-\Gamma^{N}\right)\right]=\lambda\left(N\right)\tau\left(y^{n-N}\right)\varepsilon_{H}\left(1_{H}-g^{N}\right)=0.$$

for every $n \geq N$.
Now since τ is surjective, R is generated by

$$\tau\left[\mathcal{B}\left(A\right)\right]=\left\{\varepsilon_{H}\left(h\right)y^{i}\mid 0\leq i\leq N-1,h\in\mathcal{B}\left(H\right)\right\}$$

so that $(y^n)_{0\leq n\leq N-1}$ generates R as a vector space over K and hence it is a basis.
Let us deal with the multiplication m of R. Since, by Proposition 3.4, we have $r\cdot_R s=\tau\left(r\cdot_A s\right)$, we get that

$$y^{a}\cdot_{R}y^{b}=\tau\left(y^{a}\cdot_{A}y^{b}\right)=\tau\left(y^{a+b}\right),\text{ for any }0\leq a,b\leq N-1.$$

If $0\leq a+b\leq N-1$, then $y^{a}\cdot_{R}y^{b}=y^{a+b}$ while, if $a+b\geq N$, then $y^{a}\cdot_{R}y^{b}=\tau\left(y^{a+b}\right)=0$.This entails

$$y^{\cdot_{R}n}=\begin{cases}y^{n} & \text{for } 0\leq n\leq N-1,\\ 0 & \text{for } n\geq N,\end{cases}$$

and that R is an associative algebra.
Let us deal with the comultiplication δ of R. For every $0\leq n\leq N-1$, we get

$$\delta\left(y^{n}\right)=\delta\tau\left(y^{n}\right)=\left(\tau\otimes\tau\right)\Delta_{A}\left(y^{n}\right)=\sum\nolimits_{i=0}^{n}\binom{n}{i}_{q}y^{n-i}\otimes y^{i}.$$

This tells us that R is a graded coalgebra and its homogeneous part of degree 0 is $K1_A$. Note that both the algebra and the coalgebra structures of R agree with the ones of a quantum line generated by y. In particular $R_0=K1_A$ and $P\left(R\right)=Ky$.
Let us deal with the cocycle ξ of R. Let $0\leq a,b\leq N-1$. Then, by 3.2, we have

$$\xi(y^{a}\otimes y^{b})=p(y^{a}\cdot_{A}y^{b})=p\left(y^{a+b}\right).$$

If $0\leq a+b\leq N-1$, we have $\xi(y^{a}\otimes y^{b})=p\left(y^{a+b}\right)=\delta_{a+b,0}$ while, if $N\leq a+b\leq 2N-2$, we have $\xi(y^{a}\otimes y^{b})=p\left(y^{a+b}\right)=p\left(y^{a+b-N}y^{N}\right)=\lambda\left(N\right)p\left[y^{a+b-N}\left(1_{A}-\Gamma^{N}\right)\right]=\lambda\left(N\right)\delta_{a+b,N}\left(1_{H}-g^{N}\right).$
Therefore, for any $0\leq a,b\leq N-1$, we get 3). Now, from 3.2, the Yetter-Drinfeld module structure of R is given by

$$^{h}r=\sum\sigma\left(h_{(1)}\right)r\sigma S_{H}\left(h_{(1)}\right),\qquad\rho\left(r\right)=\sum p\left(r_{(1)}\right)\otimes r_{(2)}.$$

From these equalities we get

$$\rho\left(y^{n}\right)=\sum p\left[\left(y^{n}\right)_{(1)}\right]\otimes\left(y^{n}\right)_{(2)}=g^{n}\otimes y^{n},$$

for every $0\leq n\leq N-1$, and

$$^{h}y=\sum\sigma\left(h_{(1)}\right)y\sigma S_{H}\left(h_{(2)}\right)=\sum y\sigma\left[\varphi\left(h_{(1)}\right)\right]\sigma S_{H}\left(h_{(2)}\right)=y\sigma\left[\sum\varphi\left(h_{(1)}\right)S_{H}\left(h_{(2)}\right)\right].$$

In particular we have $^{g}y=qy$. Let us prove now that $(H,g,\varepsilon_{H}\varphi)$ is a Yetter-Drinfeld datum for q and that the pre-bialgebra in ${}_{H}^{H}\mathcal{YD}$ associated to (A,p,σ) is $(R,m,u,\delta,\varepsilon)$ with corresponding cocycle ξ.

