UNIVERSITÀ DEGLI STUDI DI TORINO

JOURNAL OF NUMERICAL ANALYSIS,INDUSTRIAL AND APPLIED MATHEMATICS

Vol. *6, p.83-93, 2011*

Journal of Numerical Analysis, Industrial and Applied Mathematics (JNAIAM)

A uniformly convergent sequence of spline quadratures for Cauchy principal value integrals

C. Dagnino[1], V. Demichelis[2]

Department of Mathematics,
Faculty of Scienze Matematiche Fisiche e Naturali,
University of Torino,
10100 Torino, Italy

Received ... ; accepted in revised form ...

Abstract: We propose a new quadrature rule for Cauchy principal value integrals based on quadratic spline quasi-interpolants which have an optimal approximation order and satisfy boundary interpolation conditions. In virtue of these spline properties, we can prove uniform convergence for sequences of such quadratures and provide uniform error bounds. A computational scheme for the quadrature weights is given. Some numerical results and comparisons with other spline methods are presented.

Keywords: quadrature rule, spline quasi-interpolant

Mathematics Subject Classification: 65D30, 65D32, 65D07

PACS: 02.60.Jh

1 Introduction

In this paper we study a spline method for the numerical evaluation of Cauchy principal value integrals of the form

$$I(w_{\alpha,\beta}f;\lambda) = \fint_{-1}^{1} w_{\alpha,\beta}(x)\,\frac{f(x)}{x-\lambda}dx, \qquad \lambda \in (-1,1) = \overset{\circ}{J}, \tag{1}$$

where

$$\fint_{-1}^{1} = \lim_{\epsilon\to 0}\left\{\int_{-1}^{\lambda-\epsilon} + \int_{\lambda+\epsilon}^{1}\right\},$$

$w_{\alpha,\beta}$ is the Jacobi weight function

$$w_{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}, \quad \alpha,\beta > -1, \tag{2}$$

[1]E-mail: catterina.dagnino@unito.it
[2]Corresponding author. E-mail: vittoria.demichelis@unito.it

and f is assumed to be Hölder continuous in $J = [-1, 1]$, that is:

$$f(x) \in \mathbb{H}_\rho(J) = \{g : \omega(g; t; J) \le Bt^\rho,\ \ 0 < \rho \le 1,\ \ B > 0\},$$

where, for any $g \in C(J)$, $\omega(g; t; J)$ denotes the modulus of continuity of g on the interval J [1]

$$\omega(g; t; J) = \max_{\substack{x, x+h \in J \\ 0 \le h \le t}} |g(x+h) - g(x)|.$$

Denoting by $\{f_n\}$ a sequence of approximations to f, the following conditions [13]:

$$\|e_n\|_\infty = o(1), \quad \text{where} \quad e_n = f - f_n \quad \text{and} \quad \|g\|_\infty = \max_{x \in J} |g(x)|, \tag{3}$$

$$e_n(1) = 0 \quad \text{if} \quad \alpha \le 0, \quad e_n(-1) = 0 \quad \text{if} \quad \beta \le 0, \tag{4}$$

$$e_n \in \mathbb{H}_\sigma(J), \quad 0 < \sigma \le \rho, \quad \text{uniformly in} \quad n, \tag{5}$$

ensure that

$$I(w_{\alpha,\beta} e_n; \lambda) \to 0 \quad \text{as} \quad n \to \infty, \quad \text{uniformly in} \quad \lambda \in \overset{\circ}{J}, \tag{6}$$

if

$$\sigma + \min(\alpha, \beta) > 0. \tag{7}$$

The uniform convergence of quadratures for (1) is an important property in several applications, for instance, in Nyström methods for Cauchy singular integral equations [14].

In [3] quadrature formulas for (1) are obtained by using the spline approximation operator introduced in [11]. These splines approximate f with an order of accuracy comparable to the best one and the approximation error bounds can be made independent of any mesh ratio. However, in general, they do not satisfy the condition (4). In order to overcome this drawback, a modified spline operator is proposed in [6], but the error bounds are worse compared with the corresponding ones in [11]. A different approach is based on the technique of subtracting out the singularity [15, 17].

Quadrature rules based on optimal nodal splines are proposed in [4]. Such splines have the three properties of locality, interpolation at a subset of the knots and optimal polynomials reproduction [8] and satisfy conditions (3) - (5) [5]. However, they must have simple interior knots, so that they could not accommodate functions with discontinuities in low-order derivatives at given interior points. Moreover, their definition requires a certain computational complexity.

