A general geometric setting for the energy of the gravitational field

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Abstract

Within the geometric formulation of the Hamiltonian formalism for field theory in terms of *Hamiltonian connections* and *multisymplectic forms*, we refer to our new geometric description where a *composite fibered bundle*, involving a *line bundle*, plays the role of an *extended configuration bundle*. The concepts of *extended Legendre bundle*, *Hamiltonian connection*, *Hamiltonian form* and *covariant Hamilton equations* are exploited to provide a suitable description of the energy for the gravitational field.

Key words: jets, connections, Hamilton equations, energy, gravity.
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1 Introduction

A geometric formulation of the Hamiltonian formalism for field theory in terms of *Hamiltonian connections* and *multisymplectic forms* was developed in [12, 13, 14]. We recall that, in this framework, the covariant Hamilton equations for Mechanics and field theory are defined in terms of multisymplectic $(n+2)$–forms, where n is the dimension of the basis manifold, together with connections on the configuration bundle.

We provided in [2] a new geometric Hamiltonian description of field theory, based on the introduction of a suitable *composite fibered bundle* which plays the role of an *extended configuration bundle*. One of the main features of this approach is that one can describe the polymomenta and other objects appearing in the *polymomentum* formulation of field theory (see e.g. [1, 4, 5, 8, 9, 10] and references therein) in terms of differential forms with values in the vertical tangent bundle of an appropriate line bundle Θ. The introduction of the line bundle Θ can be here understood as a suitable way of describing the *gauge* character appearing in the Hamiltonian formalism for field theory (see e.g. [6]).

Instead of bundles over an n–dimensional base manifold X, we consider *fibrations over a line bundle* Θ *fibered over* X; the concepts of *event bundle*, *configuration bundle* and *Legendre bundle* are then introduced following the analogous setting introduced in [12, 13, 14] for Mechanics and for the polymomentum approach to field theory. Moreover, *Hamiltonian connections*, *Hamiltonian forms* and *covariant Hamilton equations* can be suitably described in this framework. This new approach takes into account the existence of more than one independent variable in field theory, but enables us to keep as far as possible most of the nice features of time–dependent Hamiltonian Mechanics. Already in the seventies, in fact, J. Kijowski stressed the prominent role of the symplectic structures in field theories [5, 6, 8, 9]. Our approach can provide a suitable geometric interpretation of the canonical theory of gravity and gravitational energy, as presented in [7], where the local line bundle coordinate τ plays the role of a *parameter* and enables one to consider the gravitational energy as a '*gravitational charge*'. In fact, proceeding in analogy with Mechanics we obtain the expression of the *'abstract' energy* for an extended version of the Hilbert–Einstein Lagrangian and we show that this quantity is conserved.

2 Jets of fibered manifolds and connections

The general framework is a fibered bundle $\pi : Y \to X$, with $dimX = n$ and $dimY = n + m$ and, for $r \geq 0$, its jet manifold J_rY. We recall the natural fiber bundles $\pi^r_s : J_rY \to J_sY$, $r \geq s$, $\pi^r : J_rY \to X$, and, among these, the *affine* fiber bundles π^r_{r-1}.

Greek indices $\lambda, \mu, \dots$ run from 1 to n and they label base coordinates, while Latin indices $i, j, \dots$ run from 1 to m and label fibre coordinates, unless otherwise specified. We denote multi–indices of dimension n by underlined Greek letters such as $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$, with $0 \leq \alpha_\mu$, $\mu = 1, l \dots, n$; by an abuse of notation, we denote with λ the multi–index such that $\alpha_\mu = 0$, if

$\mu \neq \lambda$, $\alpha_\mu = 1$, if $\mu = \lambda$. We also set $|\underline{\alpha}| \doteq \alpha_1 + \ldots + \alpha_n$. The charts induced on J_rY are denoted by $(x^\lambda, y^i_{\underline{\alpha}})$, with $0 \leq |\underline{\alpha}| \leq r$; in particular, we set $y^i_{\mathbf{0}} \equiv y^i$. The local bases of vector fields and 1–forms on J_rY induced by the coordinates above are denoted by $(\partial_\lambda, \partial^{\underline{\alpha}}_i)$ and $(d^\lambda, d^i_{\underline{\alpha}})$, respectively.