By Lemma 2.7, there is a primitive θ-th root of unity $q'\neq 1$, where $2\leq\theta\leq\dim_{K}\left(R\right)=N$, and and $g'\in H,\chi\in H^{*}$ such that

1) (H,g',χ) is a Yetter-Drinfeld datum for q',
2) $\rho(y)=g'\otimes y$ and
3) $^{h}y=\chi(h)y$ for every $h\in H$.

Let us prove that $g'=g$, $q'=q$ and $\chi=\varepsilon_{H}\varphi$.

Since $g'\otimes y=\rho(y)=g\otimes y$, we deduce $g=g'$. For every $h\in H$ we have

$$\chi(h)y=\ ^{h}y=y\sigma\left[\sum\varphi\left(h_{(1)}\right)S_{H}\left(h_{(2)}\right)\right]$$

so that $\chi(h)1_{H}=\sum\varphi\left(h_{(1)}\right)S_{H}\left(h_{(2)}\right)$ and hence

$$\varphi\left(h\right)=\sum\varphi\left(h_{(1)}\right)S_{H}\left(h_{(2)}\right)h_{(3)}=\sum\chi\left(h_{(1)}\right)h_{(2)}.$$

In particular, for $h=g$ we get $qg=\varphi\left(g\right)=\chi\left(g\right)g=\chi(g')g=q'g$ so that $q'=q$ and $\theta=N$. Note that $\varepsilon_{H}\varphi\left(h\right)=\chi\left(h\right)$, for every $h\in H$.

Thus we deduce that $R=R_q(H,g,\chi)$ is a quantum line spanned by y of dimension N. Now, as a bialgebra, A is isomorphic to $B=R\#_\xi H$ (see Remark 3.3). By Theorem 3.9, since $R_0=K1_R$, we get that A is a Hopf algebra. □

Theorem 4.2. *Let H be a Hopf algebra over a field K. Let A be a bialgebra and let $\sigma : H \to A$ be an injective morphism of bialgebras having a retraction $\pi : A \to H$ (i.e. $\pi\sigma = H$) that is an H-bilinear coalgebra map. Assume that either H is f.d. or cosemisimple and that the coalgebra in the pre-bialgebra in ${}_H^H\mathcal{YD}$ associated to (A,π,σ) is thin.*
Then there exist

- *a retraction $p : A \to H$ (i.e. $p\sigma = H$) that is an H-bilinear coalgebra map,*
- *a primitive N-th root of unit q,*
- *$g\in H,\chi\in H^*,\lambda(N)\in K$ so that $(H,g,\chi,\lambda(N))$ is a compatible datum for q*

such that the pre-bialgebra in ${}_H^H\mathcal{YD}$ associated to (A,p,σ) is $(R,m,u,\delta,\varepsilon)$ with corresponding cocycle ξ where

1) $R=R_q(H,g,\chi)$ is a braided bialgebra in ${}_H^H\mathcal{YD}$, in fact a quantum line spanned by y of dimension N and the N-th power of y in R is zero.

2) For any $0\leq n\leq N-1$, the n-th iterated power of y in R coincides with the n-th iterated power of y in A and will both be denoted by y^n.

3) For any $0\leq a,b\leq N-1$, we have

$$\xi(y^a\otimes y^b)=\begin{cases}1 & \text{for } a+b=0\\ \lambda(N)(1_H-g^N) & \text{for } a+b=N\\ 0 & \text{otherwise.}\end{cases}$$

Moreover A is a Hopf algebra with basis

$$\left\{y^i\sigma(h)\mid 0\leq i\leq N-1, h\in B(H)\right\},$$

algebra structure given by

$$\begin{aligned} y^N &= \lambda(N)\left(1_A-\Gamma^N\right),\\ \sigma(h)\,y^a &= y^a\sigma\left[\varphi_H^a(h)\right] \text{ for any } a\in\mathbb{N}, \text{ and } h\in H\end{aligned}$$

and coalgebra structure given by

$$\Delta_A(y)=y\otimes 1_A+\Gamma\otimes y.$$

Here $\varphi_H : H\to H$ denotes the algebra automorphism of H defined by $\varphi_H(h)=\sum\chi\left(h_{(1)}\right)h_{(2)}$ and $\Gamma=\sigma(g)$.
Furthermore, $\pi=p$ whenever π is a homomorphism of bialgebras.