In this paper, we propose quadrature rules for (1) based on the approximation to f by the quadratic quasi-interpolant (QI) spline operator introduced in [16]. As all QI operators, it has a simple form and the advantage of being generated without solving any linear system of equations [1], unlike interpolants. Our QI splines approximate f with the best approximation order, error bounds are independent of any mesh ratio and multiple interior knots can be used.

We show that conditions (3) - (5) are true for any sequence of such splines based on locally uniform meshes and, consequently, (6) holds. Moreover, we provide a bound for $|I(w_{\alpha,\beta} e_n; \lambda)|$ uniformly with respect to $\lambda \in \overset{\circ}{J}$.

Finally, we give a computational scheme for the weights of the proposed quadrature and we present some numerical results and comparisons with spline quadratures studied in [3, 17, 12].

2 Quadratic spline quasi-interpolants

In this section, we give the necessary background material on quadratic spline QIs based on the work in [16]. We denote by $\mathcal{S}_2(X_n)$ the space of polynomial splines of degree 2 defined on the set of knots $X_n = \{-1 = x_{-2} = x_{-1} = x_0 < x_1 \le \ldots \le x_{n-1} < x_n = x_{n+1} = x_{n+2} = 1\}$ derived from a partition $\Pi_N = \{-1 = t_0 < t_1 < \ldots < t_N = 1\}$ and a set of positive integers $d_i < 3,\ 1 \le i \le N-1$, where $n = 1+\sum_{i=1}^{N-1} d_i$ and d_i denotes the multiplicity of t_i. The condition $d_i < 3,\ 1 \le i \le N-1$, ensures that the splines in $\mathcal{S}_2(X_n)$ are continuous on J. We define

$$r_N = \max_{\substack{1\le i,j\le N\\ |i-j|=1}} \frac{t_i - t_{i-1}}{t_j - t_{j-1}}$$

and we say that the sequence of partitions $\{\Pi_N\}$ is locally uniform if the ratio between the length of two adjacent subintervals of Π_N is uniformly bounded, i.e. if there exists a constant $A \ge 1$ such that $r_N \le A$ for all N. We shall say that the sequence of spline spaces $\{\mathcal{S}_2(X_n)\}$ is locally uniform if the sequence of underlying partitions $\{\Pi_N\}$ is locally uniform.

Let $\Gamma_n = \{0, 1, \ldots, n+1\}$. A basis for $\mathcal{S}_2(X_n)$ is formed by the family of quadratic B-splines $\mathcal{B} = \{B_i,\ i \in \Gamma_n\}$, where $\mathcal{B}$ is defined on X_n and each $B_i \in \mathcal{B}$ has support $[x_{i-2}, x_{i+1}]$ [1]. We consider the set of evaluation points

$$\Theta_n = \left\{\theta_i = \frac{1}{2}(x_{i-1} + x_i),\ \ i \in \Gamma_n\right\}$$

and the following discrete QI operator S_n introduced in [16]

$$S_n f = f(x_0)B_0 + \sum_{i=1}^{n} \mu_i(f)B_i + f(x_{n+1})B_{n+1}, \tag{8}$$

whose coefficient functionals are of the form

$$\mu_i(f) = a_i f(\theta_{i-1}) + b_i f(\theta_i) + c_i f(\theta_{i+1}), \quad 1 \le i \le n.$$

Setting $h_i = x_i - x_{i-1}$, $1 \le i \le n$ and using the following notations:

$$\sigma_i = \frac{h_i}{h_{i-1} + h_i}, \qquad \sigma_i' = \frac{h_{i-1}}{h_{i-1} + h_i},$$

with $h_0 = h_{n+1} = 0$, by imposing $S_n f = f$ for all $f \in \mathcal{P}_2$, where $\mathcal{P}_2$ is the space of quadratic polynomials, we can obtain the following expression for the coefficients $a_i, b_i, c_i,\ 1 \le i \le n$, [16]

$$a_i = -\frac{\sigma_i^2 \sigma_{i+1}'}{\sigma_i + \sigma_{i+1}'}, \quad b_i = 1 + \sigma_i \sigma_{i+1}', \quad c_i = -\frac{\sigma_i \left(\sigma_{i+1}'\right)^2}{\sigma_i + \sigma_{i+1}'}.$$

Therefore, the operator S_n acts as the identity on the space $\mathcal{P}_2$.

Moreover, $S_n f$ satisfies the interpolation conditions:

$$S_n f(-1) = f(-1), \quad S_n f(1) = f(1), \tag{9}$$

the first one follows from (8), since $B_0(-1) = 1$ and $B_i(-1) = 0$, $1 \le i \le n+1$. Similarly, the interpolation condition at $x = 1$ is true.