The *contact maps* on jet spaces induce the natural complementary fibered morphisms over the affine fiber bundle $J_rY \to J_{r-1}Y$

$$D_r : J_rY \times_X TX \to TJ_{r-1}Y \,, \qquad \vartheta_r : J_rY \times_{J_{r-1}Y} TJ_{r-1}Y \to VJ_{r-1}Y \,, \qquad r \geq 1 \,, \tag{1}$$

with coordinate expressions, for $0 \leq |\underline{\alpha}| \leq r-1$, given by $D_r = d^\lambda \otimes D_\lambda = d^\lambda \otimes (\partial_\lambda + y^j_{\underline{\alpha}+\lambda} \partial^{\underline{\alpha}}_j)$, $\vartheta_r = \vartheta^j_{\underline{\alpha}} \otimes \partial^{\underline{\alpha}}_j = (d^j_{\underline{\alpha}} - y^j_{\underline{\alpha}+\lambda} d^\lambda) \otimes \partial^{\underline{\alpha}}_j$, and the natural fibered splitting $J_rY \times_{J_{r-1}Y} T^*J_{r-1}Y = J_rY \times_{J_{r-1}Y} (T^*X \oplus V^*J_{r-1}Y)$.

Definition 1 *A* connection *on the fiber bundle $Y \to X$ is defined by the dual linear bundle morphisms over Y $Y \times_X TX \to TY$, $V^*Y \to T^*Y$ which split the exact sequences*

$$0 \to VY \hookrightarrow TY \to Y \times_X TX \to 0 \,, \qquad 0 \to Y \times_X T^*X \hookrightarrow T^*Y \to V^*Y \to 0 \,. \quad \square \tag{2}$$

We recall that there is a one–to–one correspondence between the connections Γ on a fiber bundle $Y \to X$ and the global sections $\Gamma : Y \to J_1Y$ of the affine jet bundle $J_1Y \to Y$ (see e.g. [12]).

In the following a relevant role is played by the composition of fiber bundles

$$Y \to \Theta \to X \,, \tag{3}$$

where $\pi_{YX} : Y \to X$, $\pi_{Y\Theta} : Y \to \Theta$ and $\pi_{\Theta X} : \Theta \to X$ are fiber bundles. The above composition was introduced under the name of *composite fiber bundle* in [11, 13] and shown to be useful for physical applications, e.g. for the description of mechanical systems with time–dependent parameters.

2.1 Connections on composite bundles

We shall be concerned here with the description of connections on composite fiber bundles. We will follow the notation and main results stated in [12].

We shall denote by $J_1\Theta$, $J^\Theta_1 Y$ and J_1Y, the jet manifolds of the fiber bundles $\Theta \to X$, $Y \to \Theta$ and $Y \to X$ respectively.

Let γ be a connection on the composite bundle π_{YX} projectable over a connection Γ on $\pi_{\Theta X}$, i.e. such $J_1\pi_{Y\Theta} \circ \gamma = \Gamma \circ \pi_{Y\Theta}$. Let γ_Θ be a connection

on the fiber bundle $\pi_{Y\Theta}$. Given a connection Γ on $\pi_{\Theta X}$, there exists [12] a canonical morphism over Y, $\rho : J_1\Theta \times_X J_1^\Theta Y \to J_1 Y$, which sends (Γ, γ_Θ), into the *composite connection* $\gamma \doteq \gamma_\Theta \circ \Gamma$ on π_{YX}, projectable over Γ. Recall that given a composite fiber bundle 3 and a global section h of the fiber bundle $\pi_{\Theta X}$, then the restriction $Y_h \doteq h^* Y$ of the fiber bundle $\pi_{Y\Theta}$ to $h(X) \subset \Theta$ is a subbundle $i_h : Y_h \hookrightarrow Y$ of the fiber bundle $Y \to X$ [12]. Let then h be a section of $\pi_{\Theta X}$. Every connection γ_Θ induces the pull–back connection γ_h on the subbundle $Y_h \to X$. The composite connection $\gamma = \gamma_\Theta \circ \Gamma$ is reducible to γ_h if and only if h is an integral section of Γ.
We have the following exact sequences of *vector bundles over a composite bundle* Y:

$$0 \to V_\Theta Y \hookrightarrow VY \to Y \times_\Theta V\Theta \to 0 \,, \qquad 0 \to Y \times_\Theta V^*\Theta \hookrightarrow V^*Y \to V^*_\Theta Y \to 0 \,, \tag{4}$$

where $V_\Theta Y$ and $V^*_\Theta Y$ are the vertical tangent and cotangent bundles to the bundle $\pi_{Y\Theta}$.