Proof. By Theorem 3.30 we can apply Theorem 4.1.
Let us prove the last assertion. Denote by $(R',m',u',\delta',\varepsilon')$ the pre-bialgebra in ${}_H^H\mathcal{YD}$ associated to (A,π,σ) with corresponding cocycle ξ'. For every $r\in R'$ we have $\pi(r)=\sum\varepsilon_A\left(r_{(1)}\right)\pi\left(r_{(2)}\right)=\varepsilon(r)\,1_H$. Since $P(R')=Ky$, denote by y^n the n-th iterated power of y in A. Therefore, if π is an algebra homomorphism, we have

$$\pi(y^n)=\pi(y)^n=\varepsilon(y)^n\,1_H=\delta_{n,0}1_H.$$

Since A is a Hopf algebra with basis $\left\{y^i\sigma(h)\mid 0\leq i\leq N-1, h\in\mathcal{B}(H)\right\}$, and π is right H-linear, we get $\pi\left[y^n\sigma(h)\right]=\pi(y^n)\,h=\delta_{n,0}h=p\left[y^n\sigma(h)\right]$ and hence $\pi=p$. □

Corollary 4.3. *Under the hypothesis and assumptions of Theorem 4.2, the following conditions are equivalent:*

(a) $\xi=\varepsilon\otimes\varepsilon$.

(b) The compatible datum $(H,g,\chi,\lambda(N))$ is trivial.

(c) $A\simeq R\#_\xi H$ is the Radford-Majid bosonization of R.

(d) p is a bialgebra homomorphism.

Proof. $(a) \Leftrightarrow (c) \Leftrightarrow (d)$ follow by Lemma 3.8 as R is a braided bialgebra in ${}^H_H\mathcal{YD}$.

$(a) \Leftrightarrow (b)$. Since

$$\xi(y^a \otimes y^b) = \begin{cases} 1 & \text{for } a+b=0 \\ \lambda(N)(1_H - g^N) & \text{for } a+b=N \\ 0 & \text{otherwise.} \end{cases}$$

we have that $\xi = \varepsilon \otimes \varepsilon$ iff $\lambda(N)(1_H - g^N) = 0$. By Remark 3.28 we conclude. □

Definition 4.4. Recall from [AMS, Definition 2.7] that an *ad*-invariant integral for a Hopf algebra H is a linear map $\lambda : H \to K$ such that

$$\sum h_{(1)}\lambda\left(h_{(2)}\right) = 1_H \lambda(h), \qquad \lambda(1_H) = 1_K, \qquad \sum \lambda\left[h_{(1)} x S_H\left(h_{(2)}\right)\right] = \varepsilon_H(h)\lambda(x),$$

for any $h, x \in H$. From [AMS, Theorem 2.27] any semisimple and cosemisimple Hopf algebra (e.g. f.d. cosemisimple) has such an integral. Note that the group algebra, which is in general not semisimple, always admits an *ad*-invariant integral.

Theorem 4.5. *Let A be a bialgebra over a field K. Suppose that the coradical H of A is a f.d. subbialgebra of A with antipode. Then A is a Hopf algebra and there is a retraction $\pi : A \to H$ (i.e. $\pi\sigma = H$) that is an H-bilinear coalgebra map. Let $(R, m, u, \delta, \varepsilon)$ be the pre-bialgebra in ${}^H_H\mathcal{YD}$ associated to (A, π, σ) with corresponding cocycle ξ.*
Assume that R is an N-dimensional thin coalgebra where $P(R) = Ky$.
Then there exist

- *a primitive N-th root of unit q,*
- *$g \in H, \chi \in H^*, \lambda(N) \in K$ so that $(H, g, \chi, \lambda(N))$ is a compatible datum for q*

such that

1) $R = R_q(H, g, \chi)$ is a quantum line spanned by y.

2) The n-th iterated power of y in R and the n-th iterated power of y in A coincide for every $0 \leq n \leq N-1$.

3)

$$\xi(y^a \otimes y^b) = \begin{cases} 1 & \textit{for } a+b=0 \\ \lambda(N)(1_H - g^N) & \textit{for } a+b=N,\ a \neq 0, b \neq 0 \\ 0 & \textit{otherwise.} \end{cases}$$