Defining the fundamental functions of $\mathcal{S}_2(X_n)$ by

$$\tilde{B}_i(x) = c_{i-1}B_{i-1}(x) + b_i B_i(x) + a_{i+1}B_{i+1}(x), \quad i \in \Gamma_n, \tag{10}$$

with the convention:

$$c_{-1} = c_0 = a_{n+1} = a_{n+2} = 0, \quad b_0 = b_{n+1} = 1, \tag{11}$$

we can express $S_n f$ in the form

$$S_n f = \sum_{i\in\Gamma_n} f(\theta_i)\, \tilde{B}_i. \tag{12}$$

The following uniform bounds hold for f bounded on J [16]:

$$\|S_n\|_\infty \leq \begin{cases} 2.5, & \text{for any partition } \Pi_N \text{ of } J, \\ 1.66, & \text{for uniform partition } \Pi_N \text{ of } J, \end{cases} \tag{13}$$

where $\|S_n\|_\infty$ is the infinity norm of S_n

$$\|S_n\|_\infty = \max_{x\in J} \sum_{i\in\Gamma_n} \left|\tilde{B}_i(x)\right|.$$

3 Approximation error bounds

Approximation error bounds, for $f \in \mathbb{W}^{3,\infty}(J) = \left\{g : g^{(3)} \in \mathbb{L}_\infty(J)\right\}$, are provided in [16]. In order to prove conditions (3) and (5), we study here approximation error bounds for $f \in C(J)$.

We define the two integer functions:

$$p_i = \max\{0, i-2\}, \quad q_i = \min\{n+1, i+2\}, \quad i \in \Gamma_n.$$

From the definition (10) of $\tilde{B}_i$ and taking in account that each B_i has support $[x_{i-2}, x_{i+1}]$, we have

$$\tilde{B}_i(x) = 0, \quad x \notin \tilde{J}_i = [x_{p_i-1}, x_{q_i}].$$

Hence, for $x \in [x_{i-1}, x_i]$, we can express $S_n f(x)$ in the form

$$S_n f(x) = \sum_{j=p_i}^{q_i} f(\theta_j)\, \tilde{B}_j(x). \tag{14}$$

We define:

$$H_{i,n} = \max_{p_i\leq j\leq q_i} h_j, \quad \delta_{i,n} = \min_{\substack{p_i\leq j\leq q_i \\ x_{j-1}\neq x_j}} h_j, \quad \tilde{H}_n = \max_{1\leq j\leq n} h_j$$

and we shall assume that

$$\tilde{H}_n \to 0 \quad \text{as} \quad n \to \infty. \tag{15}$$

We state the following lemma proved in [11]

Lemma 1 *Suppose* $x \in [x_{j-1}, x_j)$, *with* $i-1 \leq j \leq i+1$. *If* $x = x_{j-1}$ *suppose also that* x_{j-1} *is of multiplicity at most* 1. *Then,* $B_i'(x)$ *exists and*

$$|B_i'(x)| \leq \frac{2}{\hat{\delta}_{i,2}},$$

where $\hat{\delta}_{i,2}$ *is the minimum of* $x_{\nu+2} - x_\nu$ *over* ν *such that* $x_{i-2} \leq x_\nu \leq x < x_{\nu+2} \leq x_{i+1}$.

The following lemma provides local error estimates.

Lemma 2 *If* $f \in C(\tilde{J}_i)$ *then*

$$\max_{x_{i-1}\leq x\leq x_i} |f(x) - S_n f(x)| \leq k_1 \omega(f; H_{i,n}; \tilde{J}_i), \tag{16}$$

where:

$$k_1 = \begin{cases} 7.5, & \text{for any partition } \Pi_N \text{ of } J, \\ 4.98, & \text{for uniform partition } \Pi_N \text{ of } J. \end{cases} \tag{17}$$

Moreover, if $\{\Pi_N\}$ *is locally uniform with constant* A *then*

$$\max_{x_{i-1}\le x<x_i} |D(S_n f)(x)| \le k_2 H_{i,n}^{-1}\omega(f;H_{i,n};\tilde{J}_i), \tag{18}$$

where D *denotes the derivative operator and* $0<k_2<\infty$.

Proof. The operator S_n is exact on $\mathcal{P}_0$, so it holds $\sum_{j=p_i}^{q_i}\tilde{B}_j(x)=1$. By using the inequality

$$|x-\theta_j|\le 3H_{i,n} \tag{19}$$

and the subadditivity of modulus of continuity, we write

$$|f(x)-S_nf(x)|\le\sum_{j=p_i}^{q_i}\left|(f(x)-f(\theta_j))\tilde{B}_j(x)\right|\le 3\omega(f;H_{i,n};\tilde{J}_i)\sum_{j=p_i}^{q_i}\left|\tilde{B}_j(x)\right|.$$

The thesis (16) follows from (13).