Remark 1 *Every connection γ_Θ on $\pi_{Y\Theta}$ provides the dual splittings*

$$VY = V_\Theta Y \oplus_Y \gamma_\Theta(Y \times_\Theta V\Theta) \,, \qquad V^*Y = Y \times_\Theta V^*\Theta \oplus_Y \gamma_\Theta(V^*_\Theta Y) \,, \tag{5}$$

of the above exact sequences. By means of these splittings one can easily construct the vertical covariant differential *on the composite bundle π_{YX}, i.e. a first order differential operator*

$$\Delta_{\gamma_\Theta} : J_1 Y \to T^*X \oplus_Y V^*_\Theta Y \,. \tag{6}$$

□

3 Hamiltonian formalism for field theory

We recall now that the covariant Hamiltonian field theory can be conveniently formulated in terms of Hamiltonian connections and Hamiltonian forms [13]. Here we shall construct a Hamiltonian formalism for field theory as a theory on the composite bundle $Y \to \Theta \to X$, with $\pi_{\Theta X} : \Theta \to X$ a *line bundle* having local fibered coordinates (x^λ, τ) [2].

Let us now consider the *extended Legendre bundle* $\Pi_\Theta \doteq V^*Y \wedge (\Lambda^n T^*\Theta) \to X$. There exists a canonical isomorphism

$$\Pi_\Theta \simeq \Lambda^{n+1} T^*\Theta \otimes_Y V^*Y \otimes_Y T\Theta \,. \tag{7}$$

Definition 2 *We call the fiber bundle* $\pi_{Y\Theta}: Y \to \Theta$ *the* abstract event space *of the field theory. The* configuration space *of the field theory is then the first order jet manifold* $J_1^{\Theta}Y$*. The* abstract Legendre bundle *of the field theory is the fiber bundle* $\Pi_\Theta \to \Theta$*.* □

Let now γ_Θ be a connection on $\pi_{Y\Theta}$ and Γ_Θ be a connection on $\pi_{\Theta X}$. We have the following non–canonical isomorphism

$$\Pi_\Theta \simeq_{(\gamma_\Theta,\Gamma_\Theta)} \Lambda^{n+1}T^*\Theta \otimes_Y [(Y \oplus_\Theta V^*\Theta) \oplus_Y \gamma_\Theta(V^*_\Theta Y)] \otimes_Y (V\Theta \oplus_\Theta H\Theta)\,. \quad (8)$$

In this perspective, we consider the canonical bundle monomorphism over Y providing the tangent–valued Liouville form on Π_Θ, i.e.

$$\vartheta_Y : \Pi_\Theta \hookrightarrow \Lambda^{n+2}T^*Y \otimes_Y (V\Theta \oplus_\Theta H\Theta)\,, \quad (9)$$

the coordinate expression of which is

$$\vartheta_Y = \hat{p}^\lambda_i d^i \wedge \omega \otimes \partial_\lambda \otimes \hat{\partial} \simeq \hat{p}^\lambda_i \vartheta^i \wedge \omega_\lambda \otimes \hat{\partial}\,, \quad (10)$$

where ϑ^i are generators of vertical 1–forms (i.e. contact forms) on Y, "$\simeq$" is the isomorphism defined by 8, $\hat{\partial} \doteq \partial_\tau$, ω is the volume form and $\omega_\lambda \doteq \partial_\lambda \rfloor \omega$. Following [9] we set $\hat{p}_i \doteq \hat{p}^\lambda_i \omega_\lambda$ and obtain

$$\vartheta_Y = \hat{p}_i \vartheta^i \otimes \hat{\partial}\,. \quad (11)$$

The polysymplectic form Ω_Y on Π_Θ is then intrinsically defined by $\Omega_Y \rfloor \psi = d(\vartheta_Y \rfloor \psi)$, where ψ is an arbitrary 1–form on Θ; its coordinate expression is given by

$$\Omega_Y = d\hat{p}_i \wedge d^i \wedge \omega \otimes \hat{\partial} \simeq d\hat{p}_i \wedge \vartheta^i \otimes \hat{\partial}\,. \quad (12)$$