Moreover A is a Hopf algebra with basis

$$\left\{y^i \sigma(h) \mid 0 \leq i \leq N-1, h \in B(H)\right\},$$

algebra structure given by

$$\begin{aligned} y^N &= \lambda(N)\left(1_A - \Gamma^N\right), \\ \sigma(h)\, y^a &= y^a \sigma\left[\varphi_H^a(h)\right] \ \textit{for any } a \in \mathbb{N}, \textit{ and } h \in H \end{aligned}$$

and coalgebra structure given by

$$\Delta_A(y) = y \otimes 1_A + \Gamma \otimes y.$$

Here $\varphi_H : H \to H$ denotes the algebra automorphism of H defined by $\varphi_H(h) = \sum \chi\left(h_{(1)}\right) h_{(2)}$ and $\Gamma = \sigma(g)$.
Furthermore, if $y^N = \lambda(N)\left(1_A - \Gamma^N\right) \neq 0$, then

$$\chi^N = \varepsilon_H \qquad \textit{and} \qquad g^N \in Z(H).$$

Proof. By [AMS, Theorem 2.35], the canonical injection of H in A has a retraction $\pi : A \to H$ which is an H-bilinear coalgebra map. By Theorem 3.30 we can apply Theorem 4.1. In order to conclude it is enough to prove that $\pi = p$. By the quantum binomial formula, we have $\Delta_A(y^n) = \sum_{i=0}^n \binom{n}{i}_q y^{n-i}\Gamma^i \otimes y^i$, for any $n \in \mathbb{N}$. Since π is a right H-linear coalgebra homomorphism, by applying $\pi \otimes \pi$ to both sides, we get

$$\Delta_A(\pi(y^n)) = \sum_{i=0}^n \binom{n}{i}_q \pi\left(y^{n-i}\right) g^i \otimes \pi\left(y^i\right).$$

Let $\lambda : H \to K$ be an ad-invariant integral and apply $H \otimes \lambda$ to both sides of the displayed equality to obtain

$$\lambda\pi\left(y^{n}\right)=\sum\nolimits_{i=0}^{n}\binom{n}{i}_{q}\pi\left(y^{n-i}\right)g^{i}\lambda\pi\left(y^{i}\right). \tag{49}$$

Let us prove for induction on $0 \leq n \leq N-1$ that $\lambda\pi\left(y^{n}\right)=\delta_{0,n}$. If $n=0$ there is nothing to prove. Let $n \geq 1$ and assume $\lambda\pi\left(y^{t}\right)=\delta_{0,t}$, for every $0 \leq t \leq n-1$. Let us prove that $y^{t} \in R$, for every $0 \leq t \leq n-1$:

$$\left(y^{t}\right)_{(1)}\otimes\pi\left[\left(y^{t}\right)_{(2)}\right]=\sum\nolimits_{i=0}^{t}\binom{t}{i}_{q}y^{t-i}\Gamma^{i}\otimes\pi\left(y^{i}\right)=y^{t}\otimes 1_{H}.$$

In particular $y^{n-1} \in R$ and since $y \in R$, by definition of ξ we have

$$\pi\left(y^{n}\right)=\pi\left(y\cdot_{A}y^{n-1}\right)=\xi\left(y\otimes y^{n-1}\right).$$

Since λ is ad-invariant, we have

$$\sum\lambda\xi({}^{h_{(1)}}r\otimes{}^{h_{(2)}}s)\overset{(14)}{=}\sum\lambda\left[h_{(1)}\xi(r\otimes s)Sh_{(2)}\right]=\varepsilon_{H}\left(h\right)\lambda\xi(r\otimes s)$$

Apply this equality to the case $h=g$, $r=y$ and $s=y^{n-1}$:

$$\lambda\xi({}^{g}y\otimes{}^{g}y^{n-1})=\lambda\xi(y\otimes y^{n-1}).$$

Since ${}^{g}\left(y^{t}\right)=\Gamma y^{t}S\left(\Gamma\right)=\left[\Gamma yS\left(\Gamma\right)\right]^{t}=q^{t}y^{t}$, we get

$$q^{n}\lambda\xi(y\otimes y^{n-1})=\lambda\xi(y\otimes y^{n-1}).$$

Since $1 \leq n \leq N-1$, we have $q^{n} \neq 1$ and hence $\lambda\pi\left(y^{n}\right)=\lambda\xi(y\otimes y^{n-1})=0$. Therefore we have proved that $\lambda\pi\left(y^{n}\right)=\delta_{0,n}$, for every $0 \leq n \leq N-1$. By (49), we have

$$\delta_{n,0}=\sum\nolimits_{i=0}^{n}\binom{n}{i}_{q}\pi\left(y^{n-i}\right)g^{i}\delta_{i,0}=\pi\left(y^{n}\right).$$

Since π is right H-linear, it is clear that $\pi=p$. □

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