To prove (18), we set $R(x)=f(x)-p_0(x)$, where $p_0(x)\in\mathcal{P}_0$ is any polynomial of degree 0, then

$$D(S_nR)=D(S_nf)-Dp_0=D(S_nf).$$

By setting $p_0(x)=f(t)$, $t\in[x_{i-1},x_i)$, using (14), (19) and (10), we write

$$|D(S_nf)(x)|\le\sum_{j=p_i}^{q_i}\left|(f(\theta_j)-f(t))\tilde{B}'_j(x)\right|\le 3\omega(f;H_{i,n};\tilde{J}_i)\sum_{j=p_i}^{q_i}\left|\tilde{B}'_j(x)\right|$$

$$\le 3\omega(f;H_{i,n};\tilde{J}_i)\sum_{j=i-1}^{i+1}(|a_j|+|b_j|+|c_j|)\left|B'_j(x)\right|\le 27\omega(f;H_{i,n};\tilde{J}_i)\frac{2}{\delta_{i,n}},$$

where the last inequality follows from Lemma 1, taking in account that $\delta_{i,n}\le\hat{\delta}_{j,2}$, $j=i-1,i,i+1$, and from [16]

$$|a_j|\le\frac{1}{2},\quad |b_j|\le 2,\quad |c_j|\le\frac{1}{2},\quad j\in\Gamma_n.$$

Since, for $i\in\Gamma_n$, $q_i-(p_i-1)\le 5$ and, from the local uniformity of Π_N, $\delta_{i,n}\ge A^{-4}H_{i,n}$, the thesis (18) is true. ■

The local error estimate (16) leads immediately to the following global result

Corollary 1 *Let* $f\in C(J)$ *then*

$$\|f-S_nf\|_\infty\le k_1\omega(f;\tilde{H}_n;J), \tag{20}$$

where k_1 *is given by (17).*

Proof. The assertion (20) follows immediately from (16). ■

Moreover, for $f\in\mathbb{H}_\rho(J)$ the associated error sequence is $\mathcal{O}(\tilde{H}_n^\rho)$.

For the considered spline space $\mathcal{S}_2(X_n)$, we can state the following theorem, proved in [14]:

Theorem 1 *Let* $f\in C(J)$ *and consider any sequence of locally uniform spline spaces* $\{\mathcal{S}_2(X_n)\}$. *If any spline* $S\in\mathcal{S}_2(X_n)$ *satisfies:*

1. $S\in C(J)$
2. $|f(x)-S(x)|\le k_1\omega(f;H_{i,n};\tilde{J}_i),\quad x_{i-1}\le x\le x_i$
3. $|DS(x)|\le k_2H_{i,n}^{-1}\omega(f;H_{i,n};\tilde{J}_i),\quad x_{i-1}<x<x_i.$

Then

$$\omega(S;H;J)\le k_3\omega(f;H;J).$$

4 The quadrature rule

We replace f by S_nf, defined by (12), in (1) and approximate $I(w_{\alpha,\beta}f;\lambda)$ by the quadrature sum

$$I(w_{\alpha,\beta}S_nf;\lambda)=\sum_{i\in\Gamma_n} f(\theta_i)I(w_{\alpha,\beta}\tilde{B}_i;\lambda). \tag{21}$$

Setting $w_i=I(w_{\alpha,\beta}\tilde{B}_i;\lambda)$, using (10) and (11), we can write:

$$w_i=\begin{cases} I(w_{\alpha,\beta}B_0;\lambda)+a_1I(w_{\alpha,\beta}B_1;\lambda), & i=0\\ c_{i-1}I(w_{\alpha,\beta}B_{i-1};\lambda)+b_iI(w_{\alpha,\beta}B_i;\lambda)+a_{i+1}I(w_{\alpha,\beta}B_{i+1};\lambda), & 1\le i\le n\\ c_nI(w_{\alpha,\beta}B_n;\lambda)+I(w_{\alpha,\beta}B_{n+1};\lambda), & i=n+1. \end{cases}$$

In order to compute $I(w_{\alpha,\beta}B_i;\lambda),\ \ i\in\Gamma_n$, we can consider all quadratic polynomials $p_{i,j}(x),\ \ j=i-1,i,i+1$, such that

$$p_{i,j}(x)=B_i(x),\quad x\in[x_{j-1},x_j].$$

Setting $p_{i,j}(x)=\bar{a}_1x^2+\bar{a}_2x+\bar{a}_3$, we evaluate $p_{i,j}(x)/(x-\lambda)$ using

$$p_{i,j}(x)=[\bar{a}_1x+(\bar{a}_2+\bar{a}_1\lambda)](x-\lambda)+\bar{a}_3+\bar{a}_2\lambda+\bar{a}_1\lambda^2.$$