Let $J_1\Pi_\Theta$ be the first order jet manifold of the extended Legendre bundle $\Pi_\Theta \to X$. A connection γ on the extended Legendre bundle is then in one–to–one correspondence with a global section of the affine bundle $J_1\Pi_\Theta \to \Pi_\Theta$. Such a connection is said to be a *Hamiltonian connection* iff the exterior form $\gamma \rfloor \Omega_Y$ is closed.

Let γ be a Hamiltonian connection on Π_Θ and U be an open subset of Π_Θ. Locally, we have $\gamma \rfloor \Omega_Y = d(\hat{p}_i \otimes \hat{\partial}) \wedge \vartheta^i \otimes \hat{\partial} - d\mathcal{H} \wedge \omega \doteq dH$, where $\mathcal{H} : U \subset \Pi_\Theta \to V\Theta$ and $d = \partial_i d^i + \hat{\partial}^i d\hat{p}_i + \partial_\tau d\tau$ is the total differential on $V^\Theta \Pi_\Theta$ and, of course, $d\hat{\partial} = \hat{\partial} d$.

The local mapping $\mathcal{H} : U \subset \Pi_\Theta \to V\Theta$ is called a *Hamiltonian*. The form H on the extended Legendre bundle Π_Θ is called a *Hamiltonian form*.

Every Hamiltonian form H admits a Hamiltonian connection γ_H such that the following holds: $\gamma_H \rfloor \Omega_Y = dH$.

We define the *abstract covariant Hamilton equations* [2] to be the kernel of the first order differential operator $\Delta_{\tilde{\gamma}_\Theta}$ defined as the vertical covariant differential (see Eq. 6) relative to the connection $\tilde{\gamma}_\Theta$ on the abstract Legendre bundle $\Pi_\Theta \to \Theta$.

In this case the Hamiltonian form H is the Poincaré–Cartan form of the *Lagrangian* $L_H = (\hat{p}_i \dot{y}^i - \mathcal{H})\, \omega$ on $V^\Theta \Pi_\Theta$, with values in $V\Theta$; furthermore, the *Hamilton operator* for H is defined as the Euler–Lagrange operator associated with L_H, namely $\mathcal{E}_H : V^\Theta \Pi_\Theta \to T^*\Pi_\Theta \wedge \Lambda^{n+1} T^* X$.

The kernel of the Hamilton operator, i.e. the Euler–Lagrange equations for L_H, is an affine closed embedded subbundle of $V^\Theta \Pi_\Theta \to \Pi_\Theta$, locally given by the *covariant Hamilton equations* on the extended Legendre bundle $\Pi_\Theta \to X$

$$\dot{y}^i \otimes \hat{\partial} = \hat{\partial}^i \mathcal{H} \,, \tag{13}$$

$$\dot{\hat{p}}_i \otimes \hat{\partial} = -\partial_i \mathcal{H} \,, \tag{14}$$

$$\dot{\mathcal{H}} = -\hat{\partial} \mathcal{H} \,. \tag{15}$$

In [2] we stated the relation with the standard polysymplectic approach (for a review of the topic see e.g. [1, 4, 5, 9, 10] and references quoted therein). The basic idea is that the present geometric formulation can be interpreted as a suitable generalization to field theory of the so–called *homogeneous formalism* for Mechanics. Our result can be summarized by the following Lemma and Theorem.

Lemma 1 *Let γ_H be a Hamiltonian connection on $\Pi_\Theta \to X$. Let $\tilde{\gamma}_\Theta$ and Γ be connections on $\Pi_\Theta \to Y$ and $\Theta \to X$, respectively. Let σ and h be sections of the bundles $\pi_{Y\Theta}$ and $\pi_{\Theta X}$, respectively.*

Then the standard Hamiltonian connection on $\Pi_\Theta \to X$ turns out to be the pull–back connection $\tilde{\gamma}_\phi$ induced on the subbundle $\Pi_{\Theta\,\phi} \hookrightarrow \Pi_\Theta \to X$ by the section $\phi = h \circ \sigma$ of $Y \to X$.