The evaluation of $I(w_{\alpha,\beta}B_i;\lambda),\ \ i\in\Gamma_n$, is reduced to the computation of the following integrals, for $j=i-1,i,i+1$,

$$⨍_{x_{j-1}}^{x_j} w_{\alpha,\beta}(x)\frac{p_{i,j}(x)}{x-\lambda}dx=\bar{b}_1\int_{x_{j-1}}^{x_j}w_{\alpha,\beta}(x)xdx+\bar{b}_2\int_{x_{j-1}}^{x_j}w_{\alpha,\beta}(x)dx+\bar{b}_3\tilde{I}_j(w_{\alpha,\beta};\lambda), \tag{22}$$

where, assuming that $\lambda\neq x_{j-1},x_j$,

$$\tilde{I}_j(w_{\alpha,\beta};\lambda)=\begin{cases} ⨍_{x_{j-1}}^{x_j}\frac{w_{\alpha,\beta}(x)}{x-\lambda}dx, & \lambda\in(x_{j-1},x_j)\\ \int_{x_{j-1}}^{x_j}\frac{w_{\alpha,\beta}(x)}{x-\lambda}dx, & \lambda\notin[x_{j-1},x_j] \end{cases}$$

and:

$$\begin{aligned}\bar{b}_1&=\bar{a}_1,\\ \bar{b}_2&=\bar{a}_2+\bar{b}_1,\lambda\\ \bar{b}_3&=\bar{a}_3+\bar{b}_2\lambda.\end{aligned}$$

For certain values of α,β, for example $\alpha=\beta=-1/2$, $\alpha=\beta=1/2$ and $\alpha=\beta=0$, the integrals in (22) can be evaluated exactly, otherwise a numerical method has to be used [10].

Now, we consider the quadrature error

$$E_n^{(S)}(\omega_{\alpha,\beta}f;\lambda)=I(\omega_{\alpha,\beta}e_n;\lambda),\quad e_n=f-S_nf. \tag{23}$$

Finally, we prove the following uniform convergence theorem which also gives a bound for the quadrature error.

Theorem 2 *Assume that (15) holds and $f \in \mathbb{H}_\rho(J)$ for a given order $\rho \in (0,1]$. Suppose that $\{S_n f\}$ is based on a sequence of locally uniform partitions $\{\Pi_N\}$ and $\rho + \gamma > 0$, where $\gamma = \min(\alpha, \beta)$. Then*

$$I(w_{\alpha,\beta} e_n; \lambda) \to 0 \quad \text{as} \quad n \to \infty, \quad \text{uniformly in} \quad \lambda \in \overset{\circ}{J} . \tag{24}$$

Moreover,

$$|I(w_{\alpha,\beta} e_n; \lambda)| = \begin{cases} \mathcal{O}(\tilde{H}_n^\rho |\log \tilde{H}_n|) , & \gamma \geq 0 \\ \mathcal{O}(\tilde{H}_n^{\rho+\gamma}), & \gamma < 0 , \end{cases} \tag{25}$$

where the $\mathcal{O}-$term holds uniformly with respect to $\lambda \in \overset{\circ}{J}$.

Proof. To prove (24), we verify that conditions (3) - (5) are true. The condition (3) follows from (20) since $f \in \mathbb{H}_\rho(J)$ and (15) holds. The condition (4) follows from (9). Finally, the condition (5), with $\sigma = \rho$, follows from (16), (18) and Theorem 1 since, for any $u, v \in J$,

$$|e_n(u) - e_n(v)| \leq |f(u) - f(v)| + |S_n f(u) - S_n f(v)| .$$

The three sufficient conditions, ensuring that (25) holds, are provided in [7]. Two of them are (4) and (5) with $\sigma = \rho$, respectively. The third one is

$$\|e_n\|_\infty \leq c\tilde{H}_n^\rho, \quad 0 < c < \infty. \tag{26}$$

The assertion (26) follows from (20), since $f \in \mathbb{H}_\rho(J)$. ■

5 Comparison of methods and numerical applications

Numerical evaluation of (1) has several practical applications, for instance in the solution of singular integral equations arising in aerodynamics or fluid mechanics [9, 10]. For the numerical solution of these equations it is very important to have a sequence of quadratures satisfying (6).

The majority of numerical methods proposed for (1) are global methods based on orthogonal polynomials. Even if such methods converge very fast for differentiable functions, in some practical applications one cannot always place the nodes of quadrature at the zeros of the orthogonal polynomials [10].

Recently, local methods for (1) have been introduced, mainly based on spline interpolation, see for instance [2, 9] or on spline quasi-interpolation [3, 12, 17], with simple knots inside the integration interval.

In [3, 12] the authors have proposed local methods based on quasi-interpolating splines that place no restriction on the order of the spline and a few restriction on the spacing of the knots. The sequence of rules proposed in [12] satisfies (6) for $f \in \mathbb{H}_\rho(J)$ but is exact only for $f \in \mathcal{P}_1$. Whereas, the sequence of rules proposed in [3] is exact for $f \in \mathcal{P}_m$, $m \geq 2$, but (6) holds only for $f \in C^1(J)$.

In [17] the author has proposed rules for (1) obtained by subtracting out the singularity and then applying quadrature formulas based on quasi-interpolating splines [11]. For such rules, uniform convergence properties have been proved for $f \in C^1(J)$.

Tables 1, 2 and 3 present some numerical results obtained when our rule and other known ones [3, 12, 17], based on quadratic approximating splines with the same set of knots $\{X_n\}$, are applied to the integral (1) for several functions f and different values of α, β, λ and n.

In Tables 4 and 5 we test our rule (21) in case of integrands f with a singularity in the first derivative at $x = 0$. We use both a sequence of knots $\{X_n\}$ with simple interior knots and the sequence obtained from X_n by inserting a double knot at $x = 0$.

We choose two different partitions Π_N with the norm $\tilde{H}_n$ of the corresponding X_n satisfying (15), i. e.:

$$U \,:\, \Pi_N = \{t_i = -1 + \frac{2i}{N}, \quad i = 0, 1, \ldots, N\},$$

$$P \,:\, \Pi_N = \{t_i = \cos\left(\frac{N-i}{N}\pi\right), \quad i = 0, 1, \ldots, N\}.$$

We recall that, given any initial partition Π_N, there exists an associated 4-quasi uniform partition [18].

We denote by:

$E_{(n)}^{(S)}$ the truncation error (23) of our rule (21), based on X_n with simple interior knots,

$E_{(n+r)}^{(D)}$ the truncation error (23) of the rule (21) based on the knot sequence obtained by inserting r interior double knots in X_n as the case may be,

$E_{(n)}^{(R)}$ the truncation error of the rule proposed in [12],

$E_{(n)}^{(QI)}$ the truncation error of the rule proposed in [3],

$E_{(n)}^{(SS)}$ the truncation error of the rule proposed in [17].

In case of smooth integrands f, for which the spline space is defined by using simple interior knots, we can present some comparisons with other known spline quadratures. The quadrature errors in Tables 1, 2 and 3, with smooth integrands f, show that our quadrature (21) performs better than the quadrature defined in [12], is comparable and sometimes better than that one introduced in [3]. Whereas, the rule [17], obtained by subtracting out the singularity and then applying spline quasi-interpolants, seems to perform better.

In Tables 4 and 5 we apply the rule (21) to integrands f having a singularity in the first derivative. The quadrature errors show that the rule with a double knot at the singular point of $f'(x)$ performs better than that one with simple interior knots.

Table 1: $f(x) = (x^2 + y^2)^{-1}$, $y = 5$, $\alpha = \beta = -\frac{1}{2}$, $\Pi_N = P$

λ	$I(\omega_{\alpha,\beta} f; \lambda)$	n	$\lvert E_n^{(R)}\rvert$	$\lvert E_n^{(QI)}\rvert$	$\lvert E_n^{(SS)}\rvert$	$\lvert E_n^{(S)}\rvert$
0.25	-0.0012291611	8	3.1(-5)	1.9(-6)	1.7(-8)	1.5(-6)
		16	8.1(-6)	2.0(-7)	1.3(-9)	1.4(-7)
		32	1.6(-6)	1.3(-8)	5.1(-10)	1.3(-8)
		64	4.6(-7)	3.8(-9)	2.4(-11)	5.9(-11)
0.99	-0.0046955619	8	1.3(-4)	8.4(-6)	6.6(-8)	1.5(-6)
		16	4.2(-5)	8.8(-7)	5.0(-9)	1.4(-7)
		32	1.7(-5)	1.8(-7)	7.8(-10)	9.4(-9)
		64	3.4(-6)	3.3(-9)	3.9(-11)	6.8(-10)

6 Final remarks

A new quadrature rule for the CPV integral (1) is defined by integrating a quadratic spline quasi-interpolant presented in [16]. We prove uniform convergence and provide uniform error bounds for sequences of such quadratures for integrands $f \in \mathbb{H}_\rho(J)$, $0 < \rho \leq 1$. The proposed quadrature rule includes the possibility of inserting multiple spline knots at singular points of $f'(x)$. Comparisons with other spline methods using simple interior knots are given.