Theorem 1 *Let $\Delta_{\tilde{\gamma}\,,\phi}$ be the covariant differential on the subbundle $\Pi_{\Theta\,\phi} \hookrightarrow \Pi_\Theta \to X$ relative to the pull–back connection $\tilde{\gamma}_\phi$. The kernel of $\Delta_{\tilde{\gamma}\,,\phi}$ coincides with the Hamilton–De Donder equations of the standard polysymplectic approach to field theories.*

3.1 The energy of the gravitational field

Let us now specify the above formalism for an extended version of the Hilbert–Einstein Lagrangian, i.e. essentially the Hilbert–Einstein Lagrangian for a metric g parametrized by the line coordinate τ. This enables one to deal with ambiguities in the definition of gravitational energy. The result is in fact that the energy turns out to be a 'vertical' i.e. , in a convenient sense, gauge conserved quantity in accordance with the more recent approach of [7], where the energy is derived as a quasi–local quantity, *via* a boundary integral.

In the following a 'dot' stands for $\frac{d}{d\tau}$. Let then $dimX = 4$ and X be orientable. Let $Lor(X)_\Theta$ be the bundle of Lorentzian metrics on X (provided that they exist), *parametrized by* Θ.

The *extended* Hilbert–Einstein Lagrangian is the form $\lambda_{HE} = L_{HE}\,\omega \otimes \hat{\partial}$, were $L_{HE} = r\,\sqrt{\underline{g}}$. Here $r : J_2^\Theta(Lor(X)_\Theta) \to I\!\!R$ is the function such that, for any parametrized Lorentz metric g, we have $r \circ j_2^\Theta g = s$, being s the scalar curvature associated with g, and $\underline{g}$ is the determinant of g.

If we set $G^{\alpha\beta\epsilon\gamma} \doteq g^{\alpha\epsilon}g^{\beta\gamma} + g^{\alpha\gamma}g^{\beta\epsilon} - 2g^{\alpha\beta}g^{\epsilon\gamma}$ (the De Witt metric), then we can write

$$r = \frac{1}{2}G^{\alpha\beta\epsilon\gamma}(g_{\epsilon\gamma,\alpha\beta} + g_{\mu\nu}\Gamma^{\mu}_{\alpha\beta}\Gamma^{\nu}_{\epsilon\gamma})\,; \tag{16}$$

furthermore, we set $\pi^{\mu\nu} = \sqrt{\underline{g}}g^{\mu\nu}$ and $\tilde{\phi}_{\mu\nu} \doteq \hat{\phi}_{\mu\nu}\otimes\hat{\partial}$, where

$$\hat{\phi}_{\mu\nu} \doteq \partial L_{HE}/\partial\dot{\pi}^{\mu\nu} - \frac{d}{d\tau}(\partial L_{HE}/\partial\ddot{\pi}^{\mu\nu})\,. \tag{17}$$

In analogy with Mechanics we obtain now

$$\tilde{L}_{HE} \doteq L_{HE}\otimes\hat{\partial} = \pi^{\mu\nu}R_{\mu\gamma}\otimes\hat{\partial} = \tilde{\phi}_{\mu\nu}\dot{\pi}^{\mu\nu} - \mathcal{H}\,, \tag{18}$$

where $R_{\mu\gamma}$ denotes the components of the Ricci tensor of the metric. Hence the *'abstract' energy* turns out to be

$$\mathcal{H} = (-\pi^{\mu\gamma}R_{\mu\gamma} + \hat{\phi}_{\mu\nu}\dot{\pi}^{\mu\nu})\otimes\hat{\partial}\,. \tag{19}$$

Notice that, like in Mechanics, since the Hamiltonian does not depend explicitly on τ, it is, in fact, a conserved quantity, i.e. from the covariant Hamilton equations, see Eq. 15, we have $\dot{\mathcal{H}} = 0$.

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