Applications to numerical solution of singular integral equations are in progress.

Table 2: $f(x) = (x^2 + y^2)^{-1},\ y = 0.1,\ \alpha = \beta = -\frac{1}{2},\ \Pi_N = P$

λ	$I(\omega_{\alpha,\beta} f; \lambda)$	n	$\lvert E_n^{(R)}\rvert$	$\lvert E_n^{(QI)}\rvert$	$\lvert E_n^{(SS)}\rvert$	$\lvert E_n^{(S)}\rvert$
		8	5.6(1)	2.0(1)	3.0(1)	7.0(1)
0.25	-107.79315611	32	8.5(-1)	3.9(-1)	5.3(-2)	6.8(-1)
		64	2.4(-1)	2.7(-2)	2.6(-4)	6.8(-4)
		128	5.8(-2)	5.8(-4)	1.5(-5)	2.0(-3)
		8	6.9(0)	1.0(1)	8.9(0)	1.0(1)
0.99	-31.256858013	32	3.8(-2)	2.2(-2)	1.5(-2)	1.0(-1)
		64	1.1(-2)	1.8(-4)	7.7(-5)	2.1(-4)
		128	2.5(-3)	1.0(-4)	4.5(-6)	1.9(-6)

Table 3: $f(x) = \mathrm{e}^x,\ \alpha = \beta = 0,\ \Pi_N = P$

λ	$I(\omega_{\alpha,\beta} f; \lambda)$	n	$\lvert E_n^{(R)}\rvert$	$\lvert E_n^{(QI)}\rvert$	$\lvert E_n^{(SS)}\rvert$	$\lvert E_n^{(S)}\rvert$
		8	1.2(-2)	5.7(-3)	1.6(-4)	2.7(-3)
0.1	1.99903605021	16	3.3(-3)	5.1(-4)	1.1(-5)	3.4(-4)
		32	8.3(-4)	1.5(-5)	3.1(-7)	2.1(-5)
		8	1.2(-2)	2.6(-3)	1.7(-4)	2.7(-3)
0.5	0.91378643172	16	3.3(-3)	3.0(-4)	1.2(-5)	2.6(-4)
		32	6.0(-4)	4.7(-6)	1.1(-6)	2.3(-5)
		8	4.5(-2)	5.5(-3)	1.8(-4)	1.8(-3)
0.9	-3.85323498264	16	8.1(-3)	6.7(-5)	1.3(-5)	1.0(-4)
		32	7.1(-4)	1.3(-4)	3.6(-7)	4.7(-6)

Table 4: $f(x) = x^4 + |x|,\ \alpha = \beta = -\frac{1}{2},\ \Pi_N = U$

λ	$I(\omega_{\alpha,\beta} f; \lambda)$	$n = 16$		$n = 32$		$n = 64$	
		$\lvert E_n^{(S)}\rvert$	$\lvert E_{n+1}^{(D)}\rvert$	$\lvert E_n^{(S)}\rvert$	$\lvert E_{n+1}^{(D)}\rvert$	$\lvert E_n^{(S)}\rvert$	$\lvert E_{n+1}^{(D)}\rvert$
0.01	0.12168225086258	4.7(-2)	3.1(-4)	3.3(-2)	8.3(-6)	2.0(-2)	5.3(-7)
0.1	0.76188165530404	3.7(-2)	1.3(-4)	4.7(-5)	1.4(-5)	7.1(-4)	3.4(-7)
0.2	1.27517331669127	2.6(-4)	2.9(-4)	1.4(-3)	9.3(-6)	3.9(-4)	4.0(-6)
0.4	2.19699495620963	2.5(-3)	3.0(-4)	7.1(-4)	7.8(-5)	2.0(-4)	3.0(-7)
0.6	3.26898024225450	1.6(-3)	4.4(-4)	4.0(-4)	1.3(-4)	1.3(-4)	1.9(-6)
0.8	4.71352498156708	2.2(-3)	3.7(-3)	3.1(-4)	9.3(-5)	7.2(-5)	2.9(-5)
0.9	5.63300644487096	1.1(-3)	2.5(-3)	3.2(-4)	6.8(-4)	7.3(-5)	1.7(-5)
0.99	6.59666565782882	8.0(-3)	6.7(-3)	9.4(-4)	6.1(-4)	3.8(-5)	4.4(-5)

Acknowledgement The authors are grateful to the reviewers for their constructive suggestions and comments.

References

[1] C. de Boor: *A Practical Guide to Splines*. Springer Verlag, New York (2001).

[2] C. Dagnino and E. Santi, On the evaluation of one dimensional Cauchy principal value integrals by rules based on cubic spline interpolation, *Computing* **43** 267–276 (1990).

Table 5: $f(x) = \sqrt{|x|},\ \alpha = \beta = 0,\ \Pi_N = U$

λ	$I(\omega_{\alpha,\beta} f; \lambda)$	$n = 8$		$n = 16$		$n = 32$	
		$\lvert E_n^{(S)}\rvert$	$\lvert E_{n+1}^{(D)}\rvert$	$\lvert E_n^{(S)}\rvert$	$\lvert E_{n+1}^{(D)}\rvert$	$\lvert E_n^{(S)}\rvert$	$\lvert E_{n+1}^{(D)}\rvert$
0.01	0.27415846531453	2.5(-1)	3.9(-2)	2.2(-1)	7.0(-3)	1.8(-1)	4.5(-2)
0.1	0.59265435112872	3.4(-1)	1.9(-2)	1.1(-1)	7.0(-2)	4.8(-3)	4.3(-3)
0.2	0.59841666101709	1.6(-1)	1.0(-1)	6.9(-3)	6.1(-3)	9.1(-3)	6.2(-4)
0.4	0.33059150273724	9.7(-3)	8.6(-3)	1.3(-2)	8.8(-4)	4.7(-3)	7.1(-4)
0.6	-0.18587259394768	2.2(-2)	3.9(-5)	8.8(-3)	1.1(-3)	3.1(-3)	5.2(-4)
0.8	-1.07790426887581	1.8(-2)	1.2(-3)	6.6(-3)	1.0(-3)	2.4(-3)	3.9(-4)
0.9	-1.91011754796427	1.6(-2)	1.4(-3)	5.9(-3)	9.0(-4)	2.1(-3)	3.6(-4)
0.99	-4.38851568738530	1.5(-2)	1.2(-3)	5.5(-3)	8.0(-4)	1.9(-3)	3.2(-4)

[3] C. Dagnino, V. Demichelis and E. Santi, Numerical integration based on quasi-interpolating splines, *Computing* **50** 149–163 (1993).

[4] C. Dagnino and E. Santi, Numerical evaluation of Cauchy principal value integrals by means of nodal spline approximation, *Revue d'Anal. Num. Theor. Approx.* **27** 59–69 (1998).

[5] V. Demichelis, Convergence of derivatives of optimal nodal splines, *J. Approx. Theory* **88** 370–383 (1997).

[6] V. Demichelis, Spline approximation with interpolation constraints for numerical integration, *Intern. J. Computer Math.* **70** 739–750 (1999).

[7] V. Demichelis, Uniform error bounds for Cauchy principal value integrals, *Rend. Sem. Mat. Univ. Pol. Torino* **62, 2** 155–164 (2004).

[8] J.M. De Villiers and C.H. Rohwer, Optimal local spline interpolants, *J. Comp. Appl. Math.* **18** 107–119 (1987).

[9] K. Diethelm, Uniform convergence of optimal order quadrature rules for Cauchy principal value integrals,*J. Comp. Appl. Math.* **56** 321–329 (1994).

[10] A. Gerasoulis, Piecewise-polynomial quadratures for Cauchy singular integrals, *SIAM J. Numer. Anal.* **23** 891–902 (1986).

[11] T. Lyche and L. L. Schumaker, Local spline approximation methods, *J. Approx. Theory* **15** 294–325 (1975).

[12] P. Rabinowitz, Numerical integration based on approximating splines, *J. Comput. Appl. Math.* **33** 73–83 (1990).

[13] P. Rabinowitz, Uniform convergence results for Cauchy principal value integrals, *Math. Comp.* **56** 731–740 (1991).

[14] P. Rabinowitz, Application of approximating splines for the solution of Cauchy singular integral equations, *Appl. Num. Math.* **15** 285–297 (1994).

[15] P. Rabinowitz and E. Santi, On the uniform convergence of Cauchy principal values of quasi-interpolating splines, *BIT* **35** 277–290 (1995).

[16] P. Sablonniére, On Some Multivariate Quadratic Spline Quasi-Interpolants on Bounded Domains, in: *Modern Developments in Multivariate Approximation*, W. Haussmann et al. (Eds), ISNM **145**, Bikhäuser Verlag, Basel 263–278 (2003).

[17] E. Santi, On the evaluation of Cauchy principal value integrals by rules based on quasi-interpolating splines, *J. Comp. Appl. Math.* **71** 1–14 (1996).

[18] L. L. Schumaker: *Spline functions: basic theory.* John Wiley & Sons, New York (1981).